# Type Safety

## Concepts of Programming Languages

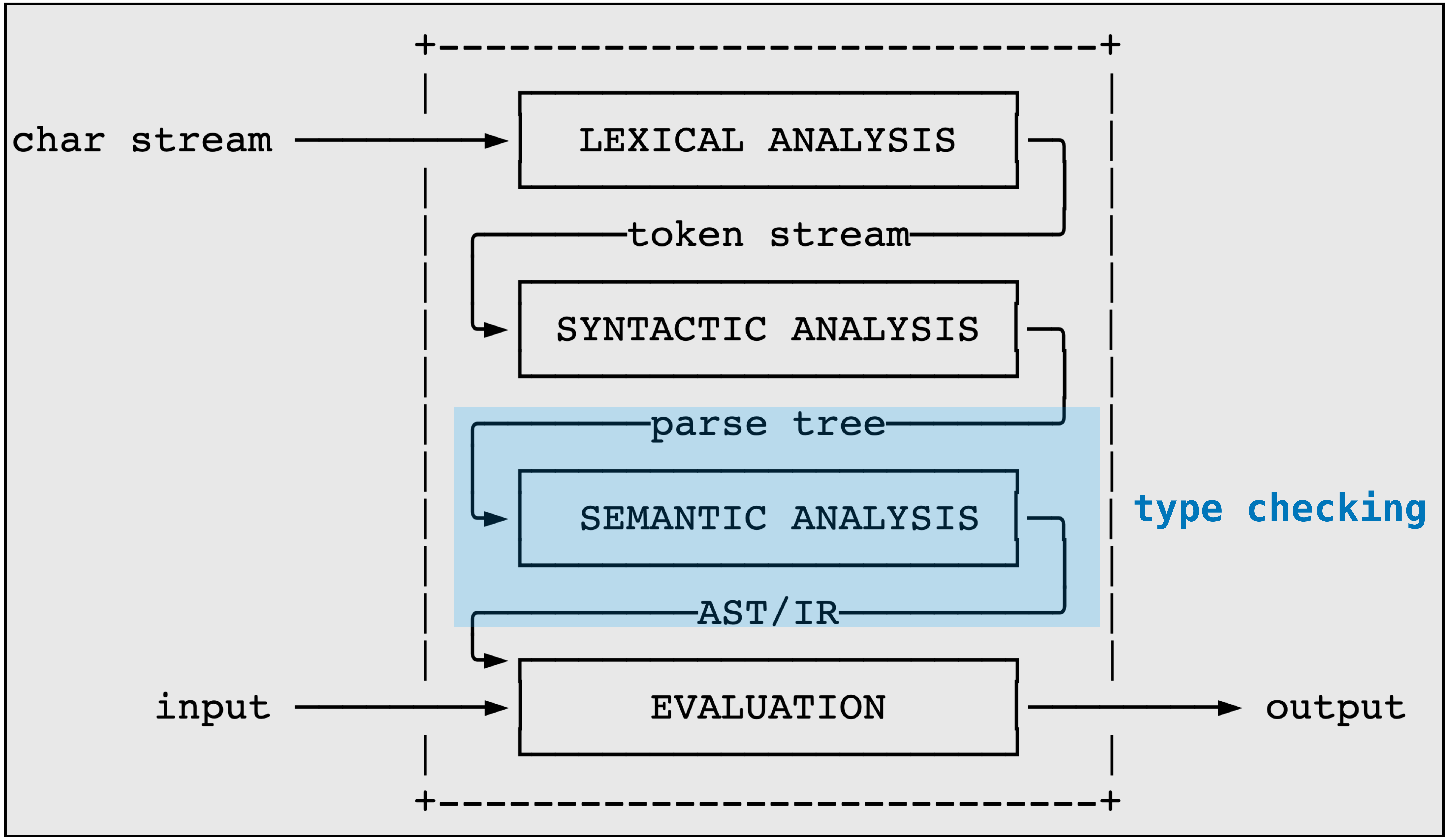
CAS CS 320

# Outline

- Demo an **implementation** of the simply typed lambda calculus
- Discuss **induction** over derivations
- Show that STLC satisfies **progress** and **preservation**

# Recap

# Recall: The Picture



# Recall: Type Checking/Inference

```
type_check : expr -> ty -> bool

type_of : expr -> ty option
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type_check : expr -> ty -> bool
type_of : expr -> ty option
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Type checking: determining whether an expression can be typed

Type inference: *synthesizing* a type for an expression

Theoretically, these two problems can be very different. *For STLC, they are both easy*

# Recall: Syntax (STLC)

$$e ::= \bullet \mid x \mid \lambda x^{\tau} . e \mid ee$$

$$\tau ::= \top \mid \tau \to \tau$$

$$x ::= \textit{variables}$$

`The syntax is the same as that of the lambda calculus except:`

- `we include a unit expression`
- `we have types, which annotate arguments`

# Recall: Typing (STLC)

$$\frac{}{\Gamma \vdash \bullet : \top}\;\textbf{unit}$$

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x^{\tau} . e : \tau \rightarrow \tau'}\;\textbf{abstraction}$$

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau}\;\textbf{variable}$$

$$\frac{\Gamma \vdash e_1 : \tau \rightarrow \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}\;\textbf{application}$$

These rules enforce that a function can only be applied if we *know* that it's a function

# Recall: Semantics (STLC)

$$\frac{e_1 \longrightarrow e_1'}{e_1 e_2 \longrightarrow e_1' e_2}\ \texttt{leftEval} \qquad \frac{}{(\lambda x\,.\,e)e' \longrightarrow [e'/x]e}\ \texttt{beta}$$

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**`This is part of the point.`** `Type-checking only determines` *`whether`* `we go on to evaluate the program`

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We can use any semantics (this is small-step CBN)

**This is part of the point.** Type-checking only determines *whether* we go on to evaluate the program

It doesn't determine *how* we evaluate the program

# Practice Problem

$$(\lambda x^{(\top \to \top) \to \top} . x(\lambda z^{\top} . x(wz)))y$$

*Determine the smallest context such that the above expression is well-typed (also give its type)*

$$\frac{}{\Gamma \vdash \bullet : \top}\ \textbf{unit}$$

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau}\ \textbf{variable}$$

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# Answer

$$(\lambda x^{(\top\to\top)\to\top}.\,x(\lambda z^{\top}.\,x(wz)))y$$

$$\{w:\top\to(\top\to\top),\ y:(\top\to\top)\to\top\}$$

$$\vdash \boxed{(\lambda x^{(\top\to\top)\to\top}.\ \underbrace{x}_{(\top\to\top)\to\top}\,(\overbrace{\lambda z^{\top}.\ \underbrace{\underbrace{x}_{(\top\to\top)\to\top}\,(\underbrace{w\,\underbrace{z}_{\top}}_{\top\to\top})}_{\top}}^{\top\to\top}))}\ y : \top$$

# demo

(STLC)

$$\frac{\Gamma \vdash e_1 : \text{int} \qquad \Gamma \vdash e_2 : \text{int}}{\Gamma \vdash e_1 + e_2 : \text{int}}$$

$$\frac{\Gamma \vdash e_1 : \text{bool} \qquad \Gamma \vdash e_2 : \tau \qquad \Gamma \vdash e_3 : \tau}{\Gamma \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 : \tau}$$

```
let sum (n:int) : int =
  if n=0
  then 0
  else sum (n-1) + n
in sum 5
```

$e_1$

$\{x : int\} \vdash e_1 : int$

$sum : int \to int$

$$\frac{\Gamma, f : \tau_a \to \tau_0, x : \tau_a \vdash e_1 : \tau_0 \qquad \Gamma, f : \tau_a \to \tau_0 \vdash e_2 : \tau}{\Gamma \vdash \text{let rec } f\ (x : \tau_a) : \tau_0 = e_1 \text{ in } e_2 : \tau}$$

hint: let rec f f = f

# Type Safety

How do we know if we've defined a "good" programming language?

# Type Safety

# Type Safety

if $e$ is terminating

> **Theorem.** If $\cdot \vdash e : \tau$ then there is a value $v$ such that $\langle \varnothing, e \rangle \Downarrow v$ and $\cdot \vdash v : \tau$

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With small-step semantics, we can give a finer-grained analysis:

no stuck terms

> **Theorem.** If $\cdot \vdash e : \tau$, then
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> » *(progress)* either $e$ is a value or there is an $e'$ such that $e \longrightarrow e'$
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These results are *fundamental*. They tell us that our PL is well-behaved (it's a "good" PL)

# Induction over Derivations

# The Key Idea

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$$
\dfrac{
  \dfrac{}{\{\} \vdash \texttt{2} : \texttt{int}}\,(\mathsf{intLit})
  \qquad
  \dfrac{
    \dfrac{}{\{\texttt{y} : \texttt{int}\} \vdash \texttt{y} : \texttt{int}}\,(\mathsf{var})
    \quad
    \dfrac{}{\{\texttt{y} : \texttt{int}\} \vdash \texttt{y} : \texttt{int}}\,(\mathsf{var})
  }{\{\texttt{y} : \texttt{int}\} \vdash \texttt{y} + \texttt{y} : \texttt{int}}\,(\mathsf{intAdd})
}{\{\} \vdash \texttt{let y = 2 in y} + \texttt{y} : \texttt{int}}\,(\mathsf{let})
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Derivations are *trees*

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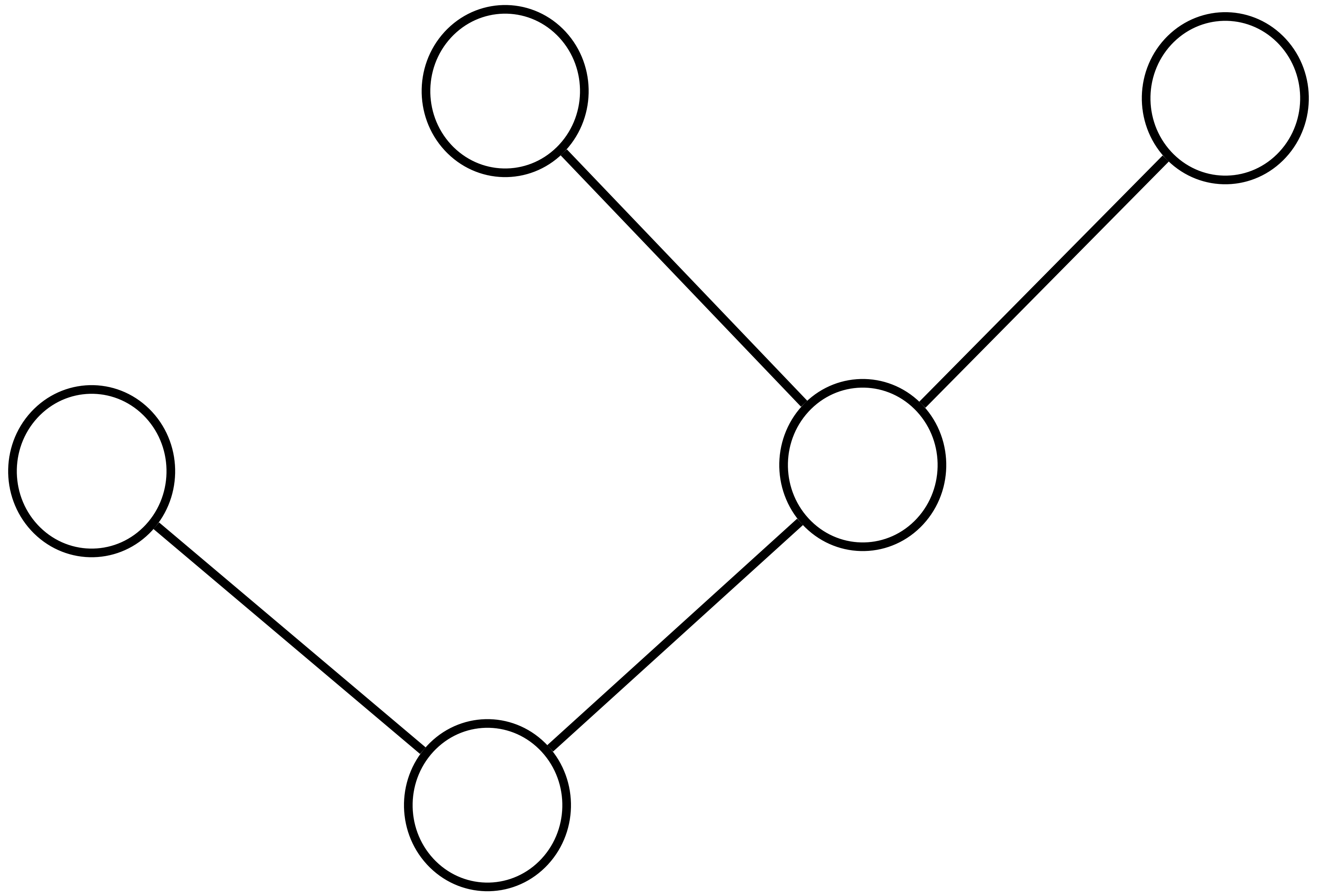
Derivations are *trees*

We can prove things about trees using induction



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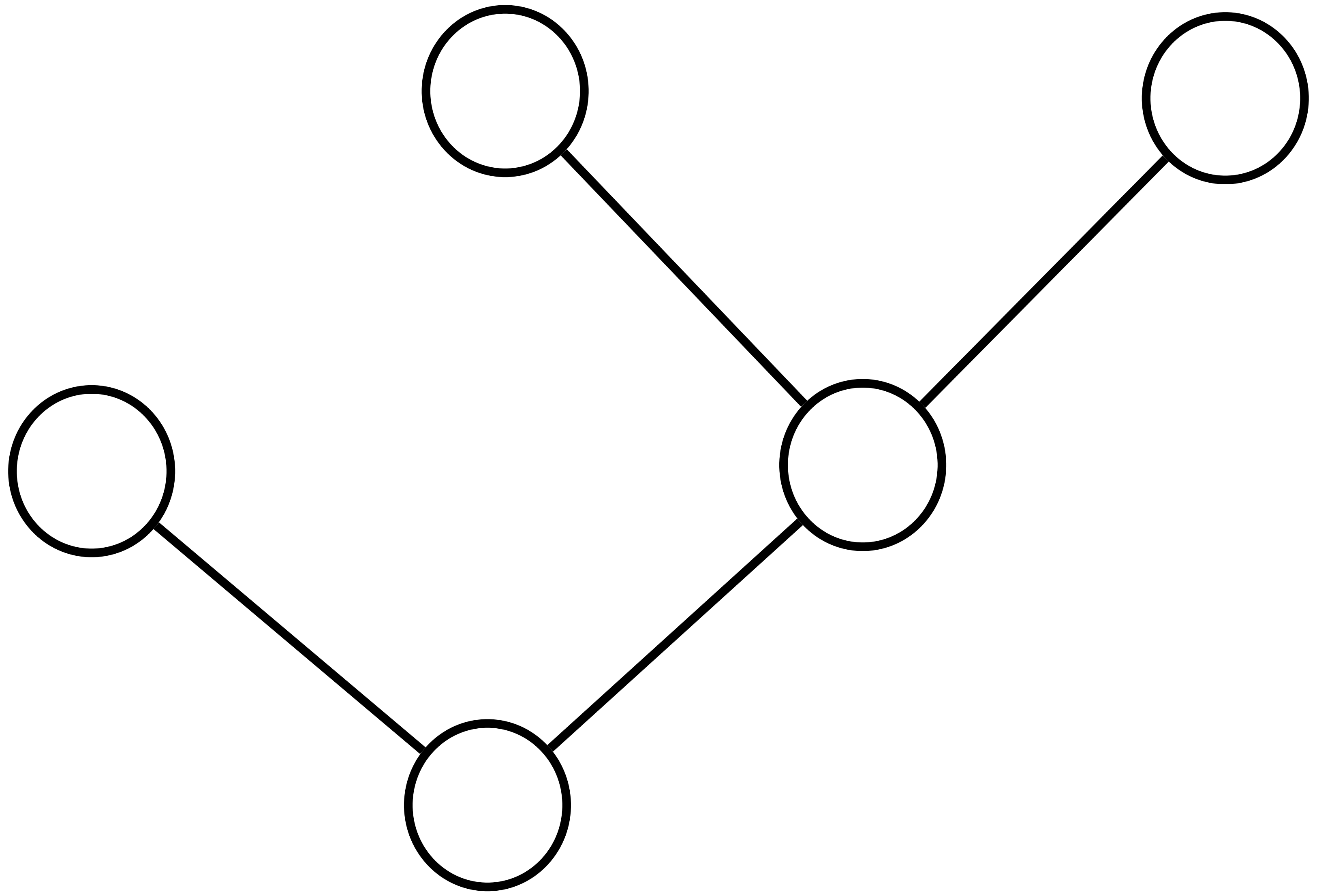
We can prove things about *derivable judgments* using induction



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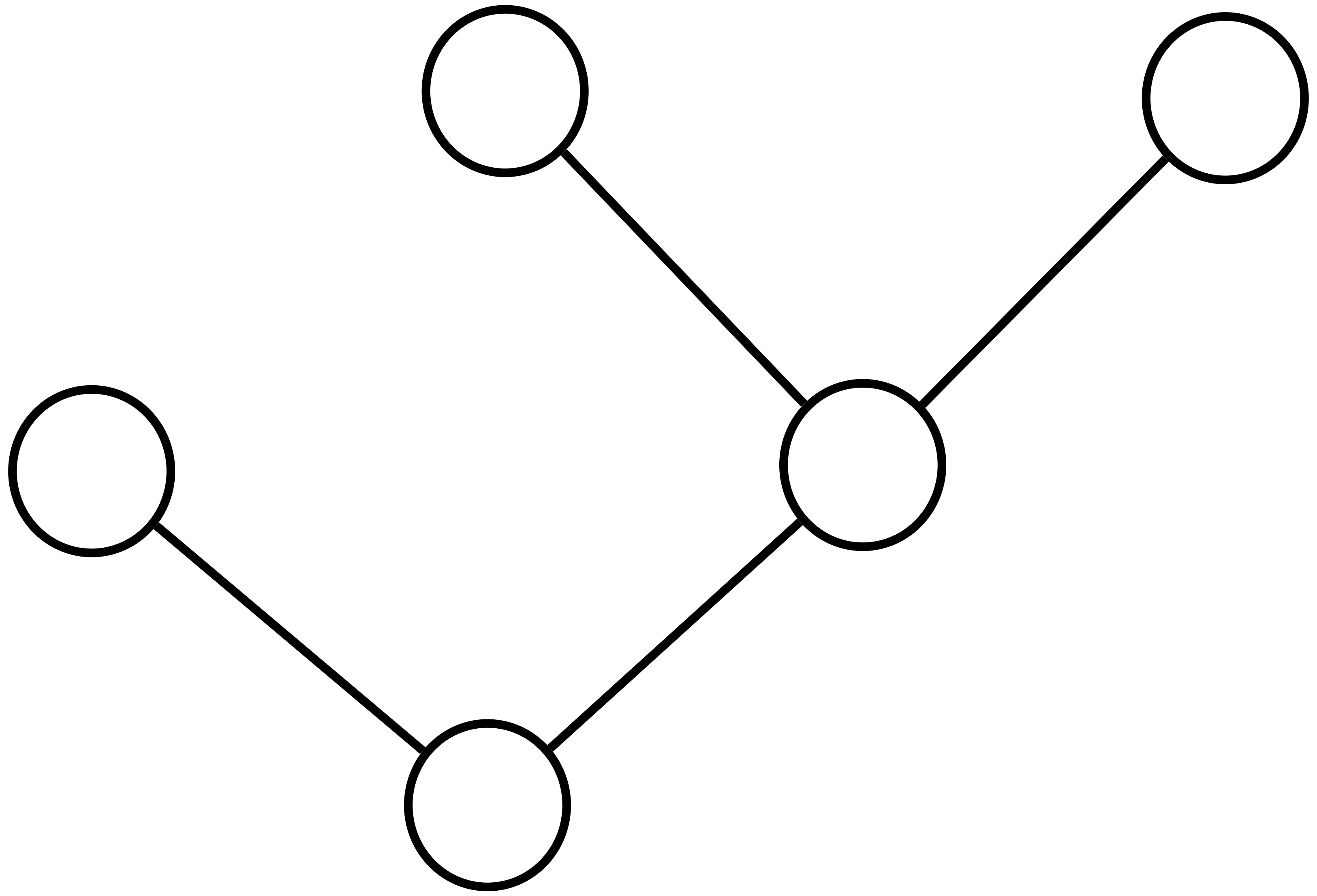
We can prove things about *derivable judgments* using induction

**Important: Every derivable judgment corresponds to a derivation**



# Warm-up: Binary Trees

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type 'a tree =
  | Empty
  | Node of 'a tree * 'a * 'a tree
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**Theorem.** $\text{size}(T) \leq 2^{\text{height}(T)} - 1$ for any tree $T$

*Proof.* By induction on the structure of trees

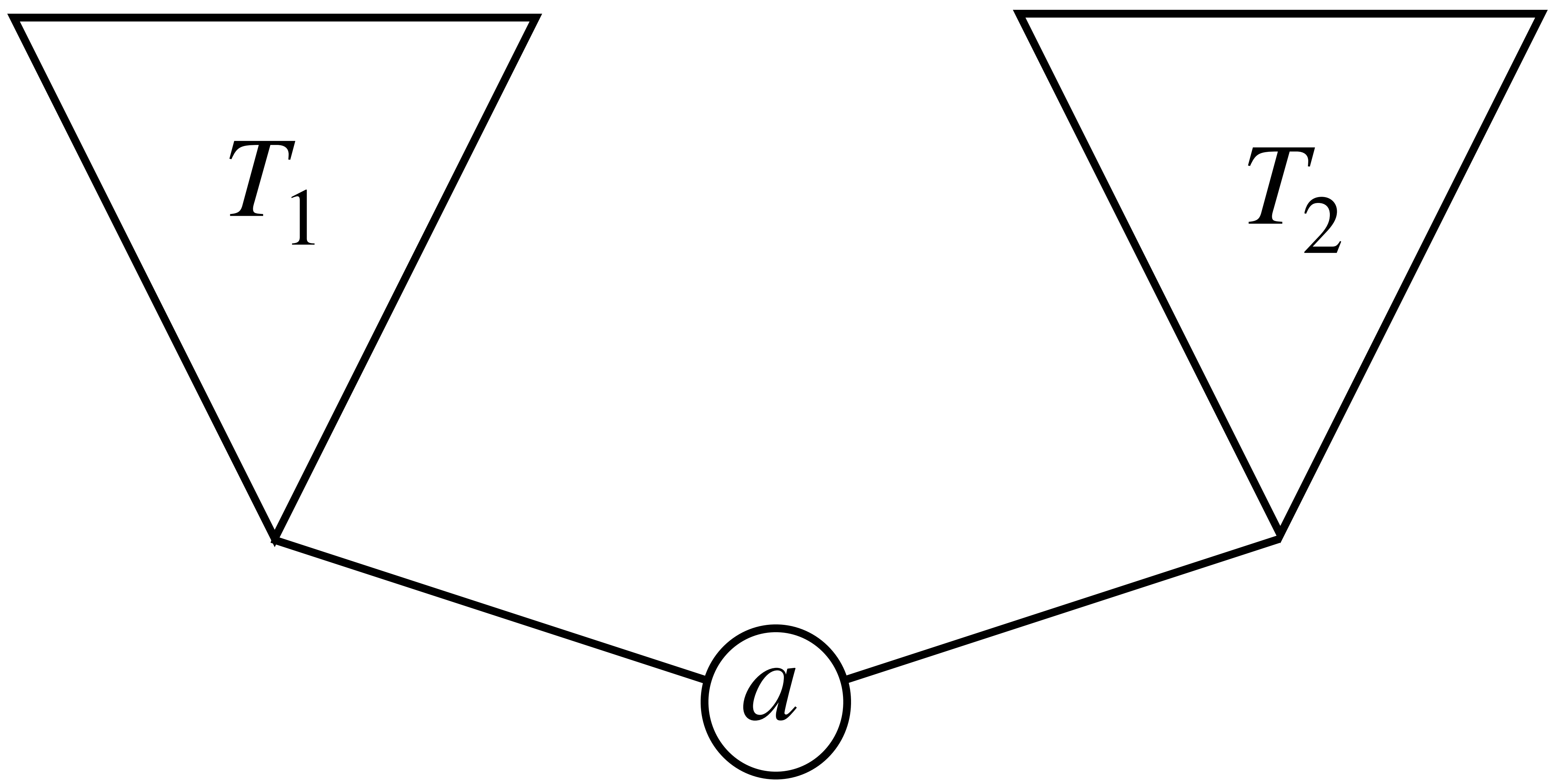
# Base Case: Empty

$\text{size}(T) = 0$

$h(T) = 0$

$0 = s(T) \leq 2^{h(T)} - 1 = 1 - 1 = 0$

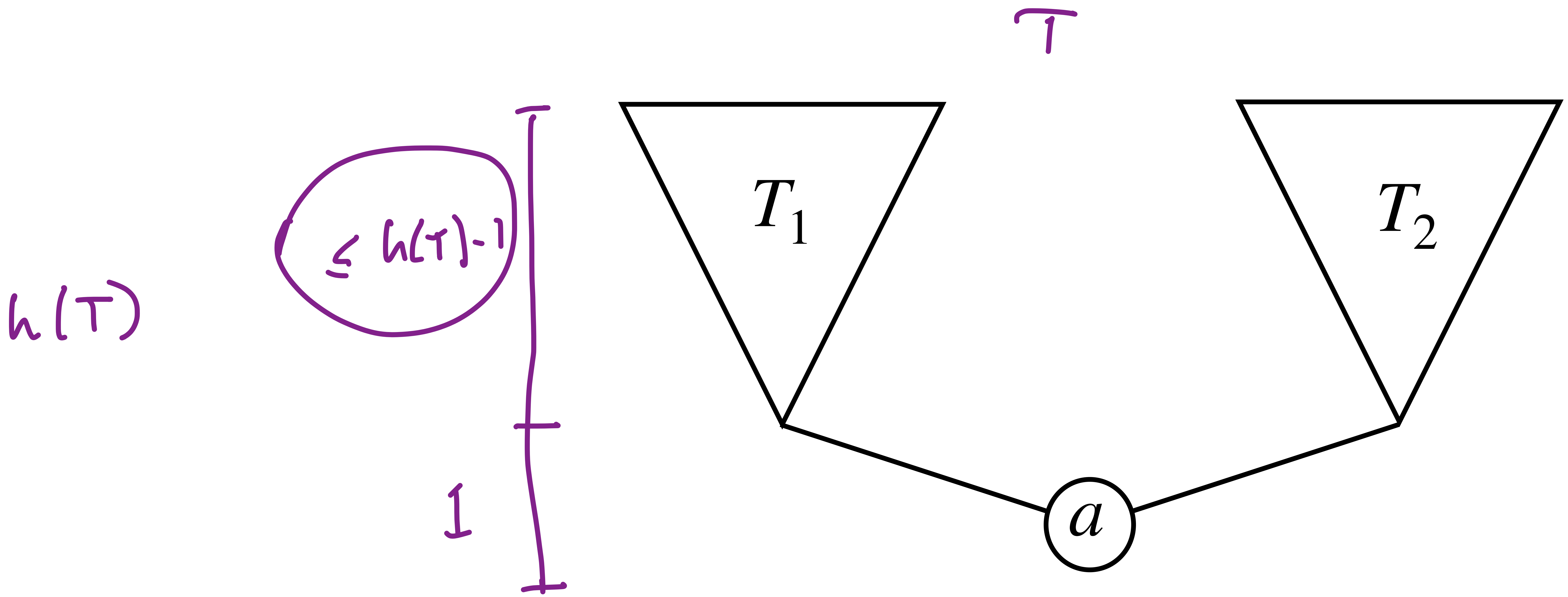
# Inductive Hypothesis




If $T$ is of the form `Node` $(T_1, a, T_2)$ then $\text{size}(T_1) \le 2^{\text{height}(T_1)} - 1$ and $\text{size}(T_2) \le 2^{\text{height}(T_2)} - 1$

*That is, we get to assume that what we want holds of our subtrees*

# Inductive Step: Nodes




$$s(T) = s(T_1) + s(T_2) + 1$$

$$\leq 2^{h(T_1)} - 1 + 2^{h(T_2)} - 1 + 1$$

$$\leq \boxed{2^{h(T)-1}} - 1 + \boxed{2^{h(T)-1}} \cancel{-1} \cancel{+1} = 2^{h(T)} - 1$$

# Another Warm-up: Well-Scopedness

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An expression $e$ is **well-scoped** with respect to a context $\Gamma$ if $x \in FV(e)$ implies $x$ appears in $\Gamma$

> Theorem. If $e$ is well-typed in $\Gamma$, then $e$ is well-scoped

*Proof.* By induction on derivations

# Base Case: Axioms

$$\frac{}{\Gamma \vdash \bullet : \top}\ \text{unit}\ \checkmark \qquad \frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau}\ \text{variable}\ \checkmark$$

*We need to show that expressions typed using just axioms satisfy well-scopedness*

# Inductive Hypothesis

$$\dfrac{\begin{array}{c}\vdots\\ \mathscr{D}_1\\ \Gamma_1 \vdash e_1 : \tau_1\end{array} \qquad \begin{array}{c}\vdots\\ \mathscr{D}_2\\ \Gamma_2 \vdash e_2 : \tau_2\end{array} \qquad \dots \qquad \begin{array}{c}\vdots\\ \mathscr{D}_k\\ \Gamma_k \vdash e_k : \tau_k\end{array}}{\Gamma \vdash e : \tau}$$

*If* $e_1, \dots, e_k$ *are well-scoped (because they are typeable in the each of their contexts)*

# Inductive Step 1: Application

$$\dfrac{\begin{array}{c}\vdots \\ \mathcal{D}_1 \\ \Gamma \vdash e_1 : \tau \to \tau'\end{array} \qquad \begin{array}{c}\vdots \\ \mathcal{D}_2 \\ \Gamma \vdash e_2 : \tau\end{array}}{\Gamma \vdash e_1 e_2 : \tau'}\ \textbf{application}$$

$FV(e_1) \subseteq dom(\Gamma)$

$FV(e_2) \subseteq dom(\Gamma)$

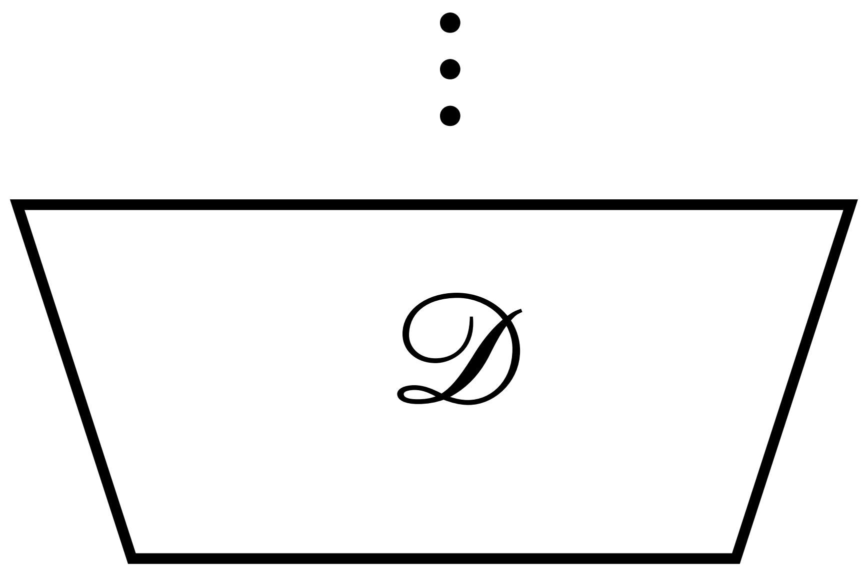
$FV(e_1 e_2) = FV(e_1) \cup FV(e_2)$

$\subseteq dom(\Gamma)$

*What if the last rule I applied was application?*

# Inductive Step 2: Abstraction

$\Gamma = \{x_1 : \tau_1, \ldots, x_k : \tau_k\}$



$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x^{\tau} . e : \tau \to \tau'} \; \textbf{abstraction}$$

$FV(e) \subseteq \{x_1, \ldots, x_k, x\}$

$FV(e) \setminus \{x\}$

$\|$

WTS $FV(\lambda x^{\tau}. e) \subseteq \{x_1, \ldots, x_k\}$ ✓

*What if the last rule I applied was abstraction?*

# Progress and Preservation

# Recall: Type Safety

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> Theorem. If $\cdot \vdash e : \tau$ then there is a value $v$ such that $\langle \varnothing, e \rangle \Downarrow v$ and $\cdot \vdash v : \tau$

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> <u>Theorem.</u> If $\cdot \vdash e : \tau$ then there is a value $v$ such that $\langle \varnothing, e \rangle \Downarrow v$ and $\cdot \vdash v : \tau$

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> Theorem. If $\cdot \vdash e : \tau$, then
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These results are *fundamental*. They tell us that our PL is well-behaved (it's a "good" PL)

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With small-step semantics, we can give a finer-grained analysis:

goal for today

> Theorem. If $\cdot \vdash e : \tau$, then
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These results are *fundamental*. They tell us that our PL is well-behaved (it's a "good" PL)

Disclaimer: We're gonna hand-wave liberally

# Recall: STLC

$$e ::= \bullet \mid x \mid \lambda x^{\tau} . e \mid ee$$
$$\tau ::= \top \mid \tau \to \tau$$
$$x ::= \mathit{variables}$$

**Typing**

$$\frac{}{\Gamma \vdash \bullet : \top}\ \texttt{unit}$$

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x^{\tau} . e : \tau \to \tau'}\ \texttt{abstraction}$$

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$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}\ \texttt{application}$$

**Semantics**

$$\frac{e_1 \longrightarrow e_1'}{e_1 e_2 \longrightarrow e_1' e_2}\ \texttt{leftEval}$$

$$\frac{}{(\lambda x . e)e' \longrightarrow [e'/x]e}\ \texttt{beta}$$

# Progress (STLC)

> **Theorem.** If $e$ is well-typed ($\cdot \vdash e : \tau$ for some type $\tau$), then $e$ is a value, or there is an expression $e'$ such that $e \longrightarrow e'$

*Proof.* By induction over derivations

# Base Case: Axioms

$$\frac{}{\cdot \vdash \bullet : \top}\ \textbf{unit}$$

$$\frac{(x : \tau) \in \varnothing}{\cdot \vdash x : \tau}\ \textbf{variable}$$

*We need to show that expressions typed using just axioms yield non-stuck terms*

# Inductive Step 1: Application

$$\frac{\cdot \vdash e_1 : \tau \to \tau' \qquad \cdot \vdash e_2 : \tau}{\cdot \vdash e_1 e_2 : \tau'}\ \texttt{application}$$

*What do we know given that $e_1$ is either a* ***value*** *or* ***reducible****?*

# Inductive Step 2: Abstraction

$$\frac{\{x : \tau\} \vdash e : \tau'}{\cdot \vdash \lambda x^{\tau}.\, e : \tau \to \tau'} \; \textbf{abstraction}$$

*Our expression already a value if the last rule we applied was abstraction!*

# Preservation (STLC)

> Theorem. If $e$ has type $\tau$ in $\Gamma$ (i.e., $\Gamma \vdash e : \tau$ is derivable) and $e \longrightarrow e'$ then so is $e'$ (i.e., $\Gamma \vdash e' : \tau$ is derivable)

*Proof.* By induction over derivations

This one is much tricker...

# Base Case: Axioms

$$\frac{}{\Gamma \vdash \bullet : \top}\,\textbf{unit}$$

$$\frac{(x : \tau) \in \Gamma}{\Gamma \vdash x : \tau}\,\textbf{variable}$$

*Expressions typed using just axioms cannot be reduced (nothing to do here)*

# Inductive Step 1: Abstraction

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x^{\tau} . e : \tau \to \tau'} \ \textbf{abstraction}$$

*Expressions derived using abstraction as the last rule is already a value (nothing to do here)*

# Inductive Step 2: Application

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}\ \texttt{application}$$

This is where the work comes in...

<u>The trick:</u> We do induction (inside our current induction) on the structure of *semantic* derivations!

*What possible ways can* $e_1 e_2$ *be reduced?*

# Inductive Step 2.1: leftEval

$$\frac{e_1 \longrightarrow e_1'}{e_1 e_2 \longrightarrow e_1' e_2}\ \texttt{leftEval}$$

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'}\ \texttt{application}$$

*What if our last rule was an application* ***and*** $e_1 e_2$ *is reducible by leftEval?*

# Inductive Step 2.1: leftEval

$$\frac{}{(\lambda x\,.\,e)e_2 \longrightarrow [e_2/x]e}\textbf{beta}$$

$$\frac{\Gamma \vdash (\lambda x\,.\,e) : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash (\lambda x\,.\,e)e_2 : \tau'}\textbf{application}$$

*What if our last rule was an application **and** $e_1 e_2$ is reducible by beta?*

# Substitution Lemma

> <u>Lemma.</u> If $\Gamma \vdash e_2 : \tau_2$ and $\Gamma, x : \tau_2 \vdash e : \tau$ then
>
> $$\Gamma \vdash [e_2/x]e : \tau$$

*That is, if* $e$ *is well-typed in a context with* $(x, \tau)$ *then we can substitute* $x$ *with anything of type* $\tau$ *and it's still the same type*

(we can prove this by, you guessed it, induction on derivations)

# The Point

```
let rec eval env e =
  match e with
  | Var x -> Env.find x env
  ...
```

Progress and preservation tell us that **terms never get stuck during evaluation**

*This is **HUGE.** I can't emphasize this enough*

Our type system ensures we only evaluate programs that make sense!

# Summary

**Progress** and **preservation** are fundamental features of good programming languages

We can prove things about well-typed expressions by performing **induction** over derivations