

Project 5

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1 Introduction

The Velocity Verlet method is a widely used method for solving coupled ordinary differential equations. In this report, we will model our solar system's dynamics, utilizing said method. The equations to solve, simply come from Newton's laws of motion in gravitational fields, although we will make a small modification down the road to account for relativistic effects and achieve greater accuracy. Due to the sheer number of variables and methods required to calculate the motion of this many bodies, we will object orient our code - simplifying the process of adding bodies and making them interact with each other.

On our way to the final model of the solar system, we are going to explore the accuracy-differences between the Velocity Verlet and the Euler Forward method. The impact the massive planet Jupiter has on Earth's orbit is explored in isolation with the sun, in addition to how Mercury's perihelion precesses when relativistic effects are accounted for. Closing everything with the final model of our solar system, we study the stability of our program at different time steps, and check that the total energy and angular momentum is conserved - making it physically accurate.

The report has a theory part explaining the physical theory and the thought behind our computational implementation of it. Following this are our results and finally a discussion of them.

2 Theory

2.1 The Earth-Sun system

We begin by looking at the problem simply using the Euler's Forward and the Velocity Verlet method. In two dimensions we have the following for the Earth-Sun system

The gravitational force F_G :

$$F_G = \frac{GM_0M_E}{r^2} \quad (1)$$

where $M_E = 6 \times 10^{24}$ kg, $M_0 = 2 \times 10^{30}$ kg, and $r = 1.5 \times 10^{11}$ m
The force acting on Earth is given by Newtons 2. law, here given in x- and y-direction

$$\frac{d^2x}{dt^2} = \frac{F_x}{M_E}, \quad \frac{d^2y}{dt^2} = \frac{F_y}{M_E}$$

By using the following equalities $x = r \cos(\theta)$, $y = r \sin(\theta)$ and $r = \sqrt{x^2 + y^2}$ we obtain

$$F_x = -\frac{GM_0M_E}{r^2} \cos(\theta) = -\frac{GM_0M_E}{r^3} x \quad (2)$$

$$F_y = -\frac{GM_0M_E}{r^2} \sin(\theta) = -\frac{GM_0M_E}{r^3} y, \quad (3)$$

This gives the following first order coupled differential equations:

$$\frac{dv_x}{dt} = -\frac{GM_0}{r^3} x, \quad \frac{dv_y}{dt} = -\frac{GM_0}{r^3} y \quad (4)$$

$$\frac{dx}{dt} = v_x, \quad \frac{dy}{dt} = v_y, \quad (5)$$

In order to simplify we will use astronomical units (AU) defined by r , which is the average distance between Earth and the sun.

Introducing astronomical units: $1 \text{ AU} = r = 1.5 \times 10^{11}$

$$\frac{M_e v^2}{r} = F = \frac{GM_0M_E}{r^2} \quad (6)$$

Since $GM_0 = v^2 r$ and the velocity of Earth, assuming it moves in circular motion:

$$v = 2\pi r / \text{years} = 2\pi \text{AU} / \text{years}$$

Then we have the following relationship

$$GM_0 = v^2 r = 4\pi^2 \frac{(\text{AU})^2}{\text{years}^2}$$

Bulding code- discretized equations:

$$\begin{aligned} v_{x,i+1} &= v_{x,i} - h \frac{4\pi^2}{r_i^3} x_i, & x_{i+1} &= x_i + h v_{x,i} \\ v_{y,i+1} &= v_{y,i} - h \frac{4\pi^2}{r_i^3} y_i, & y_{i+1} &= y_i + h v_{y,i} \end{aligned}$$

2.2 The Verlet method

Another numerical method to be used to evaluate the motion of planets in our solarsystem is the Verlet method. This is a method pretty easy to implment as well as it gives stable results. When calculating molecular dynamics, this method is one of the first choises to implement.

If we again look at Newtons second law in the form of a second order differential equation in one dimension.

$$m \frac{d^2 x}{dt^2} = F(x, t) \quad (7)$$

In coupled diffenretial equations one obtain

$$\frac{dx}{dt} = v(x, t)$$

and

$$\frac{dv}{dt} = F(x, t)/m = a(x, t)$$

Using a Taylor expansion:

$$x(t, h) = x(t) + hc^{(1)}(t) + \frac{h^2}{2}x^{(2)}(t) + O(h^3). \quad (8)$$

From Newtons second law we allready have obtained the second derivative, $x^{(2)}(t) = a(x, t)$.

Using Taylor for $x(t - h)$ and the discretized expressions $x(t_i, \pm h) = x_{i\pm 1}$ and $x_i = x(t_i)$ we obtain

$$x_{i+1} = 2x_i - x_{i-1} + h^2 x_i^{(2)} + O(h^4) \quad (9)$$

Corresponding velocity Taylor expansion is

$$v_{i+1} = v_i + hv_{(1)} + \frac{h^2}{2}v_i^{(2)} + O(h^3) \quad (10)$$

With Newtons second law:

$$v_i^{(1)} = \frac{d^2 x}{dt^2} = \frac{F(x_i, t_i)}{m}, \quad (11)$$

Adding the expansion of the derivative of the velocity

$$v_{i+1} = v_i + \frac{h}{2} \left(v_{i+1}^{(1)} + v_i^{(1)} \right) + O(h^3) \quad (12)$$

Since our error goes as $O(h^3)$ we only use the terms up to the second derivative of the velocity.

$$hv_i^{(2)} \approx v_{i+1}^{(1)} - v_i^{(1)} \quad (13)$$

Rewriting the Taylor expansions for the velocity:

$$x_{i+1} = x_i + hv_i + \frac{h^2}{2}v_i^{(1)} + O(h^3) \quad (14)$$

and

$$v_{i+1} = v_i + \frac{h}{2} \left(v_{i+1}^{(1)} + v_i^{(1)} \right) + O(h^3) \quad (15)$$

2.3 Testing of the algorithm

Before inserting all the planets in the solar system, we would like to thoroughly test the simple Sun-Earth case. This is done by finding initial values for a perfectly circular orbit, and then test stability with different stepsizes and check for conservation of energy. We will also compare the performance of Eulers forward method to the Verlet method.

2.3.1 Initial values

When using astronomical units the radius between the Sun and Earth is quite easily set to $1AU$. The mass of the Sun is also set to 1, and the Earth mass is relative to this mass. For the orbit to be circular we set the centripetal force equal and opposite to the gravitational force. For finding the velocity, the equations are formulated as the following:

$$\begin{aligned} F_g &= \frac{\gamma M_\odot M_E}{r^2} \\ F_c &= \frac{M_E v^2}{r} \\ \frac{\gamma M_\odot M_E}{r^2} &= \frac{M_E v^2}{r} \\ v &= \sqrt{\frac{\gamma M_\odot}{r}} \end{aligned}$$

Where γ is the gravitational constant commonly set to $4\pi^2$ for solar system computations. Since M_\odot and r is 1, we get:

$$v = 2\pi \frac{AU}{yr} \quad (16)$$

Thus our initial value for the velocity of Earth should be 2π . This is achieved setting x-position to 0, x-velocity to 2π , y-position to 1 and y-velocity to 0.

2.3.2 Stability with varying timestep

Changing the timestep Δt is crucial for finding a good balance between calculation speed and accuracy. We simulated over a period of a thousand years. The timesteps simulated was $\Delta t = \{0.01, 0.02, 0.05, 0.1\}$ years. The results are shown in section 3.2.1.

2.3.3 Energy and angular momentum conservation

As we have a circular orbit, we would expect the potential and kinetic energy to be conserved since the velocity is the same, and the distance from the sun should also be the same. These energies should be conserved since the only forces acting on the system is conservative forces, namely the gravitational force.

$$E_{tot} = E_k + E_p = \frac{1}{2}M_E v^2 + M_E \gamma r$$

Conservation of angular momentum is true if the system is not acted upon by a torque. Since the only force acting on the system is the gravitational pull of the sun, the angular momentum must be conserved. This can be shown by the following:

$$\begin{aligned} L &= I\omega \\ I &= r^2 m, \quad \omega = \frac{v}{r} \\ L &= r^2 m \frac{v}{r} \\ L &= rmv \end{aligned}$$

Thus the angular momentum is conserved as long as the orbit velocity is constant, which it is if the kinetic energy is conserved.

2.4 Escape velocity

The escape velocity is the minimum velocity needed by an object to be projected to overcome the pull from the gravitational force in order to escape the gravitational field and the orbit.

$$v_{esc} = \sqrt{\frac{2GM}{R}} \quad (17)$$

where G is the universal gravitational constant, $6.673 \times 10^{-11} m^2 kg^{-2}$, M is the mass of the sun and R is the radius of the sun.

We will also look at what will happen if the gravitational force is change to the following

$$F_G = \frac{GM_0 M_E}{r^\beta}$$

with $\beta \in [2, 3]$, e.g. changing the exponential from 2 towards 3 and study the difference.

2.5 The three-body problem

In order to find out how much the planet with the greatest mass, Jupiter, alters the motion of the Earth. Without Jupiter the motion will remain unchanged with time and will be stable.

This is done by simply adding the magnitude of the force between Earth and Jupiter,

$$F_{Earth-Jupiter} = \frac{GM_{Jupiter}}{r_{Earth-Jupiter}^2} \quad (18)$$

Where $M_{Jupiter}$ is the mass of Jupiter, and M_{Earth} is the mass of Earth. r is the distance between the two planets, and G is the gravitational constant.

The problem is calculated by modifying the first order differential equations to accommodate the motion of Earth and Jupiter by taking into account the distance between them, x,y and z with the Verlet algorithm. We will also study the effect of altering the mass of Jupiter by a factor of 10 and 100.

2.6 All planets

Finally we will assemble all of our previous systems, Earth, Jupiter and the Sun in motion, using the Verlet solver. Instead of setting the center of mass at the position of the Sun, it will be set at the origin. The initial velocity will be such that the total momentum of the system is zero so the center of mass is fixed. Thereafter we will include all the other planets in our system as well as moons if there is time. The initial velocities and positions are taken from NASA's webpage [1]

2.7 The perihelion precession of Mercury

When a planet is the closest to the sun, called, perihelion, it is a good measurement/test of general relativity theory by calculating the theoretical value and compare it with observations. For Mercury, after subtracting all the classical effects, the observed time is 43 seconds per century.

... We will here study the orbit of Mercury around the sun with a slightly adjusted gravitational force where the general relativistic correction is added to the Newtonian gravitational force

$$F_G = \frac{GM_{Sun}M_{Mercury}}{r^2} \left[1 + \frac{3l^2}{(rc)^2} \right] \quad (19)$$

with the distance r to the Sun and with angular momentum $l = \vec{r} \times \vec{v}$ of Mercury per unit mass and c is the speed of light. The perihelion angle ϕ_p is given by

$$\tan \phi_p = \frac{y_p}{x_p}$$

where y_p and x_p is the position of Mercury at perihelion. The speed of Mercury at perihelion is 12.44 AU/yr and the distance to the Sun is 0.3075 AU

2.8 Approach to object orientation

Object orienting our solver makes solving a system of an arbitrary number of bodies a lot more simple. The gist of our approach is having a **Body** object containing the mass, position, name and velocity of a specific body, as well as methods to calculate the distance to other bodies and the gravitational acceleration due to the presence of other bodies.

In addition, we also have a **System** object that contains all of the system's bodies. This also has a **solve**-method that applies the Velocity Verlet algorithm on all the bodies based on its current acceleration (that is calculated firsthand).

The source code to the program as explained here can be found [here](#).

3 Results

3.1 Euler and Verlet without oo

In order to make sure that our algorithm is running correctly, we will start solving the differential equation using both Euler's and Verlet's method without using object oriented(oo) code. The algorithms used to calculate the two are located in the following folders ((Euler) and (Verlet))

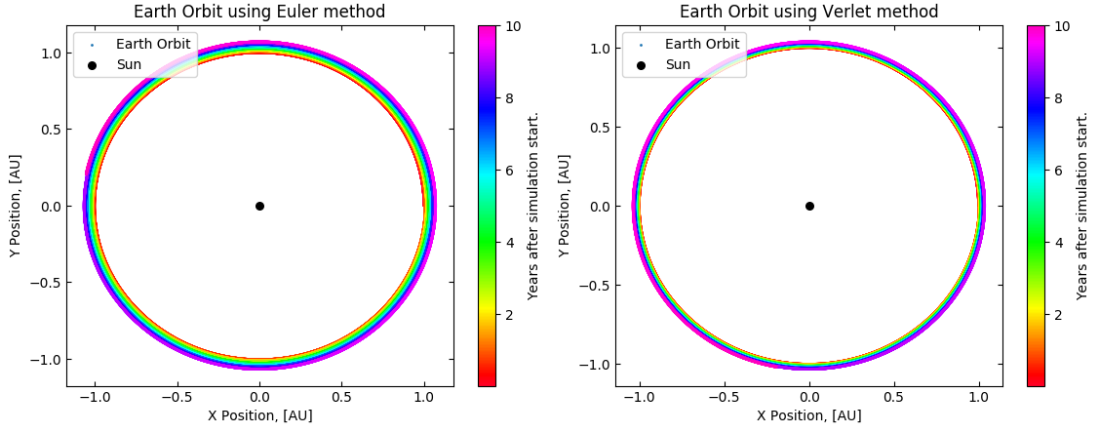


Figure 1: Earth orbit around the Sun using Eulers method and Verlets method respectively

3.2 Testing

3.2.1 Stability with varying timestep

In the figures below we plotted Earths orbit over a thousand years with different timesteps.

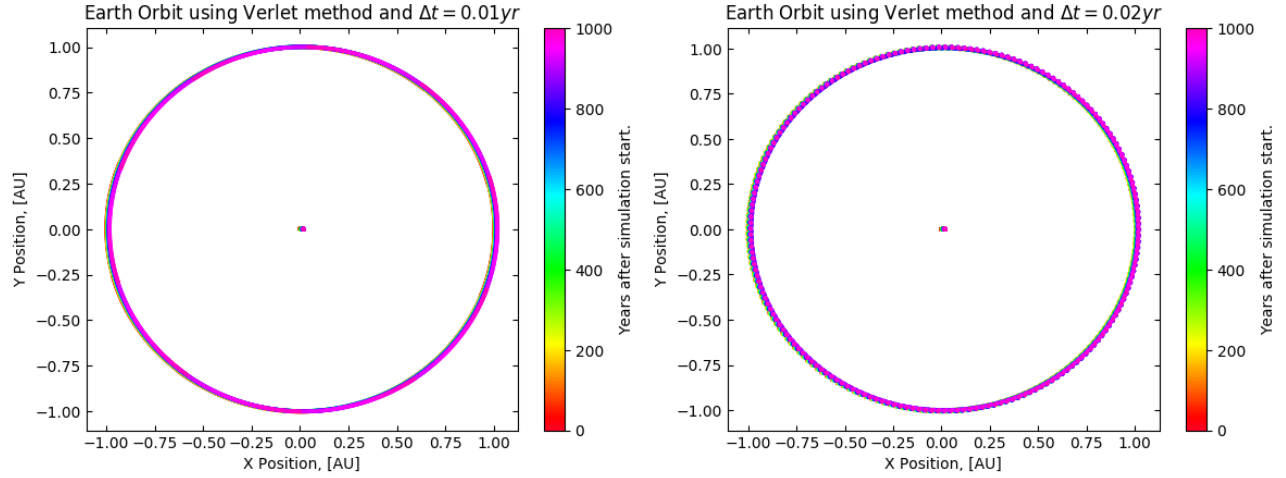


Figure 2: Earth orbit with time steps $\Delta t = 0.01$ years and 0.02 years respectively

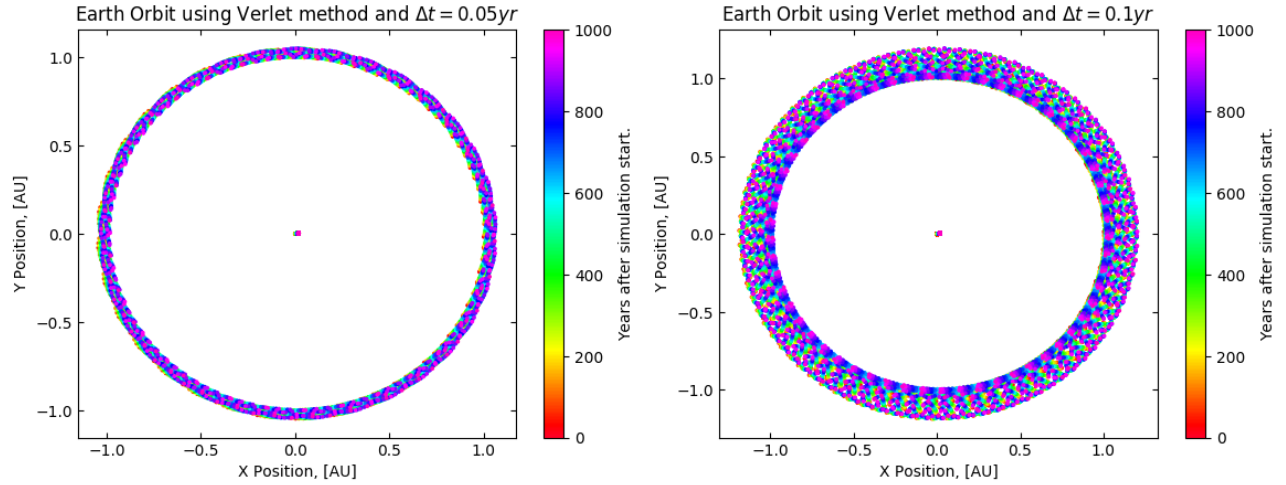


Figure 3: Earth orbit with time steps $\Delta t = 0.05$ year and 0.1 year respectively

3.2.2 Energy and angular momentum conservation

In the figures below, kinetic energy and potential energy is plotted as a function of time in the system. We chose to simulate over a thousand years with a timestep of $\Delta t = 0.01$, as this gives the most stable results.

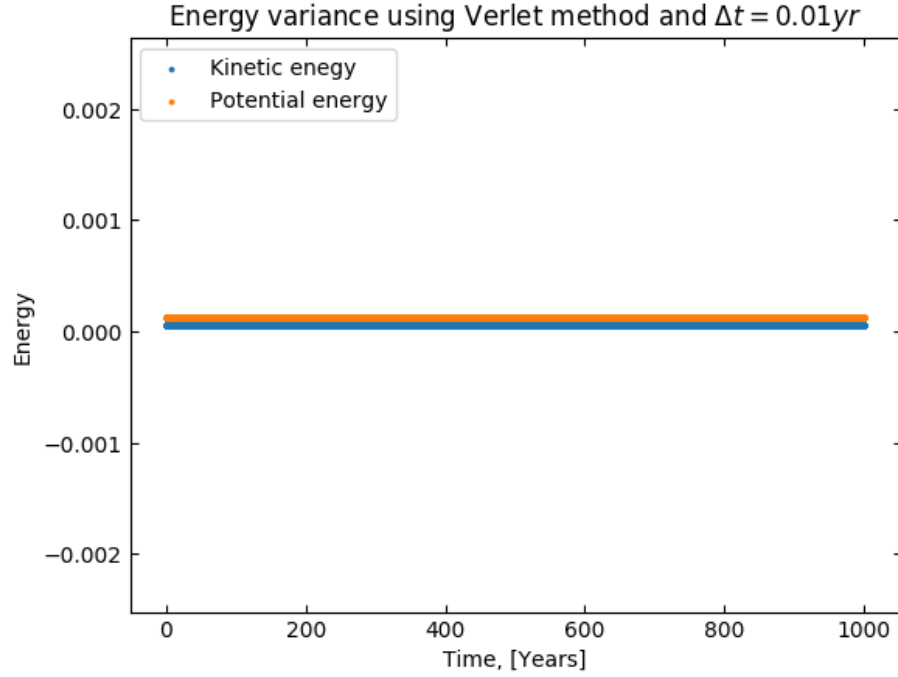


Figure 4: Kinetic and Potential energy with timestep $\Delta t = 0.01\text{year}$.

3.2.3 Verlet vs. Euler

Table 1: Comparison of flops and time for the Verlet and Euler method for 100000 iterations over a period of 10 years

	Flops	Timing
Euler's method:	10N	2580 ms
Verlet's method:	6N	2875 ms

3.3 Escape velocity

By trail and error the escape velocity of planet Earth is somewhere between 2.8π and 2.85π , which is pretty close to the theoretical value calculated below. From section ?? using equation 17 to calculate the theoretical value

$$v_{esc-theoretical} = \sqrt{\frac{2 \cdot 4\pi^2 \cdot 1}{1}} = 2.83\pi$$

We also looked at what happens when changing the exponent of the denominator the force of gravity from 2 towards 3 with initial velocity $v_{initial,x} = 2.2\pi$. This is shown in the following plot.

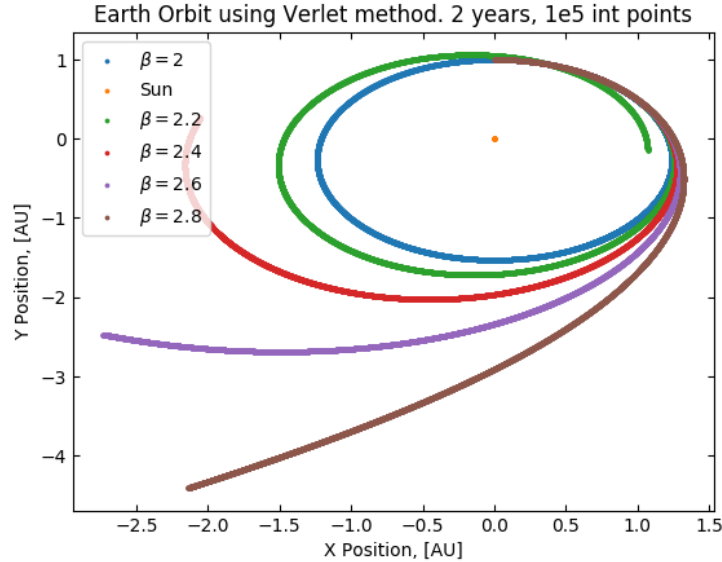


Figure 5: Escape velocity with increasing exponent of denominator of the gravitational force.

3.4 Three-body problem- Sun, Earth and Jupiter.

Stability Verlet solver with increased mass:

3.5 The Solar system

Finally we will add all the other planets to make the Solar system complete, still using the Verlet solver. Both the Earth, Jupiter and the Sun are in motion. Rather than having the origin in the position of the sun, we set the center-of-mass of the three-body system in the origin. The initial velocity of the Sun is set to ?? in order to have the total momentum of the system to exactly zero.

Sammenlign resultatene med de tidligere.

4 Discussion

4.1 Testing

As we can see from section 3.2.1, the Verlet method is quite stable. Even with a step length of 0.02 years, we see that the orbit for the first thousand years is

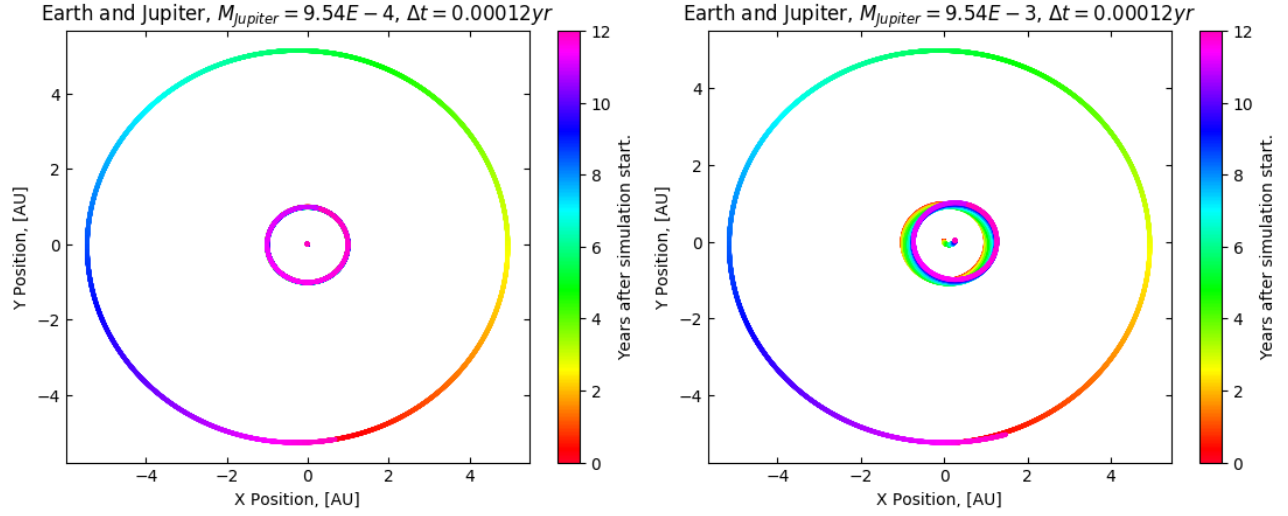


Figure 6: Positions of Sun in the middle, Earth second and outermost Jupiter calculated using the velocity Verlet method with original mass of Jupiter and an increase of mass of a factor of 10 respectively

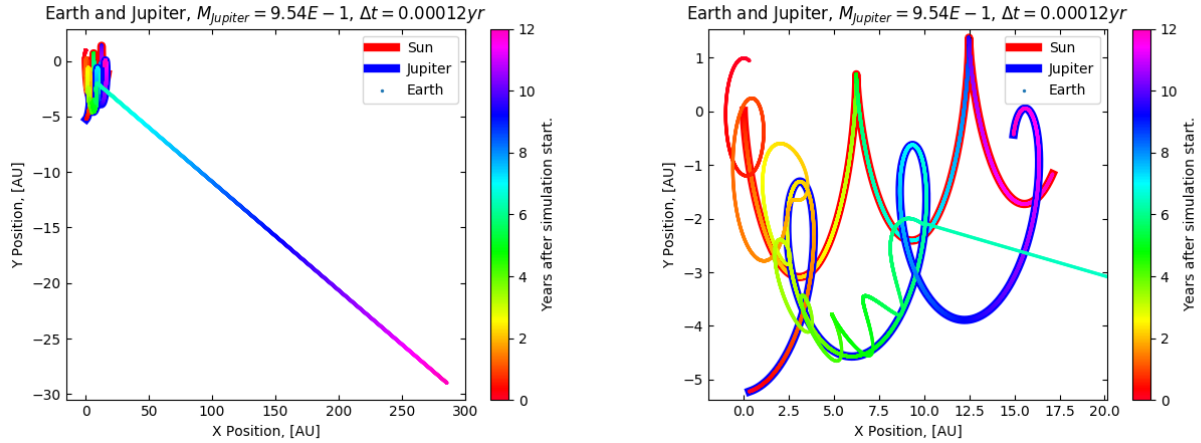


Figure 7: Positions of Earth and Jupiter using the velocity Verlet method with an increase of mass with a factor of 1000. The Sun starts at approximately (0,0)???, Jupiter starts at (0.0,-5)??? and Earth starts at (???????????). After some time, $t \approx 7\text{years}$ Earth escapes the system. Picture to the right is a zoomed version of the first plot.

still quite good. However, with greater steplengths the calculations soon become unreliable.

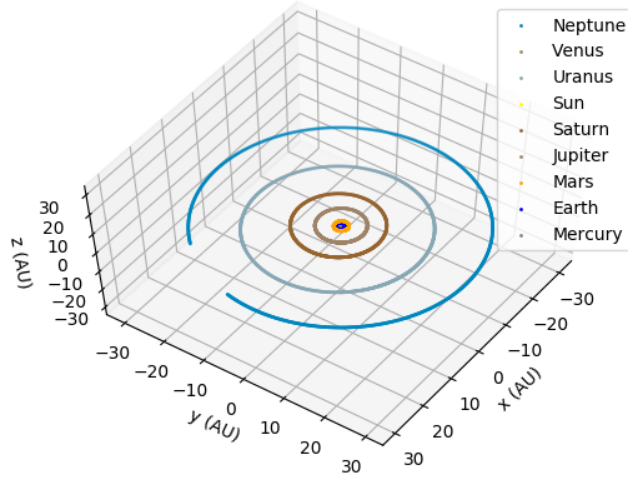


Figure 8: All planets in The Solar System. Due to the large orbits of the outer planets, it is hard to plot the orbit of the inner planets, the Sun, Venus and Mercury without losing information from the outermost planets.

From results 3.2.2 we see that both the kinetic and potential energy is constant over time, thus it is conserved. Since these values are constant, the angular momentum must also be constant, since it only depends on distance, speed and mass.

DIFFERENCES using Euler vs. Verlet.

When comparing the Verlet method with Euler(section ??), we see that Euler uses approximately 10N FLOPs, while Verlet only uses 6N FLOPs thus using slightly longer time for the same time interval(10 years). Verlet is, as we have seen, more stable and this method will be used in the rest of this study.

Discuss the stability of the solutions using your Verlet solver. + stability when altering the mass of Jupiter.

The numerical calculated escape velocity for planet Earth compared with the analytical:

With increasing β Earth escapes earlier with the same initial velocity. This is simply because the gravitational force from the Sun is decreasing with increasing β and therefore it is easier for Earth to escape from its orbit.

5 Appendix

References

- [1] NASA JPL Solar System Dynamics Group. *Horizons System*. 2017. URL: <https://ssd.jpl.nasa.gov/?horizons>.