Abstractions Over Algebraic Structures

Aditya Mukherjee

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What is Algebra?

I guess that's what we're here to figure out..

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Abstract

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Haha just kidding, this is a book about algebraic structures, not Latin.

Part I

Preliminaries

Chapter 1

Set Theory

The chapter starts off with what is a set, then I introduce you to common sets you will find along your journey, and then I show you some set operations.

1.1 Sets and Operations over Sets

A **set** is a collection of objects called elements. The elements in a set have no order and no repetition. We describe the contents of a set using { and }. An example of a set containing elements 1 and 2 called A is:

$$A = \{1, 2\} = \{2, 1\} = \{x \in \mathbb{N} : x = 1, 2\}.$$

If we want to show an element, x, is a set, A, we say: $x \in A$. Many sets can also have an infinite number of elements, for example, $\mathbb{R}, \mathbb{N}$, and $\mathbb{Z}$ all have an infinite number of elements. We can indicate this with ellipsis:

$$\mathbb{N} = \{1, 2, \dots\},\ \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

Another way of writing these is using **set builder notation**,

$$\mathbb{Q} = \left\{\frac{q}{p} : q, p \in \mathbb{Z}, p \neq 0\right\},$$

you write the structure of the set before the colon and statements about it after the colon.

$$\{x \in A : P(x)\}$$

Sets can contain all sorts of elements besides numbers, think functions, other sets, ***"Algebraic Structures"***, functions, etc. For example, the set of real functions whose value at $x = 2$ is 6 (the arrow will be explained later)

$$\{(f : \mathbb{R} \mapsto \mathbb{R}) : f(2) = 6\}$$

and the set of differentiable real functions whose derivative is $6x^2$:

$$\left\{(f : \mathbb{R} \mapsto \mathbb{R}) : f \text{ is differentiable}, \frac{df}{dx} = 6x^2\right\}$$

both functions $2x^3$ and $2x^3 + 8$ are in that set. Here is another set:

$$K = \{A = \{a\}, B = \{b\}\}$$

a in that set is described as $a \in A \in K$. You could have a set called "animals", featuring dogs and cats:

$$\text{Animals} = \{\text{Cats}, \text{Dogs}\}$$

(all the other animals are inferior).

And with that, and our new understanding of sets, comes out first definition:

Definition 1.1: Set

A **set** is a collection of objects called elements. The elements in a set have no order and no repetition.

There are many operations one can apply on sets, the most common ones are: union, intersection, and complement. The **Union** of two sets is a set containing all of the elements of both sets, for example:

$$A = \{1, 2, 3\}, B = \{3, 4, 5\}$$

$$A \cup B = \{1, 2, 3, 4, 5\}.$$

The formal definition of a union is

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The **Intersection** of two sets is a set containing all of the elements that are in both sets, for example:

$$A = \{1, 2, 3\}, B = \{3, 4, 5\}$$

$$A \cap B = \{3\}.$$

The formal definition of an intersection is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Before I can define the complement of a set, I need to define a couple more things. If every element in a set, A, is in another set, B, then A is a **subset** of B, we write this as $A \subseteq B$ or $A \subset B$. If every element in A is in B, and every element in B is in A, then A and B are **equal**, we write this as $A = B$. The **Difference** of two sets, A and B, in that order ($A \setminus B$), is the set containing all the elements of A that are not in B. For example:

$$A = \{1, 2, 3\}, B = \{3, 4, 5\}$$

$$A \setminus B = \{1, 2\}.$$

The formal definition of a set difference is

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

If $A \subseteq U$, then U can be described as the **universal set** of A. The **Complement** of A is $A^c = U \setminus A$.

Next, probably one of the most important operations on a set you will encounter in set theory is the **Cardinality** of a set. The cardinality of a set is the number of elements it contains. For example, the cardinality of $\{1, 2\}$ is 2. We write the cardinality of a set A as $|A|$.

$$|\{1, 2, \ldots, n\}| = n.$$

The next operations I will introduce are the Cartesian Product and the power set. The **Cartesian Product** of two sets is essentially each all the possible coordinates you can make with the elements of the set. The Cartesian Product of two sets, A and B, is written as $A \times B$. For example,

$$\{1, 2\} \times \{2, 3\} = (1, 2), (1, 3), (2, 2), (2, 3).$$

And now I think it is due time for our first theorem:

Theorem 1.1: Cardinality of Cartesian Product

The cardinality of the Cartesian Product of two sets is the product of the cardinalities of the two sets.

$$|A \times B| = |A| \cdot |B|.$$

Proof: This result is relatively easy to show, for each possible element, there are $|A|$ possible values for the first coordinate, and $|B|$ possible values for the second coordinate, so there are $|A| \cdot |B|$ possible coordinates, and thus $|A| \cdot |B|$ elements in the Cartesian Product. $\square$ This is more clear in this diagram:

δ	2	3
1	{1,2}	{1,3}
2	{2,2}	{2,3}

The last thing I will show in this section is **Power Sets**. The power set of a set is the set of all subsets of that set. This includes the empty set and the set itself. Each and every element of the set is also a subset of it too. We denote the power set with $\mathcal{P}(A)$, where A is the set we are operating on. An example of a power set is:

$$\mathcal{P}(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

The formal definition of the power set is:

$$\mathcal{P}(A) = \{x : x \subseteq A\}.$$

I think it is important to notice the cardinality of the power set. The original set had 3 elements, and the power set has 8 elements, which just so happens to be 2^3. This is not a coincidence, and in fact, it is true for all sets. I think this calls for another theorem!

Theorem 1.2: Cardinality of Power Set

The cardinality of the power set of a set is $2^{|A|}$.

$$|\mathcal{P}(A)| = 2^{|A|}.$$

Proof: This is a relatively easy proof, we can prove it by induction, or take a more simple approach. One way to describe the cardinality of the power set, is to understand that the power set contains each **grouping** of elements in A. In other words, out of all the elements in A, the power set contains all the groups of 0 elements, + all the groups of 1 elements, ...

$$|\mathcal{P}(A)| = \binom{|A|}{0} + \binom{|A|}{1} + \cdots + \binom{|A|}{|A|}.$$

This can easily be re-arranged into

$$\sum_{k=0}^{|A|} \binom{|A|}{k}$$

Now all we have to do is show this sum equals $2^{|A|}$, which is easy to do with the Binomial Theorem.

$$2^{|A|} = (1+1)^{|A|} = \sum_{k=0}^{|A|} \binom{|A|}{k} 1^k 1^{n-k} = \sum_{k=0}^{|A|} \binom{|A|}{k}. \square$$

And with that, I think this is a great conclusion to the section.