1. Compute QR factorization of

$$A = Q R$$
$m \times n \quad n \times n$

$$A\vec{x} = \vec{b}$$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

Gram-Schmidt: $\vec{u}_1 = \vec{v}_1 \Rightarrow \|\vec{v}_1\| = \sqrt{10}$ $\qquad \vec{e}_1 = \begin{bmatrix} \frac{3\sqrt{10}}{10} \\ 0 \\ \frac{\sqrt{10}}{10} \end{bmatrix}$

$$\vec{u}_2 = \vec{v}_2 - \frac{<\vec{v}_2, \vec{u}_1>}{\|\vec{u}_1\|^2}\vec{u}_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} - 0 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \qquad \|\vec{v}_2\| = 4$$

$$<\vec{v}_2, \vec{u}_1> = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{3\sqrt{10}}{10} \\ 0 \\ \frac{\sqrt{10}}{10} \end{bmatrix} = 0$$

$$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{3\sqrt{10}}{10} & 0 \\ 0 & 1 \\ \frac{\sqrt{10}}{10} & 0 \end{bmatrix} \qquad Q^T = \begin{bmatrix} \frac{3\sqrt{10}}{10} & 0 & \frac{\sqrt{10}}{10} \\ 0 & 1 & 0 \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} \frac{3\sqrt{10}}{10} & 0 & \frac{\sqrt{10}}{10} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 2 \end{bmatrix} = R$$

2. Find the closest line in a weighted least squares that approx (0,0), (1,2), (2,1), (3,6) w/ weights $2/10$, $3/10$, $3/10$, $2/10$

$$A^T C A \vec{x} = A^T C \vec{b}$$

$$\vec{x} = (A^T C A)^{-1} A^T C \vec{b}$$

$$C = \begin{bmatrix} 2/10 & 0 & 0 & 0 \\ 0 & 3/10 & 0 & 0 \\ 0 & 0 & 3/10 & 0 \\ 0 & 0 & 0 & 2/10 \end{bmatrix}$$

$$\begin{bmatrix}0\\2\\1\\6\end{bmatrix} = M\begin{bmatrix}0\\1\\2\\3\end{bmatrix} + b\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \Rightarrow \underset{\vec{b}}{\begin{bmatrix}0\\2\\1\\6\end{bmatrix}} = \underset{A}{\begin{bmatrix}0 & 1\\1 & 1\\2 & 1\\3 & 1\end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix}M\\b\end{bmatrix}}$$

$$A^T C = \underset{2\times 4}{\begin{bmatrix}0 & 1 & 2 & 3\\1 & 1 & 1 & 1\end{bmatrix}} \underset{4\times 4}{\begin{bmatrix}2/10 & 0 & 0 & 0\\0 & 3/10 & 0 & 0\\0 & 0 & 3/10 & 0\\0 & 0 & 0 & 2/10\end{bmatrix}} = \begin{bmatrix}0 & \frac{3}{10} & \frac{3}{5} & \frac{3}{5}\\ \frac{2}{10} & \frac{3}{10} & \frac{3}{10} & \frac{2}{10}\end{bmatrix}$$

$$A^T C A = \underset{2\times 4}{\begin{bmatrix}0 & \frac{3}{10} & \frac{3}{5} & \frac{3}{5}\\ \frac{2}{10} & \frac{3}{10} & \frac{3}{10} & \frac{2}{10}\end{bmatrix}} \underset{4\times 2}{\begin{bmatrix}0 & 1\\1 & 1\\2 & 1\\3 & 1\end{bmatrix}} = \begin{bmatrix}3.3 & 1.5\\1.5 & 1\end{bmatrix}$$

$$(A^T C A)^{-1} = \frac{1}{1.05}\begin{bmatrix}1 & -1.5\\-1.5 & 3.3\end{bmatrix} = \begin{bmatrix}0.952 & -1.429\\-1.429 & 3.143\end{bmatrix}$$

$$(A^T C A)^{-1} A^T C \vec{b} = \begin{bmatrix}0.952 & -1.429\\-1.429 & 3.143\end{bmatrix}\begin{bmatrix}4.8\\2.1\end{bmatrix} = \begin{bmatrix}1.5689\\-0.2589\end{bmatrix} = \begin{bmatrix}M\\b\end{bmatrix}$$

$$y = 1.5689x - 0.2589$$

3 $u_t = Du_{xx}$, derive G using backward Euler for the time derivative & centered difference for the second order spatial derivative. Let $u = G^n e^{ikx}$

$$\frac{du}{dt} = D\frac{d^2u}{dx^2} \qquad y^{(n+1)} = \frac{y^{(n)}}{1+\Delta t}$$

Let $A = 1$

$$u_t = \frac{u(x,t) - u(x, t-\Delta t)}{\Delta t} = \frac{G^n e^{ikx} - G^{n-1} e^{ikx}}{\Delta t}$$

Centered Differences:

$$\frac{u(x-\Delta x, t) - 2u(x,t) + u(x+\Delta x, t)}{\Delta x^2} = u_{xx}$$

$$u_{xx} = \frac{G^n e^{ik(x-\Delta x)} - 2G^n e^{ikx} + G^n e^{ik(x+\Delta x)}}{\Delta x^2}$$

$$u_t = D u_{xx}$$

$$\frac{G^n e^{ikx} - G^{n-1} e^{ikx}}{\Delta t} = D \frac{G^n e^{ik(x-\Delta x)} - 2G^n e^{ikx} + G^n e^{ik(x+\Delta x)}}{\Delta x^2}$$

$$G^n e^{ikx} - \frac{G^n}{G} e^{ikx} = D\left(\frac{\Delta t}{\Delta x^2}\right)\left[G^n e^{ikx} e^{-ik\Delta x} - 2G^n e^{ikx} + G^n e^{ikx} e^{ik\Delta x}\right]$$

$$\cancel{G^n e^{ikx}}\left(1 - \frac{1}{G}\right) = \left(\frac{\Delta t}{\Delta x^2}\right)\left[\left(\cancel{G^n e^{ikx}}\right)\left(e^{-ik\Delta x} - 2 + e^{ikx}\right)\right] D$$

$$1 - \frac{1}{G} = \frac{D\Delta t}{\Delta x^2}\left(e^{-ik\Delta x} - 2 + e^{ikx}\right)$$

$$\frac{1}{G} = 1 - \frac{D\Delta t}{\Delta x^2}\left(2\cos(k\Delta x) - 2\right)$$

$$G = \frac{1}{1 - \frac{D\Delta t}{\Delta x^2}\left(2\cos(k\Delta x) - 2\right)}$$

④ Consider $f(x) = 4 + 8x^2 - x^4$

a) $f'(x) = 16x - 4x^3$ $\quad f''(x) = 16 - 12x^2$

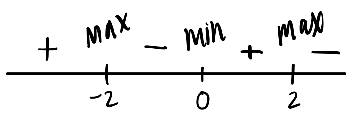
b) y-int $= (0, 4)$, maxima @ $(-2, 20) \cup (2, 20)$, minima @ $(0, 4)$

pts of inflection: $\left(-\frac{2\sqrt{3}}{3}, 12.889\right) \cup \left(\frac{2\sqrt{3}}{3}, 12.889\right)$

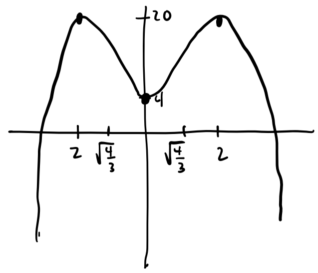
$16x - 4x^3 = 0$

$4x(4 - x^2) = 0$ $\quad x = 0, -2, 2$ $\quad 16 - 12x^2 = 0$

$x = \pm\sqrt{4/3}$



c)



even

d) $x^{(0)} = 3$ $\quad x^{(1)} = 3 - \frac{(-5)}{-60} = \frac{35}{12}$ $\quad x^{(2)} = \frac{35}{12} - \frac{(-.3125)}{(-4.8786)} = 2.853$

⑤ Consider $f(x) = x^3 - 3x - 3$ $\quad f'(x) = 3x^2 - 3$ $\quad f''(x) = 6x$

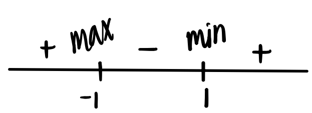
a) max: $(-1, -1)$

$3(x^2 - 1)$

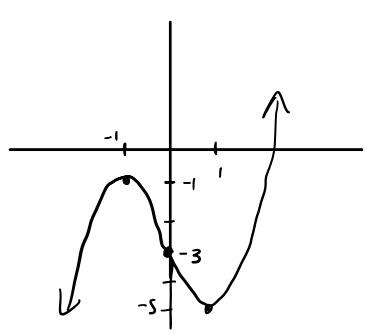
min: $(1, -5)$

inflection pt: $(0, -3)$

y-int: $(0, -3)$



b) Neither even nor odd



c) $x^{(0)} = 2$ $\qquad x^{(1)} = 2 - \frac{-1}{9} = \frac{19}{9}$ $\qquad x^{(2)} = \frac{19}{9} - \frac{.075}{10.370} = \boxed{2.104}$

⑥ Python file submitted. output below.

```
PS C:\Users\amycu> & C:/Users/amycu/AppData/Local/Microsoft/WindowsApps/python3.11.exe "c:/Users/amycu/OneDrive/Documents/Junior Year/Adv Scientific Eng/HW4/explicit bisection method.p
y"
root =
1.7320508075688776
PS C:\Users\amycu>
```