assignment 2 textbook

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1. Computer the Fourier transform of the following signals.
   (a) x(n) = δ(n) + δ(n-2) + u(n) - u(n-2)
   (b) x(n) = {-1,2,3,2,1}
   (c) $x(n) = a^n \sin(\omega_0 n)\, u(n)$, |a|<1
   (d) A signal x(n) has the Fourier transform X(w), determine the Fourier transform of $y(n) = x(n) \sin(\omega_0 n) + x(n-1)$, where $X(\omega) = \dfrac{1}{1-0.8e^{-j\omega}}$.

DTFT Def: $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

Z Def: $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$ $\quad z \to e^{j\omega}$

a) $x(n) = \overset{1}{\delta(n)} + \overset{z^{-2}}{\delta(n-2)} + \overset{\frac{1}{1-z^{-1}}}{u(n)} - \overset{\frac{z^{-2}}{1-z^{-1}}}{u(n-2)}$

$X(\omega) = 1 + e^{-j2\omega} + \dfrac{1}{1-e^{-j\omega}} - \dfrac{e^{-2j\omega}}{1-e^{-j\omega}}$

b) $x(n) = -1\,\delta(n) + 2\,\delta(n-1) + 3\,\delta(n-2) + 2\,\delta(n-3) + \delta(n-4)$

$X(\omega) = -1 + 2e^{-j\omega} + 3e^{-2j\omega} + 2e^{-3j\omega} + e^{-4j\omega}$

c) $a^n \sin(\omega_0 n)\, u(n) \quad |a|<1$

$= a^n \dfrac{1}{2j}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right] u(n)$ $\qquad \sin(\omega_0 n) = \dfrac{1}{2j}\left[e^{j\omega n} - e^{-j\omega_0 n}\right]$

Freq shifting:

modulation theorem:

$x(n) \sin(\omega_0 n) \Longleftrightarrow \dfrac{1}{2j}\left[X(\omega-\omega_0) - X(\omega+\omega_0)\right]$

$x(n)\, e^{j\omega_0 n} \Longleftrightarrow X(\omega-\omega_0)$

$= \dfrac{1}{2j}\left[\dfrac{1}{1-ae^{-j(\omega-\omega_0)}} - \dfrac{1}{1-ae^{-j(\omega+\omega_0)}}\right]$

d) $y(n) = x(n) \sin(\omega_0 n) + x(n-1)$, $X(\omega) = \dfrac{1}{1-0.8e^{-j\omega}}$

find $Y(\omega)$,

$Y(\omega) = \dfrac{1}{2j}\left[\dfrac{1}{1-0.8e^{-j(\omega-\omega_0)}} - \dfrac{1}{1-0.8e^{-j(\omega+\omega_0)}}\right] + \dfrac{e^{-j\omega}}{1-0.8e^{-j\omega}}$

2. Consider an LTI system, described by
   $$y(n) = \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2) + x(n)$$
   (a) Determine the impulse response, h(n), of the system
   (b) Sketch roughly the magnitude response |H(ω)| of this system.
   (c) What is the response of the system to the input signal 0.4^n u(n)?
   (d) Is this system stable?

$y(n) = \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2) + x(n)$ $\qquad 1 = a_1(1-\frac{1}{2}z^{-1}) + a_2(1-\frac{1}{4}z^{-1})$

$Y(z) \quad Y(z) = \frac{3}{4} z^{-1} Y(z) - \frac{1}{8} z^{-2} Y(z) + X(z)$

$y(n) = \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2) + x(n)$

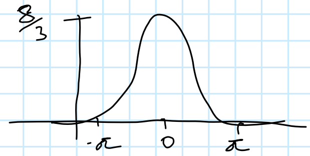
$1 = a_1(1-\frac{1}{2}z^{-1}) + a_2(1-\frac{1}{4}z^{-1})$

a) $h(n) \Leftrightarrow H(z) = \frac{Y(\omega)}{X(\omega)}$, $Y(z) = \frac{3}{4}z^{-1}\,Y(z) - \frac{1}{8}z^{-2}\,Y(z) + X(z)$

$Y(z)(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}) = X(z)$, $H(z) = \frac{1}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} = \frac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{a_1}{1-\frac{1}{4}z^{-1}} + \frac{a_2}{1-\frac{1}{2}z^{-1}}$, $a_1 = -1$, $a_2 = 2$

$\therefore$ $2 \cdot 0.5^n u(n) - (0.25)^n u(n)$

b) $H(\omega) = \frac{1}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}}$, $\omega = 0$ $|H(\omega)| = \frac{1}{1-\frac{3}{4}+\frac{1}{8}} = \frac{1}{3/8}$



c) $x(n) = 0.4^n u(n) \Leftrightarrow \frac{1}{1-0.4z^{-1}} = X(z)$

$Y(z) = \frac{1}{(1-0.4z^{-1})(1-0.25z^{-1})(1-0.5z^{-1})} = \frac{a_1}{1-0.4z^{-1}} + \frac{a_2}{1-0.25z^{-1}} + \frac{a_3}{1-0.5z^{-1}}$

$1 = a_1(1-0.25z^{-1})(1-0.5z^{-1}) + a_2(1-0.4z^{-1})(1-0.5z^{-1}) + a_3(1-0.4z^{-1})(1-0.25z^{-1})$

$a_1 = \frac{1}{(1-\frac{0.25}{0.4})(1-\frac{0.5}{0.4})} = -10.67$, $a_2 = 1.67$, $a_3 = 10$

$y(n) = -10.67 \cdot 0.4^n u(n) + 1.67(0.25)^n u(n) + 10(0.5)^n u(n)$

d) the Poles $z_p: 0.25, 0.5, 0.4$ are all in the unit circle & ROC includes the unit circle so yeah it's stable

$X_1(k) = \sum_{n=0}^{N-1} x_1(n) W_N^{kn}$, $k = 0, 1, \dots, N-1$

$X(k') = \sum_{n=0}^{2N-1} x_1(n) W_{2N}^{k'n} = \sum_{n=0}^{N-1} x_1(n) W_{2N}^{k'n} + \sum_{n=N}^{2N-1} x_1(n) W_{2N}^{k'n}$

$= \sum_{n=0}^{N-1} x_1(n) W_N^{k(\frac{k'}{2})} = \sum_{n=0}^{N-1} x_1(n) W_N^{(\frac{k'}{2})n} = X_1(\frac{k'}{2})$

$k' = 0, 1, \dots, 2N-1$

3. For the sequence, N=6: $x_1(n) = \cos(\frac{2\pi}{N}n) + \delta(n)$, and $x_2(n) = u(n) - \delta(n-1)$,

$0 \le n \le N-1$.

(a) Suppose N=6. Determine the N-point DFT of $x_1(n)$.

(b) Determine the 2N-point DFT of $x_1(n)$ by zero-padding first. What is their relationship?

(c) Determine the N-point circular convolution $x_1(n)$ Ⓝ $x_2(n)$

a) $x_1(n) = \frac{1}{2}[e^{j\frac{2\pi}{N}n} + e^{-j\frac{2\pi}{N}n}] + \delta(n)$, $N=6$

$X(k) = 0.5N[\delta((n-1))_N + \delta((n+1))_N] + u(n)$

$= \{0,3,0,0,0,3\} + \{1,1,1,1,1,1\} = \{1,4,1,1,1,4\}$

b) Zero Padding appends 0's to the end, $x_3(n) = \begin{cases} x_1(n), & 0 \le n \le N-1 \\ 0, & N \le n \le 2N-1 \end{cases}$

$x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$, $x_1(n) \xleftrightarrow[2N]{DFT} X_2(k)$

Zero Padding doesn't alter the frequency spectrum, it just interpolates information between our Points.

c) $X_1(k) = 0.5N\{\delta(k-1)_{\bmod N} + \delta(k+1)_{\bmod N}\} + U(k)$ $0 \le k \le N-1$

$x_2(n) = u(n) - \delta(n-1)$

Since this is 0 everywhere but k=0

c) $X_1(k) = 0.5N\{\delta(k-1)_{modN} + \delta(k+1)_{modN}\} + U(k) \quad 0 \le k \le N-1$

$x_2(n) = u(n) - \delta(n-1)$

$X_2(k) = N\delta(k) - e^{\frac{-j2\pi k}{N}} U(k) = 6\delta(k) - e^{-j\frac{2\pi k}{N}},$

$x_1(n) \circledN x_2(n) \Leftrightarrow X_1(k)X_2(k)$

$X_1(k)X_2(k) = (6\delta(k) - e^{\frac{-j2\pi k}{N}})(3\delta(k-1)_{Mod 6} + 3\delta(k+1)_{Mod 6} + U(k)$

$= 6\delta(k) - e^{\frac{-j2\pi k}{N}} X_1(k) \Leftrightarrow \quad u(n) - x_1(n-1)_{mod 6}$

$U(n) \Leftrightarrow N\delta(k)$

$\cos(\frac{2\pi n}{6}) + \delta(n) = [2, 0.5, -0.5, -1; 0.5, 0.5]$

$[1,1,1,1,1,1] - [0.5, 2, 0.5, -0.5, -1, -0.5] = [0.5, -1, 0.5, 1.5, 2, 1.5]$

Since this is 0 every where but $k=0$

$X_1(k)X_2(k) = 6X_1(0)\delta(k) - e^{\frac{-j2\pi k}{N}} X_1(k)$

$e^{\frac{-j2\pi k}{N}} X(k) \Leftrightarrow x(n-1)_{mod N}$

$6X_1(0) = 6[\underbrace{3\delta(0-1) + 3\delta(0+1)}_{0's} + U(0)] = 6$

4. Determine the 8-point DFTs of the following signals
   (a) x(n) = {1,0,1,0,0,0,0,0}.
   (b) $x(n) = a^n$, $|a|<1$, $0<=n<=7$.

a) $x(n) = [1, 0, 1, 0, 0, 0, 0, 0] = \delta(n) + \delta(n-2)$

$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}} = 1 + e^{-j2\pi k\frac{2}{8}} = X(k)$

b) $X(k) = \sum_{n=0}^{7} a^n e^{\frac{-j2\pi kn}{8}} = \frac{1-a^8}{1-ae^{\frac{-j2\pi k}{8}}}, \ k=0,1,\dots,7$

5. Consider the sequence $x_1(n) = \{\underset{\uparrow}{1}, 1, 0, 0\}$ and $x_2(n) = \{\underset{\uparrow}{1}, 1, 3, 6\}$.

   (a) Given the 4-point DFT of the sequence x1(n), compute the DFT of the sequence $y(n) = \{\underset{\uparrow}{1}, 0, 0, 1\}$.

   (b) Determine a sequence y(n) such that $Y(k)=X_1(k)X_2(k)$.

   (c) Calculate the linear convolution $x_1(n)*x_2(n)$ by using DFT.

a) $y(n) = x_1((n+1))_4, \quad X_1(k) = 1 + e^{\frac{-j\pi k}{2}}$

$Y(k) = X_1(k) e^{\frac{+j2\pi k}{4}} = (1 + e^{\frac{-j\pi k}{2}})(e^{\frac{+j\pi k}{2}}) = 1 + e^{\frac{j\pi k}{2}} = [2, 1+j, 0, 1-j]$

b) $Y(k) = X_1(k)X_2(k), \quad X_1(k) = 1 + e^{\frac{-j\pi k}{2}} \quad (1 + e^{\frac{-j\pi k}{2}})X_2(k) \overset{\text{Shifting}}{\Rightarrow} x_2(n) + x_2(n+1)$

$\therefore y(n) = x_2(n) + x_2((n-1))_4 = [1, 1, 3, 6] + [6, 1, 1, 3] = [7, 2, 4, 9]$

c) Linear Convolution $x_1(n) * x_2(n)$ by DFT

Zero Padding + Circular Conv = linear. $X_1^{ZP}(k) = 1 + e^{\frac{-j\pi k}{4}} \quad Y^{ZP}(k) = X_1^{ZP}(k) X_2^{ZP}(k)$

$y^{ZP}(n) = x_2^{ZP}(n) + x_2^{ZP}((n-1))_8 = [1, 1, 3, 6, 0, 0, 0, 0] + [0, 1, 1, 3, 6, 0, 0, 0]$

Zero Padding + Circular Conv = linear. $X_1^{zp}(k) = 1 + e^{-j\frac{\pi k}{4}}$ $Y^{zp}(k) = X_1^{zp}(k)\, X_2^{zp}(k)$

$y^{zp}(n) = x_2^{zp}(n) + x_2^{zp}((n-1))_8 = [1,1,3,6,0,0,0,0] + [0,1,1,3,6,0,0,0]$

$= [1,2,4,9,6,0,0,0]$ $x_1(n) * x_2(n) = [1,2,4,9,6]$

6. An LTI system has the impulse response in the frequency domain as

$$H(e^{j\omega}) = \frac{1 - 1.25e^{-j\omega}}{1 - 0.8e^{-j\omega}} = 1 - \frac{0.45e^{-j\omega}}{1 - 0.8e^{-j\omega}}.$$

(a) Specify the difference equation that is satisfied by the input x(n) and the output y(n).

(b) Determine the impulse respose h(n).

(c) Show that H(w) is an all-pass filter (i.e., $|H(w)|^2 = C$) and determine the constant C.

a) $$H(e^{j\omega}) = \frac{1 - 1.25e^{-j\omega}}{1 - 0.8e^{-j\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$X(z) - 1.25 z^{-1} X(z) = Y(z) - 0.8 z^{-1} Y(z)$$

$$x(n) - 1.25\, x(n-1) = y(n) - 0.8\, y(n-1)$$

$\therefore$ $y(n) = 0.8\, y(n-1) + x(n) - 1.25\, x(n-1)$

b) $$H(e^{j\omega}) = 1 - \frac{0.45e^{-j\omega}}{1 - 0.8e^{-j\omega}} \Leftrightarrow h(n) = \delta(n) - 0.45\,(0.8)^{n-1}\, u(n-1)$$

c) Show that $|H(\omega)|^2 = C$ & determine C

$$|H(e^{j\omega})|^2 = H(e^{j\omega})\, H^*(e^{j\omega}) = \left(\frac{1 - 1.25e^{-j\omega}}{1 - 0.8e^{-j\omega}}\right)\left(\frac{1 - 1.25e^{j\omega}}{1 - 0.8e^{j\omega}}\right)$$

$$= \frac{1 - 1.25e^{j\omega} - 1.25e^{-j\omega} + (1.25)^2}{1 - 0.8e^{j\omega} - 0.8e^{-j\omega} + (0.8)^2} = \frac{1 - 2.5\cos(\omega) + 1.5625}{1 - 1.6\cos(\omega) + 0.64}$$

$$= \frac{2.5625 - 2.5\cos(\omega)}{1.64 - 1.6\cos(\omega)} = \frac{2.5625(1 - 0.9756\cos(\omega))}{1.64(1 - 0.9756\cos(\omega))}$$

$$\frac{2.5625}{1.64} = (1.25)^2, \quad C = 1.5625$$