Project 3

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November 19, 2019

Abstract

This report adresses the Ising model applied to a ferro-magnetic crystal in two dimensions, consisting of electron-spins in one of two states. We calculate, both analytically and numerically, the expected energy, magnetization, heat capacity susceptibility and critical temperature - all the while paying attention to numerical efficiency. The results are satisfactory and the computation is parallelizable, which keeps it from becoming too time intensive.

1 Introduction

The Ising model is a statistical model of ferromagnetism. Modeling the atomic spins as discrete values (± 1), the model can identify phase transitions for a periodically repeating crystal of said spins interracting with its neighbours. In this way, the model is also relevant in other studies - modeling the way networks evolve in for example neuroscience and elections.
In this report, we will study the Ising model by applying it to a two-dimensional ferromagnetic crystal with one electron-spin at each lattice site. We model this as an $L \times L$ matrix, where L is the number of spins along one direction. To study when the most likely state is reached, we initialize the structure first with a given order, then randomly. Modeling the temperature dependence, we try to find the critical temperature at which ferromagnetism is lost. Analytical values are first found for comparison, followed by numerical modeling utilizing the Metropolis Monte Carlo algorithm.
This report will first have a theory part, showing the methods we are using and the theory behind the Ising model. Results are then shown, followed by a discussion and a conclusion. Every program referenced in this report is found in the GitHub repository, under `/code/`.

2 Theory

Using periodic boundary conditions we assume that the lattice is infinite in all directions. By simply setting the edge's neighbouring spins equal to the spin of the electron on the opposite side, the boundary will not affect the lattice.

2.1 2×2 lattice, analytical expressions

To get started we will find the analytical expression for the partition function and the corresponding expectation values for the energy E, the mean absolute value of the magnetic moment $|\mathcal{M}|$ (which we will refer to as magnetization), the specific heat C_V and the susceptibility χ as function of T using periodic boundary conditions. These calculations will serve as benchmarks for our next steps.

Partition function, Z

The partition function is used as a dividend when finding the mean of the energy and magnetization, to keep it on a per spin basis. In the canonical ensemble it is defined as:

$$Z = \sum_{i=1}^{M} e^{-\beta E_i},$$

where $\beta = \frac{1}{k_B T}$ and E_i is the energy of the system in the microstate i and M is the number of microstates ($= 2^N$ if N is the number of electrons).
We therefore have to find E_i which is defined as:

$$E_i = -J \sum_{<kl>}^{N} s_k s_l,$$

where $< kl >$ indicates that we sum only over the nearest neighbors and J is a constant for the bonding strenght. For our two dimensional system the equation reads:

$$E_{i,2D} = -J \sum_{i}^{N} \sum_{j}^{N} \left(s_{i,j} s_{i,j+1} + s_{i,j} s_{i+1,j} \right).$$

For our two-spin-state system in two dimensions we get the following table if we use periodic boundary conditions:

Number of spins up	Degeneracy	Energy	Magnetization
4	1	-8J	4
3	4	0	2
2	4	0	0
2	2	8J	0
1	4	0	-2
0	1	-8J	-4

Table 1: Number of spins up, degeneracy, energy and magnetization of the two-dimensional benchmark scenario.

where the magnetization is found by subtracting the number of spins down from the number of spins up, or in other words the sum of the spins:

$$\mathcal{M} = \sum_{j=1}^{N} s_j.$$

Getting back to the partition function, we insert all 16 of the E_i's respectively. For the degeneracies, we just multiply one iteration of the respecitve E_i with the amount of degeneracies. When the energy E_i is zero, we will just add one to the sum since $e^0 = 1$. Thus we get the following:

$$Z = e^{-\beta(-8J)} + 2 \cdot e^{-\beta(8J)} + e^{-\beta(-8J)} + 12 = 2e^{-\beta 8J} + 2e^{\beta 8J} + 12,$$

$$Z = 4\cosh(\beta 8J) + 12.$$

Energy expectation value, $\langle E \rangle$

The expectation value of the energy is defined as:

$$\langle E \rangle = \sum_{i=1}^{M} E_i P_i(\beta) = \frac{1}{Z} \sum_{i=1}^{M} E_i e^{-\beta E_i},$$

where M is the sum over all microstates. P_i is the Boltzmann probability distribution which reads:

$$P_i(\beta) = \frac{e^{-\beta E_i}}{Z}.$$

For our system, this is easily calculated by inserting the partition function and the microstate energy E_i. The mean energy is then (calculations are shown in appendix, equation (5)):

$$\langle E \rangle = -8J \frac{\sinh(\beta 8J)}{\cosh(\beta 8J) + 3}.$$

Since the variance of the mean energy (σ_E) is needed for the heat capacity later, we will calculate this as well. Full calculation is found in the appendix, equation (6).

$$\sigma_E^2 = 64J^2 \left(\frac{\cosh(\beta 8J)}{\cosh(\beta 8J) + 3} - \left(\frac{\sinh(\beta 8J)}{\cosh(\beta 8J) + 3} \right)^2 \right)$$

Magnetization expectation value, $\mathcal{M}$

In the canonical ensemble the mean absolute magnetization can be described as

$$\langle |\mathcal{M}| \rangle = \sum_i^M |\mathcal{M}_i| P_i(\beta) = \frac{1}{Z} \sum_i^M |\mathcal{M}_i| e^{-\beta E_i}.$$

We can now simply insert the magnetization and the energies for each respective microstate. This is found in table 2. Using this, we find (as shown in the appendix, equation (7)):

$$\langle |\mathcal{M}| \rangle = \frac{2e^{\beta 8J} + 4}{\cosh(\beta 8J) + 3}.$$

Since the variance of the mean magnetization (σ_M) is needed for the susceptibility later, we will calculate this here. For this we will need the mean magnetization square $\langle \mathcal{M}^2 \rangle$, and the mean magnetization $\langle \mathcal{M} \rangle$. $\langle \mathcal{M} \rangle$ is shown to be 0 in the appendix, equation (8), and $\langle \mathcal{M}^2 \rangle = \frac{8e^{\beta 8J}+8}{\cosh(\beta 8J)+3}$ (shown in the appendix, equation (9)). Thus, the variance is:

$$\sigma_{\mathcal{M}}^2 = \langle \mathcal{M}^2 \rangle - \langle \mathcal{M} \rangle^2 = \frac{8e^{\beta 8J} + 8}{\cosh(\beta 8J) + 3}$$

Specific heat capacity, C_V

The specific heat capacity is defined as

$$C_V = \frac{\sigma_E^2}{k_B T^2}.$$

Insering the value σ_E^2 we get

$$C_V = \frac{1}{k_B T^2} 64J^2 \left(\frac{\cosh(\beta 8J)}{\cosh(\beta 8J) + 3} - \left(\frac{-\sinh(\beta 8J)}{\cosh(\beta 8J) + 3} \right)^2 \right).$$

This is the main function we will be comparing to the values from our computations later.

Susceptibility, χ

The susceptibility is defined as

$$\chi = \frac{\sigma_{\mathcal{M}}^2}{k_B T^2}.$$

Inserting the value of $\sigma_{\mathcal{M}}^2$, we get

$$\chi = \frac{1}{k_B T^2} \frac{8e^{\beta 8J} + 8}{\cosh(\beta 8J) + 3}.$$

Note that all the four abovementioned characteristics ($\langle E \rangle$, $\langle |\mathcal{M}| \rangle$, C_V and χ) are temperature dependent, through the variable $\beta = \frac{1}{k_B T}$. [1].

2.2 Ising model

The Ising model is applied for the study of phase transistions at finite temperatures for magnetic systems. Energy is expressed as:

$$E = -J \sum_{<kl>}^{N} s_k s_l, \qquad s_k = \pm 1. \tag{1}$$

N is the number of spins and J is a constant expressing the interaction between neighboring spins. The sum is over the nearest neighbours only, indicated by $< kl >$ in the above equation. For $J > 0$ it is energetically favorable for neighboring spins to align. Leading to - at low temperatures T - spontaneous magnetization.
A probability distribution is needed in order to calculate the mean energy $\langle E \rangle$ and magnetization $\langle \mathcal{M} \rangle$ at a given temperature. The distribution is given by:

$$P_i(\beta) = \sum_{i=1}^{M} s_k s_l \exp -\beta E_i, \tag{2}$$

where M is the number of microstates and P_i is the probability of having the system in a state/configuration i.
We utilize the Metropolis algorithm, which checks if we get a lower energy for the system by flipping a spin. If that is the case, we flip the spin. This is repeated, in the hopes of it reaching the lowest state in total.
The pseudocode looks as follows:

```
for Temperature;

  for MonteCarlo Cycle;

    - Metropolis algorithm
    - Sum all values
```

```
    end for MonteCarlo loop

    - Divide values by MC cycles
    - Output values

end for Temperature loop
```

To show that our code has good correspondence to analytical results, we will compare calculations for a 2×2 lattice with the analytical result for the same lattice. This is shown in section 3.3.

2.3 Equlibrium - energy minimum

In order to find the equilibrium of the system (20×20 lattice) we perform a study of the time - corresponding to the number of Monte Carlo cycles - one needs to reach the equilibrium state from where we can compute expectation values. Equilibrium is reached when the thermodynamic quantities energy and magnetization reach their steady states. Furthermore, it will not be energetically favorable to flip spins when reaching equilibrium. So the number of accepted spins per Monte Carlo cycle is at this point expected to move towards a constant value.
We will graphically plot expectation values for energy and magnetization as a function of Monte Carlo cycles for different temperatures, $T = 1.0$ and $T = 2.4$ using both a random and a set starting point for the initial spin configuration.

2.4 Analyzing the probability distribution

We will also compute the probability of the energy, $P(E)$ for the system with $L = 20$ with the temperatures $T = 1$ and $T = 2.4$. This is computed by counting the number of times a given energy appears in our computation. The computation will start at a number of Monte Carlo cycles of which we know that the system is stable. From the results in section 3.4, we see that after the 10000th Monte Carlo cycle we are well beyond the stability limit, so this is what we will be using. We will also compare the variance of the probability distribution with the computed variance of the energy, σ_E^2, and discuss the behavior.

2.5 Numerical studies of phase transitions

We wish to study the behavior of the Ising model in two dimensions close to the critical temperature as a function of the lattice size $L \times L$. To do this, we will be calculating the expectation values for energy $\langle E \rangle$, the mean absolute magnetization $\langle |M| \rangle$, the specific heat capacity C_V and the susceptibility χ as a function of the temperature in the interval $T \in [2.0, 2.5]$ with a step size of $\Delta T = 0.005$. This will be done for the lattice sizes $L = 20, 40, 60, 80, 100$. From the plots we hope to see an indication of a phase transition.

The code will be parallellized using MPI, and timed using the `MPI\_Time()` function.

2.6 Extracting the critical temperature

We would like to compute the critical temperature T_C in the thermodynamic limit where $L \to \infty$. This will be done using the equation below, together with the exact result of $\nu = 1$.

$$T_C(L) - T_C(L = \infty) = aL^{-1/\nu}$$

By using two different lattice sizes, we can calculate the factor a, and then calculate T_C.

$$T_C(L_1) - T_C(L = \infty) = aL_1^{-1/\nu}$$
$$T_C(L_2) - T_C(L = \infty) = aL_2^{-1/\nu}$$

By subtracting the equations we obtain

$$\frac{T_C(L_1) - T_C(L_2)}{L_1^{-1/\nu} - L_2^{-1/\nu}} = a. \tag{3}$$

It will now be easy to use the equation below to find the critical temperature for an infinitely large lattice.

$$T_C(L = \infty) = T_C(L) - aL^{-1/\nu} \tag{4}$$

3 Results

3.1 2×2 lattice, analytical expressions

If we scale the value of β from $1/k_BT$ to $1/J$ (Scaling factor k_BT/J) in the analytical expression from section 2.1, we will get a good benchmark for computer computations to come. These values are listed in table 2 below. Note that all values are divided by four, since we want the values per bond, and not for the entire lattice.

Mean energy, $\langle \mathbf{E} \rangle$	-1.9960
Mean absolute magnetization, $\langle \lvert \mathcal{M} \rvert \rangle$	0.9987
Specific heat capacity, $\mathbf{C_V}$	0.0321
Susceptibility, χ	3.9933

Table 2: Analytically calculated benchmark for material characteristics per bond for a 2×2 lattice

3.2 2×2 lattice, numerical results

Using $T = 1.0$, like in the analytical calculations, the program /code/Ising/ gives the results listed in table 3:

	Set initialization	**Random initialization**
Mean energy, $\langle \mathbf{E} \rangle$	-1.9959	-1.9958
Mean absolute magnetization, $\langle \lvert \mathcal{M} \rvert \rangle$	0.9986	0.9986
Specific heat capacity, $\mathbf{C_v}$	0.0331	0.0336
Susceptibility, χ	3.9839	3.9929

Table 3: Computed values of material characteristics per bond for a 2×2 lattice.

The values correspond very well with each other and the analytical, this is further discussed in section 4

3.3 Ising model: simulation over temperature

We ran the program for different amounts of Monte Carlo cycles and plotted the error (analytical − simulated) in figure 1 below. Using 10^7 Monte Carlo cycles, we seem to be getting pretty accurate results.

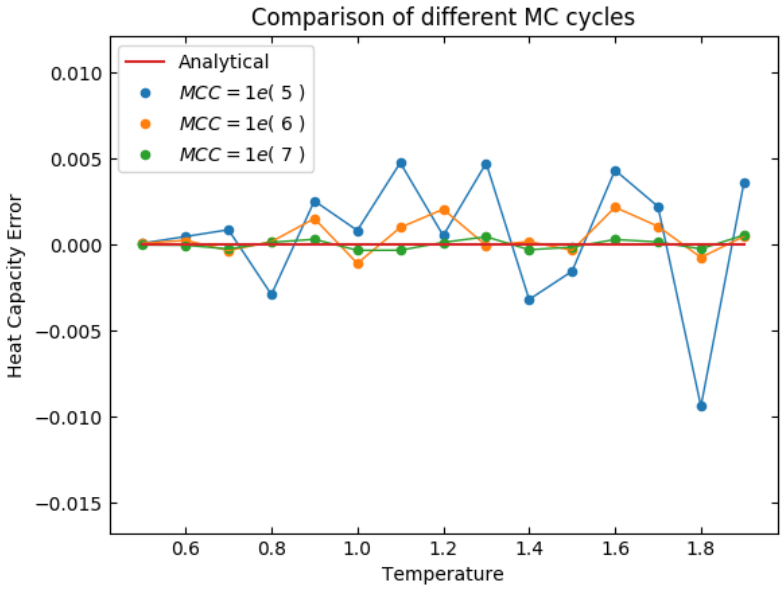

Figure 1: Shows the accuracy of different amount of MC cycles over temperature.

This shows that our computed results are quite close to our analytical results for the 2×2 lattice. This is a good indication of a successfull simulation.

3.4 20×20 lattice

Ordered spin orientation

Initializing the spin structure, we first set every spin up for $T < 1.5$ and every spin down for $T \geq 1.5$. In figure 2, the computed values for the mean magnetization and energy are plotted against the number of MC cycles, at $T = 1.0$ and $T = 2.4$:

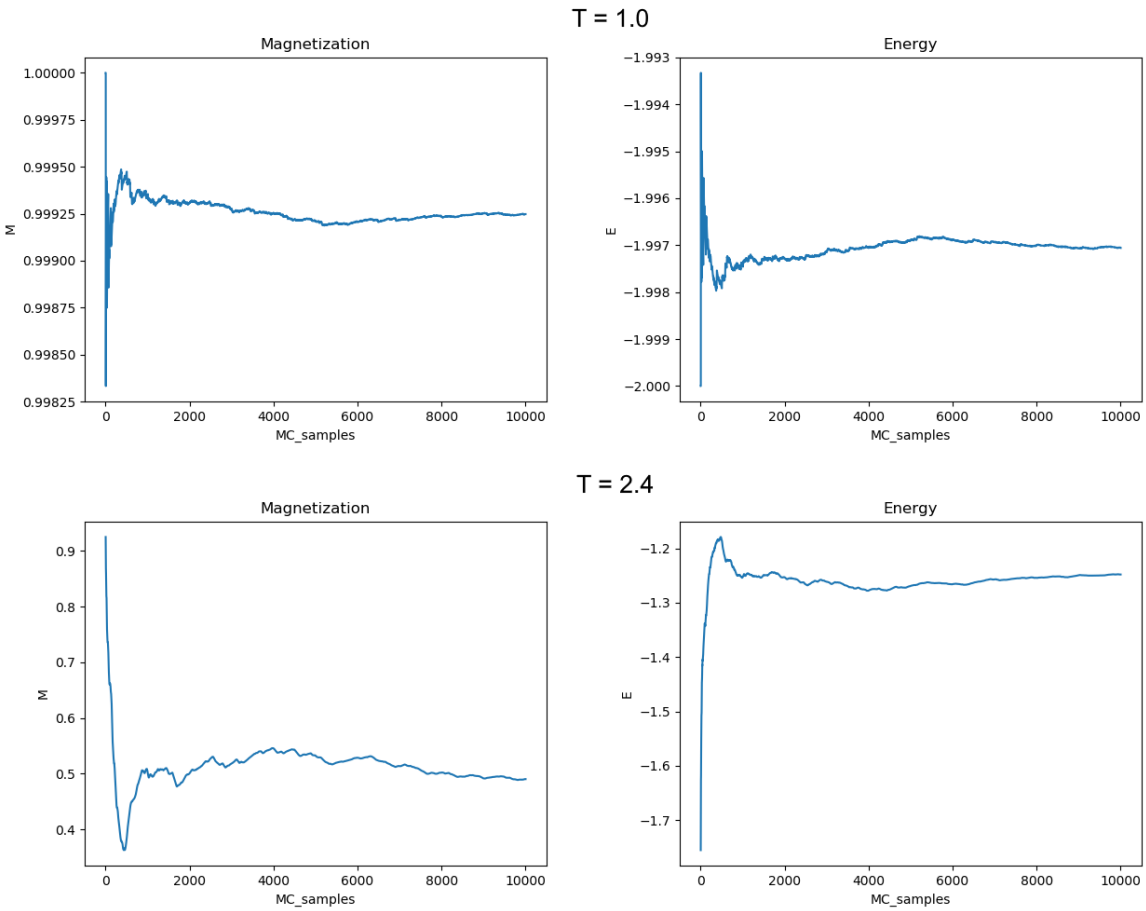

Figure 2: Shows the computed value for the mean magnetization and energy, with ordered initialization, against the number of MC cycles. The scaled temperature is $T = 1.0$ and $T = 2.4$ respectively.

All the plots pretty much stabilize into a value after 8000-10000 MC cycles. For $T = 1.0$, the magnetization stabilizes around the value 0.99950 and the energy around the value -1.997. This corresponds pretty good with the analytically calculated values. For $T = 2.4$, the magnetization stabilizes around the

value 0.5 and the energy around the value -1.25.

Random spin orientation

Following the ordered initialisation, we also initialized the crystal randomly. In figure 3, the computed values for the mean magnetization and energy are plotted against the number of MC cycles, at $T = 1.0$ and $T = 2.4$:

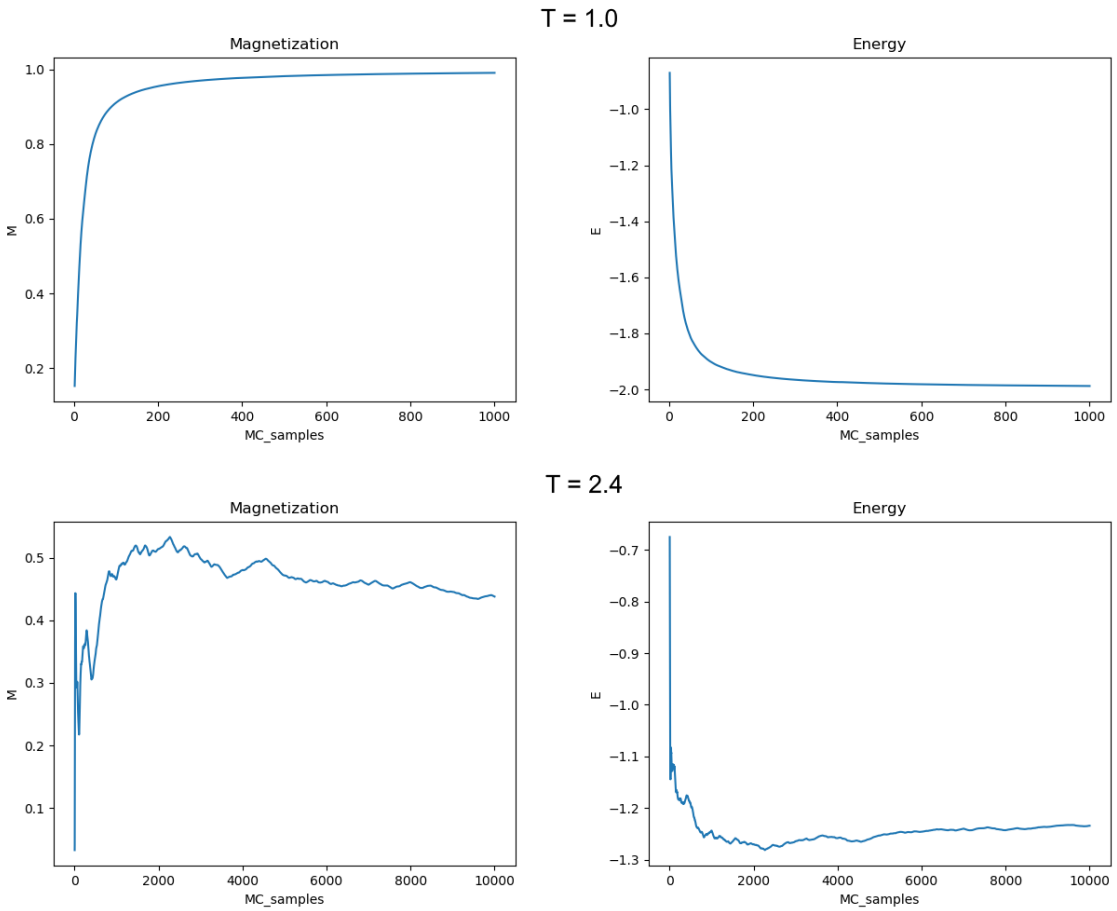

Figure 3: Shows the computed value for the mean magnetization and energy, with random initialization, against the number of MC cycles. The scaled temperature is $T = 1.0$ and $T = 2.4$ respectively.

The plots for $T = 1.0$ follow a clean exponential curve, while the other plots pretty much stabilize after 8000-10000 MC cycles - like the previous ones. For $T = 1.0$, the magnetization ends on the value 1.0 and the energy on the value -2.0. This is similar to the analytical values, but does not have the same

accuracy. For $T = 2.4$, the magnetization stabilizes around the value 0.45 and the energy around the value -1.25.

Number of accepted spins

Another good indication for when the equlibrium is reached is when the number of spins per Monte Carlo cycle does not increase further. The following figures display the number of accepted spin configurations per Monte Carlo cycle at temperatures $T = 1.0$ and $T = 2.4$, with both ordered and random initialization.

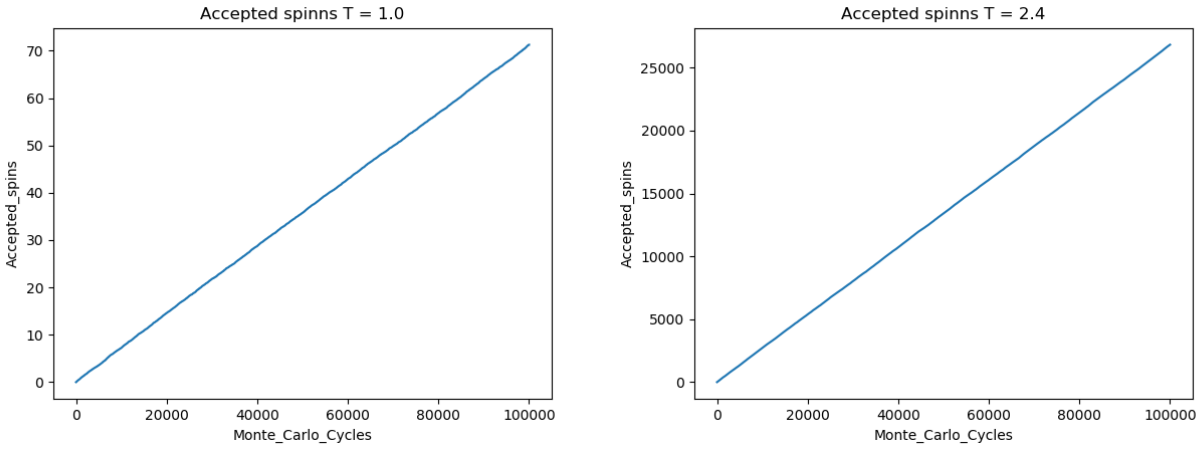

Figure 4: Cumulative sum of accepted spins on a per-spin-basis over 10^5 MC cycles. For ordered initialization at $T = 1.0(Jk/T)$ and $T = 2.4(Jk/T)$ respectively.

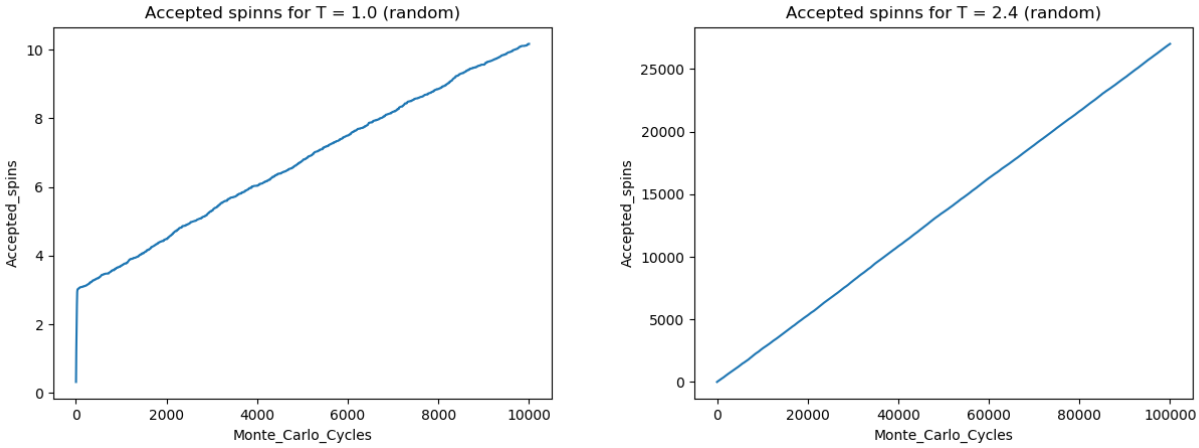

Figure 5: Cumulative sum of accepted spins on a per-spin-basis over 10^5 MC cycles. For random initialization at $T = 1.0(Jk/T)$ and $T = 2.4(Jk/T)$ respectively.

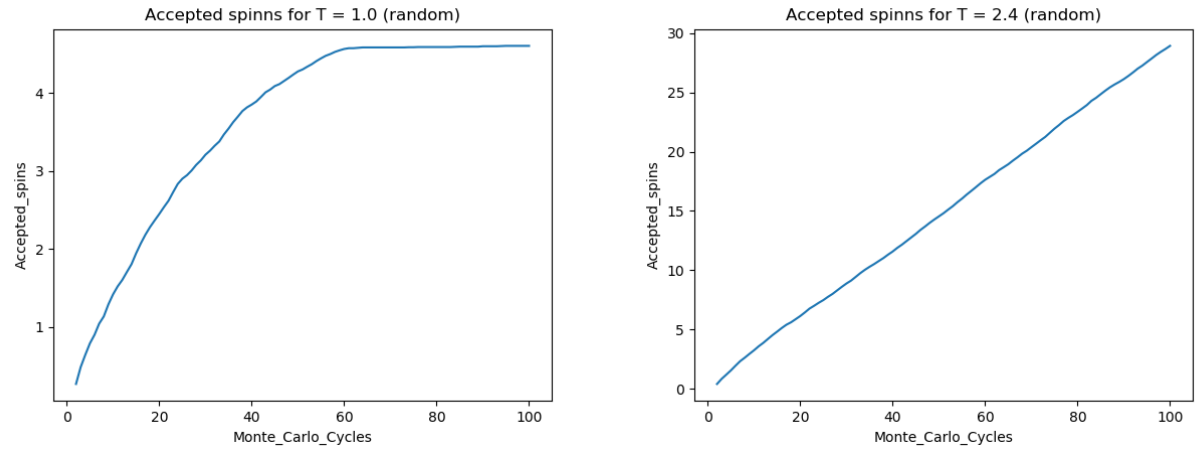

Figure 6: Cumulative sum of accepted spins on a per-spin-basis over 100 MC cycles. For random initialization at $T = 1.0(Jk/T)$ and $T = 2.4(Jk/T)$ respectively.

The number of accepted spins behave - as expected - the same way as the energies and magnetization when increasing the number of Monte Carlo cycles. This is further discussed in section 4.

3.5 Analyzing the probability distribution

In figure 7 you can see the probability distribution for low and high temperature respectively. We can see that for a low temperature, the system tends to settle in the lowest energy state, while for the higher temperature the energies are a bit more spread. In table 4 you will see the computed variance. Note that we calculated standard deviation of the histogram with `numpy.std`, and took the square root of this to get the variance.

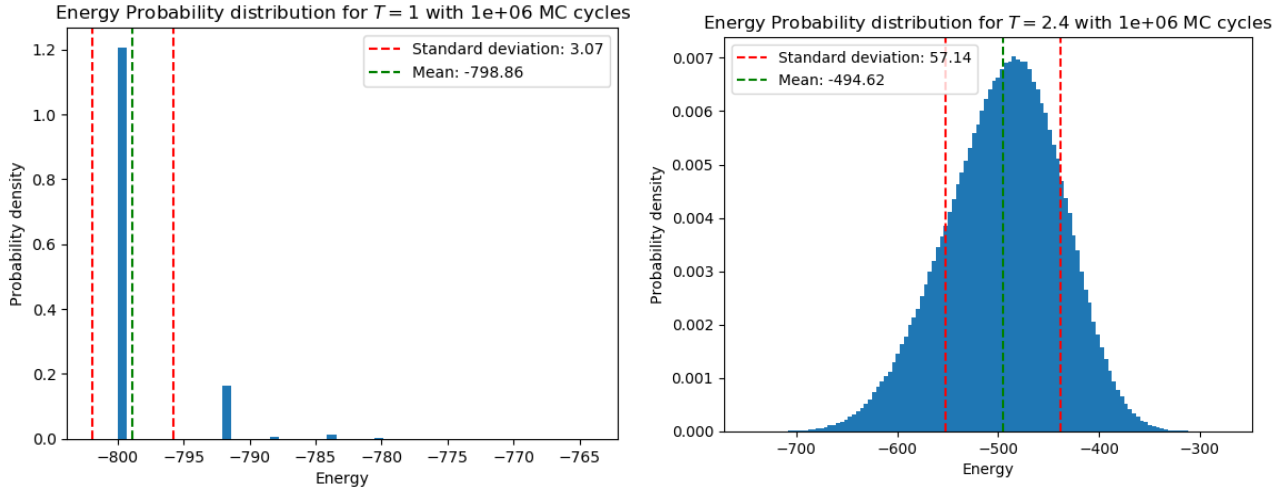

Figure 7: Shows probability distribution for low and high temperature.

Temperature	calculated variance	from histogram	deviation
1	9.375	1.752	81%
2.4	8.053	7.550	6.2%

Table 4: Computed variance

3.6 Numerical studies of phase transitions

After playing around with the domain of the temperature we found that the domain used in figure 8 and 9 nicely presents the phase change of the material. We used 10^6 Monte Carlo Cycles for each temperature step, which had a stepsize of $\Delta T = 0.005$. As shown in the figures we can clearly see that something is happening around $T = 2.3$.

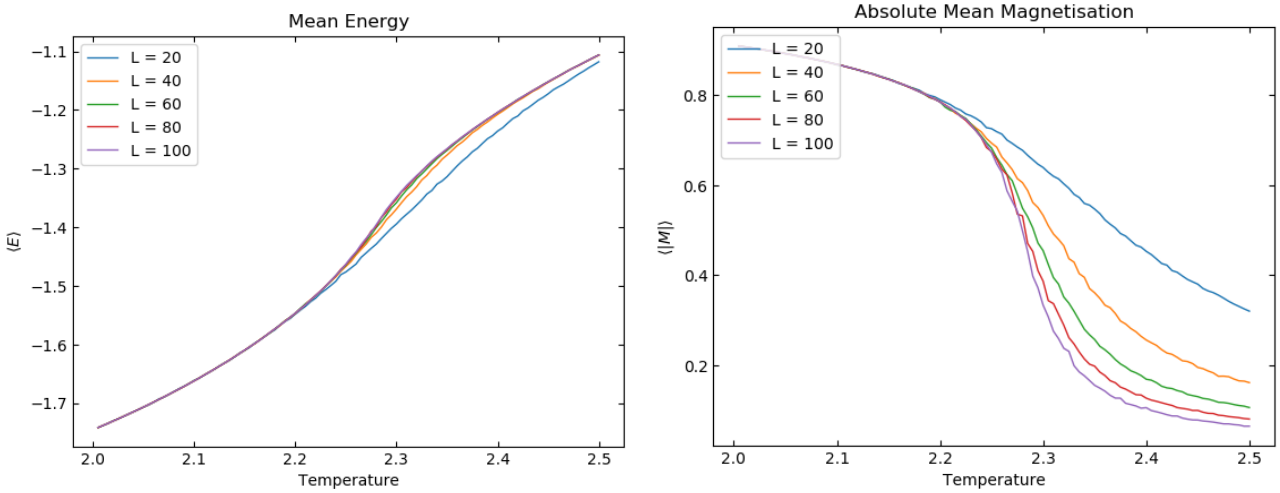

Figure 8: Mean energy and magnetization over temperature interval $T \in [2.0, 2.5]$ with lattice sizes $L = 20, 40, 60, 80, 100$.

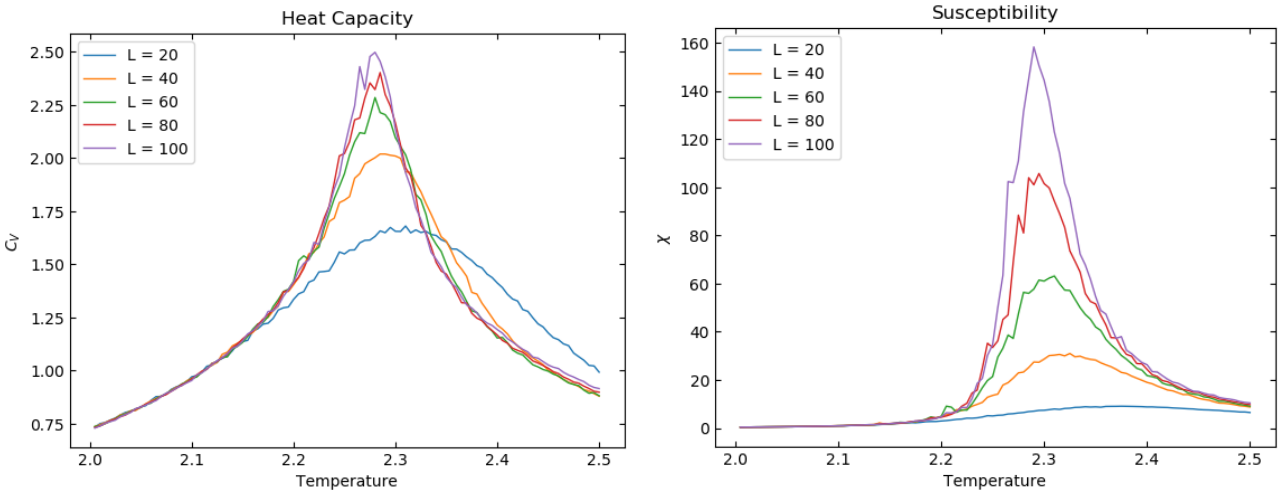

Figure 9: Heat capacity and Susceptibility over temperature interval $T \in [2.0, 2.5]$ with lattice sizes $L = 20, 40, 60, 80, 100$.

The simulations were run on an 16-core AMD Ryzen 1700x with 3.4GHz clock speed. It took approximately 3 hours to complete.
To compare parallelized time to using a single core, we ran calculations for the 20x20 lattice with 10^5 Monte Carlo cycles and $T \in [2.0, 2.5]$ with step size $\Delta T = 0.05$. The time is shown in table 5.

Cores	Time spent
1	42s
2	26s

Table 5: Time spent on calculations for different thread-counts.

3.7 Extracting the critical temperature

The critical temperatures of the different sized arrays was found by taking the average of the full width half maximum of the heat capacity and the susceptibility. This resulted in the critical temperatures found in table 6.

Lattice size, L	T_C
40	2.318
60	2.301
80	2.291
100	2.284

Table 6: Critical temperatures of different lattice sizes.

By using equation (3), and (4). we find the constant a, and thereafter the critical temperature for an infinitely large lattice.

Parameter	Value
a	2.26667
T_C	2.261

Table 7: Numerical values for the parameter a, and the critical temperature.

For reference, the exact result for the critical temperature for $L \to \infty$ is $kT/J = 2/ln(1+\sqrt{2}) \approx 2.269$ after Lars Onsager. [2]

4 Discussion

4.1 Ising model: simulation over temperature

As presented in Results (see table 3 and 1), using periodic boundary conditions, $T = 1.0(kT/J)$ and 10^6 Monte Carlo cycles gives results with a high precision. The energy and magnetization have a precision of three leading digits after the decimal point, and a little less for susceptibility and heat capacity.
When using a random starting configuration, the precision increases slightly compared with the spesific one for some of the parameters, especially for the susceptibility.

For further computations we have used $10^4 - 10^7$ Monte Carlo cycles in order to avoid time consuming computations.

4.2 20×20 lattice

From Figure 2 and 3 one sees the impact of the different parameters, initialization and temperature, before reaching equilibrium for different Monte Carlo cycles.
For $T = 1.0(kT/J)$ the predetermined starting configuration, the energy and magnetization are already stable after only a few Monte Carlo cycles(take a close look at the y-axis). It is easy to see that at the first Monte Carlo cycle, the magnetization is one (because all the spins are pointing the same way). Then one spin flips and the energy changes slightly. After a while it is almost no longer energetically favourable to flip spins.
For the random initialization it takes more time/Monte Carlo cycles before reaching a steady state (take a look at the y-axis, it is 3 orders of magnitude larger than the predetermined). For easier comparison it would be convenient to have the axes be more similar.
To summarize, the ordered starting point seems to stabilize the fastest for both temperatures (energy a bit faster than magnetization). This is expected, because at low temperatures - which we are dealing with - one would expect a high order at equilibrium (from eg. Gibbs free energy). So when we start with a high order, we are already close to equilibrium.
There is also a difference between the different temperatures, which is also expected from thermodynamics. A system of higher temperature is expected to reach equilibrium at a less ordered state. This is quite evident in both the ordered and the random initialization.
When studying the accepted spin configurations per Monte Carlo cycle (see figures 4,5 and 6) the same trend as above occurs. At low temperature and ordered starting point we are already close to equilibrium and therefore the number of accepted spins does not increase further with the increased number of Monte Carlo cycles. The same goes for the high temperature, except that the number of accepted spins is significantly higher. This is expected, as explained before, at this point the system has a lower order and higher energy due to higher temperature. Therefore the system accepts flipping of spins more often.

When initalizing randomly for $T = 1.0$(see figure 5) the number of accepted spins increases rapidly in the beginning. This is because the system is far from equilibrium, so many spins are flipped to reduce the total energy. At about 60 Monte Carlo cycles (see figure 6) the number of accepted spins decreases and the system reaches equilibrium. The random start configuration at high temperature behaves much like the ordered one with a high number of accepted spins, compared to low temperature.

4.3 Analyzing the probability distribution

While the probability distribution for the higher temperature was really good, it seemed rather odd for the low temperature. One would think that since most energies are at their lowest, the histogram would take the shape of an inverse exponential curve, and while this is the case we still have large gaps between the energies. For example, when the energy is between -800 and -794, it looks like there isnt a single lattice observed.
This must be because the energy change when flipping a spin can be either 4J or 8J. This correlates well with the histogram. Another thing to mention is that at lower temperatures, the flipping spins could very well cause higher energy, resulting in less flips, which locks the lattice in its state.
The standard deviation for the low temperature histogram does not give us much information. It would only be relevant, had it followed a gaussian curve, like the one for the higher temperature. Thus it is irrelevant to compare the standard deviation to the calculated variance. On the flipside, the variance of the higher temperature has a good correspondence to the calculated variance, with a deviation of about 6.2%.

4.4 Numerical studies of phase transitions

From the plots **??** and 9 it is clear that something is happening around $T = 2.3$. You can also see that a bigger lattice reacts more to the temperature than the smaller lattice, giving higher peaks, with a smaller width for the heat capacity and susceptibility. Also for the mean energy and absolute mean magnetisation, the larger lattices "reacts" more to temperature, giving steeper plots.
This is to be expected as the boundary conditions limit the simulations. The larger the lattice, the less of an issue the boundary conditions become.

4.5 Extracting the critical temperature

The most difficult part was to set the critical temperatures for the different lattice sizes. As explained, we used the FWHM technique, but this is also prone to error. However with this technique we got within 0.3% of Lars Onsager's exact result, which is quite good.

5 Conclusion

The Ising model does a great job of simulating a ferrous material. The results correspond well with analytical calculations. We showed what amount of Monte Carlo cycles you would need to stabilize such a system, and even though the calculations are time consuming, we are able to keep the calculation time down with parallelization. We also showed that the critical temperature for a system like this approaches $T_C = 2.261$, which is very close to Lars Onsager's result.

6 Appendix

6.1 GitHub

GitHub repository: https://github.com/kmaasrud/Project-4

6.2 Calculations

Energy expectation value, $\langle E \rangle$

$$\begin{aligned}
\langle E \rangle &= \frac{1}{2e^{-\beta 8J} + 2e^{\beta 8J} + 12} \left(2 \cdot -8J \cdot e^{\beta 8J} + 2 \cdot 8J \cdot e^{-\beta 8J} \right) \\
&= \frac{1}{2e^{-\beta 8J} + 2e^{\beta 8J} + 12} \left(-16Je^{\beta 8J} + 16Je^{-\beta 8J} \right) \\
&= 8J \frac{1}{e^{-\beta 8J} + e^{\beta 8J} + 6} \left(e^{-\beta 8J} - e^{\beta 8J} \right) \\
&= -8J \frac{sinh(\beta 8J)}{cosh(\beta 8J) + 3}
\end{aligned} \tag{5}$$

$$\begin{aligned}
\sigma_E^2 = \langle E^2 \rangle - \langle E \rangle^2 &= \frac{1}{Z} \sum E_i^2 e^{-\beta E_i} - \left(\frac{1}{Z} \sum E_i e^{-\beta E_i} \right)^2 \\
&= \frac{1}{2e^{-\beta 8J} + 2e^{\beta 8J} + 12} \left(2 \cdot (-8J)^2 \cdot e^{\beta 8J} + 2 \cdot (8J)^2 \cdot e^{-\beta 8J} \right) \\
&\qquad - \left(-8J \frac{sinh(\beta 8J)}{cosh(\beta 8J) + 3} \right)^2 \\
&= 128J^2 \frac{2cosh(\beta 8J)}{4cosh(\beta 8J) + 12} - \left(-8J \frac{sinh(\beta 8J)}{cosh(\beta 8J) + 3} \right)^2 \\
&= 64J^2 \frac{cosh(\beta 8J)}{cosh(\beta 8J) + 3} - \left(-8J \frac{sinh(\beta 8J)}{cosh(\beta 8J) + 3} \right)^2 \\
&= 64J^2 \left(\frac{cosh(\beta 8J)}{cosh(\beta 8J) + 3} - \left(\frac{sinh(\beta 8J)}{cosh(\beta 8J) + 3} \right)^2 \right)
\end{aligned} \tag{6}$$

Magnetization, $\mathcal{M}$

$$\begin{aligned}
\langle |\mathcal{M}| \rangle &= \frac{1}{4cosh(\beta 8J) + 12} \left(4 \cdot e^{\beta 8J} + 4 \cdot 2 \cdot e^0 + 4 \cdot |-2| \cdot e^0 + |-4| \cdot e^{\beta 8J} \right) \\
&= \frac{1}{cosh(\beta 8J) + 3} (e^{\beta 8J} + 2 + 2 + e^{\beta 8J}) \\
&= \frac{1}{cosh(\beta 8J) + 3} (2e^{\beta 8J} + 4) \\
&= \frac{2e^{\beta 8J} + 4}{cosh(\beta 8J) + 3}
\end{aligned} \tag{7}$$

$$
\begin{aligned}
\langle \mathcal{M} \rangle &= \frac{1}{4\cosh(\beta 8J) + 12} \left(4 \cdot e^{\beta 8J} + 4 \cdot 2 \cdot e^{0} + 4 \cdot -2 \cdot e^{0} + -4 \cdot e^{\beta 8J} \right) \\
&= \frac{1}{\cosh(\beta 8J) + 3} \left(4e^{\beta 8J} - 4e^{\beta 8J} + 8 - 8 \right) \\
&= \frac{1}{\cosh(\beta 8J) + 3} \left(0 \right) \\
\langle \mathcal{M} \rangle &= 0
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\langle \mathcal{M}^2 \rangle &= \frac{1}{Z} \sum |\mathcal{M}_\imath\rangle|^2 e^{-\beta E_i} \\
&= \frac{1}{4\cosh(\beta 8J) + 12} \left(4^2 \cdot e^{\beta 8J} + 4 \cdot 2^2 \cdot e^{0} + 4 \cdot |-2|^2 \cdot e^{0} + |-4|^2 \cdot e^{\beta 8J} \right) \\
&= \frac{1}{4\cosh(\beta 8J) + 12} \left(16 \cdot e^{\beta 8J} + 16 \cdot e^{0} + 16 \cdot e^{0} + 16 \cdot e^{\beta 8J} \right) \\
&= \frac{4}{\cosh(\beta 8J) + 3} \left(2e^{\beta 8J} + 2 \right) \\
&= \frac{8e^{\beta 8J} + 8}{\cosh(\beta 8J) + 3}
\end{aligned} \tag{9}
$$

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