Formulas-3

Fubini's Theorem

$$V = \iint_R f(x,y) dA = \int_c^d \int_a^b f(x,y) dx dy = \int_a^b \int_c^d f(x,y) dy dx$$

if R is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$ then,

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) dy dx$$

Notice: The functions $g_n(x)$ are evaluated first.

Properties

$$\iint_R cf(x,y) dA = c \iint_R f(x,y) dA$$

$$\iint_R f(x,y) \pm g(x,y) dA = \iint_R f(x,y) dA \pm \iint_R g(x,y) dA$$

$$\iint_R f(x,y) dA \geq 0 \text{ if } f(x,y) \geq 0 \text{ on } R$$

$$\iint_R f(x,y) dA \geq \iint_R g(x,y) dA \text{ if } f(x,y) \geq g(x,y) \text{ on } R$$

If R_1 and R_2 are non-overlapping regions,

$$\iint_R f(x,y) dA = \iint_{R_1} f(x,y) dA + \iint_{R_2} f(x,y) dA$$

$$A = \iint_R dA = \int_{x_1}^{x_2} \int_{y_1}^{y_2} dy dx = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} r \, dr d\theta$$

$$V = \iint_R f(x,y) dA$$

$$\text{Average Value} = \frac{1}{\text{Area of } R} \iint_R f dA \quad \text{OR} \quad \frac{1}{\text{Volume of } D} \iiint_D F dV$$

$$V = \iiint_D dV$$
$$= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} dz dy dx$$
$$= \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} \int_{z_1}^{z_2} r \, dz dr d\theta$$
$$= \int_{\rho_1}^{\rho_2} \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \rho^2 \sin(\phi) \, d\theta d\phi d\rho$$

Mass and First Moments

3D Solids

δ is the density function.

$$\text{Mass: } M = \iiint_D \delta \, dV$$

$$\text{First Moments: } M_{yz} = \iiint_D x\delta \, dV, \, M_{xz} = \iiint_D y\delta \, dV, \, M_{xy} = \iiint_D z\delta \, dV$$

$$\text{Center of mass: } \bar{x} = \frac{M_{yz}}{M}, \bar{y} = \frac{M_{xz}}{M}, \bar{z} = \frac{M_{xy}}{M}$$

2D Plates

$$\text{Mass: } M = \iint_R \delta \, dA$$

$$\text{First Moments: } M_y = \iint_R x\delta \, dA, \, M_x = \iint_R y\delta \, dA$$

$$\text{Center of mass: } \bar{x} = \frac{M_y}{M}, \bar{y} = \frac{M_x}{M}$$

Moments of Inertia

3D Solids

$$\text{About x-axis: } I_x = \iiint_D (y^2 + z^2)\delta \, dV$$

$$\text{About y-axis: } I_y = \iiint_D (x^2 + z^2)\delta \, dV$$

$$\text{About z-axis: } I_z = \iiint_D (x^2 + y^2)\delta \, dV$$

$$\text{About a line L: } I_L = \iiint_D r^2(x, y, z)\delta \, dV$$

2D Plates

$$\text{About x-axis: } I_x = \iint_R y^2 \delta \, dA$$

$$\text{About y-axis: } I_y = \iint_R x^2 \delta \, dA$$

$$\text{About a line L: } I_L = \iint_R r^2(x, y)\delta \, dA$$

$$\text{About the origin: } I_O = \iint_R (x^2 + y^2)\delta \, dA = I_x + I_y$$

Joint probability density function:

Conditions:

$$f(x, y) \geq 0$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

$$P((X, Y) \in R) = \iint_R f(x, y) dx dy$$

Mean and expected value:

$$\mu_X = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y) dx dy$$

$$\mu_Y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x, y) dx dy$$

Spherical Coordinates (ρ, ϕ, θ)

$$r = \rho \sin(\phi),$$

$$x = r \cos(\theta) = \rho \sin(\phi) \cos(\theta)$$

$$z = \rho \cos(\phi),$$

$$y = r \sin(\theta) = \rho \sin(\phi) \sin(\theta)$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$

Jacobian:

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x, y)}{\partial(u, v)}$$

$$\iint_R f(x, y) dx dy = \iint_G f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x, y, z)}{\partial(u, v, w)}$$

$$\iiint_D f(x, y, z) dx dy dz = \iiint_B f(g(u, v, w), h(u, v, w), k(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$$