



UNIVERSITY OF THE BASQUE COUNTRY

FINAL YEAR PROJECT

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# About Tree Depth

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# 1 Introduction to Graph Theory

## 1.1 Definition of a graph

A graph is defined as a pair of sets  $G = (V, E)$ , such that  $E \subseteq V^2$ . The members of  $V$  are called vertices and the ones of  $E$  edges. Take into account, that the vertices can be anything, they can even be sets themselves. The usual way to draw a graph is by representing the vertices as individual points and for each edge, draw a link between both elements of that edge. The shape in which a graph is drawn is irrelevant, it will contain the same information.

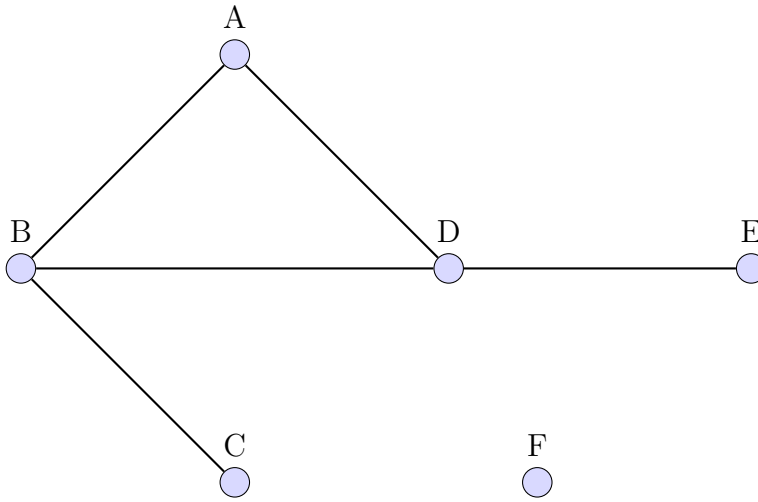


Figure 1: A graph with  $V = \{A, B, C, D, E, F\}$  and  $E = \{\{A, B\}, \{A, D\}, \{B, D\}, \{B, C\}, \{D, E\}\}$

## 1.2 Connectivity

An essential concept in graph theory is adjacency. Two vertices  $x, y \in V$  are said to be adjacent in  $G$  if and only if  $\{x, y\} \in E$ .

# 2 Introduction to Tree Depth

## 2.1 Basic definitions

Vertex  $x$  is said to be the ancestor of  $y$  in a rooted forest  $F$ , if and only if  $x$  belongs to the path between  $y$  and the root of the component to which  $x$  belongs,  $y$  included.

The closure of a rooted forest  $F$ , expressed as  $C = \text{clos}(F)$ , is defined as follows:

- $V(C) = V(F)$
- $E(C) = \{ \{x, y\} : x \text{ is an ancestor of } y \text{ in } F, x \neq y \}$

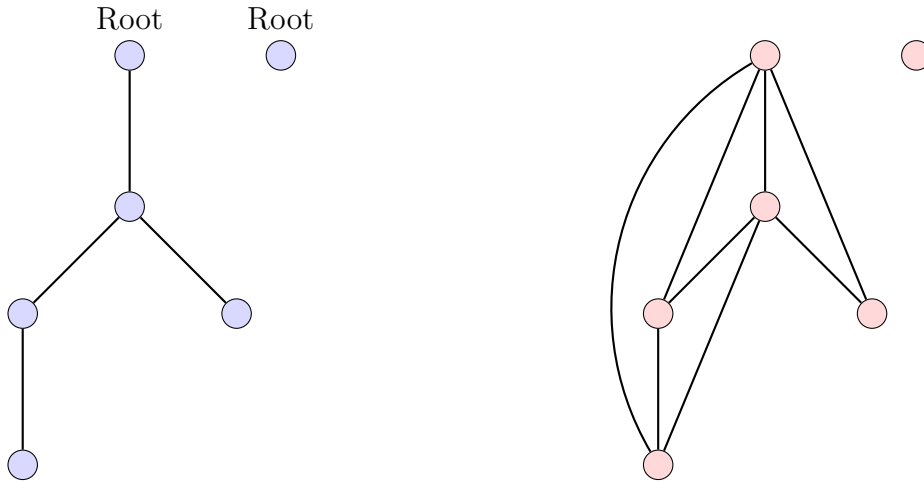


Figure 2: The blue graph at the left is a rooted forest  $F$ , the red graph at the right represents  $\text{clos}(F)$ .

## 2.2 Tree Depth

**Definition** The tree-depth  $\text{td}(G)$  of a graph  $G$  is the minimum height of a rooted forest  $F$  such that  $G \subseteq \text{clos}(F)$

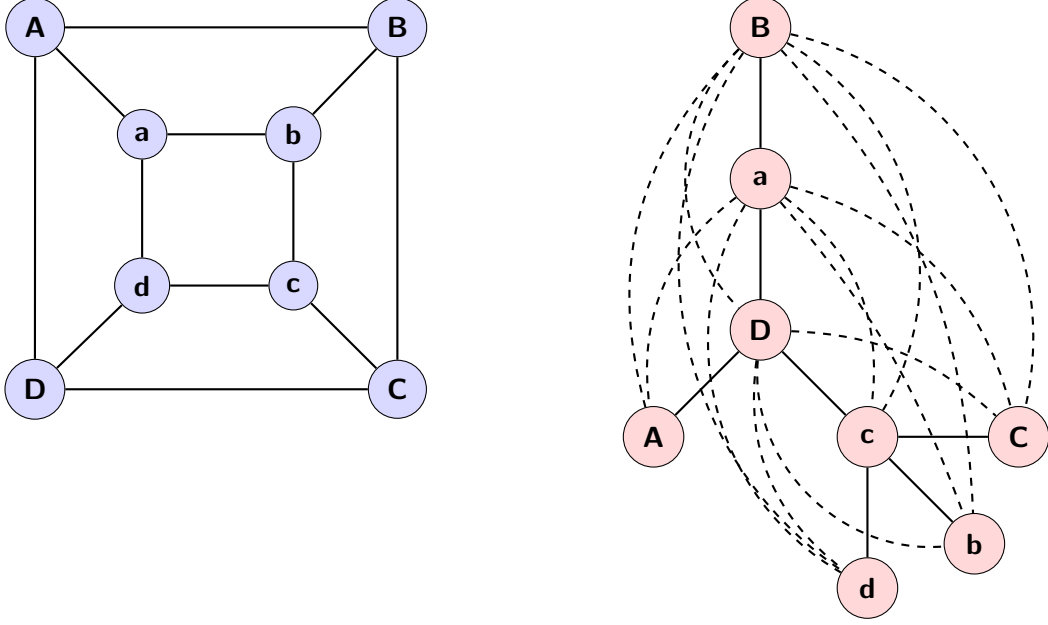


Figure 3: The graph  $G$  and tree  $T$  are in the left and right respectively. The dotted edges in  $T$ , represent the  $\text{clos}(T)$ . Because  $G \subseteq \text{clos}(T)$ , we know that  $\text{td}(G)$  is at most 4.

The tree depth of a graph  $G$  is a numerical invariant of a graph. In other words, the tree depth is a property that depends only on the abstract structure of a graph, not in its representation.

## 2.3 Elimination tree

An elimination tree  $Y$  of a connected graph  $G$  is defined recursively as follows:

- If  $V(G) = \{x\}$  then  $Y$  is just  $\{x\}$ .
- Otherwise,  $r \in V(G)$  is chosen as the root of  $Y$  and an elimination tree is created for each component of  $G - r$ .  $r$  will be connected to the root of each of these trees.

The tree  $T$  in Figure 3 is an elimination tree for the graph  $G$ .

**Lemma 2.1** *Let  $G$  be a graph and  $T$  a tree such that  $G \subseteq \text{clos}(T)$ . Then,  $Y$  exists, where  $Y$  is an elimination tree of  $G$  and  $\text{height}(Y) \leq \text{height}(T)$ .*

**Proof**

