Project 1

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Abstract

Introduction

Theory and technicalites

Conclusion and perspectives

Project 1 a)

We have the discretized version of u, v, with the boundary conditions $v_0 = v_n = 0$:
For $i = 1$

$$-\frac{v_2 + v_0 - 2v_1}{h^2} = f_1$$

For $i = 2$

$$-\frac{v_3 + v_1 - 2v_2}{h^2} = f_2$$

For $i = n - 1$

$$-\frac{v_n + v_{n-2} - 2v_{n-1}}{h^2} = f_{n-1}$$

If you multiply both sides by h^2

$$-v_2 + 2v_1 - v_0 = h^2 \cdot f_1$$

$$-v_3 + 2v_2 - v_1 = h^2 \cdot f_2$$

$$-v_n + 2v_{n-1} - v_{n-2} = h^2 \cdot f_{n-1}$$

Which you can rewrite as a linear set of equations $\mathbf{A}\mathbf{v} = \mathbf{d}$ where

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \ddots & \vdots \\ 0 & -1 & 2 & -1 & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \end{bmatrix}$$

and

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-1} \end{bmatrix}$$

with $d_i = h^2 \cdot f_i$

Project 1 b)

We have a linear set of equations $\mathbf{A}\mathbf{v} = \mathbf{d}$ we want to solve, where $\mathbf{A}$ is tridiagonal. In the general case, we can express any tridiagonal matrix

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & c_2 & \ddots & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & c_{n-2} & 0 \\ 0 & \dots & 0 & a_{n-2} & b_{n-1} & c_{n-1} \\ 0 & \dots & \dots & 0 & a_{n-1} & b_n \end{bmatrix}$$

just by the three vectors a, b and c, where b has length n, and a and c have length $n-1$.

Forward substitution

Firstly, we want to eliminate the a_i's.

$\mathbf{Av} = \mathbf{d}$ gives us these equations for the case of $i = 1$ and $i = n$

$$b_1 v_1 + c_1 v_2 = d_1, \quad i = 1 \qquad (1)$$

$$a_{n-1} v_{n-1} + b_n v_n = d_n, \quad i = n. \qquad (2)$$

For the rest, we get

$$a_1 v_1 + b_2 v_2 + c_2 v_3 = d_2, \quad i = 2. \qquad (3)$$

$$a_{i-1} v_{i-1} + b_i v_i + c_i v_{i+1} = d_i, \quad i = 2, ..., n-1.$$

We can then modify (3) by subtracting (1), like this

$$b_1 \cdot (3) - a_1 \cdot (1)$$

Which gives

$$(a_1 v_1 + b_2 v_2 + c_2 v_3) b_1 - (b_1 v_1 + c_1 v_2) a_1 = d_2 b_1 - d_1 a_1$$

$$(b_2 b_1 - c_1 a_1) v_2 + c_2 b_1 v_3 = d_2 b_1 - d_1 a_1.$$

Notice that v_1 has been eliminated (the first lower diagonal element has been eliminated).

This can be continued further - to eliminate all the a_i's - and is what we call *forward substitution.*

Its apparent that the vector elements get more and more complicated. To solve this, we make modified vectors and find their elements recursively. Furthermore, we ensure that the $\tilde{b}_i$'s are 1 by normalizing with the modified diagonal elements.

$$\tilde{b}_i = 1$$

$$\tilde{c}_1 = \frac{c_1}{b_1}$$

$$\tilde{c}_i = \frac{c_i}{b_i - \tilde{c}_{i-1}a_{i-1}}$$

$$\tilde{d}_1 = \frac{d_1}{b_1}$$

$$\tilde{d}_i = \frac{d_i - \tilde{d}_{i-1}a_{i-1}}{b_i - \tilde{c}_{i-1}a_{i-1}}$$

Backward substitution

If we look at the coefficients defined above, we see that they give these equations for every i:

$$v_n = \tilde{d}_n$$

$$v_i = \tilde{d}_i - \tilde{c}_i v_{i+1}$$

This *backwards substitution* gives us the solution $\mathbf{v}$.

Appendix

Bibliography