Simulating the Motion of the Solar System with the Velocity Verlet and Forward Euler Methods

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October 24, 2020

Abstract

The Solar System through history has been an area of great interest. Einstein's theory of gravitation in the last century brought new light to classically unexplainable observations of the universe. Here the Solar System is simulated to test theory against experiment. It is found that the Velocity Verlet is more computational efficient for the task of simulating the motion of planets over Euler's forward method. The Velocity Verlet method is found to be conserve total energy and angular momentum, conforming to expectations of a physical system. The computation for the perihelion shift of Mercury could not be compared with Einstein's theory of gravitation due to systematic error. The expected shift per century is 43 arc seconds, however the results calculated here give a shift of 0 arc seconds.

1 Introduction

The study of the Solar System has intrigued philosophers and scientists for centuries, even those who had difficulty observing it. Johannes Kepler, the assistant to Tycho Brahe, had smallpox at a young age and as a result his eye sight was never very good. Kepler's theories of the Solar System brought very different ideas to his predecessor who believed in geocentricity, that the Sun orbited the Earth. Kepler's theory that all planets follow an ellipse with the Sun at one focus, fitted the heliocentric thinking (that the Earth orbited the Sun). Modelling the Solar System since then till now has still been of great interest. Einstein's theory of general relativity for example, suggests that the observed rotation of the elliptical path that Mercury takes, cannot be explained with Newtonian mechanics alone. Modelling it must include some general relativistic correction. Here, this report builds on these theories to model the Solar System and runs experiments to test how they compare to the real physical system.

The motion of the Solar System is simulated by using two methods for solving ordinary differential equations; the Velocity Verlet method and Euler's method. The Velocity Verlet method is very popular in the field of particle dynamics for updating the position of rigid bodies [1] since the method is less sensitive to change in step size and improves the numerical accuracy compared with Euler's method at little extra computational cost [2]. Both methods entail writing a Taylor expansion which leads to the discretisation of the continuous function and, hence, adds a truncation error. The accuracy and computational efficiency of both methods are compared and discussed in this report. The code scripts used in this report are mainly developed with the Velocity Verlet method and so Euler's method is discussed minimally.

In isolated physical systems angular momentum and total energy should be conserved. The gravitational force is a conservative force and therefore the total energy is conserved if no other forces are acting on the planet [3]. To what extend the Velocity Verlet method conserves angular momentum and energy is investigated. A further test is to look at the escape velocity of the Earth and compare this with the analytic result. Following this, the Velocity Verlet method is applied to the three-body problem and eventually the whole Solar System. The three-body problem is one which appears in many contexts in nature [4]. A century ago it was still a legitimate question if the gravitational force of a third body, like the Moon or another planet, would disrupt the Earth's orbit and cause it to fall into the Sun. Or even worse, that the Moon would soon fall into the Earth. This problem was given such importance that the King of Sweden proclaimed there was a prize for the solution [4].

Finally the unusual motion of Mercury is investigated. As briefly mentioned, Newtonian physics does not explain the observed precession or rotation of Mercury's perihelion (point of closest approach). For this reason, a relativist correction term is added to the model to account for this shift. The perihelion shift is calculated analytically and the results of numerically adding the relativistic correction term is discussed.

2 Theory

2.1 Units

Since we are investigating planetary orbits, which are on astronomical scales, it's useful to use larger units in order to make the code and the results more readable. Therefore we use the length unit astronomical units, $1\,\mathrm{au} = 149\,597\,870\,700\,\mathrm{m} \approx 150$ million km, the time unit is years, $1\,\mathrm{yr} = 31\,556\,926\,\mathrm{s}$, and the mass unit is solar masses, $1M_\odot = 1.98847 \cdot 10^{30}$ kg. These are the units we have used in our program.

2.1.1 The gravitational constant

Since the units of mass, length and time have been changed, this also affects the numerical value of the gravitational constant G. By approximating Earth's orbit as perfectly circular, and therefore also assuming a constant orbital speed when taking into account Kepler's second law [3], the gravitational force from the Sun gives Earth a centripetal acceleration a_c:

$$
\begin{aligned}
F_g &= M_\mathrm{E} a_\mathrm{c} \\
\Rightarrow \frac{G M_\odot M_\mathrm{E}}{r^2} &= M_\mathrm{E} \frac{v^2}{r} \\
\Rightarrow G M_\odot &= v^2 \cdot r \\
&= \left(\frac{2\pi \cdot 1\,\mathrm{au}}{1\,\mathrm{yr}} \right)^2 \cdot 1\,\mathrm{au} \\
&= 4\pi^2 \frac{\mathrm{au}^3}{\mathrm{yr}^2} \\
\Rightarrow G &= 4\pi^2 \frac{\mathrm{au}^3}{\mathrm{yr}^2\, M_\odot},
\end{aligned}
$$

Which means that the numerical value of G in our programs will be $4\pi^2$. This can also be seen from converting G from SI units to the new units in the following way:

$$1\,\mathrm{kg} = \frac{1}{1.98847 \cdot 10^{30}}\, M_\odot = 5.02899214 \cdot 10^{-36}\, M_\odot \tag{1}$$

$$1\,\mathrm{m} = \frac{1}{149\,597\,870\,700}\,\mathrm{au} = 6.684587122 \cdot 10^{-12}\,\mathrm{au} \tag{2}$$

$$1\,\mathrm{s} = \frac{1}{31\,556\,926}\,\mathrm{yr} = 3.168876462 \cdot 10^{-8}\,\mathrm{yr}, \tag{3}$$

which gives

$$
\begin{aligned}
G &= 6.67408 \cdot 10^{-11}\, \frac{\mathrm{m}^3}{\mathrm{s}^2\,\mathrm{kg}} \\
&= 6.67408 \cdot 10^{-11} \cdot (6.684587122 \cdot 10^{-12}\,\mathrm{au})^3 \cdot (3.168876462 \cdot 10^{-8}\,\mathrm{yr})^{-2} \\
&\quad \cdot (5.02899214 \cdot 10^{-36}\, M_\odot)^{-1} \\
&= 39.47513264321821\, \frac{\mathrm{au}^3}{\mathrm{yr}^2\, M_\odot}.
\end{aligned}
$$

The numerical value of $4\pi^2$ is ≈ 39.4784176, which is nearly identical to the numerical value calculated above.

2.1.2 Energy and angular momentum

Because of the units we use in our programs, the units of energy and angular momentum will have different numerical values from the SI units. The new units will be referred to here as PU (program units). The SI unit of energy is joules, $\mathrm{J} = \mathrm{N\,m} = \mathrm{kg\,m^2\,s^{-2}}$. Substituting our new units, we get

$$\begin{aligned}
[E]_{\mathrm{SI}} = \mathrm{J} &= 1\,\mathrm{kg} \cdot (1\,\mathrm{m})^2 \cdot (1\,\mathrm{s})^{-2} \\
&= 5.02899214 \cdot 10^{-36}\, M_\odot \cdot (6.684587122 \cdot 10^{-12}\,\mathrm{au})^2 \cdot (3.168876462 \cdot 10^{-8}\,\mathrm{yr})^{-2} \\
&\approx 2.237790962 \cdot 10^{-43}\, M_\odot\,\mathrm{au}^2\,\mathrm{yr}^{-2} \\
&= 2.237790962 \cdot 10^{-43}\, [E]_{\mathrm{PU}}.
\end{aligned}$$

or

$$[E]_{\mathrm{PU}} = 4.46869264\, [E]_{\mathrm{SI}} \tag{4}$$

The SI unit of angular momentum is $[L] = \mathrm{kg\,m^2\,s^{-1}} = \mathrm{J\,s}$. In our new units this becomes

$$\begin{aligned}
[L]_{\mathrm{SI}} &= 1\,\mathrm{kg\,m^2\,s^{-1}} \\
&= 1\,\mathrm{J} \cdot 1\,\mathrm{s} \\
&= (2.237790962 \cdot 10^{-43}\, M_\odot\,\mathrm{au}^2\,\mathrm{yr}^{-2}) \cdot (3.168876462 \cdot 10^{-8}\,\mathrm{yr}) \\
&= 7.09128311 \cdot 10^{-51}\, M_\odot\,\mathrm{au}^2\,\mathrm{yr}^{-1} \\
&= 7.09128311 \cdot 10^{-51}\, [L]_{\mathrm{PU}}.
\end{aligned}$$

or

$$[L]_{\mathrm{PU}} = 1.41018203\, [L]_{\mathrm{SI}} \tag{5}$$

All of the plots of energy and angular momentum in this report uses program units, so equations (4) and (5) can be used to convert these values to SI units.

2.2 Forward Euler method

2.3 Velocity Verlet method

The velocity Verlet method assumes that the acceleration only depends on the position and not on the velocity. In this case, where gravity is the only acting force and there is no air resistance, this is true.

2.3.1 Exact solution of the escape velocity

An object is said to have escaped the orbit of the Sun (or any other gravitational object) if it will has a velocity greater than or equal to zero infinitely far away from the Sun, meaning that it will never return to the Sun in the future. Using the equation for potential energy we can calculate the necessary condition for a planet to escape orbit around the Sun.

Since gravity is assumed to be the only force acting on the planet and gravity is a conservative force, the total energy of the orbit is conserved. That is,

$$E_{\mathrm{tot}} = U + K = \text{constant} \tag{6}$$

where

$$U = U(r) = -\frac{GM_\odot m}{r} \tag{7}$$

is the gravitational potential energy of a planet with mass m at distance r from the center of the Sun, and

$$K = \frac{1}{2}mv^2 \tag{8}$$

is the kinetic energy of the planet, where v is the orbital speed of the planet. The escape velocity is defined as the lowest velocity at which the planet escapes orbit. The planet escapes orbit if it reaches a point in space where its potential energy is non-negative. Gravitational potential energy is in general negative at a finite distance away from the object, but at $r = \infty$ it is zero. By the definition of the escape velocity, the planet will have zero velocity at infinity. In mathematical terms, the equation for the escape velocity is

$$\begin{aligned} U_0 + K_0 &= U_\infty + K_\infty \\ -\frac{GM_\odot m}{r_0} + \frac{1}{2}mv_{\text{esc}}^2 &= 0 + 0 \\ \Rightarrow v_{\text{esc}}^2 &= \frac{2GM_\odot}{r_0} \\ v_{\text{esc}} &= \sqrt{\frac{2GM_\odot}{r_0}}, \end{aligned}$$

independent of the planet mass m.

2.4 Conservation of angular momentum

Although the Earth's orbit around the Sun is nearly circular it is in fact, elliptical. This is more pronounced in a very elongated orbit or an orbit with a high eccentricity. Eccentricity, e, is equal to 0 for a circle, and equal to 1 where the ellipse extends to infinity (a parabola). At the planets closest approach (perihelion) the planets has its greatest angular velocity, and at the planets furthest distance from the Sun (aphelion) the angular velocity is the smallest. Kepler's second law states that a vector $\mathbf{r} \equiv r\hat{r}$ extending from the Sun to a planet in its orbit sweeps the same amount of area at any point in the orbit for equal time intervals. The result of this directly shows that angular momentum is conserved throughout this orbit.

In an infinitesimal time dt the area dA that is swept over by the vector is approximately the area of a right triangle with base length r and height approximately equal to the arc length $ds = r(t) \cdot \omega(t)\, dt$ where $\omega(t) \equiv d\theta/dt$ is the angular frequency of the planet's orbit at time t. This triangle has the area

$$dA = \frac{r \cdot r(t)\omega(t)\, dt}{2} = \frac{1}{2}r(t)^2\omega(t)\, dt, \tag{9}$$

so the rate at which the area is swept is

$$\frac{dA}{dt} = \frac{1}{2}r(t)^2\omega(t). \tag{10}$$

where r is the radius and θ is the angle swept out by the planet. According to Kepler's second law, this area rate is constant, so we have

$$\frac{1}{2}r^2\omega = \text{constant}, \tag{11}$$

that is, this expression has the same value for all values of t.

Angular momentum is given by

$$\mathbf{L} = \mathbf{r} \times (m\mathbf{v}) = mrv\,\hat{r} \times \hat{v} = mrv\hat{\omega} \tag{12}$$

where $\mathbf{r}$ is the planet's position vector, $\mathbf{v} = v\hat{v}$ is the velocity of the planet and $\hat{\omega}$ is the direction of the angular velocity vector $\boldsymbol{\omega} = \omega\hat{\omega}$, obtained from the right hand convention for the angular velocity vector. The orbit speed v in an elliptical orbit is given by

$$v = \frac{ds}{dt} \approx \frac{r \cdot d\theta}{dt} = r\omega, \tag{13}$$

where the approximation symbol has been used since a small orbit path distance Δs is not exactly equal to $r\,\Delta\theta$ in the case of a non-circular orbit. Substituting eq. (13) into eq. (12) we get

$$\begin{aligned}\mathbf{L} &= mrv\hat{\omega} \\ &= mr \cdot (r\omega)\hat{\omega} \\ &= mr^2\omega\hat{\omega} \\ &= 2m \cdot \left(\frac{1}{2}r^2\omega\right)\hat{\omega} \\ &= (2m \cdot \text{constant})\,\hat{\omega}\end{aligned}$$

where the last equation comes from eq. (11). The mass m is assumed to be constant. Also, since the only assumed force is the gravitational pull of the Sun, there will be no forces perpendicular to the plane of the orbit, which means that the orientation of the orbit will be constant in time, so the direction $\hat{\omega}$ will be constant in time. So every quantity in the expression for $\mathbf{L}$ is constant, so we have

$$\mathbf{L} = \text{constant}. \tag{14}$$

2.5 Different forms of the gravitational force law

$$F_g = \frac{Gm_1m_2}{r^\beta}. \tag{15}$$

2.6 Perihelion Precession of Mercury

Mercury is the planet orbiting closest to the Sun in the Solar System. Its orbit also has the highest eccentricity of the planets. The ellipse that Mercury traces as it orbits the Sun, however, is not always the same. The perihelion of Mercury precesses, or rotates, around the Sun. This observation is mainly due to the other planets in the Solar System, which cause the Mercury to deviate from tracing the same path. The perihelion shift calculated due to the other planets is on average 526.7 arc seconds per century, however by observation the shift was found to be 565 arc seconds (from the 40 years of observation at the Paris observatory) and 570 arc seconds from modern data [5]. This discrepancy of 42.3 arc seconds per century was not solved until in 1915 when Einstein published his theory of gravitation which accounted for a shift of 43 arc seconds per century [6].

According to [7], the perihelion shift in radians per revolution due to Einstein theory of gravitation should be:

$$\sigma = \frac{24\pi^3L^2}{T^2c^2(1-e^2)} = \frac{6\pi GM}{c^2L(1-e^2)} \tag{16}$$

where T is the time period, L is the semi-major axis. Using a geometrical relation $L(1-e^2)$ equals

$$= \frac{2}{1/a + a/b} \tag{17}$$

where a is the aphelion and b is the perihelion. Hence the result leads to

$$\sigma = \frac{3GM\pi(1/a + 1/b)}{c^2} \tag{18}$$

which for a planet like Mercury results in $\sigma = 2.887\text{e-}5$ deg per revolution or 0.012 deg per century. When this is converted to arc seconds it is approximately 43 arc seconds. In this report in order to reproduce these calculation, a relativistic correction term is added to the gravitational force:

$$F_G = \frac{GM_{\odot}M_{Mercury}}{r^2}(1 + \frac{3l^2}{r^2c^2}) \tag{19}$$

where r is the relative distance between the Sun and Mercury, l is Mercury's angular momentum per unit mass and c is the speed of light in vacuum.

3 Method

4 Results

4.1 Stability of the Velocity Verlet & Euler Algorithm

The Velocity Verlet method was tested with various time steps between 1 yr and 5 minutes, using a final time of 10 yrs. Fig. 1 shows that when the step size is approximately 4 days the orbit appears circular. The initial velocity used to start the algorithm was $v_{init} = 2\pi$ AU/yr, since $v = 2\pi R/T$ where R is the radius of orbit and T is the time period. Time steps between 37 days and 5 minutes were used for the forward Euler method. Fig. 2 shows that when $dt \approx 5$ minutes that the orbit appears circular. This shows that the forward Euler needed much smaller time steps before reaching an orbit which is roughly circular compared with the Velocity Verlet method. Table. 1 shows the timings of each algorithm with $10^5, 10^6$ and 10^7 time steps. The table shows that neither algorithm is much computational faster than the other.

4.2 Unit tests

4.2.1 Conservation of total energy

$$E = U(t) + K(t) = U(t) + \frac{1}{2}mv(t)^2 = \text{constant} \tag{20}$$

Performing the Velocity Verlet method with the parameters: N = 1e5, dt = 1e-4 (plotted in Fig. 1g) the kinetic and potential energy were found to be conserved. As shown in the figure, the orbit is circular. It was expected that both quantities are conserved since the distance between the Earth and Sun is unchanged. In the case of an elliptical orbit, the Sun-Earth distance changes and so do the potential and kinetic energy, yet the total energy should be conserved. This result confirms that the Velocity Verlet algorithm does conserve energy.

4.2.2 Conservation of angular momentum

To test the conservation of angular momentum, the same parameters were used (N = 1e5, dt = 1e-4). As shown by the following equation, the angular momentum was computed by taking the cross product of the velocity vector and the radial vector, multiplied by the Earth's mass at each time step. The results was that angular momentum was constant throughout the simulation.

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times (m\mathbf{v}) = \text{constant} \tag{21}$$

Both these quantities, the conservation of energy and angular momentum, are important tests in the algorithm to reflect the behaviour of a physical system. If energy or momentum were to be lost during the simulation, it would lead to very different results over a period of time to the real solar system.

4.3 The varying force law

Here, results for implementing the varying force law presented in equation (15) for different values of β is presented. We have tested the force laws for $\beta \in [2, 3]$.

Fig. 3 shows the total energy of the Earth-Sun system with different values of β. In Fig. 3(c) the figure shows that when using $1/r^2$ that the total energy of the Earth is constant throughout its orbit

Table 1: Timing the Velocity Verlet and Euler forward methods with different number of time steps. The results are given in seconds and are the average of four runs each.

Number of time steps	10^5	10^6	10^7
Forward Euler	0.0781	0.883	8.803
Velocity Verlet	0.0938	0.879	8.823

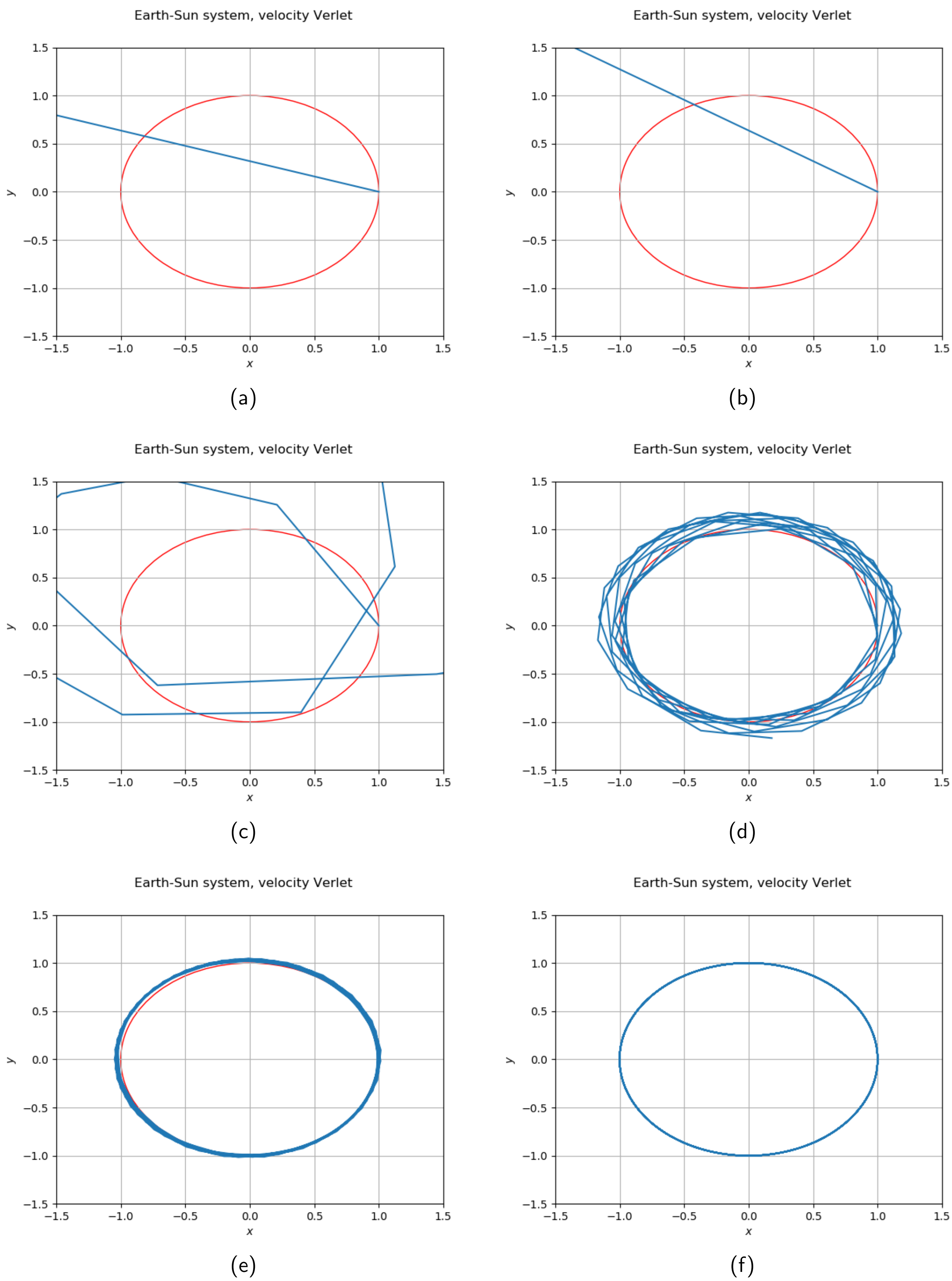
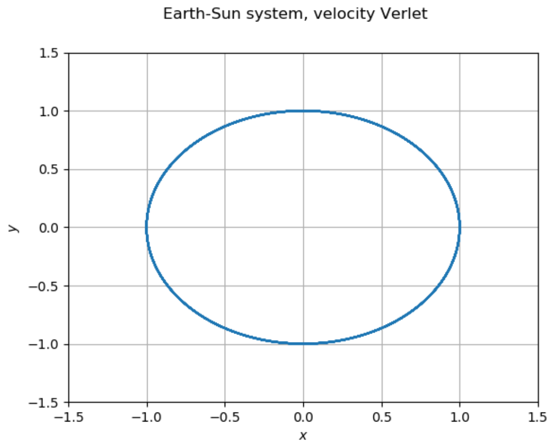
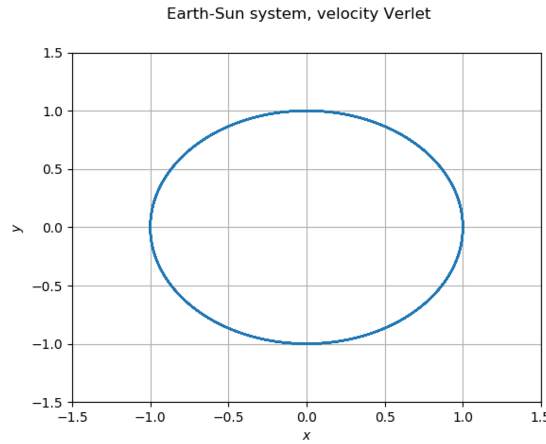


Figure 1: Plots of the the solar system solved using the Velocity Verlet method. The final time is 10 years with several time steps, dt. (a) $dt = 1$ yr and $N = 10$, (b) $dt = 0.5$ yr ≈ 183 days and $N = 20$, (c) $dt = 0.2$ yr ≈ 73 days and $N = 50$, (d) $dt = 0.1$ yr ≈ 37 days and $N = 100$, (e) $dt = 0.05$ yr ≈ 18 days and $N = 200$, (f) $dt = 0.01$ yr ≈ 4 days and $N = 1000$. The blue line indicated the orbit of the Earth, the red line is a circle plotted for comparison.



(g)



(h)

Figure 1: Plots of the the solar system solved using the Velocity Verlet method. The final time is 10 years with several time steps, dt. (g) dt = 1e-4 yr ≈ 53 minutes and $N = 1e5$, (h) dt = 1e-5 yr ≈ 5 minutes and $N = 1e6$. The blue line indicated the orbit of the Earth, the red line is a circle plotted for comparison.

Table 2: Finding the escape velocity of a planet starting at a distance 1 au from the Sun.

Initial speed, km/s	10	20	30	40	42	42.1	42.125	42.15	42.2	42.5	43	45	50	100
Escaped orbit?	no	no	no	no	no	no	yes	yes	yes	yes	yes	yes	yes	yes

around the Sun. For circular orbits the potential energy and kinetic energy were constant, although this is not plotted.

Figure 4 shows the absolute value of the angular momentum, $L = |\mathbf{L}|$, as a function of time for different values of β.

Figure 5 shows the distance between the Sun and the Earth as a function of time for different values of β. For $\beta \gtrsim 2.88$ the Earth escaped orbit in our simulation. Therefore, only results for values of β below this have been shown in the results figures.

When computing the same result for 20 years, the same oscillatory between a distance of 1 au and a distance closer to 0 continued in the same manner as for the 5 year simulation. Figure 6 shows the orbits of Earth for different values of β over a time period of 3 years.

4.4 Escape velocity

4.4.1 Trial and error

By running the velocity Verlet algorithm with initial position $\mathbf{r}_{\text{init}} = (1, 0, 0)$ and initial velocity $\mathbf{v}_{\text{init}} = v_{\text{init}} \cdot (0, 1, 0) = (0, v_{\text{init}}, 0)$ for several values for the initial speed v_{init}, the escape velocity was found to be approximately $v_{\text{esc}} \approx 42.125\,\text{km/s}$. The trial and error process was done in a bisection fashion, approximately halving the interval with each guess. Whether the planet had escaped from orbit or not was checked simply by visually inspecting the plot and having the end time sufficiently high (ranging from 500 to 10 000 years). The results can be seen in table 2 where the velocity guesses has been sorted in ascending order. Different masses were tested and the results were the same for all masses. The explanation of this is that the acceleration is $a = F_g/m$, so the factor m in the gravitational force F_g is cancelled and the acceleration is independent of the planet's mass.

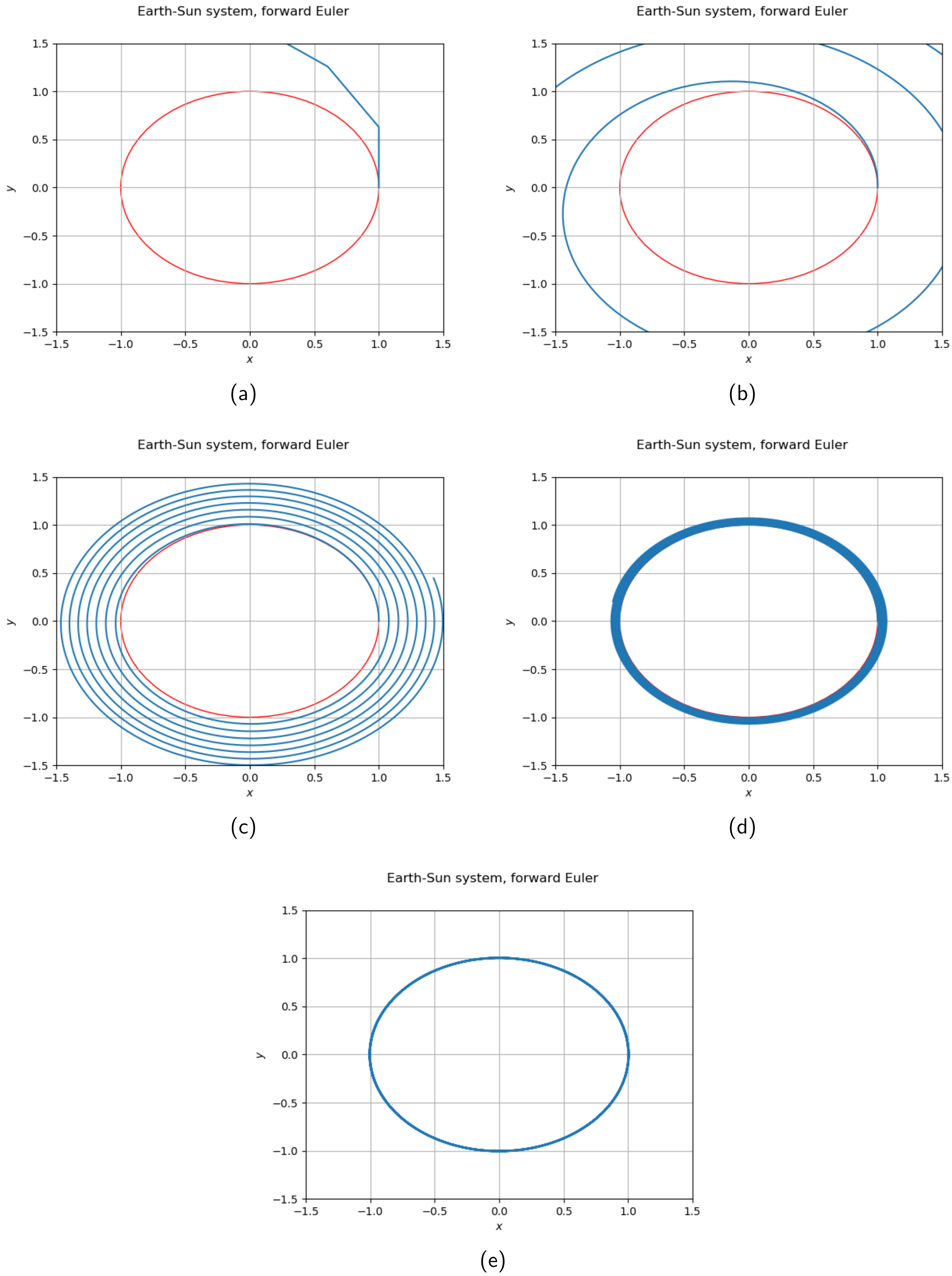
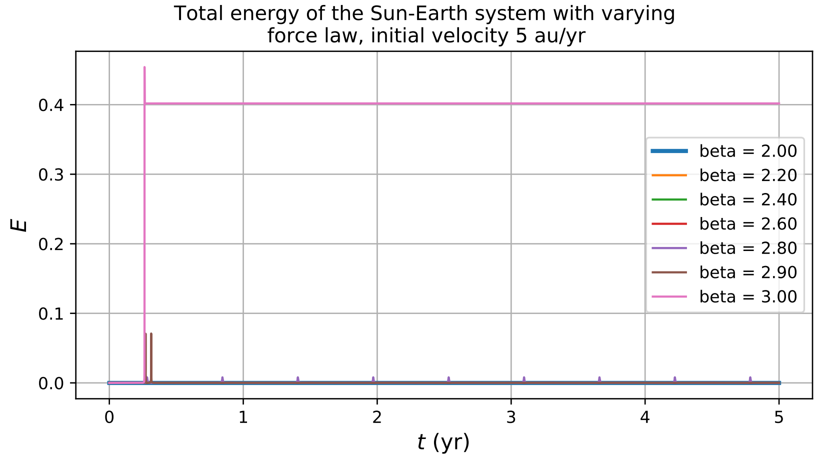
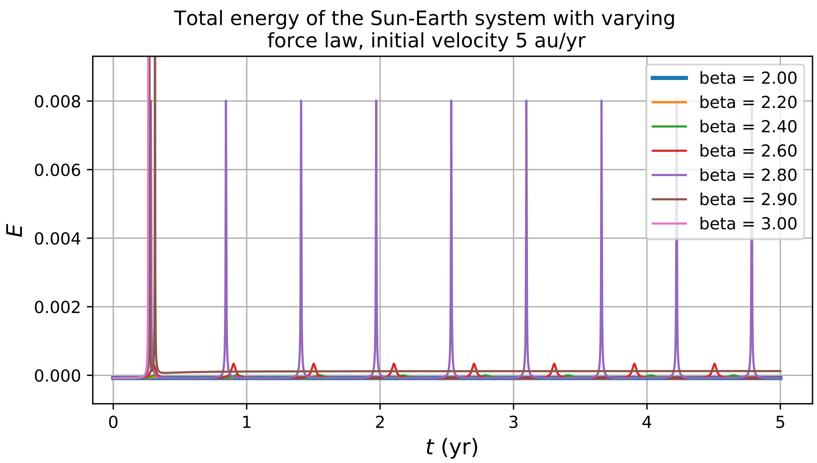


Figure 2: Plots of the the solar system solved using Euler's method. The final time is 10 years with several time steps, dt. (a) dt = 0.1 yr ≈ 37 days and $N = 100$, (b) dt = 0.01 yr ≈ 4 days and $N = 1000$, (c) dt = 1e-3 yr ≈ 9 hours and $N = 1e4$, (d) dt = 1e-4 yr ≈ 53 minutes and $N = 1e5$, (e) dt = 1e-5 yr ≈ 5 minutes and $N = 1e6$.



(a)



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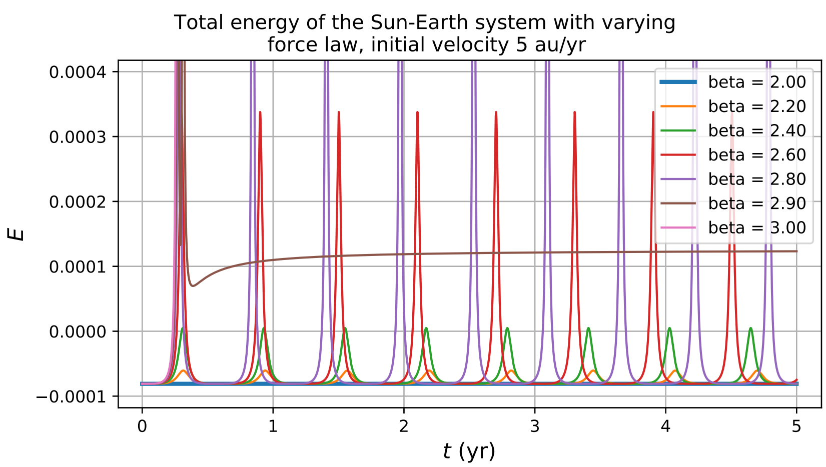
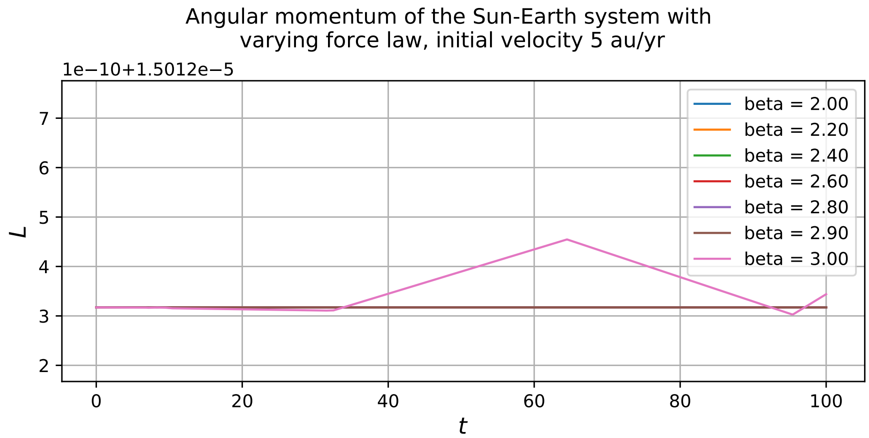
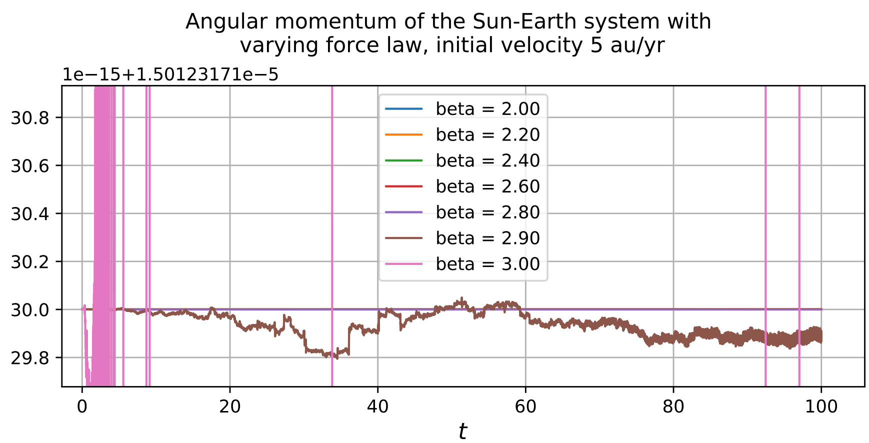


(c)

Figure 3: The total energy (kinetic plus potential energy) of Earth in orbit around the Sun for different values of β. This figure shows a simulation of a 5 year period. (a) The figure zoomed out, where all graphs are visible. (b) The same figure, zoomed in to make more details visible. (c) The same figure, further zoomed in.



(a)



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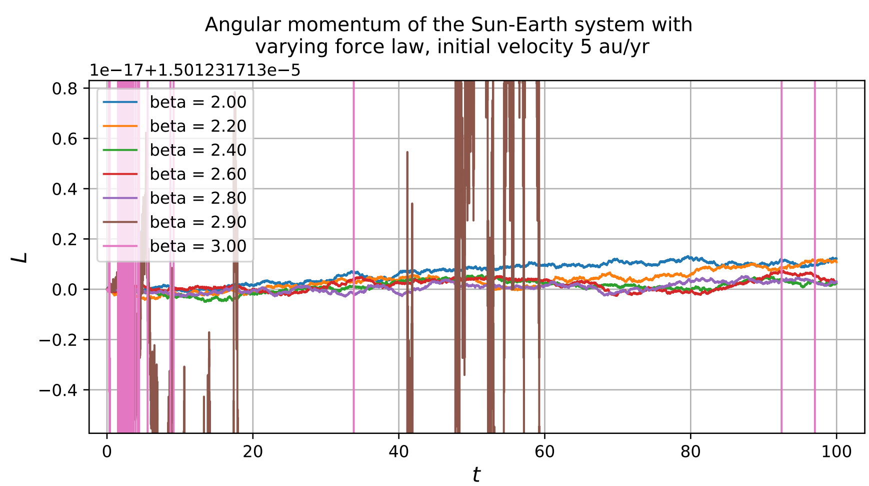


(c)

Figure 4: The magnitude of the angular momentum of Earth in orbit around the Sun for different values of β. The system is simulated over a 100 year period. (a) The figure zoomed out, where all graphs are visible. (b) The same figure, zoomed in to make more details visible. (c) The same figure, further zoomed in.

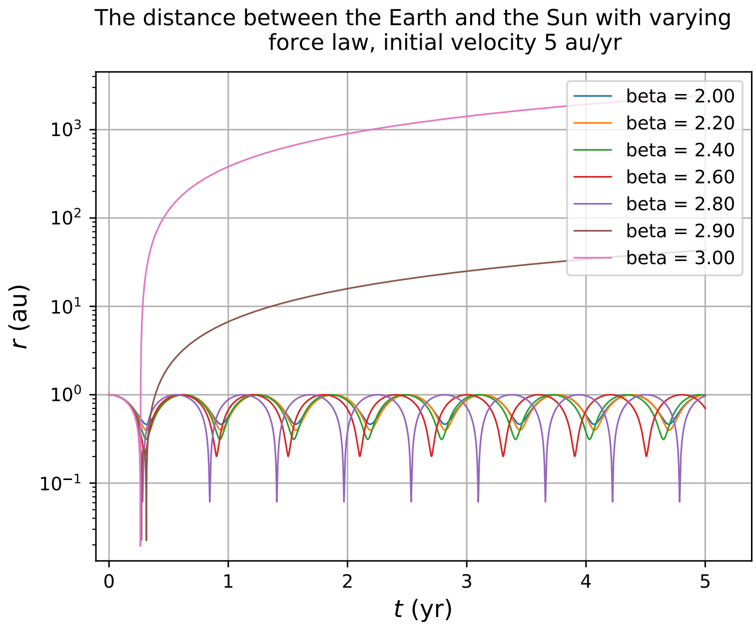


Figure 5: The distance between the Earth and the Sun simulated over a 5 year period with different values of β in the force law. The initial orbit speed is 5 au. The y-axis is on a logarithmic scale.

4.4.2 Exact solution of the escape velocity

Equation (**??**) gives the analytical expression for the escape velocity. For an initial distance of $r_0 = 1\,\mathrm{au}$ we get, using SI units for G, $M_\odot$ and r_0,

$$v_{\mathrm{esc}} = 42\,121.9\,\mathrm{m/s} \approx 42.122\,\mathrm{km/s}. \tag{22}$$

4.5 The three-body problem

In figure 7 the orbits of Earth and Jupiter is plotted. The initial conditions of Earth was position $r_{\mathrm{init,E}} = (1, 0, 0)$ in units au, and velocity $v_{\mathrm{init,E}} = (0, 2\pi, 0)$ in units au/yr. The initial conditions of Jupiter was position $r_{\mathrm{init,J}} = (5.2044, 0, 0)$, and velocity $v_{\mathrm{init,J}} = (0, 2.75522, 0)$. In figure 8 the three-body problem has been simulated with Jupiter's mass being 10 times its real mass, and in figure 9 the three-body problem has been simulated with Jupiter's mass being 1000 times its real mass.

4.6 The solar system including all planets

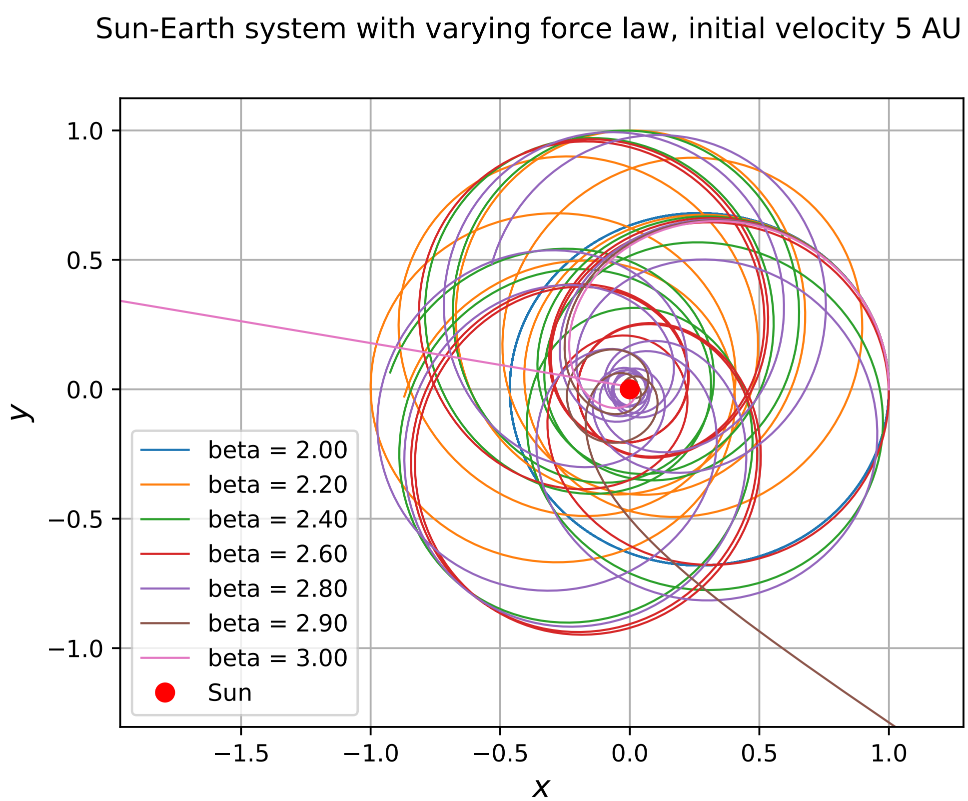


Figure 6: The orbit of the Earth around the Sun simulated over a 3 year period with different values of β in the force law. The initial orbit speed is $5\,\mathrm{au/yr}$ for all orbits.

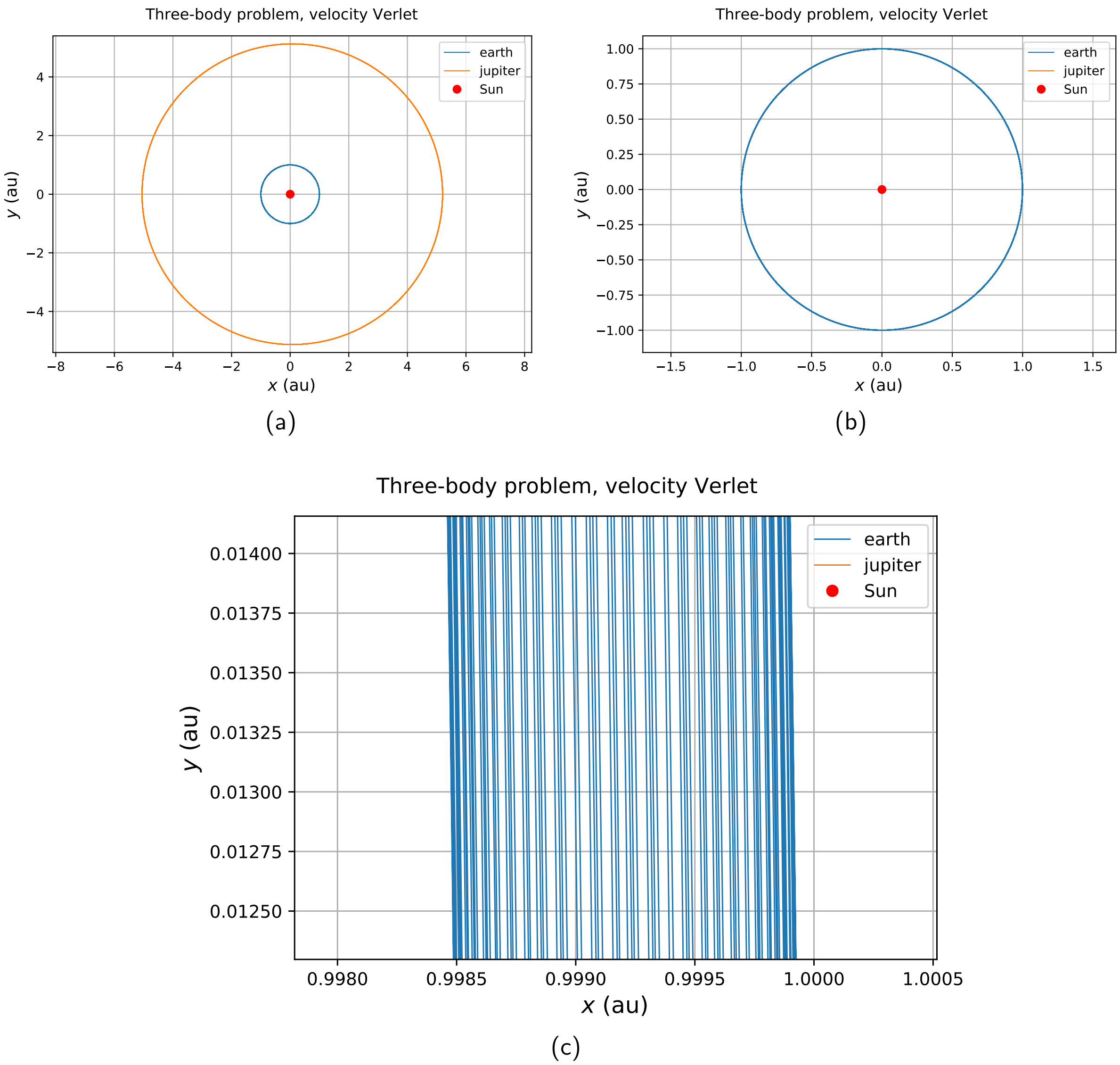
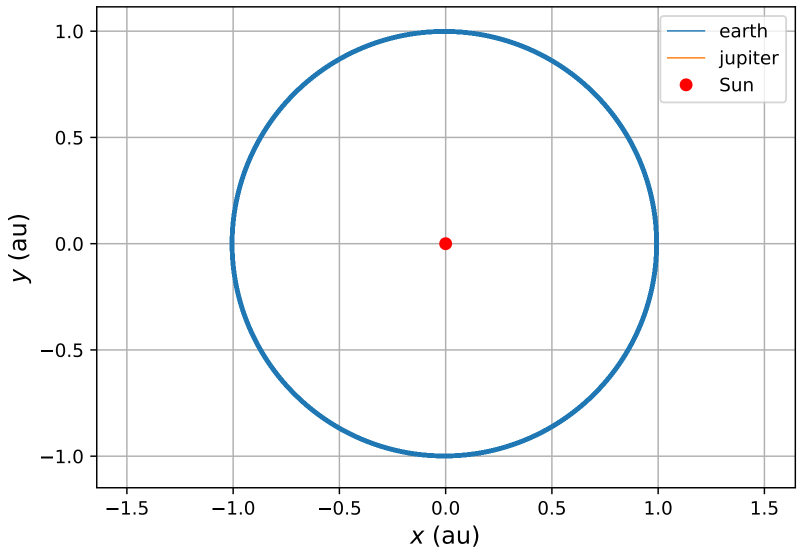
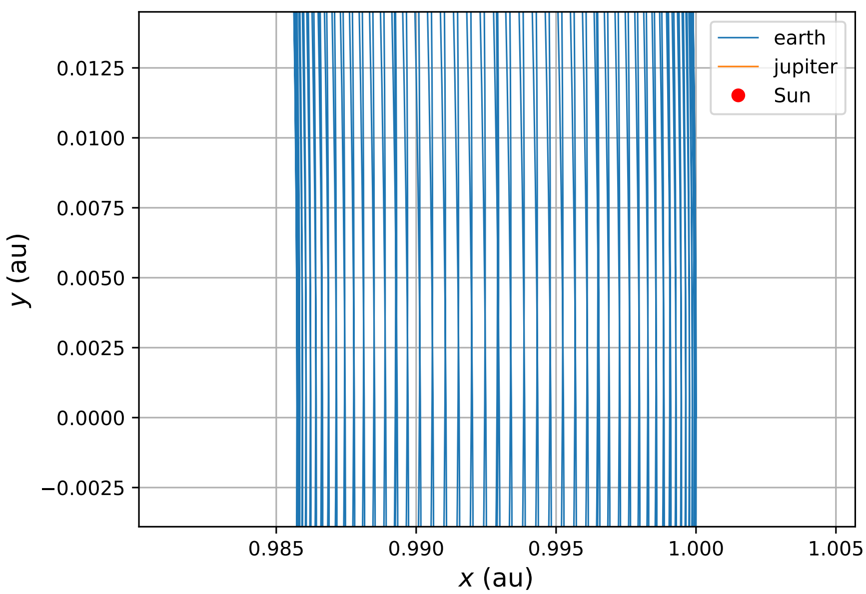


Figure 7: The orbits of Earth and Jupiter in an interacting three-body system between the Sun, the Earth and Jupiter, with Jupiter's mass being its real mass. The simulation is over a 100 year period. (a) Full view including Jupiter's orbit. (b) Zoomed in view of Earth's orbit. (c) Zoomed in view of a small portion of Earth's orbit for better visibility of Jupiter's effect on Earth's orbit.

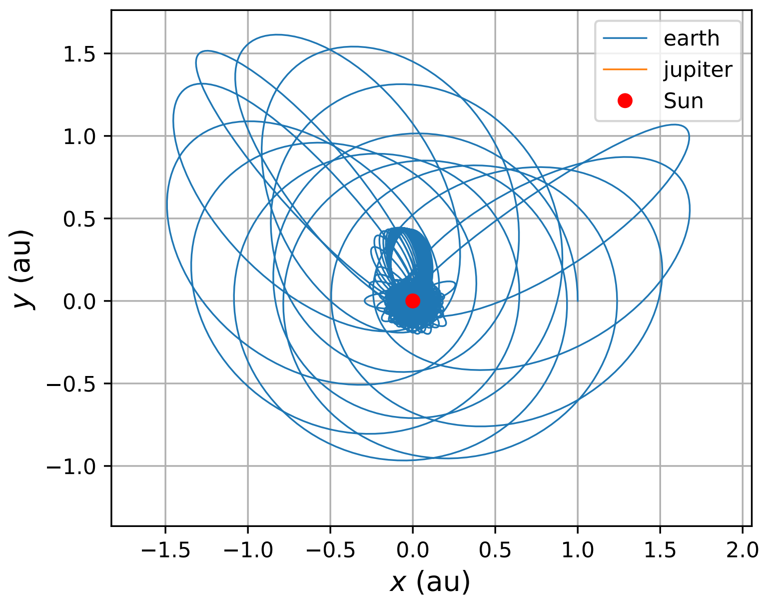


(a)

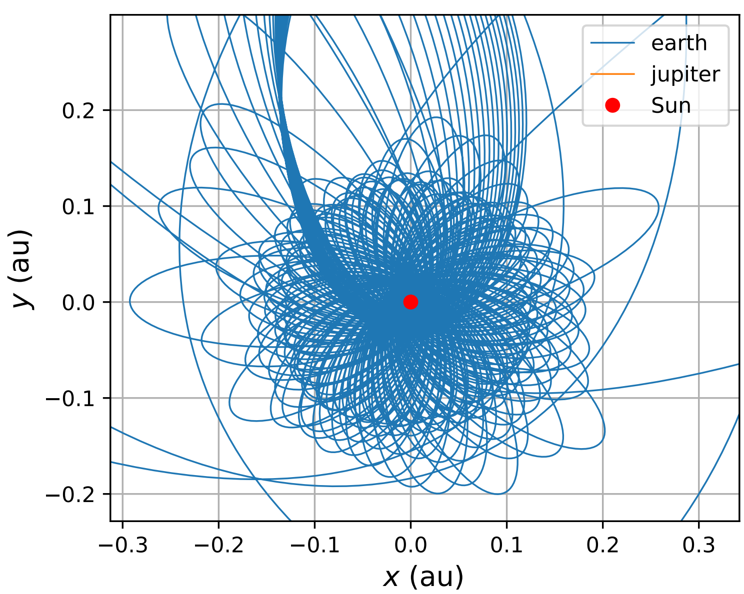


(b)

Figure 8: The orbits of Earth and Jupiter in an interacting three-body system between the Sun, the Earth and Jupiter, with Jupiter's mass being 10 times its real mass. The simulation is over a 100 year period. (a) View of Earth's orbit. (b) Zoomed in view of a small portion of Earth's orbit for better visibility of Jupiter's effect on Earth's orbit.



(a)



(b)

Figure 9: The orbits of Earth and Jupiter in an interacting three-body system between the Sun, the Earth and Jupiter, with Jupiter's mass being 1000 times its real mass. The simulation is over a 20 year period. (a) View of Earth's orbit. (b) Earth's orbit in a more zoomed in view.

4.7 Perihelion Precession of Mercury

The precession of Mercury's perihelion was first simulated without the relativistic correction term, as shown in Fig. 10. In Fig. 10(a) it was believed that Mercury was precessing, however could have been due poor time resolution. Fig. 10(b) shows at a higher time resolution, which seems to not be precessing as expected. In addition, by looking at Fig. 11, where the perihelion position has been plotted with and without the correction term, that there is some systematic error in the code which results in an unusual repetition of points during a simulation of 100 years ($dt \approx 5$ minutes. It is expected that without the relativistic correction term that in a century the perihelion shift should be approximately 527 arc seconds, and 570 arc seconds with the correction terms added. As shown in Fig. 12 the perihelion shift does not give the values expected with and without the correction term. The time resolution of the simulation was tested with dt up to 1e-6 (≈ 31 seconds) and found little change in the results.

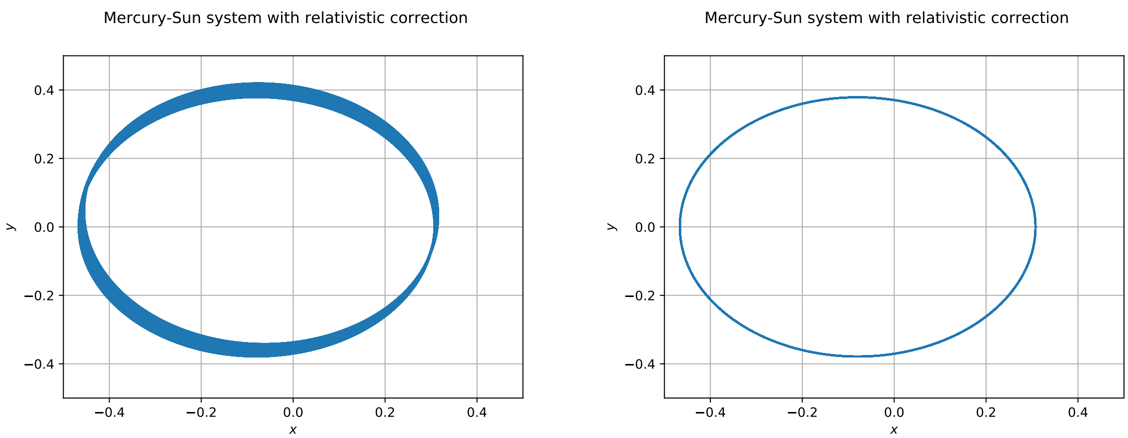


Figure 10: The motion of Mercury around the Sun over a period of a century without the relativistic correction term. (a) The simulation was run with dt = 1e-3 $\approx$ 9 hours. (b) Simulation run with dt = 1e-4 $\approx$ 53 minutes.

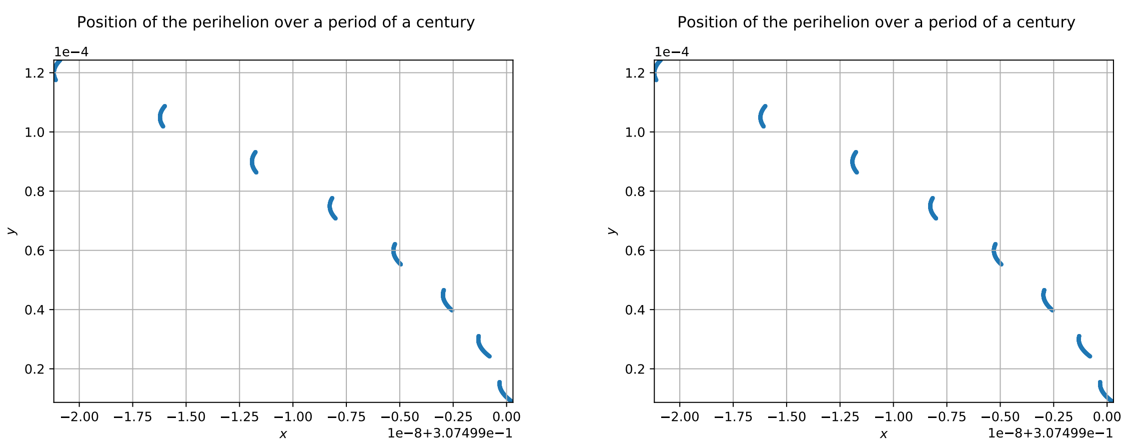


Figure 11: The change in position of the perihelion of Mercury over 100 years, dt = 1e-5 $\approx$ 5 minutes. (a) Without the relativistic correction term. (b) With relativistic correction term added to force. The perihelion motion seems unusual and therefore it is believed there is a systematic error in the code.

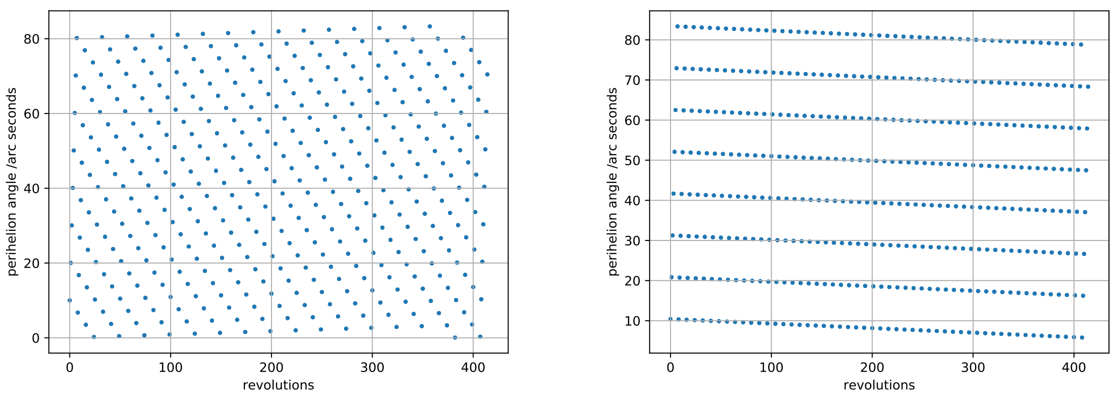


Figure 12: The angle the perihelion makes to the x axis at each orbit round the sun, simulation with dt = 1e-5 $\approx$ 5 minutes. (a) Without the relativistic correction term. (b) With relativistic correction term added to force. It is expected that (a) should reach 527 arc seconds after one century and that (b) should reached 570 arc seconds.

5 Discussion

Figs. 1 and 2 shows that the forward Euler method requires approximately 1000 times more data points before reaching an orbit which is circular. On the other hand, both methods show similar computational efficiency as shown in Table. 1 where both method take the same amount of time to perform simulations with different number of time steps. This suggests that the Velocity Verlet method is more accurate with little difference in computational cost to the forward Euler method.

5.1 Escape velocity

The trial and error result of 42.125 km/s matches the analytical result of $v_{\text{esc}} = 42.122\,\text{km/s}$ to four significant digits, which is a relatively good approximation. It is possible that the result would be even closer to the analytical value by doing more bisections.

5.2 Different forms of the gravitational force

The results seen in figure 5 show that the

The total energy of Earth's orbit for different values of β plotted in figure 3 shows that the total energy for $\beta = 2.0$, the original form of the gravitational force, is constant in time, which is expected. As β increases, however, it is seen that the total energy is not conserved. For $\beta = 3.0$, the total energy makes a large jump at $t \approx 0.26\,\text{yr}$. Comparing with figure 5 we see that this corresponds well with when the orbit for $\beta = 3.0$ is flung out of orbit, which happens close to around $t = 1/4\,\text{yr}$. The simulation was also performed to $t = 20\,\text{yr}$, and the same periodic pattern continued for the values $\beta < 3$, and $\beta = 3$ stayed constant after its sudden increase.

Fig. 7 shows that using the real masses for the Sun, Earth and Jupiter that the orbits are stable over a period of a century. In (c) the motion of the Earth does deviate slightly from tracing the exact same path. This is a positive result that the Velocity Verlet method is performing as expected, especially since the addition of Jupyter is steering the Earth off its elliptical path. The three-body problem is a more difficult problem than the binary problem, since in the binary system the lines of the force pass through the centre of mass of the system. However in the three-body problem, the motion of each body must be considered with the motion of the other two [4]. Historically, this made the three-body problem quite intractable.

The relativistic correction to the force as shown in Eq. 19 was used in order to simulate the relativistic gravitation on Mercury. However, as shown in Fig. 11 and Fig. 12 it seems that there is some systematic error which does not give the expected results. It is expect that the perihelion should reach 527 and 570 arc seconds without and with the correction, respectively. However, in the results shown in the figure the perihelion repeats at many points and gives the value 80 arc seconds in both Fig. 12(a) and (b) after one century. This means that there is zero perihelion shift between the two calculations.

6 Conclusion

The escape velocity found by trial and error with our simulation was found to match the analytical expression for the escape velocity to four significant digits. This test is considered a success, and it is one piece of evidence that the simulation is able to reconstruct the physics of gravity to a good approximation.

Using different forms of the force law with the exponend β had consequences for the conservation of total energy and angular momentum.

Using Einsteins theory of gravitation it was attempted to simulate the precession of Mercury's perihelion. However, systematic uncertainty is found in the results and the analytic perihelion shift expected (43 arc seconds) compared with Newtonian gravitation was not found.

7 Github address

Github address: `https://github.com/amundwf/comp-phys-project3`

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