Abstractions Over Algebraic Structures

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What is Algebra?

I guess that's what we're here to figure out..

Contents

Abstract

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Haha just kidding, this is a book about algebraic structures, not Latin.

Part I

Preliminaries

Chapter 1

Set Theory

The chapter starts off with what is a set, then I introduce you to common sets you will find along your journey, and then I show you some set operations.

1.1 Sets and Operations over Sets

A **set** is a collection of objects called elements. The elements in a set have no order and no repetition. We describe the contents of a set using { and }. An example of a set containing elements 1 and 2 called A is:

$$A = \{1, 2\} = \{2, 1\} = \{x \in \mathbb{N} : x = 1, 2\}.$$

If we want to show an element, x, is a set, A, we say: $x \in A$. Many sets can also have an infinite number of elements, for example, $\mathbb{R}, \mathbb{N}$, and $\mathbb{Z}$ all have an infinite number of elements. We can indicate this with ellipsis:

$$\mathbb{N} = \{1, 2, \dots\},\ \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

Another way of writing these is using **set builder notation**,

$$\mathbb{Q} = \left\{ \frac{q}{p} : q, p \in \mathbb{Z}, p \neq 0 \right\},$$

you write the structure of the set before the colon and statements about it after the colon.

$$\{x \in A : P(x)\}$$

Sets can contain all sorts of elements besides numbers, think functions, other sets, ***"Algebraic Structures"***, functions, etc. For example, the set of real functions whose value at $x = 2$ is 6 (the arrow will be explained later)

$$\{(f : \mathbb{R} \mapsto \mathbb{R}) : f(2) = 6\}$$

and the set of differentiable real functions whose derivative is $6x^2$:

$$\left\{ (f : \mathbb{R} \mapsto \mathbb{R}) : f \text{ is differentiable}, \frac{df}{dx} = 6x^2 \right\}$$

both functions $2x^3$ and $2x^3 + 8$ are in that set. Here is another set:

$$K = \{A = \{a\}, B = \{b\}\}$$

a in that set is described as $a \in A \in K$. You could have a set called "animals", featuring dogs and cats:

$$\text{Animals} = \{\text{Cats}, \text{Dogs}\}$$

(all the other animals are inferior).

And with that, and our new understanding of sets, comes out first definition:

Definition 1.1: Set

A **set** is a collection of objects called elements. The elements in a set have no order and no repetition.

There are many operations one can apply on sets, the most common ones are: union, intersection, and complement. The **Union** of two sets is a set containing all of the elements of both sets, for example:

$$A = \{1, 2, 3\}, B = \{3, 4, 5\}$$
$$A \cup B = \{1, 2, 3, 4, 5\}.$$

The formal definition of a union is

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The **Intersection** of two sets is a set containing all of the elements that are in both sets, for example:

$$A = \{1, 2, 3\}, B = \{3, 4, 5\}$$
$$A \cap B = \{3\}.$$

The formal definition of an intersection is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Before I can define the complement of a set, I need to define a couple more things. If every element in a set, A, is in another set, B, then A is a **subset** of B, we write this as $A \subseteq B$ or $A \subset B$. If every element in A is in B, and every element in B is in A, then A and B are **equal**, we write this as $A = B$. The **Difference** of two sets, A and B, in that order $(A \setminus B)$, is the set containing all the elements of A that are not in B. For example:

$$A = \{1, 2, 3\}, B = \{3, 4, 5\}$$
$$A \setminus B = \{1, 2\}.$$

The formal definition of a set difference is

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

If $A \subseteq U$, then U can be described as the **universal set** of A. The **Complement** of A is $A^c = U \setminus A$.

Next, probably one of the most important operations on a set you will encounter in set theory is the **Cardinality** of a set. The cardinality of a set is the number of elements it contains. For example, the cardinality of $\{1, 2\}$ is 2. We write the cardinality of a set A as $|A|$.

$$|\{1, 2, \ldots, n\}| = n.$$

The next operations I will introduce are the Cartesian Product and the power set. The **Cartesian Product** of two sets is essentially each all the possible coordinates you can make with the elements of the set. The Cartesian Product of two sets, A and B, is written as $A \times B$. For example,

$$\{1, 2\} \times \{2, 3\} = (1, 2), (1, 3), (2, 2), (2, 3).$$

And now I think it is due time for our first theorem:

Theorem 1.1: Cardinality of Cartesian Product

The cardinality of the Cartesian Product of two sets is the product of the cardinalities of the two sets.

$$|A \times B| = |A| \cdot |B|.$$

Proof: This result is relatively easy to show, for each possible element, there are $|A|$ possible values for the first coordinate, and $|B|$ possible values for the second coordinate, so there are $|A| \cdot |B|$ possible coordinates, and thus $|A| \cdot |B|$ elements in the Cartesian Product.

□

This is more clear in this diagram:

δ	2	3
1	{1,2}	{1,3}
2	{2,2}	{2,3}

Now that we understand Cardinality, I can introduce these theorems:

Theorem 1.2: Cardinality of Union

The cardinality of the union of two sets is the sum of the cardinalities of the two sets minus the cardinality of their intersection.

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Proof: This is a bit more complicated to show, but it is still relatively easy. We can split the union into two parts, the intersection and the union of the two sets without the intersection. The intersection is counted twice, so we subtract it once.

□

The second theorem is a corollary of the first:

Theorem 1.3: Cardinality of Intersection

The cardinality of the intersection of two sets is the sum of the cardinalities of the two sets minus the cardinality of their union.

$$|A \cap B| = |A| + |B| - |A \cup B|.$$

Proof: This is a corollary of the previous theorem, we can just switch the union and intersection. We know $|A| + |B|$ is the union + the intersection, so we subtract the union to get the intersection □

Another important theorem that involves unions and intersections is De Morgan's Laws:

Theorem 1.4: De Morgan's Laws

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

Proof: This is much harder than the others. I will only prove the first one, the second one will be left as an exercise to the reader. First I will assume $A, B \subseteq U$. So to prove the result we must show

$$(A \cup B)^c \subseteq A^c \cap B^c$$

and

$$(A^c \cap B^c) \subseteq (A \cup B)^c.$$

First, let there exist some $x \in (A \cup B)^c$, that means $x \notin (A \cup B)$. Had x been in A or B, it would have been in $A \cup B$, therefore $x \notin A$ and $x \notin B$. That is the same as saying $x \in A^c$ and $x \in B^c$. This means $x \in (A^c \cap B^c)$. And that implies

$$(A \cup B)^c \subseteq A^c \cap B^c.$$

Now to prove it in the other direction. Suppose some $x \in (A^c \cap B^c)$, this is the same as saying $x \in A^c$ and $x \in B^c$. That means $x \notin A$ and $x \notin B$. This implies $x \notin (A \cup B)$, or $x \in (A \cup B)^c$. This means that $(A^c \cap B^c) \subseteq (A \cup B)^c$, and thus

$$(A \cup B)^c = A^c \cap B^c.$$

□

The last thing I will show in this section is **Power Sets**. The power set of a set is the set of all subsets of that set. This includes the empty set and the set itself. Each and every element of the set is also a subset of it too. We denote the power set with $\mathcal{P}(A)$, where A is the set we are operating on. An example of a power set is:

$$\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

The formal definition of the power set is:

$$\mathcal{P}(A) = \{x : x \subseteq A\}.$$

I think it is important to notice the cardinality of the power set. The original set had 3 elements, and the power set has 8 elements, which just so happens to be 2^3. This is not a coincidence, and in fact, it is true for all sets. I think this calls for another theorem!

Theorem 1.5: Cardinality of Power Set

The cardinality of the power set of a set is $2^{|A|}$.

$$|\mathcal{P}(A)| = 2^{|A|}.$$

Proof: This is a relatively easy proof, we can prove it by induction, or take a more simple approach. One way to describe the cardinality of the power set, is to understand that the power set contains each **grouping** of elements in A. In other words, out of all the elements in A, the power set contains all the groups of 0 elements, + all the groups of 1 elements, ...

$$|\mathcal{P}(A)| = \binom{|A|}{0} + \binom{|A|}{1} + \cdots + \binom{|A|}{|A|}.$$

This can easily be re-arranged into

$$\sum_{k=0}^{|A|} \binom{|A|}{k}$$

Now all we have to do is show this sum equals $2^{|A|}$, which is easy to do with the Binomial Theorem.

$$2^{|A|} = (1+1)^{|A|} = \sum_{k=0}^{|A|} \binom{|A|}{k} 1^k 1^{n-k} = \sum_{k=0}^{|A|} \binom{|A|}{k}.$$

□

And with that, I think this is a great conclusion to the section.

Exercises

1. Show that $A \subseteq B$ if and only if $A \cap B = A$.
2. Show that the set $\{k \in \mathbb{Z} : k = 12a, a \in \mathbb{Z}\}$ is a subset of $\{k \in \mathbb{Z} : k = 3a, a \in \mathbb{Z}\}$.