Formulas-3

Fubini's Theorem

$$V = \iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

if R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$ then,

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

Notice: The functions $g_n(x)$ are evaluated first.

Properties

$$\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$$
$$\iint_R f(x,y) \pm g(x,y)dA = \iint_R f(x,y)dA \pm \iint_R g(x,y)dA$$
$$\iint_R f(x,y)dA \ge 0 \text{ if } f(x,y) \ge 0 \text{ on } R$$
$$\iint_R f(x,y)dA \ge \iint_R g(x,y)dA \text{ if } f(x,y) \ge g(x,y) \text{ on } R$$

If R_1 and R_2 are non-overlapping regions,

$$\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$$

$$A = \iint_R dA = \int_{x_1}^{x_2} \int_{y_1}^{y_2} dydx = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} r\ drd\theta$$

$$V = \iint_R f(x,y)dA$$

$$\text{Average Value} = \frac{1}{\text{Area of } R} \iint_R fdA \quad \text{OR} \quad \frac{1}{\text{Volume of } D} \iiint_D FdV$$

$$\begin{aligned} V &= \iiint_D dV \\ &= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} dzdydx \\ &= \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} \int_{z_1}^{z_2} r\ dzdrd\theta \\ &= \int_{\rho_1}^{\rho_2} \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \rho^2 \sin(\phi)\ \ d\theta d\phi d\rho \end{aligned}$$

Mass and First Moments

3D Solids

δ is the density function.

$$\text{Mass: } M = \iiint_D \delta \, dV$$
$$\text{First Moments: } M_{yz} = \iiint_D x\delta \, dV, M_{xz} = \iiint_D y\delta \, dV, M_{xy} = \iiint_D z\delta \, dV$$
$$\text{Center of mass: } \bar{x} = \frac{M_{yz}}{M}, \bar{y} = \frac{M_{xz}}{M}, \bar{z} = \frac{M_{xy}}{M}$$

2D Plates

$$\text{Mass: } M = \iint_R \delta \, dA$$
$$\text{First Moments: } M_y = \iint_R x\delta \, dA, M_x = \iint_R y\delta \, dA$$
$$\text{Center of mass: } \bar{x} = \frac{M_y}{M}, \bar{y} = \frac{M_x}{M}$$

Moments of Inertia

3D Solids

$$\text{About x-axis: } I_x = \iiint_D (y^2 + z^2)\delta dV$$
$$\text{About y-axis: } I_y = \iiint_D (x^2 + z^2)\delta dV$$
$$\text{About z-axis: } I_z = \iiint_D (x^2 + y^2)\delta dV$$
$$\text{About a line L: } I_L = \iiint_D r^2(x, y, z)\delta dV$$

2D Plates

$$\text{About x-axis: } I_x = \iint_R y^2 \delta dA$$
$$\text{About y-axis: } I_y = \iint_R x^2 \delta dA$$
$$\text{About a line L: } I_L = \iint_R r^2(x, y)\delta dA$$
$$\text{About the origin: } I_O = \iint_R (x^2 + y^2)\delta dA = I_x + I_y$$

Joint probability density function

Conditions

$$f(x,y) \geq 0$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$$
$$P((X,Y) \in R) = \iint_R f(x,y)dxdy$$

Mean and expected value

$$\mu_X = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dxdy$$
$$\mu_Y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dxdy$$

Spherical Coordinates (ρ, ϕ, θ)

$$r = \rho \sin(\phi),$$

$$x = r\cos(\theta) = \rho \sin(\phi)\cos(\theta)$$

$$z = \rho \cos(\phi),$$

$$y = r\sin(\theta) = \rho \sin(\phi)\sin(\theta)$$

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$$

Jacobian:

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)}$$

$$\iint_R f(x,y)dxdy = \iint_G f(g(u,v),h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

$$\iiint_D f(x,y,z)dxdydz = \iiint_B f(g(u,v,w),h(u,v,w),k(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$