

Matrix Vector Product

#matrix_multiplication

Symbol	Meaning
\in	belongs to
\mathbb{R}^n	the set of vectors with n real-valued elements
$\mathbb{R}^{m \times n}$	the set of real-valued matrices with m rows and n columns

Definition of Matrix Vector Multiplication

If $A \in \mathbb{R}^{m \times n}$ has vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ and $\mathbf{x} \in \mathbb{R}^n$, then the matrix Vector Product $A\mathbf{x}$ is a linear combination of the columns of A . It can be shown as shown:

$$A\mathbf{x} = \begin{bmatrix} | & | & \dots & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \\ | & | & \dots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \begin{bmatrix} \vdots \\ x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n \\ \vdots \end{bmatrix}$$

NOTE: $A\mathbf{x}$ is always in the span of A

Existence of Solutions

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if \mathbf{b} is a linear combination of the columns of A .

Example

For what vectors $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ does the equation have a solution?

$$\begin{bmatrix} 1 & 3 & 4 & | & b_1 \\ 2 & 8 & 4 & | & b_2 \\ 0 & 1 & -2 & | & b_3 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 3 & 4 & | & b_1 \\ 0 & 2 & -4 & | & b_2 - 2b_1 \\ 0 & 0 & 0 & | & b_3 - \frac{1}{2}b_2 + b_1 \end{bmatrix}$$

Solving for b_1 from the last row,

$$b_1 = \frac{1}{2} - b_3$$

$$\mathbf{b} = \begin{bmatrix} \frac{1}{2} - b_3 \\ b_2 \\ b_3 \end{bmatrix}$$