assignment 3 textbook

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Problem 1: Suppose a process $\{x(n)\}$ is characterized by the autocorrelation sequence

$$\gamma_{xx}(0)=\frac{4}{3},\gamma_{xx}(1)=\frac{2}{3},\gamma_{xx}(2)=\frac{1}{3},\gamma_{xx}(3)=\frac{1}{6}.$$

(a) Use the autocorrelations and calculate an AR(2) model for this process.
(b) Substituting these results into the Yule-Walker equation for k=0 to estimate the white noise variance. Then compute the power spectrum based on the AR(2) model.
(c) Repeat the above calculation for an AR(3) model. Comment on your results.

a)

$\gamma_{xx}(0)=\frac{4}{3}$, $\gamma_{xx}(1)=\frac{2}{3}$, $\gamma_{xx}(2)=\frac{1}{3}$

AR(2) Yule Walker:

$$\begin{bmatrix} \gamma_{xx}(0) & \gamma_{xx}(-1) & \gamma_{xx}(-2) \\ \gamma_{xx}(1) & \gamma_{xx}(0) & \gamma_{xx}(-1) \\ \gamma_{xx}(2) & \gamma_{xx}(1) & \gamma_{xx}(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sigma_w^2 \\ 0 \\ 0 \end{bmatrix}$$

Eq1: $\frac{4}{3}+a_1\cdot\frac{2}{3}+a_2\frac{1}{3}=\sigma_w^2$

Eq2: $\frac{2}{3}+a_1\frac{4}{3}+a_2\frac{2}{3}=0 \Rightarrow 4a_1=-2-2a_2$

Eq3: $\frac{1}{3}+a_1\frac{2}{3}+a_2\frac{4}{3}=0 \Rightarrow 2a_1=-1-4a_2$

$-\cancel{2}-2a_2=-\cancel{2}-8a_2$

$a_2=4a_2$

$\therefore a_2=0$

$\therefore a_1=-0.5$

AR(2) model $= H(\omega)=\dfrac{1}{1+a_1e^{-j\omega}+a_2e^{-2j\omega}}$

$$H(z)=\frac{1}{A(z)}=\frac{1}{1+\sum_{k=1}^{p}a_kz^{-k}}$$

b) Eq1: $\frac{4}{3}-0.5\,\frac{2}{3}+0\,\frac{1}{3}=\sigma_w^2$

$\frac{4}{3}-\frac{1}{3}+0=\sigma_w^2=1$

$$S_{xx}(\omega)=\sigma_w^2\,|H(\omega)|^2$$
$$=\frac{1}{|1+a_1e^{-j\omega}+a_2e^{-2j\omega}|^2}$$

c) Yule-Walker

$$\begin{bmatrix} 4/3 & 2/3 & 1/3 & 1/6 \\ 2/3 & 4/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & 4/3 & 2/3 \\ 1/6 & 1/3 & 2/3 & 4/3 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sigma_w^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Eq2: $a_1\,4/3+a_2\frac{2}{3}+a_3\frac{1}{3}=-\frac{2}{3}$

Eq3: $a_1\,2/3+a_2\,4/3+a_3\,2/3=-1/3$

Eq4: $a_1\,1/3+a_2\,2/3+a_3\,4/3=-1/6$

Problem 2: A process $\{x(n)\}$ is characterized by the autocorrelation sequence

$$\gamma_{xx}(0)=\frac{4}{3},\gamma_{xx}(1)=\frac{2}{3},\gamma_{xx}(2)=\frac{1}{3},\gamma_{xx}(3)=\frac{1}{6}.$$

(a) Consider the one-step forward predictor of order 2. Determine the prediction coefficients.
(b) Calculate the resulting MMSE (minimum mean-square error) for this 2nd-order predictor.
(c) Write down the difference equation.

$$\hat{x}(n)=-\sum_{k=1}^{p}a_p(k)\,x(n-k)$$

a) $\hat{x}(n)=-\sum_{k=1}^{2}a_2(k)\,x(n-k) \Rightarrow \gamma_{xx}(l)=-\sum_{k=1}^{p}a_3(k)\,\gamma_{xx}(l-k)$, $l=1,2$

$$X_n = \begin{bmatrix} x(n-1) \\ x(n-2) \end{bmatrix}, \quad h = \begin{bmatrix} -a_2(1) \\ -a_2(2) \end{bmatrix}, \quad \gamma_d = E[d(n) X_n] = E\left[x(n)\begin{bmatrix} x(n-1) \\ x(n-2) \end{bmatrix}\right] = \begin{bmatrix} \gamma_{xx}(1) \\ \gamma_{xx}(2) \end{bmatrix}$$

$$\Gamma_2 h = \gamma_d$$

$$\Gamma_2 = E[X_n X_n^T] = \begin{bmatrix} \gamma_{xx}(0) & \gamma_{xx}(1) \\ \gamma_{xx}(1) & \gamma_{xx}(0) \end{bmatrix}$$

$$\begin{pmatrix} \gamma_{xx}(0) & \gamma_{xx}(1) \\ \gamma_{xx}(1) & \gamma_{xx}(0) \end{pmatrix} \begin{bmatrix} -a_2(1) \\ -a_2(2) \end{bmatrix} = \begin{pmatrix} \gamma_{xx}(1) \\ \gamma_{xx}(2) \end{pmatrix} \qquad h = \Gamma_2^{-1} \gamma_d$$

$$\begin{bmatrix} a_2(1) \\ a_2(2) \end{bmatrix} = \frac{1}{\frac{16}{9} - \frac{4}{9}} \begin{bmatrix} 4/3 & -2/3 \\ -2/3 & 4/3 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

$$a_2(1) = \left(\frac{8}{3} - \frac{2}{3}\right) \cdot \frac{3}{4} = \frac{6}{4} = 1.5$$

$$a_2(2) = \left(-\frac{4}{3} + \frac{4}{3}\right) \cdot \frac{3}{4} = 0$$

b) $MMSE = \gamma_{dd}(0) - \gamma_d^T h, \quad d(n) = x(n)$

$$MMSE = \gamma_{xx}(0) + \sum_{k=1}^{2} a_3(k)\gamma_{xx}(k) = \frac{4}{3} + \frac{3}{2} \cdot \frac{2}{3} - 0 \cdot \frac{1}{3} = \frac{7}{3}$$

c) $e(n) = x(n) - \hat{x}(n) = x(n) + \sum_{k=1}^{2} a_2(k) x(n-k) = x(n) + 1.5\, x(n-1)$

Problem 3: Wiener filter

A zero-mean stationary process {s(n)} is characterized by the autocorrelation sequence

$$\gamma_{ss}(0) = 1, \gamma_{ss}(1) = \frac{1}{2}, \gamma_{ss}(2) = \frac{1}{8}, \gamma_{ss}(3) = \frac{1}{64}.$$

a.) Let us assume that the observed signal x(n) can be expressed as x(n) = 2 s(n) + w(n), where s(n) is a process described above, and {w(n)} is a white noise sequence with variance $\sigma_w^2 = 1$. Assume that {w(n)} is uncorrelated to {s(n)}. Determine the autocorrelation sequence $\{\gamma_{xx}(m)\}$ of the process {x(n)} for m=0,1, 2, 3.

a) $x(n) = 2\,s(n) + w(n)$

$$\gamma_{xx}(l) = E[x(n)\,x(n-l)] = E[(2s(n) + w(n))(2s(n-l) + w(n-l))] = 4\gamma_{ss}(l) + 2\underbrace{\cancel{\gamma_{sw}(l)}}_{0} + 2\underbrace{\cancel{\gamma_{ws}(l)}}_{0} + \gamma_{ww}(l)$$

$4\,s(n)\,s(n-l) + 2\,s(n)\,w(n-l) + 2\,w(n)\,s(n-l) + w(n)\,w(n-l)$

Since s(n) & w(n) are uncorrelated.

for w(n) $\gamma_{ww}(l) = \sigma_w^2 \delta(l)$, σ_w^2 given as 1

$\therefore \gamma_{ww}(l) = \delta(l)$, $\{\gamma_{ss}(l)\}$ given as $\{1, \frac{1}{2}, \ldots$

since $s(n)$ & $w(n)$ are uncorrelated.

for $w(n)$ $\gamma_{ww}(l) = \sigma_w^2 \delta(l)$, σ_w^2 given as 1

$\therefore \gamma_{ww}(l) = \delta(l)$, $\{\gamma_{ss}(l)\}$ given as $\{1, \frac{1}{2}, \frac{1}{8}, \frac{1}{64}\}$

$\therefore \{\gamma_{xx}(l)\} = \{5, 2, 0.5, \frac{1}{16}\}$

b.) Consider the signal smoothing application. Based on the signal {x(n)}, we want to design a FIR Wiener filter of length M=2 to estimate s(n+1). Write down the expression of the output y(n) in terms of the FIR filter of length M=2 with coefficients {h(0),h(1)}.

c.) Calculate the solution for this optimum FIR Wiener filter in b.), and calculate the corresponding minimum MSE (i.e. $MMSE_2$).

d.) In the above problem, suppose the observed signal x(n) can be expressed as x(n) = s(n) + s(n-1) + w(n). We want to use x(n) and x(n+1) to estimate s(n). Write down the expression of the output y(n) with coefficients {h(0),h(1)}. Calculate the solution for this optimum Wiener filter.

b) $d(n) = s(n+1)$

Filter of length M with coefficients $\{h(k)\}$

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) = \vec{x}_n^T h_M$$

Auto corr Mat: $\Gamma_M = E\{\vec{x}_n \vec{x}_n^T\}$

X-corr Vec: $\gamma_d = E\{d(n)\vec{x}_n^*\}$

MMSE solution: $\Gamma_M h_M = \gamma_d$

$h_{opt} = \Gamma_M^{-1}\gamma_d$

$MMSE_M = \min_h \mathcal{E}_M = \sigma_d^2 - \gamma_d^{*T}\Gamma_M^{-1}\gamma_d$

$$y(n) = \hat{d}(n) = \hat{s}(n+1) = \sum_{k=0}^{1} h(k)x(n-k) = h(0)x(n) + h(1)x(n-1) = x_n^T h, \quad x_n^T = [x(n) \quad x(n-1)], \quad h = \begin{bmatrix} h(0) \\ h(1) \end{bmatrix}$$

c)

$$\gamma_d = E[d(n)x_n] = E\left\{s(n+1)\begin{bmatrix} x(n) \\ x(n-1)\end{bmatrix}\right\} = E\left\{s(n+1)\begin{bmatrix} 2s(n) + w(n) \\ 2s(n-1) + w(n-1)\end{bmatrix}\right\} = \begin{bmatrix} 2\gamma_{ss}(1) \\ 2\gamma_{ss}(2)\end{bmatrix}$$

$$\Gamma_2 = E\{x_n x_n^T\} = \begin{bmatrix} \gamma_{xx}(0) & \gamma_{xx}(1) \\ \gamma_{xx}(1) & \gamma_{xx}(0)\end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

γ_{ss} given

$$\Gamma_2 h = \gamma_d \quad \therefore h = \Gamma_2^{-1}\gamma_d = \frac{1}{25-4}\begin{bmatrix} 5 & -2 \\ -2 & 5\end{bmatrix}\begin{bmatrix} 1 \\ 1/4 \end{bmatrix} = \begin{bmatrix} \frac{5}{21} - \frac{2}{4 \cdot 21} \\ \frac{-2}{21} + \frac{5}{4 \cdot 21}\end{bmatrix} = \begin{bmatrix} 0.214 \\ -0.036 \end{bmatrix}$$

$$MMSE = \gamma_{dd}(0) - \gamma_d^T h = 1 - \begin{bmatrix} 1 & 1/4 \end{bmatrix}\begin{bmatrix} 0.214 \\ -0.036\end{bmatrix} = \underline{0.795}$$

$$\gamma_{dd} = E[s(n+1)\, s(n+1)] = \gamma_{ss}(0) = 1$$

d) $d(n) = s(n)$ $\qquad x_n = \begin{bmatrix} x(n) \\ x(n+1)\end{bmatrix}$

$$x(n) = s(n) + s(n-1) + w(n)$$

$$y(n) = \hat{d}(n) = \hat{s}(n) = h(0)x(n) + h(1)x(n+1)$$

$$= x_n^T h, \quad x_n = \begin{bmatrix} x(n) \\ x(n+1)\end{bmatrix}, \quad h = \begin{bmatrix} h(0) \\ h(1)\end{bmatrix}$$

$$\gamma_d = E[d(n)x_n] = E\left\{s(n)\begin{bmatrix} x(n) \\ x(n+1)\end{bmatrix}\right\} = E\left\{s(n)\begin{bmatrix} s(n) + s(n-1) + w(n) \\ s(n+1) + s(n) + w(n+1)\end{bmatrix}\right\} = \begin{bmatrix} \gamma_{ss}(0) + \gamma_{ss}(1) \\ \gamma_{ss}(1) + \gamma_{ss}(0)\end{bmatrix}$$

$$\gamma_d = E[d(n)X_n] = E\left\{S(n)\begin{bmatrix} x(n) \\ x(n+1) \end{bmatrix}\right\} = E\left\{S(n)\begin{bmatrix} S(n)+S(n-1)+w(n) \\ S(n+1)+S(n)+w(n+1) \end{bmatrix}\right\} = \begin{bmatrix} \gamma_{ss}(0)+\gamma_{ss}(1) \\ \gamma_{ss}(1)+\gamma_{ss}(0) \end{bmatrix}$$

$$\Gamma_2 = E[X_n X_n^T] = \begin{bmatrix} \gamma_{xx}(0) & \gamma_{xx}(1) \\ \gamma_{xx}(1) & \gamma_{xx}(0) \end{bmatrix}$$

$$\gamma_{xx}(0) = 2\gamma_{ss}(0) + 2\gamma_{ss}(1) + \gamma_{ww}(0) = 2+1+1=4$$

$$\gamma_{xx}(1) = \gamma_{ss}(0) + 2\gamma_{ss}(1) + \gamma_{ss}(2) = 1+1+\tfrac{1}{8} = 2.125$$

$$\gamma_{xx}(1) = (S(n+1)+S(n)\ w(n+1))\ (S(n)+S(n-1)+w(n))$$

$$\gamma_{xx}(1) = \gamma_{ss}(1)+\gamma_{ss}(2)+0+\gamma_{ss}(0)+\gamma_{ss}(1)+0+\gamma_{ww}(1)$$

$$\gamma_{ww}(l) = \sigma_w^2\,\delta(l)$$

$$h = \Gamma_2^{-1}\gamma_d = \begin{bmatrix} 4 & 2.125 \\ 2.125 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1+2/3 \\ 1+2/3 \end{bmatrix} = \begin{bmatrix} 0.272 \\ 0.272 \end{bmatrix}$$