# Continuous Separation Axioms

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**Abstract**

Continuous separation axiom contributes to the broader field of topology by extending classical separation concepts to a continuous framework. The prerequisite knowledge about Vietoris topology is presented in the first section. Some important results about continuous separation axioms are presented in the second section. A development of continuous versions of $T_0$, $T_1$ and $T_2$ spaces is presented in the last section. A unified framework illustrating connections among all classical and continuous separation axioms can be found in Figure 2, as a key finding of this dissertation.

# Contents

# 1 Introduction

Separation axiom is an important concept in analytic topology. It is a general property of a space that describes the ability to separate points or closed sets. Besides the classical definitions, many other separation axioms are developed for various purposes by different people. For example, star separation axioms are introduced and developed by C. Chattopadhyay in [Cha11], and generalized separation axioms are introduced by V.K. Sharma and developed by S. Balasubramanian in [Bal10].

Among them, P. Zenor and G. Gruenhage introduced a continuous version of separation axioms. This development can be attributed to enhance the metrization theorem. The motivation initially came from an equivalent definition of stratifiable space in [Bor66]. In this paper, the author shows a space $X$ is stratifiable if and only if there is a function $\phi$, mapping from the space of closed subsets $2^X$ equipped with Vietoris topology, to the space of non-negative real-valued continuous functions $C^+(X)$ equipped with compact open topology. Such $\phi$ satisfies two properties:

- For each closed set $H$, $\phi(H)(x) = 0$ if and only if $x \in H$.
- If $H, K$ are two closed sets and $H \subset K$, then $\phi(K)(x) \leq \phi(H)(x)$ for all $x$.

The choice of compact open topology on $C^+(X)$ implies that such $\phi$ without the second property is exactly the definition of perfect normality operator that we will introduce formally in later section. Although we will not prove this statement here, relative information can be found in chapter 2.6 and 3.4 in [Eng89]. In addition, G. Gruenhage has showed that the second property is critical by providing an example that is continuously perfectly normal but not metrizable in [Gru76], which makes the definition of continuously perfectly normal space more significant. Following from this definition, they also defined continuously normal space and continuously completely regular space. These definitions together form the continuous separation axioms. This development makes meaningful contributions to the study of the metrization theorem and the classification of separation properties of a space.

The primary objective of this dissertation is to introduce and extend the continuous version of separation axioms. This includes discussing the prerequisite topology on space of closed subsets, known as the Vietoris topology, and presenting key results from P. Zenor and G. Gruenhage. Motivated by their definitions, we also develop lower continuous separation axioms. A unique contribution of this dissertation includes providing an overview of the relationship between all classical and continuous separation axioms, which can be viewed in Figure 2. In this dissertation, we will cite the original source of the theorem if it has been proved or mentioned in the previous papers. If a theorem is presented without citation, it is an original contribution.

We assume readers are familiar with basic definitions such as the definition of a topology, a basis and $T_0$, $T_1$, $T_2$, $T_3$, $T_{3.5}$ and normal spaces. More background information can be found in the textbook [Mun00] and [Eng89]. There are some conventions and notations in this dissertation:

- A class of points is called set, denoted by capital letters (for example $A$).
- A class of sets is called collection, denoted by cursive letters (for example $\mathcal{A}$).
- All topological space X are at least $T_1$ unless specified.

Given the above assumption, we do not differentiate the definition of a $T_4$ space and a normal space. These two words may be used interchangeably in this dissertation. Similarly, we define a topological space $X$ to be $T_6$ or **perfectly normal** if and only if for every disjoint closed sets $C, D \subset X$, there exists a continuous function $f : X \to [0,1]$ such that $C = f^{-1}(\{0\})$, $D = f^{-1}(\{1\})$. This is consistent with the more widely used definition which says that $X$ is perfectly normal if and only if it is normal and every closed subset of $X$ is $G_\delta$, as stated in Theorem 1.5.19 in [Eng89].

## 2 Vietoris Topology

To facilitate future discussions within this dissertation, we introduce definitions and fundamental findings relating to spaces of closed subsets and its extension to the space of nonempty set. These results are either only asserted but not proved, or briefly proved but we fill in more details here.

Denote:

- $\mathcal{A}X = \{E \subset X : E \text{ is not empty}\}$,
- $2^X = \{C \subset X\text{: } C \text{ is closed and not empty}\}$.

### 2.1 Basic Tools

**Definition 2.1.** *[Eng89] Let $X$ is a topological space with topology $\tau$, the* ***Vietoris topology*** *$2^\tau$ on $2^X$ is the topology generated by the collection of all the sets of the form*

$$\langle U_1, ..., U_n \rangle = \left\{ E \in 2^X : E \subset \bigcup_{i=1}^{n} U_i \text{ and } E \cap U_i \neq \emptyset \text{ for } i = 1,2,...,n \right\}.$$

**Lemma 2.2.** *[Mic51] Let $\mathcal{U}_1 = \langle U_1, ..., U_n \rangle$, $\mathcal{U}_2 = \langle V_1, ..., V_m \rangle$. Denote $U = \bigcup_{i=1}^n U_i$, $V = \bigcup_{i=1}^m V_i$, then $\mathcal{U}_1 \cap \mathcal{U}_2 = \langle U_1 \cap V, ..., U_n \cap V, V_1 \cap U, ..., V_m \cap U \rangle$.*

*Proof.* ($\subset$) First let $C \in \mathcal{U}_1 \cap \mathcal{U}_2$, then $C \subset U \cap V = (U_1 \cap V) \cup ... \cup (U_n \cap V) \cup (V_1 \cap U) \cup ... \cup (V_m \cap U)$. Since C intersects $U_i, V_j$ for all $i, j$, we have $C \cap (U_i \cap V) \neq \emptyset$ as $C \subset V$. Similarly, $C \cap (V_j \cap U) \neq \emptyset$. Thus, $C \in \langle U_1 \cap V, \ldots, U_n \cap V, V_1 \cap U, \ldots, V_m \cap U \rangle$ by definition.

($\supset$) Assume $C \in \langle U_1 \cap V, \ldots, U_n \cap V, V_1 \cap U, \ldots, V_m \cap U \rangle$, then $C \subset (U_1 \cap V) \cup \ldots \cup (U_n \cap V) \cup (V_1 \cap U) \cup \ldots \cup (V_m \cap U) = U \cap V$. $C \cap (U_i \cap V) \neq \emptyset$ implies $C \cap U_i \neq \emptyset$ for all $i = 1, 2, \ldots, n$ and similarly $C \cap V_j \neq \emptyset$ for all $j = 1, 2, \ldots, m$. Thus, $C \in \mathcal{U}_1 \cap \mathcal{U}_2$. $\square$

**Proposition 2.3.** *[Mic51] The collections of the form $\langle U_1, \ldots, U_n \rangle$, with $U_1, ..., U_n$ open, forms a basis for the Vietoris topology on $2^X$.*

*Proof.* The above lemma shows these collections satisfy the two properties of a basis, Namely, $\langle X \rangle$ contains all closed sets in $2^X$, and the intersection of two open collections is also an open collection. $\square$

We now present fundamental results concerning the Vietoris topology, which serve as essential tools for technical proofs in later sections. The first tool involves identifying two disjoint open collections, which is crucial for analyzing the separation axioms within the space of closed subsets.

**Corollary 2.3.1.** *Let $\mathcal{U}_1 = \langle U_1, ..., U_n \rangle$, $\mathcal{U}_2 = \langle V_1, ..., V_m \rangle$. Denote $U = \bigcup_{i=1}^n U_i$, $V = \bigcup_{i=1}^m V_i$. $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$ if and only if there exists $U_i$ such that $U_i \cap V = \emptyset$ or there exists $V_j$ such that $V_i \cap U = \emptyset$.*

*Proof.* It is worth observing that if one of the open sets in $\langle U_1 \cap V, ..., U_n \cap V, V_1 \cap U, ..., V_m \cap U \rangle$ is empty, then this collection is empty, because the intersection of any set and empty set is empty. With this on hand, the proof direct follows from the lemma above. $\square$

It may not be easy to see directly how a sub-collection of basic open collection looks like in $2^X$. Notably, let $\mathcal{U} = \langle U_1, ..., U_n\rangle$ be open in $2^X$, but $\langle U_1, ..., U_{n-1}\rangle$ may not be an open sub-collection of $\mathcal{U}$, if we do not have any information about $U_n$. The following lemma gives a precise method to identify a basic open sub-collection of an open collection in $2^X$ using only the knowledge on the standard space $X$.

**Lemma 2.4.** *[Mic51] For $(X, \tau)$ and space of closed subsets with Vietoris topology $(2^X, 2^\tau)$:*

$$\langle U_1, ..., U_n\rangle \subset \langle V_1, ..., V_m\rangle \iff \bigcup_{i=1}^{n} U_i \subset \bigcup_{i=1}^{m} V_i \text{ and } \forall i, \exists j \text{ such that } U_j \subset V_i.$$

*Proof.* ($\Rightarrow$) Let $\langle U_1, \dots, U_n\rangle \subset \langle V_1, \dots, V_m\rangle$. Assume $x \in \bigcup_{i=1}^n U_i$. Randomly pick $x_i \in U_i$ for each $i$. $\{x, x_1, \dots, x_n\}$ form a closed set in $\langle U_1, ..., U_n\rangle$. Then $\{x, x_1, \dots, x_n\} \in \langle V_1, ..., V_m\rangle$ and thus $x \in \bigcup_{i=1}^m V_i$. For the second part, assume $\exists V_i$ such that there does not exist a $U_j$ that satisfies $U_j \subset V_i$, $\forall j \in \{1, \dots, n\}$. Then pick $x_j$ in $U_j \backslash V_i \neq \emptyset$ for all $j$. $\{x_1, \dots, x_n\}$ is in $\langle U_1, \dots, U_n\rangle$ but not in $\langle V_1, \dots, V_m\rangle$, which arrives a contradiction.

($\Leftarrow$) Let $C \in \langle U_1, \dots, U_n\rangle$, then $C \subset \bigcup_{j=1}^n U_j \subset \bigcup_{i=1}^m V_i$. For each $i$, since $\exists j$ such that $V_i \supset U_j$, then $V_i \cap C \supset U_j \cap C \neq \emptyset$. Therefore $C \in \langle V_1, \dots, V_m\rangle$ by definition. □

**Corollary 2.4.1.** *[Mic51] For $(X, \tau)$ and space of closed subsets with Vietoris topology $(2^X, 2^\tau)$, if $\{U_\alpha\}_{\alpha \in A}$ forms a neighborhood basis of $x$ in $X$, then $\{\langle U_\alpha\rangle\}_{\alpha \in A}$ forms a neighborhood basis for $\{x\}$ in $2^X$.*

*Proof.* Let $\{U_\alpha\}_{\alpha \in A}$ be a neighborhood basis of $x \in X$, let $\langle V_1, \dots, V_m\rangle$ be a neighborhood of $\{x\}$, then $x \in \bigcap_{i=1}^m V_i$, we can find a basic neighborhood $U_\beta$ for $\beta \in A$ such that $x \in U_\beta \subset \bigcap_{i=1}^m V_i$. Applying the above lemma, $\{x\} \in \langle U_\beta\rangle \subset \langle V_1, \dots, V_m\rangle$, thus $\{\langle U_\alpha\rangle\}_{\alpha \in A}$ forms a basis for the neighborhoods of $\{x\}$ in $2^X$. □

From this lemma, we remark that an open collection $\langle U_1, ..., U_n\rangle$ is in $\langle U_1, ..., U_{n+1}\rangle$ if and only if there exists $i$ such that $U_i \subset U_{n+1}$. This will be helpful when we want to reduce the complexity of an open collection.

Given the Vietoris topology, we can use $\langle U_1, \dots, U_n\rangle$ to see what an open collection looks like, but closed collections can be hard to represent. Next, we will discuss some properties about closed collections in this topology.

**Lemma 2.5.** *[Mic51] For $(X, \tau)$ and space of closed subsets with Vietoris topology $(2^X, 2^\tau)$,*

$$\overline{\langle U_1, ..., U_n\rangle} = \left\langle \overline{U_1}, ..., \overline{U_n}\right\rangle.$$

*Proof.* ($\subset$) Assume $A$ is a closed set that is not in $\langle \overline{U_1}, \dots, \overline{U_n}\rangle$, there are two cases: the first case is $\exists i$ such that $A \cap \overline{U_i} = \emptyset$, then $\exists V \subset X$ open and $V$ contains $A$ such that $V \cap \overline{U_i} = \emptyset$ (for example, let $V = X \backslash \overline{U_i}$). Since $V$ contains $A$, $A \in \langle V\rangle$ and $\langle V\rangle \cap \langle U_1, \dots, U_n\rangle = \emptyset$ using Corollary 2.3.1. The second case is $A \not\subset \bigcup_{i=1}^n \overline{U_i}$, then exists $x \in A$ such that $x \notin \bigcup_{i=1}^n \overline{U_i}$ and an open set $V$ containing $x$ such that $V \cap \bigcup_{i=1}^n \overline{U_i} = \emptyset$, so $A \in \langle V, X\rangle$ and $\langle V, X\rangle \cap \langle U_1, \dots, U_n\rangle = \emptyset$ by Corollary 2.3.1. Therefore, in either case, we have found an open neighborhood of $A$ that does not intersect $\langle U_1, \ldots, U_n \rangle$, by theorem 17.5 in [Mun00] ($x \in \overline{B}$ if and only if every open set $U$ containing $x$ intersects $B$), we conclude $A \notin \overline{\langle U_1, \ldots, U_n \rangle}$.

($\supset$) Let $C \in \langle \overline{U_1}, \ldots, \overline{U_n} \rangle$, randomly pick an open set $\langle V_1, \ldots, V_m \rangle$ containing $C$, we will show $\langle V_1, \ldots, V_m \rangle \cap \langle U_1, ..., U_n \rangle \neq \emptyset$. Since $C \subset \bigcup_{i=1}^{n} \overline{U_j}$, and $C \cap V_i \neq \emptyset$, then for all $i$, there exists $j$ such that $V_i \cap \overline{U_j} \neq \emptyset$. Since $V_i$ is open, we can further conclude $V_i \cap U_j \neq \emptyset$. For each $i \in \{1, \ldots, m\}$, pick $x_i \in V_i \cap U_j$. Similarly, for all $j$, there exists $i$ such that $U_j \cap V_i \neq \emptyset$. Then for each $j \in \{1, \ldots, n\}$, pick $y_j \in U_j \cap V_i$. We can check $\bigcup_{i=1}^{m} \{x_i\} \cup \bigcup_{i=1}^{n} \{y_i\} \in \langle V_1, \ldots, V_m \rangle \cap \langle U_1, ..., U_n \rangle$ by Lemma 2.2. Thus, $C \in \overline{\langle U_1, ..., U_n \rangle}$. $\square$

Given this result, we can easily provide some examples of closed sets, rather than writing them as complements of open collections all the time. Additionally, it renders the unproven statements from [Mic51] more evident.

**Corollary 2.5.1.** *[Mic51] For $(X, \tau)$ and space of closed subsets with Vietoris topology $(2^X, 2^\tau)$,*

1. *$\{E \in 2^X : E \subset A\}$ is closed in $(2^X, 2^\tau)$ if $A \subset X$ is closed.*
2. *$\{E \in 2^X : E \cap A \neq \emptyset\}$ is closed in $(2^X, 2^\tau)$ if $A \subset X$ is closed.*

*Proof.* For the first statement, let $A$ be a closed set in $X$. Observe that $\{E \in 2^X : E \subset A\} = \langle A \rangle = \langle \overline{A} \rangle = \overline{\langle A \rangle}$ is closed in $2^X$ by Lemma 2.5.

Similarly, for the second statement, let $A$ be a closed set in $X$. Observe that $\{E \in 2^X : E \cap A \neq \emptyset\} = \langle A, X \rangle = \langle \overline{A}, \overline{X} \rangle = \overline{\langle A, X \rangle}$ is closed in $2^X$ again by Lemma 2.5. $\square$

## 2.2 Extended Vietoris Topology

We now extend the Vietoris topology to the space of nonempty subsets. This topology will not be used until we define continuous $T_1$ and $T_0$ operator in the last section, because for higher continuously separated spaces, the closed sets are our main consideration, and there are more interesting results on the Vietoris topology. However, in lower separated spaces, working on only closed subsets does not give us desired separation properties, instead we want to work with the space of all nonempty subsets $\mathcal{A}X$. This will be discussed with more details in the last section.

**Definition 2.6.** *[Mic51] The **extended Vietoris topology** in the space of nonempty subsets $\mathcal{A}X$ is the one generated by open collections of the form*

$$\langle U_1, \ldots U_n \rangle^+ := \left\{ E \in \mathcal{A}X : E \subset \bigcup_{i=1}^{n} U_i ; \forall i \in \{1, \ldots, n\}, E \cap U_i \neq \emptyset \right\}$$

Notice that the intersection of two collections with this form will still be a collection with the same form, this means Lemma 2.2 still hold on extended Vietoris topology, with exactly the same proof, so we can directly show the above collections form a basis of this topology by checking the two properties for a basis, as what we did in Proposition 2.3.

## 2.3 Further Properties

Next we will present some advanced properties about Vietoris topology. These results may not appear directly in the continuous Separation Axioms, but will further build more comprehensive understanding on the space of closed subsets. The next few results talk about some interesting subspaces in $2^X$. Denote:

- $\mathcal{F}X = \{E \in 2^X : E \text{ is finite}\}$,
- $\mathcal{F}_n X = \{E \in 2^X : E \text{ has at most n elements}\}$.

**Proposition 2.7.** *[Mic51] For a topological space $(X, \tau)$ and its space of closed subsets with Vietoris topology $(2^X, 2^\tau)$:*

1. *$\mathcal{F}X$ is dense in $(2^X, 2^\tau)$;*
2. *If $X$ is Hausdorff, then $\mathcal{F}_n X$ is closed in $(2^X, 2^\tau)$ for all $n \geq 1$;*
3. *The natural map $pr_n : X^n \to \mathcal{F}_n X$, defined by $pr_n(x_1, ..., x_n) = \{x_1, ..., x_n\}$ is continuous.*

*Proof.* For the first statement, we want to show that every open collection intersect $\mathcal{F}X$. Let $\langle U_1, \ldots, U_n \rangle$ be a basic open collection in $2^X$, Randomly pick one point $x_i$ from each $U_i$ for $1 \leq i \leq n$, the set $\{x_1, \ldots, x_n\}$ is contained in both $\langle U_1, \ldots, U_n \rangle$ and $\mathcal{F}X$.

For the second statement, let $X$ be Hausdorff, $n \geq 1$, let $C$ be a closed set in $X$ that contains at least $(n+1)$ points. Randomly pick $(n+1)$ points from $C$, denote them to be $x_1, \ldots, x_{n+1}$. Applying the Hausdorff property multiple times, we get $(n+1)$ open sets $U_1, \ldots, U_{n+1}$, they are mutually disjoint and each $U_i$ contains $x_i$ but not any other points in $\{x_1, \ldots, x_{n+1}\}$. Consider the open collection $\langle U_1, \ldots, U_{n+1}, X \rangle$. It contains $C$ but does not contain any element in $\mathcal{F}_n X$, since any closed set in $\langle U_1, \ldots, U_{n+1}, X \rangle$ must contain at least $(n+1)$ points to intersect with each of $U_1, \ldots, U_{n+1}$. Thus, we have shown the complement of $\mathcal{F}_n X$ is open, so $\mathcal{F}_n X$ is a closed subspace in $2^X$.

For the third statement, it can be checked directly that $pr_n$ is a well-defined map. Let $n \geq 1$, Let $\langle U_1, \ldots, U_k \rangle$ be open in $2^X$, denote $\mathcal{U} = \langle U_1, \ldots, U_k \rangle \cap \mathcal{F}_n X$, which is open in $\mathcal{F}_n X$ under subspace topology. Claim $pr_n^{-1}(\mathcal{U})$ is open. Assume $(x_1, \ldots, x_n) \in pr_n^{-1}(\mathcal{U})$, then each point $x_i$ belongs to at least one of $U_1, \ldots, U_k$, for illustration, let us say $x_i$ is contained in $U_1, U_2 \ldots,$ and $U_r$ for some $r \leq k$. Denote their intersection $\bigcap_{i=1}^r U_i$ to be $V_i$. Then $V_1 \times \ldots \times V_n$ is the desired basic open sets in $X^n$ that contains $(x_1, \ldots, x_n)$ and is contained in $pr_n^{-1}(\mathcal{U})$. □

In most cases, $\mathcal{F}_n X$ does not exhibit an obvious connection with the original space $X$. However, the situation becomes more interesting when $n = 1$.

**Lemma 2.8.** *[Eng89] $pr_1$ is a homeomorphism between $X$ and $\mathcal{F}_1(X)$.*

*Proof.* Notice $pr_1$ is a bijection. If $x_1 \neq x_2 \in X$, then $pr_1(x_1) = \{x_1\} \neq \{x_2\} = pr_1(x_2)$. For any singleton set $\{x\}$, $pr_1(x) = \{x\}$. Also, $pr_1$ is continuous by the above lemma, we are left to show $pr_1$ is an open map. Let $U$ be open in $X$, and $x \in U$, then we can find an open set $V$ such that $x \in V \subset U$. Observe that $pr_1(U) = \langle U \rangle \cap \mathcal{F}_1 X$. we can directly check that $\langle V \rangle \cap \mathcal{F}_1 X$ is the open collection in $\mathcal{F}_1 X$ that contains $\{x\}$, and it is contained in $\langle U \rangle \cap \mathcal{F}_1 X$ by Lemma 2.4. Thus $pr_1$ is open and $X$ is homeomorphic to $\mathcal{F}_1 X$. □

Using the above lemma, when we have an open collection $\mathcal{B}$ of sets in $2^X$, all the singleton sets in $\mathcal{B}$ form an open collection in $\mathcal{F}_1 X$ under subspace topology. By the homeomorphism, $\{x : \{x\} \text{ is a singleton set in } \mathcal{B}\}$ is an open set in $X$.

**Proposition 2.9.** *[Mic51] Let $X$ be a topological space,*

1. *$2^X$ is always $T_0$;*
2. *$X$ is $T_1 \implies 2^X$ is $T_1$;*
3. *$X$ is regular $\iff 2^X$ is Hausdorff.*

*Proof.* For the first statement, let $A, B$ be two distinct closed sets in $X$. $A \neq B$ means $\exists x \in B \backslash A$ or $\exists x \in A \backslash B$. Without loss of generality assume it is the first case. Then $B \in \langle X, X \backslash A \rangle$ but $A \notin \langle X, X \backslash A \rangle$.

For the second statement, let $A, B$ be two distinct closed sets in $X$. Without loss of generality assume $\exists x \in B \backslash A$. Then since $X$ is $T_1$, $\{x\}$ is closed, so $\langle X \backslash \{x\} \rangle$ is an open collection that contains $A$ but not $B$. $\langle X, X \backslash A \rangle$ is an open collection containing $B$ but not $A$.

For the third statement, ($\Rightarrow$) Let $X$ be $T_3$. $A \neq B \in 2^X$. Without loss of generality assume $\exists x \in B \backslash A$ and $\exists U, V$ disjoint open sets in $X$ such that $x \in U, A \subset V$. Then $A \in \langle V \rangle$ and $B \in \langle X, U \rangle$ and $\langle V \rangle \cap \langle X, U \rangle = \emptyset$ by Corollary 2.3.1, then $A, B$ are separated and so $2^X$ is $T_2$.

($\Leftarrow$) Let $x \in X$ and $A \subset X$ be a closed set that does not contain $x$, we want to find open sets in $X$ that separate them. Consider $A$ and $A \cup \{x\} \in 2^X$. They are both closed. Since $2^X$ is $T_2$, there exist disjoint open basic collections $\langle U_1, \dots, U_n \rangle \ni A$ and $\langle V_1, \dots, V_m \rangle \ni A \cup \{x\}$. Then we must have $A \notin \langle V_1, \dots, V_m \rangle$. However, $A \subset A \cup \{x\} \subset V_1 \cup \dots \cup V_m$, then $\exists i$ such that $A \cap V_i = \emptyset$ but $x \in V_i$. On the other hand, $A \in \langle U_1, \dots, U_n \rangle$ but $A \cup \{x\} \notin \langle U_1, \dots, U_n \rangle$. Then $x \notin U_1 \cup \dots \cup U_n$. Thus $V_i \ni x$ and $U \coloneqq U_1 \cup \dots \cup U_n \supset A$ are two open disjoint subset of $X$ that separate $A, x$. $\square$

It is worth noticing that the reverse of second statement is not true. As a counterexample, let $X$ be a space that contains infinite points, equipped with the coarsest topology $\{X, \emptyset\}$. Since $2^X$ only contains nonempty closed sets in $X$, it only contains a single element $X$, then by definition $2^X$ is $T_1$. However, clearly $X$ is not $T_1$ because any two points in $X$ are indifferent in this topology.

Using this proposition, we can find out the separation level of a space of closed subsets. Consider the following example:

**Example 1.** *Consider the $K$-topology on $\mathbb{R}$ (see section 13 in [Mun00]), where $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. This topology contains the standard topology on $\mathbb{R}$ and the sets of the form $(a, b) \backslash K$, with $a, b \in \mathbb{R}$. In this case, $K$ is a closed set because $\mathbb{R} \backslash K$ is open. This space is not $T_3$, since we cannot find disjoint open sets separating 0 and $K$. Therefore, we can conclude its space of closed subsets $2^{\mathbb{R}}$ is not Hausdorff, as we cannot find disjoint open sets separating $K$ and $K \cup \{0\}$.*

Recall that if $X$ is a topological space, $X$ is **disconnected** if there exist two disjoint proper non-empty open subsets of $X$ which cover $X$. $X$ is **connected** if it is not disconnected.

**Proposition 2.10.** *[Mic51] Let $\mathcal{B}$ be a collection of sets in $2^X$. If $\mathcal{B}$ is connected in the Vietoris topology and at least one of whose elements are connected, then $\bigcup_{B \in \mathcal{B}} B$ is connected.*

*Proof.* Let $\mathcal{B} = \left\{B \in 2^X\right\}$ be a collection of closed sets in $X$. we want to show if $\bigcup_{B \in \mathcal{B}} B$ is not connected but there is a $B \in \mathcal{B}$ that is connected, then $\mathcal{B}$ is not connected.

Let $U, V$ be the nonempty open sets separating $\bigcup_{B \in \mathcal{B}} B$. We need to consider two cases:

If $\exists B \in \mathcal{B}$ such that $B \subset U$, then $\langle V, X \rangle$ and $\langle U \rangle$ are the desired open collections that separate $\mathcal{B}$. Clearly, $\langle U \rangle$ is not empty. Since $V$ is nonempty, it must intersect some $B \in \mathcal{B}$, then $\langle V, X \rangle$ is not empty. For any element $B_0 \in B, B_0 \notin \langle U \rangle$ indicates $B_0$ is not fully contained in $U$ then $B_0$ must meet $V$, and thus $B_0 \in \langle V, X \rangle$. Thus $\mathcal{B} \subset \langle U \rangle \cup \langle V, X \rangle$, Finally, $\langle U \rangle$ and $\langle V, X \rangle$ are disjoint by Corollary 2.3.1.

On the other hand, if all closed set $B \in \mathcal{B}$ that intersects $U$ also intersects $V$, then consider $\langle U, X \rangle$ and $\langle V \rangle$. Firstly we show they are nonempty. Since $U$ is non-trivial, $U$ intersects some $B \in \mathcal{B}$, then $\langle U, X \rangle$ is non-empty. If $\langle V \rangle$ is empty, then there is not any $B$ that is fully contained in $V$, they all intersect $U$. However, in this case we assumed every $B$ that intersects $U$ must also intersects $V$. Thus each $B$ is properly partitioned by $U$ and $V$, which contradicts to our assumption that at least one of $B \in \mathcal{B}$ is connected. Thus, both $\langle U, X \rangle, \langle V \rangle$ are nonempty. We can use the exactly same method to show they are disjoint and their union contains $\mathcal{B}$ as in the first case.

Thus $\mathcal{B}$ is not connected. $\square$

## 3 Continuous Separation Axioms

Denote:

- $\mathcal{M}X = \{(H, K) \in 2^X \times 2^X : H \cap K = \emptyset\}$,
- $\mathcal{D}X = \{(x, K) \in X \times 2^X \colon x \notin K\}$.

**Definition 3.1.** *[Zen75] $T : X \times 2^X \to [0,1]$ is a **perfect normality operator (PN operator)** if and only if for any $H \in 2^X$, $H = \{x \in X : T(x, H) = 0\}$. A space $X$ is **continuously perfectly normal** ($CPN$) if it admits a continuous perfect normal operator.*

**Definition 3.2.** *[Zen75] $T : X \times \mathcal{M}X \to [0,1]$ is a **normality operator (N operator)** if and only if for any $(H, K) \in \mathcal{M}X$, $H \subset \{y \in X : T(y,(H,K)) = 0\}$ and $K \subset \{y \in X : T(y,(H,K)) = 1\}$. A space $X$ is **continuous normal** ($CN$) if it admits a continuous normal operator.*

**Definition 3.3.** *[Zen75] $T : X \times \mathcal{D}X \to [0,1]$ is a **completely regularity operator (CR operator)** if and only if for any $(x, H) \in \mathcal{D}X$, $T(x,(x,H)) = 0$ and $H \subset \{y \in X : T(y,(x,H)) = 1\}$. A space $X$ is **continuous completely regular** ($CCR$) if it admits a continuous completely regular operator.*

These three definitions are the continuous version of $T_6$, $T_4$ and $T_{3.5}$ respectively. The formal mathematical definition is abstract, but for each operator, we can understand it better by fixing all variables except the first one, which gives us classical separation axioms. For example, for a fixed closed set $C \subset X$, the continuous perfect normal operator $T(x, C)$ is just a function that maps from X to $[0,1]$ that has the same property as the continuous function $f$ such that $f^{-1}(\{0\}) = C$ in the definition of $T_6$.

In addition, as an intuitive example for these spaces, one can think about metric space. For example, the continuous $PN$ operator is the distance between a single point $x$ and a closed subset $C$, defined by $d(x, C) = \inf_{c \in C} d(x, c)$, where $d$ is the metric. This is a continuous function and the distance is zero if and only if $x \in C$. The formal statement of these two examples are proven below.

**Proposition 3.4.** *[Zen75] Let $X$ be a topological space, $CPN \implies T_6$ ; $CN \implies T_4$; $CCR \implies T_{3.5}$.*

*Proof.* Let $X$ be $CPN$, for each closed set $C \subset X$, the restriction map of the perfectly normality operator $T$ to the first coordinate, $f(x) \coloneqq T(x, C) : X \longrightarrow [0,1]$, where $C$ is fixed, is a continuous function that maps $x$ to 0 if and only if $x \in C$ by definition.

If $X$ is $CN$, let $T$ be its continuous normality operator. Then each coordinate of $T$ is continuous. Specifically, for each disjoint closed sets $C, D \subset X$, the restriction map to the first coordinate $f(x) \coloneqq T(x,(C,D)) : X \longrightarrow [0,1]$ is continuous. $f^{-1}([0,0.5))$ and $f^{-1}((0.5,1])$ are the two disjoint open sets that separates $C$ and $D$.

Similarly, if $X$ is $CCR$, for each point $x$ and closed set $C$ in $X$, where $x \notin C$, let $f(z) \coloneqq T(z,(x,C)) : X \longrightarrow [0,1]$. $f$ is the desired continuous function such that $f(x) = 0$ and $f(C) = 1$ by definition. □

**Proposition 3.5.** *[Zen75] Every metrizable space is $CPN$ and every $CPN$ is $CN$.*

*Proof.* (Metric $\Rightarrow CPN$): Let $X$ be a metric space, $d : X \times X \to \mathbb{R}$ be its metric. Without loss of generality, $d$ is bounded by 1. If not, we can define a new metric $d' : X \times X \to [0, 1]$ by $d'(x, y) = \frac{d(x,y)}{1+d(x,y)}$, $\forall x, y \in X$. Direct computation gives that $d'$ is a metric.

Define $T : X \times 2^X \longrightarrow [0, 1]$ by

$$T(x, H) = d(x, H) = \inf_{y \in H} d(x, y).$$

Firstly, if $x \in H$, then $T(x, H) = \inf_{y \in H} d(x, y) = d(x, x) = 0$. If $x \in X$ such that $T(x, H) = \inf_{y \in H} d(x, y) = 0$, then $\forall \epsilon > 0$, there is an open ball $B(x, \epsilon)$ centred at $x$ with radius $\epsilon$ such that $B(x, \epsilon) \cap H \neq \emptyset$. Then $x \in \overline{H} = H$ as $H$ is closed. Thus $H = \{x \in X : T(x, H) = 0\}$.

It suffices to prove $T$ is continuous. Let $\epsilon > 0$, $x$ be a point and $H$ be a closed set in $X$. Pick $h \in H$ such that $|d(x, h) - d(x, H)| < \frac{\epsilon}{4}$, define

$$\begin{aligned}
U &= \left\{u \in X : d\left(u, H\right) < \frac{\epsilon}{4}\right\}, \\
V &= \left\{v \in X : d(x, v) < \frac{\epsilon}{4}\right\}, \\
W &= \left\{w \in X : d(h, w) < \frac{\epsilon}{4}\right\}.
\end{aligned}$$

Notice that all of these three sets are open. $U$ is open since $\forall u \in U$, $\delta := d\left(u, H\right) < \frac{\epsilon}{4}$, then $B\left(u, \frac{\epsilon}{4} - \delta\right)$ is the open set containing $u$ and $\forall y \in B\left(u, \frac{\epsilon}{4} - \delta\right), d(y, H) \leq d\left(u, y\right) + d\left(u, H\right) < \frac{\epsilon}{4} - \delta + \delta = \frac{\epsilon}{4}$, then $y \in U$. $V, W$ are open as they are open balls centered at $x$, $h$ respectively.

Claim: $V \times \langle U, W\rangle$ is the desired open set such that $\forall v \in V, K \in \langle U, W\rangle, |T(v, K) - T(x, H)| < \epsilon$. Let $w \in K \cap W$, observe that $\forall k \in K$, $d(x, k) + \frac{3\epsilon}{4} \geq d(x, w)$, otherwise $k \notin U$. Triangle inequality gives

$$\begin{aligned}
d(v, k) + d(x, v) + \frac{3\epsilon}{4} &\geq d(v, w) - d(v, x) \\
d(v, k) + \frac{3\epsilon}{4} &\geq d(v, w) - 2d(v, x) \\
d(v, k) + \frac{3\epsilon}{4} &\geq d(v, w) - \frac{2\epsilon}{4} \\
d(v, k) + \frac{\epsilon}{4} &\geq d(v, w).
\end{aligned}$$

With this result on hand, $\forall k \in K$, we can bound $d(v, k)$ by

$$\begin{aligned}
d(v, k) &\geq d(v, w) - \frac{\epsilon}{4} \\
&\geq d(h, v) - d(h, w) - \frac{\epsilon}{4} > d(h, v) - \frac{2\epsilon}{4} \\
&\geq d(x, h) - d(x, v) - \frac{2\epsilon}{4} > d(x, h) - \frac{3\epsilon}{4} \\
&= [d(x, h) - d(x, H)] + d(x, H) - \frac{3\epsilon}{4} > d(x, H) - \epsilon.
\end{aligned}$$

Thus, $d(v, K) \geq d(x, H) - \epsilon$. On the other hand,

$$\begin{aligned} d(v, K) \leq d(v, w) \leq d(v, h) + d(h, w) &< d(v, h) + \frac{\epsilon}{4} \\ &\leq d(x, v) + d(x, h) + \frac{\epsilon}{4} < d(x, h) + \frac{2\epsilon}{4} \\ &< d(x, H) + \frac{\epsilon}{4} + \frac{2\epsilon}{4} < d(x, H) + \frac{3\epsilon}{4}. \end{aligned}$$

Thus $|T(v, K) - T(x, H)| < \epsilon$ as desired and $T$ is continuous.

($CPN \Rightarrow CN$): Let $X$ be a $CPN$ space. $T$ be its continues PN operator. Define

$$F : X \times \mathcal{M}X \to [0, 1] \text{ by } F(x, (H, K)) = \frac{T(x, H)}{T(x, H) + T(x, K)}.$$

Notice that when $x \in H, F(x, (H, K)) = \frac{0}{0+T(x,K)} = 0$. When $x \in K$, $F(x, (H, K)) = \frac{T(x,H)}{T(x,H)+0} = 1$. Since $H \cap K = \emptyset$, the denominator is never 0. $F$ is continuous as $T$ is. Thus, $F$ is a continuous normality operator by definition, and $X$ is $CN$ space. □

The following theorem about metrizability is strong and useful for detecting if a space is $CCR$. However, the original proof is technical and hard to understand, we will break them into small pieces for a better illustration.

**Lemma 3.6.** *Let $X$ be topological space, the function $L : 2^X \times 2^X \to 2^X$ given by $L(C, D) = C \cup D$ is continuous.*

*Proof.* Let $\langle U_1, \dots, U_n \rangle$ be a basic open set in $2^X$, and $\exists C$, $D$ closed set in $X$ such that $L(C, D) \in \langle U_1, \dots, U_n \rangle$. Then for $i \in \{1, \dots, n\}$, each $U_i$ intersects of at least one of $C, D$. Assume without loss of generality $U_1, \dots, U_j$ for some $j \leq n$ are all the open sets intersecting $C$, and $U_k, \dots, U_n$ for some $k \leq n$ are all the set intersecting $D$. Notice $k \leq j + 1$. Then $C \subset \bigcup_{i=1}^{j} U_i$, $D \subset \bigcup_{i=k}^{n} U_i$.

We claim that $\langle U_1, \dots, U_j \rangle \times \langle U_k, \dots, U_n \rangle$ is the desired open set containing $(C, D)$. To this end, randomly pick an element $(E, F)$ in this collection, $E \subset \bigcup_{i=1}^{j} U_i$ and $F \subset \bigcup_{i=k}^{n} U_i$, then $F \cup E \subset \bigcup_{i=1}^{j} U_i \cup \bigcup_{i=k}^{n} U_i = \bigcup_{i=1}^{n} U_i$. Since $E$ must meet with each $U_i$ for $1 \leq i \leq j$ and $F$ meets with each $U_i$ for $k \leq i \leq n$, then $E \cup F$ meets each $U_i$ for $i \in \{1, \dots, n\}$.

Therefore the function $L$ is continuous. □

**Theorem 3.7.** *[Gru76] A separable $CCR$ space is metrizable.*

*Proof.* Let $T$ be the continuous $CR$ operator of $CCR$ space $X$, $Q = \{r_1, r_2, ..., r_n, ...\}$ be the dense countable set in $X$. For a fixed point $x \in X$ and a natural number $n$, define

$$U_{x,n} = \left\{ z \in X : T\left(x, (r, H \cup \{z\})\right) > 1 - \frac{1}{n}, \forall r \in \{r_1, r_2, ..., r_n\}, H \subset \{r_1, r_2, ..., r_n\} \right\}$$

as a subset of $T^{-1}\left(\left(1 - \frac{1}{n}, 1\right]\right)$. We will firstly show $U_{x,n}$ is open, then show $U_{r,n}$ form a basis of $X$ for $r \in Q$ and $n \in \mathbb{N}$. Since both $Q$ and $\mathbb{N}$ are countable, we can conclude $X$ is metrizable by Urysohn metrization theorem (see Theorem 34.1 in [Mun00], which states every regular space with a countable basis is metrizable).

To show $U_{x,n}$ is open, fix $r \in \{r_1, r_2, ..., r_n\}, H \subset \{r_1, r_2, ..., r_n\}$, denote

$$U_{x,n,r,H} = \left\{ z \in X : T\left(x, (r, H \cup \{z\})\right) > 1 - \frac{1}{n} \right\}.$$

Observe that $U_{x,n}$ is in fact the intersection of all possible $U_{x,n,r,H}$ for $r \in \{r_1, r_2, ..., r_n\}, H \subset \{r_1, r_2, ..., r_n\}$, notationally $U_{x,n} = \bigcap_{r,H} U_{x,n,r,H}$. However, we have only finitely many choices for $r$ and for $H$, this intersection must be of finitely many sets, if we can show each $U_{x,n,r,H}$ is open, then $U_{x,n}$ is open.

Using Lemma 3.6, define $F(x, r, H, z) = T(x, (r, H \cup \{z\}))$, since $T$ is continuous, $F$ is also continuous as a composition of continuous functions. Notice the projection map is also continuous as we are considering the Tychonoff product. Denote $F_4(z)$ to be the projection of $F$ to the forth coordinate. Observe that $F_4^{-1}\left((1 - \frac{1}{n}, 1]\right)$ is exactly $U_{x,n,r,H}$, which is open as desired.

In the next following two lemmas, we want to specifically work with $U_{r,n}$, where $r \in Q$ and $n \in \mathbb{N}$. We will show the sets of this form satisfy the two properties of a basis.

**Lemma 3.8.** *$\forall x_0 \in X$, there exists a sequence $\{s_n\}_n \subset Q$ such that $\{s_n\}_n \to x_0$ and $\forall n, x_0 \in U_{s_n,n}$.*

*Proof.* Let $x_0 \in X$, if $x_0 \in Q$, then let $s_n = x_0$ for all $n$, and we are done.

Otherwise, since Q is dense, by Theorem 2.3 in [Gru76], there exists a sequence $\{q_n\}_n \subset Q$ converging to $x_0$. Let $n \in \mathbb{N}$, $r \in \{r_1, ..., r_n\}$, $H \subset \{r_1, ..., r_n\}\backslash\{r\}$. Since for any r, H, $T(x_0, (r, H \cup \{x_0\})) = 1$, there exists an open set $V_{n,r,H}$ containing $x_0$ such that

$$\forall x, y \in V_{n,r,H},\, T\left(x, (r, H \cup \{y\})\right) \geq 1 - \frac{1}{n}.$$

Such $V_{n,r,H}$ exists. If we again let $F(x, r, H, y) = T(x, (r, H \cup \{y\}))$, $F$ is continuous by the above lemma, thus both first coordinate projection $F_1$ and forth coordinate projection $F_4$ are continuous. Then $V_{n,r,H} = F_1^{-1}((1 - \frac{1}{n}, 1]) \cap F_4^{-1}((1 - \frac{1}{n}, 1])$ is the desired open set.

Let $V_n$ be the intersection of $V_{n,r,H}$ for all possible $r \in \{r_1, ..., r_n\}$ and $H \subset \{r_1, ..., r_n\}\backslash\{r\}$, which is open as finite intersection of open sets. Notice that since $x_0$ is in each $V_{n,r,H}$, $x_0$ is in $V_n$.

Since the sequence $\{q_n\}_n$ converges to $x_0$, for each $V_k$, we can find an element $q_{n_k} \in \{q_n\}_n$ in it. Then we have $T(q_{n_k}, (r, H \cup \{x_0\}) > 1 - \frac{1}{k}$, and this implies by definition $x_0 \in U_{q_{n_k},k}$. Finally let $s_k = q_{n_k}$ be the desired sequence. $\square$

**Lemma 3.9.** *$\forall x_0 \in X$, $U$ an open neighborhood of $x_0$, $\exists U_{s_n,n}$ containing $x_0$ such that $x_0 \in U_{s_n,n} \subset U$.*

*Proof.* Assume not, then we can find a neighborhood $U$ of a point $x_0$ that there is not any $U_{s_n,n}$ contained in $U$, which mean $\forall n$, $U_{s_n,n}\backslash U \neq \emptyset$. Since X is $T_3$, we can find an open set V such that $x_0 \in V \subset \overline{V} \subset U$, for any $n$, $U_{s_n,n}\backslash\overline{V}$ is open and nonempty.

Let $p_1 \in \mathbb{N}$ such that $\forall n \geq p_1, s_n \in V$. Since $\{s_n\}_n \subset Q$, we can represents $s_{p_1}$ using index in $Q$, $s_{p_1} = r_{m_1}$ for some $r_{m_1} \in Q$. Recall by definition $s_{p_1} \in U_{s_{p_1},p_1}$, so we actually have

$$s_{p_1} = r_{m_1} \in V \cap U_{s_{p_1},p_1}.$$

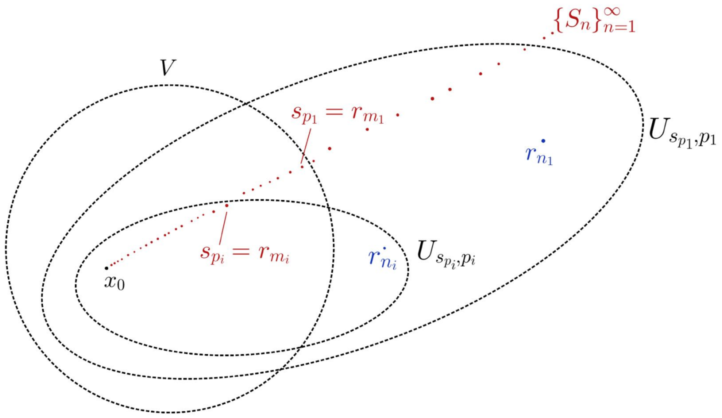


Figure 1: Relative positions between points in the induction of Lemma 3.9

Also, there exists $n_1 \in \mathbb{N}$ such that

$$r_{n_1} \in (U_{s_{p_1},p_1} \backslash \overline{V}) \cap Q.$$

Such $r_{n_1}$ exists as $Q$ dense. Pick $p_2$ large enough (i.e. $p_2 = \max\{m_1, p_1, n_1\} + 1$) such that $s_{p_1}(= r_{m_1}), r_{n_1} \in \{r_1, \ldots, r_{p_2}\}$. Figure 1 shows the relationship between each points.

By induction, when we have $p_i \in \mathbb{N}$, notice $s_{p_i} \in V \cap U_{s_{p_i},p_i}$. $\exists m_i$ such that $s_{p_i} = r_{m_i}$. There exists $n_i \in \mathbb{N}$ such that $r_{n_i} \in (U_{s_{p_i},p_i} \backslash \overline{V}) \cap Q$. Pick $p_{i+1}$ large enough such that $\{s_{p_1}, \ldots, s_{p_i}\} \cup \{r_{n_1}, \ldots r_{n_i}\} \subset \{r_1, ..., r_{p_{i+1}}\}$.

Finally, we have

$$T\left(s_{p_i}, \left(s_{p_{i-1}}, \{r_{n_1}, \ldots, r_{n_{i-1}}\} \cup \{r_{n_i}\}\right)\right) \geq 1 - \frac{1}{i} \to 1 \text{ as } i \to \infty$$

because $r_{n_i}, s_{p_i} \in U_{s_{p_i},p_i}$. On the other hand, as $n \to \infty$,

$$T\left(s_{p_i}, \left(s_{p_{i-1}}, \{r_{n_1}, ..., r_{n_i}\}\right)\right) \to T\left(x_0, \left(x_0, \overline{\bigcup_{j=1}^{\infty}\{r_{n_j}\}}\right)\right) = 0,$$

which arrives a contradiction. Notice that $x_0 \notin \overline{\bigcup_{j=1}^{\infty}\{r_{n_j}\}}$ since $V$ contains $x$ but does not contain $r_{n_j}$ for any $j$. □

By definition, we have shown $\{U_{r,n} : r \in Q, n \in \mathbb{N}\}$ forms a countable basis for $X$. Thus, $X$ is second countable. Since X is CCR thus $T_3$, by Urysohn metrization theorem, we conclude $X$ is metrizable. □

A countable product of metrizable space is metrizable, but it has been shown that an uncountable product of metrizable space is not necessarily metrizable (Example 2, Section 21 in [Mun00]). Thus we only know $\mathbb{R}^{\mathbb{R}}$ is somewhere between metrizable and $T_{3.5}$ (as the product of $T_{3.5}$ space is $T_{3.5}$, but this is not necessarily true for $T_4$). However, Theorem 16.4 in [Wil70] implies $\mathbb{R}^{\mathbb{R}}$ is separable, as each copy $\mathbb{R}$ is Hausdorff and contains more than two points, the product $\mathbb{R}$ has cardinality equal to $2^{\aleph_0}$ ($\aleph_0$ denotes the cardinality of $\mathbb{N}$). Utilizing the theorem above, we can deduce that $\mathbb{R}^{\mathbb{R}}$ is not $CCR$, thus providing further insight into its nature.

**Example 2.** *[Gru76] There exist a $CCR$ space which is not $CN$.*

*Proof.* Let $\Omega$ be the first uncountable ordinal, which means for every $\alpha < \Omega$, the space of all ordinals from 0 to $\alpha$, $[0, \alpha]$ is a countable space but $[0, \Omega]$ is uncountable. Let $X = [0, \Omega]$, we equip this space with the topology generated by basic open set $U$, where $U = \{\alpha\}, \forall \alpha \in [0, \Omega)$ and $U$ is uncountable if it contains $\Omega$.

Let $H \subseteq [0, \Omega]$ be closed. Define $\sup H = \inf\{\alpha \in [0, \Omega] : \alpha \geq h \text{ for all } h \in H\}$. Construct $T : X \times \mathcal{M}X \to [0, 1]$ by

$$T(x, (y, H)) = \begin{cases} 0 \text{ if } x = y \text{ or } (y \geq SupH \text{ and } x \notin H) \\ 1 \text{ if } (x \neq y \text{ and } y < SupH) \text{ or } x \in H. \end{cases}$$

Notice that $T$ is a function that only maps to $\{0, 1\}$, and the condition for $T(x, (y, H)) = 1$ is exactly the negation of the condition for $T(x, (y, H)) = 0$. Thus for each element in $X \times \mathcal{M}X, T$ only maps to either 0 or 1, $T$ is well defined. Also, $T(x, (x, H)) = 0$ and $\forall x \in H, T(x, (y, H)) = 1$, this satisfies the condition for a CR operator. Now, we show $T$ is continuous by showing $T^{-1}(\{0\})$ and $T^{-1}(\{1\})$ are open:

Let $T(x, (y, H)) = 0$ for some $(x, (y, H)) \in X \times \mathcal{M}X$. Consider the following cases:

1. If $x = y \neq \Omega$, pick $\{x\} \times \{y\} \times \langle X \rangle$ to be the open set that contains $(x, (y, H))$ in $T^{-1}(\{0\})$.

2. If $x = y = \Omega$, pick $(\sup H, \Omega] \times (\sup H, \Omega] \times \langle [0, \sup H] \rangle$, since $y \notin H$, then $\sup H < \Omega$, $[\sup H, \Omega]$ is not empty. Also, $[0, \sup H]$ is open as it is the union of countably many open sets.

3. If $x \neq y, x, y < \Omega$, we must have $x \notin H, y \geq \sup H$, then pick $\{x\} \times \{y\} \times \langle [0, \sup H) \setminus \{x\} \rangle$.

4. If $x = \Omega, y < \Omega$, pick $(\sup H, \Omega] \times \{y\} \times \langle [0, \sup H] \setminus \{y\} \rangle$.

5. If $x < \Omega, y = \Omega$, pick $\{x\} \times (\sup H, \Omega] \times \langle [0, \sup H] \setminus \{x\} \rangle$.

It can be checked by definition that in each case, any points in the open sets mentioned above are mapped to 0. This are all the cases for $T^{-1}(\{0\})$. Now assume $T(x, (y, H)) = 1$,

1. If $y < \sup H$, $x \neq y \neq \Omega$, pick $\{x\} \times \{y\} \times \langle H, (y, \Omega] \rangle$.

2. If $y < \sup H, x = \Omega, y < \Omega$, pick $(y, \Omega] \times \{y\} \times \langle H, (y, \Omega] \rangle$.

3. If $y < \sup H, x < \Omega, y = \Omega$ is impossible since $\sup H$ cannot be strictly greater than $\Omega$.

4. If $y \geq \sup H$, then $x \in H$, if $x = \Omega$, then $y \geq \sup H \geq x = \Omega$, which is impossible as $x \neq y$.

5. If $y \geq \sup H, x \in H, x < \Omega,$ pick $\{x\} \times (X \times \langle \{x\}, X \rangle) \cap \mathcal{M}X$.

Again we can check by definition that any points in these open sets are mapped to 1. Thus $T$ is continuous and $[0, \Omega]$ is $CCR$.

Assume $[0, \Omega]$ is $CN$, and let $N$ be its continuous N operator.

**Lemma 3.10.** *$\forall a < \Omega$, $\exists x_a$ depending on $a$, such that $\forall y \geq x_a$, $H \subset [x_a, \Omega]$ closed, $\forall F, G \subset [0, a]$ closed disjoint,*

$$N(y, (H \cup F, G)) = 0 \qquad N(y, (F, G \cup H)) = 1.$$

*Proof.* let $a < \Omega$, we first fix $F, G \subset [0, a]$. Claim: Given $a, F, G, \exists x_n$ such that $\forall y > x_n, H \subseteq [x_n, \Omega], H$ closed, $N(y, (F \cup H, G)) < \frac{1}{n}$ and $N(y, (F, G \cup H)) > 1 - \frac{1}{n}$.

If this claim is true, we pick $x_a$ to be the supremum of all $x_n$ over all possible disjoint $F, G$, i.e.

$$x_a = \sup_n \{x_n : n \in \mathbb{N}, \forall F, G \subset [0, a] \text{ disjoint closed}\}$$

which finishes the proof.

To prove the claim, define $L$ by $L(H, F) = H \cup F$, let

$$K(y, H, F, G) = N(y, (L(H, F), G)).$$

$K$ is continuous by Lemma 3.6. Observe $K^{-1}([0, \frac{1}{n}))$ is open and contains $(\Omega, \{\Omega\}, F, G)$. Then there exists a basic open set $A \times \mathcal{B} \times \mathcal{C} \times \mathcal{D} \subset K^{-1}([0, \frac{1}{n}))$ that contains $(\Omega, \{\Omega\}, F, G)$. Without loss of generality, $\mathcal{B} = \langle E \rangle$ for some $E$ (if $\mathcal{B} = \langle U_1, ..., U_n \rangle$, we can take $E = \bigcap_{i=1}^n U_i$). Since $\mathcal{B}$ is an open set containing $\{\Omega\}$, $E$ must be an open set containing $\Omega$. Since $A$ is an open set containing $\Omega$, $A \cap E$ is open and contains $\Omega$. Let

$$x'_n = \inf A \cap E = \sup\{x \in [0, \Omega] : \forall y \in A \cap E, y \geq x\}.$$

Thus $\forall y \geq x'_n, y \in A$, and $\forall H \subseteq [x'_n, \Omega], H \subseteq E$, then $K(x'_n, H, F, G) = N(x'_n, (F \cup H, G)) < \frac{1}{n}$. Similarly, we can get a $x''_n$ for the $N(x_n, (F, G \cup H)) > 1 - \frac{1}{n}$ part, and picking $x_n = max\{x'_n, x''_n\}$ finishes the claim. ☐

Let $b_1 \neq c_1 < \Omega$, denote $a_1 = \max\{b_1, c_1\}$, $F_1 = \{b_1\}, G_1 = \{c_1\}$. Apply the above lemma, $\exists x_{a_1}$ such that $\forall b_2 \in [x_{a_1}, \Omega], b_2 \neq b_1 \neq c_1$, since $\Omega > x_{a_1}$,

$$N(\Omega, (F_1 \cup \{b_2\}, G_1)) = 0.$$

Similarly, pick $c_2 \in [x_{a_1}, \Omega], c_2 \neq b_2 \neq b_1 \neq c_1$,

$$N(\Omega, (F_1, G_1 \cup \{c_2\})) = 1.$$

Constructing inductively, if we have $F_{n-1} = \{b_1, \ldots, b_{n-1}\}$, $G_{n-1} = \{c_1, \ldots, c_{n-1}\}$, denote $a_{n-1} = \max\{b_1, \ldots, b_{n-1}, c_1, \ldots, c_{n-1}\}$. Apply the above lemma, $\exists x_{a_{n-1}}$ such that randomly pick $b_n \neq c_n \in [x_{a_{n-1}}, \Omega]$ distinct from all previous $b_i, c_i$ that we have picked,

$$N(\Omega, (F_{n-1} \cup \{b_n\}, G_{n-1})) = 0 \qquad N(\Omega, (F_{n-1}, G_{n-1} \cup \{c_n\})) = 1.$$

However, when $n \to \infty$,

$$N\left(\Omega,\left(F_{n-1} \cup \{b_n\}, G_{n-1}\right)\right) \to N\left(\Omega,\left(\{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}\right)\right)=0,$$
$$N\left(\Omega,\left(F_{n-1}, G_{n-1} \cup \{c_n\}\right)\right) \to N\left(\Omega,\left(\{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}\right)\right)=1,$$

which arrives a contradiction. Therefore, $[0, \Omega]$ is $CCR$ but not $CN$. □

**Theorem 3.11.** *[Zen75] The one-point compactification of an uncountable discrete space is not $CCR$.*

*Proof.* Let $M$ be the uncountable discrete space and $\infty$ be its limit. Suppose $T$ is the continuous CR operator for contradiction. Take $x_1 \in M$, $\exists U_1$ open and containing $\infty$, such that

$$\forall x, y \in U_1, T(x,(y,\{x_1\}))<\frac{1}{2}.$$

To construct such $U_1$, observe $T^{-1}([0, \frac{1}{2}))$ is open and contains $(\infty,(\infty,\{x\}))$, then there exists a basic open set $A \times B \times \mathcal{C} \subset T^{-1}([0, \frac{1}{2}))$ containing $(\infty,(\infty,\{x\}))$, where $A, B$ are open in $M \cup\{\infty\}$, $\mathcal{C}$ is open in $2^{M \cup\{\infty\}}$. Picking $U_1 \subset (A \cap B) \backslash\{x\}$ suffices.

Constructing inductively, if we have $x_1, \ldots, x_n \in M$, and $U_1, \ldots, U_n$. Pick $x_{n+1}$ distinct from $x_1, \ldots, x_n$ and not $\infty$. We construct $U_{n+1}$ using the above strategy. Since $U_i$'s are open sets containing $\infty$, they are co-finite, thus they must intersect. Without loss of generality, let $U_{n+1} \subset \bigcap_{i=1}^{n} U_i$ and $\forall x, y \in U_{n+1}$, $T\left(x,\left(y,\left\{x_{n+1}\right\}\right)\right)<\frac{1}{2^{n+1}}$.

Again since each $U_i$ is co-finite, then the countable intersection $\bigcap_{i=1}^{\infty} U_i$ is uncountable. we can pick $z \in \bigcap_{i=1}^{\infty} U_i$ and $z \neq \infty$.

Claim: $\exists V$ open containing $\infty$ such that $\forall x, y \in V, T(x,(z,\{y\}))>\frac{1}{2}$.

To prove this claim, since $T^{-1}((\frac{1}{2}, 1])$ is open, then there exists a basic open set $D \times E \times \mathcal{F}$ that contains $(\infty,(z,\{\infty\}))$, Denote $\mathcal{F}'$ to be the collection of singleton set in $\mathcal{F}$. Then pick $V=\{b:\{b\} \in \mathcal{F}'\} \cap C$ to be the intersection of $D$ and the elements in sets in $\mathcal{F}'$. Then $x, y \in V$, we have $\{y\} \in \mathcal{F}$ and $x \in D$, then $T(x,(z,\{y\}))>\frac{1}{2}$ follows. We notice that $\mathcal{F}'$ is the intersection of $\mathcal{F}$ and space of singleton $\mathcal{F}_1 X$, thus $\{b:\{b\} \in \mathcal{F}'\}$ is open in $M$ by Lemma 2.8 and $V$ is open as an intersection of two open sets.

Since V contains $\infty$ and it is open, it must be co-finite. We have constructed infinitely many $x_n$, then there exists some large $n \in \mathbb{N}$ such that $x_n \in V$. However, consider $(\infty,(z,\{x_n\}))$, $T(\infty,(z,\{x_n\}))>\frac{1}{2}$ as $\infty, x_n \in V$, and $T(\infty,(z,\{x_n\}))<\frac{1}{2^n}$ as $\infty, x_n \in U_n$, which arrives a contradiction. □

One direct application of this theorem is concluding the one point compactification of $\mathbb{R}$ is not $CCR$.

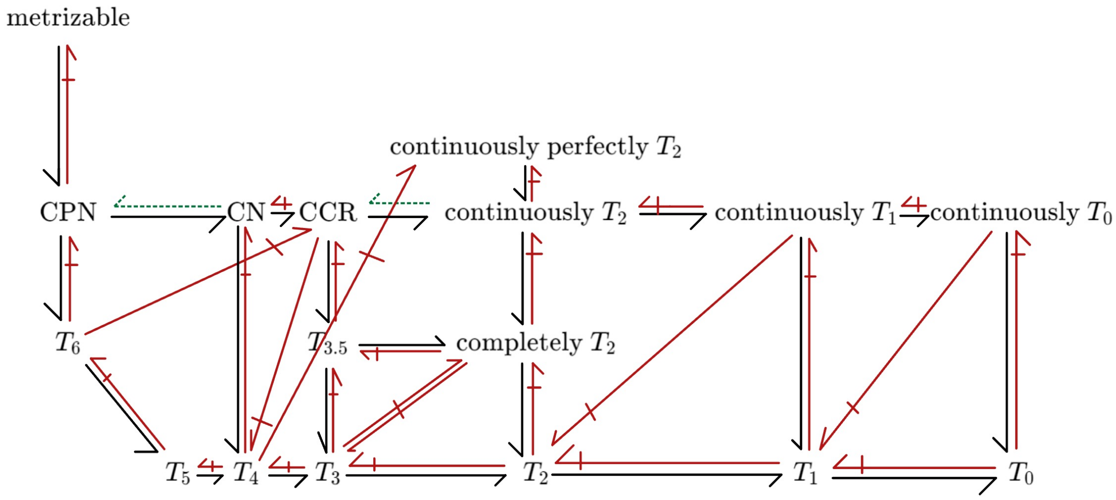


Figure 2: Relationships Between Spaces in Separation Axioms.

Black arrows denote implications; red arrows denote instances where counterexamples are presented; green dashed arrows denote instances where results remain unknown.

# 4 Classification of Spaces

In this final chapter, we extend upon prior research findings to develop a comprehensive classification framework for the spaces under investigation. We first provide a few new definitions about continuously $T_2, T_1$ and $T_0$ spaces, then put all spaces together and summarize their relationships, by providing either proofs or counter examples. The final visual result can be found in Figure 2.

## 4.1 Lower Separation Axioms

The objective of this subsection is to define continuously $T_i$ spaces, where $i = 0, 1, 2$, such that they have separations properties stronger than classical $T_i$ spaces, ideally stronger than completely $T_i$ spaces if there is any, but weaker than continuously $T_{i+1}$ spaces.

### 4.1.1 Continuously $T_2$ Space

**Definition 4.1** ([JS78] definition of Urysohn)**.** *A topological space $X$ is* ***completely Hausdorff*** *if and only if $\forall x, y \in X$ distinct, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.*

Examples 88 and 90 in [JS78] demonstrate that neither completely $T_2$ implies $T_3$, nor does $T_3$ imply completely $T_2$. Both $T_{3.5}$ and completely $T_2$ are complete version of separation axioms, by definition $T_{3.5}$ implies completely $T_2$, but the opposite is not true because complete $T_2$ does not imply $T_3$ and $T_{3.5}$ is strictly stronger than $T_3$.

It is also clear that completely $T_2$ implies $T_2$ by taking $f^{-1}([0, 0.5))$ and $f^{-1}((0.5, 1])$ for any pair of points $x, y$ and the continuous function $f$ between them such that $f(x) = 0$, $f(y) = 1$. The opposite direction is not true by Example 80 in [JS78].

Finally, we also have $T_2 \implies T_1 \implies T_0$ and the converses are not true, with Example 8, 18 provided in [JS78]. Now, we have an implication chain in the complete and classical version of lower separation axioms, as shown in the right bottom in Figure 2.

**Definition 4.2.** *Let $X$ be a topological space, $T : X \times ((X \times X)\backslash\{(x, x) : x \in X\}) \to [0, 1]$ is a $T_2$* ***operator*** *if and only if for any distinct points $x, y \in X$, $T(x, x, y) = 0$ and $T(y, x, y) = 1$. A space $X$ is* ***continuously*** *$T_2$ if it admits a continuous $T_2$ operator.*

This definition of continuous $T_2$ fits our intuition of defining a space that is stronger than the complete (and thus classical) version of a $T_2$ space. If $X$ is continuously $T_2$, for each pair of distinct points $y, z \in X$, the restriction of $T$ to its first coordinate $T_1(x) = T(x, y, z)$ that maps from $X$ to $[0, 1]$ is a continuous function separating $y, z$. Thus $X$ is completely $T_2$.

Nevertheless, the following example demonstrates our definition is strictly stronger than completely $T_2$.

**Example 3.** *The one point compactification of $\mathbb{R}, \mathbb{R} \cup \{\infty\}$ is completely $T_2$ but not continuously $T_2$.*

*Proof.* Denote $X = \mathbb{R} \cup \{\infty\}$, let $x, y$ be two distinct points in $X$. Then at least one of them must be in $\mathbb{R}$, without loss of generality let $x \in \mathbb{R}$. Define $f : X \longrightarrow [0, 1]$ by mapping $x$ to 0 and all the rest elements to 1. $f$ is a discrete function, to check it is continuous, we only need to observe that $f^{-1}(0) = \{x\}$ and $f^{-1}(1) = X\backslash\{x\}$ are both open. Thus, $X$ is completely $T_2$.

To show $X$ is not continuously $T_2$, we use the same strategy in the proof of Theorem 3.11. Assume there is a continuous $T_2$ operator $T$. Take $x_1 \in \mathbb{R}$, then $\exists U_1$ open and containing $\infty$ such that $\forall x, y \in U_1, T(x, y, x_1) < \frac{1}{2^1}$. We can use the same claim in the proof of Theorem 3.11 to show such $U_1$ exists. Constructing inductively, we admit a sequence $\{x_n\}_n^\infty$ and a sequence of open sets $\{U_n\}_n^\infty$ such that $\forall x, y \in U_n$, $T(x, y, x_n) < \frac{1}{2^n}$. Since each $U_n$ is open and contains $\infty$, it is co-finite, its intersection $\bigcap_{i=1}^\infty U_i$ contains infinitely many elements including $\infty$. Pick $z \neq \infty \in \bigcup_{i=1}^\infty U_i$.

Let $V$ be an open set containing $\infty$ such that $\forall x, y \in V$, $T(x, z, y) > \frac{1}{2}$. Such $V$ exists again by the proof of Theorem 3.11. Notice that $V$ is also co-finite. Since we constructed infinitely many $x_n$, there exists a large $n \in \mathbb{N}$ such that $x_n \in V$.

Finally, we achieve the same contradiction by observing

$$T(\infty, z, x_n) > \frac{1}{2},$$
$$T(\infty, z, x_n) < \frac{1}{2^n} \leq \frac{1}{2},$$

as $\infty, x_n \in V$ and $\infty, x_n \in U_n$ respectively. □

Our intuition also requires continuously $T_2$ space to be weaker than $CCR$ space, we will discuss this in Section 4.1.4.

In the definition of continuously $T_2$, we only require $T^{-1}(\{0\}) \supset \{(x, x, y) : x, y \in X, x \neq y\}$, but they are not necessarily equal. If instead we define $T^{-1}(\{0\}) = \{(x, x, y) : x, y \in X, x \neq y\}$, the definition is much stronger than the current one. In that case, if we fix the last two variables in $T$ to be $y, z$ and only consider the restriction map to the first coordinate $T_1(x) = T(x, y, z)$, $T_1(x) = 0 \iff x = y$. The fact that $\{0\}$ is $G_\delta$ (the intersection of countable open sets) forces

$T_1^{-1}(\{0\}) = \{y\}$ must be $G_\delta$. Since we pick $y$ randomly, each point in $X$ must be $G_\delta$, which is too strong to align with our intuition of continuously $T_2$ space, because the following two examples show that even $CCR$ or $T_4$ spaces do not necessarily satisfy this property. Thus we give a new definition to the spaces that satisfy such property.

**Definition 4.3.** *Let $X$ be a topological space, $T : X \times ((X \times X)\backslash\{(x, x) : x \in X\}) \to [0, 1]$ is a **perfect** $T_2$ **operator** if and only if for any points $x, y, z \in X$ with $y \neq z$, $T(x, y, z) = 0 \iff x = y$ and $T(x, y, z) = 1 \iff x = z$. A space $X$ is **continuously perfectly** $T_2$ if it admits a continuous perfect $T_2$ operator.*

Notice that if $X$ is continuously perfectly $T_2$, $X$ is continuously $T_2$ by definition. However the converse is not true, as shown below.

**Example 4.** *The space $[0, \Omega]$ in Example 2 is continuously $T_2$ but not continuously perfectly $T_2$.*

*Proof.* Example 2 shows $[0, \Omega]$ is $CCR$ thus continuously $T_2$ (we will prove the formal statement in Section 4.1.4), but $\{\Omega\}$ is not $G_\delta$. Assume it is, $\exists\{U_n\}_{n=1}^{\infty}$ open neighborhood of $\Omega$ such that $\bigcap_{n=1}^{\infty} U_n = \{\Omega\}$. However, since each $U_n$ contains $\Omega$, its complement $[0, \Omega]\backslash U_n$ must be countable. Then,

$$\bigcup_{n=1}^{\infty} ([0, \Omega] \backslash U_n) = [0, \Omega] \backslash \bigcap_{n=1}^{\infty} U_n$$

is countable, which means $\bigcap_{n=1}^{\infty} U_n$ is still uncountable and thus cannot be a singleton set $\{\Omega\}$. By our observation above, $[0, \Omega]$ is not continuously perfectly $T_2$. □

**Example 5.** *Let $\{0, 1\}$ be equipped with poset topology $\{\{0\}, \{1\}, \{0, 1\}\}$. Its uncountable product $\{0, 1\}^{[0,1]}$ is compact $T_2$ but not continuously perfectly $T_2$.*

*Proof.* Let $X = \{0, 1\}^{[0,1]}$, since $\{0, 1\}$ is compact $T_2$, and any arbitrary product of compact space is compact, any arbitrary product of $T_2$ space is $T_2$, then $X$ is compact $T_2$. If $X$ is continuously perfectly $T_2$, then each point must be $G_\delta$ by our observation above. However, assume if there exists $x \in X$ is $G_\delta$ for contradiction. Then $\exists\{U_i\}_{i\in\mathbb{N}}$ sequence of open sets such that $\{x\} = \bigcap_{i=1}^{\infty} U_i$. For each $i$, $U_i$ can be written as a product of open sets in $\{0, 1\}$, $\prod_{\alpha\in[0,1]} U_{i_\alpha}$ where $U_{i_\alpha} = \{0, 1\}$ except finitely many. Then denote

$$U = \bigcap_{i=1}^{\infty} U_i = \prod_{\alpha\in[0,1]} U_\alpha$$

where $U_\alpha = \{0, 1\}$ except maybe countably many. However, we have uncountable indices, $U$ must contain some points other than $x$, which arrives a contradiction. Thus, $x$ is not $G_\delta$ and $\{0, 1\}^{[0,1]}$ is not continuously perfectly $T_2$. □

To incorporate this result into our summary Figure 2, it indicates that $T_4$ does not imply continuously perfectly $T_2$, as every compact $T_2$ space is $T_4$, by Theorem 32.3 in [Mun00].

### 4.1.2 Continuously $T_1$ Space

It is hard to define continuously $T_1$ space in a similar fashion, due to the fact that $[0, 1]$ is $T_2$. If there exists a continuous map $T$ that sends three elements in a $T_1$ space $X$ to $[0, 1]$, and satisfies

$T(x,x,y) = 0, T(y,x,y) = 1$, for each pair of $x, y \in X$, we can always use $T_1^{-1}([0,0.5))$ and $T_1^{-1}((0.5,1])$ to separate them, where $T_1$ is the projection of $T(z,x,y)$ to the first coordinate with $x, y$ fixed. This means $X$ has to be $T_2$. This cannot reflect the unique property of $T_1$ space, so we have to define $T_1$ in another way.

**Definition 4.4.** *Let $X$ be a topological space, $T : (X \times X)\backslash\{(x,x) : x \in X\} \to \mathcal{A}X$ is a $T_1$* ***operator*** *if and only if for any distinct points $x, y \in X$, $T(x,y)$ is an open set in $X$ containing $x$ but not $y$. A space $X$ is* ***continuously*** *$T_1$ if it admits a continuous $T_1$ operator (on extended Vietoris topology for $\mathcal{A}X$).*

In the previous chapter we have shown that $\mathcal{A}X$, the space of nonempty subsets, is a topological space equipped with extended Vietoris topology. Thus this definition is logically coherent and by definition, a continuous $T_1$ space is $T_1$. At the first glance, it may be better to define the operator by sending two distinct points in $X$ to $2^X$, the space of closed subsets. However, it can be checked that this definition is equivalent to $T_1$. If we always pick $T(x,y) = \{x\}$, essentially we project $(x,y)$ to $x$, since $X$ is homeomorphic to the space of singleton sets, as shown in Lemma 2.8. We can check that $T$ is always continuous. This definition does not align with our original objective of constructing a space that is strictly stronger than $T_1$ space.

The following two examples illustrate continuously $T_1$ is weaker than $T_2$ and thus continuously $T_2$, but it is a stronger than $T_1$.

**Example 6.** *The natural number $\mathbb{N}$ equipped with co-finite topology is continuous $T_1$ but not $T_2$.*

*Proof.* Recall that in the co-finite topology, an open set is defined as a set whose complement is finite. Let $a, b \in \mathbb{N}$. Define $T(a,b) = \{c \in \mathbb{N} : c > b\} \cup \{a\}$, which is open. We will show that this it is a continuous $T_1$ operator. Clearly, $T$ is a $T_1$ operator because $T(a,b)$ is open and contains $a$.

Let $\langle U_1, \ldots, U_n \rangle^+$ be an open basis of $\mathcal{A}\mathbb{N}$ and contains $T(a,b)$ for some $a, b \in \mathbb{N}$. Claim

$$\bigcup_{i=1}^{n} U_i \times \{c \in \mathbb{N} : c \geq b\} \subset T^{-1}\left(\langle U_1, \ldots, U_n \rangle^+\right)$$

is the desired open set containing $(a,b)$.

Let $x \in \bigcup_{i=1}^n U_i$ and $y \geq b$. $T(x,y) = \{c \in \mathbb{N} : c > y\} \cup \{x\}$. Since $x \in \bigcup_{i=1}^n U_i$ and $\{c \in \mathbb{N} : c > y\} \subset \{c \in \mathbb{N} : c > b\} \subset \bigcup_{i=1}^n U_i$, then $T(x,y) \subset \bigcup_{i=1}^n U_i$. Since $T(x,y)$ is co-finite and each $U_i$ is co-finite, $T(x,y) \cap U_i \neq \emptyset$ for all $i$.

Therefore $T(x,y) \in \langle U_1, \ldots, U_n \rangle^+$ and $T$ is a continuous $T_1$ operator.

$\mathbb{N}$ is not $T_2$ since for any two number $a, b$, they cannot be separated by open sets as their neighborhoods are co-finite, thus must intersect. □

**Example 7.** *The uncountable product of natural number $\mathbb{N}$ equipped with co-finite topology is $T_1$ but not continuous $T_1$.*

*Proof.* Let $M$ be an uncountable set of indices. Denote $X = \mathbb{N}^M$. $X$ is $T_1$ because $\mathbb{N}$ with co-finite topology is $T_1$ and the product of $T_1$ space is $T_1$. Assume there is a continuous $T_1$ operator. Randomly pick $x_1 = \{x_{1_\alpha}\}_{\alpha \in M}, y_1 = \{y_{1_\alpha}\}_{\alpha \in M}$ in $X$. Then $T(x_1, y_1) = U_1$, which is a basic open set (without loss of generality) containing $x_1$ but not $y_1$ . $U_1 = \prod_{\alpha \in M} U_{1_\alpha}$, where each coordinate

$$U_{1_\alpha} = \begin{cases} U_{1_\alpha} \subsetneq \mathbb{N} & \text{if } \alpha = \alpha_1, \ldots, \alpha_{n_1} \text{ for finitely many indices} \\ \mathbb{N} & \text{otherwise.} \end{cases}$$

Claim that there exists an open set $V_1 \subset X$ such that for all distinct $x, y$ in $V_1$, $T(x, y) \in \langle U_1 \rangle^+$. Since $T$ is continuous, $T^{-1}\left(\langle U_1 \rangle^+\right)$ is open containing $(x_1, y_1)$, there exists a basic open set $W_1 \times W_2$ in $X \times X$ such that $\forall x \in W_1, y \in W_2, x \neq y$ and $T(x, y) \in \langle U_1 \rangle^+$. Pick $V_1 = W_1 \cap W_2$ finishes this claim. Notice that for $i \in \{1, \ldots, n_1\}, V_{\alpha_i} \subset U_{\alpha_i}$, otherwise we can pick $x, y \in V$ such that $x_{\alpha_i} \in V_{\alpha_i} \backslash U_{\alpha_i}$, since $T(x, y)$ contains $x$, $T(x, y) \not\subset U_1$, which arrives a contradiction. In addition, $V_1$ is uncountable because it has uncountable coordinates and each of its coordinates contains infinite points.

Pick distinct points $x_2, y_2 \in V_1$ such that

$$x_{2_{\alpha_i}} = y_{2_{\alpha_i}} \forall i \in \{1, \ldots, n_1\}.$$

There exists $\beta \notin \{1, \ldots, n_1\}$ such that $x_{2_\beta} \neq y_{2_\beta}$. By construction, $T(x_2, y_2) = U_2 \in \langle U_1 \rangle^+$. Then $U_2$ is a subset of $U_1$, so $U_{2\alpha_i} \subset U_{1\alpha_i} \subsetneq \mathbb{N}$ for $i \in \{1, \ldots, n_1\}$. Other than that, we must also have $U_{2_\beta} \neq \mathbb{N}$ because $U_{2_\beta}$ contains $x_{2_\beta}$ but not $y_{2_\beta}$. Then $U_2$ has more coordinates that are proper open subset of $\mathbb{N}$ than $U_1$. We can find a index set $\{1, \ldots, n_1, \ldots, n_2\} \supset \{1, \ldots, n_1\}$ such that

$$U_{2_{\alpha_i}} = \begin{cases} U_{2_{\alpha_i}} \subsetneq \mathbb{N} & i \in \{1, \ldots, n_1, \ldots, n_2\} \\ \mathbb{N} & \text{otherwise.} \end{cases}$$

Again, there exists an open set $V_2 \subset V_1$ without loss of generality (if not, then pick $V_2' = V_2 \cap V_1$) such that for all distinct $x, y \in V_2$, $T(x, y) \in \langle U_2 \rangle^+ = \langle U_2 \rangle^+ \cap \langle U_1 \rangle^+$.

Constructing inductively, if we have $x_1 \ldots, x_m, y_1 \ldots, y_m$ and they are all distinct. Let $x_{m+1} \neq y_{m+1} \in V_m$ such that $x_{m+1_{\alpha_i}} = y_{m+1_{\alpha_i}}$ for all $i \in \{1, \ldots, n_m\}$ and there exists a coordinate $\gamma$ such that $x_{m+1} \neq y_{m+1}$ on this coordinate. By construction $T\left(x_{m+1}, y_{m+1}\right) = U_{m+1} \in \bigcap_{i=1}^{m} \langle U_i \rangle^+$. Thus $U_{m+1_{\alpha_i}} \subsetneq \mathbb{N}$ for all $i \in \{1, \ldots, n_m, \gamma\}$. There exists a $V_{m+1}$ such that for all $x, y \in V_{m+1}$, $T(x, y) \in \bigcap_{i=1}^{m+1} \langle U_i \rangle^+$.

Keep doing this we get infinite sequences $\{x_n\}_n, \{y_n\}_n, \{V_n\}_n, \{U_n\}_n$. Let $V \subset \bigcap_{i=1}^{\infty} V_i$. This may not be open but is still an uncountable set, let $x, y \in V$ be two distinct points that are not equal to any $x_i, y_i$ in previous. Then $T(x, y) = U \in \bigcap_{i=1}^{\infty} \langle U_i \rangle^+$ for all $i$. However, $U \in \langle U_i \rangle^+$ means $U \subset U_i$, then its projection to $\alpha$-th coordinate $U_\alpha \neq \mathbb{N}$ as long as $U_{i_\alpha} \neq \mathbb{N}$ for any $i$. In the previous construction, we have shown that each $U_i$ is distinct from others. If $U \subset \bigcap_{i=1}^{\infty} U_i$, U must have infinitely many coordinates that are not $\mathbb{N}$, but then $U$ is not open because we cannot find a basic open set that is contained in $U$, which arrives a contradiction. $\square$

#### 4.1.3 Continuously $T_0$ Space

**Definition 4.5.** *Let $X$ be a topological space, $T : (X \times X) \backslash \{(x, x) : x \in X\} \to \mathcal{A}X$ is a $T_0$* ***operator*** *if and only if for any distinct points $x, y \in X$, $T(x, y)$ is an open set in $X$ containing* *exactly one of $x$ and $y$. A space $X$ is* ***continuously*** *$T_0$ if it admits a continuous $T_0$ operator.*

Continuous $T_0$ implies $T_0$ by definition, but the reverse is not true. Similar as continuously $T_1$, we provide two examples that show this definition fits our goal.

**Example 8.** *The Sierpinsky space $\mathbb{S}$ is continuous $T_0$ but not $T_1$. $\mathbb{S} = \{0,1\}$ equipped with the topology $\{\{0,1\},\{1\},\emptyset\}$.*

*Proof.* $\mathbb{S}$ is not $T_1$ since we cannot find an open set containing 0 but not 1.

Define $T(0,1) = T(1,0) = \{1\}$. To show $T$ is continuous, let $\langle U_1,\ldots,U_n\rangle$ be an open collection in $\mathcal{A}\mathbb{S}$. If $\{1\} \in \langle U_1,\ldots,U_n\rangle$, then $T^{-1}(\langle U_1,\ldots,U_n\rangle) = \{(0,1),(1,0)\} = (\mathbb{S}\times\mathbb{S})\backslash\{(0,0),(1,1)\}$ which is the whole domain and thus is open. Otherwise $T^{-1}(\langle U_1,\ldots,U_n\rangle) = \emptyset$, which is also open. Thus $T$ is continuous. □

**Example 9.** *The uncountable product of $\mathbb{S}$ equipped with co-finite product topology is $T_0$ but not continuous $T_0$.*

*Proof.* Let M be an uncountable set of indices. Denote $X = \mathbb{S}^M$. Again $X$ is $T_0$ as the product of $T_0$ spaces is $T_0$.

To show it is not continuously $T_0$, let $x_1$, $y_1 \in X$. Assume there exists a continuous $T_0$ operator $T$ such that $T(x_1,y_1) = U_1$ open set containing exactly one of $x_1$ and $y_1$. $U_1 = \prod_{\alpha\in M} U_{1\alpha}$, where $U_{1_\alpha} = \{0,1\}$ for all except finitely many indices $\alpha_1,\ldots,\alpha_{n_1}$, and in these coordinates, the projection of $U$, $U_{\alpha_i} = \{1\}$. Clearly $U_1 \in \langle U_1\rangle^+$, which is a basic open set in $\mathcal{A}X$. There exists an open set $V_1 \subset X$ such that for all distinct $x,y$ in $V_1$, $T(x,y) \in \langle U_1\rangle^+$. The proof will be the same as Example 7. In addition, on the coordinates $\alpha = \alpha_1,\ldots,\alpha_{n_1}$, the projection of $V$, $V_\alpha$ is $\{1\}$. It is also worth to notice that $V$ is an uncountable set.

Constructing inductively, if we have $x_1,\ldots,x_m,y_1,\ldots,y_m$, pick $x_{m+1},y_{m+1} \in V_m$ that are distinct from all of the previous $x_i$'s and $y_i$'s, $T(x_{m+1},y_{m+1}) = U_{m+1}$ is open and $T^{-1}(\langle U_{m+1}\rangle)$ is open. We can find $V_{m+1} \subseteq \bigcap_{i=1}^m V_i$ such that $\forall x,y \in V_{m+1}, T(x,y) \in \langle U_{m+1}\rangle \cap \ldots \cap \langle U_1\rangle$. The coordinate of $V_{m+1}$, $V_{m+1_\alpha} = \{1\}$ if $U_{i_\alpha} = \{1\}$ for any $i$.

Keep doing this, we get an infinite sequence. Let $V = \bigcap_{i=1}^\infty V_i$, this is not necessarily an open set, but is still an uncountable set, because its coordinates $V_\alpha = \{0,1\}$ for all except maybe countably many indices. Enumerate the excepted indices to be $\beta_i$ for $i \in \mathbb{N}$, so $V_{\beta_i} = \{1\}$. Pick $x,y \in V$ to be two distinct points that are not equal to any $x_i,y_i$ in previous. $T(x,y) = U$ which is open and contains exactly one of $x,y$, since $x,y \in V_i$ for all $i$, then $U \in \bigcap_{i=1}^\infty \langle U_i\rangle^+$ for all $i$. However, $U \in \langle U_i\rangle^+$ means its projection to $\alpha$-th coordinate $U_\alpha = \{1\}$ as long as $\exists i$ such that $U_{i_\alpha} = \{1\}$. Then $U$ must have infinitely many coordinates to be $\{1\}$, which is not open, and arrives a contradiction.

Therefore, such continuous operator does not exist and $\mathbb{S}^M$ is not continuous $T_0$. □

After establishing an alternative method to define continuous $T_1$ and continuous $T_0$ operators, we encounter other choices for continuous $T_2$ operator. One such approach involves mapping two points, $x$ and $y$, in the space $X$ to two disjoint open sets, $U$ and $V$ such that $U$ contains $x$ but not $y$, $V$ contains $y$ but not $x$. While this presents an intriguing alternative, a comprehensive analysis of this definition extends beyond the intended scope of this dissertation. We will adhere to the definition outlined in Definition 4.2 within this paper. This decision is motivated by its alignment with the definitions of $CCR$, $CN$, and $CPN$ as proposed in [Zen75]. In addition, our objective is to define a space that is stronger than completely $T_2$ spaces. It is not direct to see if the aforementioned new choice implies completely $T_2$, but we can clearly see that Definition 4.2 implies completely $T_2$.

#### 4.1.4 The Continuous Chain

To connect all continuous separated spaces, we expect them to behave like classical separation axioms, form a chain flowing from highly separated spaces to weakly separated spaces. Remarkably, the following proposition shows that they do form such chain, as demonstrated in the second row of Figure 2.

**Proposition 4.6.** *$X$ is a topological space, the following chain holds: $X$ is metric space $\Rightarrow$ $CPN \Rightarrow CN \Rightarrow CCR \Rightarrow$ continuously $T_2$ $\Rightarrow$ continuously $T_1$ $\Rightarrow$ continuously $T_0$.*

*Proof.* The first and second implications are proved in Proposition 3.5. Each $CN$ space is $CCR$ space due to the fact that a singleton set $\{x\}$ is closed in $T_1$ space $X$.

($CCR \Rightarrow$ continuously $T_2$ ): Let $T$ be a continuous CR operator, since $\mathcal{F}_1(X)$ is homeomorphic to $X$ by Proposition 2.8, define $F : X \times X \times X \to [0,1]$ by $F(x, y, z) = T(x, (y, \{z\}))$. For $x, y \in X, x, y$ distinct, $F(x, x, y) = T(x, (x, \{y\})) = 0$ and $F(y, x, y) = T(y, (x, \{y\})) = 1$. $F$ is equivalent to the restriction of $T$ on $X \times \{(x, \{y\}) \in X \times \mathcal{F}_1 X : x \neq y\}$, which is continuous under subspace topology.

(Continuously $T_2$ $\Rightarrow$ continuously $T_1$): Let $T$ be a continuous $T_2$ operator. Let $x \neq y \in X$ be two random points, Denote $T_{x,y} : X \to [0, 1]$ to be the function $T_{x,y}(z) = T(z, x, y)$. Notice that $T$ is continuous so its projection to the first coordinate $T_{x,y}$ is continuous. $\forall z, w \in X, \exists a, b \in [0, 1]$ such that $T(z, x, y) = a, T(w, x, y) = b$, define $F : X \times X \to \mathcal{A}X$ by:

$$F(z, w) = \begin{cases} T_{x,y}^{-1}\left([0, \frac{a+b}{2})\right) & \text{if } a \leq b \\ T_{x,y}^{-1}\left((\frac{a+b}{2}, 1]\right) & \text{if } a > b. \end{cases} \tag{1}$$

We can cheek that $F(z, w)$ is an open set in $X$ containing $z$ in either case. Now, we are left to show $F$ is continuous. Let $\langle U_1, \ldots, U_n\rangle^+$ be a basic open set in $\mathcal{A}X$ and $\exists z, w \in X$ such that $F(z, w) \in \langle U_1, \ldots, U_n\rangle^+$.

Assume $a \leq b$, denote

$$V = \bigcap_{i=1}^{n} U_i \cap T_{x,y}^{-1}\left([0, a)\right),$$
$$W = T_{x,y}^{-1}((a, b)).$$

They are both open. Let $p \in V, q \in W$, we want to show $F(p, q) \in \langle U_1, \ldots, U_n\rangle^+$. Denote $T_{x,y}(p) = c$, $T_{x,y}(q) = d$. By our construction, $c < a < d < b$,

$$F(p, q) = T_{x,y}^{-1}\left(\left[0, \frac{c+d}{2}\right)\right) \subset T_{x,y}^{-1}\left(\left[0, \frac{a+b}{2}\right)\right) \subset \bigcup_{i=1}^{n} U_i.$$

Since $p \in U_i$ for each $i, F(p, q)$ contains $p$, so $F(p, q) \cap U_i \neq \emptyset$ for each $i$.

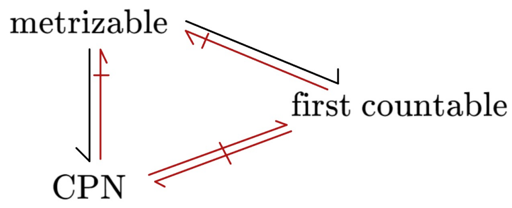


Figure 3: Relationships Between Metrizable Spaces and $CPN$ Spaces.

Black arrows denote implications; red arrows denote instances where counterexamples are presented.

On the other hand, if $a > b$, consider

$$p \in \bigcap_{i=1}^{n} U_i \cap T_{x,y}^{-1}\left((a, 1]\right),$$
$$q \in T_{x,y}^{-1}((b, a)).$$

Denote $T_{x,y}(p) = c$ and $T_{x,y}(q) = d$. We have $b < d < a < c$, then $F(p, q) = T_{x,y}^{-1}\left(\left(\frac{c+d}{2}, 1\right]\right) \subset T_{x,y}^{-1}\left(\left(\frac{a+b}{2}, 1\right]\right) \subset \bigcup_{i=1}^{n} U_i$. Since $p \in U_i$ for each $i$, $F(p, q) \cap U_i \neq \emptyset$ for all $i$.

Thus $F$ is continuous $T_1$ operator.

Lastly, continuously $T_1$ implies continuously $T_0$ directly by definition. □

For the reverse direction, Example 4.1 and 4.2 in [Gru76] provides spaces that are $CPN$ but not metrizable; it is still unknown if every $CN$ space is $CPN$; Example 2 shows that not every $CCR$ space is $CN$; we have not found an example that is continuously $T_2$ but not $CCR$ yet; we show $\mathbb{N}$ with co-finite topology is continuously $T_1$ but not $T_2$ in Example 6 and Sierpinsky space is continuously $T_0$ but not $T_1$ in Example 8.

## 4.2 Higher Separation Axioms

### 4.2.1 The Triangular Relationships between Metrizable, $CPN$ and First Countable Spaces

**Definition 4.7.** *[Mun00] The **Sorgenfrey line**, Denoted as $\mathbb{R}_l$, is real number $\mathbb{R}$ equipped with the topology generated by all unions of intervals of the form $\{[a, b) : a, b \in \mathbb{R}, a < b\}$.*

*Its product space $\mathbb{R}_l \times \mathbb{R}_l$ is called the **Sorgenfrey plane**.*

Sorgenfrey plane is an extremely important example in separation axioms as it not normal but Sorgenfrey line is. The proof can be found in Example 3, Section 31 in [Mun00].

**Example 10.** *$\mathbb{R}_l$ is first countable but not $CPN$.*

*Proof.* Claim for each $x \in \mathbb{R}_l$, its countable open neighborhood basis is $\{[x, b) : b \in \mathbb{Q}, b > x\}$, where $\mathbb{Q}$ is rational number. For each interval $[a, c)$ containing $x$, $\exists b \in \mathbb{Q}$ such that $x < b \leq c$, thus $[x, b) \subset [a, c)$. Thus $\mathbb{R}_l$ is first countable.

To show it is not $CPN$, the General Reference Chart in [JS78] asserts that $\mathbb{R}_l$ is separable. However, it is not metrizable, as its product $\mathbb{R}_l \times \mathbb{R}_l$ is not $T_4$ and we know that finite product of metrizable space is metrizable. Finally, Theorem 3.7 (a separable $CCR$ space is metrizable) concludes $\mathbb{R}_l$ is not $CCR$, thus it is not $CPN$. $\square$

It is not hard to see that every metrizable space is first countable. G. Gruenhage has provided an example that is first countable but not metrizable, and an example that is $CPN$ but not first countable in his work [Gru76]. Thus not every $CPN$ space is metrizable. The conclusion between metric spaces, $CPN$ spaces and first countable spaces is summarized in Figure 3.

### 4.2.2 Relationships between Traditional Separation Axioms and Continuous Separation Axioms

In the classical Separation Axioms, we have a clear implication chain $T_6 \implies T_5 \implies T_4 \implies T_3 \implies T_2$. Relative counter examples that show the converses are not true can be found in Example 43, 86, 90 and 75 in [JS78].

Proposition 3.4 shows that the continuous version of separation axioms always imply the corresponding non-continuous version of separation axioms. However, except that, the inclusion of the continuity property within the spaces does not significantly correlate with their non-continuous equivalents. For instance, the last example provided in [Zen75] shows a space that is $CCR$ but not $T_4$. Moreover, highly separated spaces are not guaranteed to be even $CCR$. For example, page 47 in [Eng89] illustrates that $\mathbb{R}_l$ is $T_6$, but in the last example we show it is not $CCR$. This example contributes to three red arrows in Figure 2 simultaneously, which are $T_{3.5} \not\Longrightarrow CCR$, $T_4 \not\Longrightarrow CN$ and $T_6 \not\Longrightarrow CPN$.

As a conclusion of this chapter, Figure 2 gives a comprehensive framework for continuous and classical separation axioms. The bottom row illustrates the connections between classically separated spaces. Above them are completely (functionally) separated spaces. Notice that we include $T_6$ in this row because its definition in this dissertation is equivalent to two closed sets being perfectly separated by a continuous function. Also, we ignored functionally $T_4$ for simplicity as it is equivalent to $T_4$ by Urysohn lemma (see Theorem 33.1 in [Mun00]). Above the completely separated spaces are continuously separated spaces, including the new definitions introduced in this work. Finally on the top of all spaces are metric space. This graph shows the strength of separability from top to bottom and from left to right. The black arrows and red arrows indicates implication and counter examples that we have discussed in this section. The green dashed arrows represent results that are still unknown yet.

# 5 Conclusion and Future Work

The construction of $CPN$, $CN$ and $CCR$ space gives a class of spaces between classically separated spaces and metric space. In our work, we have introduced this classification to readers by giving intuitive illustrations and providing more detailed proofs for important results. Following them, we have also finished the puzzle of continuous separation axioms by defining continuously $T_2$, continuously $T_1$ and continuously $T_0$ spaces and providing examples and results. Most importantly, Figure 2 summarizes all of the these results about classical and continuous separation axioms and provides a comprehensive visualization for readers to understand the connections between them.

However, as illustrated in Figure 2, it remains unknown whether there exists a space that is $CN$ but not $CPN$, or continuously $T_2$ but not $CCR$. Here we hypothesize that such spaces exist in both cases. Furthermore, there are other intriguing variations of the definition of continuously $T_2$ that merit consideration. We briefly mentioned these alternatives in the last section without delving into details. These could all serve as immediate next steps for one interested in this area.

For the future study, one can delve into the connectivity or compactness of these spaces, or revisit metrizability to explore its potential relationships with continuously separated spaces, because this is why these spaces are invented.

The lower continuously separated spaces introduced in this work are original contributions. While their direct connection to metrizability may not be apparent yet, it is still hard to give a definite evaluation about these definitions for now. Time will tell whether they can enrich our broader understanding of separation axioms as fundamental building blocks in topology. [1]

[1]Word count: 7474, according to overleaf

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