

# Formulas-1

Credit Aarush Magic

$$\begin{aligned}\vec{a} \cdot \vec{b} &= a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n \\ \vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \\ &= (a_2b_3 - b_2a_3)\vec{i} - (a_1b_3 + b_1a_3)\vec{j} + (a_1b_2 + b_1a_2)\vec{k}\end{aligned}$$

## Properties:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos(\theta)$$

$$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \sin(\theta)$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$\vec{a} \cdot \vec{0} = 0$$

$$\vec{a} \times \vec{0} = \vec{0}$$

$$(c\vec{a}) \cdot \vec{b} = \vec{a} \cdot (c\vec{b})$$

$$(c\vec{a}) \times \vec{b} = \vec{a} \times (c\vec{b})$$

$$\vec{a} \cdot \vec{a} = ||\vec{a}||^2$$

$$\vec{a} \times \vec{a} = \vec{0}$$

$$\text{If } \vec{a} \perp \vec{b} \text{ then } \vec{a} \cdot \vec{b} = 0$$

$$\text{If } \vec{a} \parallel \vec{b} \text{ then } \vec{a} \times \vec{b} = \vec{0}$$

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w} \quad \text{nbsp;}$$

## Projectile Motion:

$$\text{Max Height} = \frac{(v_0 \sin(\theta))^2}{2g}$$

$$\text{Range} = \frac{v_0^2 \sin(2\theta)}{g}$$

$$\text{Flight Time} = \frac{2v_0 \sin(\theta)}{g}$$

# Graphs

Type	Equations
Elliptical Paraboloid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$
Elliptical Cone	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Hyperboloid of 1 sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
Hyperboloid of 2 sheets	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Hyperbolic Paraboloid	$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}, c > 0$

## Other Formulas

Line through  $P(p_1, p_2, p_3)$  and parallel to  $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$  when  $t \in \mathbb{R}$ :

$$x = at + p_1 \quad y = bt + p_2 \quad z = ct + p_3$$

$$\langle at + p_1, bt + p_2, ct + p_3 \rangle = \langle a, b, c \rangle t + \langle p_1, p_2, p_3 \rangle$$

Line through  $P(p_1, p_2, p_3)$  and perpendicular to  $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ :

$$a(x - p_1) + b(y - p_2) + c(z - p_3) = 0$$

Distance between line and point:

$$d = \frac{||\vec{PS} \times v||}{||v||}$$

Distance between Point  $S$  and a Plane,

$$d = \left| \vec{PS} \cdot \frac{n}{||n||} \right|$$

Projection,

$$\text{proj}_b a = \left( \frac{a \cdot b}{||b||} \right) \frac{b}{||b||}$$

Angle between planes or vectors:

$$\theta = \cos^{-1} \left( \left| \frac{\vec{n}_1 \cdot \vec{n}_2}{||\vec{n}_1|| \cdot ||\vec{n}_2||} \right| \right)$$

Arc Length ( $s(t)$ ):

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_a^b ||\vec{r}'(t)|| dt$$

$$s(t) = \int_{t_0}^t \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2} d\tau = \int_{t_0}^t ||\vec{r}'(\tau)|| d\tau$$

Speed:

$$\frac{ds}{dt} = ||\vec{v}(t)||$$

The unit tangent vector ( $T(t)$ ):

$$\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||} = \frac{\vec{v}(t)}{||\vec{v}(t)||}$$

The curvature function ( $\kappa(t)$ ):

$$\begin{aligned} \kappa &= \left\| \frac{d\vec{T}}{ds} \right\| \\ &= \frac{||\vec{T}'(t)||}{||\vec{v}(t)||} \\ &= \frac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||^3} \end{aligned}$$

Radius of curvature:

$$p = \frac{1}{\kappa}$$

Principal Normal Vector ( $N(t)$ ):

$$\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||}$$

Binormal vector ( $B(t)$ ):

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

# Formulas-3

Fubini's Theorem

$$V = \iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$$

if  $R$  is defined by  $a \leq x \leq b$  and  $g_1(x) \leq y \leq g_2(x)$  then,

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

Notice: The functions  $g_n(x)$  are evaluated first.

## Properties

$$\begin{aligned} \iint_R c f(x, y) dA &= c \iint_R f(x, y) dA \\ \iint_R f(x, y) \pm g(x, y) dA &= \iint_R f(x, y) dA \pm \iint_R g(x, y) dA \\ \iint_R f(x, y) dA &\geq 0 \text{ if } f(x, y) \geq 0 \text{ on } R \\ \iint_R f(x, y) dA &\geq \iint_R g(x, y) dA \text{ if } f(x, y) \geq g(x, y) \text{ on } R \end{aligned}$$

If  $R_1$  and  $R_2$  are non-overlapping regions,

$$\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA$$

$$A = \iint_R dA = \int_{x_1}^{x_2} \int_{y_1}^{y_2} dy dx = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} r dr d\theta$$

$$V = \iint_R f(x, y) dA$$

$$\text{Average Value} = \frac{1}{\text{Area of } R} \iint_R f dA \quad \text{OR} \quad \frac{1}{\text{Volume of } D} \iiint_D F dV$$

$$\begin{aligned} V &= \iiint_D dV \\ &= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} dz dy dx \\ &= \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} \int_{z_1}^{z_2} r dz dr d\theta \\ &= \int_{\rho_1}^{\rho_2} \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \rho^2 \sin(\phi) d\theta d\phi d\rho \end{aligned}$$

# Mass and First Moments

## 3D Solids

$\delta$  is the density function.

$$\text{Mass: } M = \iiint_D \delta dV$$

$$\text{First Moments: } M_{yz} = \iiint_D x\delta dV, M_{xz} = \iiint_D y\delta dV, M_{xy} = \iiint_D z\delta dV$$

$$\text{Center of mass: } \bar{x} = \frac{M_{yz}}{M}, \bar{y} = \frac{M_{xz}}{M}, \bar{z} = \frac{M_{xy}}{M}$$

## 2D Plates

$$\text{Mass: } M = \iint_R \delta dA$$

$$\text{First Moments: } M_y = \iint_R x\delta dA, M_x = \iint_R y\delta dA$$

$$\text{Center of mass: } \bar{x} = \frac{M_y}{M}, \bar{y} = \frac{M_x}{M}$$

# Moments of Inertia

## 3D Solids

$$\text{About x-axis: } I_x = \iiint_D (y^2 + z^2)\delta dV$$

$$\text{About y-axis: } I_y = \iiint_D (x^2 + z^2)\delta dV$$

$$\text{About z-axis: } I_z = \iiint_D (x^2 + y^2)\delta dV$$

$$\text{About a line L: } I_L = \iiint_D r^2(x, y, z)\delta dV$$

## 2D Plates

$$\text{About x-axis: } I_x = \iint_R y^2\delta dA$$

$$\text{About y-axis: } I_y = \iint_R x^2\delta dA$$

$$\text{About a line L: } I_L = \iint_R r^2(x, y)\delta dA$$

$$\text{About the origin: } I_O = \iint_R (x^2 + y^2)\delta dA = I_x + I_y$$

Joint probability density function:

Conditions:

$$f(x, y) \geq 0$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

$$P((X, Y) \in R) = \iint_R f(x, y) dx dy$$

Mean and expected value:

$$\mu_X = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y) dx dy$$

$$\mu_Y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x, y) dx dy$$

## Spherical Coordinates $(\rho, \phi, \theta)$

$$r = \rho \sin(\phi),$$

$$x = r \cos(\theta) = \rho \sin(\phi) \cos(\theta)$$

$$z = \rho \cos(\phi),$$

$$y = r \sin(\theta) = \rho \sin(\phi) \sin(\theta)$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$

Jacobian:

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x, y)}{\partial(u, v)}$$

$$\iint_R f(x, y) dx dy = \iint_G f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x, y, z)}{\partial(u, v, w)}$$

$$\iiint_D f(x, y, z) dx dy dz = \iiint_B f(g(u, v, w), h(u, v, w), k(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$$

# Formulas-2

$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L$  if for every  $\epsilon > 0$ , there exists a corresponding  $\delta > 0$ , such that for all  $(x,y)$  in the domain of  $f$ ,  $|f(x,y) - L| < \epsilon$  whenever  $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$

$$\left. \frac{\partial f}{\partial x} \right|_{(x_0,y_0)} = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = f_x(x_0, y_0)$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right)$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Directional Derivative of the unit vector  $u = u_1 \mathbf{i} + u_2 \mathbf{j}$

$$\begin{aligned} f'_u(x, y) &= D_u f(x, y) = \lim_{s \rightarrow 0} \frac{f(x + su_1, y + su_2) - f(x, y)}{s} \\ &= \nabla f(x, y) \cdot u \\ &= \|\nabla f\| \cos \theta \end{aligned}$$

The Gradient

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

Tangent line to a level curve of the form  $f(x, y) = 0$  at a point  $(x_0, y_0)$ ,

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$$

$$\frac{d}{dt}(f(r(t))) = \nabla f(r(t)) \cdot r'(t)$$

Tangent plane to  $f(x, y, z) = c$  at the point  $P(x_0, y_0, z_0)$ ,

$$f_x(P)(x - x_0) + f_y(P)(y - y_0) + f_z(P)(z - z_0) = 0$$

Or when  $f(x, y) = z$  at the point  $P(x_0, y_0)$ ,

$$f_x(P)(x - x_0) + f_y(P)(y - y_0) - (z - z_0) = 0$$

Normal line to  $f(x, y, z) = c$  at the point  $P(x_0, y_0, z_0)$ ,

$$x = x_0 + f_x(P)t$$

$$y = y_0 + f_y(P)t$$

$$z = z_0 + f_z(P)t$$

Linear Approximation ( $f(x, y) \approx L(x, y)$ ) of  $f(x, y)$  at  $(x_0, y_0)$ ,

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Total Differential,

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

$$df = (\nabla f(P_0) \cdot u)ds$$

Standard Linear Approximation Error where  $M$  is the upper bound of the second partials on a rectangle centered at point  $P$ ,

$$|E| \leq \frac{1}{2}M(|x - x_0| + |y - y_0|)^2$$

Second Partial Test at  $(x_0, y_0)$  assuming  $\nabla f = 0$ ,

$$A = f_{xx}(x_0, y_0), \quad B = f_{xy}(x_0, y_0), \quad C = f_{yy}(x_0, y_0)$$

$$D = AC - B^2$$

$$D < 0 \text{ Saddle point}$$

$$D > 0 \text{ Relative Extrema}$$

$$D = 0 \text{ Indecisive}$$

For the relative extrema, if  $D > 0$  and  $A > 0$  then you have a local min, if  $A < 0$  you have a local max.

Lagrange Multipliers when  $g(x, y) = 0$ ,

$$\nabla f = \lambda \nabla g$$