① a) <u>Exact Solution</u> $\frac{dy}{dt} = -\alpha y \quad y(0)=1,\ \alpha>0,\ 0 \le t \le T$

$$\int \frac{1}{y}\,dy = \int -\alpha\,dt$$

$$\ln y = -\alpha t + C$$

$$y = e^{-\alpha t}$$

b) Derive the <u>update eqn</u> using forward Euler time discretization.

$$y^{(n+1)} = (1-\alpha\Delta t)y^n = (1-\alpha\Delta t)^n$$

c) <u>Stability condition</u> for a general $\alpha, \Delta t$

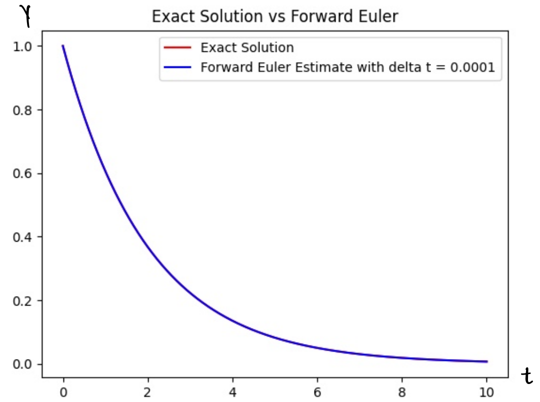
if $|1-\alpha\Delta t| > 1 \Rightarrow$ unstable $\therefore$ $0 < \alpha\Delta t < 2$

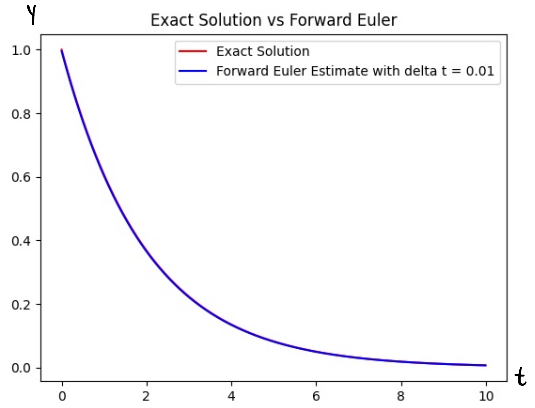
$$1-\alpha\Delta t > -1 \qquad\qquad 1-\alpha\Delta t > 1$$

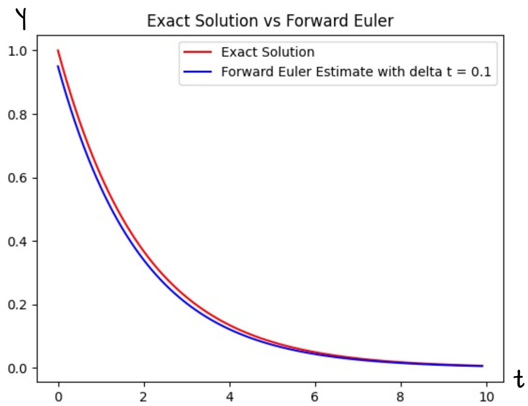
$$-\alpha\Delta t > -2 \qquad\qquad -\alpha\Delta t > 0$$

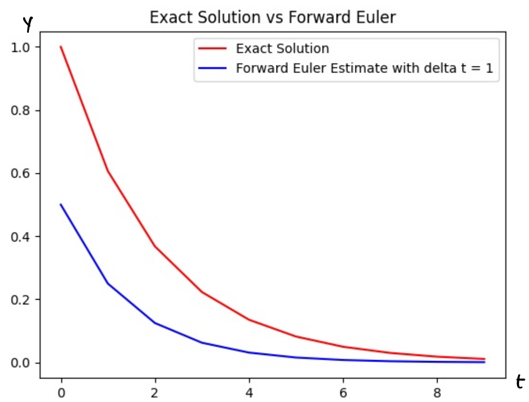
$$\alpha\Delta t < 2$$

d & e) Python file for code submitted. Graphs can be found here.






Exact Solution vs Forward Euler
Exact Solution
Forward Euler Estimate with delta t = 0.1
Y
t
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10


Exact Solution vs Forward Euler
Exact Solution
Forward Euler Estimate with delta t = 1
Y
t
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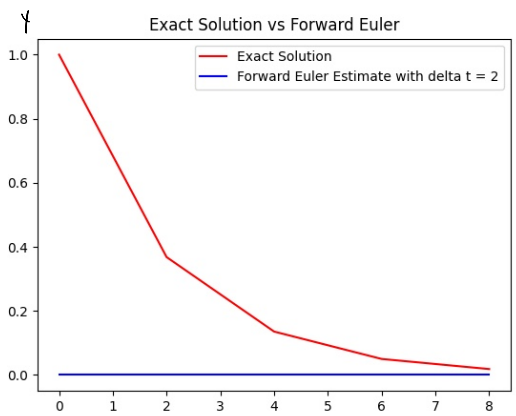
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8


Exact Solution vs Forward Euler
Exact Solution
Forward Euler Estimate with delta t = 2
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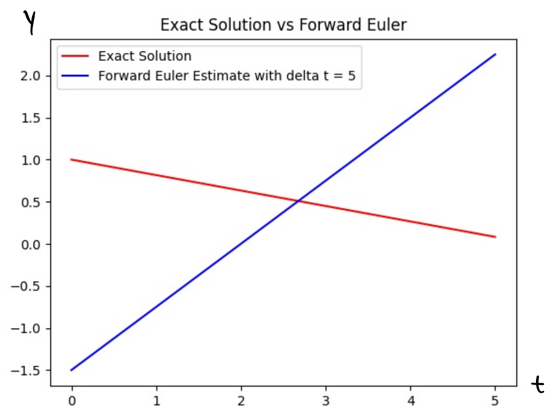
0.6
0.4
0.2
0.0
0
1
2
3
4
5
6
7
8


Exact Solution vs Forward Euler
Exact Solution
Forward Euler Estimate with delta t = 5
Y
t
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
1
2
3
4
5

DIVERGES

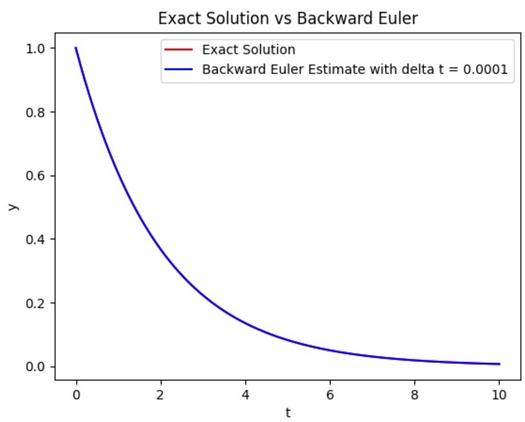
2 For the same IVP

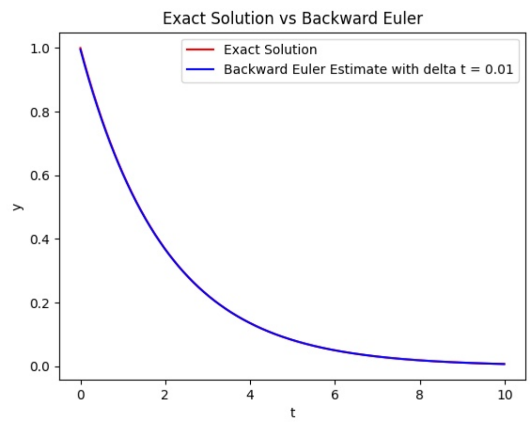
a) Derive the update equation using Backward Euler Time discretization

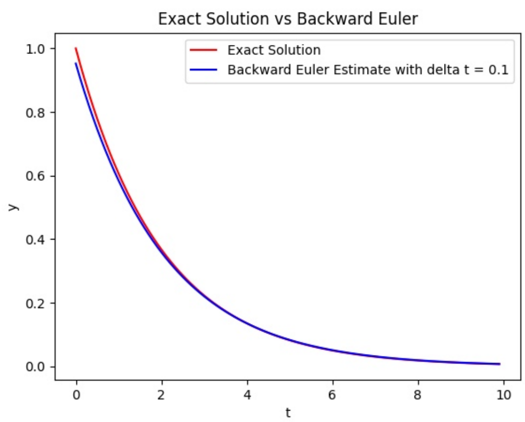
$$\frac{dy}{dt} = -\alpha y \quad y(0) = 1 \ , \ \alpha > 0 \ , \ 0 \le t \le T$$

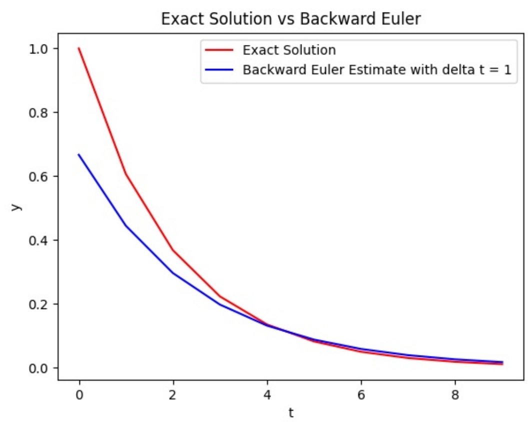
$$y^{(n+1)} = \frac{y^{(n)}}{1 + \alpha \Delta t}$$

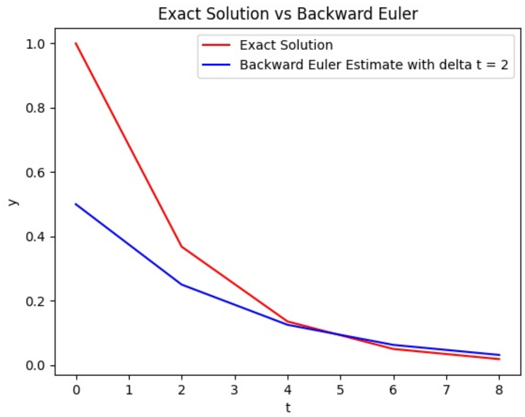
b & c) Code will be submitted as a separate file, graphs can be found here.

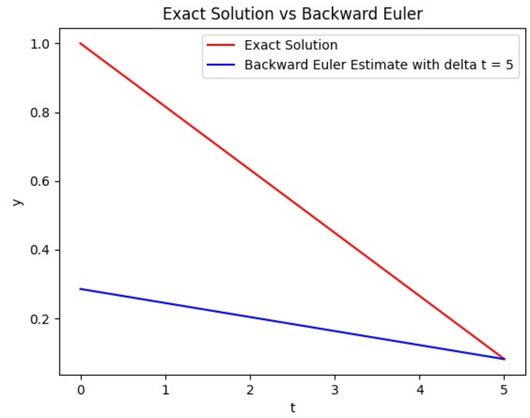










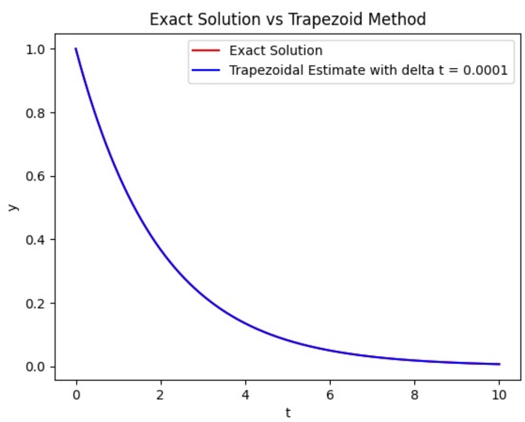


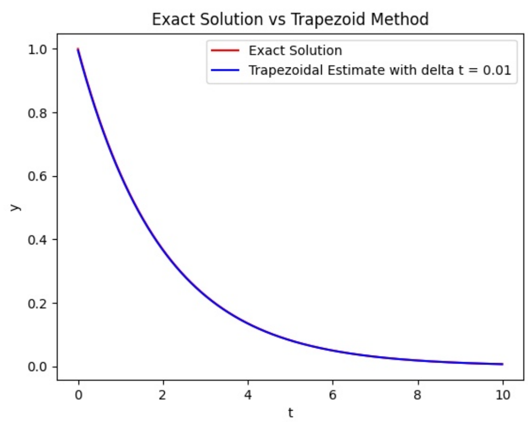
③ For the same IVP $\frac{dy}{dt} = -\alpha y,\ y(0)=1,\ \alpha>0,\ 0 \le t \le T$

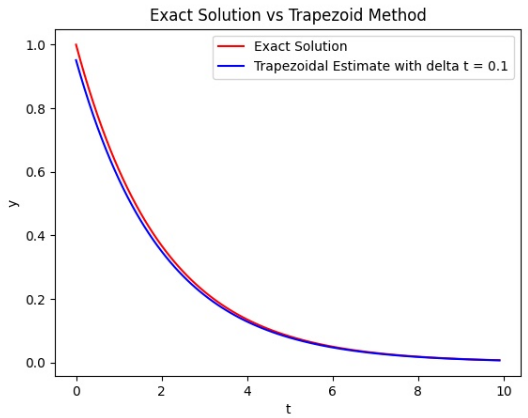
a) Derive update eqn using trapezoid method

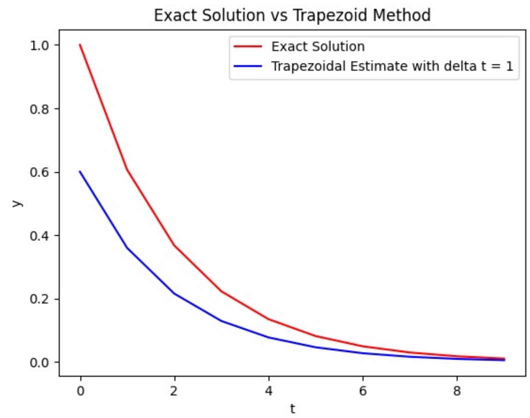
$$y^{(n+1)} = \frac{y^{(n)}\left(1 - \frac{\Delta t}{2}\alpha\right)}{1 + \frac{\Delta t}{2}\alpha}$$

b & c) Code will be attached. Graphs can be found here.






Exact Solution vs Trapezoid Method
Exact Solution
Trapezoidal Estimate with delta t = 0.1
y
t
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10


Exact Solution vs Trapezoid Method
Exact Solution
Trapezoidal Estimate with delta t = 1
y
t
1.0
0.8
0.6
0.4
0.2
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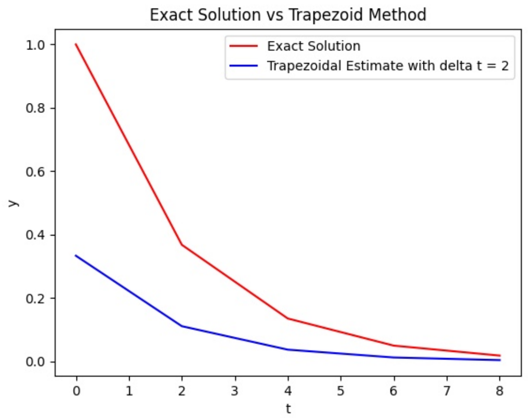
0
2
4
6
8


Exact Solution vs Trapezoid Method
Exact Solution
Trapezoidal Estimate with delta t = 2
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t
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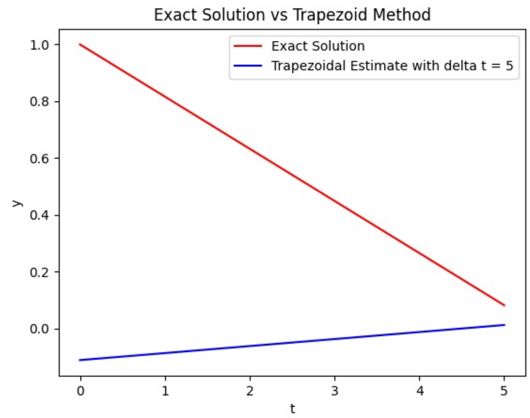
0.6
0.4
0.2
0.0
0
1
2
3
4
5
6
7
8


Exact Solution vs Trapezoid Method
Exact Solution
Trapezoidal Estimate with delta t = 5
y
t
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5