Unification

Concepts of Programming Languages

CAS CS 320

Outline

- Finish up our discussion of **Hindley–Milner Light** (HM^-)
- Describe the **unification** algorithm used to determine the "actual" type of our expression, given a collection of constraints

Recap

Recall: Parametric Polymorphism

```
let rec rev = function
  | [] -> []
  | x :: xs -> rev xs @ [x]
```

Parametric polymorphism allows for functions which are agnostic to the types of its inputs

For example, we can write a single reverse function and use it in multiple contexts

Recall: Quantification

```
let id : 'a . 'a -> 'a = fun x -> x
```

`In reality, types variables in OCaml are` **`quantified`**

`We read this "`**`id`**` has type `**`t -> t`**` for any type `**`t`**`"`

Recall: Hindley-Milner Light

$$
\begin{aligned}
e ::= {} & \lambda x . e \mid e e \\
& \mid \text{let } x = e \text{ in } e \\
& \mid \text{if } e \text{ then } e \text{ else } e \\
& \mid e + e \mid e = e \\
& \mid n \mid x \\
\sigma ::= {} & \text{int} \mid \text{bool} \mid \alpha \mid \sigma \to \sigma \\
\tau ::= {} & \sigma \mid \forall \alpha . \tau
\end{aligned}
$$

type variable

type quant.

Recall: Type Schemes

$$\sigma ::= \mathsf{int} \mid \mathsf{bool} \mid \alpha \mid \sigma \to \sigma$$

$$\tau ::= \sigma \mid \forall \alpha \,.\, \tau$$

$\alpha \to \beta$

monotype (σ): type with no quantification

$\mathsf{int} \to \mathsf{bool}$

monomorphic type: monotype with no type variables

$\forall \beta .\ \alpha \to \beta$

type scheme (τ): type with zero or more quantified type variables

polymorphic type: *closed* type scheme $\forall \alpha \forall \beta .\ \alpha \to \beta$

Recall: Constraint-Based Inference

$$\Gamma \vdash e : \tau \dashv \mathscr{C}$$

Our typing rules well need to keep track of a set of **constraints**, which tell use what must hold for e to be well-typed

The idea: We're formalizing the idea of "collecting together" our constraints, as in our intuitive example

Recall: Constraints

$$\tau_1 \doteq \tau_2$$

"τ_1 should be the same as τ_2"

Enforcing this constraint means **unifying** τ_1 and τ_2

Recall: HM$^-$ (Typing)

$$\frac{n \text{ is an integer}}{\Gamma \vdash n : \mathsf{int} \dashv \varnothing}\ (\mathtt{int})$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathscr{C}_1 \qquad \Gamma \vdash e_2 : \tau_2 \dashv \mathscr{C}_2 \qquad \Gamma \vdash e_3 : \tau_3 \dashv \mathscr{C}_3}{\Gamma \vdash \mathsf{if}\ e_1\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3 : \tau_3 \dashv \tau_1 \doteq \mathsf{bool}, \tau_2 \doteq \tau_3, \mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3}\ (\mathtt{if})$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathscr{C}_1 \qquad \Gamma \vdash e_2 : \tau_2 \dashv \mathscr{C}_2}{\Gamma \vdash e_1 = e_2 : \mathsf{bool} \dashv \tau_1 \doteq \tau_2, \mathscr{C}_1, \mathscr{C}_2}\ (\mathtt{eq})$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathscr{C}_1 \qquad \Gamma \vdash e_2 : \tau_2 \dashv \mathscr{C}_2}{\Gamma \vdash e_1 + e_2 : \mathsf{int} \dashv \tau_1 \doteq \mathsf{int}, \tau_2 \doteq \mathsf{int}, \mathscr{C}_1, \mathscr{C}_2}\ (\mathtt{add})$$

$$\frac{\boxed{\alpha \text{ is fresh}} \qquad \Gamma, x : \alpha \vdash e : \tau \dashv \mathscr{C}}{\Gamma \vdash \lambda x . e : \alpha \to \tau \dashv \mathscr{C}}\ (\mathtt{fun})$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathscr{C}_1 \qquad \Gamma \vdash e_2 : \tau_2 \dashv \mathscr{C}_2 \qquad \boxed{\alpha \text{ is fresh}}}{\Gamma \vdash e_1 e_2 : \alpha \dashv \tau_1 \doteq \tau_2 \to \alpha, \mathscr{C}_1, \mathscr{C}_2}\ (\mathtt{app})$$

Practice Problem

$$\{f : \alpha \to \alpha\} \vdash f\,(f\;2 = 2) : \;\tau \dashv \mathscr{C}$$

Determine the type τ and constraints $\mathscr{C}$ such that the above judgment is derivable

$$\frac{n \text{ is an integer}}{\Gamma \vdash n : \mathsf{int} \dashv \varnothing}\ (\texttt{int}) \qquad \frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathscr{C}_1 \qquad \Gamma \vdash e_2 : \tau_2 \dashv \mathscr{C}_2}{\Gamma \vdash e_1 = e_2 : \mathsf{bool} \dashv \tau_1 \doteq \tau_2, \mathscr{C}_1, \mathscr{C}_2}\ (\texttt{eq})$$

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Answer

not polymorphic

C

$\{f : \alpha \to \alpha\} \vdash f\,((f\ 2) = 2) : \gamma \dashv \boxed{\alpha \to \alpha \doteq bool \to \gamma,\ \alpha \to \alpha \doteq int \to \beta,\ \beta \doteq int}$

- $\{f : \alpha \to \alpha\} \vdash f : \alpha \to \alpha \dashv \emptyset$
- $\{f : \alpha \to \alpha\} \vdash (f\ 2) = 2 : bool \dashv \alpha \to \alpha \doteq int \to \beta,\ \beta \doteq int$
  - $\{f : \alpha \to \alpha\} \vdash f\ 2 : \beta \dashv \alpha \to \alpha \doteq int \to \beta$
    - $\{f : \alpha \to \alpha\} \vdash f : \alpha \to \alpha \dashv \emptyset$
    - $\{f : \alpha \to \alpha\} \vdash 2 : int \dashv \emptyset$
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HM^{-} (Typing Variables)

$$\frac{(x : \forall \alpha_1 . \forall \alpha_2 \ldots \forall \alpha_k . \tau) \in \Gamma \qquad \beta_1, \ldots, \beta_k \ \texttt{are fresh}}{\Gamma \vdash x : [\beta_1 / \alpha_1] \ldots [\beta_k / \alpha_k] \tau \dashv \varnothing} \ \texttt{(var)}$$

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If x is declared in Γ, then x can be given the type τ *with all free variables replaced by **fresh variables***

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If x is declared in Γ, then x can be given the type τ *with all free variables replaced by **fresh variables***

This is where the polymorphism magic happens

fresh variables can be unified with anything

Example

$\{f : \forall\alpha\,.\,\alpha \to \alpha\} \vdash f\,(f\ 2 = 2) : \varepsilon \dashv \beta \to \beta \doteq bool \to \varepsilon, \ldots$

$\vdash \{f : \forall\alpha\,.\,\alpha \to \alpha\} \vdash f : \beta \to \beta \dashv \emptyset$

$\vdash \{\ldots\} \vdash f\ 2 = 2 : bool \dashv \gamma \to \gamma \doteq int \to \delta,\ \delta \doteq int$

$\vdash \{\ldots\} \vdash f\ 2 : \delta \dashv \gamma \to \gamma \doteq int \to \delta$

$\vdash \{f : \forall\alpha\,.\,\alpha \to \alpha\} \vdash f : \gamma \to \gamma \dashv \emptyset$

$\vdash \{\ldots\} \vdash 2 : int$

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C:

$\beta \to \beta \doteq bool \to \varepsilon$

$\gamma \to \gamma \doteq int \to \delta$

$\delta \doteq int$

HM⁻ (Typing Let-Expressions)

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathscr{C}_1 \qquad \Gamma, x : \tau_1 \vdash e_2 : \tau_2 \dashv \mathscr{C}_2}{\Gamma \vdash \textsf{let } x = e_1 \textsf{ in } e_2 : \tau_2 \dashv \mathscr{C}_1, \mathscr{C}_2} \text{ (let)}$$

The type of a let-expression is the same as the type of its body, relative to the constraints of typing the let-binding and the body (wordy...)

Aside: Let-Polymorphism

```
let f = fun x -> x in
let y = f 2 in
f true
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The Takeaway: We will have to treat typing of top-level let-expressions as *different* from local let-expressions

Unification

High Level

$$a \doteq d \to e$$
$$c \doteq \mathsf{int} \to d$$
$$\mathsf{int} \to \mathsf{int} \to \mathsf{int} \doteq b \to c$$

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`Unification` is the process of solving a system of equations over *symbolic* expressions

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Unification is the process of solving a system of equations over *symbolic* expressions

e.g., we could solve a system of equations over *variables* and *ADT constructors*

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ADT Unification Problem

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where $s_1, \ldots, s_k$ and $t_1, \ldots, t_k$ are element of the ADT possibly with variables

Example

```
type ty =
    | TInt
    | TBool
    | TFun of ty * ty
    | TVar of string
```

$$\text{TVar } \alpha \doteq \text{TFun}(\text{TInt}, \text{TInt})$$

$$\text{TVar } \beta \doteq \text{TFun}(\text{TBool}, \text{TVar } \alpha)$$

$$\text{TVar } \beta \doteq \text{TFun}(\text{TVar } \alpha, \text{TInt})$$

$$\alpha \doteq \text{int} \to \text{int}$$

$$\beta \doteq \text{bool} \to \alpha$$

$$\beta \doteq \alpha \to \text{int}$$

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$$\mathcal{S} = \{x_1 \mapsto t_1, x_2 \mapsto t_2, \ldots, x_i \mapsto t_i\}$$

incr i

We write $\mathcal{S}t$ for $[t_i/x_i]\ldots[t_1/x_1]t$

incr i

Unifiers (2)

A solution must have the property that it **satisfies** every equation

$$
\begin{gathered}
\mathcal{S} t_1 = \mathcal{S} s_1 \\
\mathcal{S} s_2 = \mathcal{S} t_2 \\
\vdots \\
\mathcal{S} s_k = \mathcal{S} t_k
\end{gathered}
$$

Example

$$S = \{ \\ b \mapsto \text{int}, \\ c \mapsto \text{int} \to \text{int}, \\ d \mapsto \text{int} \\ a \mapsto \text{int} \to e \\ \}$$

$$a \doteq d \to e$$
$$c \doteq \text{int} \to d$$
$$\text{int} \to (\text{int} \to \text{int}) \doteq b \to c$$

$$S\,a = \text{int} \to e$$
$$\parallel$$
$$S(d \to e) = \text{int} \to e$$

Exercise: check others

Unification may Fail

$$a \doteq b \to c$$
$$b \doteq a \to \text{int}$$

Not all unification problems have solutions...

Cyclic

Most General Unifiers

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<u>Ex.</u>

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\end{aligned}
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$S = \{$
$b \mapsto int$
$c \mapsto int \to int$
$d \mapsto int$
$a \mapsto int \to e$
$\}$

$S'' = \{ e \mapsto bool \}$

$S' = \{$
$b \mapsto int$
$c \mapsto int \to int$
$d \mapsto int$
$a \mapsto int \to bool$
$e \mapsto bool$
$\}$

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Process each equation, updating *the collection of equations*, **FAIL** if we reach an unsatisfiable equation

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When we see an assignment, it *becomes part of our solution*

And we're guaranteed to get a most general unifier

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      $\mathcal{U} \leftarrow \mathcal{U} \backslash \{eq\}$

      perform the substitution $\alpha \mapsto t$ to *every* equation in $\mathcal{U}$

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OTHERWISE $\implies$ **FAIL**

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WHILE eq ∈ U:   // U is not empty

  MATCH eq:

    t₁ ≐ t₂ when t₁ = t₂ ⟹ U ← U\{eq}   // t₁ and t₂ are syntactically equal, remove eq

    s₁ → t₁ ≐ s₂ → t₂ ⟹ U ← U\{eq} ∪ {s₁ ≐ s₂, t₁ ≐ t₂} // remove eq and add s₁ ≐ s₂ and t₁ ≐ t₂

    α ≐ t or t ≐ α where α ∉ FV(t) ⟹ // type variable α does not appear free in t

      S ← S ∪ {α ↦ t}   // add α ↦ t to S

      U ← U\{eq}

      perform the substitution α ↦ t to every equation in U

    OTHERWISE ⟹ FAIL

RETURN S
```

Example

$$\cancel{a \doteq d \to e} \quad \text{(asgn)}$$

$$\cancel{c \doteq \mathsf{int} \to d} \quad \text{(asgn)}$$

$$\cancel{\mathsf{int} \to (\mathsf{int} \to \mathsf{int}) \doteq b \to \cancel{x}} \quad \text{(fun)}$$

$$\cancel{(\mathsf{int} \to d)}$$

$$\cancel{\mathsf{int} \doteq b} \quad \text{(asgn)}$$

$$\cancel{\mathsf{int} \to \mathsf{int} \doteq \mathsf{int} \to d} \quad \text{(fun)}$$

$$\cancel{\mathsf{int} \doteq \mathsf{int}} \quad \text{(eq)}$$

$$\cancel{\mathsf{int} \doteq d} \quad \text{(asgn)}$$

$$S = \{$$

$$a \mapsto d \to e$$

$$c \mapsto \mathsf{int} \to d$$

$$b \mapsto \mathsf{int}$$

$$d \mapsto \mathsf{int}$$

$$\}$$

$$a$$

$$\Downarrow$$

$$S\,a = d \to e$$

$$\Downarrow$$

$$\mathsf{int} \to e$$

Example

$S = \{$

$a \mapsto b \to c$

(asg) $\cancel{a \doteq b \to c}$

(asg) $b \doteq \cancel{a} \to \text{int}$

$(b \to c)$

$b \in FV((b \to c) \to \text{int})$

$\Downarrow$

FAIL

Summary

Unification is used to solve a collection of constraints generated by constraint-based inference

Not all unification problems have solutions. In the type unification problem, this indicates a type error