# Classification of Functions

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**Abstract**

In function analysis, continuous functions can be classified by their degree of differentiability. When a function $f$ is $k$ times differentiable, it belongs to a class of functions $C^k$, having $C^{(0)}$ be the class of all continuous functions and $C^\infty$ be the class of all infinite differentiable functions. Additionally, there is an important function class known as analytic denoted $C^\omega$. This functions are not only infinite differentiable but also have their functions be equivalent to the convergence of its Taylor series centered on a point and with its corresponding interval of convergence. In this write-up we analyze the concepts of differentiability and continuity from an analysis perspective, and explore the different relations and properties that these classes hold. First we begin the generalization for real numbers as we delve later into the complex functions and their differentiability classes.

## 1 Introduction

The concept of categorization is necessary when dealing with mathematical objects to understand their properties and perform mathematical proofs [12]. We have learned the categorization of numbers (i.e. $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$, etc.) and sets (i.e. $\emptyset$, power set, singleton set, etc.) which allow us to understand their innate properties and relations with members of other categories. Here we introduce the idea of classes of functions that one may encounter in analysis. Recall that in set theory, a function $f : X \longmapsto Y$ is a map of points from one set to another, where each point gets mapped exactly to one other point.

This definition of function serves adequately until one starts employing this concepts in other areas. In analysis, a function may not necessarily be defined on how it acts on points but also on how it acts on other objects such as sets or other functions [11]. We will explain the categorization of functions into classes, which has as its defining characteristic its degree of differentiability. After explaining this classification we will review closely one class in particular; analytic functions.

# 2 Characterization of Functions

## 2.1 Continuity and Differentiability

Even though one may be acquainted with the concept of continuity and differentiability, it is imperative to understand the implications and differences between each other before categorizing functions by these concepts. We will restrict our definitions of continuity and differentiability to functions between sets of one dimension. A similar definition exists for sets of higher dimensions but will not be included in this write-up.

We define continuity on a point $x_0$ the following way. Let $f : X \longmapsto Y$ be a function and $x_0 \in X$. We say $f$ is continuous at $x_0$ if and only if

$$\lim_{x \to x_0; x \in X} f(x) = f(x_0).$$

and $f(x)$ is defined [9]. We say that a function is continuous if it has continuity at every point $x_0 \in X$.

Moreover, we define differentiability at a point $x \in X$ if the limit

$$\lim_{x \to x_0; x \in X} \frac{f(x) - f(x_0)}{x - x_0}$$

converges to a point $f'(x) \in W$ [9]. We say that a function $f$ is differentiable at $x$ on $X$ with derivative $f' \in \mathbb{R}$. A function is differentiable if it is differentiable at every point $x \in X$.

If the function does not converge to a point $f'(x)$, then the limit does not exists. We define this as the function $f$ is not differentiable.

It is important to know that differentiability implies continuity. We can prove this using our definitions of differentiability and continuity

*Proof.* Let $f : X \longmapsto Y$ be differentiable at some $x_0 \in X$. Then from our definition of a differentiable function we get that

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

is defined and exists. Thus we get

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} [f(x) - f(x_0) - f(x_0)] = \lim_{x \to x_0} [f(x) - f(x_0)] + \lim_{x \to x_0} f(x_0)$$

$$= \lim_{x \to x_0} [\frac{(f(x) - f(x_0)(x - x_0))}{(x - x_0)}] + f(x_0)$$

which we know exists [4]. It follows that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) + f(x_0) = f(x_0).$$

Therefore this results in $f(x) = f(x_0)$ where $f$ is continuous at $x_0 \in X$. $\square$

The reciprocal is not necessarily true, which is that continuity implies differentiability. It is easy to see this, for example, in the function $f(x) = |x|$. It is not differentiable at $x = 0$, even though it is continuous. Such function is not differentiable in one point $x = 0$, but it is differentiable at every other point.

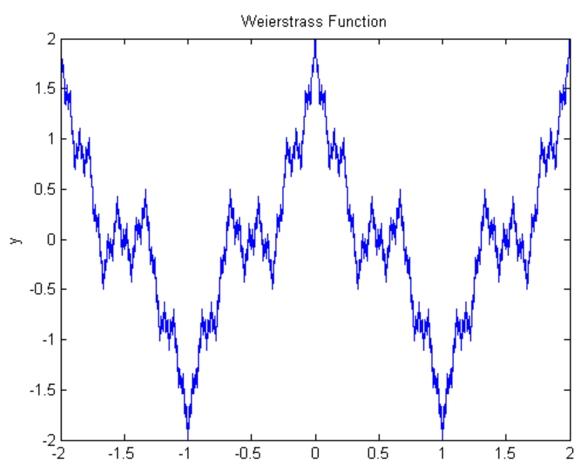
Furthermore, there exist functions that are continuous everywhere but are not differentiable anywhere. We use as an example a Weierstrass function, which were discovered by mathematician Karl Weierstrass.

In Weierstrass's original text, he specifies a function $f : X \longmapsto Y$ as the Fourier series

$$f(x) = \sum_{n=0}^{\infty} a^n cos(b^n \pi x)$$

where $0 < a < 1$, $b \geq 0$, and $ab > 1 + \frac{3}{2}\pi$.

Figure 1: Weierstrass's function graph with $a = \frac{1}{2}$ and $b = 3$ [6].




[3]. Without much desire to explain the construction motivation, it follows that while $f$ is continuous everywhere, it is not differentiable anywhere. We are going to omit the proof in this paper. Thus there exists functions in $f \in C^{\infty}$ such that $f \notin C\omega$.

## 2.2 $k$ Differentiable Functions

Let us define a set of continuous functions $C$. From the definitions above, observe that a function $f$ with derivative $f'(x)$ is a function in a set $C$ which defines all functions from one set $X$ to another set $Y \subseteq \mathbb{R}$. Similarly, the resulting differentiation maps points from $X$ to a point $f'(x_0) \in \mathbb{R}$, thus we have $f' : X \longmapsto \mathbb{R}$. Thus $f'$ is also a function in a set $C$ that maps elements from $V$ to $\mathbb{R}$. We define a function $f$ to be once differentiable if $f'$ is also defined. We denote $f'$ as $f^{(1)}$, and $f^{(0)}$ is used for all continuous functions, regardless of their differentiability. For $k \geq 2$ we define a function to be $k$ times differentiable if the derivative $f^{(k)}$ is defined, where recursively $f^{(k)} = f^{(k-1)'}$ is defined with base case $f$. We define a function to be infinite differentiable if it is $k$ times differentiable for all $k \geq 0$ [10].

With the set of definitions explained above we move into categorizing continuous functions by classes defined by how many times are they differentiable. We define $C^n$ as a class of all functions $f : X \longmapsto Y$ where $f$ is $n$ times differentiable. We define $n$ as the order or degree of such function or a class.

From the definition of an infinite differentiable function we have a class of functions denoted as $C^\infty$, which contains all functions that are infinite differentiable. Here we introduce the concept of *smoothness*. If derivatives of a function $f : X \longmapsto Y$ exist for all orders and are continuous in $X$, then $f$ is considered a smooth function as well as infinite differentiable [7]. If a function $f$ belongs to a class of a higher order that $g$, then $f$ is also considered to be smoother than $g$.

## 2.3 Relations among Function Classes

We defined a class $C^m$ as a class containing all continuous functions that are $m$ times differentiable. Observe that if a function $f$ is $m$ times differentiable, it is also $n$ times differentiable for $m > n$. This means that function $m$ belongs to all classes $C^k$ where $m \geq k \geq 0$. Thus we have that $C^m \subset C^n$ for $m > n$, which gives us a total ordering of continuous function classes, where $C^{k+1} \subset C^k$ for any $k \geq 0$ [10]. One important notion is that $C^\infty \subset C^m$ for any $m \geq 0$ .

Moreover, $C^0$ contains all functions that are continuous, regardless of how many times are they differentiable. Thus we settle that all functions belonging to the class $C^0$ can be integrated, as well as perform with them algebraic operations such as multiplication and composition, but they are not specifically regular enough to perform operations such as differentiation [8]. They are still considered as one of the smoother classes of functions in analysis, as those functions that are not continuous are the rougher types of functions.

# 3 Real Analytic Functions

## 3.1 Power Series and Taylor's Series

In this section we will deviate from the previous concepts and review our knowledge on power series to define a particular class of functions. This class is known as real analytic functions, which happen to be a subset of $C^\infty$.

Recall a power series centered at $x_0$ is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n(x-x_0)^n = c_0 + c_1(x-x_0)^1 + c_2(x-x_0)^2 + \dots$$

where $c_n$ represents a coefficient of the $nth$ term and $x_0$ is a constant. We know that any power series has a radius of convergence $r$, where the series converges when $x \in (x_0 - r, x_0 + r)$ .

Assume $r > 0$, and let $f : (x_0 - r, x_0 + r) \to \mathbb{R}$ where we have the power series $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ converge to a value $y \in \mathbb{R}$. Therefore the function $f$ is differentiable on $(x_0 - r, x_0 + r)$. Furthermore, the series $\sum_{n=1}^{\infty} nc_n(x-x_0)^{n-1}$ converges uniformly to $y$ on the interval $(x_0 - r, x_0 + r)$ [1]. This is known as radius of convergence, which remains the same for derivatives of $f$ (for complex functions we have a complex plane instead of an open interval, which we will discuss briefly in **Section 5**).

We define a Taylor's Series as

$$y = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

which is a power series that is expressed in terms of the function's derivatives at a single point $x_0$, which converges to a point which we denote $y$.

## 3.2 Class of Analytic Functions

Not every real or complex-valued function $f(x)$ is equal to its corresponding Power Series, even if the series converges. With that said we introduce the following classification of a function. A function is considered analytic if its Taylor series about a point $x$ converges to it for some interval within its neighborhood, determined by its radius of convergence. The class of all analytic functions is denoted as $C^\omega$. Thus we express

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

where $x \in (x_0 - r, x_0 + r)$ [8]. From this result we can observe that every $f^{(k)}$ of $f$ has an equivalent convergent Taylor series expansion for all $x \in X$ and $k \geq 1$. This assumption follows from the differentiation of the power series $k$ times, which is equivalent to

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

which is also analytic. Since the $k-1$ values of the series have an order lower than $k$, these values get dropped as they get differentiated until they become 0. It follows that $f^{(k)}$ is also real analytic for any $k \geq 1$.

It is worth mentioning that each analytic function has a Taylor series that converge to it which is unique. Also, any operators or linear maps such as sums, products, and compositions of analytic functions are analytic.

# 4 $C^\omega$ and $C^\infty$

We now focus on the relationship between $C^\infty$ and $C^\omega$. It occurs that all analytic functions are smooth; in other words, $C^\omega \subset C^\infty$.

*Proof.* Let $X$ be a subset of $\mathbb{R}$, where $x \in (x_0 - r, x_0 + r)$ for some $x \in \mathbb{R}$ and some radius of convergence $r > 0$. Let $f \in C^\omega$ such that

$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

It follows that $f$ is $k$ times differentiable for every $k \geq 0$, as the Taylor series expansion is well defined. We get that $f^k$ is equal to

$$f^{(k)}(x) = \sum_{n=0}^{\infty} c_{n+k} \frac{(n+k)!}{n!}(x - x_0)^n.$$

Therefore since $f$ is $k$ times differentiable for all $k \geq 0$, then $f$ is infinite differentiable. We conclude that $C^\omega \subset C^\infty$ [10]. $\square$

Even though all analytic functions are smooth, there is no similar implication for the reciprocal. That is, not all smooth functions are analytical. Take in for example the functions known as "bump" functions. Let $f : X \to Y$ such that

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & if x > 0, \\ 0 & if x \leq 0, \end{cases}$$

where $x \in X$ [5]. Observe that the function is both always continuous and differentiable. We also get that its derivative is $f'(x) = \frac{e^{-\frac{1}{x}}}{x^2}$, which is defined. We can continue taking derivatives and the function remains defined. Thus $f$ is infinite differentiable.

Moreover, observe that all derivatives at $x = 0$ are 0. Thus the Taylor series expansion for $f : X \longmapsto Y$ at $x_0 = 0$ is expressed as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} 0 \cdot x^n = 0.$$

Thus we have that the Taylor series converges to 0 when it is centered at $x = 0$. It follows that that the series does not equal $f(x)$ for $x > 0$. We conclude that $f$ is not analytic. as we have showed a case where a function $f$ is smooth but not analytic.

# 5 Complex Analytic Functions

In the complex number space we have that real analytic functions and complex analytic functions behave in pretty much the same way [8]. In this section we will just cover lightly some properties of complex functions and their function classes, omitting any proofs for the sake of simplicity. For function classes $C^0$ to $C^\infty$, we have that complex functions are assigned the same way to this classes as real continuous functions. Given this, there still exists some differences between functions being real or complex analytic.

As its real counterpart, a complex function is said to be analytic if it has a Taylor series expansion that converges to the value of the function centered at a point within some open interval [5]. We have that $f \in C^\omega$ is complex differentiable if

$$f(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

for some $z \in \mathbb{C}$ [2] . In this case, since the complex space $\mathbb{C}$ can be seen as an $\mathbb{R}^2$ space, then we instead have a disk of convergence determined by a radius $r$ of convergence.

Here is where the properties of complex analytic functions start to change. Any smooth complex function is analytic. That is because all of infinite differentiable functions are equivalent to their corresponding Taylor series, and any differentiable complex function implies that it is also infinite differentiable. Thus any differentiable complex function is analytic [2].

From the properties of a real analytic function $f$, we know that if $f$ is analytic then it is smooth. There is no proof for the reciprocal. In the other hand, for a complex valued function $g$ we have that $g$ is analytic if and only if $g$ is complex differentiable [5]. This leads to the technique of analytic continuation, which tries to extend the domain of a given analytic function into the complex space [8]. The term "holomorphic" for complex analytic functions is often used interchangeably.

# 6 Conclusion

Classification of functions by differentiability allows us to better understand their properties and how they relate with other objects. For example, it comes with great benefit to be able to express a large range of functions by their Taylor expansion; and also know the distinction between infinite differentiable and analytic functions.

The conditions of categorization used in function analysis are mere tools to better understand functions, and a particular distinction is not exclusive as many characterizations may exist for the same object [12]. There is a tempting possibility of delving deeper into the properties of complex analytic functions which might lead into interesting theorems and results, but will be stalled until another write-up centered on complex analysis.

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