

# **Principle Types**

## **Concepts of Programming Languages**

# Outline

- » Demo an implementation of **unification**
- » Discuss **principle types**

# Practice Problem

$$\cdot \vdash \lambda x . xx : \tau \dashv \mathcal{C}$$

*Determine the type  $\tau$  and constraints  $\mathcal{C}$  such that the above judgment is derivable*

$$\frac{(x : \forall \alpha_1 . \forall \alpha_2 \dots \forall \alpha_k . \tau) \in \Gamma \quad \beta_1, \dots, \beta_k \text{ are fresh}}{\Gamma \vdash x : [\beta_1/\alpha_1] \dots [\beta_k/\alpha_k] \tau \dashv \emptyset} \text{ (var)} \qquad \frac{\alpha \text{ is fresh} \quad \Gamma, x : \alpha \vdash e : \tau \dashv \mathcal{C}}{\Gamma \vdash \lambda x . e : \alpha \rightarrow \tau \dashv \mathcal{C}} \text{ (fun)}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathcal{C}_1 \quad \Gamma \vdash e_2 : \tau_2 \dashv \mathcal{C}_2 \quad \alpha \text{ is fresh}}{\Gamma \vdash e_1 e_2 : \alpha \dashv \tau_1 \doteq \tau_2 \rightarrow \alpha, \mathcal{C}_1, \mathcal{C}_2} \text{ (app)}$$

**Answer**

$$\cdot \vdash \lambda x . xx : \tau \dashv \mathcal{C}$$

# Recap

# Recall: Unification

$$\begin{aligned}a &\doteq d \rightarrow e \\c &\doteq \text{int} \rightarrow d \\ \text{int} \rightarrow \text{int} \rightarrow \text{int} &\doteq b \rightarrow c\end{aligned}$$

**Unification** is the process of solving a system of equations over *symbolic* expressions

# Recall: Type Unification Problem

A **unification problem** is a collection of equations of the form

$$\begin{array}{c} s_1 \doteq t_1 \\ s_2 \doteq t_2 \\ \vdots \\ s_k \doteq t_k \end{array}$$

where  $s_1, \dots, s_k$  and  $t_1, \dots, t_k$  are **types**

# Recall: Unifiers

A **unifier** is a sequence of substitutions to variables, typically written

$$\mathcal{S} = \{x_1 \mapsto t_1, x_2 \mapsto t_2, \dots, x_n \mapsto t_n\}$$

s.t.

$$\mathcal{S}t_1 = \mathcal{S}s_1$$

$$\mathcal{S}s_2 = \mathcal{S}t_2$$

$$\vdots$$

$$\mathcal{S}s_k = \mathcal{S}t_k$$



# Recall: Most General Unifiers

A **most general unifier** of a unification problem is a solution  $\mathcal{S}$  such that, for any solution  $\mathcal{S}'$ , there is another solution  $\mathcal{S}''$  such that  $\mathcal{S}' = \mathcal{S}\mathcal{S}''$

In other words,  $\mathcal{S}'$  is  $\mathcal{S}$  *with more substitutions*

# **An Algorithm (Pseudocode)**

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**RETURN**  $\mathcal{S}$

# Another Practice Problem

$$\beta \doteq \eta$$

$$\alpha \rightarrow \beta \doteq \alpha \rightarrow \gamma$$

$$\alpha \rightarrow \beta \doteq \gamma \rightarrow \eta$$

$$\alpha \rightarrow \beta \doteq \text{int} \rightarrow \eta$$

*Determine a most general unifier to the above type unification problem using the algorithm we just gave*

# Answer

$$\beta \doteq \eta$$

$$\alpha \rightarrow \beta \doteq \alpha \rightarrow \gamma$$

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demo  
(unification)

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$$\text{principle}(\tau, \mathcal{C}) = \forall \alpha_1 \dots \forall \alpha_k. \mathcal{S}\tau \text{ where } \text{FV}(\mathcal{S}\tau) = \{\alpha_1, \dots, \alpha_k\}$$

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i.e, the **principle type** of  $e$  (note: it may not exist). Every type we *could* give  $e$  is a *specialization* of  $\forall \alpha_1, \dots, \alpha_k. \mathcal{S}\tau$

# Example

*Determine the principle type of  $\lambda f. \lambda x. f(x + 1)$*

# Example

Show that  $\text{let } f = \lambda x. x \text{ in } f (f\ 2 = 2)$  has no principle type



**Putting everything together (is\_well\_typed)**

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3. *Generalization*: Quantify over the free variables in  $\mathcal{S}\tau$  to get the principle type  $\forall \alpha_1 \dots \forall \alpha_k. \mathcal{S}\tau$  of  $e$

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4. Add  $(x : \forall \alpha_1 \dots \forall \alpha_k. \mathcal{S}\tau)$  to  $\Gamma$



# Example

```
let id = fun x -> x  
let _ = id (id 2 = 2)
```

# As a Type System

$$\frac{}{\Gamma \vdash \epsilon} \text{ (emptyProg)}$$

$$\frac{\Gamma \vdash e : \tau \dashv \mathcal{C} \quad \tau' = \text{principle}(\tau, \mathcal{C}) \quad \Gamma, x : \tau' \vdash P}{\Gamma \vdash \text{let } x = e \text{ } P} \text{ (topLet)}$$

We can also express this as a type system with judgments of the form  $\Gamma \vdash P$ , where  $P$  is a program (note there is no `":"`)

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$\emptyset \vdash \text{let id} = \text{fun x} \rightarrow \text{x} \text{ let a} = \text{id (id 2 = 2)}$  (topLet)

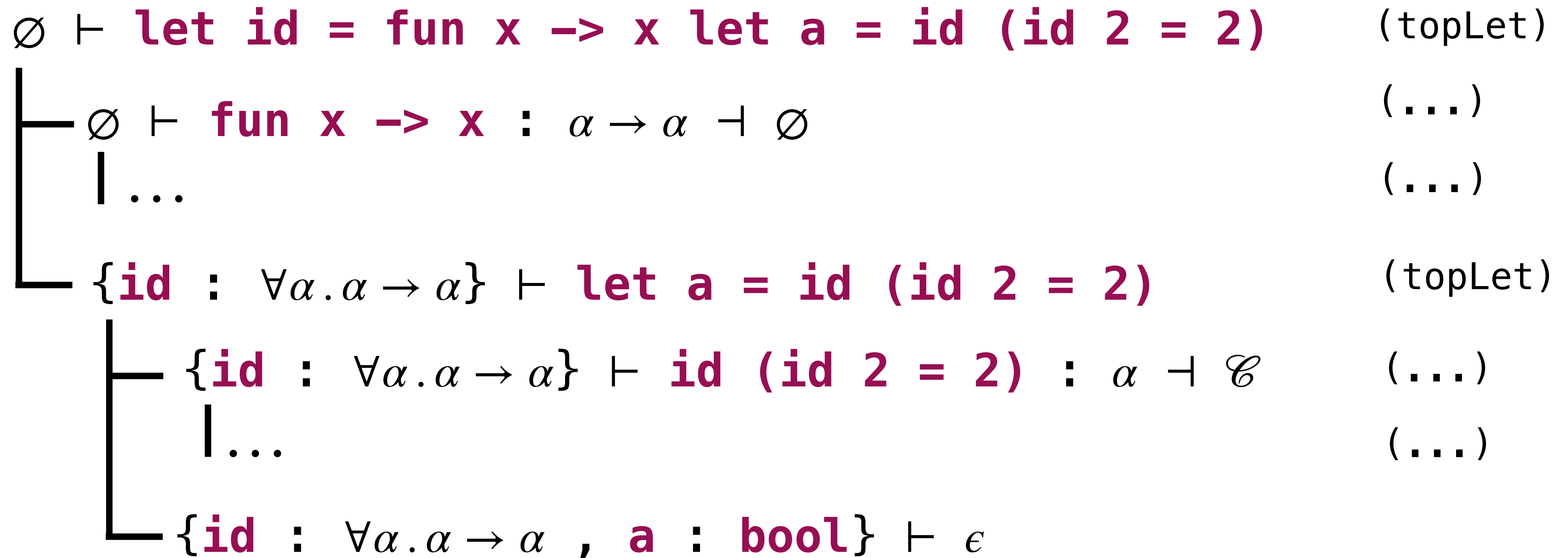
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$\emptyset \vdash$	<b>let id = fun x -&gt; x let a = id (id 2 = 2)</b>	(topLet)	
┌	$\emptyset \vdash$	<b>fun x -&gt; x</b> : $\alpha \rightarrow \alpha \dashv \emptyset$	(...)
		...	(...)
	└	<b>{id : <math>\forall \alpha. \alpha \rightarrow \alpha</math>}</b> $\vdash$ <b>let a = id (id 2 = 2)</b>	

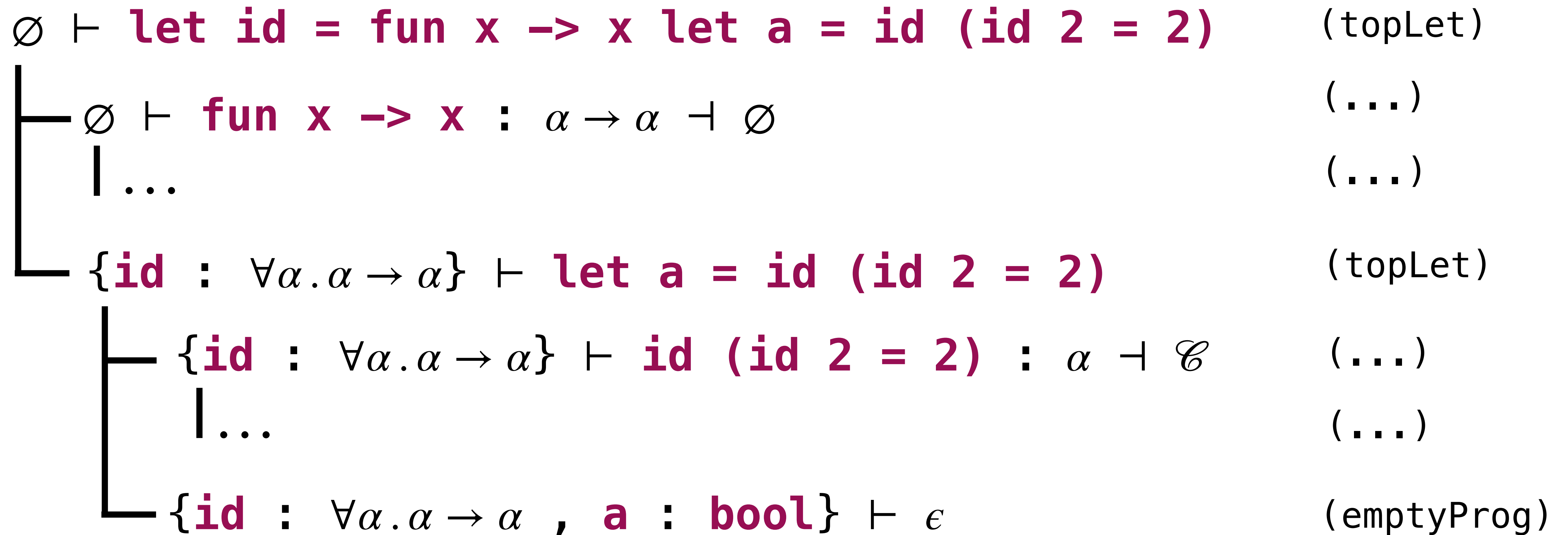
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# Summary

The **principle type** of an expression is the most general type we could give it

Our unification algorithm gives us a **most general unifier**, which we use to get the principle type of an expression