Homework #4

Problem 1

$$y_i = ax_i^2 + bx_i + c + e_i$$

$$e_i \sim N(0, \sigma^2)$$

prior $\quad$ prob$(a, b, c, \log\sigma) \propto$ const

$\hookrightarrow$ jeffrey's prior

$$\text{prob}(a, b, c, \sigma) \propto \frac{1}{\sigma}$$

$$\text{prob}(\log\sigma)\, d\log\sigma = \text{prob}(\sigma)\, d\sigma$$

$$\left| \frac{d\log\sigma}{d\sigma} \right| = \frac{1}{\sigma}$$

posterior

$$\text{prob}(a, b, c, \sigma \mid \text{data}) \propto \frac{\text{likelyhood} \cdot \text{prior}}{\text{prob(data)}}$$

$$y = A\beta + c$$

$$\beta = \begin{bmatrix} c \\ b \\ a \end{bmatrix} \qquad A = \begin{bmatrix} 1 & t_1 & t_1^2 \\ \vdots & \vdots & \vdots \\ 1 & t_{10} & t_{10}^2 \end{bmatrix}$$

$$\text{prob}(\text{data} \mid \underbrace{a, b, c, \sigma}_{\text{model}}) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2} \left(\frac{y_i - at_i^2 + bt_i + c)^2}{\sigma^2} \right) \right)$$

$$y = A\beta + c$$

$$e = y - A\beta$$

$$= \frac{1}{2\pi^5 \sigma^{10}} \exp\left(\frac{1}{2\sigma^2} (y - Ab)^T (y - Ab) \right)$$

likelyhood

likelihood

$$= \frac{1}{2\pi^5 \sigma^{10}} \exp\left(\frac{-1}{2\sigma^2} e^T e\right)$$

$$\text{posterior} \propto \frac{1}{2\pi^5 \sigma^{10}} \exp\left(-\frac{1}{2\sigma^2} e^T e\right) \cdot \frac{1}{\sigma}$$

prior

(and drop parts proportional)

$$\text{posterior} \propto \frac{1}{2\pi \sigma^{11}} \exp\left(-\frac{1}{2\sigma^2} e^T e\right)$$

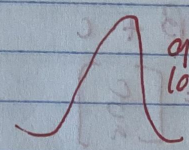
$$\text{prob}(\underline{B}|\text{data}) = \int_{-\infty}^{\infty} d\log\sigma \; \text{prob}(B, \log\sigma | \text{data})$$

$$= \int_{-\infty}^{\infty} d\sigma \left\{ \text{prob}(\underline{B}\sigma | \text{data}) \right\}$$

$$\propto V(B)^{-k/2} \quad \text{where } V(B) = (\underline{\underline{A}}B - y)^T (\underline{\underline{A}}B - y)$$

length(B)

$$\text{prob}(B|\text{data}) \propto \frac{1}{(\underline{e}^T \underline{e})^{3/2}}$$



gaussian look

$$\text{prob}(B, \sigma | \text{data}) \propto \exp\left(-\frac{1}{2\sigma^2}(B - \hat{B})(A^T A(B - \hat{B}))\right)$$

$$\mathcal{L} = \log(\text{prob}(B, \sigma | \text{data})) = -\frac{1}{2\sigma^2}(AB - y)^T(AB - y) - (N+1)\log\sigma$$

$$\left.\frac{\partial \mathcal{L}}{\partial B}\right|_{B=\hat{B}} = -\frac{1}{\sigma^2}(AB - y)^T(A) = 0 \qquad \hat{\sigma}^2 = \frac{(A\hat{B} - y)^T(A\hat{B} - y)}{N+1}$$

$$\hat{B} = (A^T A)^{-1} A^T y$$

(lets solve in Matlab)!

problem 2

likeyhood

$$\text{prob}(\text{data} \mid m, b, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(y_i - mx_i + b)\right)$$

prior $\frac{1}{\sigma}$

prostrior $\propto$ likelyhood $\cdot$ prior

(in matlab code)

Problem 3

$y_{ij} = e_{ij} \exp(-bt_j)$

$\log(e_{ij}) \sim N(\mu, \sigma^2)$

estimate b, μ, σ

$\log(y_{ij}) = \log(e_{ij}) - bt$

$x_{ij} = \mu - bt_j + e_{ij}$

$\langle \log e_{ij} \rangle = \mu \rightarrow \langle \log e_{ij} - \mu \rangle = 0$

$t = [1 \longrightarrow 5]$

$A = [[1, \rightarrow 1], [t \longrightarrow t]]$

prior

$\text{prob}(\mu, b, \sigma) \propto \frac{1}{\sigma}$

likelyhood

$\text{prob}(\text{data} \mid \mu, b, \sigma) \propto \frac{1}{\sigma^{10}}$

$\propto \frac{1}{\sigma^{10}} \exp\left(\frac{-1}{2\sigma^2} \underline{\varepsilon}^T \underline{\varepsilon}\right)$

$\varepsilon = y - AB$

$B = \begin{bmatrix} \mu \\ b \end{bmatrix}$

$\propto \frac{1}{\sigma^{10}} \exp\left(-\frac{1}{2}(AB-y)^T(AB-y)\right)$

proportional

$$\text{prob}(\sigma, N, b \mid y) \propto \frac{1}{\sigma^n} \exp\left[-\frac{1}{2}(AB-y)^T(AB-y)\right]$$

$$\log \text{proportional} = \text{const} - 11 \log \sigma - \frac{1}{2\sigma^2}\left[-\frac{1}{2}(AB-y)^T(AB-y)\right]$$

$$\left.\frac{\partial \mathcal{L}}{\partial B}\right|_{B=\hat{\beta}} = 0 \qquad \frac{\partial \mathcal{L}}{\partial B} = -\frac{1}{\sigma} 2 (A\hat{B}-y)^T(A)$$

$$\underbrace{A^T A \hat{B} = A^T y}_{\text{normal}}$$

$$\frac{\partial \mathcal{L}}{\partial \sigma} = \frac{-11}{\sigma} + \sigma^{-3}\left[(\hat{B}-B)A^T A(\hat{B}-B) + (y-A\hat{B})^T(y-A\hat{B})\right] = 0$$

$$\hat{\sigma}^2 = \frac{\left[(y-A\hat{B})^T(y-A\hat{B})\right]}{11}$$

$$\text{prob}(B \mid y) = \int_0^\infty d\sigma \, \text{prob}(B, \sigma \mid y)$$

$$\text{prob}(N \mid y) = \int_{-\infty}^{\infty} dp \int_0^\infty d\sigma \, \text{prob}(B, \sigma \mid y)$$

$$\log \text{prob}(N \mid y) = \frac{-1}{2(N-K+1)} \log\left[v + (N-\hat{b})^2\right]$$

$v = N-K$

$s^2 = e^T e$

$H = s^2 A^T A$

$$\log[\text{prob}(b \mid y)] = \text{const} - \frac{1}{2(N-K+1)} \log\left[v + \frac{(b-\hat{b})^2}{C(2,2)}\right]$$

$$\text{prob}(b \mid y) = \frac{1}{Z}\left[v - \frac{(b-\hat{b})}{C(2,2)}\right]^{2(N-K+1)}$$

$$Z = \int_{-\infty}^{\infty} db \left[v + \frac{(b-\hat{b})^2}{C(2,2)}\right]^{2/(N-K+1)}$$

$$N = \langle N \rangle \pm \sqrt{\langle N - \langle N \rangle^2 \rangle}$$

$$\langle b \rangle = \int_{-\infty}^{\infty} dp \, \text{prob}(b \mid y) b$$

Problem 1

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%% Problem 4

clear, close all

%% Problem 1

t = [1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0]';
y = [-2.73, -2.71, -2.65, -0.87, -3.10, -1.03, 0.63, 1.46, 5.90, 8.38]';

A = [ones(size(t)), t, t.^2];

beta= (A' * A) \ A' * y;

AB = A * beta;

e = AB - y;

N = length(y);

sigma_squared = (e' * e) / (N + 1);

disp(sigma_squared)

sigma = sqrt(sigma_squared);

disp(sigma)

%% estimate of y(t = 12)

t_12 = 12;
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y_12 = [1, t_12, t_12^2];

y_12_est = y_12 * beta;

fprintf('Estimated value of y(t = 12) = %.4f\n', y_12_est);

%%
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Problem 2

```
%%
clear, close all

%% HMW 4 problem 2
%
% x = [0, 1, 2, 3, 4];
% y = [0.0434, 1.0343, -0.2588, 3.68622, 4.3188];
%sigma = [0.1, 0.1 NaN NaN NaN];

x = [2, 3, 4]';
y = [-0.2588, 3.68622, 4.3188]';

% x = x(~isnan(sigma));
% y = y(~isnan(sigma));
% sigma = sigma(~isnan(sigma));

A = [ones(size(x)), x];

beta = (A' * A) \ A' * y;

AB = A * beta;
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e = AB - y;

N = length(y);

sigma_squared = (e' * e) / (N + 1);

disp(sigma_squared)

sigma_final = sqrt(sigma_squared);

disp(sigma_final)

%%
```

Problem 3

```
%%
clear, close all

%% Problem 3

t = [1.0, 2.0, 3.0, 4.0, 5.0]';
y = [3.75, 0.36, 0.58, 2.06;
    0.93, 0.32, 0.67, 1.01;
    0.38, 0.11, 0.12, 0.60;
    0.05, 0.15, 0.05, 0.11;
    0.04, 0.03, 0.08, 0.06]';
n = numel(t);

A = [ones(n, 1), t(:)];

beta = (A' * A) \ A' * y;
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```
AB = A * beta;

e = AB - y;

%%
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