Vector Functions

Definition: The Limit of a Vector Function

Let $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ be a vector function with domain D, and let **L** be a vector. We say that **r** has **limit L** as t approaches t_0 and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that for all $t \in D$

$$\| \mathbf{r}(t) - \mathbf{L} \| < \epsilon \text{ whenever } 0 < | t - t_0 | < \delta$$

(Just take the lim of each of the funtions)

Limit Rules

Let **f** and **g** be vector functions and let u be a real-valued function. If we are given that as $t \to t_0$

$$\mathbf{f}(t) \to \mathbf{L}, \quad \mathbf{g}(t) \to \mathbf{M}, \quad u(t) \to U$$

Then,

(1) $\mathbf{f}(t) + \mathbf{g}(t) \to \mathbf{L} + \mathbf{M}$,
(2) $\alpha\mathbf{f}(t) \to \alpha\mathbf{L}$,
(3) $u(t)\mathbf{f}(t) \to U\mathbf{L}$,
(4) $\mathbf{f}(t) \cdot \mathbf{g}(t) \to \mathbf{L} \cdot \mathbf{M}$,
(5) $\mathbf{f}(t) \times \mathbf{g}(t) \to \mathbf{L} \times \mathbf{M}$.

Derivative

Definition: Derivative of a Vector Function

The **derivative** of a vector function is found by:

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

providing this limit exists. When the limit exists, we say that **r** is *differentiable* at t and $\mathbf{r}'(t)$ is called the *derivative* of **r** at t.

(1) $(\mathbf{f}+\mathbf{g})'(t) = \mathbf{f}'(t) + \mathbf{g}'(t)$

(2) $(\alpha\mathbf{f})'(t) = \alpha\mathbf{f}'(t)$

(3) $(\mathbf{f}\cdot\mathbf{g})'(t) = [\mathbf{f}(t)\cdot\mathbf{g}'(t)] + [\mathbf{f}'(t)\cdot\mathbf{g}(t)]$

(4) $(\mathbf{f}\times\mathbf{g})(t) = [\mathbf{f}(t)\times\mathbf{g}'(t)] + [\mathbf{f}'(t)\times\mathbf{g}(t)]$

(5) $(u\mathbf{f})'(t) = u(t)\mathbf{f}'(t) + u'(t)\mathbf{f}(t)$

(6) $(\mathbf{f}\circ u)'(t) = \mathbf{f}'(u(t))\,u'(t)$

The curve is smooth of the derivative of the curve is continuous and never 0.

The direction of motion is the direction of the velocity vector.

Integrals

Definition: Indefinite Integral

A differentiable vector function $\mathbf{R}(t)$ is an **antiderivative** of $\mathbf{r}(t)$ on some interval I if $\dfrac{d\mathbf{R}}{dt} = \mathbf{r}$ at each point of I. The **indefinite integral** of $\mathbf{r}$ with respect to t is the set of all antiderivatives of $\mathbf{r}$, denoted by $\int \mathbf{r}\,dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int \mathbf{r}\,dt = \mathbf{R}(t) + \mathbf{C}$$

Definition: The Definite Integral

For $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ continuous on $[a, b]$,

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}$$

Properties of the Integral

(1) $\displaystyle\int_a^b [\mathbf{f}(t) + \mathbf{g}(t)]\,dt = \int_a^b \mathbf{f}(t)\,dt + \int_a^b \mathbf{g}(t)\,dt$

(2) $\displaystyle\int_a^b [\alpha\mathbf{f}(t)]\,dt = \alpha\int_a^b \mathbf{f}(t)\,dt \quad (\alpha \textit{ a constant scalar})$

(3) $\displaystyle\int_a^b [\mathbf{c}\cdot\mathbf{f}(t)]\,dt = \mathbf{c}\cdot\int_a^b \mathbf{f}(t)\,dt \quad (\mathbf{c} \textit{ a constant vector})$

(4) $\displaystyle\left\|\int_a^b \mathbf{f}(t)\,dt\right\| \leq \int_a^b ||\,\mathbf{f}(t)\,||\ dt$