Homogeneous systems

Linear systems of the form $A\mathbf{x} = \mathbf{0}$ are homogeneous.
Linear systems of the form $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} \neq \mathbf{0}$, are inhomogeneous.

Here the **trivial**/**nontrivial** refers to $\mathbf{x}$.

If $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ it is **trivial** solution. Otherwise it is **nontrivial**

Observations

$$
\begin{aligned}
&A\vec{x} = \vec{0} \text{ has a nontrivial solution} \\
&\quad \iff \text{there is a free variable} \\
&\quad \iff A \text{ has a column with no pivot.}
\end{aligned}
$$

Example

Identify the free variables, and the solution set, of the system.

$$
\begin{aligned}
x_1 + 3x_2 + x_3 &= 0 \\
2x_1 - x_2 - 5x_3 &= 0 \\
x_1 - 2x_3 &= 0
\end{aligned}
$$

In Echelon Form:

$$
\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

$$
\begin{aligned}
x_1 + -2x_3 &= 0 \\
x_2 + x_3 &= 0 \\
0 &= 0
\end{aligned}
$$

$$x_1 = 2x_3$$
$$x_2 = -x_3$$
$$x_3 = x_3$$

$$\text{Solution set} = \mathbf{x} = \begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$