# Formulas-1

Credit Aarush Magic

$$\begin{aligned}
\vec{a}\cdot\vec{b} &= a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n \\
\vec{a}\times\vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & n_2 \end{vmatrix}\mathbf{k} \\
&= (a_2b_3 - b_2a_3)\vec{i} - (a_1b_3 + b_1a_3)\vec{j} + (a_1b_2 + b_1a_2)\vec{k}
\end{aligned}$$

# Properties:

$$\begin{aligned}
\vec{u}\cdot(\vec{v}\times\vec{w}) &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\
\vec{a}\cdot\vec{b} &= \vec{b}\cdot\vec{a} \\
\vec{a}\times\vec{b} &= -\vec{b}\times\vec{a} \\
\vec{a}\cdot\vec{b} &= |\vec{a}|\cdot|\vec{b}|\cos(\theta) \\
|\vec{a}\times\vec{b}| &= |\vec{a}|\cdot|\vec{b}|\sin(\theta) \\
\vec{a}\cdot(\vec{b}+\vec{c}) &= \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} \\
\vec{a}\times(\vec{b}+\vec{c}) &= \vec{a}\times\vec{b} + \vec{a}\times\vec{c} \\
\vec{a}\cdot\vec{0} &= 0 \\
\vec{a}\times\vec{0} &= \vec{0} \\
(c\vec{a})\cdot\vec{b} &= \vec{a}\cdot(c\vec{b}) \\
(c\vec{a})\times\vec{b} &= \vec{a}\times(c\vec{b}) \\
\vec{a}\cdot\vec{a} &= ||\vec{a}||^2 \\
\vec{a}\times\vec{a} &= \vec{0} \\
\text{If } \vec{a}\perp\vec{b} \text{ then } \vec{a}\cdot\vec{b} &= 0 \\
\text{If } \vec{a}\parallel\vec{b} \text{ then } \vec{a}\times\vec{b} &= \vec{0} \\
\vec{u}\times(\vec{v}\times\vec{w}) &= (\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w} \qquad nbsp;
\end{aligned}$$

# Projectile Motion:

$\text{Max Height} = \frac{(v_0\sin(\theta))^2}{2g}$
$\text{Range} = \frac{v_0^2\sin(2\theta)}{g}$
$\text{Flight Time} = \frac{2v_0\sin(\theta)}{g}$

## Graphs

| Type | Equaions |
|---|---|
| Elliptical Paraboloid | $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ |
| Elliptical Cone | $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$ |
| Ellipsoid | $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ |
| Hyperboloid of 1 sheet | $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ |
| Hyperboloid of 2 sheets | $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ |
| Hyperbolic Paraboloid | $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}, c > 0$ |

## Other Formulas

Line through $P(p_1, p_2, p_3)$ and parallel to $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ when $t \in \mathbb{R}$:

$$x = at + p_1 \quad y = bt + p_2 \quad z = ct + p_3$$

$$\langle at + p_1, bt + p_2, ct + p_3 \rangle = \langle a, b, c \rangle t + \langle p_1, p_2, p_3 \rangle$$

Line through $P(p_1, p_2, p_3)$ and perpendicular to $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ :

$$a(x - p_1) + b(y - p_2) + c(z - p_3) = 0$$

Distance between line and point:

$$d = \frac{||\vec{PS} \times v||}{||v||}$$

Distance between Point $S$ and a Plane,

$$d = \left| \vec{PS} \cdot \frac{n}{||n||} \right|$$

Projection,

$$\text{proj}_b a = \left( \frac{a \cdot b}{||b||} \right) \frac{b}{||b||}$$

Angle between planes or vectors:

$$\theta = \cos^{-1} \left( \left| \frac{\vec{n_1} \cdot \vec{n_2}}{||\vec{n_1}|| \cdot ||\vec{n_2}||} \right| \right)$$

Arc Length $(s(t))$:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \qquad = \int_a^b ||\vec{r}'(t)|| dt$$

$$s(t) = \int_{t_0}^t \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2} d\tau \qquad = \int_{t_0}^t ||\vec{r}'(\tau)|| d\tau$$

Speed:

$$\frac{ds}{dt} = ||\vec{v}(t)||$$

The unit tangent vector $(T(t))$:

$$\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||} = \frac{\vec{v}(t)}{||\vec{v}(t)||}$$

The curvature function $(\kappa(t))$:

$$\begin{aligned} \kappa &= \left\| \frac{d\vec{T}}{ds} \right\| \\ &= \frac{||\vec{T}'(t)||}{||\vec{v}(t)||} \\ &= \frac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||^3} \end{aligned}$$

Radius of curvature:

$$p = \frac{1}{\kappa}$$

Principal Normal Vector $(N(t))$:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||}$$

Binormal vector $(B(t))$:

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

# Formulas-2

$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if for every ε>0, there exists a corresponding δ>0, such that for all $(x,y)$ in the domain of f, $|f(x,y) - L| < \epsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

$$\frac{\partial f}{\partial x}|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h} = f_x(x_0,y_0)$$

$$\begin{aligned}
f_{xx} &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \\
f_{yy} &= \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \\
f_{xy} &= \frac{\partial^2 f}{\partial yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\
f_{yx} &= \frac{\partial^2 f}{\partial xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \\
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt} \\
\frac{dy}{dx} &= -\frac{F_x}{F_y}
\end{aligned}$$

Directional Derivative of the unit vector $u = u_1\mathbf{i} + u_2\mathbf{j}$

$$\begin{aligned}
f'_u(x,y) = D_u f(x,y) &= \lim_{s\to 0} \frac{f(x+su_1, y+su_2) - f(x,y)}{s} \\
&= \nabla \mathrm{f}(x,y)\cdot u \\
&= ||\nabla f||\cos\theta
\end{aligned}$$

The Gradient

$$\nabla\ f(x,y,z) = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$$

Tangent line to a level curve of the form $f(x,y) = 0$ at a point $(x_0, y_0)$,

$$f_x(x_0,y_0)\,(x-x_0) + f_y(x_0,y_0)\,(y-y_0) = 0$$

$$\frac{d}{dt}(f(r(t)) = \nabla f(r(t))\cdot r'(t)$$

Tangent plane to $f(x,y,z) = c$ at the point $P(x_0,y_0,z_0)$,

$$f_x(P)\,(x-x_0) + f_y(P)\,(y-y_0) + f_z(P)\,(z-z_0) = 0$$

Or when $f(x,y) = z$ at the point $P(x_0,y_0)$,

$$f_x(P)\,(x-x_0)+f_y(P)\,(y-y_0)-(z-z_0)=0$$

Normal line to $f\,(x,y,z)=c$ at the point $P(x_0,y_0,z_0)$,

$$\begin{aligned} x &= x_0 + f_x(P)t \\ y &= y_0 + f_y(P)t \\ z &= z_0 + f_z(P)t \end{aligned}$$

Linear Approximation ($f(x,y)\approx L(x,y)$) of $f(x,y)$ at $(x_0,y_0)$,

$$L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

Total Differential,

$$\begin{aligned} df &= f_x\,(x_0,y_0)dx + f_y\,(x_0,y_0)dy \\ df &= (\nabla f(P_0)\cdot u)ds \end{aligned}$$

Standard Linear Approximation Error where $M$ is the upper bound of the second partials on a rectangle centered at point $P$,

$$|E|\leq\frac{1}{2}M(|x-x_0|+|y-y_0|)^2$$

Second Partials Test at $(x_0,y_0)$ assuming $\nabla f=0$,

$$\begin{gathered} A=f_{xx}(x_0,y_0),\quad B=f_{xy}(x_0,y_0)\quad ,C=f_{yy}(x_0,y_0) \\ D=AC-B^2 \\ D<0 \text{ Saddle point} \\ D>0 \text{ Relative Extrema} \\ D=0 \text{ Indecisive} \end{gathered}$$

For the relative extrema, if $D>0$ and $A>0$ then you have a local min, if $A<0$ you have a local max.

Lagrange Multipliers when $g(x,y)=0$,

$$\nabla f=\lambda\nabla g$$

# Formulas-3

Fubini's Theorem

$$V=\iint_R f(x,y)dA=\int_c^d\int_a^b f(x,y)dxdy=\int_a^b\int_c^d f(x,y)dydx$$

if $R$ is defined by $a\le x\le b$ and $g_1(x)\le y\le g_2(x)$ then,

$$V=\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

Notice: The functions $g_n(x)$ are evaluated first.

# Properties

$$\begin{gathered}
\iint_R cf(x,y)dA=c\iint_R f(x,y)dA\\
\iint_R f(x,y)\pm g(x,y)dA=\iint_R f(x,y)dA\pm\iint_R g(x,y)dA\\
\iint_R f(x,y)dA\ge 0 \text{ if } f(x,y)\ge 0 \text{ on } R\\
\iint_R f(x,y)dA\ge\iint_R g(x,y)dA \text{ if } f(x,y)\ge g(x,y) \text{ on } R
\end{gathered}$$

If $R_1$ and $R_2$ are non-overlapping regions,

$$\iint_R f(x,y)dA=\iint_{R_1} f(x,y)dA+\iint_{R_2} f(x,y)dA$$

$$A=\iint_R dA=\int_{x_1}^{x_2}\int_{y_1}^{y_2} dydx=\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} r\,drd\theta$$

$$V=\iint_R f(x,y)dA$$

$$\text{Average Value}=\frac{1}{\text{Area of } R}\iint_R fdA \quad \text{OR} \quad \frac{1}{\text{Volume of } D}\iiint_D FdV$$

$$\begin{aligned}
V&=\iiint_D dV\\
&=\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} dzdydx\\
&=\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\int_{z_1}^{z_2} r\,dzdrd\theta\\
&=\int_{\rho_1}^{\rho_2}\int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2} \rho^2\sin(\phi)\;d\theta d\phi d\rho
\end{aligned}$$

# Mass and First Moments

## 3D Solids

$\delta$ is the density function.

$$\text{Mass: } M = \iiint_D \delta \, dV$$

$$\text{First Moments: } M_{yz} = \iiint_D x\delta \, dV, M_{xz} = \iiint_D y\delta \, dV, M_{xy} = \iiint_D z\delta \, dV$$

$$\text{Center of mass: } \bar{x} = \frac{M_{yz}}{M}, \bar{y} = \frac{M_{xz}}{M}, \bar{z} = \frac{M_{xy}}{M}$$

## 2D Plates

$$\text{Mass: } M = \iint_R \delta \, dA$$

$$\text{First Moments: } M_y = \iint_R x\delta \, dA, M_x = \iint_R y\delta \, dA$$

$$\text{Center of mass: } \bar{x} = \frac{M_y}{M}, \bar{y} = \frac{M_x}{M}$$

# Moments of Inertia

## 3D Solids

$$\text{About x-axis: } I_x = \iiint_D (y^2 + z^2)\delta dV$$

$$\text{About y-axis: } I_y = \iiint_D (x^2 + z^2)\delta dV$$

$$\text{About z-axis: } I_z = \iiint_D (x^2 + y^2)\delta dV$$

$$\text{About a line L: } I_L = \iiint_D r^2(x, y, z)\delta dV$$

## 2D Plates

$$\text{About x-axis: } I_x = \iint_R y^2 \delta dA$$

$$\text{About y-axis: } I_y = \iint_R x^2 \delta dA$$

$$\text{About a line L: } I_L = \iint_R r^2(x, y)\delta dA$$

$$\text{About the origin: } I_O = \iint_R (x^2 + y^2)\delta dA = I_x + I_y$$

# Joint probability density function

## Conditions

$$f(x,y) \geq 0$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$$
$$P((X,Y) \in R) = \iint_R f(x,y)dxdy$$

## Mean and expected value

$$\mu_X = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dxdy$$
$$\mu_Y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dxdy$$

## Cylindrical Coordinates $(r, \theta, z)$

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}$$
$$\iiint_T dV = \iiint r\ drd\theta dz$$

## Spherical Coordinates $(\rho, \phi, \theta)$

$$r = \rho\sin(\phi)$$
$$x = r\cos(\theta) = \rho\sin(\phi)\cos(\theta)$$
$$y = r\sin(\theta) = \rho\sin(\phi)\sin(\theta)$$
$$z = \rho\cos(\phi)$$
$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$$
$$\iiint_T dV = \iiint \rho^2\sin\phi\ d\rho d\phi d\theta$$

## Jacobian

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)}$$

$$\iint_R f(x,y)dxdy = \iint_G f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$$

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

$$\iiint_D f(x,y,z)dxdydz = \iiint_B f(g(u,v,w),h(u,v,w),k(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$

# Formulas-4

Module 4 Formulas

$$\text{Line Integral:}$$

$$ds = ||\vec{r}'(t)||dt$$

$$\int_C f(x,y,z)ds = \int_a^b f(g(t),h(t),k(t))|\vec{v}(t)|dt$$

$$\text{Mass (Thin wire)} = \int_C \delta ds$$

$$\text{First Moments (Thin wire):}$$

$$M_{yz} = \int_C x\delta ds$$

$$M_{xz} = \int_C y\delta ds$$

$$M_{xy} = \int_C z\delta ds$$

$$\text{Moment of inertia (Thin wire):}$$

$$I_x = \int_C (y^2+z^2)\delta ds$$

$$I_y = \int_C (x^2+z^2)\delta ds$$

$$I_z = \int_C (x^2+y^2)\delta ds$$

$$I_L = \int_C r^2\delta ds \text{ where r(x,y,z) = distance from point (x,y,z) to line L}$$

$$\text{Line Integral of vector field } \vec{F} \text{ along } C:$$

$$\int_C \vec{F}\cdot\vec{T}ds = \int_C \vec{F}\cdot\frac{d\vec{r}}{ds}ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\frac{d\vec{r}}{dt}dt$$

$$\int_C Mdx + Ndy + Pdz = \int_C M(x,y,z)dx + \int_C N(x,y,z)dy + \int_C P(x,y,z)dz$$

$$\text{Where } \int_C M(x,y,z)dx = \int_C M(g(t),h(t),k(t))g'(t)dt$$

$$\text{Work} = \int_C \vec{F}\cdot\vec{T}ds$$

$$\text{Flow} = \int_C \vec{F} \cdot \vec{T} ds = \int_C M dx + N dy$$

$$\text{Flux} = \int_C \vec{F} \cdot \vec{T} ds = \oint_C M dy - N dx$$

Conservative Fields:

Fields are conservative if $\int_C \vec{F} \cdot d\vec{r}$ is path independent

Potential Function: If $\vec{F} = \nabla f$ then $f$ is the potential function for $\vec{F}$

$\vec{F}$ is conservative if and only if $\vec{F}$ is a gradient field $\nabla f$ for differentiable function $f$

$\vec{F}$ is conservative if and only if $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$ and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

For a conservative vector field, $\vec{F}$, $\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$

$\oint \vec{F} \cdot d\vec{r} = 0 \Leftrightarrow$ The field $\vec{F}$ is conservative on $D$

Differential form: $M(x, y, z)dx + N(x, y, z)dy + P(x, y, z)dz$

Exact form: $Mdx + Ndy + Pdz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$ ($\vec{F}$ is conservative)

Circulation density (k-component of curl): $(\text{curl } \vec{F}) \cdot \vec{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

Flux density (divergence): $\text{div } \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$

Green's Theorem:

Circulation-Curl (counterclockwise circulation): $\oint_C \vec{F} \cdot \vec{T} ds = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$

Flux Divergence (outward flux of F): $\oint_C \vec{F} \cdot \vec{n} ds = \iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$

$$\text{Area of R} = \frac{1}{2} \oint xdy - ydx$$

Surface Area:

$$SA = \iint_S d\sigma$$

$$d\sigma = ||\vec{r_u} \times \vec{r_v}||dudv = \frac{||\nabla F||}{\nabla F \cdot \vec{p}} dA = \sqrt{{f_x}^2 + {f_y}^2 + 1} dxdy$$

Surface Integral:

$$\iint_S G(x, y, z) d\sigma$$

Surface integral of F over S (Flux): $\iint_S \vec{F}\cdot\vec{n}d\sigma = \iint_S \vec{F}\cdot(\vec{r_u}\times\vec{r_v})dudv$

$$\text{Mass (Thin shell)} = \iint_S \delta d\sigma$$

First Moments (Thin shell):

$$M_{yz} = \iint_S x\delta d\sigma$$

$$M_{xz} = \iint_S y\delta d\sigma$$

$$M_{xy} = \iint_S z\delta d\sigma$$

Moment of inertia (Thin shell):

$$I_x = \iint_S (y^2+z^2)\delta d\sigma$$

$$I_y = \iint_S (x^2+z^2)\delta d\sigma$$

$$I_z = \iint_S (x^2+y^2)\delta d\sigma$$

$$I_L = \iint_S r^2\delta d\sigma \text{ where r(x,y,z) = distance from point (x,y,z) to line L}$$

$$\text{The del operator: } \nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\text{curl } \vec{F} = \nabla\times\vec{F}$$

$$\text{div } \vec{F} = \nabla\cdot\vec{F}$$

$$\text{Important properties: curl(grad } f) = \nabla\times\nabla f = \vec{0}$$

,

$$\text{div(curl } \vec{F}) = \nabla\cdot(\nabla\times\vec{F}) = 0$$

Stoke's Theorem and Divergence Theorem:

$$\text{Circulation of } \vec{F} = \text{Flux of curl } \vec{F}\text{: } \oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}d\sigma$$

$$\nabla\times\vec{F} = \vec{0} \text{ at every point } \Leftrightarrow \oint_C \vec{F}\cdot d\vec{r} = 0 \Leftrightarrow \text{ The field } f \text{ is conservative}$$

Outward flux of $\vec{F}$ along surface $S = \iint_S \vec{F}\cdot\vec{n}d\sigma = \iiint_D(\nabla\cdot\vec{F})dV$