The Substitution Model

Concepts of Programming Languages

CAS CS 320

Outline

» Discuss **substitution** and the pitfalls to avoid

» Demo an **implementation** of the lambda calculus

» *If we have time:* Discuss the difference between **lexical** and **dynamic** scoping

Recap

Recall: Lambda Calculus

(fun x -> e)

```
<expr> ::= λ<var>.<expr>
         | <var>
         | <expr><expr>
```

syntax

f (fun x -> x)

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syntax

$$\frac{e_1 \longrightarrow e_1'}{e_1 e_2 \longrightarrow e_1' e_2} \qquad \frac{e_2 \longrightarrow e_2'}{(\lambda x \,.\, e_1) e_2 \longrightarrow (\lambda x \,.\, e_1) e_2'}$$

$$\frac{}{(\lambda x \,.\, e)(\lambda y \,.\, e') \longrightarrow [(\lambda y \,.\, e')/x]e}$$

small-step call-by-value

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$$(\lambda x\,.\,x + x + x + x)e$$

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This is good if the variable appears several times in the body of our function

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This is also called **eager**, or **applicative**, or **strict** evaluation (and is what OCaml does)

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$$(\lambda x\,.\,\lambda y\,.\,x)e_1e_2$$

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If a variables doesn't appear in our function, then the argument is *not evaluated at all*

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If a variables doesn't appear in our function, then the argument is *not evaluated at all*

Or if an argument is only seldomly used, it will only be computed when it is used (e.g, if its computed in a branch of an if-expression that is almost never reached)

Practice Problem

$$(\lambda x\,.\,\lambda y\,.\,y)((\lambda z\,.\,z)(\lambda q\,.\,q)) \Downarrow \lambda y\,.\,y$$

```
Give a derivation of the above judgment in both
versions of the big-step semantics
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Answer

$$\dfrac{}{\lambda x\,.\,e \Downarrow \lambda x\,.\,e}\ (1)$$

$$\dfrac{e_1 \Downarrow \lambda x\,.\,e \qquad e_2 \Downarrow v_2 \qquad [v_2/x]e \Downarrow v}{e_1 e_2 \Downarrow v}\ (2)$$

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$$(\lambda x\,.\,\lambda y\,.\,y)((\lambda z\,.\,z)(\lambda q\,.\,q)) \Downarrow \lambda y\,.\,y$$

$$[\lambda q.q / z]\, z = \lambda q.q$$

$$[\lambda q.q / x]\,(\lambda y.y) = \lambda y.y$$

$$\dfrac{\dfrac{}{\lambda \overset{x}{x}.\overset{e}{(\lambda y.y)} \Downarrow \lambda \overset{x}{\boxed{x}}\overset{e}{\boxed{\lambda y.y}}}\ (1) \qquad \dfrac{\dfrac{}{\lambda z.z \Downarrow \lambda \overset{x}{\boxed{z}}.\overset{e}{\boxed{z}}}\ (1) \qquad \dfrac{}{\lambda q.q \Downarrow \overset{v_2}{\boxed{\lambda q.q}}}\ (1) \qquad \dfrac{}{\lambda q.q \Downarrow \lambda q.q}\ (1)}{\overset{e_1}{(\lambda z.z)}\overset{e_2}{(\lambda q.q)} \Downarrow \overset{v_2}{\boxed{\lambda q.q}}}\ (2) \qquad \dfrac{}{\lambda y.y \Downarrow \lambda y.y}\ (1)}{\overset{e_1}{(\lambda x.\lambda y.y)}\overset{e_2}{((\lambda z.z)(\lambda q.q))} \Downarrow \overset{v}{\lambda y.y}}\ (2)$$

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Ex. CBN deriv.

Substitution

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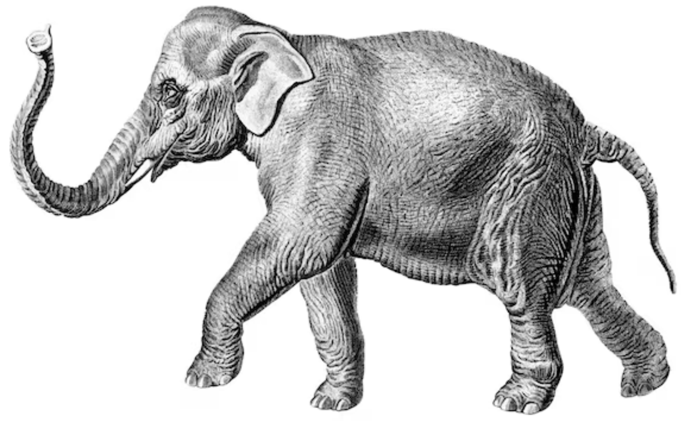

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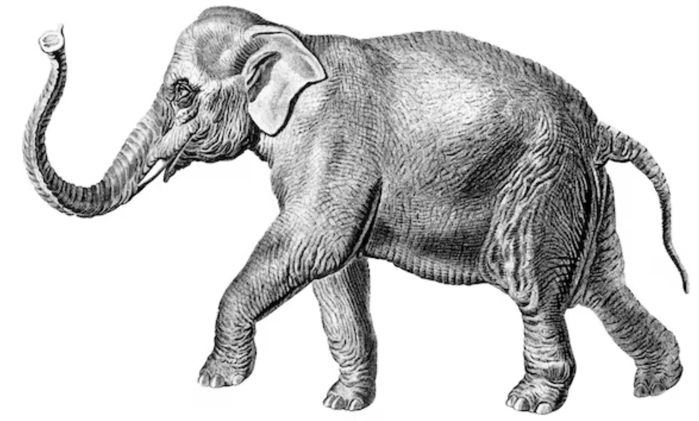

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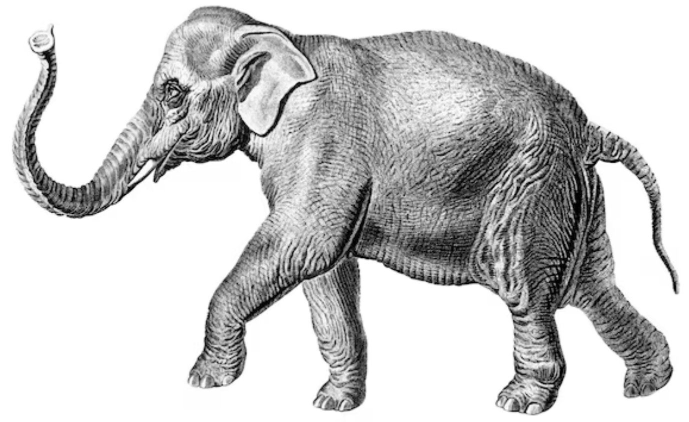

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We've been able to get by on our intuitions for a while, but our intuitions won't help us *implement* substitution (which is *difficult*)

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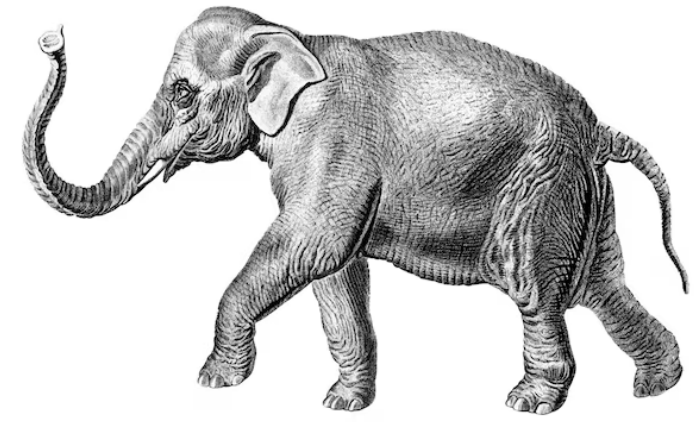

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It's time to get more formal about **substitution**

We've been able to get by on our intuitions for a while, but our intuitions won't help us *implement* substitution (which is *difficult*)

We need to understand why...

Recall: Notation

$$[y/x](\lambda x\,.\,y)$$

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Informally. *Replace every instance of* x *with* v

Already things start to break down with this informal definition, e.g., consider the above substitution...

Our Primary Concern

$$[y/x](\lambda x\,.\,y)$$

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However we define substitution shouldn't *change the underlying behavior of a function*

The Key Point: A function does not depend on our choice of variable names

α-Equivalence

```
let x = 2 in x + 1
```

$=_\alpha$

```
let z = 2 in z + 1
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OCaml

$$\lambda x\,.\,\lambda y\,.\,x =_\alpha \lambda v\,.\,\lambda w\,.\,v$$

λ–calculus

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The **principle of name irrelevance** says that any two programs that are the same up to "renaming of variables" should behave exactly the same way (they are α-**equivalent**)

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We say that a variable x is **bound** is an expression if it appears in the expression as $(\ldots \lambda x . e \ldots)$

Substitution should preserve this

Preserving α-equivalent

$$[y/x](\lambda x\,.\,y) \quad =_\alpha [y/x](\lambda z\,.\,y)$$

$$\lambda y\,.\,y \quad \neq_\alpha \quad \lambda z\,.\,y$$

What does it mean to *preserve α–equivalence*?

The idea: If two expressions are α–equivalent, then the should ***remain*** α–equivalent after any substitution

Definition (First Attempt)

$$[v/y]x = \begin{cases} v & x = y \\ x & \texttt{else} \end{cases} \tag{1}$$

$$[v/y](\lambda x\,.\,e) = \lambda x\,.\,[v/y]e \tag{2}$$

$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2) \tag{3}$$

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1. Replace every y with v, leave other variables
2. Replace y with v in the body of a function
3. Replace y with v in both subexpressions of an application

(This is an example of an *inductive definition*)

$$[v/y]x = \begin{cases} v & x = y \\ x & \texttt{else} \end{cases}$$

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Example

$$[\lambda z . z / y](\lambda x . y(xy))$$

$$\cancel{[y/\lambda z \,.\, z]}(\lambda x \,.\, y(xy))$$

$$\lambda x . [\lambda z . z / y]\, y(xy) =$$

$$\lambda x . ([\lambda z . z / y]\, y)([\lambda z . z / y](xy)) =$$

$$\lambda x . (\lambda z . z)(x(\lambda z . z))$$

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Problem Case I

$$[y/x](\lambda x\,.\,x) = \lambda x.\,y$$

$$\|_\alpha \qquad \not\|_\alpha$$

$$[y/x](\lambda z.\,z) = \lambda z.\,z$$

We shouldn't be allowed to substitute x if it's the argument of a function

This may *change the behavior* of a function

Definition (Second Attempt)

$$[v/y]x = \begin{cases} v & x = y \\ x & \texttt{else} \end{cases}$$

$$[v/y](\lambda x\,.\,e) = \boxed{\begin{cases} \lambda x\,.\,e & x = y \\ \lambda x\,.\,[v/y]e & \texttt{else} \end{cases}}$$

$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2)$$

We can handle the problem case directly in our definition. *Check the bound variable before we substitute in the body of a function*

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Is there still a problem?

$$[v/y]x = \begin{cases} v & x = y \\ x & \texttt{else} \end{cases}$$

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Problem Case II

$$[y/x](\lambda y\,.\,x) = \lambda y.y$$

$$\Vert_\alpha \qquad\qquad \not\Vert_\alpha$$

$$[y/x]\,\lambda z.x = \lambda z.x$$

We're not replacing a bound variable, but we *are* substituting an expression that has variables which *became* bound

The variable y is said to be **captured** in this (incorrect) substitution

Free and Bound Variables

$$FV(x) = \{x\} \tag{1}$$

$$FV(\lambda x \,.\, e) = FV(e) \setminus \{x\} \tag{2}$$

$$FV(e_1 e_2) = FV(e_1) \cup FV(e_2) \tag{3}$$

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1. x is free in x

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Free and Bound Variables

$$FV(\lambda x . yx) = FV(yx) \setminus \{x\} = (FV(y) \cup FV(x)) \setminus \{x\} = (\{y\} \cup \{x\}) \setminus \{x\} = \{y, x\} \setminus \{x\} = \{y\}$$

set sub.

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<u>Definition.</u> A variable x is **free** in e if it does not appear **bound** by a λ. Formally:

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Definition. A variable x is **free** in e if $x \in FV(e)$ as above

Definition (Third Attempt)

rename bound variable

$$[v/y]x = \begin{cases} v & x = y \\ x & \texttt{else} \end{cases}$$

$$[v/y](\lambda x\,.\,e) = \begin{cases} \lambda x\,.\,e & x = y \\ \lambda z\,.\,[v/y][z/x]e & x \in FV(v) \\ \lambda x\,.\,[v/y]e & \texttt{else} \end{cases}$$

$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2)$$

Since we're interested in α–equivalence, we can first *replace* the bound variable and *substitute* it in the body of the function. This is called **α–renaming**

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Is there still a problem?

Problem Case III

$$FV(x) = \{x\}$$
$$FV(\lambda x . e) = FV(e) \backslash \{x\}$$
$$FV(e_1 e_2) = FV(e_1) \cup FV(e_2)$$

$$[v/y]x = \begin{cases} v & x = y \\ x & \texttt{else} \end{cases}$$

$$[v/y](\lambda x . e) = \begin{cases} \lambda x . e & x = y \\ \lambda z . [w/z][z/x]e & x \in FV(v) \\ \lambda x . [v/y]e & \texttt{else} \end{cases}$$

$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2)$$

$$[x/y](\lambda \overset{z}{\cancel{x}} . \overset{z}{\cancel{x}} yz) =$$

$$\lambda z . z x z$$

free

bound

This isn't exactly a problem, but *we have to be careful about which variable to replace the bound variable x with*

If we choose z, then we capture a *different* variable!

"Correct" Definition

$$[v/y]x = \begin{cases} v & x = y \\ x & \texttt{else} \end{cases}$$

$$[v/y](\lambda x\,.\,e) = \begin{cases} \lambda x\,.\,e & x = y \\ \lambda z\,.\,[v/y][z/x]e & x \in FV(v), z \texttt{ is fresh} \\ \lambda x\,.\,[v/y]e & \texttt{else} \end{cases}$$

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Finally a definition, that works. Sort of...

"Correct" Definition

$$[v/y]x = \begin{cases} v & x = y \\ x & \texttt{else} \end{cases}$$

$$[v/y](\lambda x\,.\,e) = \begin{cases} \lambda x\,.\,e & x = y \\ \lambda z\,.\,[v/y][z/x]e & x \in FV(v), z \text{ is fresh} \\ \lambda x\,.\,[v/y]e & \texttt{else} \end{cases}$$

$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2)$$

Finally a definition, that works. Sort of...

The only problem with this definition is that it now poses an *implementation issue*. **How do we come up with z?**

Well-Scopedness/Closedness

open

$$\lambda x \,.\, y$$

closed

$$\lambda x \,.\, \lambda y \,.\, y$$

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Closed terms are well-scoped

Our Solution: Well-Scopedness Check

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$$[v/y](e_1 e_2) = ([v/y]e_1)([v/y]e_2)$$

If we only work with closed (well-scoped) expressions, then we don't need to worry about captured variables. The condition requiring α-renaming never holds!

(**Hint:** In mini-project 1, you should check if the expression has a free variable *before* you evaluate it)

demo

(lambda calculus)

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Variable Scoping

Two Major Concerns

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<u>OCaml variables are:</u>

» immutable

» binding defined

» lexically scoped

Mutability

```ocaml
let x = 0
let f () =
  let x = 1 in
  ()
print_int x
```

Immutable (OCaml)

```python
x = 0
def f():
    global x
    x = 1
print(x)
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Mutable (Python)

Definition. *(informal)* A variable is **mutable** if we are allowed to change its value after it has been declared

We think of variables as:

» names if they're immutable

» (abstract) memory locations when they're mutable

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» the scope of a function

Dynamic Scoping

```bash
f() { x=23; g; }
g() { y=$x; }
f
echo $y
```

Bash

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Dynamic scoping refers to when bindings are determined at runtime based on *computational context*

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Bash

Dynamic scoping refers to when bindings are determined at runtime based on *computational context*

This is a *temporal view*, i.e., what a computation done beforehand which affected the value of a variable

Lexical Scoping

```python
x = 0
def f():
    x = 1
    return x
assert(f() == 1)
assert(x == 0)
```

Python

```ocaml
let x = 0
let f () =
  let x = 1 in
  x
let _ = assert (f () = 1)
let _ = assert (x = 0)
```

OCaml

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<u>There are two common ways lexical scope is determined:</u>

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OCaml

Lexical (static) scoping refers to the use of textual delimiters to define the scope of a binding

<u>There are two common ways lexical scope is determined:</u>

» The binding defines it's own scope (let-bindings)

» A block defines the scope of a variable (python functions)

Tradeoffs

```
f() { x=23; g; }
g() { y=$x; }
f
echo $y
```

dynamic

vs.

```
let x = 0
let f () =
  let x = 1 in
  x
let _ = assert (f () = 1)
let _ = assert (x = 0)
```

lexical

Implementing dynamic scoping is *way* easier...(we'll see this in lab)

But **every modern programming language** implements lexical scoping

Looking Ahead: Didn't we do this?

```
let x = v in ...
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We've already implemented lexical scoping using the substitution model (mini-project 1) *Why do it again?*

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Answer. The substitution model is inefficient

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We've already implemented lexical scoping using the substitution model (mini-project 1) *Why do it again?*

Answer. The substitution model is inefficient

Each substitution has to "crawl" through the *entire remainder of the program*

Next Time: The Environment Model

$$\langle \mathcal{E}, e \rangle \Downarrow v$$

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`Idea.` `We keep track of their values in an` *`environment`*

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Idea. We keep track of their values in an *environment*

And evaluate *relative* to the environment, *lazily* filling in variable values along the way

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*The **configurations** in our semantics will have nonempty state*

Summary

Substitution is a bit tricky to define correctly but any definition must preserve α−equivalence

The **scoping** paradigm of a PL determines when/where variable bindings are available