Project 1 b)

General algorithm

We have a linear set of equations $\mathbf{Av} = \mathbf{d}$

In the general case, we can express any tridiagonal matrix

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & c_2 & \ddots & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & c_{n-2} & 0 \\ 0 & \dots & 0 & a_{n-2} & b_{n-1} & c_{n-1} \\ 0 & \dots & \dots & 0 & a_{n-1} & b_n \end{bmatrix}$$

just by the three vectors $a_, b$ and c, where b has length n, and a and c have length $n - 1$.

Forward substitution

Firstly, we want to eliminate the a_i's.

$\mathbf{Av} = \mathbf{d}$ gives us these equations for the case of $i = 1$ and $i = n$

$$b_1 v_1 + c_1 v_2 = d_1, \quad i = 1 \qquad (1)$$
$$a_{n-1} v_{n-1} + b_n v_n = d_n, \quad i = n. \qquad (2)$$

For the rest, we get

$$a_1 v_1 + b_2 v_2 + c_2 v_3 = d_2, \quad i = 2. \qquad (3)$$

$$a_{i-1} v_{i-1} + b_i v_i + c_i v_{i+1} = d_i, \quad i = 2, ..., n - 1.$$

We can then modify (3) by subtracting (1), like this

$$b_1 \cdot (3) - a_1 \cdot (1)$$

Which gives

$$(a_1 v_1 + b_2 v_2 + c_2 v_3) b_1 - (b_1 v_1 + c_1 v_2) a_1 = d_2 b_1 - d_1 a_1$$
$$(b_2 b_1 - c_1 a_1) v_2 + c_2 b_1 v_3 = d_2 b_1 - d_1 a_1.$$

Notice that v_1 has been eliminated (the first lower diagonal element has been eliminated).

This can be continued further - to eliminate all the a_i's - and is what we call *forward substitution.*

Its apparent that the vector elements get more and more complicated. To solve this, we make modified vectors and find their elements recursively. Furthermore, we ensure that the $\tilde{b}_i$'s are 1 by normalizing with the modified diagonal elements.

$$\tilde{b}_i = 1$$

$$\tilde{c}_1 = \frac{c_1}{b_1}$$

$$\tilde{c}_i = \frac{c_i}{b_i - \tilde{c}_{i-1}a_{i-1}}$$

$$\tilde{d}_1 = \frac{d_1}{b_1}$$

$$\tilde{d}_i = \frac{d_i - \tilde{d}_{i-1}a_{i-1}}{b_i - \tilde{c}_{i-1}a_{i-1}}$$

Backward substitution

If we look at the coefficients defined above, we see that they give these equations for every i:

$$v_n = \tilde{d}_n$$

$$v_i = \tilde{d}_i - \tilde{c}_i v_{i+1}$$

This is the *backward substitution* necessary to find the solution.

$a_i = c_i = -1/h^2 \text{and}\, b_i = 2$

Then we can write the linear set of equations as

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & 0 & \cdots & \cdots & \cdots \\ a_2 & b_2 & c_2 & 0 & & \\ 0 & a_3 & b_3 & c_3 & 0 & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & & & & a_n & b_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ \cdots \\ \cdots \\ u_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \cdots \\ \cdots \\ \cdots \\ f_n \end{bmatrix}$$

In the 4×4 case you will get

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 \\ 0 & a_3 & b_3 & c_3 \\ 0 & 0 & a_4 & b_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

Forward substitution

If you apply Gaussian elimination by $\mathrm{II} - \frac{a_2 \cdot \mathrm{I}}{b_1}$ you will get

$b_1u_1 + c_1u_2 = f_1$
$a_2u_1 - (b_1u_1 \cdot \frac{a_2}{b_1}) + b_2u_2 - (c_1u_2 \cdot \frac{a_2}{b_1}) + c_2u_3 - 0 = f_2 - \frac{a_2 f_1}{b_1}$
$a_3u_2 + b_3u_3 + c_3u_4 = f_3$
$a_4u_3 + b_4u_4 = f_4$

Then we set

$\tilde{b}_2 = b_2 - \frac{a_2 c_1}{\tilde{b}_1}$ and $\tilde{f}_2 = f_2 - \frac{a_2 f_1}{\tilde{b}_1}$.For $i = 1$ and $i = n$, $\tilde{b}_i = b_i$.

which gives

$b_1u_1 + c_1u_2 = f_1$
$0 + \tilde{b}_2u_2 + c_2u_3 = \tilde{f}_2$
$a_3u_2 + b_3u_3 + c_3u_4 = f_3$
$a_4u_3 + b_4u_4 = f_4$

If we apply Gaussian elimination on the rest of the set and assign new variables (tilde) to the "complicated" expressions, you will end up with the following set of linear equations

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & 0 & 0 \\ 0 & \tilde{b}_2 & c_2 & 0 \\ 0 & 0 & \tilde{b}_3 & c_3 \\ 0 & 0 & 0 & \tilde{b}_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \\ f_4 \end{bmatrix}$$

From the elimination you can notice a pattern which can be generalized as

$\tilde{b}_i = b_i - \frac{a_i c_{i-1}}{\tilde{b}_i}$ and $\tilde{f}_i = f_i - \frac{a_i f_{i-1}}{\tilde{b}_i}$

Backward substitution

$\tilde{b}_1u_1 + c_1u_2 = \tilde{f}_1$
$\tilde{b}_2u_2 + c_2u_3 = \tilde{f}_2$
$\tilde{b}_3u_3 + c_3u_4 = \tilde{f}_3$
$\tilde{b}_4u_4 = \tilde{f}_4$

$u_4 = \frac{\tilde{f}_4}{\tilde{b}_4}$
$u_3 = \frac{\tilde{f}_3 - c_3u_4}{\tilde{b}_3}$
$u_2 = \frac{\tilde{f}_2 - c_2u_3}{\tilde{b}_2}$

$u_1 = \frac{\tilde{f}_1 - c_1 u_2}{\tilde{b}_1}$

In general

$u_{i-1} = \frac{\tilde{f}_{i-1} - c_{i-1} u_i}{\tilde{b}_{i-1}}$

Precise number of floating point operations