



UNIVERSITY OF THE BASQUE COUNTRY

FINAL YEAR PROJECT

About Tree Depth

Author:

Asier MUJICA

Supervisor:

Dr. Hubert CHEN

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1 Introduction to Graph Theory

1.1 Definition of a graph

A graph is defined as a pair of sets $G = (V, E)$, such that $E \subseteq V^2$. The members of V are called vertices and the ones of E edges. Take into account, that the vertices can be anything, they can even be sets themselves. The usual way to draw a graph is by representing the vertices as individual points and for each edge, draw a link between both elements of that edge. The shape in which a graph is drawn is irrelevant, it will contain the same information.

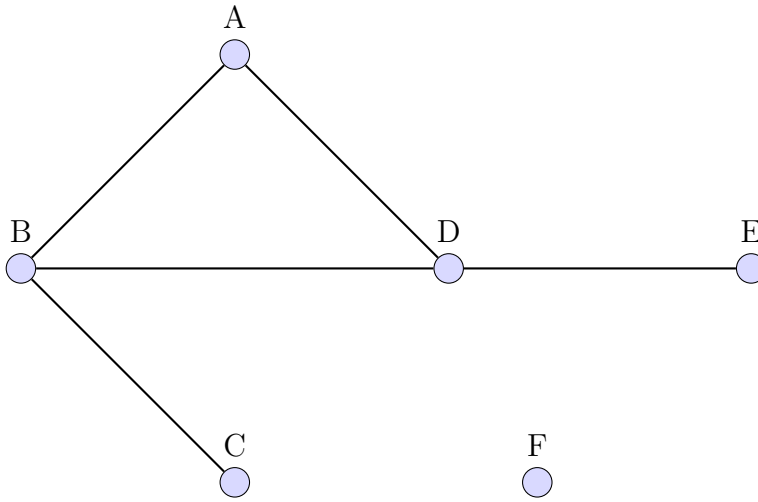


Figure 1: A graph with $V = \{A, B, C, D, E, F\}$ and $E = \{\{A, B\}, \{A, D\}, \{B, D\}, \{B, C\}, \{D, E\}\}$

1.2 Connectivity

An essential concept in graph theory is adjacency. Two vertices $x, y \in V$ are said to be adjacent in G if and only if $\{x, y\} \in E$.

2 Introduction to Tree Depth

2.1 Basic definitions

Vertex x is said to be the ancestor of y in a rooted forest F , if and only if x belongs to the path between y and the root of the component to which x belongs, y included.

The closure of a rooted forest F , expressed as $C = \text{clos}(F)$, is defined as follows:

- $V(C) = V(F)$
- $E(C) = \{ \{x, y\} : x \text{ is an ancestor of } y \text{ in } F, x \neq y \}$

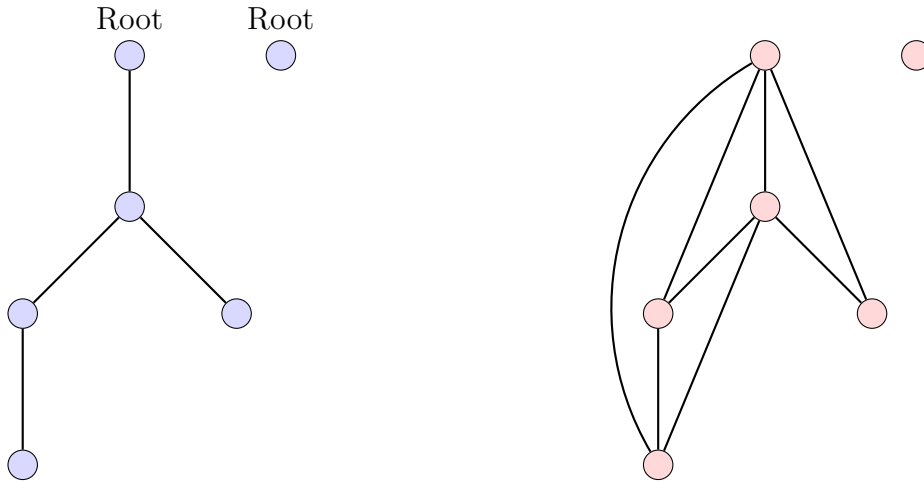


Figure 2: The blue graph at the left is a rooted forest F , the red graph at the right represents $\text{clos}(F)$.

2.2 Tree Depth

Definition The tree-depth $\text{td}(G)$ of a graph G is the minimum height of a rooted forest F such that $G \subseteq \text{clos}(F)$

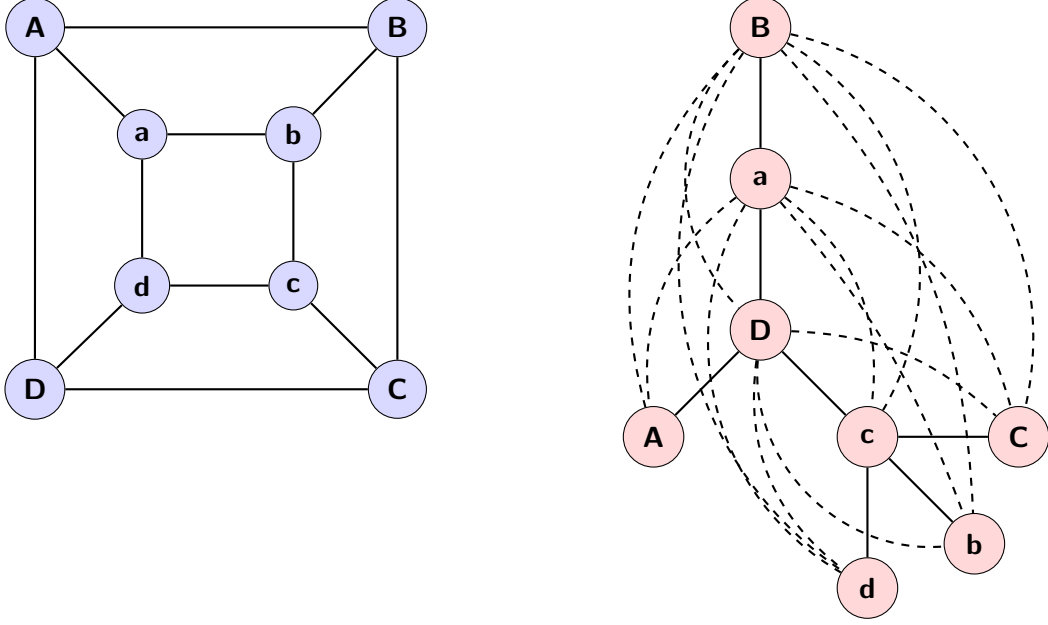


Figure 3: The graph G and tree T are in the left and right respectively. The dotted edges in T , represent the $\text{clos}(T)$. Because $G \subseteq \text{clos}(T)$, we know that $\text{td}(G)$ is at most 4.

The tree depth of a graph G is a numerical invariant of a graph. In other words, the tree depth is a property that depends only on the abstract structure of a graph, not in its representation.

2.3 Elimination tree

An elimination tree Y of a connected graph G is defined recursively as follows:

- If $V(G) = \{x\}$ then Y is just $\{x\}$.
- Otherwise, $r \in V(G)$ is chosen as the root of Y and an elimination tree is created for each component of $G - r$. r will be connected to the root of each of these trees.

The tree T in Figure 3 is an elimination tree for the graph G .

Lemma 2.1. *Let G be a graph and T a tree such that $G \subseteq \text{clos}(T)$. Then, Y exists, where Y is an elimination tree of G and $\text{height}(Y) \leq \text{height}(T)$.*

Proof. @TODO

□

3 Game Theoretic approach to Tree-Depth

3.1 Defining the game

The k -step selection-deletion game is played by Alice and Bob on a graph G . The game is played by turns as follows:

- First, Alice selects a connected component of the graph, and the rest of the components are deleted.
- Then, Bob deletes a node from the remaining graph and the next round is played with this graph.

If Bob deletes the last node at the k -th round or earlier, he is said to win the k -step selection-deletion game. Otherwise, Alice wins.

3.2 Bob's winning strategy

Lemma 3.1. *Let G be a graph and let F be a rooted forest of height at most t such that $G \subseteq \text{clos}(F)$. Then Bob has a winning strategy for the $(t+1)$ -step selection-deletion game.*

Proof. Because of lemma 2.1 we know an elimination forest Y exists such that $\text{height}(Y) \leq \text{height}(F)$. We will prove this by induction over the height of Y .

- **Base case:** If $\text{height}(Y) = 0$, then every component of G will have a single vertex, so it's clear that Bob will win the 1-step selection-deletion game.
- **Induction:** Let $G_i \subseteq G$ be the component Alice chooses, then Y_i exists such that Y_i is an elimination tree belonging to Y , $G_i \subseteq \text{clos}(Y_i)$ and obviously $\text{height}(Y_i) \leq \text{height}(Y) \leq t$. Bob will delete v , the root of Y_i . This will leave us with $G' = G_i - v$ as the new graph. If we consider the sons of v the new roots in $Y' = Y_i - v$, then $G' \subseteq \text{clos}(Y')$ because of how the elimination trees are built. As $\text{height}(Y') \leq t - 1$, we can assume by induction that Bob has a winning strategy in t rounds for G' , which together with the strategy for the first round we have just defined makes a winning strategy for Bob in the $(t+1)$ -step selection-deletion game on the graph G .

□

3.3 Alice's winning strategy

Definition A shelter S in a graph G is a partially ordered set with the next properties:

- $\forall H \in S, H \subseteq G$ and H is connected.
- $H < V$ if and only if $H \subset V$.
- If $H \in S$ and H is not minimal, then $\forall v \in V(H), \exists H'$ such that $H' = H - v$ and H covers H' .

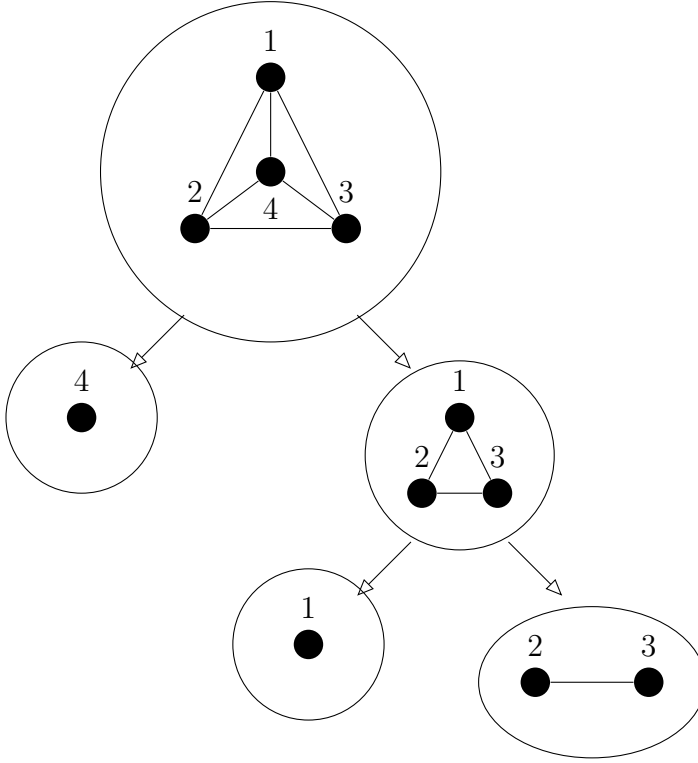


Figure 4: An example of a shelter. The arrows represent the covering relation.

The thickness of a shelter S is the shortest chain from a maximal element in S to a minimal element in S . The thickness of the shelter in figure 4 is 1.

Lemma 3.2. *Let G be a graph, S a shelter in G , and t the thickness of S . Then, S encodes a winning strategy for Alice in the t -step selection-deletion game.*

Proof. We will proof this by induction over t .

- **Base case:** If $t = 0$, then clearly Alice wins the 0-step selection-deletion game.
- **Induction:** Let H be a maximal element in S . Then, Alice picks the connected component G_i of G , such that $H \subseteq G_i$. Because $t > 0$, H is not minimal, so for any vertex v that Bob removes, if $v \in H$ there exists $H' \in S$ that is covered by H and $v \notin H'$. Otherwise, H is still a subgraph of $G_i - v$. Let $S' = \{X \mid X \in S \text{ and } X \subseteq G_i - v\}$. It is clear that S' is a shelter for $G_i - v$ and that the thickness of S' is greater than or equal to $t-1$. By induction we can assume S' encodes a winning strategy in $t-1$ steps.

□

3.4 Relation to Tree-Depth

It is clear that if Alice has a winning strategy in the t -step selection deletion game, Bob can't have a winning strategy in that same game. Because of this and lemmas 3.1 and 3.2 we can state the following:

Theorem 3.3. *Let G be a graph, S a shelter in G of thickness x and F a rooted forest of height y such that $G \subseteq \text{clos}(F)$. Then the following is true.*

1. *Every rooted forest who's closure contains G has an height higher than or equal to x .*
2. *Every shelter in G has a thickness smaller than or equal to y .*
3. $x \leq \text{td}(G) \leq y$.

With this theorem we can now proof the exact value of the tree-depth of a graph, instead of just an upper bound.

Figure 5: This is shelter of thickness 4 for the graph in figure 3. With this and the rooted forest from figure 3 we can say that $\text{td}(G) = 4$.