Discrete Morse Theory for Sequence of Cosheaves

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1 Introduction

Topological Data Analysis leverages persistent homology to study the shape of data. In a dataset that contains n points, when we convert it to a big simplicial complex, there are $\binom{n}{1}$ 0-simplices, $\binom{n}{2}$ 1-simplices, $\binom{n}{3}$ 2-simplices, and so on. In total we have $2^n - 1$ simplices. Our eventual goal is to calculate the dimension of its image and kernel of the boundary operator in each dimension. The complexity using Gaussian elimination will be $O((2^n-1)^3)$, which blows up when n increase. A possibly efficient way of solve this issue is to reduce the number of simplices that we need to consider. Morse Theory gives an impressive result on this. Instead of dealing with all the simplices, it shows we can only working on a subset of the original simplicial complex. This has been illustrated already in the lecture notes [2].

Recently, additional techniques such as sheaf and cosheaf theory are developed in TDA [1]. Similar as the normal case, it still faces a big computation cost. Fortunately, the discrete Morse theory can still be applied to reduce the (co)sheaf (co)homology computation. In this project, we prove the discrete Morse theory can simplify computation of persistent homology for sequences of cosheaves over finite simplicial complexes. It worth to notice that we only consider (real) finite dimensional vector spaces in this essay. In the end, we also present an algorithm that can be implemented in software and illustrating with an example.

2 Cosheaf

Let K be a simplicial complex, denote $(K, \leq)$ to be the poset of simplices in K ordered by face relation.

Definition 1. *A* ***cosheaf*** *over a finite simplicial complex K is a functor $\mathscr{C} : (K, \leq) \to \mathrm{Vect}_{\mathbb{F}}$, where $\mathscr{C}$ assigns:*

1. *to each simplex τ in K a (real) finite dimensional $\mathbb{F}$-vector space $\mathscr{C}(\tau)$, called* ***costalk****.*
2. *to each face relation $\tau \geq \tau'$ in K a linear map $\mathscr{C}(\tau \geq \tau') : \mathscr{C}(\tau) \to \mathscr{C}(\tau')$, called* ***extension map****,*

subject to the usual categorical axioms:

- *identity: the map $\mathscr{C}(\tau \geq \tau)$ is the identity map on $\mathscr{C}(\tau)$*
- *associativity: for all triple of simplices $\tau \geq \tau' \geq \tau''$, $\mathscr{C}(\tau' \geq \tau'') \circ \mathscr{C}(\tau \geq \tau') = \mathscr{C}(\tau \geq \tau'')$.*

Definition 2. *For each dimension $k \geq 0$, the vector space of* ***k-chains of K wtih $\mathscr{C}$-coefficients*** *is the product*

$$C_k(K, \mathscr{C}) = \prod_{dim\ \tau = k} \mathscr{C}(\tau)$$

of the costalks of $\mathscr{C}$ over all the k-dimensional simplices of K.

For each $k \geq 0$, the ***kth boundary map of K with $\mathscr{C}$-coefficients*** *is the linear map*

$$\partial_k^{\mathscr{C}} : C_k(K, \mathscr{C}) \longrightarrow C_{k-1}(K, \mathscr{C})$$

defined via the following block-action: for each pair of simplices $\tau \geq \tau'$ with $\dim \tau = k$, the $\mathscr{C}(\tau) \to \mathscr{C}(\tau')$ component of $\partial_k^{\mathscr{C}}$ is given by:

$$\partial_k^{\mathscr{C}}\big|_{\tau', \tau} = [\tau' : \tau] \cdot \mathscr{C}(\tau \geq \tau')$$

where

$$[\tau':\tau] = \begin{cases} +1 & \tau' = \tau_{-i} \textit{ for } i \textit{ even} \\ -1 & \tau' = \tau_{-i} \textit{ for } i \textit{ odd} \\ 0 & \textit{otherwise.} \end{cases}$$

As an analogous to sheaf, every cosheaf $\mathscr{C}$ on K induces a chain complex

$$\ldots \xrightarrow{\partial^{\mathscr{C}}_{k+1}} C_k(K,\mathscr{C}) \xrightarrow{\partial^{\mathscr{C}}_{k}} C_{k-1}(K,\mathscr{C}) \xrightarrow{\partial^{\mathscr{C}}_{k-1}} \ldots \xrightarrow{\partial^{\mathscr{C}}_{2}} C_1(K,\mathscr{C}) \xrightarrow{\partial^{\mathscr{C}}_{1}} C_0(K,\mathscr{C}) \xrightarrow{\partial^{\mathscr{C}}_{0}} 0$$

The columns and rows of $\partial^{\mathscr{C}}_k$ are indexed by (the costalks of) all $k-1$ simplices and k simplices respectively. Before defining the homology group, we must check this is a chain complex over $\mathbb{F}$.

Proposition 1. *Let $\mathscr{C}$ be a cosheaf over simplicial complex K, for any $k \geq 0$, $\partial^{\mathscr{C}}_k \circ \partial^{\mathscr{C}}_{k+1} = 0$.*

Proof. We want to show $\partial^{\mathscr{C}}_k \circ \partial^{\mathscr{C}}_{k+1}|_{\tau'',\tau} = 0$ for any $(k+1)$-simplex τ and $(k-1)$-simplex τ''. To this end, let $v \in \mathscr{C}(\tau)$ be a vector,

$$\begin{aligned} \partial^{\mathscr{C}}_k \circ \partial^{\mathscr{C}}_{k+1} &= \sum_{\dim \tau' = k} \partial^{\mathscr{C}}_k|_{\tau',\tau''} \circ \partial^{\mathscr{C}}_{k+1}|_{\tau,\tau'}(v) && (1) \\ &= \sum_{\dim \tau' = k} [\tau'':\tau'][\tau':\tau]\mathscr{C}(\tau' \geq \tau'') \circ \mathscr{C}(\tau \geq \tau')(v) && (2) \\ &= \left(\sum_{\dim \tau' = k} [\tau'':\tau'][\tau':\tau] \right) \mathscr{C}(\tau \geq \tau'')(v) && (3) \end{aligned}$$

Notice $\sum_{\dim \tau'=k} [\tau'':\tau'] [\tau':\tau] = \partial_k \circ \partial_{k+1}|_{\tau'',\tau}$ in the definition 3.4 in [2], where ∂_k is the standard boundary operator of K. In prop 3.5 in the lecture notes, we have proved this is indeed a boundary operator, thus this sum equals to zero. Therefore, $\partial^{\mathscr{C}}_k \circ \partial^{\mathscr{C}}_{k+1}(v) = 0$ for any v, and $\partial^{\mathscr{C}}_k \circ \partial^{\mathscr{C}}_{k+1}$ is a zero map as desired. $\square$

We now have chain complex and boundary operator, it is then natural to define the homology on cosheaf.

Definition 3. *For each dimension $k \geq 0$, the k**th homology group on** K **with coefficients in** $\mathscr{C}$ is the quotient vector space*

$$H_k(K,\mathscr{C}) = \ker \partial^{\mathscr{C}}_k / \operatorname{im} \partial^{\mathscr{C}}_{k+1}$$

where $\ker \partial^{\mathscr{C}}_k$ means the kernel of $\partial^{\mathscr{C}}_k$ and $\operatorname{im} \partial^{\mathscr{C}}_{k+1}$ means the image of $\partial^{\mathscr{C}}_{k+1}$ map.

We denote $H^{\mathscr{C}}_k = H_k(K,\mathscr{C})$ for simplicity.

Example 1. *Let V be a $\mathbb{F}$-vector space. The **constant cosheaf** $\underline{V}_k$ is the cosheaf $\mathscr{C}$ such that $\forall \tau, \tau' \in K, \mathscr{C}(\tau) = \mathscr{C}(\tau') = V$ and $\mathscr{C}(\tau \geq \tau') : \mathscr{C}(\tau) \to \mathscr{C}(\tau')$ is the identity map from V to V.*

3 Morphisms

Definition 4. *A **morphism of cosheaves** $\phi_\bullet : \mathscr{C} \to \mathscr{C}'$ over K consists of linear maps $\phi_\tau : \mathscr{C}(\tau) \to \mathscr{C}'(\tau)$ indexed by simplices $\tau \in K$, such that for each $\tau \geq \tau'$, $\mathscr{C}'(\tau \geq \tau') \circ \phi_\tau = \phi_{\tau'} \circ \mathscr{C}(\tau \geq \tau')$, equivalently, it means the following diagram of vector spaces commutes:*

$$\begin{tikzcd} \mathscr{C}(\tau) \arrow[r, "\phi_\tau"] \arrow[d, "\mathscr{C}(\tau \geq \tau')"'] & \mathscr{C}'(\tau) \arrow[d, "\mathscr{C}'(\tau \geq \tau')"] \\ \mathscr{C}(\tau') \arrow[r, "\phi_{\tau'}"'] & \mathscr{C}'(\tau') \end{tikzcd}$$

The map between chain groups $C_k^{\mathscr{C}}$ and $C_{k-1}^{\mathscr{C}}$ is the product of the morphism indexed by the k-simplices. For example, $\phi_k = (\phi_{\tau_1}, \ldots, \phi_{\tau_n})$, where $\tau_1, \ldots, \tau_n$ are all k-simplices. Since each ϕ_τ is a morphism, for each dimension $k \geq 0$, $\partial_k^{\mathscr{C}'} \circ \phi_k = \phi_{k-1} \circ \partial_k^{\mathscr{C}}$. Therefore, $\phi_\bullet$ is a chain map, where this ϕ is $\mathbb{N}$-indexed. To simplify our notation, denote $(C_\bullet(K, \mathscr{C}), \partial_\bullet^{\mathscr{C}}) = (C_\bullet^{\mathscr{C}}, \partial_\bullet^{\mathscr{C}})$.

Proposition 2. *Let $\phi_\bullet : (C_\bullet^{\mathscr{C}}, \partial_\bullet^{\mathscr{C}}) \longrightarrow \left(C_\bullet^{\mathscr{C}'}, \partial_\bullet^{\mathscr{C}'}\right)$ be a chain map, for all dimension $k \geq 0$, there is a well-defined $\mathbb{F}$-map $H_k\phi$: $H_k\left(C_\bullet^{\mathscr{C}}, \partial_\bullet^{\mathscr{C}}\right) \longrightarrow H_k\left(C_\bullet^{\mathscr{C}'}, \partial_\bullet^{\mathscr{C}'}\right)$ induced by ϕ_k.*

Proof. To show $\phi_\bullet$ induces a map on quotient vector space $H_k\left(C_\bullet^{\mathscr{C}}, \partial_\bullet^{\mathscr{C}}\right) = \ker \partial_k^{\mathscr{C}} / \operatorname{im} \partial_{k+1}^{\mathscr{C}}$. We suffice to show ϕ_k maps $\ker \partial_k^{\mathscr{C}}$ to $\ker \partial_k^{\mathscr{C}'}$ and $\operatorname{im} \partial_{k+1}^{\mathscr{C}}$ to $\operatorname{im} \partial_{k+1}^{\mathscr{C}'}$ for all k. Consider $\xi \in C_k^{\mathscr{C}}$ such that $\partial_k^{\mathscr{C}}(\xi) = 0$. Since ϕ_{k-1} is a linear map, then $\phi_{k-1} \circ \partial_k^{\mathscr{C}}(\xi) = \phi_{k-1}(0) = 0$. By the commutative property of ϕ_k, We have

$$\partial_k^{\mathscr{C}'} \circ \phi_k(\xi) = \phi_{k-1} \circ \partial_k^{\mathscr{C}}(\xi) = 0$$

Therefore $\phi_k(\xi) \in \ker \partial_k^{\mathscr{C}'}$ as desired.

Next, let $\alpha \in \operatorname{im} \partial_{k+1}^{\mathscr{C}} \subset C_k^{\mathscr{C}}$, then $\exists \zeta \in C_{k+1}^{\mathscr{C}}$ such that $\partial_{k+1}^{\mathscr{C}}(\zeta) = \alpha$. The commutative square gives

$$\phi_k^{\mathscr{C}'}(\alpha) = \phi_k^{\mathscr{C}'} \circ \partial_{k+1}^{\mathscr{C}}(\zeta) = \partial_{k+1}^{\mathscr{C}'} \circ \phi_{k+1}(\zeta)$$

Thus $\phi_k^{\mathscr{C}'}(\alpha) \in \operatorname{im} \partial_{k+1}^{\mathscr{C}'}$. Therefore $\forall \xi \in \ker \partial_k^{\mathscr{C}}, H_k\phi$ sends $\xi + \operatorname{im} \partial_{k+1}^{\mathscr{C}}$ to $\phi(\xi) + \operatorname{im} \partial_{k+1}^{\mathscr{C}'}$ is a well-defined function induced by ϕ_k. $\square$

For each $\tau \geq \tau' \in K$, essentially $\phi_\tau : \mathscr{C}(\tau) \longrightarrow \mathscr{C}'(\tau)$ is a linear map between vector spaces, ϕ is a monomorphism indicates that each ϕ_τ is injective and $\dim \mathscr{C}(\tau) \leq \dim \mathscr{C}'(\tau)$. If ϕ_τ is injective, ϕ_k between chain groups should also be injective. Then the induced map $H_k\phi$ is injective, directly follows from definition.

Similarly, ϕ is an epimorphism indicates that each ϕ_τ is surjective and $\dim \mathscr{C}(\tau) \geq \dim \mathscr{C}'(\tau)$. If ϕ_τ is surjective, the induced map $H_k\phi$ is surjective directly follows from definition.

ϕ is an isomorphism indicates that each ϕ_τ is bijective and $\dim \mathscr{C}(\tau) = \dim \mathscr{C}'(\tau)$. ϕ_k and $H_k\phi$ are also isomorphism. In this case, the inverse of ϕ_τ, denote $\phi_\tau^{-1} : \mathscr{C}'(\tau) \longrightarrow \mathscr{C}(\tau)$ satisfies that $\phi_\tau^{-1} \circ \phi_\tau(v) = v$, for all $v \in \mathscr{C}(\tau)$, and $\phi_\tau \circ \phi_\tau^{-1}(w) = w$, for all $w \in \mathscr{C}'(\tau)$. It can be easily checked that ϕ_τ^{-1} must also be a morphism. Similarly, $\phi_\bullet^{-1}$ is also a chain map. For each k, the induced map between homology $H_k\phi^{-1}$ is the inverse of $H_k\phi$.

4 Examples

In this section we gives a few examples:

Example 2. *Let $K = \Delta(2), \underline{\mathbb{F}}_K$ be the constant cosheaf and $\mathscr{C}$ be a sheaf that assigns $\tau \mapsto \mathbb{F}$ if dim τ = 0 or 1 and $\tau \mapsto 0$ otherwise. Let $\mathscr{C}(\tau \geq \tau') : \mathscr{C}(\tau) \to \mathscr{C}(\tau')$ be identity map on $\mathbb{F}$ if $\mathscr{C}(\tau) = \mathscr{C}(\tau') = \mathbb{F}$ and 0 otherwise. Then the constant cosheaf has the homology of a single point but $\mathscr{C}$ does not.*

Proof. Firstly consider $\underline{\mathbb{F}}_K$, by definition $C_0^{\underline{\mathbb{F}}_K} = \mathbb{F}^3$, $C_1^{\underline{\mathbb{F}}_K} = \mathbb{F}^3$, $C_2^{\underline{\mathbb{F}}_K} = \mathbb{F}$. Since each extension map is identity, the boundary maps are products of $id_{\mathbb{F}}$, $-id_{\mathbb{F}}$, or 0. Direct calculation shows these boundary maps are the exactly the same as the standard boundary map of $\Delta(2)$ in chapter 3 in lecture notes [2]. Then the homology of K with coefficients in $\underline{\mathbb{F}}_K$ is the same as the standard homology of $\Delta(2)$, which is the same as the homology of a single point. For the cosheaf $\mathscr{C}$, by definition $C_0^{\mathscr{C}} = \mathbb{F}^3$, $C_1^{\mathscr{C}} = \mathbb{F}^3$, $C_2^{\mathscr{C}} = 0$. Again the boundary maps are products of $id_{\mathbb{F}}$, $-id_{\mathbb{F}}$, or 0. Direct calculation using Definition 2 give $\partial_0^{\mathscr{C}} = \partial_2^{\mathscr{C}} = 0$, $\partial_1^{\mathscr{C}} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and Definition

3 gives $H_0^{\mathscr{C}} = H_1^{\mathscr{C}} = \mathbb{F}$ and $H_2^{\mathscr{C}} = 0$, which is not the same as the homology of a contractible point. $\square$

Using the above example, we observe that for the same simplicial complex K, its homology with coefficients in a cosheaf $\mathscr{C}$ can be different by how this cosheaf $\mathscr{C}$ assigns weight (the dimension of each costalk) to each simplex τ. When $\mathscr{C}$ is $\underline{\mathbb{F}}_K$, the weights of all costalks are one. Each row and column in the corresponding boundary map has width one. Also, since the extension maps in this cases are all identities. Each block $\partial_k|_{\tau',\tau} \in \{0, -1, 1\}$ depending on the index of faces. Moreover, the definition of the face indexing is exactly the same as the coefficients in standard boundary map. Therefore k-th homology group of K with coefficients in $\underline{\mathbb{F}}_K$ is always the same as (or isomorphism to) the standard k-th homology group of K.

On the other hand, we are free to pick any cosheaf and change the weight of each simplex. In this example, the 2-simplex has zero weight under the cosheaf $\mathscr{C}$. Then the chain complex of K with coefficients in $\mathscr{C}$ is the same as the chain complex of $\partial\Delta(2)$, so does the homology group.

We will use this observation in further examples.

Example 3. *Let $K = \partial\Delta(2)$, where the three vertices are named v_0, v_1, v_2 and the 1-simplex that connects v_0 and v_2 is called v_{02}, etc. Let $\underline{\mathbb{F}}_K$ be the constant cosheaf and $\mathscr{C}$ be a sheaf that assigns:*

$$\tau \mapsto \begin{cases} \mathbb{F} & \textit{if dim } \tau = v_0,\ v_2 \textit{ or } v_{02} \\ 0 & \textit{otherwise} \end{cases}$$

$$\tau \geq \tau' \mapsto \begin{cases} f = id_{\mathbb{F}} & \textit{if dim } \tau = v_{02} \textit{ and } \tau' = v_0 \textit{ or } v_1 \\ 0 & \textit{otherwise} \end{cases}$$

Then the $\underline{\mathbb{F}}_K$ does not have the homology of a point but $\mathscr{C}$ does.

Proof. Using the above observation, the k-th homology group $H_k^{\underline{\mathbb{F}}_K}$ is the same as the k-th homology group of $\partial\Delta(2)$, which is not the same as a single point. On the other hand, for any dimension k, $H_k^{\mathscr{C}} = H_k(\Delta(1))$, which is contractible. We can also achieve the same result by direct computation from definitions in the previous sections. $\square$

In the content below, we define a strict monomorphism to be a monomorphism that is not an isomorphism, a strict epimorphism to be an epimorphism that is not an isomorphism.

Example 4. *Let $K = \Delta(1)$, denote the two 0-simplex to be v_0 and v_1 and the 1-simplex to be v_{01}. Define $\mathscr{C}$ that maps*

$$\tau \mapsto \begin{cases} \mathbb{F}^2 & \tau = v_0, v_{01} \\ \mathbb{F} & \tau = v_1 \end{cases} \qquad \tau \geq \tau' \mapsto \begin{cases} id_{\mathbb{F}} & \textit{if } \tau = v_{01},\ \tau' = v_0 \textit{ or } v_{01} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} & \textit{if } \tau = v_{01}, \tau' = v_1 \end{cases}$$

then there exists a strict monomorphism $\underline{\mathbb{F}}_K \longrightarrow \mathscr{C}$ which induces an isomorphism between $H_\bullet^{\underline{\mathbb{F}}_K}$ and $H_\bullet^{\mathscr{C}}$.

Proof. Let $\phi : \underline{\mathbb{F}}_K \longrightarrow \mathscr{C}$ be a map between cosheaves by letting $\phi_\tau = id_{\mathbb{F}}$ if $\mathscr{C}(\tau) = \mathbb{F}$ and the projection map to the first coordinate $(1, 0)$ otherwise. Firstly, to check that ϕ is a morphism, we need to show $\phi_{\tau'} \circ \underline{\mathbb{F}}_K(\tau \geq \tau') = \mathscr{C}(\tau \geq \tau') \circ \phi_\tau$ for any $\tau \geq \tau'$, and we have 4 cases: $\mathscr{C}(\tau) = \mathbb{F}$ or $\mathbb{F}^2$ and $\mathscr{C}(\tau') = \mathbb{F}$ or $\mathbb{F}^2$. Notice that by our construction of $\mathscr{C}$, $\mathscr{C}(\tau) = \mathbb{F}$, $\mathscr{C}(\tau') = \mathbb{F}^2$ can never happen, and for the left three cases the equality holds directly by definition, and we will not show the calculation here. Thus, ϕ is a morphism.

In Proposition 2, we know ϕ induces a well-defined map between homology groups. By our observation stated above, this homology of K with $\underline{\mathbb{F}}_K$-coefficients is the same as the homology of a point. The homology of K with $\mathscr{C}$-coefficients is not obvious from observation, but by direct computation, we get

$$\partial_1^{\mathscr{C}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and $\partial_k^{\mathscr{C}} = 0$ for any other k. A short computation gives that it also has the homology of a point. Therefore, the two homologies are the same. Since ϕ_k is monomorphism for each k, then $H_k\phi$ is also monomorphism, Notice H_k maps from a space to itself, then $H_k\phi$ must be an isomorphism and it is induced by ϕ_k. □

Lemma 1. *There does not exist a strict epimorphism $\underline{\mathbb{F}}_K \longrightarrow \mathscr{C}$ which induces an isomorphism on homology.*

Proof. Assume there exists such morphism $\phi_\bullet$ and a simplicial complex K. Then since ϕ is an epimorphism, $\phi_\tau : \underline{\mathbb{F}}_K(\tau) \longrightarrow \mathscr{C}(\tau)$ is a surjection, for a linear map between vector spaces, this mean $\mathscr{C}(\tau) = \mathbb{F}$ or 0. If $\forall\tau$, $\mathscr{C}(\tau) = \mathbb{F}$ and ϕ_τ is surjective so it must be full rank, then ϕ is an isomorphism; if $\forall\tau, \mathscr{C}(\tau) = 0$, then two homology must be different as no simplicial complex has the same homology as an empty set. So $\mathscr{C}$ must be some nontrivial cosheaf.

Observe that if $\tau \geq \tau'$ but $\mathscr{C}(\tau) < \mathscr{C}(\tau')$, ϕ is not a morphism, as $\phi_{\tau'} \circ \underline{\mathbb{F}}_K(\tau \geq \tau') = id_{\mathbb{F}}$ but $\mathscr{C}(\tau \geq \tau') \circ \phi_\tau = 0$. To make sure ϕ is a morphism, we must have $\mathscr{C}(\tau) \geq \mathscr{C}(\tau')$ for all $\tau \geq \tau'$.

Therefore, there exists a k-simplex such that for any τ with dimension greater than k, $\mathscr{C}(\tau) = \underline{\mathbb{F}}_K(\tau) = \mathbb{F}$, and for τ with dimension less than k, $\mathscr{C}(\tau) = 0$. $C_k^{\underline{\mathbb{F}}_K} = \mathbb{F}^n$ but $C_k^{\mathscr{C}} = \mathbb{F}^m$ for some $0 < m \leq n$. Then for any $j > k$, $\dim C_j^{\underline{\mathbb{F}}_K} = \dim C_j^{\mathscr{C}}$, for any $j < k$, $C_j^{\mathscr{C}} = 0 \neq C_j^{\underline{\mathbb{F}}_K}$.

By Rank Nullity Theorem, we must have for $i \geqslant 0$,

$$\dim im\partial_i^{\underline{\mathbb{F}}_K} + \dim ker\partial_i^{\underline{\mathbb{F}}_K} = \dim C_i^{\underline{\mathbb{F}}_K} \qquad \dim im\partial_i^{\mathscr{C}} + \dim ker\partial_i^{\mathscr{C}} = \dim C_i^{\mathscr{C}} \tag{4}$$

Assumption states $H_i^{\underline{\mathbb{F}}_K} = H_i^{\mathscr{C}}$ for any $k \geq 0$, then,

$$\dim ker\partial_i^{\underline{\mathbb{F}}_K} - \dim im\partial_{i+1}^{\underline{\mathbb{F}}_K} = \dim ker\partial_i^{\mathscr{C}} - \dim im\partial_{i+1}^{\mathscr{C}} \tag{5}$$

In this case, $\partial_j^{\underline{\mathbb{F}}_K} = \partial_j^{\mathscr{C}}$ when $j \geq k+2$, plunging in $i = k, k+1$ into equation 5 and $i = k+1$ into equation 4 , we get $\dim ker\partial_k^{\underline{\mathbb{F}}_K} = \dim ker\partial_k^{\mathscr{C}} = m$. If $k = 0$, we arrive a contradiction because $\dim ker\partial_k^F = n$. If $m = n, \phi$ is not a strict monomorphism. If $k = 1$, plug in $i = k-1$ into equation 5, we get dim $C_0^{\underline{\mathbb{F}}_K} = n - m$ with some calculations. If $m \neq 0$, it is impossible for a simplicial complex to have larger dimension in first chain group than 0-th chain group. If $m = 0$, we also arrive a contradiction as $\dim ker\partial_1^{\underline{\mathbb{F}}_K} = 0$ but dim $im\partial_2^{\underline{\mathbb{F}}_K} > 0$, which gives a negative dimensional homology group. If $k \geqslant 2$, to let equation 4 and 5 hold for all dimensions, $C_i^{\mathbb{F}} = 0$ for all $i \leq k-2$, which also arrives a contradiction. Therefore such ϕ does not exist. □

5 Filtration of Cosheaves

In lecture notes [2] Chapter 1, the filtration of a simplex is a nested sequence of subcomplexes, and there exists an inclusion map between each consecutive pair of filtered simplicial complex. As an analogous, we can define $\{\mathscr{C}^i\}$, a filtration of a cosheaf $\mathscr{C}$ as a nested sequence of (sub)cosheaves $\mathscr{C}^0 \subset \mathscr{C}^1 \subset \ldots \subset \mathscr{C}^l = \mathscr{C}$, and we can always find a (strict) monomorphism between each consecutive pair. This leads to the following definition.

Definition 5. *Let $\mathscr{C}$ be a cosheaf. A **filtration** $\{\mathscr{C}^i\}$ of $\mathscr{C}$ over simplicial complex K is a nested sequence of cosheaf such that $\forall i \geq 0$, there exists a (strict) monomorphism $g_{i,\bullet} : \mathscr{C}^i \longrightarrow \mathscr{C}^{i+1}$*

indexed by simplex in K such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{C}^i(\tau) \arrow[r, "\mathscr{C}^i(\tau\geq\tau')"] \arrow[d, "g_{i,\tau}"'] & \mathscr{C}^i(\tau') \arrow[d, "g_{i,\tau'}"] \\ \mathscr{C}^{i+1}(\tau) \arrow[r, "\mathscr{C}^{i+1}(\tau\geq\tau')"'] & \mathscr{C}^{i+1}(\tau') \end{tikzcd}$$

Similar to the filtration of simplicial complex, each g_i induces a linear map $H_k g_i : H_k(\mathscr{C}^i) \longrightarrow H_k(\mathscr{C}^{i+1})$ on homology for all filtration index i. Since homology is functorial by Prop 4.8 in [2], for any pair of integers $0 \leq i \leq j \leq l$, the map induced on homology $H_k g_{i\to j}$ by the inclusion $g_{i\to j} : \mathscr{C}^i \longrightarrow \mathscr{C}^j$ is the composition $H_k g_{j-1} \circ H_k g_j \circ \ldots \circ H_k g_{i+1} \circ H_k g_i$. Therefore, we get a persistence module:

$$H_k(\mathscr{C}^0) \xrightarrow{H_k g_0} H_k(\mathscr{C}^1) \xrightarrow{H_k g_1} \ldots \xrightarrow{H_k g_{i-1}} H_k(\mathscr{C}^i) \xrightarrow{H_k g_i} \ldots \xrightarrow{H_k g_{l-1}} H_k(\mathscr{C}^l)$$

Definition 6. *For $i \leq j$ pair of integers, the associated **persistent homology group of** K **with coefficients in** $\{\mathscr{C}^i\}$ is $PH_{i\to j}(H_k(K, \{\mathscr{C}^i\}), H_k g_\bullet) = \operatorname{im}(H_k g_{i\to j})$ for $i, j \leq l$ integers.*

6 Acyclic Partial Matching on K Compatible with Cosheaf Filtration

For a simplicial complex K, recall from definition 8.1 in [2], a partial matching on K is a collection of pairs of simplices with codimension one. This will remain the same when we want to describe the discrete Morse theory for cosheaves. The only difference is we will write each pair as $\{\tau \rhd \sigma\}$ instead of $\{\sigma \lhd \tau\}$. Also, recall from definition 8.2 from [2], A Σ-path is a sequence of distinct simplices $\rho = (\tau_1 \rhd \sigma_1 \lhd \tau_2 \rhd \sigma_2 \ldots \tau_m \rhd \sigma_m)$. Again, only the notation is changed a little to fit the definition of cosheaf. ρ is cycle if $m > 1$ and $\tau_1 \rhd \sigma_m$, and ρ is gradient otherwise. Σ is called acyclic if all its paths are gradient. The unmatched simplices in K that are not in Σ are called Σ-critical.

For now, all the definitions remain the same as they are in Chapter 8, because they are all based on a simplicial complex K and have nothing to do with cosheaves. In order to define a weight, we need to "divide" a block of matrix, which leads to the following definition.

Definition 7. *An acyclic partial matching Σ on simplicial complex K is $\{\mathscr{C}^i\}$**-compatible** if the extension map $\mathscr{C}^i(\tau \geq \sigma)$ is an isomorphism for each i, for every pair $\{\tau \rhd \sigma\}$ in Σ.*

Example 5. *Let $K = \Delta(1)$, consider the cosheaf $\mathscr{C}$ in example 4. Each filtered cosheaf $\mathscr{C}^i$ is defined as a restriction of $\mathscr{C}$. For $i = 0, 1, 2$, the costalks of $\mathscr{C}^i$ are stated in the figure 1 . Let the extension map be the natural map, i.e. one of identity, injection, projection maps depends on the dimension of its domain and range.*

Define $\Sigma = \{(v_{01} \rhd v_0)\}$. Since $\mathscr{C}^0(v_{01} \geq v_0)$ is not invertible, then this partial matching is not $\{\mathscr{C}^i\}$-compatible.

On the other hand, consider the cosheaf filtration in figure 2. Again the costalks of $\mathscr{C}^i$ are stated in the figure, and the extension map is one of identity, injection, projection maps depends on the dimension of its domain and range. This partial matching is $\{\mathscr{C}^i\}$-compatible because $\mathscr{C}^i(v_{01} \geq v_0)$ is the identity map for all $i = 0, 1, 2$.

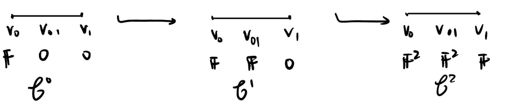


Figure 1: Cosheaf filtration with none $\{\mathscr{C}^i\}$-compatible Σ

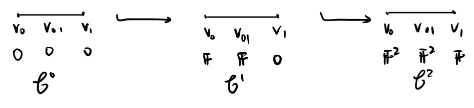


Figure 2: Cosheaf filtration with $\{\mathscr{C}^i\}$-compatible Σ

7 The Discrete Morse Theory

Definition 8. *The **weight** of a Σ-path $\rho = (\tau_1 \rhd \sigma_1 \lhd \tau_2 \rhd \sigma_2 \ldots \tau_m \rhd \sigma_m)$ is the product*

$$\omega(\rho) = (-1)^m \cdot \left[\mathscr{C}^{-1}_{\sigma_1,\tau_1} \circ \mathscr{C}_{\sigma_1,\tau_2} \circ \mathscr{C}^{-1}_{\sigma_2,\tau_2} \circ \cdots \cdot \mathscr{C}^{-1}_{\sigma_m,\tau_m}\right]$$

where $\mathscr{C}_{\alpha,\beta} : \mathscr{C}(\beta) \to \mathscr{C}(\alpha)$ is defined by

$$\mathscr{C}_{\alpha,\beta} = [\beta : \alpha] \cdot \mathscr{C}(\beta \geq \alpha) = \begin{cases} +\mathscr{C}(\beta \geq \alpha) \text{ if } \alpha = \beta_{-i} \text{ for even } i \\ -\mathscr{C}(\beta \geq \alpha) \text{ if } \alpha = \beta_{-i} \text{ for odd } i \\ 0 \text{ otherwise} \end{cases}$$

Definition 9. *Let $\mathscr{C}$ be a cosheaf over the simplicial complex K and Σ be an $\mathscr{C}$-compatible acyclic partial matching. The **Morse complex of** Σ **with coefficients in** $\mathscr{C}$ is a chain complex $\left(C^{\Sigma}_{\bullet}(K,\mathscr{C}), \partial^{\mathscr{C},\Sigma}_{\bullet}\right) := \left(C^{\mathscr{C},\Sigma}_{\bullet}, \partial^{\mathscr{C},\Sigma}_{\bullet}\right)$*

$$\cdots \xrightarrow{\partial^{\mathscr{C},\Sigma}_{k+2}} C^{\mathscr{C},\Sigma}_{k+1} \xrightarrow{\partial^{\mathscr{C},\Sigma}_{k+1}} C^{\mathscr{C},\Sigma}_{k} \xrightarrow{\partial^{\mathscr{C},\Sigma}_{k}} \cdots \xrightarrow{\partial^{\mathscr{C},\Sigma}_{2}} C^{\mathscr{C},\Sigma}_{1} \xrightarrow{\partial^{\mathscr{C},\Sigma}_{1}} C^{\mathscr{C},\Sigma}_{0} \xrightarrow{\partial^{\mathscr{C},\Sigma}_{0}} 0$$

defined by, $\forall k \geq 0$,

1. *The vector space $C^{\mathscr{C},\Sigma}_k = \prod_\alpha \mathscr{C}(\alpha)$ where indexed by all the Σ-critical k-simplex.*
2. *The linear map $\partial^{\mathscr{C},\Sigma}_k : C^{\mathscr{C},\Sigma}_k \to C^{\mathscr{C},\Sigma}_{k-1}$ is represented by a block matrix whose entry in the column of a critical k-simplex α and the row of a critical $(k-1)$-simplex ω is given by*

$$\partial^{\Sigma,\mathscr{C}}_k\Big|_{\omega,\alpha} = \mathscr{C}_{\omega,\alpha} + \sum_\rho \mathscr{C}_{\omega,\tau_\rho} \circ w(\rho) \circ \mathscr{C}_{\sigma_\rho,\alpha}$$

where ρ ranges over all the Σ-paths.

To complete the above definition, we must show $\left(C^{\mathscr{C},\Sigma}_{\bullet}, \partial^{\mathscr{C},\Sigma}_{\bullet}\right)$ is a chain complex. This will be done by induction.

Lemma 2. *Let $\Sigma = \{(\tau \rhd \sigma)\}$ contains only a single pair, then $\partial^{\Sigma,\mathscr{C}}_k \circ \partial^{\Sigma,\mathscr{C}}_{k+1} = 0$ for all k.*

Proof. For each α's row and ω's column in the map $\partial^{\Sigma,\mathscr{C}}_k \circ \partial^{\Sigma,\mathscr{C}}_{k+1}$, by direct matrix multiplication we get

$$B := \partial^{\Sigma,\mathscr{C}}_k \circ \partial^{\Sigma,\mathscr{C}}_{k+1}\Big|_{\alpha,w} = \sum_\xi \partial^{\Sigma,\mathscr{C}}_k|_{\alpha,\xi} \circ \partial_{k+1}|_{\xi,\omega}$$

where ξ are all the Σ-critical k-simplices in K. Since $\rho = \tau \rhd \sigma$ is the only path, and $w(\rho) = -\mathscr{C}^{-1}_{\sigma,\tau}$, by definition, for each ξ, we define

$$B_\xi = (\mathscr{C}_{\alpha,\xi} \circ \mathscr{C}_{\xi,\omega}) - \mathscr{C}_{\alpha,\tau} \circ \mathscr{C}^{-1}_{\sigma,\tau} \circ \mathscr{C}_{\sigma,\xi} \circ \mathscr{C}_{\xi,\omega} - \mathscr{C}_{\alpha,\xi} \circ \mathscr{C}_{\xi,\tau} \circ \mathscr{C}^{-1}_{\sigma,\tau} \circ \mathscr{C}_{\sigma,\omega} - \mathscr{C}_{\alpha,\tau} \circ \mathscr{C}^{-1}_{\sigma,\tau} \circ \mathscr{C}_{\sigma,\xi} \circ \mathscr{C}_{\xi,\tau} \circ \mathscr{C}^{-1}_{\sigma,\tau} \circ \mathscr{C}_{\sigma,\omega}$$

Notice that the last term is always zero due to the dimension problem, and the second and third term may be nonzero when dimension is correct. When $\dim \tau = k, \dim \sigma = k-1$, the third term is zero. Since $\partial_\bullet^{\mathscr{C}}$ is a boundary operator, and the only non-critical K-simplex in this case is τ. Then $\partial_k^{\mathscr{C}} \circ \partial_{k+1}^{\mathscr{C}} = 0$ gives

$$\sum_{\xi \text{ critical}} \mathscr{C}_{\alpha,\xi} \circ \mathscr{C}_{\xi,\omega} = -\mathscr{C}_{\alpha,\tau} \circ \mathscr{C}_{\tau,\omega}$$

Notice this is the for any $(k-1)$-simplex α. Then $B = \sum_\xi B_\xi = \sum_\xi \mathscr{C}_{\alpha,\xi} \circ \mathscr{C}_{\xi,\omega} - \mathscr{C}_{\alpha,\tau} \circ \mathscr{C}_{\sigma,\tau}^{-1} \circ \left(\sum_\xi \mathscr{C}_{\sigma,\xi} \circ \mathscr{C}_{\xi,\omega}\right) = -\mathscr{C}_{\alpha,\tau} \circ \mathscr{C}_{\tau,\omega} + \mathscr{C}_{\alpha,\tau} \circ \mathscr{C}_{\sigma,\tau}^{-1} \circ \mathscr{C}_{\sigma,\tau} \circ \mathscr{C}_{\tau,\omega} = 0$ as desired.

When $\dim \tau = k+1, \dim \sigma = k$, the second term is 0, and the calculation will be almost the same. We can still use the trick that $\partial_\bullet^{\mathscr{C}}$ is a boundary map to simplify the summation $B = -\mathscr{C}_{\alpha,\sigma} \circ \mathscr{C}_{\sigma,\omega} + \mathscr{C}_{\alpha,\sigma} \circ \mathscr{C}_{\sigma,\tau} \circ \mathscr{C}_{\sigma,\tau}^{-1} \circ \mathscr{C}_{\sigma,\omega}$ and conclude $B = 0$. When neither of the 2 cases above happen, $B = 0$ automatically since $\partial_k^{\mathscr{C}} \circ \partial_{k+1}^{\mathscr{C}}\big|_{\alpha,w} = 0$. Therefore, $\partial_\bullet^{\mathscr{C},\Sigma}$ is a boundary operator. $\square$

Proposition 3. *Let Σ be an acyclic partial matching and $\mathscr{C}$ be a cosheaf on a simplicial complex K. $\partial_k^{\Sigma,\mathscr{C}} \circ \partial_{k+1}^{\Sigma,\mathscr{C}} = 0$ for all k.*

Proof. We will show this by induction. Let Σ_i be any acyclic partial matching that contains i pairs, the induction hypothesis gives $\partial_k^{\mathscr{C},\Sigma_i} \circ \partial_{k+1}^{\mathscr{C},\Sigma_i} = 0$, we want to show $\partial_k^{\mathscr{C},\Sigma} \circ \partial_{k+1}^{\mathscr{C},\Sigma} = 0$ for any Σ that contains $i+1$ pairs of simplices.

Let α be $(k-1)$-simplex, ω be $(k+1)$-simplex. As before, define $B = \sum_\xi B_\xi = \partial_k^{\Sigma,\mathscr{C}} \circ \partial_{k+1}^{\Sigma,\mathscr{C}}\Big|_{\alpha,w}$ and $B_\xi = \left(\mathscr{C}_{\alpha,\xi} + \sum_\rho \mathscr{C}_{\alpha,\tau_\rho} \cdot w(\rho) \cdot \mathscr{C}_{\sigma_\rho,\xi}\right) \circ \left(\mathscr{C}_{\xi,\omega} + \sum_\eta \mathscr{C}_{\xi,\tau_\eta} \circ w(\eta) \cdot \mathscr{C}_{\sigma_\eta,\omega}\right)$ where ρ, η are Σ-paths. Observe that for since we have acyclic partial matching, for each ξ, there is at most one ρ that flows from α to ξ and at most one η that flows from ξ to ω, otherwise these paths form a cycle. Expand this equation, and eliminate the terms that are necessarily zero due to dimension problem. We only need to consider 2 cases: Firstly, $B_\xi = \mathscr{C}_{\alpha,\xi} \circ \mathscr{C}_{\xi,\omega} + \mathscr{C}_{\alpha,\tau_\rho} \circ w(\rho) \circ \mathscr{C}_{\sigma_\rho,\xi} \circ \mathscr{C}_{\xi,\omega}$ for all Σ-critical k-simplex ξ. Using the fact that $\partial_\bullet^{\mathscr{C}}$ is a boundary operator, we get

$$\sum_{\substack{\dim \xi = k \\ \xi \text{ critical}}} \mathscr{C}_{\alpha,\xi} \circ \mathscr{C}_{\xi,\omega} = -\sum_{\substack{\dim \tau = k \\ \tau \in \Sigma}} \mathscr{C}_{\alpha,\tau} \circ \mathscr{C}_{\tau,\omega}$$

and denote $\rho = \tau_1 \triangleright \sigma_1 \ldots \triangleleft \tau_m \triangleright \sigma_m$, thus

$$B = \sum_{\xi \text{ critical}} B_\xi = -\sum_{\substack{\dim \tau = k \\ \tau \in \Sigma}} \mathscr{C}_{\alpha,\tau} \circ \mathscr{C}_{\tau,\omega} - \mathscr{C}_{\alpha,\tau_\rho} \circ w(\rho) \circ \left(\sum_{\substack{\dim \tau = k \\ \tau \in \Sigma}} \mathscr{C}_{\sigma_\rho,\tau} \circ \mathscr{C}_{\tau,\omega}\right) \tag{6}$$

$$= \left(-\sum_{\substack{\dim \tau = k \\ \tau \in \Sigma, \tau \neq \tau_m}} \mathscr{C}_{\alpha,\tau} \circ \mathscr{C}_{\tau,\omega}\right) - \mathscr{C}_{\alpha,\tau_1} \circ \ldots \mathscr{C}_{\sigma_m,\tau_m}^{-1} \circ \mathscr{C}_{\sigma_m,\tau_m} \circ \mathscr{C}_{\tau_m,\omega} \tag{7}$$

$$= \left(-\sum_{\substack{\dim \tau = k \\ \tau \in \Sigma, \tau \neq \tau_m}} \mathscr{C}_{\alpha,\tau} \circ \mathscr{C}_{\tau,\omega}\right) - \mathscr{C}_{\alpha,\tau_1} \circ \ldots \mathscr{C}_{\sigma_{m-1},\tau_m} \circ \mathscr{C}_{\tau_m,\omega} \tag{8}$$

$$= \left(-\sum_{\substack{\dim \tau = k \\ \tau \in \Sigma, \tau \neq \tau_m}} \mathscr{C}_{\alpha,\tau} \circ \mathscr{C}_{\tau,\omega}\right) - \mathscr{C}_{\alpha,\tau_1} \circ w(\rho') \circ \mathscr{C}_{\sigma_{m-1},\tau_m} \circ \mathscr{C}_{\tau_m,\omega} \tag{9}$$

where $\rho' = \tau_1 \triangleright \sigma_1 \ldots \triangleleft \tau_{m-1} \triangleright \sigma_{m-1}$. We can get from equation 6 to 7 because if $\tau \in \Sigma$ and τ is not the last k-simplex in ρ, then $\mathscr{C}_{\alpha,\tau_\rho} \circ w(\rho) \circ \mathscr{C}_{\sigma_\rho,\tau} \circ \mathscr{C}_{\tau,\omega} = 0$ otherwise it forms a cycle, so the only nonzero term is the when $\tau = \tau_m$, but if $\tau = \tau_m$, the first term $\mathscr{C}_{\alpha,\tau_m} \circ \mathscr{C}_{\tau,\omega} = 0$, otherwise it forms a cycle again. Finally 9 gives 0 by induction, because this is exactly the case for acyclic partial matching $\Sigma \backslash \{(\tau_m \triangleright \sigma_m)\}$ which contains i pairs of simplices.

The second case will be $B_\xi = \mathscr{C}_{\alpha,\xi} \circ \mathscr{C}_{\xi,\omega} + \left(\sum_\eta \mathscr{C}_{\alpha,\xi} \circ \mathscr{C}_{\xi,\tau_\eta} \circ w(\eta) \circ \mathscr{C}_{\sigma_\eta,\omega}\right)$. The calculation will be the same as the first case, except for a little changes in notations and order of the maps. We can still use the fact that $\partial^{\mathscr{C}}_\bullet$ is a boundary operator to conclude $B = 0$. Thus, $\partial^{\mathscr{C},\Sigma}$ is a boundary operator. $\square$

The next step is to show the Morse chain complex $(C^{\mathscr{C},\Sigma}_\bullet, \partial^{\mathscr{C},\Sigma}_\bullet)$ is homotopy equivalent to the standard chain complex $(C^{\mathscr{C}}_\bullet, \partial^{\mathscr{C}}_\bullet)$, so that we can work with the Morse chain complex instead of the standard chain complex to reduce calculation. This will also be done by induction.

To avoid writing large amount of matrices indexed by simplices, and taking the advantage that we are working on the finite simplex K, we will use the summation of k-simplices to represent a vector in $C^{\mathscr{C}}_k$. For example, if $v \in C^{\mathscr{C}}_k$ is a vector, we write

$$v = \sum_{\substack{\alpha \in K \\ \dim \alpha = k}} v_\alpha \alpha$$

instead of writing a long column vector. In this case, each v_α is not a scalar, but an n-dimensional vector, where this n is the dimension of the costalk $\mathscr{C}(\alpha)$.

Lemma 3. *Let $\Sigma = \{(\tau \triangleright \sigma)\}$ be an acyclic partial matching on K that contains only one pair of partial matching. $\mathscr{C}$ be a cosheaf on K. Then, the simplicial chain complex with $\mathscr{C}$-coefficients $(C^{\mathscr{C}}_\bullet, \partial^{\mathscr{C}}_\bullet)$ is chain homotopy equivalent to the Morse complex with $\mathscr{C}$-coefficients $(C^{\mathscr{C},\Sigma}_\bullet, \partial^{\mathscr{C},\Sigma}_\bullet)$.*

Proof. Let $k \geq 0$, for each k-simplex α and Σ-critical k-simplex ω, define $\psi_k : C^{\mathscr{C}}_k \longrightarrow C^{\mathscr{C},\Sigma}_k$ by the following block of matrices: for each α's column and ω's row,

$$\psi_k|_{\omega,\alpha} = \begin{cases} -\mathscr{C}_{\omega,\tau} \circ \mathscr{C}^{-1}_{\sigma,\tau} & \text{if } \alpha = \sigma \\ id & \text{if } \alpha = \omega \neq \tau \\ 0 & \text{otherwise} \end{cases}$$

where id is identity map. Similarly, we define $\phi_k : C^{\mathscr{C},\Sigma}_k \longrightarrow C^{\mathscr{C}}_k$ by

$$\phi_k|_{\alpha,\omega} = \begin{cases} -\mathscr{C}^{-1}_{\sigma,\tau} \circ \mathscr{C}_{\sigma,\omega} & \text{if } \alpha = \tau \\ id & \text{if } \alpha = \omega \neq \sigma \\ 0 & \text{otherwise} \end{cases}$$

Before showing they are the chain homotopy, we must make sure that they are chain maps, which means the following equations hold for all $k \geq 0$:

$$\partial^{\mathscr{C},\Sigma}_k \circ \psi_k = \psi_{k-1} \circ \partial^{\mathscr{C}}_k \tag{10}$$

$$\phi_{k-1} \circ \partial^{\mathscr{C},\Sigma}_k = \partial^{\mathscr{C}}_k \circ \phi_k \tag{11}$$

To do this we need to consider 3 cases: when dim $\sigma = k$, dim $\tau = k+1$; when dim $\sigma = k-1$, dim $\tau = k$, any when σ, τ have other dimensions. Commutativity can be checked algebraically using matrix multiplication for all these 3 cases. It is worth to notice that when dim $\sigma = k$, we need the fact that $\partial^{\mathscr{C},\Sigma}_\bullet$ is a boundary operator to show 10 holds, and when dim $\tau = k$, we need this fact to show 11 holds. The other 2 cases are just directly computation.

Next, we want to find chain homotopies $\eta_\bullet : id_{C^{\mathscr{C}}_\bullet} \Rightarrow \phi_\bullet \circ \psi_\bullet$ and $\eta'_\bullet : \psi_\bullet \circ \phi_\bullet \Rightarrow id_{C^{\mathscr{C},\Sigma}_\bullet}$ that satisfy definition 4.12 from [2], where $id_{C^{\mathscr{C}}_\bullet}$ is the identity map on $C^{\mathscr{C}}_\bullet$, etc. Observe that regardless of the dimension of τ and σ, direct computation gives $\psi_\bullet \circ \phi_\bullet = id_{C^{\mathscr{C},\Sigma}_\bullet}$. We can define η'_k to be the zero map so it satisfies the requirement of a chain homotopy: $\psi_\bullet \circ \phi_\bullet - id_{C^{\mathscr{C},\Sigma}_\bullet} = \eta'_{k-1} \circ \partial_k + \partial_{k+1} \circ \eta'_k = 0$. On the other hand, for a k-simplex α and $(k+1)$-simplex γ, let

$\eta_k : C_k^{\mathscr{C}} \longrightarrow C_{k+1}^{\mathscr{C}}$ be defined by: for each γ's row and α's column,

$$\eta_k|_{\gamma,\alpha} = \begin{cases} \mathscr{C}_{\sigma,\tau}^{-1} & \text{if } \alpha = \sigma, \gamma = \tau \\ 0 & \text{otherwise} \end{cases}$$

Given this η_k, we want to check $id_{C_\bullet^{\mathscr{C}}} - \phi_k \circ \psi_k = \eta_{k-1} \circ \partial_k^{\mathscr{C}} + \partial_{k+1}^{\mathscr{C}} \circ \eta_k$.

As before, we need to consider when $\dim \sigma = k$ and $\dim \tau = k+1$; when $\dim \sigma = k-1$ and $\dim \tau = k$, and when σ and τ are other dimensions. All of the three cases can be done by direct calculation. For example, when $\dim \sigma = k-1$ and $\dim \tau = k$, the only nontrivial basic element is $v_\tau \tau$, for v_τ being a vector in $\mathscr{C}(\tau)$, any other basic elements are mapped to 0 on both sides. For $v_\tau \tau$, left side evaluates $id_{C_\bullet^{\mathscr{C}}}(v_\tau \tau) - \phi_k \circ \psi_k = v_\tau \tau$, and for right side, assume $\beta_1, \dots, \beta_m$ are the $(k-1)$-simplices, since η_k is a zero map, $\eta_{k-1} \circ \partial_k^{\mathscr{C}} + \partial_{k+1}^{\mathscr{C}} \circ \eta_k = \eta_{k-1}(\sum_{j=1}^m \left[\mathscr{C}_{\beta_j,\tau}(v_\tau \tau)\right] \beta_j) = v_\tau \tau$ gives the same value.

Therefore, $(C_\bullet^{\mathscr{C},\Sigma}, \partial_\bullet^{\mathscr{C},\Sigma})$ is homotopy equivalent to $(C_\bullet^{\mathscr{C}}, \partial_\bullet^{\mathscr{C}})$. □

Proposition 4. *Let Σ be an acyclic partial matching and $\mathscr{C}$ be a cosheaf on K. Then, the simplicial chain complex with $\mathscr{C}$-coefficients $(C_\bullet^{\mathscr{C}}, \partial_\bullet^{\mathscr{C}})$ is chain homotopy equivalent to the Morse complex with $\mathscr{C}$-coefficients $(C_\bullet^{\mathscr{C},\Sigma}, \partial_\bullet^{\mathscr{C},\Sigma})$.*

Proof. The base case $\Sigma_1 = \{(\tau \rhd \sigma)\}$ is proved in the above lemma. Notice that Σ_1 is always acyclic. Assume $\Sigma_i = \{(\tau_\bullet \rhd \sigma_\bullet)\}$ contains i pairs of simplices and $\left(C^{\mathscr{C},\Sigma_i}, \partial^{\mathscr{C},\Sigma_i}\right)$ is chain homotopic equivalent to $\left(C^{\mathscr{C}}, \partial^{\mathscr{C}}\right)$ and their chain homotopies are $\psi_{\bullet,i}$ and $\phi_{\bullet,i}$ respectively. Let $\Sigma = \Sigma_i \cup \{(\tau' \rhd \sigma')\}$

Define $\psi_k : C_k^{\mathscr{C}} \longrightarrow C_k^{\mathscr{C},\Sigma}$, for all k-simplex α, Σ-critical k-simplex ω, the α's columns and w's rows is

$$\psi_k|_{\omega,\alpha} = \begin{cases} -\mathscr{C}_{\omega,\tau} \circ \mathscr{C}_{\sigma,\tau}^{-1} & \text{if } \alpha = \sigma \text{ for some pair } (\tau \rhd \sigma) \subset \Sigma \\ id & \text{if } \alpha = \omega \neq \tau \text{ for some pair } (\tau \rhd \sigma) \subset \Sigma \\ 0 & \text{otherwise} \end{cases}$$

Similarly, we define $\phi_k : C_k^{\mathscr{C},\Sigma} \longrightarrow C_k^{\mathscr{C}}$ by

$$\phi_k|_{\alpha,\omega} = \begin{cases} -\mathscr{C}_{\sigma,\tau}^{-1} \circ \mathscr{C}_{\omega,\tau} & \text{if } \alpha = \tau \text{ for some pair } (\tau \rhd \sigma) \subset \Sigma \\ id & \text{if } \alpha = \omega \neq \sigma \text{ for some pair } (\tau \rhd \sigma) \subset \Sigma \\ 0 & \text{otherwise} \end{cases}$$

First, observe that for since we have acyclic partial matching, there is at most one Σ-path between each pair of simplices, otherwise these paths form a cycle. Now we need to check they are both morphism.

Consider the case that the new added $(\tau' \rhd \sigma')$ is k and $k-1$ dimensional simplex respectively. Then by directly computation, for any $k-1$-simplex β and k-simplex α, we have

$$\psi_{k-1} \circ \partial_k^{\mathscr{C}}|_{\beta,\alpha} = \psi_{k-1,i} \circ \partial_k^{\mathscr{C}}|_{\beta,\alpha} + \mathscr{C}_{\beta,\tau'} \circ \mathscr{C}_{\sigma',\tau'}^{-1} \circ \mathscr{C}_{\sigma',\alpha} \tag{12}$$

$$\partial_k^{\mathscr{C},\Sigma} \circ \psi_k|_{\beta,\alpha} = \mathscr{C}_{\beta,\alpha} + \sum_\rho \mathscr{C}_{\beta,\tau_\rho} \circ w(\rho) \circ \mathscr{C}_{\sigma_\rho,\alpha} \tag{13}$$

Notice that if the last term $\mathscr{C}_{\beta,\tau'} \circ \mathscr{C}_{\sigma',\tau'}^{-1} \circ \mathscr{C}_{\sigma',\alpha}$ in equation 12 is 0, then the new added pair $\tau' \rhd \sigma'$ does not form a path from β to α, then all the Σ-paths flowing from β to α remains the same as the Σ_i-paths flowing from β to α. Therefore, equation 13 equals to $\partial_k^{\mathscr{C},\Sigma_i} \circ \psi_{k,i}|_{\beta,\alpha} = \psi_{k-1,i} \circ \partial_k^{\mathscr{C}}|_{\beta,\alpha}$ by induction. On the other hand, if $\mathscr{C}_{\beta,\tau} \circ \mathscr{C}_{\sigma,\tau}^{-1} \circ \mathscr{C}_{\sigma,\alpha}$ is not 0, this means the $\tau' \rhd \sigma'$ form a new path from β to α, by our observation above, it should be the only path and there should not be any Σ_i-path between them in the previous inductions, then $\sum_\rho \mathscr{C}_{\beta,\tau_\rho} \circ w(\rho) \circ \mathscr{C}_{\sigma_\rho,\alpha} = \mathscr{C}_{\beta,\tau'} \circ \mathscr{C}_{\sigma',\tau'}^{-1} \circ \mathscr{C}_{\sigma',\alpha}$ and $\mathscr{C}_{\beta,\alpha} = \partial_k^{\mathscr{C},\Sigma_i} \circ \psi_{k,i}|_{\beta,\alpha} = \psi_{k-1,i} \circ \partial_k^{\mathscr{C}}|_{\beta,\alpha}$. In either case, these two equation are equal.

Then consider the case when the new added $(\tau' \rhd \sigma')$ is $k+1$ and k dimensional simplex respectively. If $\alpha = \sigma'$, then computation gives

$$\psi_{k-1} \circ \partial_k^{\mathscr{C}}|_{\beta,\sigma'} = \mathscr{C}_{\beta,\sigma'} + \psi_{k-1,i} \circ \partial_k^{\mathscr{C}}|_{\beta,\alpha} = \mathscr{C}_{\beta,\sigma'} + \partial_k^{\mathscr{C},\Sigma_i} \circ \psi_{k,i}|_{\beta,\alpha} = \partial_k^{\mathscr{C},\Sigma} \circ \psi_k|_{\beta,\alpha}$$

as desired. For any other α, it will be exactly the same as the induction hypothesis. Other than these two cases, the new added pair does not change the matrices from induction hypothesis, so the desired equality holds. We can also show $\phi_\bullet$ is a morphism with similar computation.

The next step is to find the chain homopoties $\eta_\bullet : id_{C_\bullet^{\mathscr{C},\Sigma}} \Rightarrow \psi_\bullet \circ \phi_\bullet$ and $\eta'_\bullet : \phi_\bullet \circ \psi_\bullet \Rightarrow id_{C_\bullet^{\mathscr{C}}}$. Same as the base case, we can trivially let $\eta'_\bullet$ be zero maps, because $\phi_\bullet \circ \psi_\bullet = id_{C_\bullet^{\mathscr{C}}}$. Define η_k by: for a k-simplex α and $(k+1)$-simplex γ, for each γ's row and α's column,

$$\eta_k|_{\gamma,\alpha} = \begin{cases} \mathscr{C}_{\sigma,\tau}^{-1} & \text{if } \alpha = \sigma, \gamma = \tau \text{ for some } (\tau \rhd \sigma) \in \Sigma \\ 0 & \text{otherwise} \end{cases}$$

Given this η_k, we can check $\phi_k \circ \psi_k - id_{C_\bullet^{\mathscr{C}}} = \eta_{k-1} \circ \partial_k^{\mathscr{C}} + \partial_{k+1}^{\mathscr{C}} \circ \eta_k$ is satisfied with the same computation as in the base case. Therefore, $(C_\bullet^{\mathscr{C},\Sigma}, \partial_\bullet^{\mathscr{C},\Sigma})$ is homotopy equivalent to $(C_\bullet^{\mathscr{C}}, \partial_\bullet^{\mathscr{C}})$. $\square$

We denote $H_k^{\{\mathscr{C}^i\},\Sigma}$ to be the k-th homology group of the Morse chain complex $(C_\bullet^{\{\mathscr{C}^i\},\Sigma}, \partial_\bullet^{\Sigma})$.

Theorem 7.1 (Discrete Morse Theory)**.** *Let $\{\mathscr{C}^i\}$ be a cosheaf filtration on a simplicial complex K and Σ be a $\{\mathscr{C}^i\}$-compatible acyclic partial matching on K. Then for each dimension $k \geq 0$, and pair of integers $m \leq n \leq l$, the persistent homology group of K with coefficients in $\{\mathscr{C}^i\}$ is isomorphic to the persistent homology group of Morse chain complex of Σ with coefficients in $\{\mathscr{C}^i\}$, which is*

$$PH_{m\to n}(H_\bullet^{\{\mathscr{C}^i\}}, H_\bullet g_\star) \cong PH_{m\to n}(H_\bullet^{\{\mathscr{C}^i\},\Sigma}, H_\bullet g_\star^{\Sigma})$$

where $\bullet$ indicates indexing by dimension, and $\star$ means indexing by the index of filtration.

Proof. Essentially we want to show for each pair of homology groups in the the above graph, for any $j \geq 0$ and dimension $k \geq 0$, $H_k^{\mathscr{C}^j,\Sigma}$ and $H_k^{\mathscr{C}^j}$, are isomorphic. In addition, we must show if two points a and b are correspondent in $H_k^{\mathscr{C}^j,\Sigma}$ and $H_k^{\mathscr{C}^j}$, then they will be mapped into corresponding points as well, i.e., $H_k g_j^{\Sigma}(a)$ and $H_k g_j(b)$ should also be correspondent. This means we need to find an isomorphism between $H_k^{\mathscr{C}_j}$ and $H_k^{\mathscr{C}_j,\Sigma}$ for each dimension k such that each of the following square commutes in both directions:

$$\begin{array}{ccccccc} H_k^{\mathscr{C}_0} & \xrightarrow{H_k g_0} & H_k^{\mathscr{C}_1} & \xrightarrow{H_k g_1} & \dots & \xrightarrow{H_k g_{l-1}} & H_k^{\mathscr{C}_l} \\ \updownarrow & & \updownarrow & & & & \updownarrow \\ H_k^{\mathscr{C}_0,\Sigma} & \xrightarrow{H_k g_0^{\Sigma}} & H_k^{\mathscr{C}_1,\Sigma} & \xrightarrow{H_k g_1^{\Sigma}} & \dots & \xrightarrow{H_k g_{l-1}^{\Sigma}} & H_k^{\mathscr{C}_l,\Sigma} \end{array}$$

Immediately after we proved the chain homotopic equivalence. By Prop 4.14 from [2] $\phi_\bullet \circ \psi_\bullet, id_{C_\bullet^{\mathscr{C}}}$: $(C_\bullet^{\mathscr{C}}, \partial_\bullet^{\mathscr{C}}) \to (C_\bullet^{\mathscr{C}}, \partial_\bullet^{\mathscr{C}})$ are chain homotopic, and $\psi_\bullet \circ \phi_\bullet, id_{C_\bullet^{\mathscr{C},\Sigma}}$: $\left(C_\bullet^{\mathscr{C},\Sigma}, \partial_\bullet^{\mathscr{C},\Sigma}\right) \to \left(C_\bullet^{\mathscr{C},\Sigma}, \partial_\bullet^{\mathscr{C},\Sigma}\right)$ are chain homotopic. Then their induced map on homology coincide, and since homology is functorial, for any dimension $k \geq 0$

$$H_k\phi \circ \psi = H_k\phi \circ H_k\psi = H_k id_{C^{\mathscr{C}}} = id_{H_k^{\mathscr{C}}}$$
$$H_k\psi \circ \phi = H_k\psi \circ H_k\phi = H_k id_{C^{\mathscr{C},\sigma}} = id_{H_k^{\mathscr{C},\Sigma}}$$

This is exactly the definition of $H_k^{\mathscr{C}} \cong H_k^{\mathscr{C},\Sigma}$.

For a cosheaf filtration, our definition guarantees each $\mathscr{C}^j$ is a cosheaf, so $\forall 0 \leq j \leq l$, we have $H_k^{\mathscr{C}^j} \cong H_k^{\mathscr{C}^j,\Sigma}$, we are left to show the above homology square commutes. Firstly, we abuse notation in the proof below, we write ϕ_j to denote $\phi_{k,j} : C_k^{\mathscr{C}^j} \longrightarrow C_k^{\mathscr{C}^j,\Sigma}$ maps from the k-th chain group with $\mathscr{C}^j$-coefficients to the corresponding Morse chain group, because the dimension k in this proof does not play an important role. Next, it suffices to show it commutes in the chain complex level as shown in the diagram below, which means if we can show $g_k^\Sigma \circ \phi_j = \phi_{j+1} \circ g_j$, then we are done.

$$\begin{tikzcd}
C_k^{\mathscr{C}^j} \arrow[r, "g_j"'] \arrow[d, "\phi_j"] & C_k^{\mathscr{C}^{j+1}} \arrow[d, "\phi_{j+1}"'] \\
C_k^{\mathscr{C}^j,\Sigma} \arrow[r, "g_j^\Sigma"] & C_k^{\mathscr{C}^{j+1},\Sigma}
\end{tikzcd}$$

Assume α_i is a Σ-critical simplex. Consider basic element $v = v_i\alpha_i$, notice that $\psi_\bullet$ does not do anything on it, and both g_j^Σ and g_j have the same effect on $v_i\alpha_i$, which is injecting $v_i \in \mathscr{C}^j(\alpha_i)$ to the vector space $\mathscr{C}^{j+1}(\alpha_i)$ by adding 0 on new dimensions. Thus, $g_j^\Sigma \circ \psi_j(v_i\alpha_i) = g_j^\Sigma(v_i\alpha_i) = w_i\alpha_i = \psi_{j+1}(w_i\alpha_i) = \psi_{j+1} \circ g_j(v_i\alpha_i)$ as desired.

If α_i is not a Σ-critical simplex, then

$$\psi_{j+1} \circ g_j(v) = \sum_{\substack{\alpha_s=1 \\ \alpha_s \text{ critical}}}^{n} \left[\mathscr{C}^{j+1}_{\alpha_s,\tau_\rho} \circ w(\rho) \circ g_j(v_i\alpha_i)\right] \alpha_s \text{ for some corresponding path } \rho$$

$$g_j^\Sigma \circ \psi_j(v) = g_j^\Sigma \circ \sum_{\substack{\alpha_s=1 \\ \alpha_s \text{ critical}}}^{n} \left[\mathscr{C}_{\alpha_s,\tau_\rho} \circ w(\rho)(v_i\alpha_i)\right] \alpha_s \text{ for some corresponding path } \rho$$

We claim this two equations are equal, but when we have a complicated Σ, it is hard to illustrate this algebraically. Here, we will only illustrate the case that $\Sigma = \{(\tau \rhd \alpha_i)\}$, but the idea is same. Assume $\alpha_1, \ldots, \alpha_n$ are all the k-simplices,

$$\begin{aligned}
\psi_{j+1} \circ g_j(v) = &\sum_{\substack{\alpha_s=1 \\ \alpha_s \text{ critical}}}^{n} \left[\mathscr{C}^{j+1}_{\alpha_s,\tau} \circ \mathscr{C}^{j+1^{-1}}_{\alpha_i,\tau} \circ g_j(v_i\alpha_i)\right] \alpha_s \\
= &\sum_{\substack{\alpha_s=1 \\ \alpha_s \text{ critical}}}^{n} \left[\mathscr{C}^{j+1}_{\alpha_s,\tau} \circ \mathscr{C}^{j+1^{-1}}_{\alpha_i,\tau} \circ \left(g_{j,\alpha_i}(v_i\alpha_i) + \sum_{t=1,t\neq i}^{n} g_{j,\alpha_t}(v_i\alpha_i)\right)\right] \alpha_s \\
= &\sum_{\substack{\alpha_s=1 \\ \alpha_s \text{ critical}}}^{n} \left[\mathscr{C}^{j+1}_{\alpha_s,\tau} \circ \mathscr{C}^{j+1^{-1}}_{\alpha_i,\tau} \circ g_{j,\alpha_i}(v_i\alpha_i)\right] \alpha_s \\
= &\sum_{\substack{\alpha_s=1 \\ \alpha_s \text{ critical}}}^{n} \left[\mathscr{C}^{j+1}_{\alpha_s,\tau} \circ \mathscr{C}^{j+1^{-1}}_{\alpha_i,\tau} \circ g_{j,\alpha_i}(v_i\alpha_i)\right] \alpha_s \qquad (14) \\
= &\sum_{\substack{\alpha_s=1 \\ \alpha_s \text{ critical}}}^{n} \left[\mathscr{C}^{j+1}_{\alpha_s,\tau} \circ g_{j,\tau} \circ \mathscr{C}^{j^{-1}}_{\alpha_i,\tau}(v_i\alpha_i)\right] \alpha_s \qquad (15) \\
= &\sum_{\substack{\alpha_s=1 \\ \alpha_s \text{ critical}}}^{n} \left[g_{j,\alpha_s} \circ \mathscr{C}^{j}_{\alpha_s,\tau} \circ \mathscr{C}^{j^{-1}}_{\alpha_i,\tau}(v_i\alpha_i)\right] \alpha_s \qquad (16)
\end{aligned}$$

$$\begin{aligned}
g_j^\Sigma \circ \psi_j(v) = &g_j^\Sigma \left(\sum_{\substack{\alpha_s=1 \\ \alpha_s \text{ critical}}}^{n} \left[\mathscr{C}_{\alpha_s,\tau} \circ \mathscr{C}^{j^{-1}}_{\alpha_i,\tau}(v_i\alpha_i)\right] \alpha_s\right) \\
= &\sum_{\substack{\alpha_s=1 \\ \alpha_s \text{ critical}}}^{n} \left[g^\Sigma_{j,\alpha_s} \circ \mathscr{C}_{\alpha_s,\tau} \circ \mathscr{C}^{j^{-1}}_{\alpha_i,\tau}(v_i\alpha_i)\right] \alpha_s \qquad (17)
\end{aligned}$$

Notice that equation 14 is obtained from the last line because each $g_{j,\alpha_t}(v_i\alpha_i) = 0$ if $t \neq i$, and equation 15 and equation 16 are obtained from the last lines because $g_{j,\bullet}$ is a morphism, where g_j is indexed by k-simplices. Notice 17 = 16 because we are only summing over all the critical k-simplices.

For a general Σ, things will work at exactly the same way: $g_{j,\bullet}$ will follow out of the chain, and its index will change each time until it becomes g_{j,α_s} at the outmost of the composition, and summing over all of these g_{j,α_s} for critical α_s gives exactly g^{Σ}_{j,α_s}. Thus, we will have $g^{\Sigma}_{k} \circ \phi_j = \phi_{j+1} \circ g_j$. □

8 Bonus: Algorithm

In this section, we present an algorithm that produces an acyclic partial matching that is $\mathscr{C}$-compatible, shown in Algorithm 1. The idea comes from week 8 lecture notes [2] and the paper [1]. Before proceed, we define $relation^{+}$ to be a dictionary (hash table) such that each key is a simplex τ and the correspondent value is the set $relation^{+}(\tau)$= $\{\alpha \in K : \tau$ is a face of $\alpha\}$, define $relation^{-}$ to be a dictionary (hash table) such that each key is a simplex τ and the correspondent value is the set $\{\alpha \in K : \alpha$ is a face of $\tau\}$. Notice that each value of these two dictionary is set, not a list, to reduce the computation cost.

The algorithm inputs: a simplicial complex K, which will be stored as a set. Each simplex τ has unique representation (such as the coordinate of its vertices), and the two dictionary $relation^{+}$ and $relation^{-}$.The algorithm outputs: An acyclic partial matching, called *Sigma*, and a set of *critical* simplices, called *critical*. Both of them are stored as a set.

Algorithm 1 Find $\mathscr{C}$-compatible acyclic partial matching

Initiate set *que*, *Sigma*, *critical*.
while *Sigma* + *critical* $\neq K$ **do** ▷ the union of Sigma and *critical* is not K
  Let τ to be the unclassified simplex with lowest available dimension such that the length of $ralation^{+}(\tau)$ is largest
  add τ to *que*, *critical*
  while *que* is not empty **do**
    Pick α in *que*
    for β in $relation^{+}(\alpha)$ **do**
      $relation^{-}(\beta) = relation^{-}(\beta)$ - *Sigma* - *critical*
      if length of $relation^{-}(\beta) = 1$ and $\mathscr{C}_{relation^{-}(\beta)[0],\beta}$ is invertible **then**:
        $\gamma = relation^{-}(\beta)[0]$ ▷ call the only element in $relation^{-}(\beta)$ γ
        add γ, β to *Sigma*
        $relation^{+}(\gamma) = relation^{+}(\gamma)$ - *Sigma* - *critical*
        $relation^{+}(\beta) = relation^{+}(\beta)$ - *Sigma* - *critical*
        $que = que + relation^{+}(\gamma) + relation^{+}(\beta)$
      else if length of $relation^{-}(\beta) = 0$ **or** $\mathscr{C}_{\gamma,\beta}$ not invertible for all γ in $relation^{-}(\beta)$ **then**:
        add β to *critical*
      end if
    end for
  end while
end while

Although this algorithm is used to find $\{\mathscr{C}\}$-compatible partial matching, we can also use it to find a $\{\mathscr{C}^i\}$-compatible partial matching for a cosheaf filtration $\{\mathscr{C}^i\}$, by changing $\mathscr{C}_{\gamma,\beta}$ *is invertible* to be $\mathscr{C}^j_{\gamma,\beta}$ *is invertible* for all indices j and so on in the *if* and *else if* statement respectively.

The set up of τ to be the simplex with lowest dimension and the fact that we move a pair of simplices $(\sigma \lhd \tau)$ to Σ if and only if σ is the only unclassified face of τ guarantees our output partial matching is acyclic, by proposition 4.2 in [1].

Although we have not proved the output acyclic partial matching *Sigma* is the optimal one yet, the fact that we pick a specified τ in the third line of algorithm gives a better result than randomly

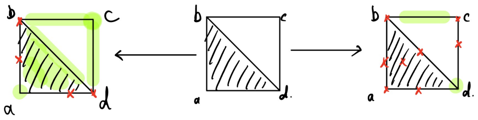
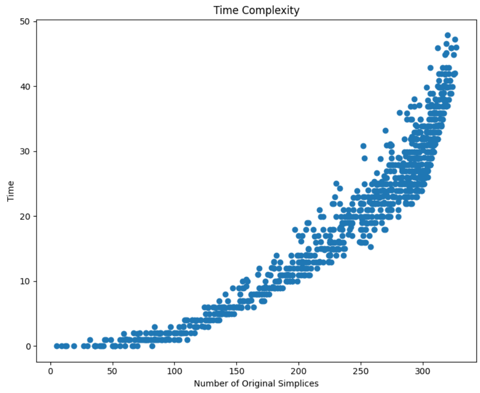


Figure 3: Results of algorithm 1 with $K = \{a, b, c, d, ab, bc, cd, da, bd, abd\}$



pick a τ. For example, if the simplicial complex K is the one shown in graph 3 and $\mathscr{C}$ is the constant cosheaf $\underline{\mathbb{F}}_K$, randomly picking vertex a to be the first simplex with the lowest dimension results in the left most graph, where $Sigma = \{b, ab, d, ad\}$ contains the simplices crossed using red pen, the critical simplices are highlighted in green, while algorithm 1 picks one of b, d to be the first simplex, and results in the right most graph, where $Sigma = \{b, bd, a, ad, c, cd, ab, abd\}$, which is the optimal one.

Finally, there are three loops in this algorithm, and in the *else if* line, we need to iterate $relation^{-}(\beta)$ to check if all the $\mathscr{C}_{\gamma,\beta}$ is invertible. However, this line can be removed if we only consider constant cosheaf on K. All other lines have computation cost $O(1)$. Thus the complexity for K with constant cosheaf is $O(m^3)$. The right figure above shows the time cost of this algorithms for varying K with constant cosheaves. The x-coordinate is the number of simplices in K, and the y-coordinate is the time used for that iteration. It can be seen the cost is approximately $O(m^3)$, which aligns with our analysis.

9 Conclusion

This project demonstrates that when calculating the persistent homology groups of a simplicial complex K with coefficients in a cosheaf filtration, identifying an acyclic partial matching Σ allows for more efficient computation by focusing solely on the Σ-critical simplices. However, certain limitations persist in our proofs, such as the assumption of K being a finite simplicial complex, the cosheaf filtration consisting only a finite sequence of cosheaves, and our considerations limited to real finite-dimensional vector spaces. Although these assumptions are practical and align with typical scenarios, it is possible to weaken some of these assumptions further.

In the final section, an algorithm for finding an acyclic partial matching and an illustrative example were presented, with a brief analysis to its computational costs. However, the optimality of the acyclic partial matching produced by this algorithm remains uncertain, pointing to an avenue for future investigation. Nonetheless, an immediate next step could involve providing a closer analysis to the computation cost and demonstrating its utility through real-world examples, offering valuable insights and potential improvements.

References

[1] Justin Curry, Robert Ghrist , Vidit Nanda. “Discrete Morse Theory for Computing Cellular Sheaf Cohomology”. In: *Foundations of Computational Mathematics* (2016), pp. 875–897.

[2] Vidit Nanda. *Computational Algebraic Topology Lecture Notes*.