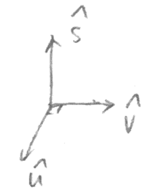
4.) the eigenvalues 1, $\cos\phi + i\sin\phi$, $\cos\phi - i\sin\phi$

related eigenvectors $\hat{s}$, $\frac{\sqrt{2}}{2}\hat{u} - \frac{\sqrt{2}}{2}i\hat{v}$, $\frac{\sqrt{2}}{2}\hat{u} + \frac{\sqrt{2}}{2}i\hat{v}$



Verify below:

$$(I + \sin\phi S + (1-\cos\phi)S^2)\hat{u} = \hat{u} + \sin\phi\hat{v} + (1-\cos\phi)(-\hat{u})$$
$$= \sin\phi\hat{v} + \cos\phi\hat{u}$$

$$(I + \sin\phi S + (1-\cos\phi)S^2)\hat{v} = \hat{v} + \sin\phi(-\hat{u}) + (1-\cos\phi)(-\hat{v})$$
$$= -\sin\phi\hat{u} + \cos\phi\hat{v}$$

① $(I + \sin\phi S + (1-\cos\phi)s^2)\hat{s} = I\hat{s} + 0 + 0 = \hat{s}$

② $(I + \sin\phi S + (1-\cos\phi)S^2)(\frac{\sqrt{2}}{2}\hat{u} - \frac{\sqrt{2}}{2}i\hat{v}) =$

$$\frac{\sqrt{2}}{2}(\sin\phi\hat{v} + \cos\phi\hat{u}) - \frac{\sqrt{2}}{2}i(-\sin\phi\hat{u} + \cos\phi\hat{v})$$
$$= (\cos\phi + i\sin\phi)(\frac{\sqrt{2}}{2}\hat{u} - \frac{\sqrt{2}}{2}i\hat{v})$$

③ $(I + \sin\phi S + (1-\cos\phi)S^2)(\frac{\sqrt{2}}{2}\hat{u} + \frac{\sqrt{2}}{2}i\hat{v}) =$

$$\frac{\sqrt{2}}{2}(\sin\phi\hat{v} + \cos\phi\hat{u}) + \frac{\sqrt{2}}{2}i(-\sin\phi\hat{u} + \cos\phi\hat{v})$$
$$= (\cos\phi - i\sin\phi)(\frac{\sqrt{2}}{2}\hat{u} + \frac{\sqrt{2}}{2}i\hat{v})$$

* when $\phi = 2\pi n$

eigenvalues: 1

eigenvectors: $\hat{s}$, $\hat{u}$, $\hat{v}$

* when $\phi = (2n+1)\pi$

eigenvalues: 1, -1

eigenvectors: $\hat{s}$, $\hat{u}$, $\hat{v}$

5.) $\text{trace}(R) = 1 + \cos\phi + i\sin\phi + \cos\phi - i\sin\phi$
$$= 1 + 2\cos\phi$$
$$\Rightarrow \cos\phi = \frac{1}{2}(\text{trace}(R) - 1)$$