Higher Homotopy Groups

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1 Introduction

This paper will give a brief introduction to the higher homotopy groups (those of $n > 1$) with some definitions and a presentation on a few fundamental results of the higher groups. All spaces are topological spaces and will be noted as such. We assume all functions are continuous maps between topological spaces unless otherwise stated. Appendix A lists the notations I have chosen for this paper, Appendix B recalls the information presented in class.

1.1 Brief History on Homotopy Theory

During the late 1800's, separate from group theory, french mathematician, Camille Jordan introduced the first ideas of homotopy. In 1895, another french mathematician, Henri Poincaré, published a set of papers titled *Analysis Situs* giving homotopy more rigor where he introduced the fundamental group (as well as the related ideas of homology). It was not until 1932 that the higher homotopy groups were put forth by Czech mathematician Eduard Čech.

2 Excitement

Why is Homotopy theory interesting? This is a perfectly reasonable question to ask. One default answer is that it is so interesting because it is a complicated and complex. To this end it is very difficult to compute the higher homotopy groups of spaces. One set of spaces of interest is $\pi_n(S^m)$, the homotopy groups on the m-sphere, which has no trivial discernible pattern for all n,m![1] In the 50's American mathematician George Whitehead showed the existence of a metastable range of homotopy groups of spheres and today people are still trying to compute the higher homotopy groups of spheres.

3 Higher Groups

Equipped with the basic notions of homotopy theory and the fundamental group from lecture, we can begin with the definition of the n^{th} homotopy group on a topological space X, generalizing the fundamental group $\pi_1(X, x_0)$. The following definition follows that of reference [3].

[1] Save for when $n = m, n > 0$ and for some $n > m$

3.1 Definitions

Definition 3.1. *n^{th} homotopy group*: Let $I^n = I \times ... \times I$ be the n-cube, X a topological space with basepoint $x_0 \in X$. Then

$$\pi_n(X, x_0) := \{[\lambda]_{\partial I^n} | \lambda \in X^{I^n}, \lambda(\partial I^n) = \{x_0\}\}$$

is the n^{th} *homotopy group of* X at x_0. The group structure of $\pi_n(X, x_0)$ is as follows: for $\alpha, \beta \in X^{I^n}$ and $\alpha(\partial I^n) = \{x_0\} = \beta(\partial I^n)$, we define the group operation $*$ as,

$$(\alpha * \beta)(s, t_2, ..., t_n) = \begin{cases} \alpha(2s, t_2, ..., t_n) & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1, t_2, ..., t_n) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

where $t_1 = s$ is the deformation parameter. Next, We define the neutral element of the group, the constant map, $\epsilon_{x_0} : I^n \longrightarrow X,\ (s, t_2, ..., t_n) \mapsto x_0$. Lastly to make $\pi_n(X, x_0)$ a group, we define the inverse $[\alpha]^{-1}_{\partial I^n}$ in which we adjust the deformation parameter in the function to: $\alpha(1-s, t_2, ..., t_n)$.

Claim: *For $n \geq 2$ the group $\pi_n(X, x_0)$ is abelian.*

This will be presented in the next section after we have put forth more notions.

As we have defined in class, $\pi_1(X, x_0)$ is the fundamental group of the topological space X. By higher homotopy groups we refer specifically for $n \geq 2$. Definition 3.1 gives us an idea of the n-cube from the Cartesian product of n unit intervals, I. If we collapse the boundary of the n-cube we will obtain the n-sphere which leads us to definition 3.2; another understanding of what the higher homotopy groups are.

Definition 3.2. *n^{th} homotopy group* (version 2): Let X be topological space with basepoint x_0 then we have the n^{th} *homotopy group* of X defined as:

$$\pi_n(X, x_0) := \{[\lambda] | \lambda \in X^{S^n}, \lambda|_{int} = x_0 = \lambda|_{fin}\}$$

where S^n is the n-sphere.

In definition 3.2 we have made explicit from definition 3.1 the collapsing of the boundary, ∂I to arrive at n-spheres in a topological space X. One can understand the n^{th} *homotopy group* simply as the group of the n-sphere in a topological space.

Definition 3.3. *Nullhomotopic*: A function, f is said to be *nullhomtopic*, if it is homotopy equivalent to the constant map, ϵ.

To show that the $\pi_n(X, x_0)$ is abelian we will need the notions of *suspensions* and *H-cogroups*[2] leading to our next several definitions.

Definition 3.4. *Wedge Sum*: The *wedge sum* of two topological spaces X and Y with basepoints x_0 and y_0 respectively is defined as:

$$X \vee Y := \{(x, y) \in X \times Y | x = x_0 \text{ or } y = y_0\}.$$

[2]for definitions of *H-cogroups* and its structure see appendix B.2

Note $X \vee Y \subset X \times Y$. Intuitively one can think of a wedge sum as taking two (or more) objects and gluing them together at a single point such as four circles giving you the rose with four petals.

Definition 3.5. Suspension: If A is a topological pointed space, the *reduced suspension* is the quotient:

$$\Sigma A = A \times I / A \times \{0\} \cup A \times \{1\} \cup \{a_0\} \times I,$$

where the basepoint becomes the image of $A \times \{0\} \cup A \times \{1\} \cup \{a_0\} \times I$ after it has been collapsed to a point.

Definition 3.6. comultiplication: let A be a topological space with basepoint a_0, Then a map η is a *comultiplication* defined by:

$$\eta : \Sigma A \longrightarrow \Sigma A \vee \Sigma A$$

$$\eta(a(t)) = \begin{cases} (a(t), a_0) & 0 \leq t \leq \frac{1}{2} \\ (a_0, a(2t-1)) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Comultiplication can be thought of as "pinching" the equator of a ΣA see figure 1.

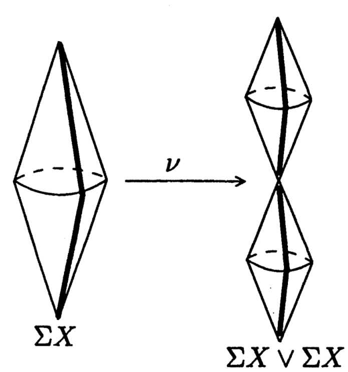


Figure 1: Comuliplication results in 'pinching' equator. Here $\nu = \eta, X = A$. Figure obtained directly from reference [1]

3.2 Fundamental Results

Lemma 3.1. *Let K be a set armed with two operations $*, \star$ s.t.*

1. *$*, \star$ have a common unit, $\epsilon \in K$, and*
2. *$*, \star$ distribute over one another in the following way:*

$$(a * b) \star (x * y) = (a \star b) * (x \star y)$$

then $*, \star$ coincide and are both commutative and associative.

Proof. Let $a, b, x, y \in K$ with unit element $\epsilon \in K$. Then we have from 1 the following chain of equalities:

$$\epsilon * x = x * \epsilon = x = \epsilon \star x = x \star \epsilon.$$

By 1 we have then: $a * b = (a \star \epsilon) * (\epsilon \star b)$. Then by 2 we obtain: $(a \star \epsilon) * (\epsilon \star b) = (a * \epsilon) \star (\epsilon * b)$, which in the end gives us: $(a * \epsilon) \star (\epsilon * b) = a \star b$, thus showing $a * b = a \star b$ and $*, \star$ coincide. Furthermore we also get:

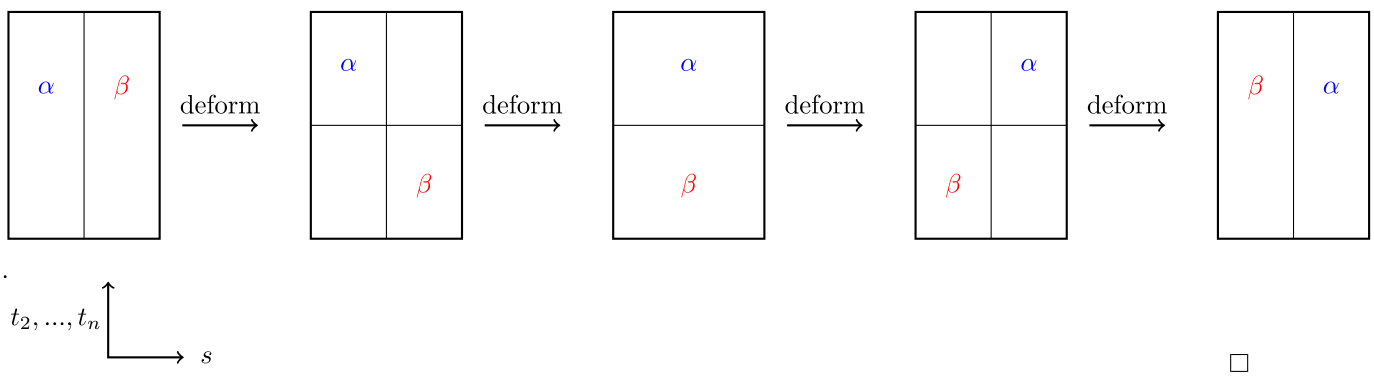
$$a * b = (a \star \epsilon) * (\epsilon \star b) = (\epsilon * b) \star (a * \epsilon) = b \star a = b * a,$$

thus showing the operation $*$ (and equivalently $\star$) is commutative. $\square$

Corollary 3.1.1. *If the set A and B are H-cogroups then the set $[A, B]_*$ is an abelian group.*[3]

Theorem 3.2. *all homotopy groups, $\pi_n(X, x_0)$ with $n \geq 2$ are abelian groups.*

Proof. $\pi_n(X, x_0)$ the group with inverse element and neutral element as defined in definition 3.1, but with group operation $(*)$ as *H-comultiplication* the group $\pi_n(X, x_0)$ becomes abelian. We present the picture:



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$\square$

Remark. In general $\pi_1(X, x_0)$ is non-abelian. For example The figure eight which is $S^1 \vee S^1$ is a non-abelian $\pi_1(X, x_0)$ group.

3.3 Examples

Here I list two examples of $\pi_1(X, x_0)$ found in [1].

1. If $\mathbb{R}^n$ is our topological space with basepoint $\{0\}$, then the fundamental group, $\pi_1(\mathbb{R}^n, \{0\})$, is the trivial group (that is $\pi_1(\mathbb{R}^n, \{0\}) = 0 = \epsilon_{\{0\}}$ the constant map). This is because of the *deformation retract* of $\mathbb{R}^n$, which can be generalized to say that if a space is *contractible* it is *nullhomotopic* and therefore the fundamental group of every contractile space to it's basepoint is the trivial group.
2. the fundamental group of the circle is $\mathbb{Z}$.[4]

3.4 Groups of S^n

Due to the time allowed for this paper I will not give proper proofs of the following claims but present ideas to each of why the claim holds.

[3]definition of H-cogroup is defined in appendix B following reference [1].
[4]As was shown in lecture.

The previous lemma was stated and proved in rigorous detail in lecture. However, we can take this notion of $\pi_1(S^1)$ to a generalization that $\pi_n(S^n) = \mathbb{Z}$ as stated in Theorem 3.5.

Theorem 3.3. *$\pi_n(S^m) = 0$ for $n < m$.*

Proof. In the case for $n = 1$, $m = 2$ ($\pi_1(S^2)$), we have mapping of the circle onto the sphere, and let s_0 denote the basepoint of the sphere. The circle can be continuously deformed around the sphere to a point. Since the circle is *contractible* to the basepoint we have that all $\lambda \in \pi_1(S^2) \sim \epsilon_{s_0}$ and therefore the $\pi_1(S^2) = 0$ is the trivial group, where zero denotes ϵ_{s_0} the constant path. ☐

Theorem 3.4. *$\pi_n(S^m) = \mathbb{Z}$ for $n = m$.*

Proof. Intuitively we can think of this group as wrapping the n-sphere around itself. Consider the ordinary 2-sphere. We could take a bag and wrap it around the sphere then twist it and wrap it around a second time. In this way we arrive at natural numbers n, where n is a twist of the bag around the sphere. We call the total number of twists the *degree*. If we were to then twist the bag in the opposite direction we say $-n$ and then have $\mathbb{Z}$. ☐

π_n Homotopy Groups						
	π_1	π_2	π_3	π_4	π_5	π_6
S^0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$

Table 1: Table of m-spheres and corresponding homotopy groups up to $n = m = 6$

4 Some Applications and Other Examples

In this section I will just list some of the interesting applications that follow from the higher homotopy groups of spheres.

1. The *degree* or *winding number* can be used to prove that every non-constant complex polynomial has a root; the so called fundamental theorem of algebra.

2. We can use the fact that $\pi_m(S^m) = \mathbb{Z}$ to obtain the Brouwer Fixed point Theorem which is a very interesting result; every continuous map from n-sphere to itself has a fixed point.

3. Singularity theory which studies the structure of singular points of smooth maps takes large value in the stable homotopy groups of spheres. Singularities arise as critical points of smooth maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. The geometry near a critical point can be described from the elements of $\pi_{m-1}(S^{n-1})$ by considering how the $m-1$-sphere is mapped into the $n-1$-sphere about the critical point.

Appendix A Some Notation

1. B^A denotes the set of all continuous functions, f, from set A to set B ($f : A \longrightarrow B$).
2. $F|_\alpha$ denotes the function F evaluated at α.
3. ∂I denotes the boundary of I.
4. $[f]$ denotes the equivalence class of all functions homotopic equivalent to f. and $[f]_{\partial I}$ denotes the homotopy class of f relative to to ∂I.
5. $A \sim_{\{0,1\}} B$ denotes A is homotopy equivalent to B relative $\{0,1\}(= \partial I)$
6. $I = [0,1]$ the unit interval.

Appendix B Definitions

B.1 From Lecture

Definition B.1. Homotopy: Let X,Y be topological spaces and $f, g : X \longrightarrow Y$ continuous maps, Then a *homotopy* is a continuous map $H : X \times I \longrightarrow Y$ s.t. $H|_0 = f$, $H|_1 = g$, and $H|_t = H(,t)$, where $t \in I = [0, 1]$. If f is homoptopic to g we write $f \sim_H g$.

Definition B.2. Homotopy relative ∂I: Let X be a topological space and $\gamma : I \longrightarrow X$ a path. We say $\tilde{\gamma} : I \longrightarrow X$ is homotopy equivalent to γ relative to $\{0,1\}$ if $\exists$ a homotopy $\gamma \sim_H \tilde{\gamma}$ s.t. $H(0, t) = \gamma(0)$ and $H(1, t) = \gamma(1)\ \forall\ t \in I$. We say $\gamma(0) = x_0$ is the initial point and $\gamma(1) = x_1$ is the endpoint.

Definition B.3. the fundamental group: Let X be a topological space, $x_0 \in X$ it's basepoint. *the fundamental group* of X at x_0 is the set $\pi_1(X, x_0) = \{[\lambda] | \lambda \text{ is a loop based at } x_0\}$[1].

Definition B.4. operation $*$ on the fundamental group: Let $\delta, \gamma : I \longrightarrow X$ be loops at x_0. We define $\varrho : I \longrightarrow X$ as the loop path $\varrho = \delta * \gamma$,

$$t \mapsto \begin{cases} \delta(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

that *appends* γ to δ.

Definition B.5. Lift: Let $f : X \longrightarrow Y$, $g : A \longrightarrow Y$ be maps with topological spaces A, X, and Y. If $\exists \hat{f} : X \longrightarrow A$ s.t. $f = g \circ \hat{f} : X \longrightarrow Y$ then $\hat{f}$ is a *lift (or lifting) of f.* If $g \circ \hat{f} \sim f$ then we say $\hat{f}$ is a *homotopy lifting of f.*

B.2 From Resources

Definition B.6. *H-cospace*: Let A be a topological space with basepoint a_0. We say A is a *H-cospace* if it is furnished with a continuous map

$$\rho : A \longrightarrow A \vee A,$$

s.t. if $\epsilon : A \longrightarrow A$ is the constant map to the basepoint ($\epsilon(A) = a_0$) then the composites,

$$A \xrightarrow{\rho} A \vee A \xrightarrow{\langle id, \epsilon \rangle} A,\ A \xrightarrow{\rho} A \vee A \xrightarrow{\langle \epsilon, id \rangle} A$$

are homotopic to the identity of A. Where ρ is called an *H-comultiplication.*

Definition B.7. *H-coassociative*: Let A be a topological space as in the previous definition, We say that A is *H-coassociative* if $(\rho \vee id) \circ \rho, (id \vee \rho) \circ \rho : A \longrightarrow A \vee A \vee A$ are homotopic.

Definition B.8. *H-coinverse*: if the composites:

$$A \xrightarrow{\rho} A \vee A \xrightarrow{\langle id, b \rangle} A,\ A \xrightarrow{\rho} A \vee A \xrightarrow{\langle b, id \rangle} A$$

Where ρ and A are as in definition B.6, are homotopic to ϵ (the identity function on A) with $b : A \longrightarrow A$ then b determines coinverses up to homotopy called *H-coinverses.*

Definition B.9. *H-cogroup*: an *H-coassociative, H-cospace* with a map deciding *H-coinverses* is called an *H-cogroup.*

B.3 From Paper

Definition B.10. *n^{th} homotopy group*: Let $I^n = I \times ... \times I$ be the n-cube, X a topological space with basepoint $x_0 \in X$. Then

$$\pi_n(X, x_0) := \{[\lambda]_{\partial I^n} | \lambda \in X^{I^n}, \lambda(\partial I^n) = \{x_0\}\}$$

is the n^{th} *homotopy group of* X at x_0. The group structure of $\pi_n(X, x_0)$ is as follows: for $\alpha, \beta \in X^{I^n}$ and $\alpha(\partial I^n) = \{x_0\} = \beta(\partial I^n)$, we define the group operation $*$ as,

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where $t_1 = s$ is the deformation parameter. Next, We define the neutral element of the group, the constant map, $\epsilon_{x_0} : I^n \longrightarrow X$, $(s, t_2, ..., t_n) \mapsto x_0$. Lastly to make $\pi_n(X, x_0)$ a group, we define the inverse $[\alpha]^{-1}_{\partial I^n}$ in which we adjust the deformation parameter in the function to: $\alpha(1-s, t_2, ..., t_n)$.

Definition B.11. *n^{th} homotopy group* (version 2): Let X be topological space with basepoint x_0 then we have the n^{th} *homotopy group* of X defined as:

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where S^n is the n-sphere.

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Definition B.13. *Wedge Sum*: The *wedge sum* of two topological spaces X and Y with basepoints x_0 and y_0 respectively is defined as:

$$X \vee Y := \{(x, y) \in X \times Y | x = x_0 \text{ or } y = y_0\}.$$

Definition B.14. *Suspension*: If A is a topological pointed space, the *reduced suspension* is the quotient:

$$\Sigma A = A \times I / A \times \{0\} \cup A \times \{1\} \cup \{a_0\} \times I,$$

where the basepoint becomes the image of $A \times \{0\} \cup A \times \{1\} \cup \{a_0\} \times I$ after it has been collapsed to a point.

Definition B.15. *comultiplication*: let A be a topological space with basepoint a_0, Then a map η is a *comultiplication* defined by:

$$\eta : \Sigma A \longrightarrow \Sigma A \vee \Sigma A$$

$$\eta(a(t)) = \begin{cases} (a(t), a_0) & 0 \leq t \leq \frac{1}{2} \\ (a_0, a(2t-1)) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

References

[1] M. Aguilar, S. Gitler, C. Prieto, *Algebraic Topology from a Homotopical Viewpoint*, Springer, 2002.

[2] M. Arkowitz, *Introduction to Homotopy Theory*, Springer, 2011.

[3] C. Bär, *Algebraic Topology, Lecture Notes*, Christian Bär, 2017.

[4] J. Rotman, *Introduction to Algebraic Topology*, Springer-Verlang, 1988.

Some of the qualitative descriptions, applications, and history was adapted from the homotopy groups of spheres Wikipedia page on `https://en.wikipedia.org/wiki/Homotopy_groups_of_spheres`.