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Homework

Study of a nonlinear system

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I. Introduction

We will work on a simplified longitudinal flight model described by the following equations :

$$m\dot{v} = F_x - mgsin(\theta) - mqw$$

$$m\dot{w} = F_z - mgcos(\theta) - mqv$$

$$\dot{\theta} = q$$

$$\dot{q} = \frac{M}{I_y}$$

Besides, the studied outputs are $Y = \begin{pmatrix} \theta \\ q \end{pmatrix}$ and the inputs will be $u = \begin{pmatrix} F_x \\ F_z \\ M \end{pmatrix}$

Parameters :

v : Vertical flight speeds

w : Horizontal flight speed

θ : Pitch angle

q : Pitch rate

F_x : Total horizontal force

F_z : Total vertical force

M : Pitch torque

m : Mass of the plane

I_y : Inertia of the plane

g : Gravitational acceleration

Therefore, this study will focus on the pitch angle and pitch rate response to the inputs induced by the external forces applied to the aircraft.

II. Analysis

A. Defining the system

1. Parameters

The system studied will be a medium size aircraft such as the airbus A320. The mass considered is at MTOW with $m = 78\ 000$kg and the inertia using the inertia of a rectangle is $I_y = 78000 * \frac{37^2+6^2}{12} \approx 9.0 * 10^6\ kg/m^2$.

2. State-space representation

We consider the state-space vector $X = [v \quad w \quad \theta \quad q]$, so we obtain the following state-space representation thanks to the initial equations.

$$\dot{X} = \begin{bmatrix} \dot{v} \\ \dot{w} \\ \dot{\theta} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{F_x}{m} - gsin(\theta) - qw \\ \frac{F_z}{m} - gcos(\theta) + qv \\ q \\ \frac{M}{I_y} \end{bmatrix}$$

It is not possible yet to put the system into a matrix form.

3. Simulations

We consider the following parameters for our first simulation

$$X_0 = \begin{bmatrix} v_0 = 800\ km/h \\ w_0 = 0\ m/s \\ \theta_0 = 4.5° \\ q = 0 \end{bmatrix} \text{ and } u \quad = \begin{bmatrix} F_x = 60\ 000\ N \\ F_z = 9.81 * 78000 \\ 0 \end{bmatrix}$$

The Simulink made is as desrcribed below :

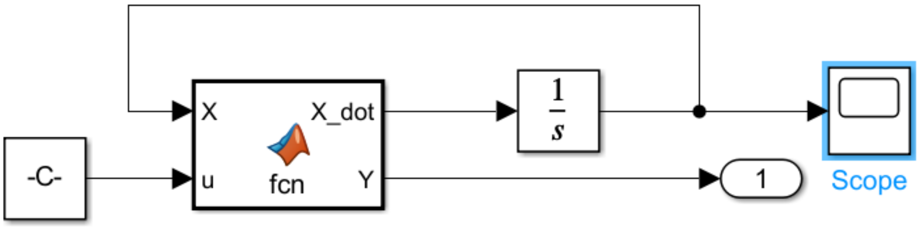

Figure 1: Simulink for non-linear simulation

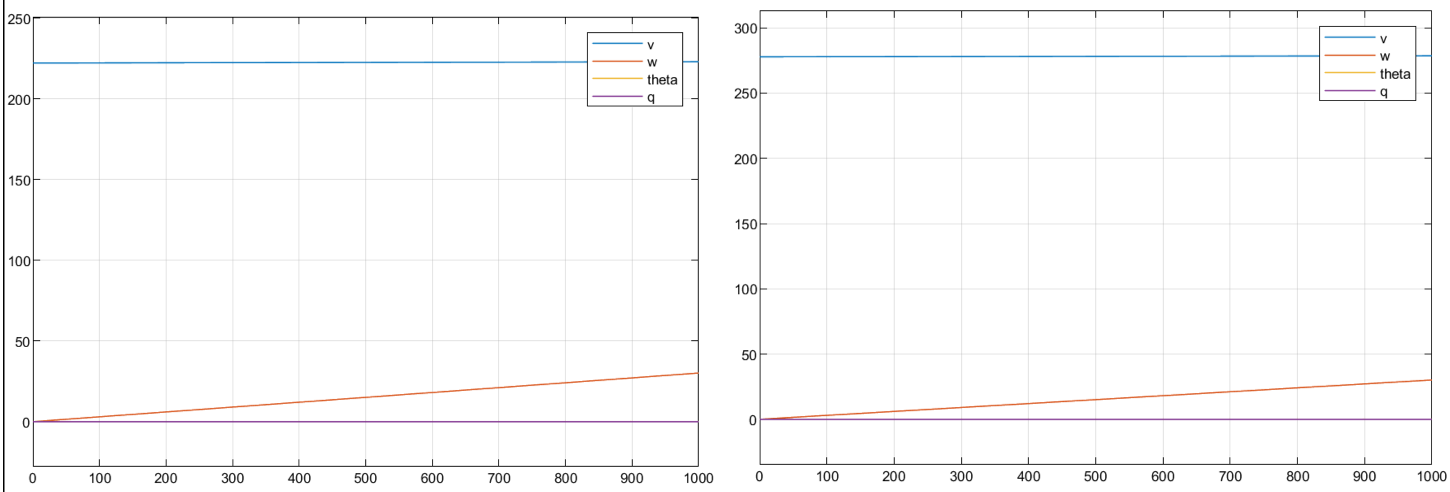

Figure 2: Simulatiosn for v = 800 and 1000 km/h

With a small angle θ, we observe that vertical speed is increasing which is a normal behavior. It is the same for the speed which should not variate.

Besides, we observe that the system is sensible to the initial conditions because it varies according to it.

As a conclusion, we can say that the model is respecting a logical behavior and it seems that it is marginally stable.

B. Equilibrium points

1. Analytical equilibrium points

To study the system further, we try to find the equilibrium points by making the assumption that we have constant speed and constant pitch angle which means :

$$\dot{\theta} = 0 = q$$

$$\dot{v} = 0$$

$$\dot{w} = 0$$

We deduce now from the equations that :

$$\sin(\theta) = \frac{F_x}{mg}$$

$$\cos(\theta) = \frac{F_z}{mg}$$

Taking into consideration the small angle approximation, we have

$$\sin\theta = \theta$$

$$\cos\theta = 1$$

So, now

$$\theta_0 = \frac{F_x}{mg}$$

$$F_z = mg$$

This can be summarized by the following matrices :

$$X_0 = \begin{bmatrix} v_0 \\ w_0 \\ \theta_0 = \dfrac{F_x}{mg} \\ q = 0 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} F_x = mg\theta_0 \\ F_z = mg \\ 0 \end{bmatrix}$$

$$Y_0 = \begin{bmatrix} \theta_0 \\ q = 0 \end{bmatrix}$$

2. Equilibrium points in simulations

We use the same parameters as before for our initial conditions and input and we obtain the exact same results because the previous chosen points were equilibrium points.

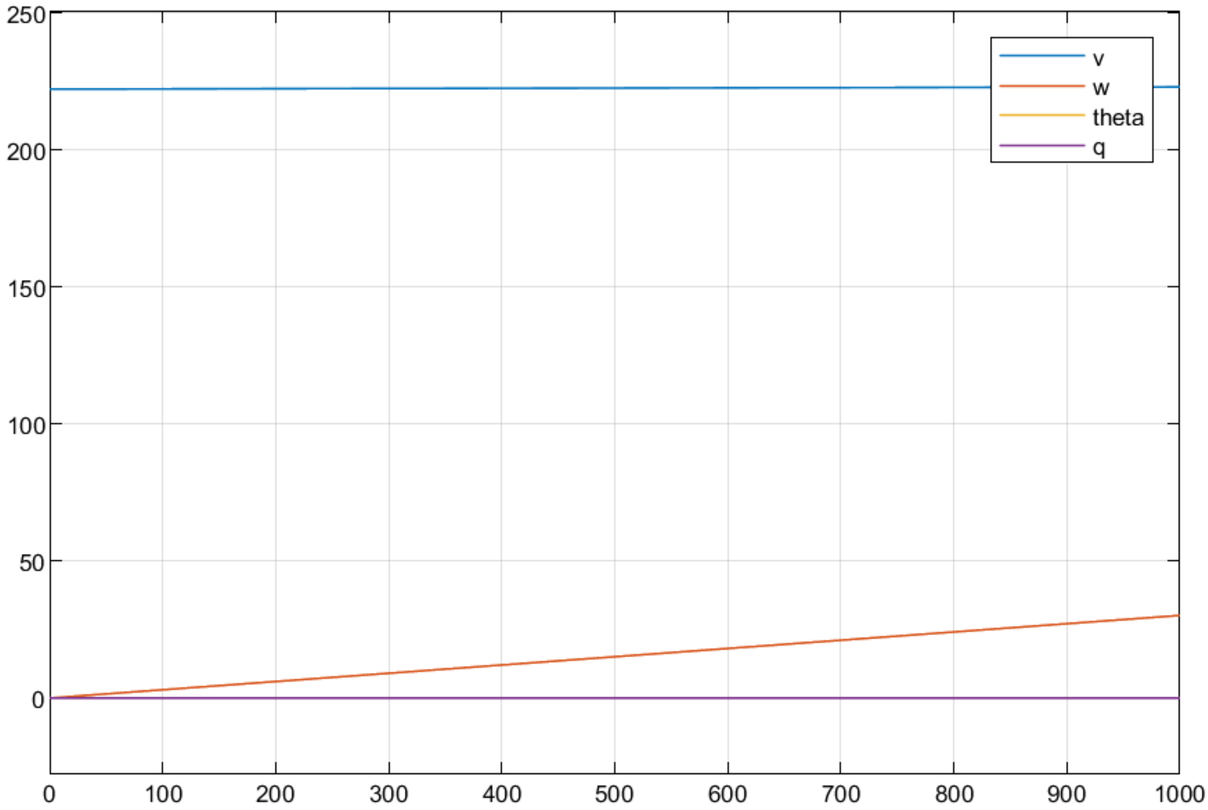

Figure 2: Equilibrium point simulation

We make the same conclusions as before around that equilibrium point.

C. Linearized system study

In this part, we will linearize the system around an equilibrium point to obtain a system in the form of

$$\dot{\tilde{X}} = A\tilde{X} + B\tilde{u}$$

$$\tilde{Y} = C\tilde{X} + D\tilde{u}$$

1. Linearization

Using the equilibrium points seen before and the following change of variable :

$$\tilde{X} = X - X_0$$

$$\tilde{Y} = Y - Y_0$$

$$\tilde{u} = u - u_0$$

We compute $A, B, C \; and \; D$ such that :

$$A = \left.\frac{\partial f}{\partial X}\right|_{X=X_0, u=u_0} = \begin{bmatrix} 0 & -q_0 & -gcos(\theta_0) & -w_0 \\ q_0 & 0 & gsin(\theta_0) & v_0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{X=X_0, u=u_0} = \begin{bmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{I_y} \end{bmatrix}$$

$$C = \left.\frac{\partial h}{\partial X}\right|_{X=X_0, u=u_0} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D = \left.\frac{\partial h}{\partial u}\right|_{X=X_0, u=u_0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. Stability

To determine if the system is stable, we compute

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda & q_0 & g\cos(\theta_0) & w_0 \\ -q_0 & \lambda & -g\sin(\theta_0) & -v_0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda \end{vmatrix} = 0$$

$$\det(\lambda I - A) = \lambda^2(\lambda^2 + q_0^2)$$

This means that $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = iq \; and \; \lambda_4 = -iq$.

We have four roots with null real part which means that the system is marginally stable. This means that the aircraft will stay in the same position as the perturbation it is receiving.

3. Controllability

We remind that the controllability matrix for our system is

$$C = [B \quad AB \quad A^2B \quad A^3B]$$

The system is controllable if $rank(C) = 4$.

We computed numerically through MATLAB the matrix at the equilibrium point which gave us :

C =

```
  1.0e-04 *

    0.1282         0         0         0         0         0         0         0   -0.0001         0         0         0
         0    0.1282         0         0         0    0.0259         0         0    0.0001         0         0         0
         0         0         0         0         0    0.0001         0         0         0         0         0         0
         0         0    0.0001         0         0         0         0         0         0         0         0         0
```

Figure 3: C matrixs

The system is controllable because $rank(C) = number\ of\ states$ which means that we can reach any pitch angle or pitch rate with the given inputs for any initial state.

4. Observability

We remind that the observability matrix for our system is

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}$$

The system is considered observable if $rank(O) = 4$.

We computed numerically the matrix at the equilibrium point which gave us :

$$O = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, $rank(O) = 2$ which is inferior to 4 which means that the system is not obervable. This means that we cannot determine all of the internal workings of our system by looking only at the outputs. To make the system observable, we must add more outputs (more sensors) that measure the unobserved states, so that the observability matrix reaches full rank.

5. Pole placement and simulation

In this part, we work on controlling the system so we introduce a pole placement strategy. By using the place function with poles

$$p_1 = -1, p_2 = -2, p_3 = -3, p_4 = -4.$$

This gives us K :

```
1.0e+08 *

 0.0023         0   -0.0076    0.0000
      0    0.0031    0.0006    0.1820
 0.0000         0    1.8000    2.7000
```

Figure 4: Values of K

Now, the system designed in MATLAB is as following :

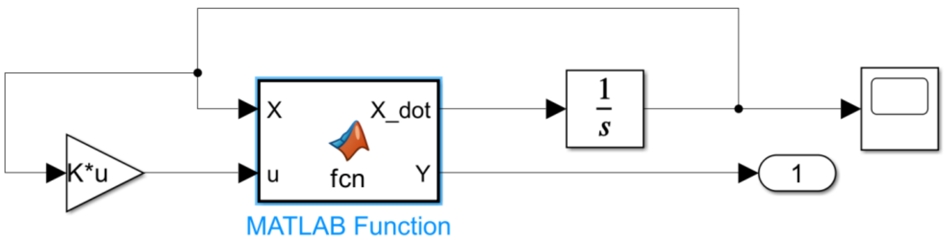

Figure 5: Simulink with control loop

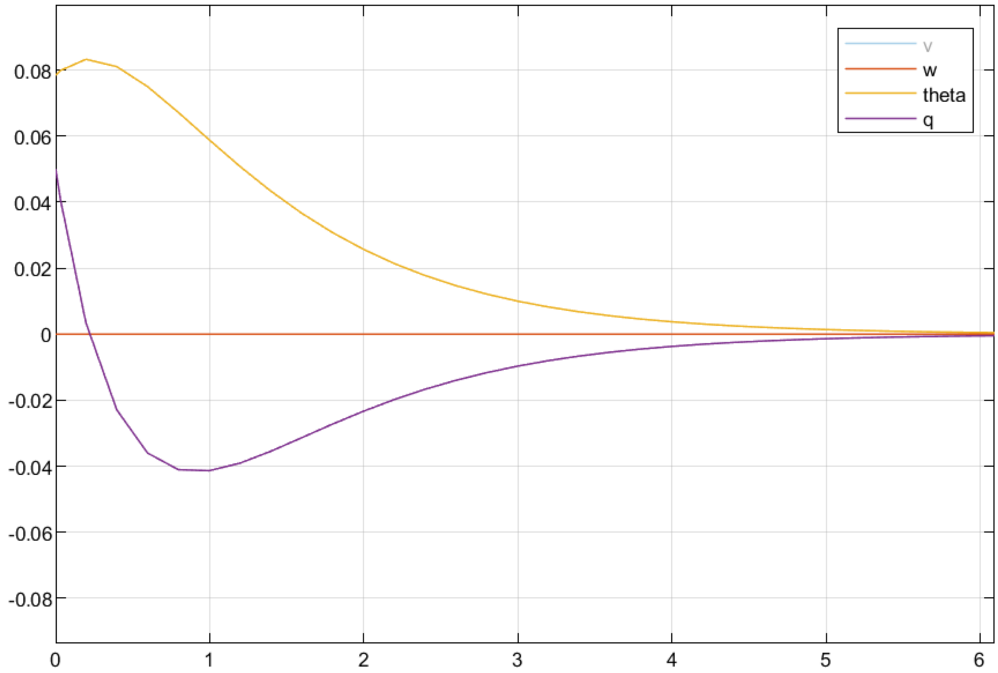

Figure 6: Controlled response

Finally, in the graph above, we observe that for a certain perturbation q, the system's θ angle is variating which is logical and then those values tend to zero which is the controlled behavior we want. However, w and v(not shown here) tend to go back to 0, for w its normal but should at least oscillate. But this is a behavior not conveniant for v. We could work on that in a later part.

6. Saturations

In this final part of the study of our linearized system, we are going to saturate our system. Saturation happens when a control signal u reaches its physical limit, the actuator can't go beyond a certain max or min value. As saturation values, we take

$$u_{min} = \begin{bmatrix} 0 \\ 600000 \\ -300000 \end{bmatrix} \leq u \leq u_{max} = \begin{bmatrix} 200000 \\ 900000 \\ 300000 \end{bmatrix}$$

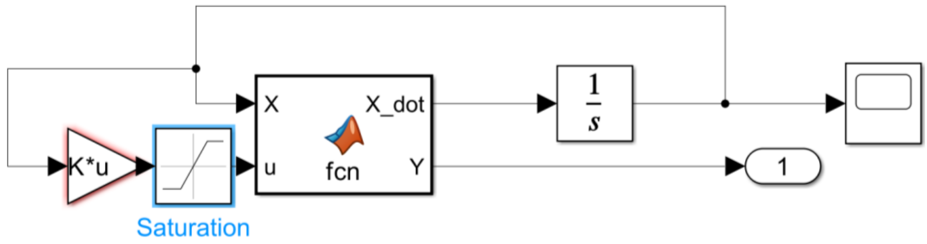

Figure 7: Saturated system

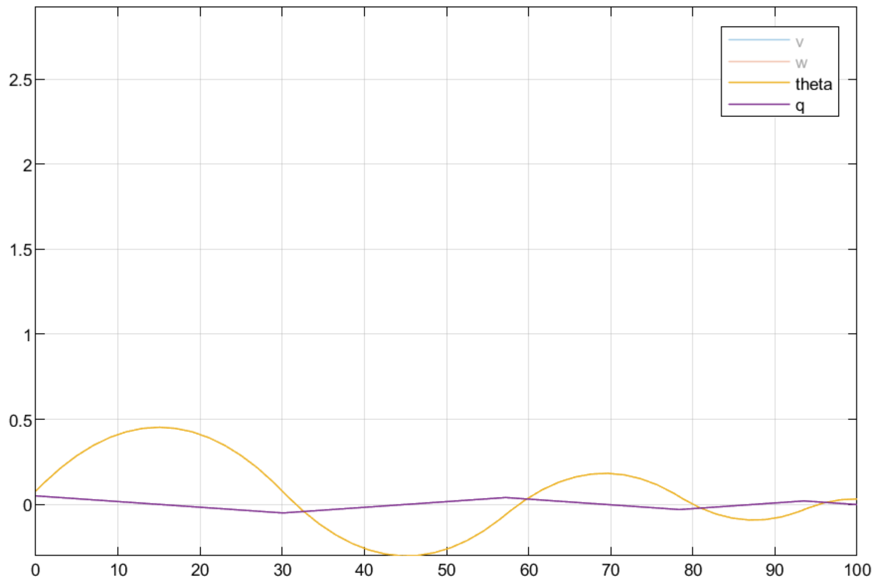

Figure 8: Saturated response

In the saturated system, we see high θ and q oscillations which is normal. The system returns to the initial position but with a really long period of time. This oscillation behavior is dur to the marginal stability of our system.

III. Conclusion

In this project, we studied a simplified model of an aircraft's longitudinal motion, using the Airbus A320 as a reference. After simulating the nonlinear system, we found that it behaves correctly and is marginally stable.

We identified the equilibrium points, then linearized the system around them. The system was shown to be controllable (we can reach any state with inputs), but not observable (we can’t reconstruct all states from the outputs).

Using pole placement, we designed a controller that brings the system back to equilibrium after a disturbance. Finally, we added saturation limits to simulate real actuator constraints. This made the response slower and more oscillatory, but the system still returned to its original state.

This study shows that linear control can stabilize a nonlinear system, but observability and actuator limits must be considered for realistic behavior.