Heavy Tailed Linear Bandits

As a sanity check, consider the case in $\mathbb{R}^d$ where$\{\eta_s\}_{s=1}^t \in [0,1]$ are bounded, uncorrelated, mean-zero noise, and we choose the actions $X_s = e_{s \mod d}$.

Let $V_t = \sum_{s=1}^t X_s X_s^T$, $V = \lambda I$, $X = (X_1, \ldots, X_t)$, and $\eta = (\eta_1, \ldots, \eta_t)$.

We want to estimate the quantity $\mathbb{P}[(X\eta)^T (V + V_t)^{-1} X\eta > \epsilon]$.

By our choice of X_s, we have that $V_t = \sum_{s=1}^t X_s X_s^T = kI$, such that $V + V_t = (\lambda + k)I$ and $(V + V_t)^{-1} = \frac{1}{\lambda+k} I$.

Moreover, we also have that

$$\begin{aligned}
X\eta &= \sum_{s=1}^t X_s \eta_s \\
&= \sum_{j=1}^d \sum_{s=j \mod d} X_s \eta_s \\
&= (\sum_{s:s=1 \mod d} \eta_s, \ldots, \sum_{s:s=d \mod d} \eta_s)^T
\end{aligned}$$

For simplicity, let $\bar{\eta}_{j \mod d} = \sum_{s:s=j \mod d} \eta_s$ This implies that

$$\begin{aligned}
(X\eta)^T (V + V_t)^{-1} X\eta &= \left(\sum_{s:s=1 \mod d} \eta_s, \ldots, \sum_{s:s=d \mod d} \eta_s \right) \frac{1}{\lambda + k} I \left(\sum_{s:s=1 \mod d} \eta_s, \ldots, \sum_{s:s=d \mod d} \eta_s \right)^T \\
&= \sum_{j=1}^d (\bar{\eta}_{j \mod d})^2 \frac{1}{\lambda + k} \\
&=: \psi
\end{aligned}$$

By our definition for $\psi = \psi(\eta_1, \ldots, \eta_t)$, we can write

$$\begin{aligned}
\mathbb{P}[(X\eta)^T (V + V_t)^{-1} X\eta > \epsilon] &= \mathbb{P}[\psi - \mathbb{E}[\psi] > \epsilon - \mathbb{E}[\psi]] \\
&\leq \exp - \frac{2(\epsilon - \mathbb{E}[\psi])^2}{\sum_{k=1}^t c_k^2} \text{ by McDiarmid's inequality}
\end{aligned}$$

To apply McDiarmid, we can compute that for $\eta_j \neq \tilde{\eta}_j$, we have

$$\begin{aligned}
\psi(\eta) - \psi(\tilde{\eta}) &= [(\bar{\eta}_{j \mod d})^2 - (\bar{\tilde{\eta}}_{j \mod d})^2] \frac{1}{\lambda + k} \\
&= \eta_j^2 - \tilde{\eta}_j^2 + \eta_j \sum_{i \neq j} \eta_i - \tilde{\eta}_j \sum_{i \neq j} e\tilde{t}a_j \\
&\leq \frac{k}{\lambda + k}
\end{aligned}$$

which implies that

$$\mathbb{P}[(X\eta)^T (V + V_t)^{-1} X\eta > \epsilon] \leq \exp - \frac{2(\epsilon - \mathbb{E}[\psi])^2}{t \left(\frac{k}{\lambda+k} \right)^2}$$

Setting the RHS equal to δ and solving yields

$$\begin{aligned}
&\exp -\frac{2(\epsilon - \mathbb{E}[\psi])^2}{t\left(\frac{k}{\lambda+k}\right)^2} = \delta \\
&\Leftrightarrow \frac{(\epsilon - \mathbb{E}[\psi])^2}{t\left(\frac{k}{\lambda+k}\right)^2} = \frac{\log(1/\delta)}{2} \\
&\Leftrightarrow \epsilon = \sqrt{\frac{\log(1/\delta) t\left(\frac{k}{\lambda+k}\right)^2}{2}} + \mathbb{E}[\psi]
\end{aligned}$$

so that with probability $\geq 1-\delta$, we have

$$\psi \leq \sqrt{\frac{\log(1/\delta) t\left(\frac{k}{\lambda+k}\right)^2}{2}} + \mathbb{E}[\psi]$$

and we can compute that

$$\begin{aligned}
\mathbb{E}[\psi] &= \mathbb{E}[\sum_{j=1}^{d} (\bar{\eta}_{j \mod d})^2 \frac{1}{\lambda+k}] \\
&\leq \frac{1}{\lambda+k}\mathbb{E}[\sum_{j=1}^{d} \sum_{s:s=j \mod d} \eta_s^2] \text{ (by uncorrelated and mean zero of } \eta) \\
&= \frac{1}{\lambda+k}\mathbb{E}[\sum_{s=1}^{t} \eta_s^2] \\
&\leq \frac{kd}{\lambda+k}\mathbb{E}[\eta^2]
\end{aligned}$$

so that with probability $\geq 1-\delta$, we have

$$\begin{aligned}
\psi &\leq \sqrt{t\frac{\log(1/\delta)\left(\frac{k}{\lambda+k}\right)^2}{2}} + \frac{kd}{\lambda+k}\mathbb{E}[\eta^2] \\
&= \mathcal{O}(\sqrt{t} + d)
\end{aligned}$$

More general scenario

$X = (X_1, \ldots, X_t)$ is $d \times t$, where $t = kd$.

Let A be a $d \times d$ matrix such that $kAA^T = XX^T$. This is possible because if we write $XX^T = Q\Lambda Q^T$, Q $d \times d$, then we can let $A = \frac{1}{\sqrt{k}}Q\sqrt{\Lambda}$.

This implies that

$$(X\eta)^T(\lambda I + XX^T)^{-1}(X\eta) = (X\eta)^T(\lambda I + kAA^T)^{-1}(X\eta)$$

Now, we want to write $X\eta = A\sum_{i=1}^{t} P_i\eta_i$ for vectors P_i. If we can do this, then the above expression can be written as

$$(X\eta)^T(\lambda I + kAA^T)^{-1}(X\eta) = (\sum_{i=1}^{t} P_i\eta_i)A^T(\lambda I + kAA^T)^{-1}A(\sum_{i=1}^{t} P_i\eta_i)$$

Now, notice that

$$A^T(\lambda I + kAA^T)^{-1}A = A^TA(\lambda I + kAA^T)^{-1}$$

and

$$A^TA = \frac{1}{\sqrt{k}}\sqrt{\Lambda}Q^T\frac{1}{\sqrt{k}}Q\sqrt{\Lambda} = \frac{1}{k}\Lambda$$

This implies that

$$\begin{aligned}
A^TA(\lambda I + kAA^T)^{-1} &= \frac{1}{k}\Lambda(\lambda I + \Lambda)^{-1} \\
&= [\frac{1}{k}\Lambda_j\frac{1}{\lambda + \Lambda_j}]_j \\
&\leq \frac{1}{k}\frac{\Lambda^*}{\lambda + \Lambda^*}I \text{ (where } \Lambda^* \text{ is the eigenvalue that maximizes this quantity)}
\end{aligned}$$

Going back to the claim $X\eta = A\sum_{i=1}^{t}P_i\eta_i$, we see that this is possible because $X = Q\sqrt{\Lambda}R^T$ (SVD of X using the known spectral decomposition of XX^T) and $A = \frac{1}{\sqrt{k}}Q\sqrt{\Lambda}$ imply that to get $X = A(P_1, \ldots, P_t)$, we can set

$$(P_1, \ldots, P_t) = \sqrt{k}R^T$$

This implies that

$$\begin{aligned}
\left|(X\eta)^T(\lambda I + XX^T)^{-1}(X\eta)\right| &\leq \left|\frac{1}{k}\frac{\Lambda^*}{\lambda + \Lambda^*}\eta^TRR^T\eta k\right| \\
&= \frac{\Lambda^*}{\lambda + \Lambda^*}\left|\eta^T(X\Lambda^{-1/2})^T(X\Lambda^{-1/2})\eta\right| \text{ (since } X = Q\sqrt{\Lambda}R^T\text{)} \\
&= \frac{\Lambda^*}{\lambda + \Lambda^*}\left|\eta^T\Lambda^{-1/2}X^TX\Lambda^{-1/2}\eta\right|
\end{aligned}$$