CSC-591

ADBI

Homework - II

1. Posterior Distribution of μ.

→ Using Bayes Theorem:

$$P(\mu|D) = \frac{P(X|\mu) * P(\mu)}{P(X)}$$

where X: univariate data

μ: Mean.

2.3. Derivation and the final estimate for μ_n and $1/\sigma_n^2$

→ The posterior distribution of μ :

$$P(\mu|X) = \frac{P(X|\mu) * P(\mu)}{P(X)}$$

$$\Rightarrow P(\mu|X) = \frac{\prod_{i=0}^{n-1} P(X_i|\mu) * P(\mu)}{P(X)}$$

$$\Rightarrow P(\mu|X) = a * \prod_{i=0}^{n-1} P(X_i|\mu) . P(\mu) \quad \text{[Ignoring } P(X) \text{ which is a const.]}$$

$$\Rightarrow P(\mu|X) = a * \prod_{i=0}^{n-1} \frac{e^{-1/2 * \left(\frac{x_i - \mu}{\sigma}\right)^2}}{\sqrt{2\pi\sigma^2}} \times \frac{e^{-1/2 * \left(\frac{\mu - \mu_0}{\sigma_0}\right)^2}}{\sqrt{2\pi\sigma_0^2}}$$

Removing the constants a and σ by replacing it with b.

So, if $b = a/\sqrt{2\pi}\,\sigma$.

$$\Rightarrow P(\mu|X) = b * \prod_{i=0}^{n-1} e^{-1/2 * \left(\frac{x_i - \mu}{\sigma}\right)^2} * e^{-1/2 * \left(\frac{\mu - \mu_0}{\sigma_0}\right)^2}$$

Since $e^a \times e^b \times e^c \ldots e^n = e^{a+b+c+\ldots+n}$.

$$\Rightarrow P(\mu|X) = b * e^{-1/2 * \sum_{i=0}^{n-1} \left(\frac{x_i^2 + \mu^2 - 2x_i\mu}{\sigma^2}\right) - 1/2\left(\frac{\mu^2 + \mu_0^2 - 2\mu\mu_0}{\sigma_0^2}\right)}$$

Ignoring constants : X_i

Also,

$$\sum_{i=0}^{n-1} X_i = n.X$$

$$\Rightarrow P(\mu|X) = b * e^{\left(-1/2 \times \left[\frac{n\mu^2}{\sigma^2} - 2\mu\frac{\sum_{i=0}^{n-1} x_i}{\sigma^2}\right] - 1/2 \times \left[\frac{\mu^2}{\sigma_0^2} - 2\frac{\mu_0\,\mu}{\sigma_0^2}\right]\right)}$$

$$P(\mu|X) = b \times e^{-\frac{1}{2}\left[\left(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}\right)\mu^2 - 2\mu\left(\frac{\sum x_i}{\sigma^2} + \frac{\mu_0}{\sigma_0^2}\right)\right]}$$

Substituting.

$$X = \frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}$$

$$Y = \frac{\sum x_i}{\sigma^2} + \frac{\mu_0}{\sigma_0^2}.$$

$$\Rightarrow P(\mu|X) = b \times e^{-\frac{1}{2}\left[X\mu^2 - 2\mu Y\right]}$$

$$\Rightarrow P(\mu|X) = b \times e^{-\frac{X}{2}\left[\mu^2 - 2\mu Y/X\right]}$$

$$\Rightarrow P(\mu|X) = b \times e^{-\frac{1}{2 \times 1/X}\left[\mu^2 - \frac{2Y}{X}\mu + \left(\frac{Y}{X}\right)^2 - \left(\frac{Y}{X}\right)^2\right]}$$

$$\Rightarrow P(\mu|X) = b \times e^{-\frac{1}{2\times 1/X}\left[\left(\mu - Y/X\right)^2\right]}$$

$$= b \times e$$

Now

$$\frac{1}{X} = \left[\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}\right]^{-1}$$

$$\boxed{\mu_n = \frac{\left(\frac{n\bar{x}}{\sigma} + \frac{\mu_0}{\sigma_0^2}\right)}{\left(\frac{n}{\sigma^2} + \frac{1}{\sigma^2}\right)} = \frac{\sigma_0^2 + n\bar{x} + \mu_0\sigma^2}{n\sigma_0^2 + \sigma^2}}$$

4 As seen from the last answer,

$$\mu_n = \frac{\sigma_0^2 n \bar{x} + \mu_0 \sigma^2}{n\sigma_0^2 + \sigma^2}$$

$$W_1 = \frac{\sigma_0^2 n}{\sigma^2 + n\sigma_0^2}$$

$$\boxed{W_2 = \frac{\sigma^2}{\sigma^2 + n\sigma_0^2}}$$

5. Since the variance is effectively in the denominator, the weights are inversely proportional to it.

6. $$W_1 + W_2 = \frac{\sigma_0^2 n}{\sigma^2 + n\sigma_0^2} + \frac{\sigma^2}{\sigma^2 + n\sigma_0^2} = \frac{\cancel{\sigma_0^2 n + \sigma^2}}{\cancel{\sigma^2 + n\sigma_0^2}} = 1$$

Yes, as seen from above, the weights, W_1 and W_2 do sum upto 1.

7

$$W_1 = \frac{\sigma_0^2 n}{\sigma^2 + n\sigma_0^2}$$

$$W_2 = \frac{\sigma^2}{\sigma^2 + n\sigma_0^2}$$

$$W_1 = \frac{1}{1 + \frac{n\sigma^2}{\sigma_0^2}}$$

and

$$W_2 = \frac{1}{1 + \frac{n\sigma_0^2}{\sigma^2}}$$

$$0 < 1 + \frac{n\sigma^2}{\sigma_0^2} < \infty$$

$$\boxed{\text{So, } W_1 = (0,1)}$$

$$0 < 1 + \frac{n\sigma_0^2}{\sigma^2} < \infty$$

$$\boxed{0 < W_2 < 1}$$

8 μ_n wrt μ_0 and $\bar{X}$

μ_n: Weighted Average of Mean (sample) and Mean (Prior)

$$\boxed{\mu_n \propto \frac{1}{\text{Sample Mean and Prior Mean}} \quad [\text{Inversely Proportional}].}$$

9. $p(X^{new} | X) \sim N(\mu_n, \sigma_n^2 + \sigma^2)$

$X^{new} = X^{new} - \mu + \mu$

$X^{new} \cong N(\mu, \sigma^2)$

$X^{new} - \mu \sim N(0, \sigma^2)$

$\mu \sim N(\mu_0, \sigma_0^2)$

Since, $X^{new} - \mu$ and μ are independent.

$E[X^{new}] = E[X^{new} - \mu] + E[\mu]$

$var[X^{new}] = var[X^{new} - \mu] + var[\mu]$

$var[X^{new}] = \sigma^2 + \sigma_0^2 \Rightarrow$ $X^{new} \in$ Normal Distribution $N(\mu, \sigma^2 + \sigma_0^2)$

10. We have, $n = 20$

$$P(\mu) \sim N(\underbrace{4}_{\mu_0}, \underbrace{0.8^2}_{\sigma_0})$$

$$P(x) \sim N(6, 1.5^2)$$

$$\sigma = 1.5$$

$$\mu_n = \frac{\sigma_0^2 \times n}{n\sigma_0^2 + \sigma^2} \bar{x} + \frac{\sigma^2}{n\sigma_0^2 \sigma^2} \mu_0.$$

$$= \frac{(0.8)^2 \times 20}{20 \times (0.8)^2 + 1.5^2} * 5.84 + \frac{1.5^2}{20 \times (0.8)^2 * 1.5^2} \times 4$$

$$\boxed{\mu_n = 5.287}$$

$$\sigma_n^2 = \left(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}\right)^{-1} = \left(\frac{20}{1.5^2} + \frac{1}{(0.8)^2}\right)^{-1} = 0.096.$$

$$\boxed{\sigma_n^2 = 0.096}$$