

# Project 3

Anna Stray Rongve  
Knut Magnus Aasrud  
Amund Midtgard Raniseth

November 8, 2019

## Abstract

## 1 Introduction

## 2 Theory

### 2.1 The problem

WRITE a bit about the system we want to solve! What is it?

### 2.2 $2 \times 2$ lattice, analytical expressions

To get started we will find the analytical expression for the partition function and the corresponding expectation values for the energy  $E$ , the mean absolute value of the magnetic moment  $|M|$  (which we will refer to as magnetization), the specific heat  $C_V$  and the susceptibility  $\chi$  as function of  $T$  using periodic boundary conditions. These calculations will serve as benchmarks for our next steps.

#### Partition function, $Z$

The partition function in the canonical ensemble is defined as:

$$Z = \sum_{i=1}^M e^{-\beta E_i}$$

Where  $\beta = \frac{1}{k_B T}$  and  $E_i$  is the energy of the system in the microstate  $i$  and  $M$  is the number of microstates ( $= 2^N$  if  $N$  is number of electrons).

We therefore have to find  $E_i$  which is defined as:

$$E_i = -J \sum_{\langle kl \rangle}^N s_k s_l$$

Where  $\langle kl \rangle$  indicates that we sum only over the nearest neighbors and  $J$  is a constant for the bonding strenght. For our two dimensional system the equation reads:

$$E_{i,2D} = -J \sum_i^N \sum_j^N (s_{i,j} s_{i,j+1} + s_{i,j} s_{i+1,j})$$

Four our two-spin-state system with two dimensions we get the following table if we use periodic boundary conditions:

Number of spins up	Degeneracy	Energy	Magnetization
4	1	-8J	4
3	4	0	2
2	4	0	0
2	2	8J	0
1	4	0	-2
0	1	-8J	-4

Table 1: Number of spins up, degeneracy, energy and magnetization of the two-dimensional benchmark scenario.

Where the magnetization is found by subtracting the number of spin downs from the number of spin up, or in other words the sum of the spins:

$$\mathcal{M} = \sum_{j=1}^N s_j$$

Getting back to the partition function, we insert all 16 of the  $E_i$  respectively. For the degeneracies, we just multiply one iteration of the respective  $E_i$  with the amount of degeneracies. When the energy  $E_i$  is zero, we will just add one to the sum since  $e^0 = 1$ . Thus we get the following:

$$Z = e^{-\beta(-8J)} + 2 \cdot e^{-\beta(8J)} + e^{-\beta(-8J)} + 12 = 2e^{-\beta 8J} + 2e^{\beta 8J} + 12$$

$$Z = 4 \cosh(\beta 8J) + 12$$

#### Energy expectation value, $\langle E \rangle$

The expectation value of the energy is defined as:

$$\langle E \rangle = \sum_{i=1}^M E_i P_i(\beta) = \frac{1}{Z} \sum_{i=1}^M E_i e^{-\beta E_i}$$

Where  $M$  is the sum over all microstates.  $P_i$  is the Boltzmann probability distribution which reads:

$$P_i(\beta) = \frac{e^{\beta E_i}}{Z}$$

For our system, this is easily calculated by inserting the partition function and the microstate energy  $E_i$ .

$$\begin{aligned}
\langle E \rangle &= \frac{1}{2e^{-\beta 8J} + 2e^{\beta 8J} + 12} (2 \cdot -8J \cdot e^{\beta 8J} + 2 \cdot 8J \cdot e^{-\beta 8J}) \\
&= \frac{1}{2e^{-\beta 8J} + 2e^{\beta 8J} + 12} (-16J e^{\beta 8J} + 16J e^{-\beta 8J}) \\
&= 8J \frac{1}{e^{-\beta 8J} + e^{\beta 8J} + 6} (e^{-\beta 8J} - e^{\beta 8J}) \\
&= -8J \frac{\sinh(\beta 8J)}{\cosh(\beta 8J) + 3}
\end{aligned}$$

Since the variance of the mean energy ( $\sigma_E$ ) is needed for the heat capacity later, we will calculate this here.

$$\begin{aligned}
\sigma_E^2 &= \langle E^2 \rangle - \langle E \rangle^2 = \frac{1}{Z} \sum E_i^2 e^{-\beta E_i} - \left( \frac{1}{Z} \sum E_i e^{-\beta E_i} \right)^2 \\
&= \frac{1}{2e^{-\beta 8J} + 2e^{\beta 8J} + 12} (2 \cdot (-8J)^2 \cdot e^{\beta 8J} + 2 \cdot (8J)^2 \cdot e^{-\beta 8J}) \\
&\quad - \left( -8J \frac{\sinh(\beta 8J)}{\cosh(\beta 8J) + 3} \right)^2 \\
&= 128J^2 \frac{2\cosh(\beta 8J)}{4\cosh(\beta 8J) + 12} - \left( -8J \frac{\sinh(\beta 8J)}{\cosh(\beta 8J) + 3} \right)^2 \\
&= 64J^2 \frac{\cosh(\beta 8J)}{\cosh(\beta 8J) + 3} - \left( -8J \frac{\sinh(\beta 8J)}{\cosh(\beta 8J) + 3} \right)^2 \\
\sigma_E^2 &= 64J^2 \left( \frac{\cosh(\beta 8J)}{\cosh(\beta 8J) + 3} - \left( \frac{\sinh(\beta 8J)}{\cosh(\beta 8J) + 3} \right)^2 \right)
\end{aligned}$$

### **Magnetization, $\mathcal{M}$**

In the canonical ensemble the absolute mean magnetization can be described as

$$\langle \mathcal{M} \rangle = \sum_i^M |\mathcal{M}_i| P_i(\beta) = \frac{1}{Z} \sum_i^M |\mathcal{M}_i| e^{-\beta E_i}$$

We can now simply insert the magnetization and the energies for each respective

microstate. This is found in table 2.

$$\begin{aligned}
\langle |\mathcal{M}| \rangle &= \frac{1}{4\cosh(\beta 8J) + 12} (4 \cdot e^{\beta 8J} + 4 \cdot 2 \cdot e^0 + 4 \cdot |-2| \cdot e^0 + |-4| \cdot e^{\beta 8J}) \\
&= \frac{1}{\cosh(\beta 8J) + 3} (e^{\beta 8J} + 2 + 2 + e^{\beta 8J}) \\
&= \frac{1}{\cosh(\beta 8J) + 3} (2e^{\beta 8J} + 4) \\
&= \frac{2e^{\beta 8J} + 4}{\cosh(\beta 8J) + 3}
\end{aligned}$$

Since the variance of the mean magnetization ( $\sigma_M$ ) is needed for the susceptibility later, we will calculate this here. For this we will need  $\langle \mathcal{M}^2 \rangle$ , and the regular mean absolute  $\langle \mathcal{M} \rangle$ .

$$\begin{aligned}
\langle \mathcal{M} \rangle &= \frac{1}{4\cosh(\beta 8J) + 12} (4 \cdot e^{\beta 8J} + 4 \cdot 2 \cdot e^0 + 4 \cdot -2 \cdot e^0 + -4 \cdot e^{\beta 8J}) \\
&= \frac{1}{\cosh(\beta 8J) + 3} (4e^{\beta 8J} - 4e^{\beta 8J} + 8 - 8) \\
&= \frac{1}{\cosh(\beta 8J) + 3} (0) \\
\langle \mathcal{M} \rangle &= 0
\end{aligned}$$

$$\begin{aligned}
\langle \mathcal{M}^2 \rangle &= \frac{1}{Z} \sum |\mathcal{M}_i|^2 e^{-\beta E_i} \\
&= \frac{1}{4\cosh(\beta 8J) + 12} (4^2 \cdot e^{\beta 8J} + 4 \cdot 2^2 \cdot e^0 + 4 \cdot |-2|^2 \cdot e^0 + |-4|^2 \cdot e^{\beta 8J}) \\
&= \frac{1}{4\cosh(\beta 8J) + 12} (16 \cdot e^{\beta 8J} + 16 \cdot e^0 + 16 \cdot e^0 + 16 \cdot e^{\beta 8J}) \\
&= \frac{4}{\cosh(\beta 8J) + 3} (2e^{\beta 8J} + 2) \\
\langle \mathcal{M}^2 \rangle &= \frac{8e^{\beta 8J} + 8}{\cosh(\beta 8J) + 3}
\end{aligned}$$

And we get the variance:

$$\begin{aligned}
\sigma_{\mathcal{M}}^2 &= \langle \mathcal{M}^2 \rangle - \langle \mathcal{M} \rangle^2 \\
&= \frac{8e^{\beta 8J} + 8}{\cosh(\beta 8J) + 3} - (0)^2 \\
\sigma_{\mathcal{M}}^2 &= \frac{8e^{\beta 8J} + 8}{\cosh(\beta 8J) + 3}
\end{aligned}$$

### Specific heat capacity, $C_V$

The specific heat capacity is defined as

$$C_V = \frac{\sigma_E^2}{k_B T^2}$$

Inserting the value  $\sigma_E^2$  we get

$$C_V = \frac{1}{k_B T^2} 64J^2 \left( \frac{\cosh(\beta 8J)}{\cosh(\beta 8J) + 3} - \left( \frac{-\sinh(\beta 8J)}{\cosh(\beta 8J) + 3} \right)^2 \right)$$

This is the main function we will be comparing to the values from our computations later.

### Susceptibility, $\chi$

The susceptibility is defined as

$$\chi = \frac{\sigma_M^2}{k_B T^2}$$

Inserting the value  $\sigma_M^2$  we get

$$\chi = \frac{1}{k_B T^2} \frac{8e^{\beta 8J} + 8}{\cosh(\beta 8J) + 3}$$

Note that these four characteristics are temperature dependent through  $\beta = \frac{1}{k_B T}$ . Ref [1].

## 2.3 Ising model

The Ising model is applied for the study of phase transitions at finite temperatures for magnetic systems. Energy is expressed as:

$$E = -J \sum_{\langle kl \rangle}^N s_k s_l \quad s_k = \pm 1 \quad (1)$$

$N$  is the number of spins and  $J$  is a constant expressing the interaction between neighboring spins. The sum is over the nearest neighbours only, indicated by  $\langle kl \rangle$  in the above equation. For  $J \neq 0$  it is energetically favorable for neighboring spins to align. Leading to, at low temperatures,  $T$ , spontaneous magnetisation.

A probability distribution is needed in order to calculate the mean energy  $\langle E \rangle$  and magnetization  $\langle M \rangle$  at a given temperature. The distribution is given by:

$$P_i(\beta) = \sum_{i=1}^M s_k s_l \exp -\beta E_i, \quad (2)$$

where  $M$  is all the microstates,  $P_i$  is the probability of having the system in a state/configuration  $i$ .

CONFIGURATIONS

## 3 Results

### 3.1 $2 \times 2$ lattice, analytical expressions

If we scale the value of  $\beta$  from  $1/k_B T$  to  $1/J$  (Scaling factor  $k_B T/J$ ) in the analytical expression from section 2.2, we will get a good benchmark for computer computations to come. These values are listed in table 2 below. Note that all values are divided by four, since we want the values per bond, and not for the entire lattice.

Mean energy, $\langle E \rangle$ :	-1.9960
Mean absolute magnetization, $\langle  \mathcal{M}  \rangle$ :	0.9987
Specific heat capacity, $C_V$ :	0.0321
Susceptibility, $\chi$ :	3.9933

Table 2: Benchmark for material characteristics per bond for a  $2 \times 2$  lattice

No. of MC-cycles before reaching good results:

## 4 Discussion

## 5 Conclusion

## References

- [1] Morten Hjorth-jensen. *Computational Physics Lectures: Statistical physics and the Ising Model*. 2019.