

Type Inference

Concepts of Programming Languages

Outline

- » Discuss **polymorphism** in general
- » Discuss **type inference** with eye towards **Hindley-Milner typing**
- » Look at a set of typing rules for **constraint-based inference**
- » Walk through some **examples**

Practice Problem

```
fun f -> fun x -> f (x + 1)
```

```
let rec f x = f (f (x + 1)) in f
```

What are the types of the above OCaml expressions?

Answer

① $\text{fun } f \text{ :int} \rightarrow \alpha \rightarrow \text{fun } x \text{ :int} \rightarrow \boxed{f \ (x + 1)}$

② $\text{let rec } f \text{ :int} \ x = f \ (f \ (x + 1)) \text{ in } f$

① $(\text{int} \rightarrow \alpha) \rightarrow \text{int} \rightarrow \alpha$

② $\text{int} \rightarrow \text{int}$

Polymorphism

Explicit Typing

```
let add (x : int) (y : int) : int = x + y
let k (x : int) (y : bool) : int = x
let _ : unit = assert(add 2 3 = k 5 false)
```

Explicit Typing

```
let add (x : int) (y : int) : int = x + y
let k (x : int) (y : bool) : int = x
let _ : unit = assert(add 2 3 = k 5 false)
```

In mini-project 2, we're implementing a PL with **explicit typing**

Explicit Typing

```
let add (x : int) (y : int) : int = x + y
let k (x : int) (y : bool) : int = x
let _ : unit = assert(add 2 3 = k 5 false)
```

In mini-project 2, we're implementing a PL with **explicit typing**

Every function argument and let-expression is annotated with typing information

Explicit Typing

```
let add (x : int) (y : int) : int = x + y
let k (x : int) (y : bool) : int = x
let _ : unit = assert(add 2 3 = k 5 false)
```

In mini-project 2, we're implementing a PL with **explicit typing**

Every function argument and let-expression is annotated with typing information

This is closer to what is done in a PL like **Java**

Implicit Types

```
let add x y = x + y
let k x y = x
let _ = assert(add 2 3 = k 5 false)
```

Implicit Types

```
let add x y = x + y  
let k x y = x  
let _ = assert (add 2 3 = k 5 false)
```

We rarely have to specify types in OCaml

Implicit Types

```
let add x y = x + y
let k x y = x
let _ = assert (add 2 3 = k 5 false)
```

We rarely have to specify types in OCaml

Type inference, or type *reconstruction* is the process of determining what type we *could* have annotated our program with

Implicit Types

```
let add x y = x + y
let k x y = x
let _ = assert (add 2 3 = k 5 false)
```

We rarely have to specify types in OCaml

Type inference, or type *reconstruction* is the process of determining what type we *could* have annotated our program with

*But what type should we give **k**?*

High Level

```
let rec rev_int (l : int list) : int list =  
  match l with  
  | [] -> []  
  | x :: l -> rev l @ [x]
```

```
let rec rev_string (l : string list) : string list =  
  match l with  
  | [] -> []  
  | x :: l -> rev l @ [x]
```

```
let _ = assert (rev_int [1;2;3] = [3;2;1])  
let _ = assert (rev_string ["1";"2";"3"] = ["3";"2";"1"])
```

High Level

```
let rec rev_int (l : int list) : int list =  
  match l with  
  | [] -> []  
  | x :: l -> rev l @ [x]
```

```
let rec rev_string (l : string list) : string list =  
  match l with  
  | [] -> []  
  | x :: l -> rev l @ [x]
```

```
let _ = assert (rev_int [1;2;3] = [3;2;1])  
let _ = assert (rev_string ["1";"2";"3"] = ["3";"2";"1"])
```

Copy/pasting code is *time consuming* and *error prone*

High Level

```
let rec rev_int (l : int list) : int list =  
  match l with  
  | [] -> []  
  | x :: l -> rev l @ [x]  
  
let rec rev_string (l : string list) : string list =  
  match l with  
  | [] -> []  
  | x :: l -> rev l @ [x]  
  
let _ = assert (rev_int [1;2;3] = [3;2;1])  
let _ = assert (rev_string ["1";"2";"3"] = ["3";"2";"1"])
```

Copy/pasting code is *time consuming* and *error prone*

Polymorphism allows for better code reuse. The *same* function can be applied in multiple contexts

Basic Example

```
let id = fun x -> x
let a = id 0
let b = id (0 = 0)
let c = id id
```

Basic Example

```
let id = fun x -> x
let a = id 0
let b = id (0 = 0)
let c = id id
```

We want to be able to define functions that can be used in multiple contexts *and* that we can type check

Basic Example

```
let id = fun x -> x
let a = id 0
let b = id (0 = 0)
let c = id id
```

We want to be able to define functions that can be used in multiple contexts *and* that we can type check

Important: We can evaluate this if we *don't* type check

Basic Example

```
let id = fun x -> x
let a = id 0
let b = id (0 = 0)
let c = id id
```

We want to be able to define functions that can be used in multiple contexts *and* that we can type check

Important: We can evaluate this if we *don't* type check

*But if we type-check, what should be the type of **id**?*

Polymorphism

Polymorphism

There are two common kinds of polymorphism

Polymorphism

There are two common kinds of polymorphism

1. **Ad Hoc Polymorphism:** The ability to overload function names so that different types can share interfaces

Polymorphism

There are two common kinds of polymorphism

1. **Ad Hoc Polymorphism:** The ability to overload function names so that different types can share interfaces
2. **Parametric polymorphism:** The ability to define functions that are *agnostic* to (parts of) the types, giving it more reusability

Polymorphism

There are two common kinds of polymorphism

1. **Ad Hoc Polymorphism:** The ability to overload function names so that different types can share interfaces

2. **Parametric polymorphism:** The ability to define functions that are *agnostic* to (parts of) the types, giving it more reusability

our focus

Aside: Ad Hoc Polymorphism

```
let add (x : float) (y : float) = x +. y  
let add (x : string) (y : string) = x ^ y  
(* This doesn't work in OCaml... *)
```

Aside: Ad Hoc Polymorphism

```
let add (x : float) (y : float) = x +. y  
let add (x : string) (y : string) = x ^ y  
(* This doesn't work in OCaml... *)
```

Ad hoc polymorphism is essentially **function overloading**

Aside: Ad Hoc Polymorphism

```
let add (x : float) (y : float) = x +. y  
let add (x : string) (y : string) = x ^ y  
(* This doesn't work in OCaml... *)
```

Ad hoc polymorphism is essentially **function overloading**

Functions can be defined and used for different types of inputs

Aside: Ad Hoc Polymorphism

```
let add (x : float) (y : float) = x +. y
let add (x : string) (y : string) = x ^ y
(* This doesn't work in OCaml... *)
```

Ad hoc polymorphism is essentially **function overloading**

Functions can be defined and used for different types of inputs

Then we can define code against *interfaces* (this is common in object oriented programming)

Parametric Polymorphism

```
let id = fun x -> x
let a = id 0
let b = id (0 = 0)
let c = id id
```

Parametric Polymorphism

```
let id = fun x -> x
let a = id 0
let b = id (0 = 0)
let c = id id
```

Parametric polymorphism allows for functions which are agnostic to the types of its inputs (this is what OCaml does)

Parametric Polymorphism

```
let id = fun x -> x
let a = id 0
let b = id (0 = 0)
let c = id id
```

Parametric polymorphism allows for functions which are agnostic to the types of its inputs (this is what OCaml does)

For example, we can write a single identity function and use it in multiple contexts

There are many subtleties
to this...

Subtlety 1: Type Annotations

```
let rec rev ('a list) : 'a list =  
  match l with  
  | [] -> []  
  | x :: l -> rev l @ [x]
```

```
let id : 'a -> 'a = fun x -> x
```

Subtlety 1: Type Annotations

```
let rec rev ('a list) : 'a list =  
  match l with  
  | [] -> []  
  | x :: l -> rev l @ [x]
```

```
let id : 'a -> 'a = fun x -> x
```

Parametric polymorphism is *not* just removing type annotations

Subtlety 1: Type Annotations

```
let rec rev ('a list) : 'a list =  
  match l with  
  | [] -> []  
  | x :: l -> rev l @ [x]
```

```
let id : 'a -> 'a = fun x -> x
```

Parametric polymorphism is *not* just removing type annotations

It's also not the same as having no type system

Subtlety 1: Type Annotations

```
let rec rev ('a list) : 'a list =  
  match l with  
  | [] -> []  
  | x :: l -> rev l @ [x]
```

```
let id : 'a -> 'a = fun x -> x
```

Parametric polymorphism is *not* just removing type annotations

It's also not the same as having no type system

There are type systems *with* polymorphism that *require* type annotations

Subtlety 2: Type Inference

```
let rec rev ('a list) : 'a list =  
  match l with  
  | [] -> []  
  | x :: l -> rev l @ [x]
```

```
let id : 'a -> 'a = fun x -> x
```

Subtlety 2: Type Inference

```
let rec rev ('a list) : 'a list =  
  match l with  
  | [] -> []  
  | x :: l -> rev l @ [x]
```

```
let id : 'a -> 'a = fun x -> x
```

Polymorphism is *not* the same as having type inference

Subtlety 2: Type Inference

```
let rec rev ('a list) : 'a list =  
  match l with  
  | [] -> []  
  | x :: l -> rev l @ [x]
```

```
let id : 'a -> 'a = fun x -> x
```

Polymorphism is *not* the same as having type inference

In OCaml, polymorphism is deeply connected with its type inference system, but they are distinct (we can choose to annotate all our OCaml code)

Subtlety 3: Dispatch

```
let to_string (x : 'a) : string = ...  
(* This is not possible in OCaml *)
```

Parametric polymorphism cannot be used for *dispatch*

We can't write a polymorphic function that "checks the type" to see what to do

Implementing Polymorphism

Implementing Polymorphism

There are a couple approaches to implementing parametric polymorphism:

Implementing Polymorphism

There are a couple approaches to implementing parametric polymorphism:

» **OCaml (Hindley-Milner):** Infer the "most general" polymorphic type

Implementing Polymorphism

There are a couple approaches to implementing parametric polymorphism:

- » **OCaml (Hindley-Milner):** Infer the "most general" polymorphic type
- » **System F (2nd-Order λ -Calculus):** take types as arguments!

Implementing Polymorphism

There are a couple approaches to implementing parametric polymorphism:

- » **OCaml (Hindley-Milner)**: Infer the "most general" polymorphic type
- » **System F (2nd-Order λ -Calculus)**: take types as arguments!

Either way, we have to introduce the notion of a *type variable*

Type Variables

```
let id : 'a -> 'a = fun x -> x
```

Type Variables

```
let id : 'a -> 'a = fun x -> x
```

The "parametric" part is the fact that types have *variables*

Type Variables

```
let id : 'a -> 'a = fun x -> x
```

The "parametric" part is the fact that types have *variables*

Type variables are instantiated at particular types according to the context

Quantification

```
let id : 'a . 'a -> 'a = fun x -> x
```

Quantification

```
let id : 'a . 'a -> 'a = fun x -> x
```

In reality, types variables in OCaml are **quantified**

Quantification

```
let id : 'a . 'a -> 'a = fun x -> x
```

In reality, types variables in OCaml are **quantified**

Just like with expression variables, we don't like *unbound* type variables

Quantification

$$\forall \alpha. \alpha \rightarrow \alpha$$

```
let id : 'a . 'a -> 'a = fun x -> x
```

In reality, types variables in OCaml are **quantified**

Just like with expression variables, we don't like *unbound* type variables

We read this "**id** has type **t -> t** for any type **t**"

Interlude: Compact Derivations

The Problem

Derivations take up a lot of horizontal space

We've been careful to choose expressions with short derivations in lecture

We won't be able to do this moving forward

The Problem

$$\frac{\frac{}{\{\} \vdash 2 : \text{int}} \text{(intLit)} \quad \frac{\frac{}{\{y : \text{int}\} \vdash y : \text{int}} \text{(var)} \quad \frac{\frac{}{\{y : \text{int}\} \vdash y : \text{int}} \text{(var)}}{\{y : \text{int}\} \vdash y + y : \text{int}} \text{(intAdd)}}{\{\} \vdash \text{let } y = 2 \text{ in } y + y : \text{int}} \text{(let)}$$

Derivations take up a lot of horizontal space

We've been careful to choose expressions with short derivations in lecture

We won't be able to do this moving forward

Visualizing Trees

```
.
├── bin
│   ├── dune
│   └── main.ml
├── dune-project
├── interp2.opam
├── lib
│   ├── dune
│   ├── interp2.ml
│   ├── lexer.ml
│   ├── parser.mly
│   └── utils.ml
├── spec.pdf
├── test
│   ├── dune
│   └── test_interp2.ml
```

There are many ways of drawing trees.
Finding a "good" visualization of
trees is an art

Moving forward we'll use the *file-tree*
format for writing derivations (this
is what is done in the textbook)

It's more horizontally space-efficient

Example

Example

$\frac{}{\{\} \vdash 2 : \text{int}}$ (intLit)	$\frac{}{\{y : \text{int}\} \vdash y : \text{int}}$ (var)	$\frac{}{\{y : \text{int}\} \vdash y : \text{int}}$ (var)
	$\frac{}{\{y : \text{int}\} \vdash y + y : \text{int}}$ (intAdd)	
$\frac{}{\{\} \vdash \text{let } y = 2 \text{ in } y + y : \text{int}}$ (let)		

$\{\} \vdash \text{let } y = 2 \text{ in } y + y : \text{int} \quad (\text{let})$

$\vdash \{\} \vdash 2 : \text{int} \quad (\text{int})$
 $\vdash \{y : \text{int}\} \vdash y + y : \text{int} \quad (\text{intAdd})$

$\vdash \{y : \text{int}\} \vdash y : \text{int} \quad (\text{var})$
 $\vdash \{y : \text{int}\} \vdash y : \text{int} \quad (\text{var})$

Hindley-Milner

High Level

High Level

Hindley-Milner type systems are typed λ -calculi with parametric polymorphism

High Level

Hindley-Milner type systems are typed λ -calculi with parametric polymorphism

They underlie nearly all functional PLs currently in use (e.g., OCaml, Haskell, Elm)

High Level

Hindley-Milner type systems are typed λ -calculi with parametric polymorphism

They underlie nearly all functional PLs currently in use (e.g., OCaml, Haskell, Elm)

They allow for a *restricted* form of type quantification, in which quantifiers always appear in the "outermost" position

High Level

Hindley-Milner type systems are typed λ -calculi with parametric polymorphism

They underlie nearly all functional PLs currently in use (e.g., OCaml, Haskell, Elm)

They allow for a *restricted* form of type quantification, in which quantifiers always appear in the "outermost" position

Type inference is decidable and (fairly) efficient

Type Inference (High Level)

$$\Gamma \vdash e : \tau \dashv \mathcal{C}$$

Type Inference (High Level)

$$\Gamma \vdash e : \tau \dashv \mathcal{C}$$

The type inference process follows the rough procedure:

Type Inference (High Level)

$$\Gamma \vdash e : \tau \dashv \mathcal{C}$$

The type inference process follows the rough procedure:

1. Derive $\Gamma \vdash e : \tau$ *relative to some constraints* \mathcal{C}

Type Inference (High Level)

$$\Gamma \vdash e : \tau \dashv \mathcal{C}$$

The type inference process follows the rough procedure:

1. Derive $\Gamma \vdash e : \tau$ *relative to some constraints* \mathcal{C}
2. Use the constraints \mathcal{C} to determine the "actual" type of e in Γ

Type Inference (High Level)

$$\Gamma \vdash e : \tau \dashv \mathcal{C}$$

The type inference process follows the rough procedure:

1. Derive $\Gamma \vdash e : \tau$ *relative to some constraints* \mathcal{C} today
2. Use the constraints \mathcal{C} to determine the "actual" type of e in Γ

Example (by Intuition)

$$(int \rightarrow ?_3) \Rightarrow int \rightarrow ?_3$$

fun f -> fun x -> f (x + 1)

$$\{ \} \vdash \text{fun } f \rightarrow \text{fun } x \rightarrow f (x + 1) : ?_1 \rightarrow ?_2 \rightarrow ?_3$$

$$\vdash \{ f : ?_1 \} \vdash \text{fun } x \rightarrow f (x + 1) : ?_2 \rightarrow ?_3$$

$$\vdash \{ f : ?_1, x : ?_2 \} \vdash f (x + 1) : ?_3 \quad ?_1 = int \rightarrow ?_3$$

$$\vdash \{ f : ?_1, x : ?_2 \} \vdash f : ?_1$$

$$\vdash \{ f : ?_1, x : ?_2 \} \vdash x + 1 : int$$

$$\vdash \{ \dots \} \vdash x : ?_2$$

$$\vdash \{ \dots \} \vdash 1 : int$$

$$?_2 = int$$

$$?_1 \rightarrow ?_2 \rightarrow ?_3$$

$$?_1 = int \rightarrow ?_3$$

$$?_2 = int$$

Hindley-Milner Light (Syntax)

$\langle \text{expr} \rangle ::= \text{fun } \langle \text{var} \rangle \rightarrow \langle \text{expr} \rangle \mid \langle \text{expr} \rangle \langle \text{expr} \rangle$
 $\mid \text{let } \langle \text{var} \rangle = \langle \text{expr} \rangle \text{ in } \langle \text{expr} \rangle$
 $\mid \text{if } \langle \text{expr} \rangle \text{ then } \langle \text{expr} \rangle \text{ else } \langle \text{expr} \rangle$
 $\mid \langle \text{expr} \rangle + \langle \text{expr} \rangle \mid \langle \text{expr} \rangle = \langle \text{expr} \rangle$
 $\mid \langle \text{int} \rangle \mid \langle \text{var} \rangle$

$\langle \text{mt}y \rangle ::= \text{int} \mid \text{bool} \mid \langle \text{tyvar} \rangle \mid \langle \text{mt}y \rangle \rightarrow \langle \text{mt}y \rangle$
 $\langle \text{ty} \rangle ::= \langle \text{tyvar} \rangle . \langle \text{ty} \rangle \mid \langle \text{mt}y \rangle$

type variables

type quantifier

Hindley-Milner Light (Mathematical)

for any type α, τ

$$e ::= \lambda x. e \mid ee$$
$$\mid \text{let } x = e \text{ in } e$$
$$\mid \text{if } e \text{ then } e \text{ else } e$$
$$\mid e + e \mid e = e$$
$$\mid n \mid x$$
$$\sigma ::= \text{int} \mid \text{bool} \mid \alpha \mid \sigma \rightarrow \sigma$$
$$\tau ::= \sigma \mid \forall \alpha. \tau$$

$\lambda x. \lambda y. x : \forall \alpha \forall \beta. \alpha \rightarrow \beta \rightarrow \alpha.$

type variables

type quantification.

As usual, we'll often use concise mathematical notation for writing down inference rules and derivations

Type Variables and Type Schemes

$$\sigma ::= \text{int} \mid \text{bool} \mid \alpha \mid \sigma \rightarrow \sigma$$

$$\tau ::= \sigma \mid \forall \alpha . \tau$$

Type Variables and Type Schemes

$$\sigma ::= \text{int} \mid \text{bool} \mid \alpha \mid \sigma \rightarrow \sigma$$

$$\tau ::= \sigma \mid \forall \alpha. \tau$$

σ represents **monotypes**, types with *no quantification*. A type is **monomorphic** if it is a monotype with no type variables

mono type: int \rightarrow α *monomorphic: int \rightarrow bool*

Type Variables and Type Schemes

$$\sigma ::= \text{int} \mid \text{bool} \mid \alpha \mid \sigma \rightarrow \sigma$$

$$\tau ::= \sigma \mid \forall \alpha. \tau$$

σ represents **monotypes**, types with *no quantification*. A type is **monomorphic** if it is a monotype with no type variables

τ represents **type schemes**, which are types with some number of quantified type variables

Type Variables and Type Schemes

$$\sigma ::= \text{int} \mid \text{bool} \mid \alpha \mid \sigma \rightarrow \sigma$$

$$\tau ::= \sigma \mid \forall \alpha. \tau$$

σ represents **monotypes**, types with *no quantification*. A type is **monomorphic** if it is a monotype with no type variables

τ represents **type schemes**, which are types with some number of quantified type variables

We say a type is **polymorphic** if it is a *closed* type scheme

type scheme. $\forall \alpha. \alpha \rightarrow \boxed{\beta}$

poly: $\forall \alpha. \forall \beta. \alpha \rightarrow \beta$

Free Variables (Monotypes)

$$FV(\lambda \alpha. \lambda \beta. \text{int}) = \{\alpha, \beta\}$$

$$FV(\text{int}) = \emptyset$$

$$FV(\text{bool}) = \emptyset$$

$$FV(\alpha) = \{\alpha\}$$

$$FV(\tau_1 \rightarrow \tau_2) = FV(\tau_1) \cup FV(\tau_2)$$

Once we introduce variables, we have to again talk about free and bound variables

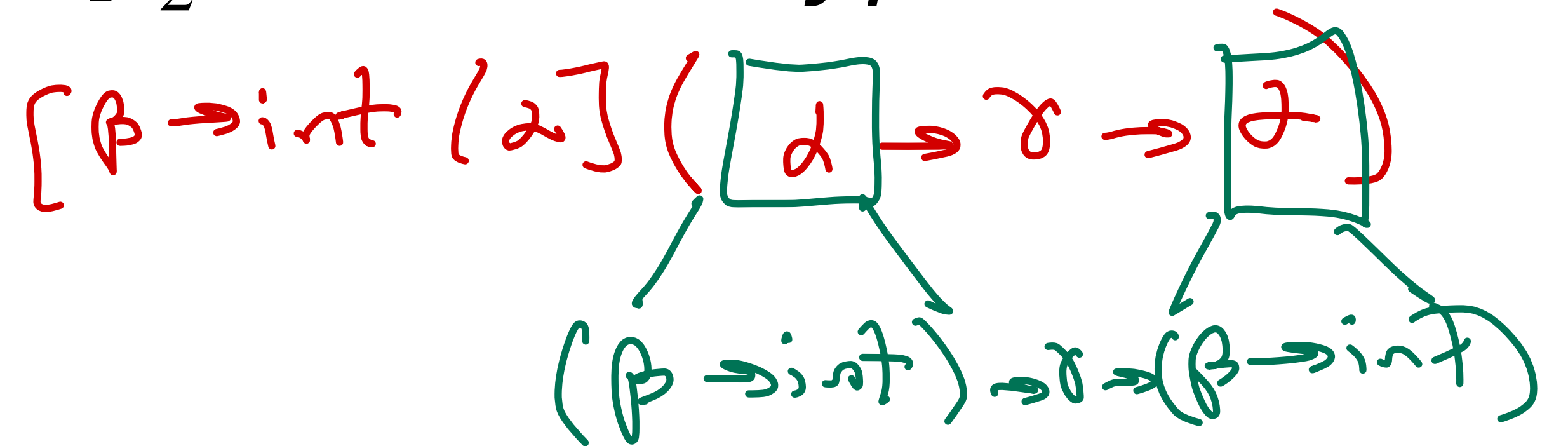
~~Unlike in System F,~~ we will only need to consider free variables of **monotypes** so there is *no issue with variable capture*

Understanding Check

Define substitution $[\tau_1/\alpha]\tau_2$ for monotypes

$$[\tau_1/\alpha] \text{int} = \text{int}$$

$$[\tau_1/\alpha] \text{bool} = \text{bool}$$



$$[\tau_1/\alpha](\tau_2 \rightarrow \tau_3) = [\tau_1/\alpha]\tau_2 \rightarrow [\tau_1/\alpha]\tau_3$$

$$[\tau_1/\alpha]\beta = \begin{cases} \tau_1 & \alpha = \beta \\ \beta & \text{o.w.} \end{cases}$$

Constraint-Based Inference

$$\Gamma \vdash e : \tau \dashv \mathcal{C}$$

Constraint-Based Inference

$$\Gamma \vdash e : \tau \dashv \mathcal{C}$$

Our typing rules will need to keep track of a set of **constraints**, which tell us what must hold for e to be well-typed

Constraint-Based Inference

$$\Gamma \vdash e : \tau \dashv \mathcal{C}$$

Our typing rules will need to keep track of a set of **constraints**, which tell us what must hold for e to be well-typed

Contexts are collections of variable declaration, i.e., mapping of variables to **type schemes**

Constraint-Based Inference

$$\Gamma \vdash e : \tau \dashv \mathcal{C}$$

Our typing rules will need to keep track of a set of **constraints**, which tell us what must hold for e to be well-typed

Contexts are collections of variable declaration, i.e., mapping of variables to **type schemes**

The idea: We're formalizing the idea of "collecting together" our constraints, as in our intuitive example

What is a constraint?

$$\tau_1 \doteq \tau_2$$

In general, a **type constraint** is a predicate on types. The only kind we will consider:

" τ_1 should be the same as τ_2 "

Enforcing a constraint like this is called **unifying** τ_1 and τ_2

Constraint-Based Inference

$$\Gamma \vdash e : \tau \dashv \mathcal{C}$$

Constraint-Based Inference

$$\Gamma \vdash e : \tau \dashv \mathcal{C}$$

The idea: For each rule, we need to determine:

Constraint-Based Inference

$$\Gamma \vdash e : \tau \dashv \mathcal{C}$$

The idea: For each rule, we need to determine:

» What is the *most general* type τ we could give e ?

Constraint-Based Inference

$$\Gamma \vdash e : \tau \dashv \mathcal{C}$$

The idea: For each rule, we need to determine:

- » What is the *most general* type τ we could give e ?
- » What must be true of τ , i.e., what *constrains* τ ?

Constraint-Based Inference

$$\Gamma \vdash e : \tau \dashv \mathcal{C}$$

The idea: For each rule, we need to determine:

- » What is the *most general* type τ we could give e ?
- » What must be true of τ , i.e., what *constrains* τ ?

If we don't know what type something should be, *we create a fresh type variable for it*

Let's see some typing rules...

HM⁻ (Typing Literals)

$$\frac{n \text{ is an integer}}{\Gamma \vdash n : \text{int} \dashv \emptyset} \quad (\text{int})$$

$$\Gamma \vdash 2 : \text{int} \dashv \emptyset$$

Literals have their expected types *without any constraints*

HM⁻ (Typing Operators)

$$\frac{\Gamma \vdash e_1 : \text{int} \quad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 + e_2 : \text{int}}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathcal{C}_1 \quad \Gamma \vdash e_2 : \tau_2 \dashv \mathcal{C}_2}{\Gamma \vdash e_1 + e_2 : \text{int} \dashv \tau_1 \doteq \text{int}, \tau_2 \doteq \text{int}, \mathcal{C}_1, \mathcal{C}_2} \text{ (add)}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathcal{C}_1 \quad \Gamma \vdash e_2 : \tau_2 \dashv \mathcal{C}_2}{\Gamma \vdash e_1 = e_2 : \text{bool} \dashv \tau_1 \doteq \tau_2, \mathcal{C}_1, \mathcal{C}_2} \text{ (eq)}$$

$e_1 + e_2$ is an **int** if the types of e_1 and e_2 can be *unified* to **int**

We don't require that τ_i is *exactly* **int**, e.g., it may be a type variable!

$$\begin{array}{c}
 \Gamma \vdash e_1 : \tau_1 \rightarrow C_1 \quad \Gamma \vdash e_2 : \tau_2 \rightarrow C_2 \quad \Gamma \vdash e_3 : \tau_3 \rightarrow C_3 \\
 \hline
 \Gamma \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 : \tau_2 \rightarrow \tau_2 \doteq \tau_3, \tau_1 \doteq \text{bool}, \\
 C_1, C_2, C_3
 \end{array}$$

HM⁻ (Typing If-Expressions)

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathcal{C}_1 \quad \Gamma \vdash e_2 : \tau_2 \dashv \mathcal{C}_2 \quad \Gamma \vdash e_3 : \tau_3 \dashv \mathcal{C}_3}{\Gamma \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 : \tau_3 \dashv \tau_1 \doteq \text{bool}, \tau_2 \doteq \tau_3, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3} \quad (\text{if})$$

An if-expression has the same type as its else-case when:

- » the type of the condition can be *unified* with **bool**
- » the types of the then-case and else-case can be *unified to each other*

Example $\{x : \alpha, y : \beta\} \vdash \text{if } x \text{ then } x \text{ else } y : \tau \dashv \mathcal{C}$

$\{x : \alpha, y : \beta\} \vdash \text{if } x \text{ then } x \text{ else } y : \alpha \dashv \alpha \doteq \text{bool}$

$\alpha \doteq \beta$

$\vdash \{ \dots \} \vdash x : \alpha \dashv \emptyset$

$\vdash \{ \dots \} \vdash x : \alpha \dashv \emptyset$

$\vdash \{ \dots \} \vdash y : \beta \dashv \emptyset$

HM⁻ (Typing Functions)

$$\frac{\alpha \text{ is fresh} \quad \Gamma, x : \alpha \vdash e : \tau \dashv \mathcal{C}}{\Gamma \vdash \lambda x. e : \alpha \rightarrow \tau \dashv \mathcal{C}} \quad (\text{fun})$$

The input type of a function is some type α and it's output type is the type of the body

We don't know the input type, so we give it the most general form, i.e., a fresh type variable with no constraints

$$\Gamma \vdash e_1 : \tau_1 \rightarrow C_1 \quad \Gamma \vdash e_2 : \tau_2 \rightarrow C_2 \quad \alpha \text{ fresh}$$

$$\Gamma \vdash e_1, e_2 : \alpha \rightarrow \tau_1 \doteq \tau_2 \rightarrow \alpha, C_1, C_2$$

HM⁻ (Typing Application)

$$\frac{\Gamma \vdash e_1 : \tau_1 \dashv \mathcal{C}_1 \quad \Gamma \vdash e_2 : \tau_2 \dashv \mathcal{C}_2 \quad \alpha \text{ is fresh}}{\Gamma \vdash e_1 e_2 : \alpha \dashv \tau_1 \doteq \tau_2 \rightarrow \alpha, \mathcal{C}_1, \mathcal{C}_2} \quad (\text{app})$$

The type of an application is some type α , such that the type of the function unifies to a function type with output type α , and the input type matches the type of the argument (wordy...)

HM⁻ (Typing Variables)

$$\frac{(x : \forall \alpha_1 . \forall \alpha_2 \dots \forall \alpha_k . \tau) \in \Gamma \quad \beta_1, \dots, \beta_k \text{ are fresh}}{\Gamma \vdash x : [\beta_1 / \alpha_1] \dots [\beta_k / \alpha_k] \tau \dashv \emptyset} \quad (\text{var})$$

HM⁻ (Typing Variables)

$$\frac{(x : \forall \alpha_1 . \forall \alpha_2 \dots \forall \alpha_k . \tau) \in \Gamma \quad \beta_1, \dots, \beta_k \text{ are fresh}}{\Gamma \vdash x : [\beta_1 / \alpha_1] \dots [\beta_k / \alpha_k] \tau \dashv \emptyset} \quad (\text{var})$$

If x is declared in Γ , then x can be given the type τ *with all free variables replaced by **fresh variables***

HM⁻ (Typing Variables)

$$\frac{(x : \forall \alpha_1 . \forall \alpha_2 \dots \forall \alpha_k . \tau) \in \Gamma \quad \beta_1, \dots, \beta_k \text{ are fresh}}{\Gamma \vdash x : [\beta_1 / \alpha_1] \dots [\beta_k / \alpha_k] \tau \dashv \emptyset} \quad (\text{var})$$

If x is declared in Γ , then x can be given the type τ *with all free variables replaced by **fresh variables***

This is where the polymorphism magic happens

HM⁻ (Typing Variables)

$$\frac{(x : \forall \alpha_1 . \forall \alpha_2 \dots \forall \alpha_k . \tau) \in \Gamma \quad \beta_1, \dots, \beta_k \text{ are fresh}}{\Gamma \vdash x : [\beta_1 / \alpha_1] \dots [\beta_k / \alpha_k] \tau \dashv \emptyset} \quad (\text{var})$$

If x is declared in Γ , then x can be given the type τ *with all free variables replaced by **fresh variables***

This is where the polymorphism magic happens

fresh variables can be unified with anything

Example

```
fun f -> fun x -> f (x + 1)
```

Up Next

We still need to:

- » introduce a **unification algorithm** to determine the "actual" type given a collection of constraints
- » Discuss **let-expressions** (and top-level let expressions)
- » introduce **type annotations**

We wont:

- » deal with **type errors** (tricker with unification-based inference)

Summary

By restricting our type quantification, we get a system that has decidable and efficient **type inference**

Hindley–Milner style type inference requires us to figure out a collection of **constraints** that need to be unified