1 Theory

1.1 Wavefunction of Helium

. The single-particle wave function of an electron i in the $1s$ state is given in terms of a dimensionless variable (the wave function is not normalized)

$$\vec{r_i} = x_i\vec{e}_x + y_i\vec{e}_y + z_i\vec{e}_z$$

as

$$\psi_{1s}(\vec{r_i}) = e^{-\alpha r_i}$$

Where α is a parameter set to 2, due to the two electrons, and the length r_i is defined by

$$r_i = \sqrt{x_i^2 + y_i^2 + z_i^2}$$

For our system with two electrons, we have the product of the two $1s$ wave functions defined as

$$\Psi(\vec{r}_1, \vec{r}_2) = e^{-\alpha(r_1+r_2)}$$

This leads to the integral, REFERANCE, which will be solved nummericaly with the three different methods mentioned earlier. The value of the integral corresponds to the energy between the two electrons repelling each other due to Columb interactions.

$$\langle\frac{1}{|\vec{r}_1 - \vec{r}_2|}\rangle = \int_{\infty}^{\infty} d\vec{r}_1 d\vec{r}_2 e^{-2\alpha(r_1+r_2)} \frac{1}{\vec{r_1} - \vec{r_2}}$$

The analytical result is $5\pi/16^2$.

1.2 Gaussian Quadrature

The main idea of Gaussian quadrature is to integrate over a set of points x_i not equally spaced with weights w_i, which are calculated in the program Gauleg.cpp and Gauss Legendre.cpp). The weights are found throug ortogonal polynomials(Laguerre and Legendre polynomials) in a set interval. The points x_i are chosen in a optimal sense and lie in the interval.

The intgral is approximated as

$$\int_a^b W(x)f(x) \approx \sum_{i=1}^{n} \omega_i f(x_i)$$

For a more detalied derivation and explenation?? of Gaussian qudrature see (Hjort-Jensen, 2015)

1.2.1 Gauss-Legendre

Using Gauss-Legendre quadrature with Legendre polynomials will make it possible to utilize the integral numerically. The first step is to change the integration limits from $-\infty$ and ∞ to $-\lambda$ and λ. The λ's are found by inserting it for r_i in the expression $e^{-\alpha r_i}$ because $r_i \approx \lambda$ when $e^{-\alpha r_i} \approx 0$. From FIGURE 1 $\lambda \in [-5, 5]$ is therefor a good approximation for the integration limits.

expfunc_plot.png

Figure 1: Plot of wavefunction in one dimension

Furthermore, the weights and mesh points are computed using "gauleg"(see program exampleprog.cpp????).

Eventually ending up with a sixdimensional integral, where all six integration limits are the same.

$$\int_a^b \int_a^b \int_a^b \int_a^b \int_a^b \int_a^b e^{-x} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

1.2.2 Improved Gauss-Quadrature- Laguerre

Gauss-Legendre quadrature gets the job done, but it is unstable and unsatisfactory. By changing to spherical coordinates and replacing Legendre- with Laguerre polynomials an improvement in accuracy is expected. The Laguerre polynomials are defined for $x \in [0, \infty)$,and in spherical coordinates:

$$d\vec{r}_1 d\vec{r}_2 = r_1^2 dr_1 r_2^2 dr_2 dcos(\theta_1) dcos(\theta_2) d\phi_1 d\phi_2$$

with

$$\frac{1}{r_{12}} = \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 cos(\beta)}}$$

and

$$cos(\beta) = cos(\theta_1)cos(\theta_2) + sin(\theta_1)sin(\theta_2)cos(\phi_1 - \phi_2)$$

For numerical integration, the deployment of the following relation is nessecary:

$$\int_0^\infty e^{-x} f(x) dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

where x_i is the i-th root of the Laguerre polynomial $L_n(x)$ and the weight w_i is given by

$$w_i = \frac{x_i}{(n+1)^2 [L_{n+1}(x_i)]^2}$$

The Laguerre polynomials are defined by Rodrigues formula:

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} \left(e^{-x} x^n\right) = \frac{1}{n!} \left(\frac{d}{dx} - 1\right)^n x^n$$

or recursively relations:

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= 1 - x \\ L_{n+1}(x) &= \frac{(2n+1-x)L_n(x) - nL_{n-1}(x)}{n+1} \end{aligned}$$

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1.3 Monte Carlo

KILDE:https://cs.dartmouth.edu/ wjarosz/publications/dissertation/appendixA.pdf Monte Carlo is numerical methods dependent of a random samlping from a function in order to approximate the integral.

In general the integral, F, of a function, $f(x), x \in [a, b]$

$$F = \int_a^b f(x) dx$$

can be approximated by taking avarage samples of f with a uniform distrubution of points in the interaval. Having N uniform random variables $x_i \in [a, b)$ with probability distrubution function, PDF $\frac{1}{b-a}$ the Monte-Carlo approximation of F is

$$\langle F^N \rangle = (b-a) \frac{1}{N-1} \sum_{i=0}^{N} f(x_i)$$

$$x_i$$

is constructed