What is Multivariate Calculus?

Single-variate calculus is the study of functions with one input variable and one output variable:

$$f : \mathbb{R} \to \mathbb{R}.$$

domain codomain

Example: $f(x) = x^2$.

Multivariate calculus is the study of functions with n input variables and m output variables:

$$f : \mathbb{R}^n \to \mathbb{R}^m,$$

where $\mathbb{R}^n$ is n-dimensional space: $\mathbb{R}^n = \{(x_1, \ldots, x_n)\}$.
Example: $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by $f(x, y) = (x + y, y, x^2 + 2y^2)$.

As in the single-variate case, we will approximate functions by their derivatives, which are linear functions: this is why we will need tools from linear algebra.

Multivariate calculus is the study of functions with n input variables and m output variables: $f : \mathbb{R}^n \to \mathbb{R}^m$.

In this class:

(§5,6: Review of integration for functions $\mathbb{R} \to \mathbb{R}$)

What is the area under the curve $y = x^2$ for $0 < x < 1$?

§14: Integration for functions $\mathbb{R}^n \to \mathbb{R}$

What is the volume under the surface $z = x^2 + y^2$ over the triangle $0 < x < y < 1$?

§12 Differentiation for functions $\mathbb{R}^n \to \mathbb{R}^m$

What is the tangent plane at $(4, 1/2, 1)$ to the surface $2x + 2\ln y = 9 - z^2$?

§13 Stationary points and extrema for functions $\mathbb{R}^n \to \mathbb{R}$

What is the largest value of $2x^2 + y^2 - y + 3$ on the unit disc $x^2 + y^2 \leq 1$?

Our domains will mostly be $\mathbb{R}^2$ or $\mathbb{R}^3$ (i.e. $n = 2, 3$ usually).

In Math3415 Vector Calculus (m, n are usually 2 or 3):

§11 Curves, i.e. functions $\mathbb{R} \to \mathbb{R}^m$

§15 Integration along curves and surfaces

§16 Relating differentiation and integration for functions $\mathbb{R}^n \to \mathbb{R}^n$.

In addition to computation, a very important skill in this class is visualisation in two and three dimensions. From the official syllabus:

Course Intended Learning Outcomes (CILOs):

Upon successful completion of this course, students should be able to:

No.	Course Intended Learning Outcomes (CILOs)
1	Visualize and sketch geometrical objects in 2- and 3-dimension, to manipulate the related issues of the chosen topics as outlined in "course content."

On homeworks and exams, you will be asked to draw.

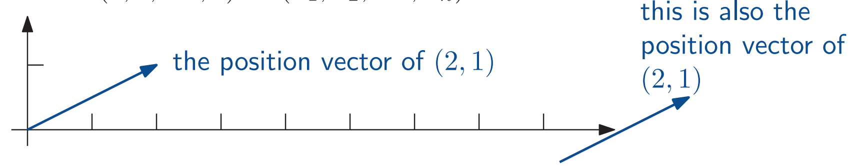
Before we start analysing functions, we will spend 1-2 weeks on some geometry in $\mathbb{R}^n$.

§10.2-10.4: Vectors, Lines and Planes

A *vector* is a quantity with a length and a direction (in n-dimensional space $\mathbb{R}^n$). Vectors are usually represented by arrows.

To distinguish between a number (a *scalar*) and a vector, we type vectors in bold ($\mathbf{v}$) and hand-write vectors with a arrow on top ($\vec{v}$).

Each point $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ is associated with a *position vector*, whose arrow goes from $(0, 0, \ldots, 0)$ to $(x_1, x_2, \ldots, x_n)$.



Vectors do not generally have a position - that is, two arrows represent the same vector if they are parallel and have the same length, even if they are in different places.

We will meet 4 operations on vectors:

i. Vector addition $\mathbf{u}+\mathbf{v}$ (p6, §10.2 definition 1 in textbook);
ii. Scalar multiplication $t\mathbf{u}$ (p7, §10.2 definition 2 in textbook);
iii. Dot product $\mathbf{u} \bullet \mathbf{v}$ and length $|\mathbf{v}| = \sqrt{\mathbf{v} \bullet \mathbf{v}}$ (p13-15, §10.2 definition 3 in textbook).

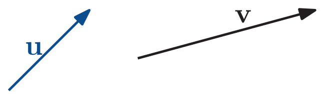
Using these operations, we can describe some simple geometric objects:

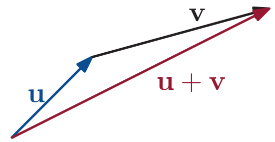
a. Vector parametric equation and scalar parametric equation of a line (p10-12, §10.4 p590 (8E) p588 (7E) in textbook);
b. Standard form of a plane (p16-18, §10.4 p588 (8E) p586 (7E) in textbook);
c. Spheres, cylinders, etc. (p19-35, §10.1 examples 2-5, 10.5 in textbook).

(There are many many other concepts in these sections of the textbook, which we will not need.)

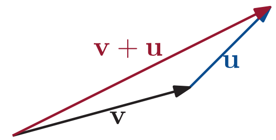
i. Vector addition

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. To calculate $\mathbf{u}+\mathbf{v}$, put the tail of $\mathbf{v}$ at the head of $\mathbf{u}$. Then $\mathbf{u}+\mathbf{v}$ is the vector going from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$.





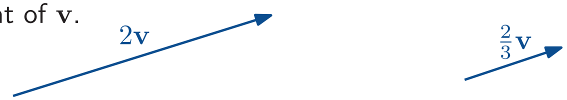
It is easy to check that $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ and $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$.



ii. Scalar multiplication

Let $\mathbf{v}$ be a vector and t be a scalar (i.e. a number).

- If $t > 0$, then $t\mathbf{v}$ is the vector in the same direction as $\mathbf{v}$ whose length is t times that of $\mathbf{v}$.



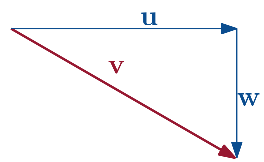
- If $t < 0$, then $t\mathbf{v}$ is the vector in the opposite direction as $\mathbf{v}$ whose length is $|t|$ times that of $\mathbf{v}$.

- If $t = 0$, then $t\mathbf{v} = 0\mathbf{v} = \mathbf{0}$, the zero vector, which has length 0 and therefore no particular direction.

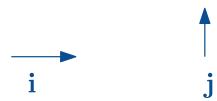
It is easy to check that $\mathbf{v} + (-1)\mathbf{v} = \mathbf{0}$ and $t(\mathbf{u}+\mathbf{v}) = t\mathbf{u} + t\mathbf{v}$.

These two operations allow us to describe all vectors in $\mathbb{R}^2$ in the following way:

Every vector in $\mathbb{R}^2$ can be written as the sum of a "horizontal" vector and a "vertical" vector.



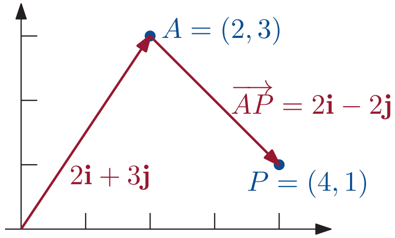
Let $\mathbf{i}$ denote the position vector of $(1, 0)$, and $\mathbf{j}$ denote the position vector of $(0, 1)$. These vectors are called the *standard basis vectors*.



Every 'horizontal" vector is a scalar multiple of $\mathbf{i}$, and every "vertical" vector is a scalar multiple of $\mathbf{j}$, so every vector in $\mathbb{R}^2$ can be written as $x\mathbf{i} + y\mathbf{j}$ for some scalars x, y. Such an expression is called a *linear combination of* $\mathbf{i}$ *and* $\mathbf{j}$.

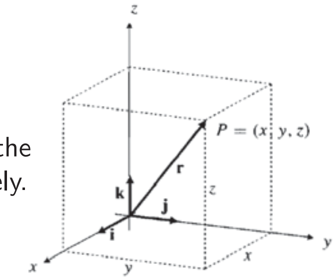
$$\mathbf{v} = \mathbf{u} + \mathbf{w}$$
$$= \frac{7}{2}\mathbf{i} - 2\mathbf{j}$$

Example: The position vector of a point (a,b) is $a\mathbf{i}+b\mathbf{j}$.
Example: The vector going from $A=(a,b)$ to $P=(p,q)$ is $\overrightarrow{AP}=(p-a)\mathbf{i}+(q-b)\mathbf{j}$ (difference of position vectors).



Addition and scalar multiplication are easy when vectors are written as linear combinations of $\mathbf{i}$ and $\mathbf{j}$:

$$(u_1\mathbf{i}+u_2\mathbf{j})+(v_1\mathbf{i}+v_2\mathbf{j})=(u_1+v_1)\mathbf{i}+(u_2+v_2)\mathbf{j};$$
$$t(u_1\mathbf{i}+u_2\mathbf{j})=(tu_1)\mathbf{i}+(tu_2)\mathbf{j}.$$



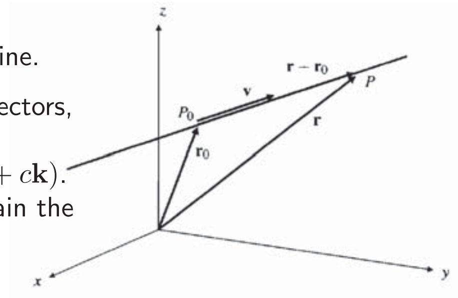
Similarly, in $\mathbb{R}^3$, the *standard basis vectors* are $\mathbf{i},\mathbf{j},\mathbf{k}$, the position vectors of $(1,0,0),(0,1,0),(0,0,1)$ respectively.
The standard basis vectors in $\mathbb{R}^n$ are usually called $\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_n$, and these are the position vectors of $(1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,\dots,0,1)$.

a. Parametric equation of a line

Let $\mathbf{r}_0=x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k}$ be the position vector of a point P_0 in $\mathbb{R}^3$, and $\mathbf{v}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ a vector in $\mathbb{R}^3$. There is a unique line passing through P_0 parallel to $\mathbf{v}$.
To find a description for this line: if P is any other point on this line, with position vector $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then $\overrightarrow{P_0P}=\mathbf{r}-\mathbf{r}_0$ is parallel to $\mathbf{v}$, i.e. is a multiple of $\mathbf{v}$. So $\mathbf{r}-\mathbf{r}_0=t\mathbf{v}$, i.e.

$$\mathbf{r}=\mathbf{r}_0+t\mathbf{v}.$$

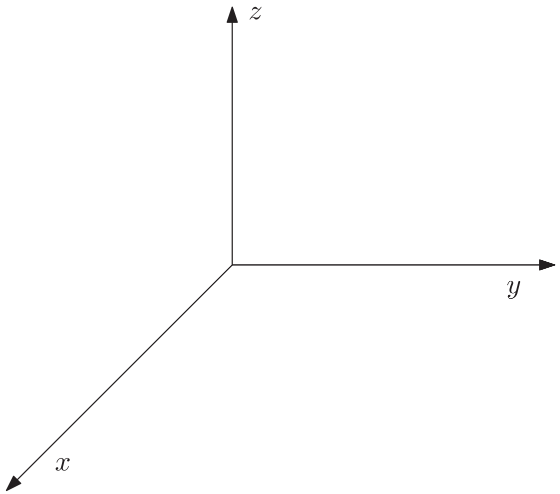
This is the *vector parametric equation* of the line.



As linear combinations of the standard basis vectors, the vector parametric equation says $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=(x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k})+t(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})$.
Equating the coefficients of $\mathbf{i},\mathbf{j}$ and $\mathbf{k}$, we obtain the *scalar parametric equations*:

$$x=x_0+at,$$
$$y=y_0+bt,$$
$$z=z_0+ct.$$

Example: Find the vector and scalar parametric equations for the line through $(1,0,-1)$ parallel to $-\mathbf{i}-\mathbf{j}+3\mathbf{k}$, and sketch this line.



Vector parametric equation: $\mathbf{r}=\mathbf{r}_0+t\mathbf{v}$
Scalar parametric equations: $x=x_0+at,$

$$y=y_0+bt,$$
$$z=z_0+ct.$$

These are call parametric or explicit equations because they give the coordinates of each point on the line as a function of the parameter t. Each value of t in $\mathbb{R}$ corresponds to one point on the line. We can think of t as time.

The same construction works in $\mathbb{R}^n$: if $\mathbf{r}_0$ is the position vector of a point P_0 in $\mathbb{R}^n$ and $\mathbf{v}$ is a vector in $\mathbb{R}^n$, then $\mathbf{r}=\mathbf{r}_0+t\mathbf{v}$ describes the "line" in $\mathbb{R}^n$ through P_0 parallel to $\mathbf{v}$.

We can similarly obtain parametric equations for a plane in $\mathbb{R}^3$: if $\mathbf{r}_0$ is the position vector of a point P_0 in $\mathbb{R}^3$ and $\mathbf{v},\mathbf{w}$ are two vectors in $\mathbb{R}^3$, then $\mathbf{r}=\mathbf{r}_0+t\mathbf{v}+s\mathbf{w}$ describes the plane through P_0 parallel to $\mathbf{v}$ and $\mathbf{w}$.
But because a plane is 2-dimensional in 3-dimensional space, and $2+1=3$, it is easier to work with implicit equations for a plane.

To obtain an implicit equation for a plane in $\mathbb{R}^3$, we first need to consider:

iii. Dot product

Given vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ in $\mathbb{R}^2$, their *dot product* (or scalar product) is the scalar

$$\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2.$$

Given vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ in $\mathbb{R}^3$, their *dot product* is the scalar

$$\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

(The definition is similar for other $\mathbb{R}^n$.)

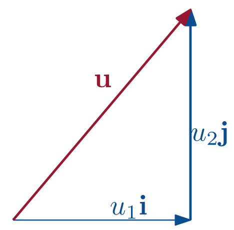
Example: If $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, then $\mathbf{u} \bullet \mathbf{v} = 3 \cdot 2 + 4 \cdot -1 - 5 \cdot 2 = -8$.

It is easy to check that:
$\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$;
$\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$;
$(t\mathbf{u}) \bullet \mathbf{v} = \mathbf{u} \bullet (t\mathbf{v}) = t(\mathbf{u} \bullet \mathbf{v})$.

By Pythagoras's Theorem, the *length* of a vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is $\sqrt{u_1^2 + u_2^2}$, i.e.

$$|\mathbf{u}| = \sqrt{\mathbf{u} \bullet \mathbf{u}}.$$

The same formula works also in $\mathbb{R}^3$:
$|\mathbf{u}| = \sqrt{\mathbf{u} \bullet \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + u_3^2}$.

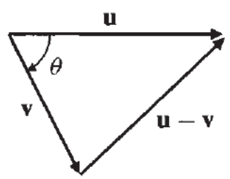


For many applications, we will be interested in vectors of length 1.
Definition: A *unit vector* is a vector whose length is 1.

Given $\mathbf{v}$, to create a unit vector in the direction of $\mathbf{v}$, we divide $\mathbf{v}$ by its length $|\mathbf{v}|$. This process is called normalising.
Example: If $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, then $|\mathbf{v}| = \sqrt{\mathbf{v} \bullet \mathbf{v}} = \sqrt{2 \cdot 2 - 1 \cdot -1 + 2 \cdot 2} = 3$, so a unit vector in the same direction as $\mathbf{v}$ is $\dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{2}{3}\mathbf{i} - \dfrac{1}{3}\mathbf{j} + \dfrac{2}{3}\mathbf{k}$.

To see why the dot product is important, recall the cosine law:
$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\,|\mathbf{u}|\,|\mathbf{v}| \cos\theta$.
We can "expand" the left hand side using dot products:

$$\begin{aligned}|\mathbf{u} - \mathbf{v}|^2 &= (\mathbf{u} - \mathbf{v}) \bullet (\mathbf{u} - \mathbf{v}) \\ &= \mathbf{u} \bullet \mathbf{u} - \mathbf{u} \bullet \mathbf{v} - \mathbf{v} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{v} \\ &= |\mathbf{u}|^2 - 2\mathbf{u} \bullet \mathbf{v} + |\mathbf{v}|^2.\end{aligned}$$



Comparing with the cosine law, we see $\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$.

In particular, two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\theta = \frac{\pi}{2}$, i.e. when $\cos\theta = 0$. This is equivalent to $\mathbf{u} \cdot \mathbf{v} = 0$.

b. Standard form of a plane

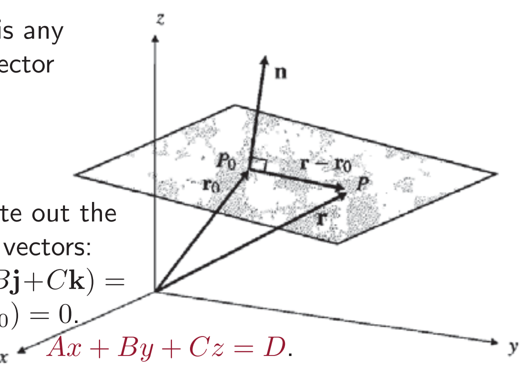
Definition: A *normal* vector to a plane is a vector perpendicular to it.

Let $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ be the position vector of a point P_0 in $\mathbb{R}^3$, and $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ a vector in $\mathbb{R}^3$. There is a unique plane passing through P_0 perpendicular to $\mathbf{n}$.

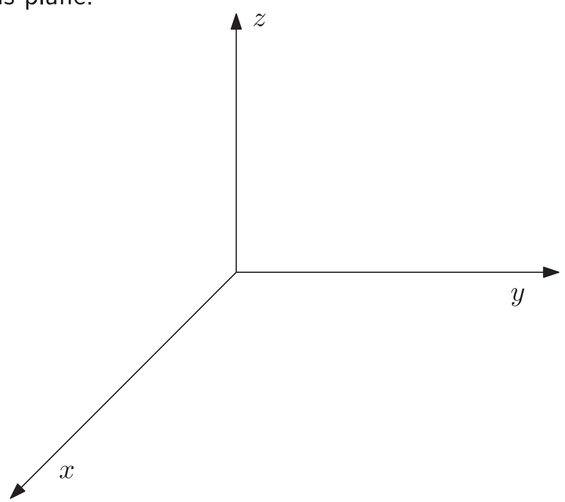
To find a description for this plane: if P is any other point on this plane, with position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0$ is perpendicular to $\mathbf{n}$. So

$$(\mathbf{r} - \mathbf{r}_0) \bullet \mathbf{n} = 0.$$

To obtain a scalar equation, we again write out the linear combinations of the standard basis vectors:
$\big((x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) - (x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k})\big) \bullet (A\mathbf{i}+B\mathbf{j}+C\mathbf{k}) = 0$, i.e. $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$.
We can rearrange this into *standard form* $Ax + By + Cz = D$.



Example: Find the standard form of the plane through $(0,0,1)$ with normal vector $\mathbf{i}+3\mathbf{j}-2\mathbf{k}$, and sketch this plane.



The standard form $Ax+By+Cz=D$ is an implicit description of the plane - it is an equation that all points on the plane must satisfy.

To obtain an explicit description (i.e. write x, y, z each as a function of parameters), we can solve for one of the variables in terms of the others: e.g. a parametrisation of $x+3y-2z=-2$ is
$$x = x,\quad y = y,\quad z = \frac{1}{2}(x+3y+2).$$

Question: What is the set satisfying the inequality $x+3y-2z<-2$? (Hint: how is the set satisfying $z<0$ related to the set satisfying $z=0$?)

Answer: The inequalities $x-3y-2z<-2$ and $x-3y-2z>-2$ describe the two sides of the plane $x-3y-2z=-2$. To find out which inequality describes which side: given a point on the plane, in order to achieve $x-3y-2z<-2$, I can fix x, y and increase z (because the coefficient of z is negative). So the inequality is the region above the plane. (See p33 for another method.)

§10.5: Quadric Surfaces

In general, the set of points in $\mathbb{R}^n$ satisfying a single equation is an $n-1$ dimensional object, a "hypersurface".

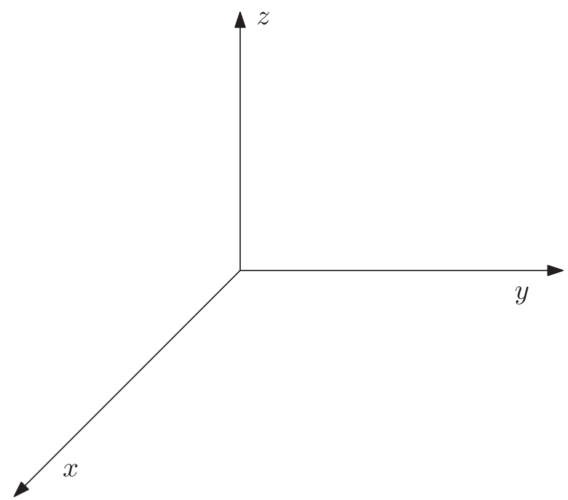
Here, we identify and sketch some sets defined by simple cases of a quadratic equation in $\mathbb{R}^3$:

$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz=J.$$

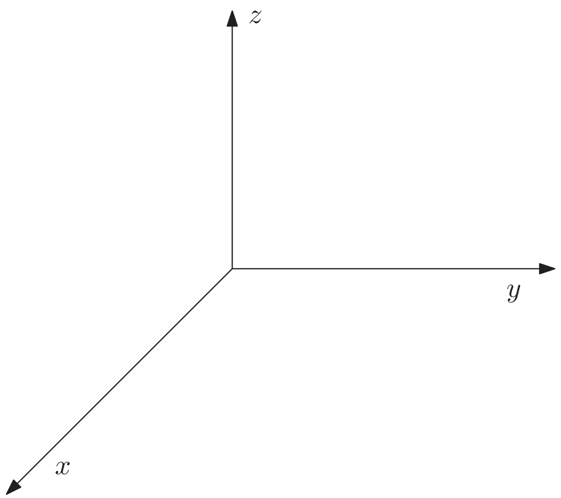
These usually (but not always, see p34-35) describe a 2-dimensional surface. We will also consider when the equals sign in the above equation is replaced by an inequality ($<$ or $>$), which will usually describe one side of these surfaces.

We begin with the simplest case, where one of the variables does not appear in the equation.

Example: Describe and sketch the set in $\mathbb{R}^3$ satisfying $x^2+y^2=1$.



Example: Describe and sketch the set satisfying $y^2 + 4z^2 = 4$.

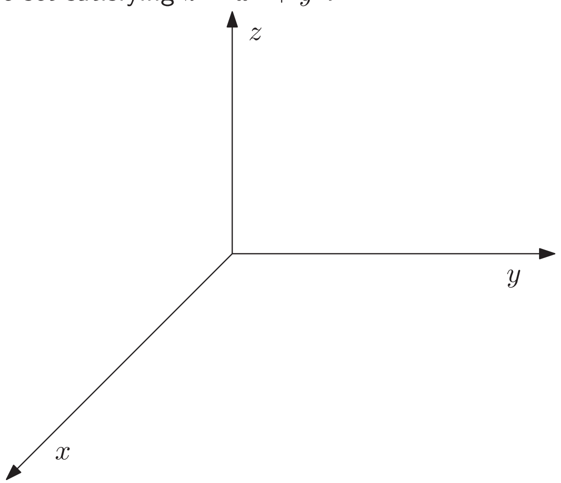


Recall that it is useful to parametrise a surface, i.e. write x, y, z explicitly as functions of a parameter.

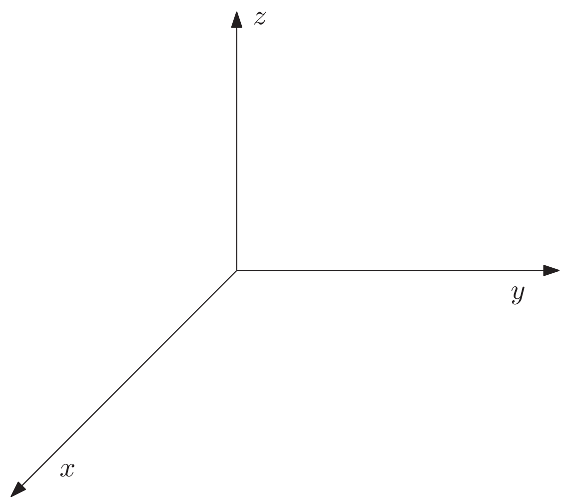
Example: Parametrise the cylinder $y^2 + 4z^2 = 4$.

The next simplest quadric surface is when one of the variables only has degree 1.

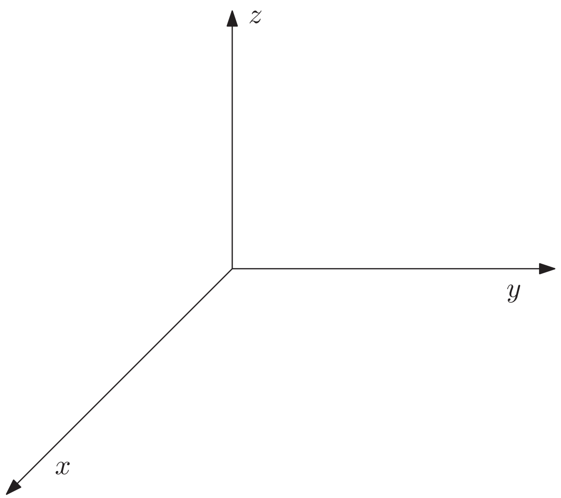
Example: Describe and sketch the set satisfying $z = x^2 + y^2$.



Example: Describe and sketch the set satisfying $y = x^2 - 2x + z^2$.



Example: Describe and sketch the set satisfying $z = x^2 - y^2$.

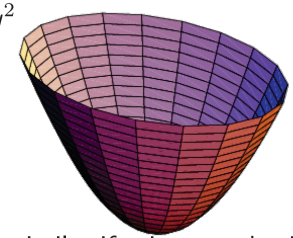


To summarise p23-25: an equation of the form

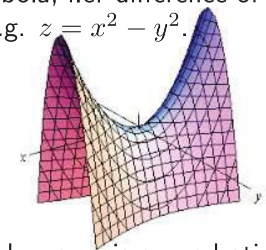
$$z = Ax^2 + By^2 + Dxy + Gx + Hy - J$$

describes either:

an elliptic paraboloid, if the right hand side is the equation of an ellipse, i.e. sum of two squares, e.g. $z = x^2 + y^2$



a hyperbolic paraboloid (or a saddle), if the right hand side is the equation of a hyperbola, i.e. difference of two squares, e.g. $z = x^2 - y^2$.



The case is similar if y is a quadratic function of x and z, or x is a quadratic function of y and z.

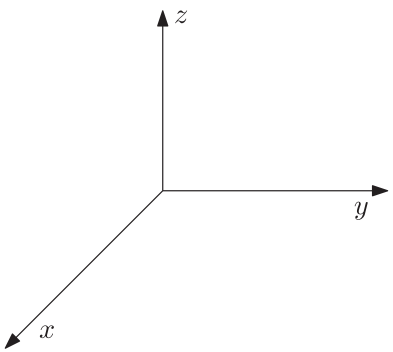
It is easy to parametrise a paraboloid, since one of the variables is an explicit function of the other two.

(pictures from Wolfram MathWorld, Paul's online math notes)

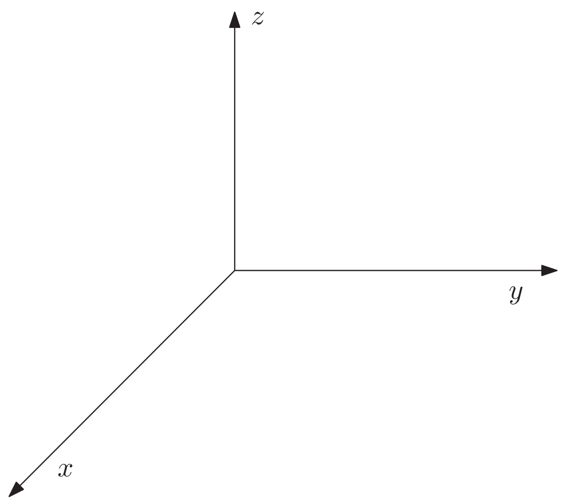
Now we consider the most general case, where (after completing the square to remove cross terms and linear terms) we have $Ax^2 + By^2 + Cz^2 = J$ and $A, B, C \neq 0$.

First consider the case where A, B, C have the same sign:

Example: Describe and sketch the set satisfying $x^2 + y^2 + z^2 = 1$.



Example: Describe and sketch the set satisfying $x^2 + y^2 + 4z^2 = 4$.



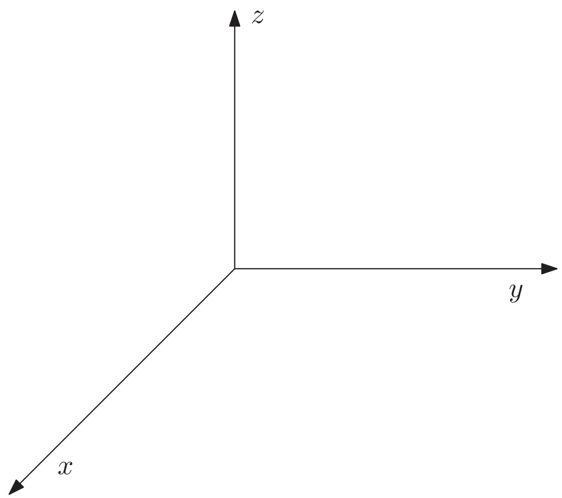
The standard way to parametrise an ellipsoid is to use two angles - this is the idea of spherical coordinates, which we will meet when we integrate over three-dimensional regions (§10.6, §14.6).

Now suppose $Ax^2 + By^2 + Cz^2 = J$ and A, B, C don't all have the same sign, e.g. $Ax^2 + By^2 - z^2 = J$ with $A, B > 0$, which we can rearrange as

$$z^2 = Ax^2 + By^2 - J, \quad A, B > 0.$$

Now there are three possibilities depending on the sign of J (zero, positive, negative).

Example: Describe and sketch the set satisfying $z^2 = x^2 + y^2$ (i.e. $J = 0$).



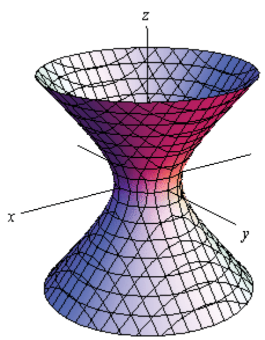
If

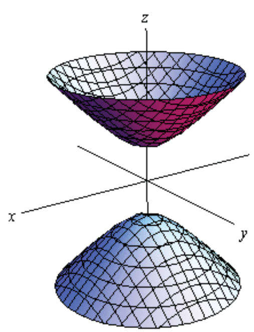
$$z^2 = Ax^2 + By^2 - J, \quad A, B > 0, \; J \neq 0,$$

then the equation describes a hyperboloid - drawing these is NOT examinable.

$J > 0$, e.g. $z^2 = x^2 + y^2 - 1$: hyperboloid of one sheet;

$J < 0$, e.g $z^2 = x^2 + y^2 + 1$: hyperboloid of two sheets.





(pictures from Paul's online math notes)

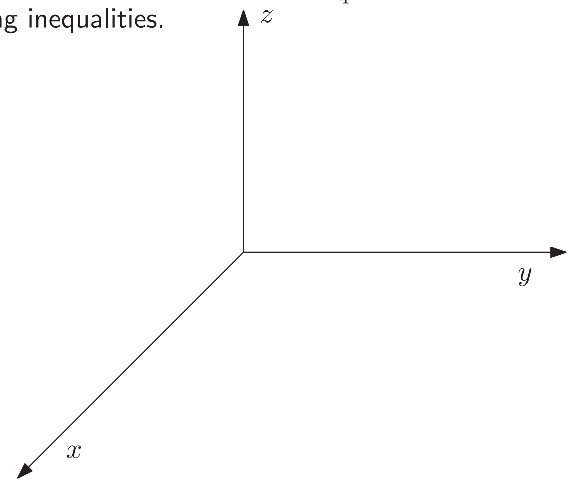
Summary:
To describe and sketch the quadric defined by

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz = J:$$

- First, complete the square to remove the cross terms $Dxy + Exz + Fyz$ (see week 2 p11).
- If one variable does not appear in the equation, then the set is a cylinder (see p20-21, ex. sheet #2 q2). $x^2 + y^2 = 1$
- If one variable only has degree one, then the set is a paraboloid: the paraboloid is elliptic if the two quadratic variables have the same sign, and hyperbolic if they have different signs (see p25). $z = x^2 + y^2$; $z = x^2 - y^2$
- If all three variables have degree two:
  - If the coefficients of x^2, y^2, z^2 have the same sign, then the set is an ellipsoid; $x^2 + y^2 + z^2 = 1$
  - If the coefficients of x^2, y^2, z^2 have different signs, then it is a cone (if there is no constant term),or a hyperboloid. $z^2 = x^2 + y^2$; $z^2 = x^2 + y^2 - 1$; $z^2 = x^2 + y^2 + 1$

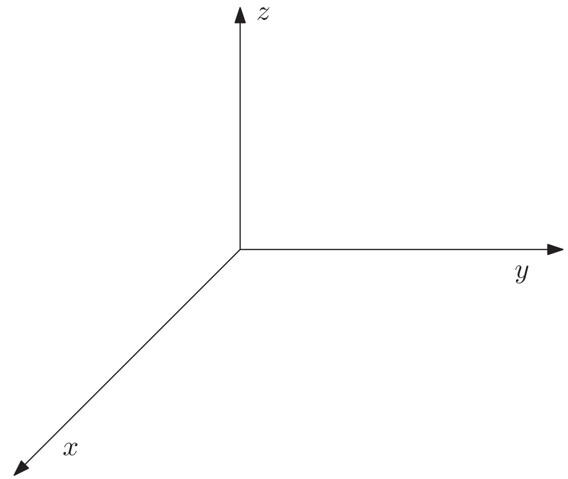
Regions bounded by surfaces and inequalities

Example: Describe and sketch the larger region bounded by $\frac{1}{4}x^2 + y^2 + z^2 = 1$ and $z = -\frac{1}{5}$, and describe it using inequalities.



Degenerate cases

Example: Describe and sketch the set satisfying $x^2 + y^2 + z^2 + 1 = 0$.



Example: Describe and sketch the set in $\mathbb{R}^3$ satisfying $x^2 - y^2 = 0$.