Simulating the motion of the Solar System with the velocity Verlet and forward Euler methods

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Abstract

Planetary motion was simulated using the velocity Verlet method for solving ordinary differential equations. The forward Euler method and the velocity Verlet method were tested and compared on the Earth-Sun system. The velocity Verlet method performed with higher accuracy and conservation of total energy and without much increase in computation time.

Using a trial and error method using the Verlet solver, the escape velocity of an object at distance 1 au was found to be 42.125 km/s, which is correct to four significant digits compared to the analytical value of 42.122 km/s.

Different forms of the gravitational force were tested with varying the exponent β of the distance r from the Sun's center, compared to the usual inverse square law. Values of β between 2 and 3 were tested. It was found that for $\beta = 2$ the total energy was conserved, and for $\beta > 2$ the total energy was not conserved. The angular momentum was conserved for $\beta \leq 2.9$ and not conserved for $\beta = 3$. Both the total energy and the distance from the center of the Sun followed a periodic pattern that coincided for $\beta \leq 2.8$ with a period of just under one year, and even shorter for larger values of β. The angular momentum for $\beta \leq 2.9$ was approximately constant. For $\beta = 3$ the angular momentum showed some variation.

The three-body problem of the Sun, the Earth and Jupiter were simulated for three cases of Jupiter's mass: Jupiter's original mass, 10 times original and 1000 times original mass. The larger the mass of Jupiter, the more Earth's orbit was affected in the simulation.

The simulation gave erroneous results when adding more planets to the system. Also, when adding the correction term from general relativity to the gravitational force, the simulation did not show the expected precession of Mercury's orbit.

1 Introduction

The study of astronomy and planetary motion is an endeavor that is both useful and interesting. Whether it's for simply understanding why the planets move the way that we observe they do, or if it's for a practical purpose, such as calculating the required velocity for a GPS satellite to enter orbit at a certain altitude, understanding the mechanics behind gravity and having the ability and tools to predict where objects in space will move in time is a necessary prerequisite.

A two-body problem in physics is quite easy to solve using the equations of Newtonian gravity. When a third body is added, however, the system has no analytical solution, and numerical methods are required.

The motion of the Solar System is simulated by using two methods for solving ordinary differential equations: the velocity Verlet method and Euler's method. The velocity Verlet method is very popular in the field of particle dynamics for updating the position of rigid bodies [1] since the method is less sensitive to change in step size and improves the numerical accuracy compared with Euler's method at little extra computational cost [2]. Both methods entail writing a Taylor expansion which leads to the discretisation of the continuous function and, hence, adds a truncation error. The accuracy and computational efficiency of both methods are compared and discussed in this report. The code scripts used in this report are mainly developed with the velocity Verlet method and so Euler's method is discussed minimally.

To what extend the Velocity Verlet method conserves angular momentum and energy is investigated. A further test is to look at the escape velocity of the Earth and compare this with the analytic result. Following this, the Velocity Verlet method is applied to the three-body problem and eventually the whole Solar System. The three-body problem is one which appears in many contexts in nature [3]. A century ago it was still a legitimate question if the gravitational force of a third body, like the Moon or another planet, would disrupt the Earth's orbit and cause it to fall into the Sun. Or even worse, that the Moon would soon fall into the Earth. This problem was given such importance that the King of Sweden proclaimed there was a prize for the solution [3].

Finally the observed precession of Mercury's orbit is investigated. Newtonian physics does not explain this phenomenon, and for this reason a relativistic correction term is added. The perihelion shift is calculated analytically and the results of numerically adding the relativistic correction term is discussed.

2 Theory

2.1 Units

Since we are investigating planetary orbits, which are on astronomical scales, it's useful to use larger units in order to make the code and the results more readable. Therefore we use the length unit astronomical units, $1\,\mathrm{au} = 149\,597\,870\,700\,\mathrm{m} \approx 150$ million km, the time unit is years, $1\,\mathrm{yr} = 31\,556\,926\,\mathrm{s}$, and the mass unit is solar masses, $1M_\odot = 1.98847 \cdot 10^{30}\,\mathrm{kg}$. These are the units we have used in our program.

2.1.1 The gravitational constant

Since the units of mass, length and time have been changed, this also affects the numerical value of the gravitational constant G. By approximating Earth's orbit as perfectly circular, and therefore also assuming a constant orbital speed when taking into account Kepler's second law [4], the gravitational force from the Sun gives Earth a centripetal acceleration a_c:

$$\begin{aligned} F_g &= M_\mathrm{E} a_\mathrm{c} \\ \Rightarrow \frac{G M_\odot M_\mathrm{E}}{r^2} &= M_\mathrm{E} \frac{v^2}{r} \\ \Rightarrow G M_\odot &= v^2 \cdot r \\ &= \left(\frac{2\pi \cdot 1\,\mathrm{au}}{1\,\mathrm{yr}}\right)^2 \cdot 1\,\mathrm{au} \\ &= 4\pi^2 \frac{\mathrm{au}^3}{\mathrm{yr}^2} \\ \Rightarrow G &= 4\pi^2 \frac{\mathrm{au}^3}{\mathrm{yr}^2\, M_\odot}, \end{aligned}$$

Which means that the numerical value of G in our programs will be $4\pi^2$. This can also be seen from converting G from SI units to the new units in the following way:

$$1\,\mathrm{kg} = \frac{1}{1.98847 \cdot 10^{30}}\, M_\odot = 5.02899214 \cdot 10^{-36}\, M_\odot \tag{1}$$

$$1\,\mathrm{m} = \frac{1}{149\,597\,870\,700}\,\mathrm{au} = 6.684587122 \cdot 10^{-12}\,\mathrm{au} \tag{2}$$

$$1\,\mathrm{s} = \frac{1}{31\,556\,926}\,\mathrm{yr} = 3.168876462 \cdot 10^{-8}\,\mathrm{yr}, \tag{3}$$

which gives

$$\begin{aligned} G &= 6.67408 \cdot 10^{-11}\, \frac{\mathrm{m}^3}{\mathrm{s}^2\,\mathrm{kg}} \\ &= 6.67408 \cdot 10^{-11} \cdot (6.684587122 \cdot 10^{-12}\,\mathrm{au})^3 \cdot (3.168876462 \cdot 10^{-8}\,\mathrm{yr})^{-2} \\ &\quad \cdot (5.02899214 \cdot 10^{-36}\, M_\odot)^{-1} \\ &= 39.47513264321821\, \frac{\mathrm{au}^3}{\mathrm{yr}^2\, M_\odot}. \end{aligned}$$

The numerical value of $4\pi^2$ is ≈ 39.4784176, which is nearly identical to the numerical value calculated above.

2.1.2 Energy and angular momentum

Because of the units we use in our programs, the units of energy and angular momentum will have different numerical values from the SI units. The new units will be referred to here as PU (program units). The SI unit of energy is joules, $\mathrm{J} = \mathrm{N\,m} = \mathrm{kg\,m^2\,s^{-2}}$. Substituting our new units, we get

$$\begin{aligned}
[E]_{\mathrm{SI}} = \mathrm{J} &= 1\,\mathrm{kg} \cdot (1\,\mathrm{m})^2 \cdot (1\,\mathrm{s})^{-2} \\
&= 5.02899214 \cdot 10^{-36}\, M_\odot \cdot (6.684587122 \cdot 10^{-12}\,\mathrm{au})^2 \cdot (3.168876462 \cdot 10^{-8}\,\mathrm{yr})^{-2} \\
&\approx 2.237790962 \cdot 10^{-43}\, M_\odot\,\mathrm{au^2\,yr^{-2}} \\
&= 2.237790962 \cdot 10^{-43}\,[E]_{\mathrm{PU}}.
\end{aligned}$$

or

$$[E]_{\mathrm{PU}} = 4.46869264\,[E]_{\mathrm{SI}} \tag{4}$$

The SI unit of angular momentum is $[L] = \mathrm{kg\,m^2\,s^{-1}} = \mathrm{J\,s}$. In our new units this becomes

$$\begin{aligned}
[L]_{\mathrm{SI}} &= 1\,\mathrm{kg\,m^2\,s^{-1}} \\
&= 1\,\mathrm{J} \cdot 1\,\mathrm{s} \\
&= (2.237790962 \cdot 10^{-43}\, M_\odot\,\mathrm{au^2\,yr^{-2}}) \cdot (3.168876462 \cdot 10^{-8}\,\mathrm{yr}) \\
&= 7.09128311 \cdot 10^{-51}\, M_\odot\,\mathrm{au^2\,yr^{-1}} \\
&= 7.09128311 \cdot 10^{-51}\,[L]_{\mathrm{PU}}.
\end{aligned}$$

or

$$[L]_{\mathrm{PU}} = 1.41018203\,[L]_{\mathrm{SI}}. \tag{5}$$

All of the plots of energy and angular momentum in this report uses program units, so equations (4) and (5) can be used to convert these values to SI units.

2.2 Forward Euler method

The forward Euler method is a first order solver for ordinary differential equations. It is used to find the values of an unknown function $y(t)$ given an initial value $y_0 \equiv y(t_0)$ and an expression for the derivative of $y(t)$,

$$\frac{dy(t)}{dt} \equiv y'(t) = f(t, y(t)). \tag{6}$$

Here we are assuming that $y(t)$ is a continuous and differentiable function. The Taylor expansion of a continuous and differentiable function is given by

$$\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} \frac{y^{(n)}(a)}{n!}(t-a)^n \\
&= y(a) + y'(a)(t-a) + \mathcal{O}((t-a)^2)
\end{aligned}$$

where $y^{(n)}$ is the n-th derivative. By approximating $y(t)$ to the first order in the Taylor series we get the forward Euler method. The goal is to have an expression of the value of y at the next timestep of t from the current timestep. A timestep is defined as $h \equiv t_{i+1} - t_i$ assuming a constant discrete interval for all points in time. Defining $y_i \equiv y(t_i) \equiv y(t_0 + ih)$, the expression we are interested in is

$$
\begin{aligned}
y_{i+1} = y(t_{i+1}) &= y(t_i) + y'(t_i)(t_{i+1} - t_i) + \mathcal{O}((t_{i+1} - t_i)^2) && (7)\\
&= y(t_i) + y'(t_i)h + \mathcal{O}(h^2) && (8)\\
&\approx y_i + hy'_i && (9)
\end{aligned}
$$

where we have chosen $a = t_i$ and neglected terms of order 2 and above. This is the mathematical expression for the Euler method.

Applying the Euler method on our solar system model, we wish to solve a gravitational system for the time evolution of the positions of planets. We extend to three spatial dimensions for position $\mathbf{r}(t) = [x(t), y(t), z(t)]$ and velocity $\mathbf{v}(t) = [v_x(t), v_y(t), v_z(t)]$. Since each component $x_i(t)$ and $v_i(t)$ are not explicitly dependent on each other, each component can be treated separately as in equation (9). Noting that $\mathbf{v}(t) = \mathbf{r}'(t)$, the Euler method approximation for the position on the next timestep is

$$\mathbf{r}_{i+1} = \mathbf{r}_i + h\mathbf{v}_i. \tag{10}$$

With initial conditions $\mathbf{r}_0$ and $\mathbf{v}_0$ this gets us from $\mathbf{r}_0$ to $\mathbf{r}_1$. To get the next timestep, what is needed is an expression of $\mathbf{v}_{i+1}$, and this is retrieved from the acceleration of Newton's second law. The only forces acting on the planets is the gravitational force,

$$\mathbf{F}_{12} = -\frac{Gm_1m_2}{r^2}\hat{r}_{12} \tag{11}$$

where the force $\mathbf{F}_{12}$ is the force acting on object 1 from the object 2 and $r \equiv |\mathbf{r}_2 - \mathbf{r}_1|$ and $\hat{r}_{12} \equiv (\mathbf{r}_2 - \mathbf{r}_1)/r$ is the direction away from object 1 and towards object 2. From Newton's third law we know that object 2 experiences an equally strong gravitational force in the opposite direction, towards object 1. We will now rewrite the symbols in the following way: $m_1 = m$, $m_2 = M$, $\mathbf{F}_{12} = \mathbf{F}_g$, $\hat{r}_{12} = \hat{r}$. In the case of a two-body problem the sum of the forces on a planet is just the force from the other planet, so Newton's second law becomes

$$
\begin{aligned}
\Sigma\mathbf{F} = \mathbf{F}_g &= m\mathbf{a}\\
&\Downarrow\\
\mathbf{a} &= \frac{\mathbf{F}_g}{m}\\
&= -\frac{GM}{r^2}\hat{r}
\end{aligned}
$$

where m_1 is the mass of our planet (planet 1) and $\mathbf{a}$ is its acceleration. Acceleration is the time-derivative of velocity, so we can apply the Euler method again:

$$
\begin{aligned}
\mathbf{v}_{i+1} &= \mathbf{v}_i + h\mathbf{a}_i && (12)\\
&= \mathbf{v}_i - h \cdot \frac{GM}{r_i^2}\hat{r}_i, && (13)
\end{aligned}
$$

where once again i denotes timestep i. Since we have the expression for the gravitational force, all $\mathbf{a}_i = a_i(r_i)$ are always known for the current timestep, and thus if we supply initial conditions $\mathbf{r}_0, \mathbf{v}_0$ we can calculate the two-body system to any desired time.

2.2.1 N-body system

For a system of N bodies or planets, the acceleration at each timestep is given by the sum of the forces from all objects in the system:

$$\Sigma F = \sum_{b \neq c} \mathbf{F}_{g,b}$$

$$\Downarrow$$

$$m\mathbf{a} = -Gm \sum_{b \neq c} \frac{M_b}{r_b^2} \hat{r}_b$$

$$\Downarrow$$

$$\mathbf{a} = -G \sum_{b \neq c} \frac{M_b}{r_b^2} \hat{r}_b,$$

where the summation is over all bodies b that is not the current body c, M_b is the mass of the body b, r_b is the distance between the current body and b, and $\hat{r}_b$ is the direction from the current object towards b. The discretised version of the N-body acceleration is

$$\mathbf{a}_i = -G \sum_{b \neq c} \frac{M_b}{r_{b,i}^2}, \tag{14}$$

which can be used to calculate the next timestep of the velocity in an N body system exactly as in equation (12).

2.2.2 Pseudocode for forward Euler, two-body problem

The pseudocode for the forward Euler method for the two-body problem in three dimensions is presented in algorithm 1.

2.3 Velocity Verlet method

The velocity Verlet method assumes that the acceleration only depends on the position and not on the velocity. In this case, where gravity is the only acting force and there is no air resistance, this is true.

The velocity Verlet method is another explicit solver as the forward Euler method. It is quite similar to the Euler method, but it contains a second-order correction term in the equation for the next timestep in the position. Also, when calculating the next timestep in the velocity, the average acceleration between the current and next timestep in the acceleration is used. The mathematical expressions for the velocity Verlet method is as follows:

$$\mathbf{r}_{i+1} = \mathbf{r}_i + h\mathbf{v}_i + \frac{h^2}{2}\mathbf{a}_i \tag{15}$$

and

$$\mathbf{v}_i = \mathbf{v}_{i-1} + h\frac{\mathbf{a}_i + \mathbf{a}_{i-1}}{2}. \tag{16}$$

$\mathbf{a}_i$ is given by the sum of the gravitational forces from all the bodies in the system exactly as in equation (14). Thus, given initial conditions $\mathbf{r}_0$ and $\mathbf{v}_0$ we can calculate the position r_i to any desired timestep.

The upside of the forward Euler method is that it is very intuitive and the most simple to implement. The main drawback however is that the Euler method does not conserve energy and angular momentum. The velocity Verlet method is a symplectic method, so it does conserve energy and angular momentum. Therefore the Verlet method is a good choice for simulating the solar system, since the gravitational force is a conservative force.

The number of FLOPs between the Euler method and the Verlet method will now be compared. For the Euler method calculating the next step in position, $\mathbf{r}_{i+1}$, requires one multiplication and one addition

Algorithm 1: Forward Euler method: ODE solver.

```
Input: initialPosition, initialVelocity, tFinal
Output: A matrix, results, containing the positions and velocities for all timesteps
/* Initialise the position and velocity of the current planet and the Sun:  */
position = initialPosition
velocity = initialVelocity
sunPosition = [0,0,0]
sunVelocity = [0,0,0]
/* Calculate the initial force and acceleration:  */
force = gForceVector(mass, mass_Sun, initialPosition, sunPosition)
acceleration = force/mass
dt = length of timestep
N = number of timesteps
results(all rows, column 1) = list of N timesteps with stepsize dt
for i = 1,2,..., N do
    /* Update the position and velocity using the forward Euler method:  */
    position = position + dt*velocity
    velocity = velocity + dt*acceleration
    /* Store the results at the current timestep:  */
    results(row i, columns 2 to 4) = position
    results(row i, columns 5 to 7) = velocity
    /* Get the next timestep of the acceleration:  */
    force = gForceVector(mass, mass_Sun, position, sunPosition)
    acceleration = force/mass
return results
```

for each coordinate, as seen in equation (10). The acceleration term requires two multiplications and one division, three FLOPs, for each coordinate and each planet in the system for both methods. The next timestep for the velocity requires one multiplication and one addition for each coordinate for the Euler method, as seen in equation (12). This all adds up to a total of 7 FLOPs each timestep for each coordinate, including the FLOPs for calculating the acceleration term, assuming one planet.

For calculating the next timestep in position the Verlet method requires two additions, three multiplications and one division, a total of 6 FLOPs each timestep. Calculating the next velocity step requires two additions, one multiplication and one division, a total of 4 FLOPs. This adds up to 13 FLOPs per timestep per coordinate including the three FLOPs for the acceleration term.

2.3.1 Pseudocode for velocity Verlet, two-body problem

The pseudocode for the velocity Verlet method for the two-body problem in three dimensions is presented in algorithm 2. It is quite similar to the pseudocode for the forward Euler method, the difference lying in the updating of the position and velocity terms.

2.4 Exact solution of the escape velocity

An object is said to have escaped the orbit of the Sun (or any other gravitational object) if it will has a velocity greater than or equal to zero infinitely far away from the Sun, meaning that it will never return to the Sun in the future. Using the equation for potential energy we can calculate the necessary condition for a planet to escape orbit around the Sun.

Since gravity is assumed to be the only force acting on the planet and gravity is a conservative force, the total energy of the orbit is conserved. That is,

Algorithm 2: Velocity Verlet method: ODE solver.

```
Input: initialPosition, initialVelocity, tFinal
Output: A matrix, results, containing the positions and velocities for all timesteps
/* Initialise the position and velocity of the current planet and the Sun: */
position = initialPosition
velocity = initialVelocity
sunPosition = [0,0,0]
sunVelocity = [0,0,0]
/* Calculate the initial force and acceleration: */
force = gForceVector(mass, mass_Sun, initialPosition, sunPosition)
acceleration = force/mass
dt = length of timestep
N = number of timesteps
results(all rows, column 1) = list of N timesteps with stepsize dt
for i = 1,2,..., N do
    /* Temporarily save the current timestep of the acceleration: */
    previous_acceleration = acceleration
    /* Get the next timestep of the acceleration: */
    force = gForceVector(mass, mass_Sun, position, sunPosition)
    acceleration = force/mass
    /* Update the position and velocity using the velocity Verlet method: */
    velocity = velocity + dt*acceleration
    position = position + dt*velocity + 0.5*dt*dt*previous_acceleration
    /* Get the next timestep of the velocity: */
    velocity = velocity + 0.5*dt*(acceleration + previous_acceleration)
    /* Store the results at the current timestep: */
    results(row i, columns 2 to 4) = position
    results(row i, columns 5 to 7) = velocity
return results
```

$$E_{\text{tot}} = U + K = \text{constant} \tag{17}$$

where

$$U = U(r) = -\frac{GM_\odot m}{r} \tag{18}$$

is the gravitational potential energy of a planet with mass m at distance r from the center of the Sun, and

$$K = \frac{1}{2}mv^2 \tag{19}$$

is the kinetic energy of the planet, where v is the orbital speed of the planet. The escape velocity is defined as the lowest velocity at which the planet escapes orbit. The planet escapes orbit if it reaches a point in space where its potential energy is non-negative. Gravitational potential energy is in general negative at a finite distance away from the object, but at $r = \infty$ it is zero. By the definition of the escape velocity, the planet will have zero velocity at infinity. In mathematical terms, the equation for the escape velocity is

$$\begin{aligned} U_0 + K_0 &= U_\infty + K_\infty \\ -\frac{GM_\odot m}{r_0} + \frac{1}{2}mv_{\text{esc}}^2 &= 0 + 0 \\ \Rightarrow v_{\text{esc}}^2 &= \frac{2GM_\odot}{r_0} \end{aligned}$$

$$v_{\text{esc}} = \sqrt{\frac{2GM_\odot}{r_0}}, \tag{20}$$

independent of the planet mass m.

2.5 Conservation of angular momentum

Although the Earth's orbit around the Sun is nearly circular it is in fact, elliptical. This is more pronounced in a very elongated orbit or an orbit with a high eccentricity. Eccentricity, e, is equal to 0 for a circle, and equal to 1 where the ellipse extends to infinity (a parabola). At the planets closest approach (perihelion) the planets has its greatest angular velocity, and at the planets furthest distance from the Sun (aphelion) the angular velocity is the smallest. Kepler's second law states that a vector $\mathbf{r} \equiv r\hat{r}$ extending from the Sun to a planet in its orbit sweeps the same amount of area at any point in the orbit for equal time intervals. The result of this directly shows that angular momentum is conserved throughout this orbit.

In an infinitesimal time dt the area dA that is swept over by the vector is approximately the area of a right triangle with base length r and height approximately equal to the arc length $ds = r(t) \cdot \omega(t)\,dt$ where $\omega(t) \equiv d\theta/dt$ is the angular frequency of the planet's orbit at time t. This triangle has the area

$$dA = \frac{r \cdot r(t)\omega(t)\,dt}{2} = \frac{1}{2}r(t)^2\omega(t)\,dt, \tag{21}$$

so the rate at which the area is swept is

$$\frac{dA}{dt} = \frac{1}{2}r(t)^2\omega(t). \tag{22}$$

where r is the radius and θ is the angle swept out by the planet. According to Kepler's second law, this area rate is constant, so we have

$$\frac{1}{2}r^2\omega = \text{constant}, \tag{23}$$

that is, this expression has the same value for all values of t.

Angular momentum is given by

$$\mathbf{L} = \mathbf{r} \times (m\mathbf{v}) = mrv\,\hat{r} \times \hat{v} = mrv\hat{\omega} \tag{24}$$

where $\mathbf{r}$ is the planet's position vector, $\mathbf{v} = v\hat{v}$ is the velocity of the planet and $\hat{\omega}$ is the direction of the angular velocity vector $\boldsymbol{\omega} = \omega\hat{\omega}$, obtained from the right hand convention for the angular velocity vector. The orbit speed v in an elliptical orbit is given by

$$v = \frac{ds}{dt} \approx \frac{r \cdot d\theta}{dt} = r\omega, \tag{25}$$

where the approximation symbol has been used since a small orbit path distance Δs is not exactly equal to $r\,\Delta\theta$ in the case of a non-circular orbit. Substituting eq. (25) into eq. (24) we get

$$\begin{aligned}
\mathbf{L} &= mrv\hat{\omega} \\
&= mr \cdot (r\omega)\hat{\omega} \\
&= mr^2\omega\hat{\omega} \\
&= 2m \cdot \left(\frac{1}{2}r^2\omega\right)\hat{\omega} \\
&= (2m \cdot \text{constant})\,\hat{\omega}
\end{aligned}$$

where the last equation comes from eq. (23). The mass m is assumed to be constant. Also, since the only assumed force is the gravitational pull of the Sun, there will be no forces perpendicular to the plane of the orbit, which means that the orientation of the orbit will be constant in time, so the direction $\hat{\omega}$ will be constant in time. So every quantity in the expression for $\mathbf{L}$ is constant, so we have

$$\mathbf{L} = \text{constant}. \tag{26}$$

2.6 Different forms of the gravitational force law

Although the gravitational force is usually assumed to be given by the inverse square law

$$F_g = \frac{Gm_1m_2}{r^2}. \tag{27}$$

between two objects with masses m_1 and m_2 respectively, and a distance r between them, other forms of the force can also be experimented with. One possibility is to replace the exponent 2 of r by a variable real number β, so that the gravitational force strength becomes

$$F_g = \frac{Gm_1m_2}{r^\beta}. \tag{28}$$

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2.7 Perihelion precession of Mercury

Mercury is the planet orbiting closest to the Sun in the Solar System. Its orbit also has the highest eccentricity of the planets. The ellipse that Mercury traces as it orbits the Sun, however, is not always the same. The perihelion of Mercury precesses, or rotates, around the Sun. This observation is mainly due to the other planets in the Solar System, which cause the Mercury to deviate from tracing the same path. The perihelion shift calculated due to the other planets is on average 526.7 arc seconds per century, however by observation the shift was found to be 565 arc seconds (from the 40 years of observation at the Paris observatory) and 570 arc seconds from modern data [5]. This discrepancy of 42.3 arc seconds per century was not solved until in 1915 when Einstein published his theory of relativity which accounted for a shift of 43 arc seconds per century [6].

According to [7], the perihelion shift in radians per revolution due to Einstein theory of gravitation should be:

$$\sigma = \frac{24\pi^3 L^2}{T^2 c^2 (1-e^2)} = \frac{6\pi GM}{c^2 L(1-e^2)} \tag{29}$$

where T is the time period, L is the semi-major axis. Using a geometrical relation $L(1-e^2)$ equals

$$= \frac{2}{1/a + a/b} \tag{30}$$

where a is the aphelion and b is the perihelion. Hence the result leads to

$$\sigma = \frac{3GM\pi(1/a + 1/b)}{c^2} \tag{31}$$

which for a planet like Mercury results in σ = 2.887e-5 deg per revolution or 0.012 deg per century. When this is converted to arc seconds it is approximately 43 arc seconds. In this report in order to reproduce these calculation, a relativistic correction term is added to the gravitational force:

$$F_G = \frac{GM_\odot M_{\mathrm{Mercury}}}{r^2}\left(1 + \frac{3l^2}{r^2c^2}\right) \tag{32}$$

where r is the relative distance between the Sun and Mercury, l is Mercury's angular momentum per unit mass and c is the speed of light in vacuum.

3 Method

The simulations were written in C++ and Python. Mainly C++ was used to perform the heavy-duty calculations involved in performing the ODE solvers, namely forward Euler and the velocity Verlet algorithm. The C++ linear algebra library Armadillo was used for handling of vectors and matrices. The results for the time evolution of positions and velocities was saved in `.csv` files, one file for each planet in the system, excluding the Sun. The plotting was done in Python using the Matplotlib library. We tested the usage of the forward Euler method in the beginning of the project, but mainly we used the velocity Verlet method because of its superior accuracy and conservation of energy and angular momentum.

The Sun was kept stationary in the origin for all simulations. This is a good approximation since the Sun has a very large mass compared to any of the other planets in the solar system. The Sun is around 1000 times more massive than Jupiter, and Jupiter alone is more massive than all the other planets in the solar system combined.

To simulate a circular orbit of Earth around the Sun, the initial values of Earth were chosen as position $\mathbf{r}_0 = [1, 0, 0]$ and velocity $\mathbf{v}_0 = [0, 2\pi, 0]$. This is because in units au/yr an orbit speed of 2π au/yr gives a circular motion.

For the three-body problem we used initial conditions $\mathbf{r}_{0,E} = [1, 0, 0]$ and $\mathbf{v}_{0_E} = [0, 2\pi, 0]$ for the Earth and $\mathbf{r}_{0,J} = [5.2044, 0, 0]$ and $\mathbf{v}_{0,J} = [0, 2.75522, 0]$ for Jupiter.

For the simulation of the entire solar system, the initial conditions were retrieved from NASA's webpage [8]. The masses of the planets were retrieved from Wikipedia. Note that the velocities given at the NASA web page are in units au/d (astronomical units per day), so the velocities needed to be converted to au/yr to be used in our program.

When performing simulations for different versions of the gravitational force, that is, different values of the exponent β, this was done by storing the desired values of β in a list and then running through that list in a for-loop and saving one text file for each value of β.

When finding the escape velocity for a planet starting at a distance 1 au from the Sun we gave the planet an arbitrary mass of one Earth mass (the planet's mass doesn't affect its escape velocity, as seen in eq. (20) and an initial position of $\mathbf{r}_0 = [1, 0, 0]$. The initial velocity was of the form $\mathbf{v}_0 = [0, v, 0]$, where v is the chosen initial speed of the planet. The escape velocity was found by systematic guesswork using the bisection method. By guessing a small value of v where the planet did not escape and then a very large value of v where the planet did escape, the interval was halved or close to halved, giving two intervals of v. If the planet did not escape with this middle value, this middle value became the new bottom value of the interval, and the middle of this interval was bisected to further close in on the escape velocity. Determining whether the planet had escaped or not was done by visual inspection and doing simulations for many years, at the most 500 years, to make sure the planet had in fact escaped.

When performing simulations with the goal of observing the precession of Mercury's perihelion, the Verlet solver was used as in the other cases, the only difference being the correction term in the gravitational force as seen in eq. (32). This change in the force affects the acceleration in each timestep, which should affect the orbit as well.

4 Results

4.1 Stability of the Velocity Verlet & Euler Algorithm

Fig. 1 and fig. 2 show plots of the the Earth-Sun system solved using the velocity Verlet method and the forward Euler method respectively. The final time is 10 years with several different values of the time step dt.

Table. 1 shows the timings of each algorithm with $10^5, 10^6$ and 10^7 time steps.

4.2 Unit tests

4.2.1 Conservation of total energy

Performing the Velocity Verlet method with the parameters: N = 1e5, dt = 1e-4 (plotted in Fig. 1g) the kinetic and potential energy were found to be conserved. As shown in the figure, the orbit is circular. It was expected that both quantities are conserved since the distance between the Earth and Sun is unchanged. In the case of an elliptical orbit, the Sun-Earth distance changes and so do the potential and kinetic energy, yet the total energy should be conserved. This result confirms that the Velocity Verlet algorithm does conserve energy.

4.2.2 Conservation of angular momentum

To test the conservation of angular momentum, the same parameters were used (N = 1e5, dt = 1e-4). As shown by the following equation, the angular momentum was computed by taking the cross product of the velocity vector and the radial vector, multiplied by the Earth's mass at each time step. The results was that angular momentum was constant throughout the simulation.

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times (m\mathbf{v}) = \text{constant} \tag{33}$$

Both these quantities, the conservation of energy and angular momentum, are important tests in the algorithm to reflect the behaviour of a physical system. If energy or momentum were to be lost during the simulation, it would lead to very different results over a period of time to the real solar system.

4.3 The varying force law

Here, results for implementing the varying force law presented in equation (28) for different values of β is presented. We have tested the force laws for $\beta \in [2, 3]$.

Fig. 3 shows the total energy of the Earth-Sun system with different values of β. In Fig. 3(c) the figure shows that when using $1/r^2$ that the total energy of the Earth is constant throughout its orbit around the Sun. For circular orbits the potential energy and kinetic energy were constant, although this is not plotted.

Fig. 4 shows the absolute value of the angular momentum, $L = |\mathbf{L}|$, as a function of time for different values of β.

Table 1: Timing the Velocity Verlet and Euler forward methods with different number of time steps. The results are given in seconds and are the average of four runs each.

Number of time steps	10^5	10^6	10^7
Forward Euler	0.0781	0.883	8.803
Velocity Verlet	0.0938	0.879	8.823

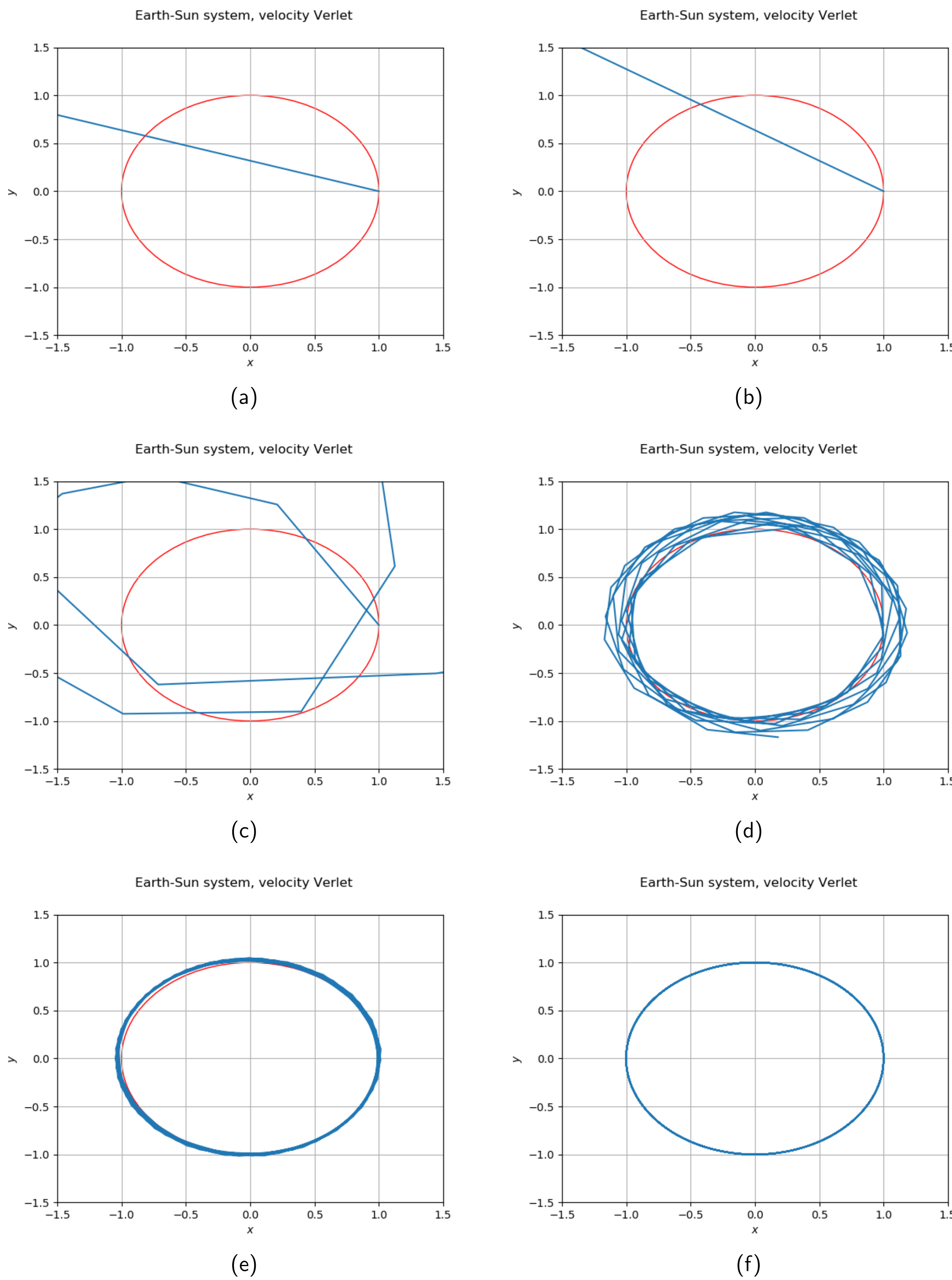


Figure 1: Plots of the the solar system solved using the velocity Verlet method. The final time is 10 years with several different time steps dt, and number of time steps N. (a) $dt = 1$ yr and $N = 10$, (b) $dt = 0.5$ yr ≈ 183 days and $N = 20$, (c) $dt = 0.2$ yr ≈ 73 days and $N = 50$, (d) $dt = 0.1$ yr ≈ 37 days and $N = 100$, (e) $dt = 0.05$ yr ≈ 18 days and $N = 200$, (f) $dt = 0.01$ yr ≈ 4 days and $N = 1000$. The blue line indicated the orbit of the Earth, the red line is a circle plotted for comparison.

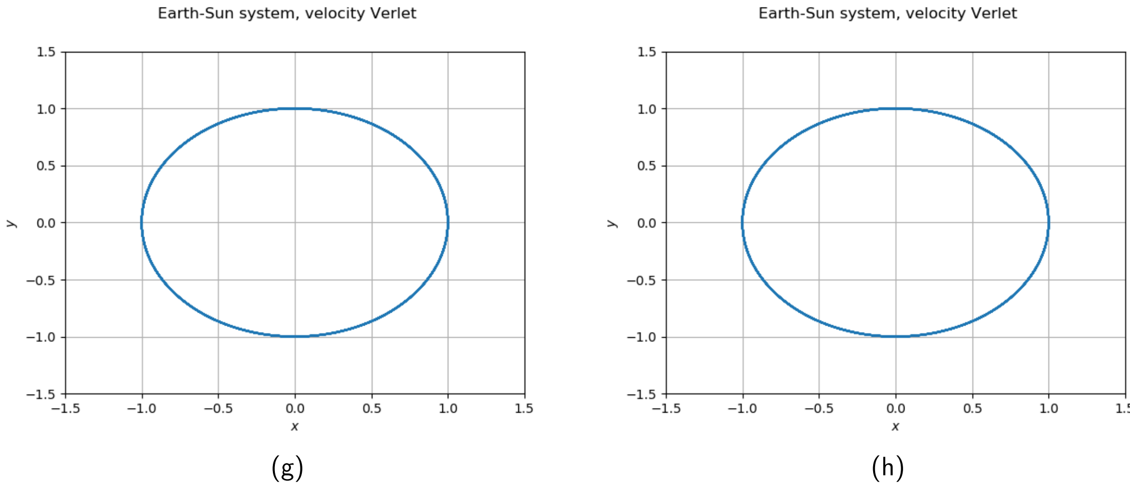


Figure 1: Plots of the the solar system solved using the velocity Verlet method. The final time is 10 years with several different time steps dt, and number of time steps N. (g) dt = 1e-4 yr ≈ 53 minutes and N = 1e5, (h) dt = 1e-5 yr ≈ 5 minutes and N = 1e6. The blue line indicated the orbit of the Earth, the red line is a circle plotted for comparison.

Table 2: Finding the escape velocity of a planet starting at a distance 1 au from the Sun.

Initial speed, km/s	10	20	30	40	42	42.1	42.125	42.15	42.2	42.5	43	45	50	100
Escaped orbit?	no	no	no	no	no	no	yes	yes	yes	yes	yes	yes	yes	yes

Fig. 5 shows the distance between the Sun and the Earth as a function of time for different values of β. For $\beta \gtrsim 2.88$ the Earth escaped orbit in our simulation. Therefore, only results for values of β below this have been shown in the results figures.

When computing the same result for 20 years, the same oscillatory between a distance of 1 au and a distance closer to 0 continued in the same manner as for the 5 year simulation. Fig. 6 shows the orbits of Earth for different values of β over a time period of 3 years.

4.4 Escape velocity

4.4.1 Trial and error

By running the velocity Verlet algorithm with initial position $\mathbf{r}_{\text{init}} = (1, 0, 0)$ and initial velocity $\mathbf{v}_{\text{init}} = v_{\text{init}} \cdot (0, 1, 0) = (0, v_{\text{init}}, 0)$ for several values for the initial speed v_{init}, the escape velocity was found to be approximately $v_{\text{esc}} \approx 42.125\,\text{km/s}$. The trial and error process was done in a bisection fashion, approximately halving the interval with each guess. Whether the planet had escaped from orbit or not was checked simply by visually inspecting the plot and having the end time sufficiently high (ranging from 500 to 10 000 years). The results can be seen in table 2 where the velocity guesses has been sorted in ascending order. Different masses were tested and the results were the same for all masses. The explanation of this is that the acceleration is $a = F_g/m$, so the factor m in the gravitational force F_g is cancelled and the acceleration is independent of the planet's mass.

4.4.2 Exact solution of the escape velocity

Equation (20) gives the analytical expression for the escape velocity. For an initial distance of $r_0 = 1\,\text{au}$ we get, using SI units for G, $M_\odot$ and r_0,

$$v_{\text{esc}} = 42\,121.9\,\text{m/s} \approx 42.122\,\text{km/s}. \tag{34}$$

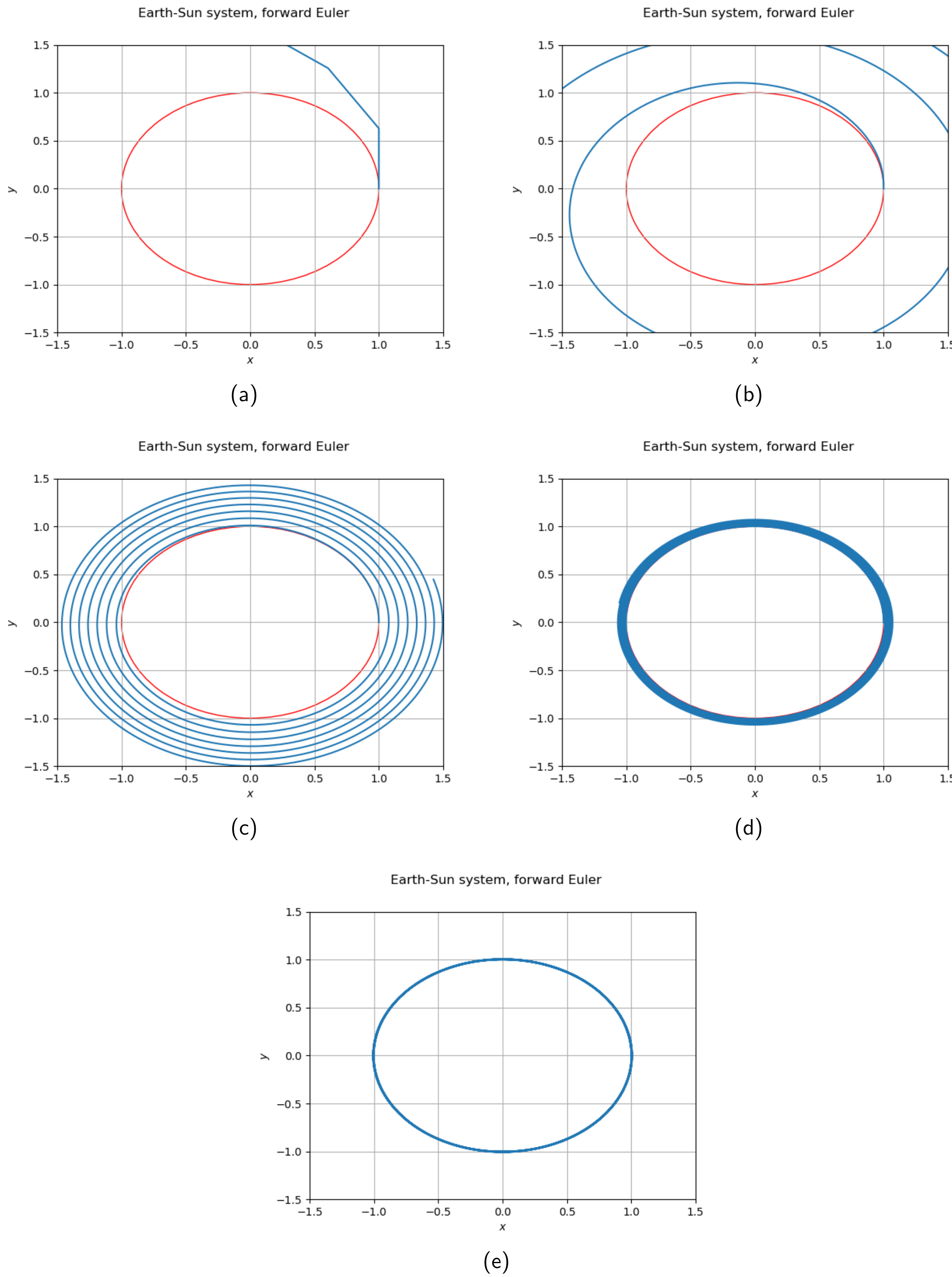
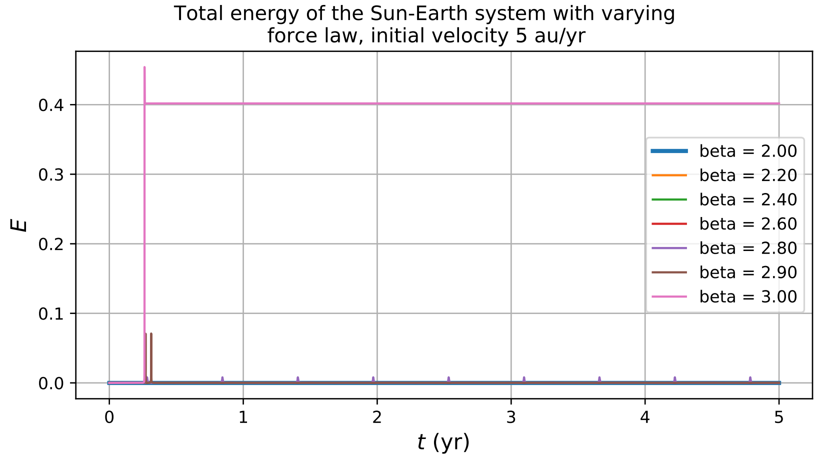
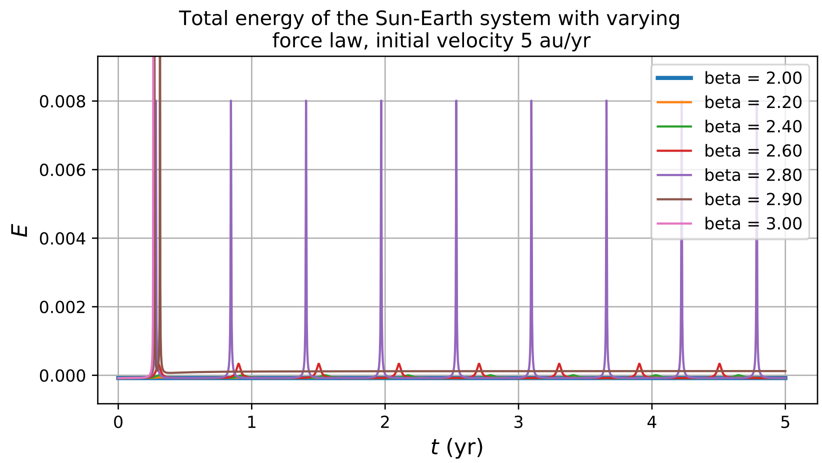


Figure 2: Plots of the the solar system solved using Euler's method. The final time is 10 years with several different time steps dt, and number of time steps N. (a) $dt = 0.1$ yr ≈ 37 days and $N = 100$, (b) $dt = 0.01$ yr ≈ 4 days and $N = 1000$, (c) $dt =$ 1e-3 yr ≈ 9 hours and $N = 1e4$, (d) $dt =$ 1e-4 yr ≈ 53 minutes and $N =$ 1e5, (e) $dt =$ 1e-5 yr ≈ 5 minutes and $N =$ 1e6.



(a)



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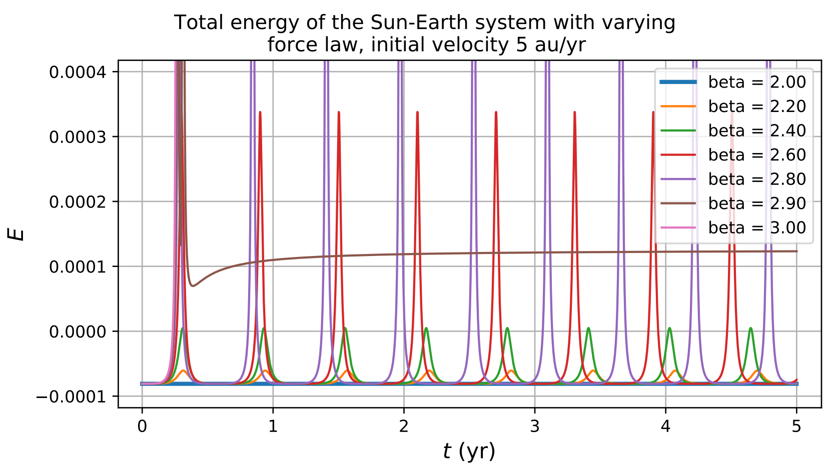
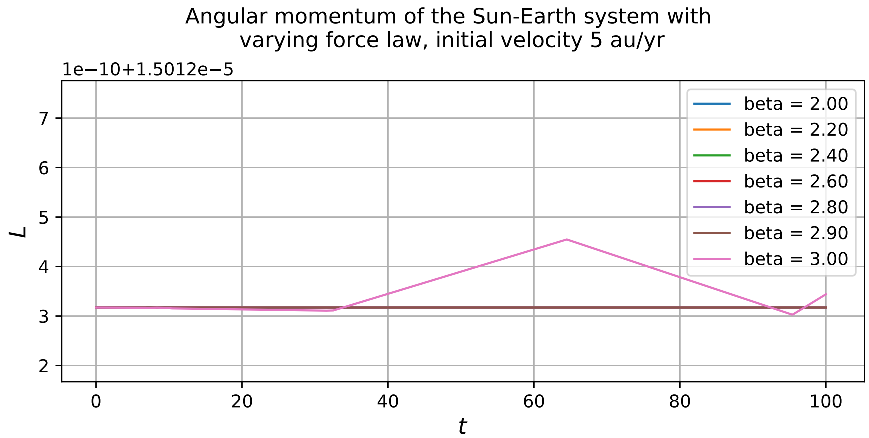
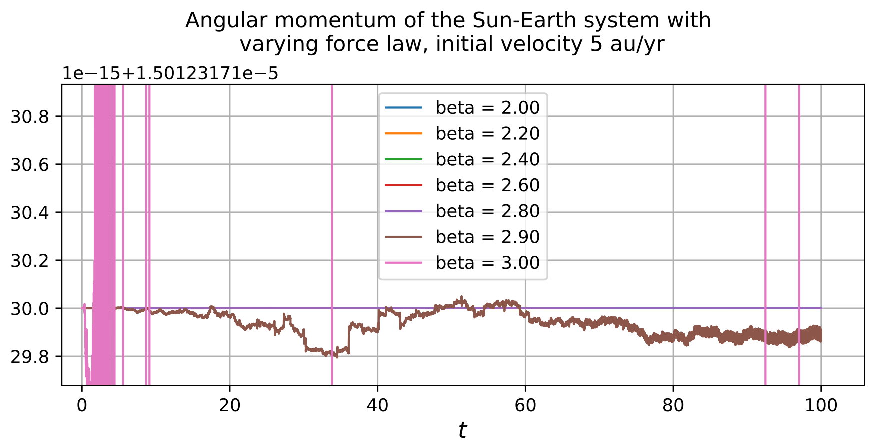


(c)

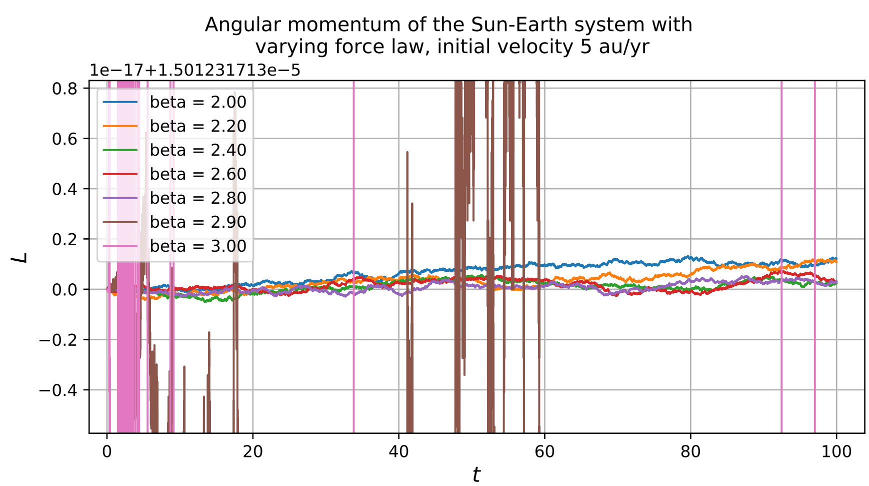
Figure 3: The total energy (kinetic plus potential energy) of Earth in orbit around the Sun for different values of β. This figure shows a simulation of a 5 year period. (a) The figure zoomed out, where all graphs are visible. (b) The same figure, zoomed in to make more details visible. (c) The same figure, further zoomed in.



(a)



(b)



(c)

Figure 4: The magnitude of the angular momentum of Earth in orbit around the Sun for different values of β. The system is simulated over a 100 year period. (a) The figure zoomed out, where all graphs are visible. (b) The same figure, zoomed in to make more details visible. (c) The same figure, further zoomed in.

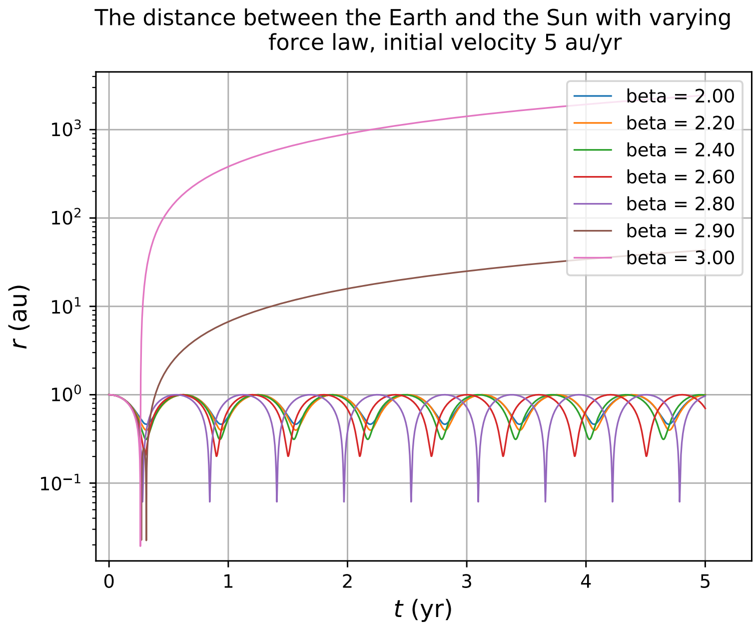


Figure 5: The distance between the Earth and the Sun simulated over a 5 year period with different values of β in the force law. The initial orbit speed is 5 au. The y-axis is on a logarithmic scale.

4.5 The three-body problem

In Fig. 7 the orbits of Earth and Jupiter is plotted. The initial values of Earth and Jupiter are in the Method section. In Fig. 8 the three-body problem has been simulated with Jupiter's mass being 10 times its real mass, and in Fig. 9 the three-body problem has been simulated with Jupiter's mass being 1000 times its real mass.

4.6 The solar system including all planets

In Fig. 10 the simulation of the solar system including all planets is plotted.

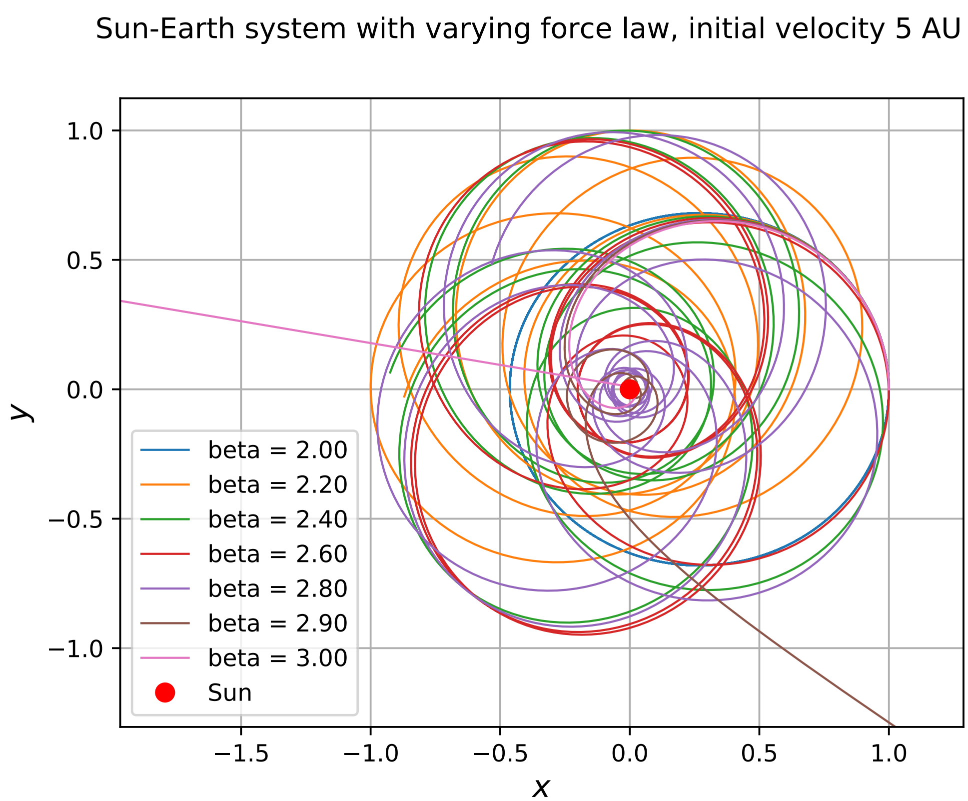


Figure 6: The orbit of the Earth around the Sun simulated over a 3 year period with different values of β in the force law. The initial orbit speed is 5 au/yr for all orbits.

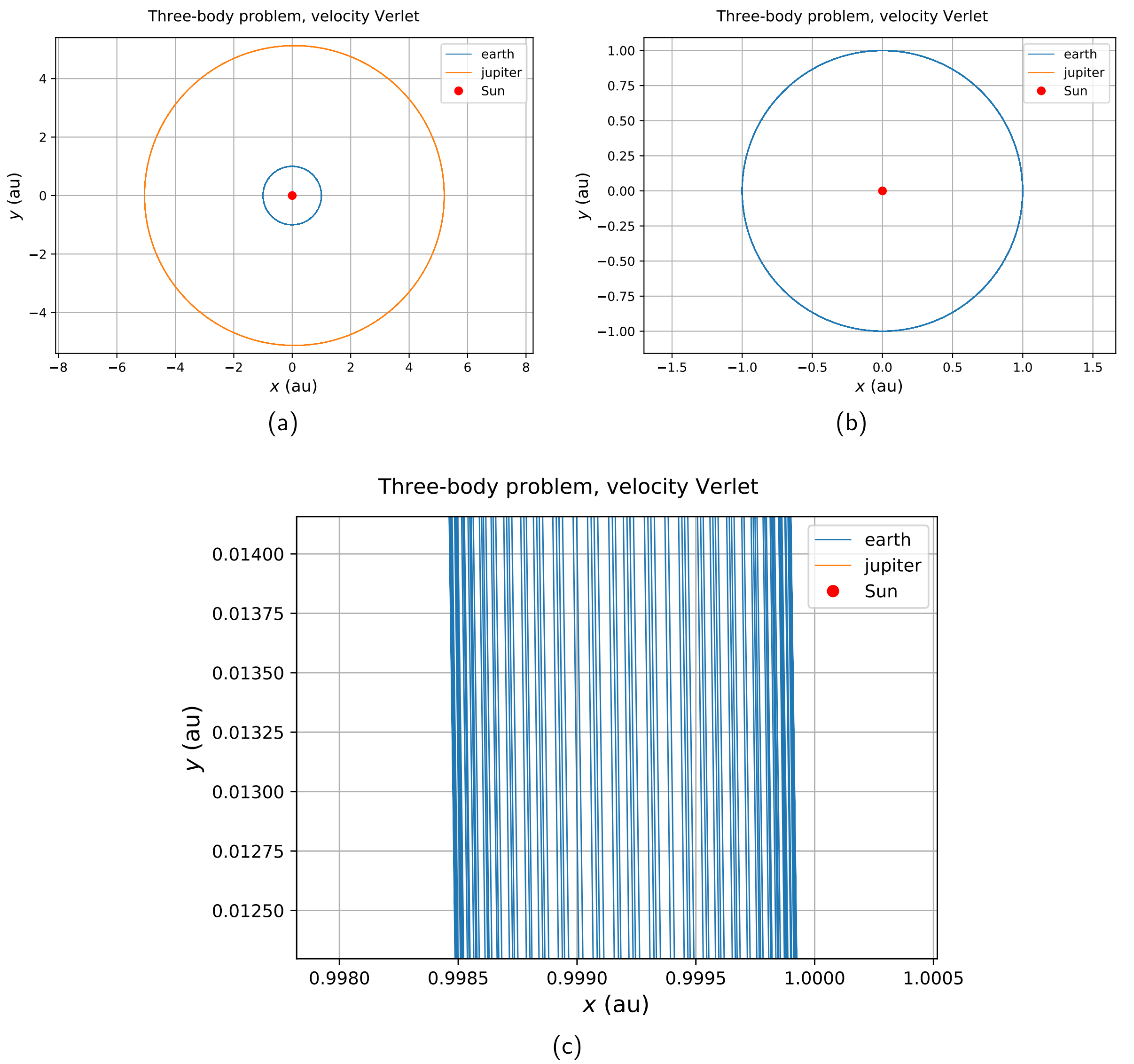


Figure 7: The orbits of Earth and Jupiter in an interacting three-body system between the Sun, the Earth and Jupiter, with Jupiter's mass being its real mass. The simulation is over a 100 year period. (a) Full view including Jupiter's orbit. (b) Zoomed in view of Earth's orbit. (c) Zoomed in view of a small portion of Earth's orbit for better visibility of Jupiter's effect on Earth's orbit.

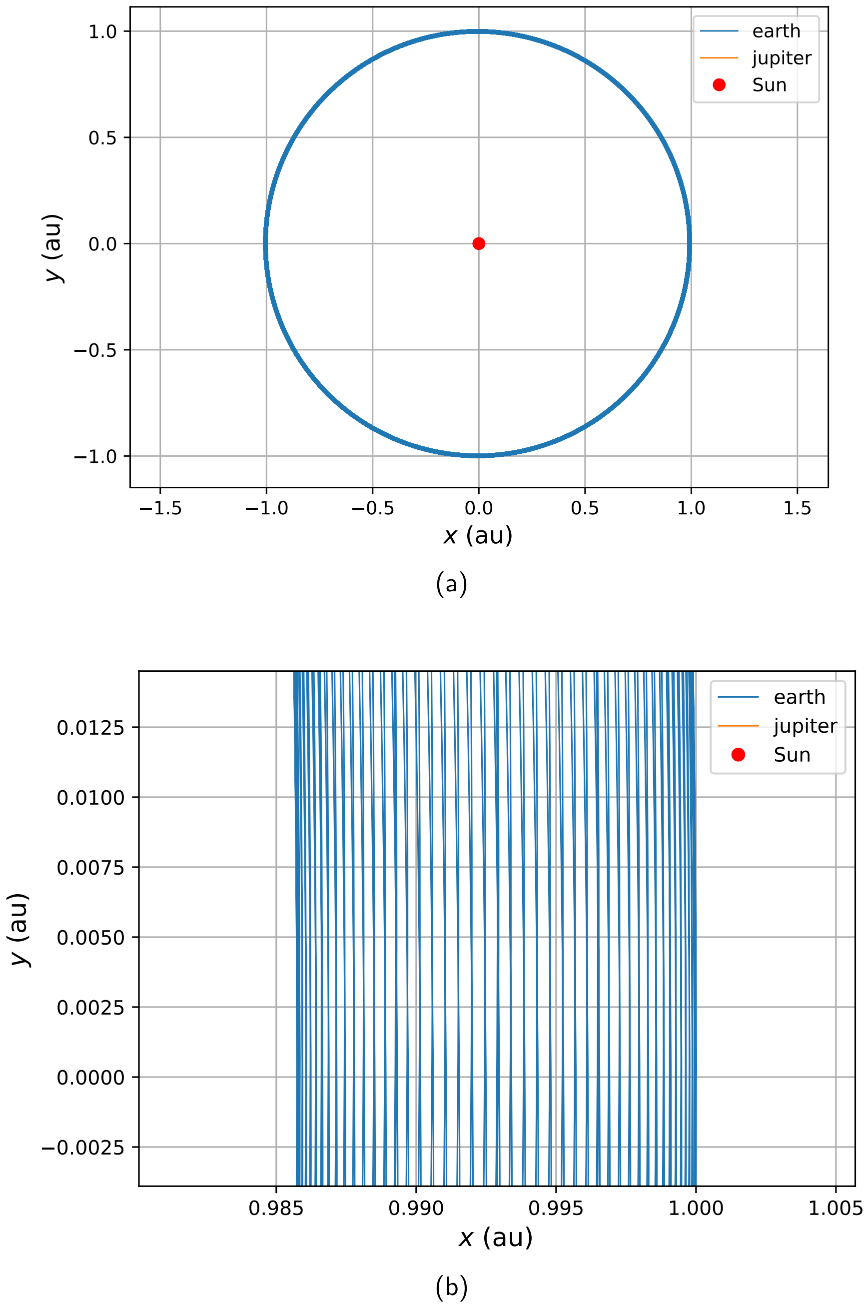
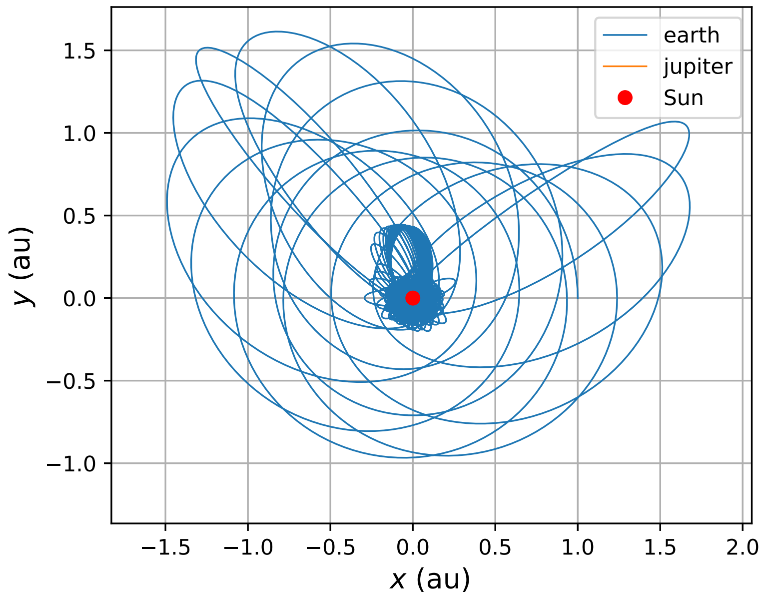
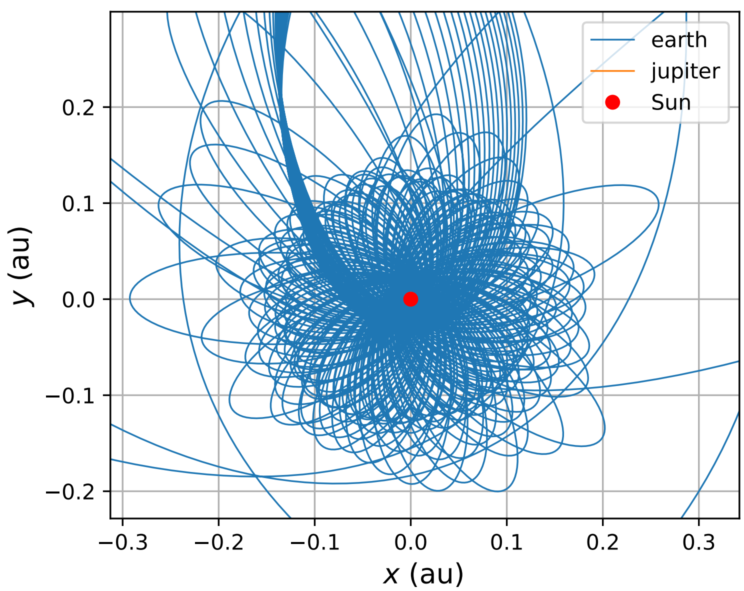


Figure 8: The orbits of Earth and Jupiter in an interacting three-body system between the Sun, the Earth and Jupiter, with Jupiter's mass being 10 times its real mass. The simulation is over a 100 year period. (a) View of Earth's orbit. (b) Zoomed in view of a small portion of Earth's orbit for better visibility of Jupiter's effect on Earth's orbit.

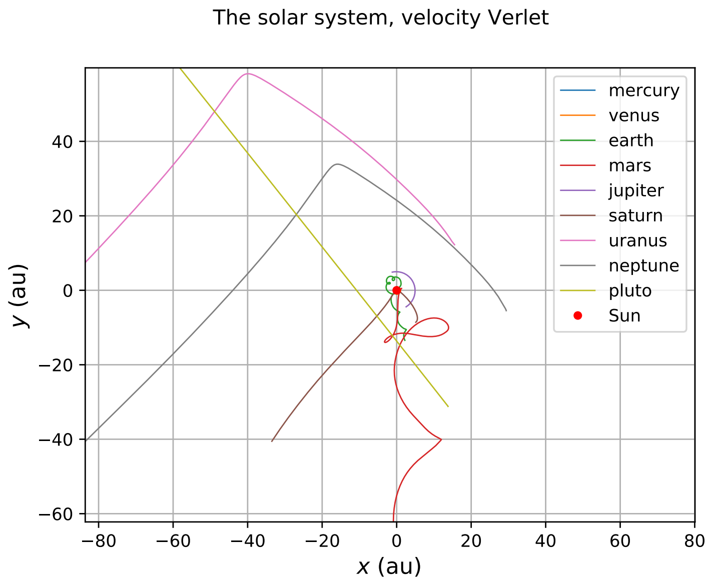


(a)

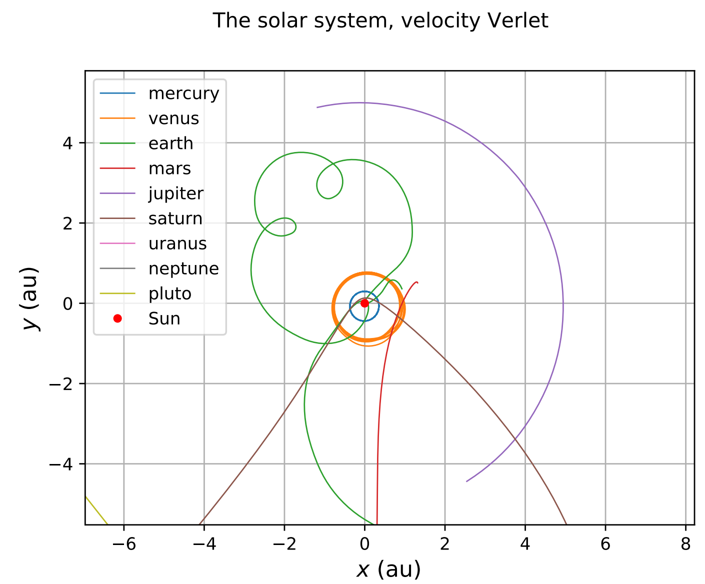


(b)

Figure 9: The orbits of Earth and Jupiter in an interacting three-body system between the Sun, the Earth and Jupiter, with Jupiter's mass being 1000 times its real mass. The simulation is over a 20 year period. (a) View of Earth's orbit. Jupiter's orbit is further out. (b) Earth's orbit in a more zoomed in view.



(a)



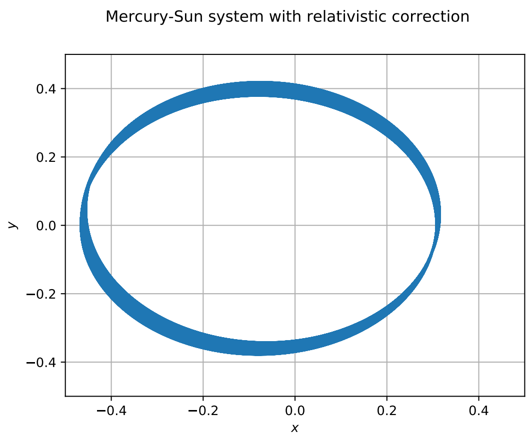
(b)

Figure 10: The results from running the simulation of the solar system including all planets for a time period of 5 years. (a) Zoomed out view. (b) Zoomed in view.

4.7 Perihelion Precession of Mercury

Fig. 11 shows a simulation of Mercury's orbit without the relativistic correction, for different values of the timestep dt. Fig. 12 shows a simulation of Mercury's orbit when including the relativistic correction term. In fig. 13 the perihelion shift does not give the values expected with and without the correction term. The time resolution of the simulation was tested with dt up to 1e-6 ($\approx$ 31 seconds) and found little change in the results.

The precession of Mercury's perihelion was first simulated without the relativistic correction term, as shown in Fig. 11. In Fig. 11(a) it was believed that Mercury was precessing, however could have been due poor time resolution. Fig. 11(b) shows at a higher time resolution, which seems to not be precessing as expected. In addition, by looking at Fig. 12, where the perihelion position has been plotted with and without the correction term, that there is some systematic error in the code which results in an unusual repetition of points during a simulation of 100 years ($dt \approx 5$ minutes). It is expected that without the relativistic correction term that in a century the perihelion shift should be approximately 527 arc seconds, and 570 arc seconds with the correction terms added. As shown in Fig. 13 the perihelion shift does not give the values expected with and without the correction term. The time resolution of the simulation was tested with dt up to 1e-6 ($\approx$ 31 seconds) and found little change in the results.



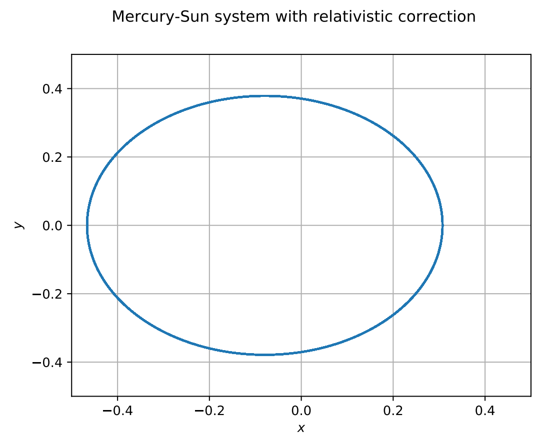
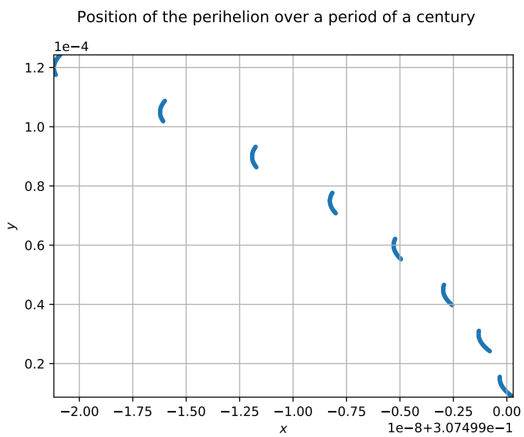


Figure 11: The motion of Mercury around the Sun over a period of a century without the relativistic correction term. (a) The simulation was run with dt = 1e-3 yr $\approx$ 9 hours. (b) Simulation run with dt = 1e-4 yr $\approx$ 53 minutes.



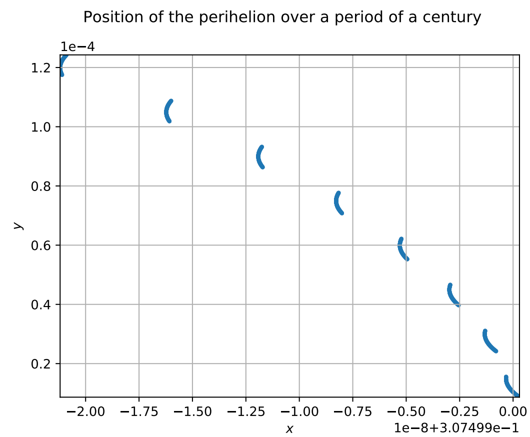


Figure 12: The change in position of the perihelion of Mercury over 100 years, dt = 1e-5 yr $\approx$ 5 minutes. (a) Without the relativistic correction term. (b) With relativistic correction term added to force. The perihelion motion seems unusual and therefore it is believed there is a systematic error in the code.

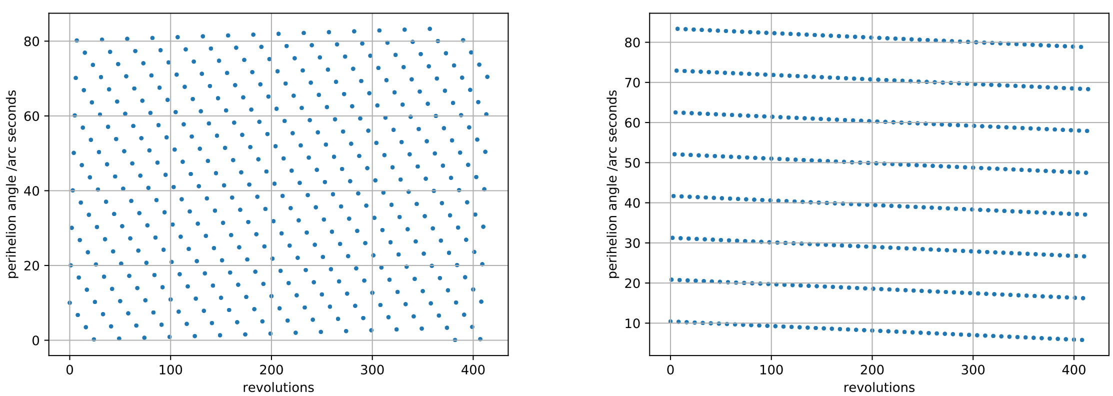


Figure 13: The angle the perihelion makes to the x axis at each orbit round the sun, simulation with $dt = 1\text{e-}5\,\text{yr} \approx 5\,\text{minutes}$. (a) Without the relativistic correction term. (b) With the relativistic correction term.

5 Discussion

5.1 Stability of the velocity Verlet & Euler Algorithm

Fig. 1 shows that when the step size dt is approximately 4 days the orbit appears circular. Fig. 2 shows that when $dt \approx 5$ minutes that the orbit appears circular. This means that the forward Euler method requires around 1000 times more data points before reaching an orbit which is approximately circular. On the other hand, both methods show similar computational efficiency as shown in table 1 where both method take the same amount of time to perform simulations with different number of time steps. This suggests that the velocity Verlet method is more accurate with little difference in computational cost compared to the forward Euler method.

5.2 Escape velocity

The trial and error result of 42.125 km/s seen in table 2 matches the analytical result of $v_{\text{esc}} = 42.122$ km/s to four significant digits, which is a relatively good approximation. It is possible that the result would be even closer to the analytical value by doing more bisections.

5.3 Different forms of the gravitational force

The results seen in Fig. 5 show that for $\beta \leq 2.8$ the Earth stays in orbit. The distance oscillates between 1 au and a point that is closer to the Sun, differing for each value of β. For $\beta = 2.9$ and $\beta = 3$ it seems that the Earth is flung out of orbit.

One possible way to test if nature (the real gravitational force) deviates from a perfect inverse square law would be to see if there are discrepancies between the observations of planetary motions and simulations that are based on the gravitational force being the perfect inverse square law. However, one must be careful that it is only the gravitational force acting in the system, and it would be best to include as many gravitational bodies in the solar system as possible and include them in the simulations, since if not there could be deviations just based on the fact that the simulation is missing the gravitational effects of many bodies.

The total energy of Earth's orbit for different values of β plotted in fig. 3 shows that the total energy for $\beta = 2.0$, the original form of the gravitational force, is constant in time, which is expected. As β increases, however, it is seen that the total energy is not conserved. For $\beta = 3.0$, the total energy makes a large jump at $t \approx 0.26$ yr. Comparing with Fig. 5 we see that this corresponds well with when the orbit for $\beta = 3.0$ is flung out of orbit, which happens close to around $t = 1/4$ yr. This also seems to be the case for $\beta = 2.9$. We can also see from comparing these graphs that the periodic motion for $\beta \leq 2.8$ of total energy and distance from the sun are similar, that is, the maxima of total energy coincide with the minima of distance. The frequency of the periodic motion seems to be slightly less than one year, but it also seems to be shorter the larger β is. This can suggest that the kinetic energy is highly increased by the $1/r^{\beta}$ force which is stronger close to the Sun, which might explain the instability in the total energy. And that the higher β is the shorter the orbital period of Earth becomes. The simulation was also performed to $t = 20$ yr, and the same periodic pattern continued for the values $\beta < 3$, and $\beta = 3$ stayed constant after its sudden increase.

While this is rather speculative, if Kepler's second law holds for a general $1/r^{\beta}$ force then this could be a reasonable result in the following sense: Kepler's second law states that the area swept over in any given time span is equal at any starting point in the orbit, so for an increased β which would more strongly accelerate and pull in the planet closer to the Sun, the planet would have to move faster in order to sweep over the same area since it is closer to the Sun. But in order to test whether Kepler's laws actually hold for this simulation, further testing would be required.

The angular momentum for the same simulation, plotted in fig. 4, shows that the angular momentum for $\beta = 3$ is unstable. Though the y-axis shows that the variation is of the order 1e-10 and the numerical value is $L \approx 1.5012$e-5, the variation in L for $\beta = 3$ is not confined and looks like it could increase further in time. In 4 (b) is is seen that the angular momentum for $\beta = 2.9$ shows variation on the scale of 1e-15, and in 4 (c) the angular momenta for $\beta \leq 2.8$ show some variation on the scale of 1e-17. But this is negligible compared to the actual magnitude of ∼1e-5, so it is reasonable to say that for $\beta \leq 2.9$ the angular momentum is constant according to our simulations.

5.4 The three-body problem

Fig. 7 shows that using the real mass of Jupiter that the orbit of Earth is stable over the entire time of the simulation, which was 100 years. In (c) it is seen that the motion of the Earth does deviate slightly from tracing the exact same path. This is something we would expect to observe since the addition of Jupiter is steering the Earth off its original elliptical orbit. This is a positive result for the velocity Verlet method. Fig. 8, where Jupiter has a mass 10 times its real mass, the orbit is slightly less stable compared to fig. 7(c), suggesting that the Earth's orbit will more noticably deviate from its original orbit in time. Since fig. 8 (b) shows a higher line density at the edges this might suggest that there is some kind of oscillatory pattern of the orbit between each edge of the lines in the figure. Fig. 9, where Jupiter has a mass 1000 times its real mass, the orbit of the Earth is very much affected by Jupiter's presence. Earth's orbit is highly unstable. It is worth mentioning that 1000 Jupiter masses is almost the same mass as the Sun, so it affects Earth's orbit with as much force as the Sun does. In this case the assumption of a stationary Sun is unrealistic because of Jupiter's high mass, and if the Sun was not fixed in this simulation then the current system should end up looking more like a binary solar system with Earth following a more complex path compared to its original elliptical or approximately circular orbit.

5.5 The solar system including all planets

The results seen in Fig. 10 clearly shows that the results of our simulation is incorrect. We would expect all of the planets to have approximately elliptical orbits, but this was not the result when we ran our simulation. It shows, at several points in many of the planet orbits, sudden, drastic changes in direction. This should not happen in a gravitational system such as this one. The points at which this happens for the different orbits seem random. It is possible that the calculation of the acceleration in the Verlet solver were incorrect at certain points. The masses of the planets were checked to make sure that there was no division of zero when calculating the acceleration from the force, and all the planets had their correct values (in solar masses). It was attempted to remove some of the lighter planets (Mercury, Venus and Pluto), but the remaining orbits were still incorrect. Still, it is essentially the same code used for the three-body problem. When running the code with adding just Earth and Jupiter, the results from the simulations showed expected, approximately circular orbits. When performing the simulation with only Earth, Mars and Jupiter the orbits were incorrect. When performing the simulation with Earth, Jupiter and Uranus, the orbits were also incorrect. When attempting to simulate the three-body problem with Saturn and Uranus only, the result was correct for Saturn but incorrect for Uranus. We suspect that is has something to do with adding the forces from multiple planets to the sum of forces acting on a planets, since the code works when running for single planets, but we were unable to fix the bug in time.

5.6 Precession of Mercury's perihelion

The relativistic correction to the force as shown in Eq. 32 was used in order to simulate the relativistic gravitation on Mercury. In Fig. 11 (a) it was at first believed that Mercury was precessing; however, when making the timestep smaller as seen in fig. 11 (b), the orbit is stable. In addition, fig. 12, where the perihelion position has been plotted with and without the correction term, show unexpected results. This suggests that there is some error in the code which results in an unusual repetition of points during a simulation of 100 years ($dt \approx 5$ minutes). It is expected that without the relativistic correction term that in a century the perihelion shift should be approximately 527 arc seconds, and 570 arc seconds with the correction terms added. As shown in Fig. 13 the perihelion shift does not give the values expected with and without the correction term. The perihelion repeats points and only reaches a maximum angle of 80 arc seconds in both Fig. 13(a) and (b) after one century. This numerically suggests that there is zero perihelion shift between Newtonian gravitation and general relativistic gravitation. The time resolution of the simulation was tested with dt up to 1e-6 ($\approx$ 31 seconds) and found little change in the results.

6 Conclusion

In conclusion, the velocity Verlet method was found to be more numerically accurate and not much more computationally expensive compared to the forward Euler method for simulating the motion of the Earth in the Earth-Sun system. Approximately the forward Euler method required 1000 times more points than the velocity Verlet method to produce circular orbits.

The escape velocity of a body at a distance 1 au from the Sun was found by trial and error with our simulation, and it was found to match the analytical expression for the escape velocity to four significant digits. This test is considered a success, and it is one piece of evidence that the simulation is accurate.

Using different forms of the force law with the exponent β had consequences for the conservation of total energy and angular momentum. For $\beta \gtrsim 2.9$ the Earth was flung out of orbit, while for $\beta \lesssim 2.8$ the Earth remained in orbit, and it was seen that the total energy increased at the closest points in the orbit, and larger β values gave slightly shorter orbital periods.

The angular momentum for the same simulation, plotted in fig. 4, shows that the angular momentum for $\beta = 3$ is unstable. Though the y-axis shows that the variation is of the order 1e-10 and the numerical value is $L \approx 1.5012$e-5, the variation in L for $\beta = 3$ is not confined and looks like it could increase further in time. In 4 (b) is is seen that the angular momentum for $\beta = 2.9$ shows variation on the scale of 1e-15, and in 4 (c) the angular momenta for $\beta \leq 2.8$ show some variation on the scale of 1e-17. But this is negligible compared to the actual magnitude of $\sim$1e-5, so it is reasonable to say that for $\beta \leq 2.9$ the angular momentum is constant according to our simulations.

After successfully simulating the binary system of the Earth and Sun, the model was applied to the three-body problem (Earth, Jupyter and the Sun). This produced expected results as the orbit of the Earth and Mercury was stable for the duration of the one century simulation. When moving to more bodies and the entire Solar System, the model gave unexpected results. None of the planetary orbits were stable and many showed sudden changes in direction, suggesting systematic error in the algorithms. Applying Einsteins theory of gravitation it was attempted to simulate the perihelion precession of Mercury. However, systematic uncertainty is found in the results and the analytic perihelion shift expected (43 arc seconds) compared with Newtonian gravitation was not found. Rather, the calculations in this report produced the result that there is no perihelion shift.

7 Github address

Github address: `https://github.com/amundwf/comp-phys-project3`

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