Project 5

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1 Introduction

The Velocity Verlet method is a widely used method for solving coupled ordinary differential equations. In this report, we will model our solar system's dynamics, utilizing said method. The equations to solve, simply come from Newton's laws of motion in gravitational fields, although we will make a small modification down the road to account for relativistic effects and achieve greater accuracy. Due to the sheer number of variables and methods required to calculate the motion of this many bodies, we will object orient our code - simplifying the process of adding bodies and making them interact with each other.
On our way to the final model of the solar system, we are going to explore the accuracy-differences between the Velocity Verlet and the Euler Forward method. The impact the massive planet Jupiter has on Earth's orbit is explored in isolation with the sun, in addition to how Mercury's perihelion precesses when relativistic effects are accounted for. Closing everything with the final model of our solar system, we study the stability of our program at different time steps, and check that the total energy and angular momentum is conserved - making it physically accurate.
The report has a theory part explaining the physical theory and the thought behind our computational implementation of it. Following this are our results and finally a discussion of them.

2 Theory

2.1 The Earth-Sun system

We beging by looking at the problem simply using the Eulers Forward and the Velocity Verlet method. In two dimensions we have the following for the Earth-Sun system

The gravitational force F_G:

$$F_G = \frac{GM_0 M_E}{r^2} \tag{1}$$

where $M_E = 6 \times 10^{24}$ kg, $M_0 = 2 \times 10^{30}$ kg, and $r = 1.5 \times 10^{11}$ m
The force acting on Earth is given by Newtons 2. law, here given in x- and y-direction

$$\frac{d^2x}{dt^2} = \frac{F_x}{M_E}, \quad \frac{d^2y}{dt^2} = \frac{F_y}{M_E}$$

By using the following equalities $x = r\cos(\theta)$, $y = r\sin(\theta)$ and $r = \sqrt{x^2 + y^2}$ we obtain

$$F_x = -\frac{GM_0 M_E}{r^2}\cos(\theta) = -\frac{GM_0 M_E}{r^3}x \tag{2}$$

$$F_y = -\frac{GM_0 M_E}{r^2}\sin(\theta) = -\frac{GM_0 M_E}{r^3}y, \tag{3}$$

This gives the following first order coupled differential equations:

$$\frac{dv_x}{dt} = -\frac{GM_0}{r^3}x, \qquad \frac{dv_y}{dt} = -\frac{GM_0}{r^3}y \tag{4}$$

$$\frac{dx}{dt} = v_x, \qquad \frac{dy}{dt} = v_y, \tag{5}$$

In order to simplify we will use astronomical units (AU) defined by r, which is the avarage distance between Earth and the sun.
Introducing astronomical units: 1 AU $= r = 1.5 \times 10^{11}$

$$\frac{M_e v^2}{r} = F = \frac{GM_0 M_E}{r^2} \tag{6}$$

Since $GM_0 = v^2 r$ and the velocity of Earth, assuming it moves in circular motion:

$$v = 2\pi\mathrm{r}/\mathrm{years} = 2\pi\mathrm{AU}/\mathrm{years}$$

Then we have the following relationship

$$GM_0 = v^2 r = 4\pi^2 \frac{(\mathrm{AU})^2}{\mathrm{years}^2}$$

Bulding code- discretized equations:

$$v_{x,i+1} = v_{x,i} - h\frac{4\pi^2}{r_i^3}x_i, \qquad x_{i+1} = x_i + hv_{x,i}$$

$$v_{y,i+1} = v_{xy,i} - h\frac{4\pi^2}{r_i^3}y_i, \qquad y_{i+1} = y_i + hv_{y,i}$$

2.2 The Verlet method

Another numerical method to be used to evalute the motion of planets in our solarsystem is the Verlet method. This is a method pretty easy to implment as well as it gives stable results. When calculating molecular dynamics, this method is one of the first choises to implement.

If we again look at Newtons second law in the form of a second order differential equation in one dimension.

$$m\frac{d^2x}{dt^2} = F(x,t) \tag{7}$$

In coupled diffenretial equations one obtain

$$\frac{dx}{dt} = v(x,t)$$

and

$$\frac{dv}{dt} = F(x,t)/m = a(x,t)$$

Using a Taylor expansion:

$$x(t,h) = x(t) + hc^{(1)}(t) + \frac{h^2}{2}x^{(2)}(t) + O(h^3). \tag{8}$$

From Newtons second law we allready have obtained the second derivative, $x^{(2)}(t) = a(x,t)$.

Using Taylor for $x(t-h)$ and the discretized expressions $x(t_i, \pm h) = x_{i\pm1}$ and $x_i = x(t_i)$ we obtain

$$x_{i+1} = 2x_i - x_{i-1} + h^2 x_i^{(2)} + O(h^4) \tag{9}$$

Corresponding velocity Taylor expansion is

$$v_{i+1} = v_i + hv_{(1)} + \frac{h^2}{2}v_i^{(2)} + O(h^3) \tag{10}$$

With Newtons second law:

$$v_i^{(1)} = \frac{d^2x}{dt^2} = \frac{F(x_i,t_i)}{m}, \tag{11}$$

Adding the expansion of the derivative of the velocity

$$v_{i+1} = v_i + \frac{h}{2}\left(v_{i+1}^{(1)} + v_i^{(1)}\right) + O(h^3) \tag{12}$$

Since our error goes as $O(h^3)$ we only use the terms up to the second derivative of the velocity.

$$hv_i^{(2)} \approx v_{i+1}^{(1)} - v_i^{(1)} \tag{13}$$

Rewriting the Taylor expansions for the velocity:

$$x_{i+1} = x_i + hv_i + \frac{h^2}{2}v_i^{(1)} + +O(h^3) \tag{14}$$

and

$$v_{i+1} = v_i + \frac{h}{2}\left(v_{i+1}^{(1)} + v_i^{(1)}\right) + O(h^3) \tag{15}$$

2.3 Testing of the algorithm

Before inserting all the planets in the solar system, we would like to thoroughly test the simple Sun-Earth case. This is done by finding initial values for a perfectly circular orbit, and then test stability with different stepsizes and check for conservation of energy. We will also compare the performance of Eulers forward method to the Verlet method.

2.3.1 Initial values

When using astronomical units the radius between the Sun and Earth is quite easily set to $1AU$. The mass of the Sun is also set to 1, and the Earth mass is relative to this mass. For the orbit to be circular we set the centripetal force equal and opposite to the gravitational force. For finding the velocity, the equations are formulated as the following:

$$\begin{aligned} F_g &= \frac{GM_\odot M_E}{r^2} \\ F_c &= \frac{M_E v^2}{r} \\ \frac{GM_\odot M_E}{r^2} &= \frac{M_E v^2}{r} \\ v &= \sqrt{\frac{GM_\odot}{r}} \end{aligned}$$

Where G is the gravitational constant commonly set to $4\pi^2$ for solar system computations. Since $M_\odot$ and r is 1, we get:

$$v = 2\pi\frac{AU}{yr} \tag{16}$$

Thus our initial value for the velocity of Earth should be 2π. This is achieved setting x-position to 0, x-velocity to 2π, y-position to 1 and y-velocity to 0.

2.3.2 Stability with varying timestep

Changing the timestep Δt is crucial for finding a good balance between calculation speed and accuracy. We simulated over a period of a thousand years. The timesteps simulated was $\Delta t = \{0.01, 0.02, 0.05, 0.1\}$ years. The results are shown in section 3.2.1.

2.3.3 Energy and angular momentum conservation

As we have a circular orbit, we would expect the potential and kinetic energy to be conserved since the velocity is the same, and the distance from the sun should also be the same. These energies should be conserved since the only forces acting on the system is conservative forces, namely the gravitational force.

$$E_{tot} = E_k + E_p = \frac{1}{2}M_E v^2 + M_E \gamma r$$

Conservation of angular momentum is true if the system is not acted upon by a torque. Since the only force acting on the system is the gravitational pull of the sun, the angular momentum must be conserved. This can be shown by the following:

$$\begin{aligned} L &= I\omega \\ I &= r^2 m, \qquad \omega = \frac{v}{r} \\ L &= r^2 m \frac{v}{r} \\ L &= rmv \end{aligned}$$

Thus the angular momentum is conserved as long as the orbit velocity and radius is constant, which it is if the kinetic and potential energy is conserved.

2.4 Escape velocity

The escape velocity is the minimum veocity needed by an object to be projected to overcome the pull from the gravitational force in order to escape the gravitanonal field and the orbit. We will try to find this velocity by changig the initial conditions of Earth until it looks like it has escaped the gravitational field of the Sun. Then we will compare this to the analytical result found from this equation:

$$v_{esc} = \sqrt{\frac{2GM_\odot}{r}} \tag{17}$$

where G is the unversal gravitatonal constant, $4\pi^2$, $M_\odot$ is the mass of the sun and r is the distance between the Sun and Earth.

We will also look at what will happen if we let the gravitational force deviate from the original by changing the exponential of the distance:

$$F_G = \frac{GM_\odot M_E}{r^\beta}$$

with $\beta \in [2, 3]$, e.i. changing the exponetial from 2 towards 3 and study the difference. This is visualized in plot 5.

2.5 The three-body problem

In order to find out how much the planet with the greatest mass, Jupiter, alters the motion of the Earth, we will include the planet in our solar system.

This is done by simply adding the magnitude of the force between Earth and Jupiter,

$$F_{Earth-Jupiter} = \frac{GM_{Jupiter}}{r^2_{Earth-Jupiter}} \tag{18}$$

Where $M_{Jupiter}$ is the mass of Jupiter, and M_{Earth} is the mass of Earth, r is the distance between the two planets, and G is the gravitational constant.

The problem is solved by modifying the first order differential equations to accomodate the motion of Earth and Jupiter. The way we do this is to also consider the force from equation (18) when calculating the acceleration of Earth. We will also study the effect of altering the mass of Jupiter by a factor of 10 and 1000, which gives Jupiter roughly the same mass as the Sun.

Finally we would like to check if the fixed-sun approximation was a good approximation. To check this, will see what changes when we let the sun also be a free body. The initial velocity of the sun is set such that the total momentum of the system is zero, and thus the center of mass is fixed.

2.6 All planets

After testing what timestep is needed for the Velocity Verlet algorithm, as well as testing our solver thoroughly for borth conservation of energy, and angular momentum, and figuring out if it is really necessary to let the sun move as a free body, we will run our computations with all planets of the solar system present. The initial velocities and positons are taken from NASA's webpage [1] which means that we are using the solar systems Barycenter as our origin.

2.7 The perihelion precession of Mercury

The observed perihelion precession of Mercury is 43 arc seconds per century, which translates to roughly 0.01194° per century. The perihelion of a planet means, in its simplicty, where it is at its closest position to the sun. Assuming a planar orbit of Mercury, we want to see how this perihelion position changes in regards to the Sun over time. This is what is called the perihelion precession, and is often measured in arc seconds per century, of which 3600 equals one degree per century.

As closed elliptical orbits is a special feature of the Newtonian $\frac{1}{r^2}$ force, we would assume the perihelion precession of the Sun-Mercury scenario is zero when we only use this force. However, by adding a general relativistic correction to the Newtonian gravitational force, we would expect our computed perihelion precession of Mercury to be very close to the observed 43 arc seconds per century. This is a good test of the general theory of relativity. In this case we will look at the system with no other bodies than the Sun and Mercury itself.

The new gravitational force including the relativistic correction is as follows:

$$F_G = \frac{GM_{Sun}M_{Mercury}}{r^2}\left[1 + \frac{3l^2}{r^2c^2}\right] \tag{19}$$

Where $l = |\vec{r} \times \vec{v}|$ is the absolute angular momentum of Mercury and c is the speed of light. The perihelion angle θ_p is given by

$$\tan\theta_p = \frac{y_p}{x_p}$$

Where y_p and x_p is the position of Mercury at perihelion. Our simulation will run over a century, where we choose our y_p and x_p values as the last time in the simulation where Mercury reaches perihelion. The speed of Mercury at perihelion is set to $12.44 AU/yr$ and the distance to the Sun is set to $0.3075 AU$ for our calculations.

2.8 Approach to object orientation

Object orienting our solver makes solving a system of an arbitrary number of bodies a lot more simple. The gist of our approach is having a `Body` object containing the mass, position, name and velocity of a specific body, as well as methods to calculate the distance to other bodies and the gravitational acceleration due to the presence of other bodies.
In addition, we also have a `System` object that contains all of the system's bodies. This also has a `solve`-method that applies the Velocity Verlet algorithm on all the bodies based on its current acceleration (that is calculated firsthand).
The source code to the program as explained here can be found here.

3 Results

3.1 Euler and Verlet without oo

In order to make sure that our algorithm is running correctly, we will start solving the differential equation using both Euler's and Verlet's method without using object oriented(oo) code. The algorithms used to calculate the two are located in the following folders ((Euler) and (Verlet))

3.2 Testing

3.2.1 Stability with varying timestep

In the figures below we plotted Earths orbit over a thousand years with different timesteps.

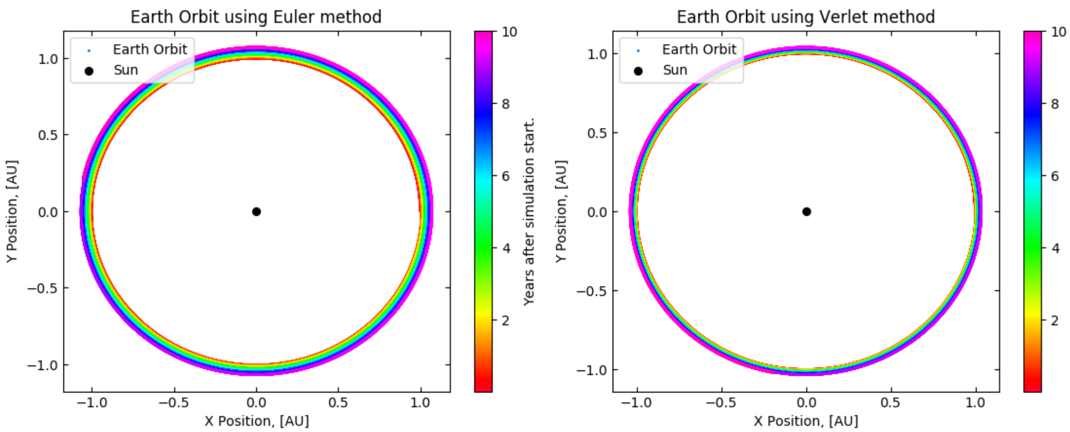

Figure 1: Earth orbit around the Sun using Eulers method and Verlets method respectively

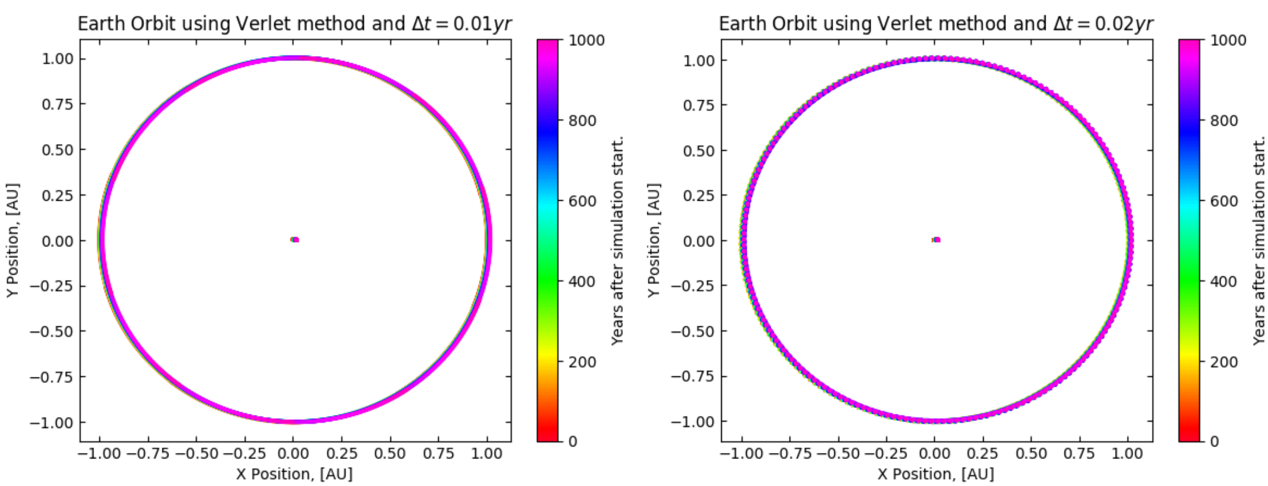

Figure 2: Earth orbit with time steps $\Delta t = 0.01$years and 0.02years respectively

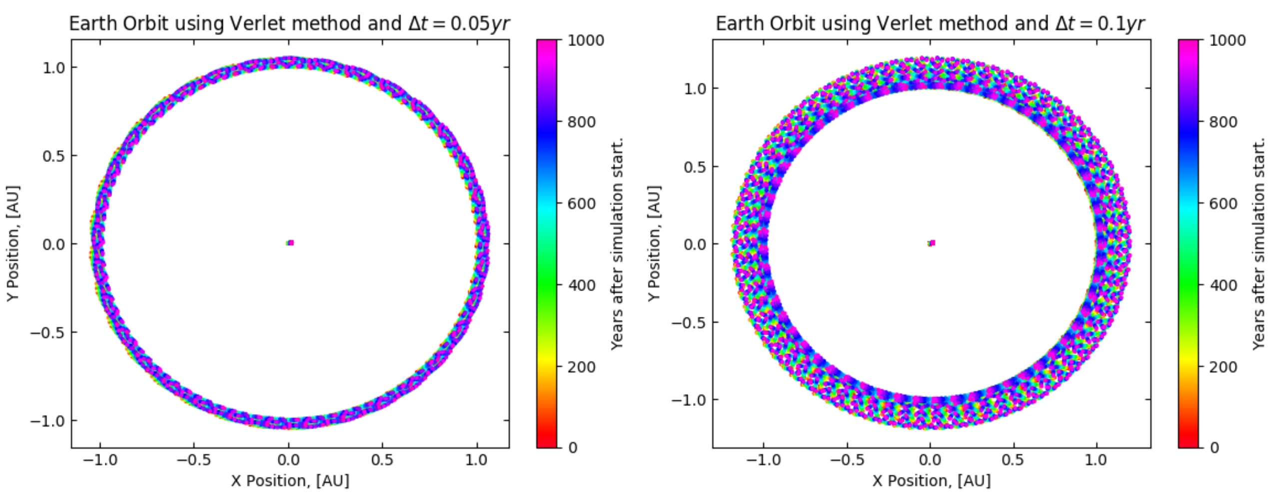

Figure 3: Earth orbit with time steps $\Delta t = 0.05$year and 0.1year respectively

3.2.2 Energy and angular momentum conservation

In the figures below, kinetic energy and potential energy is plotted as a function of time in the system. We chose to simulate over a thousand years with a timestep of $\Delta t = 0.01$, as this timestep is sufficient.

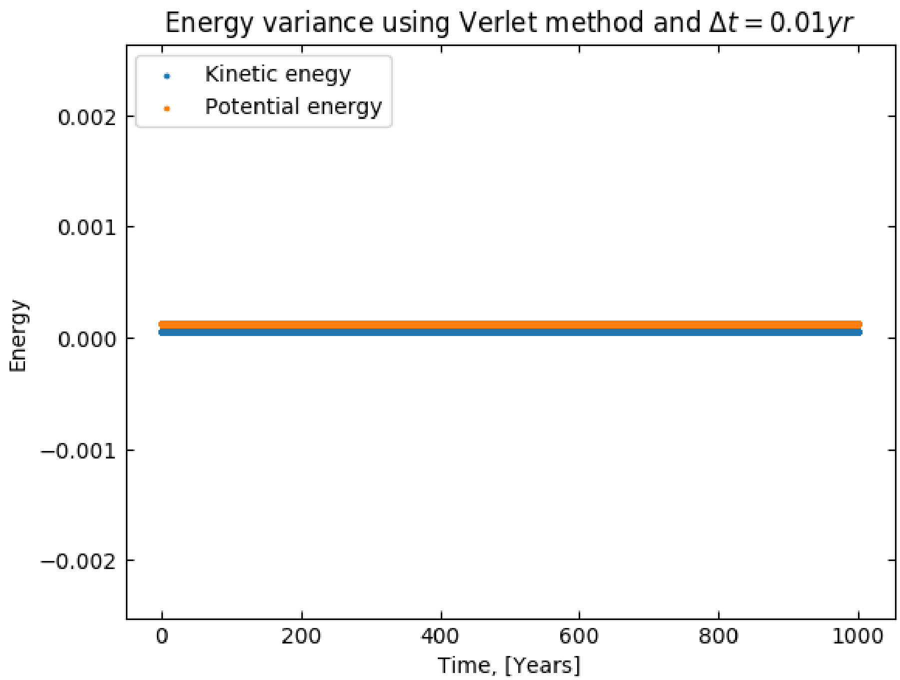

Figure 4: Kinetic and Potential energy with timestep $\Delta t = 0.01$year.

3.2.3 Verlet vs. Euler

Table 1: Comparison of flops and time for the Verlet and Euler method for 100000 iterations over a period of 10 years

	Flops	**Timing**
Euler's method:	10N	2580 ms
Verlet's method:	6N	2875 ms

3.3 Escape velocity

By trail and error we found that the escape velocity of planet Earth is somewhere around 2.828π, which is pretty close to the theoretical value calculated below.

From section **??** using equation 17 to calculate the theoretical value

$$v_{esc-theoretical} = \sqrt{\frac{2 \cdot 4\pi^2 \cdot 1}{1}} \approx 2.8284\pi$$

We also looked at what happens when changing the exponent of the denominator the force of gravity from 2 towards 3 with initial velocity $v_{initial,x} = 2.2\pi$. This is shown in the following plot.

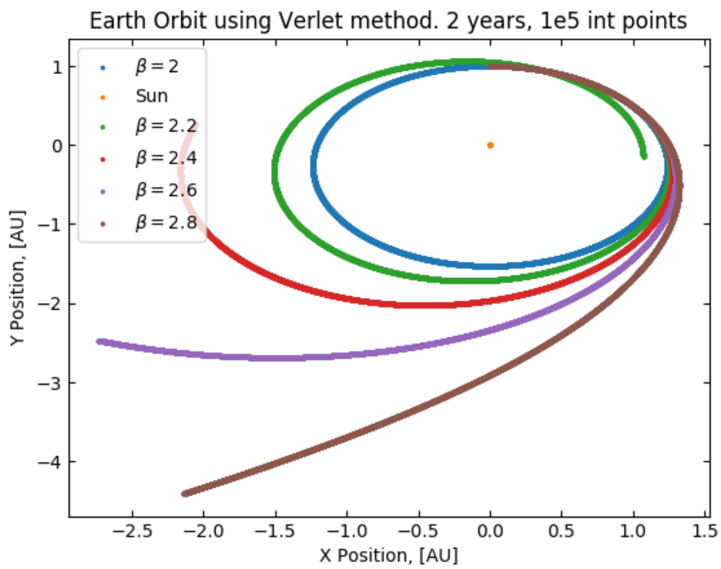

Figure 5: Escape velocity with increasing exponent of denominator of the gravitational force.

All this was found using the initializer found in /src/initializer_test.cpp inside our complete_solar-system code folder by changeing the initial velocity of the Earth. Then the plotter found in /Plotter/2d-plot_beta.py was used to plot the figure above.

3.4 Three-body problem: Sun, Earth and Jupiter.

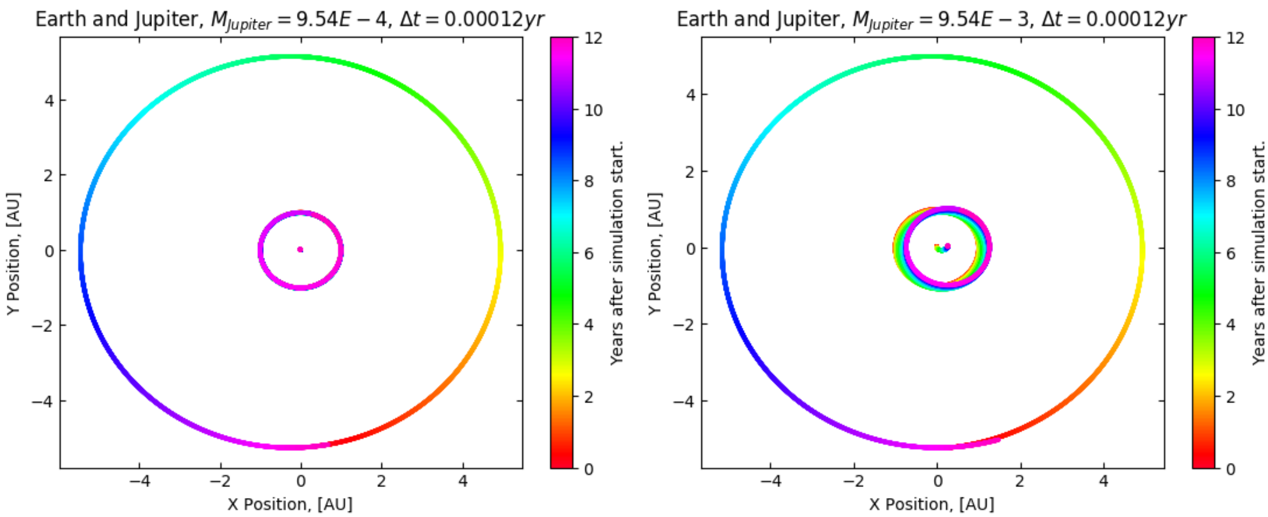

Figure 6: Positions of Sun in the middle, Earth second and outermost Jupiter calculated using the velocity Verlet method with with original mass of Jupiter and an increase of mass of a factor of 10 respectively

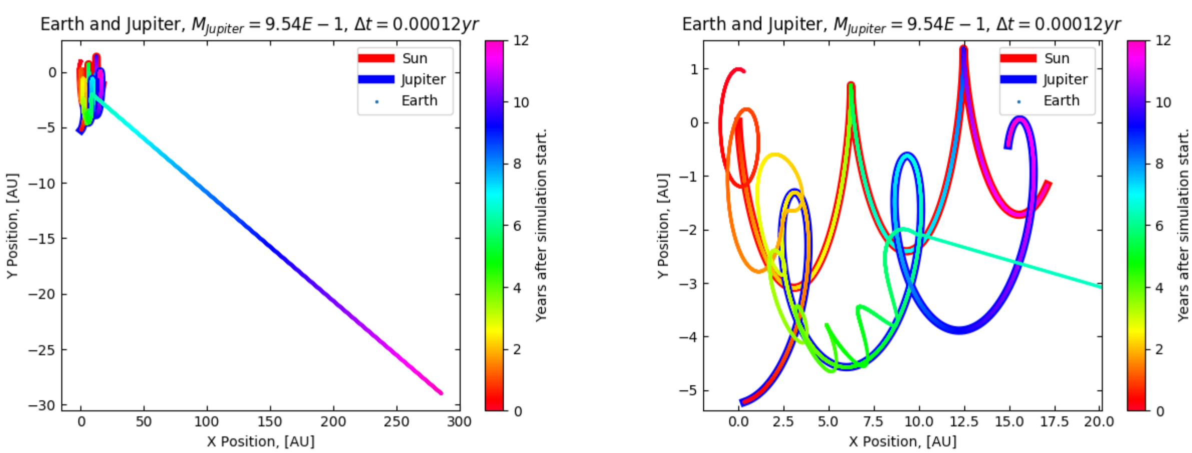

Figure 7: Positions of Earth and Jupiter using the velocity Verlet method with an increase of mass with a factor of 1000. The The initial conditions are from NASA's HORIZONS web interface at December 4. After approximately 7 years, Earth escapes the system. The illustration to the right is a magnified version of the first plot and shows the beautiful interaction between two equally heavy objects.

Stability Verlet solver with increased mass:

3.5 The Solar system

Finally we will add all the other planets to make the Solar system complete, still using the Verlet solver. Rather than having the origin in the postion of the sun, we are using the Barycenter of our solar system. All initial conditions were taken from NASA's HORIZONS web interface[1] with a date of December 4.

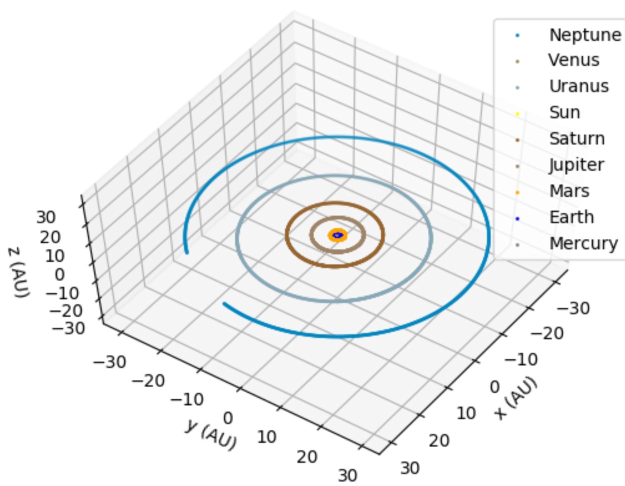

Figure 8: All planets in The Solar System. Due to the large orbits of the outer planets, it is hard to plot the orbit of the inner planets, the Sun, Venus and Mercury without loosing information from the outermost planets.

Sammenlign resultetene med de tidligere.

3.6 The perihelion precession of Mercury

By using initial conditions of $2 \cdot 10^8$ integration points over a century, we are using a timestep $\Delta t = 5 \cdot 10^{-7}$, which is really small. We also changed the code to only save the position of Mercury to file in the last year of simulation to save space. The initializer used is found in /SRC/INITIALIZER_PERIHELION.CPP inside our COMPLETE_SOLAR-SYSTEM code folder.

	θ_p(Degrees)
Newtonian	$-0.000426°$
Relativistic	$0.0117318°$

Table 2: Table of Mercury's calculated perihelion precession with and without relativistic correction.

4 Discussion

4.1 Testing

As we can se from section 3.2.1, the Verlet method is quite stable. Even with a step length of 0.02 years, we see that the orbit for the first thousand years is still quite good. However, with greater steplengths the calculations soon become inreliable.

From results 3.2.2 we see that both the kinetic and potential energy is constant over time, thus it is conserved. Since these values are constant, the angular momentum must also be constant, since it only depends on distance, speed and mass.

DIFFERENCES using Euler vs. Verlet.

When comparing the Verlet method with Euler(section **??**), we see that Euler uses approximately 10N FLOPs, while Verlet only uses 6N FLOPs thus using slightly longer time for the same time interval(10 years). V erlet is, as we have seen, more stable and this method will be used in the rest of this study.

Discuss the stability of the solotiuns using your Verlet solver. + stability when altering the mass of Jupiter.

The numerical calculated escape velocity for planet Earth compared with the analytical:

With increasing β Earth escapes earlier with the same initial velocity. This is simply beacuse the gravitational force from the Sun is decreasing with increasing β and therefore it is easier for Earth to escape from its orbit.

4.2 Escape velocity

As shown in section 3.3, the escape velocity found by trail and error on the initial conditions resulted in an incredibly good result compared to our analytical expression. This is yet another proof that our code is solid.

When modifying the exponent of the distance in the force formula, we get the expected result in figure 5. When β increases we would expect all objects with orbit radius greater than $1AU$ to feel a weaker force, which the trajectories in the mentioned figure clearly shows.

5 Appendix

References

[1] NASA JPL Solar System Dynamics Group. *Horizons System.* 2017. URL: `https://ssd.jpl.nasa.gov/?horizons`.