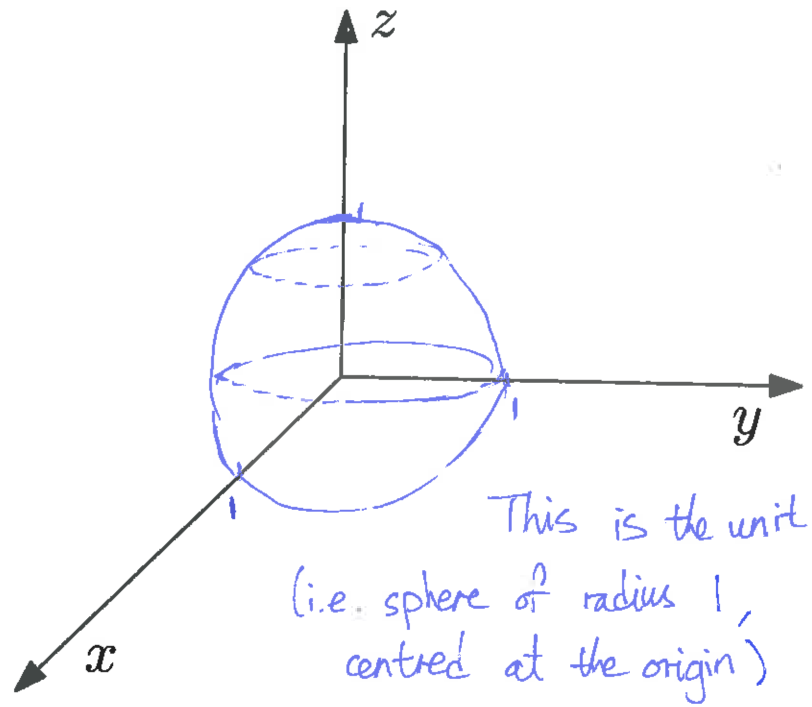
Now we consider the most general case, where (after completing the square to remove cross terms and linear terms) we have $Ax^2 + By^2 + Cz^2 = J$ and $A, B, C \neq 0$.

First consider the case where A, B, C have the same sign:
Example: Describe and sketch the set satisfying $x^2 + y^2 + z^2 = 1$.

$x^2+y^2+z^2 = (\text{length of position vector})^2$
So $x^2+y^2+z^2=1$ describes all points of distance 1 from the origin



This is the unit sphere (i.e. sphere of radius 1, centred at the origin)

Example: Describe and sketch the set satisfying $x^2 + y^2 + 4z^2 = 4$.

$x^2 + y^2 + (2z)^2 = 2^2$

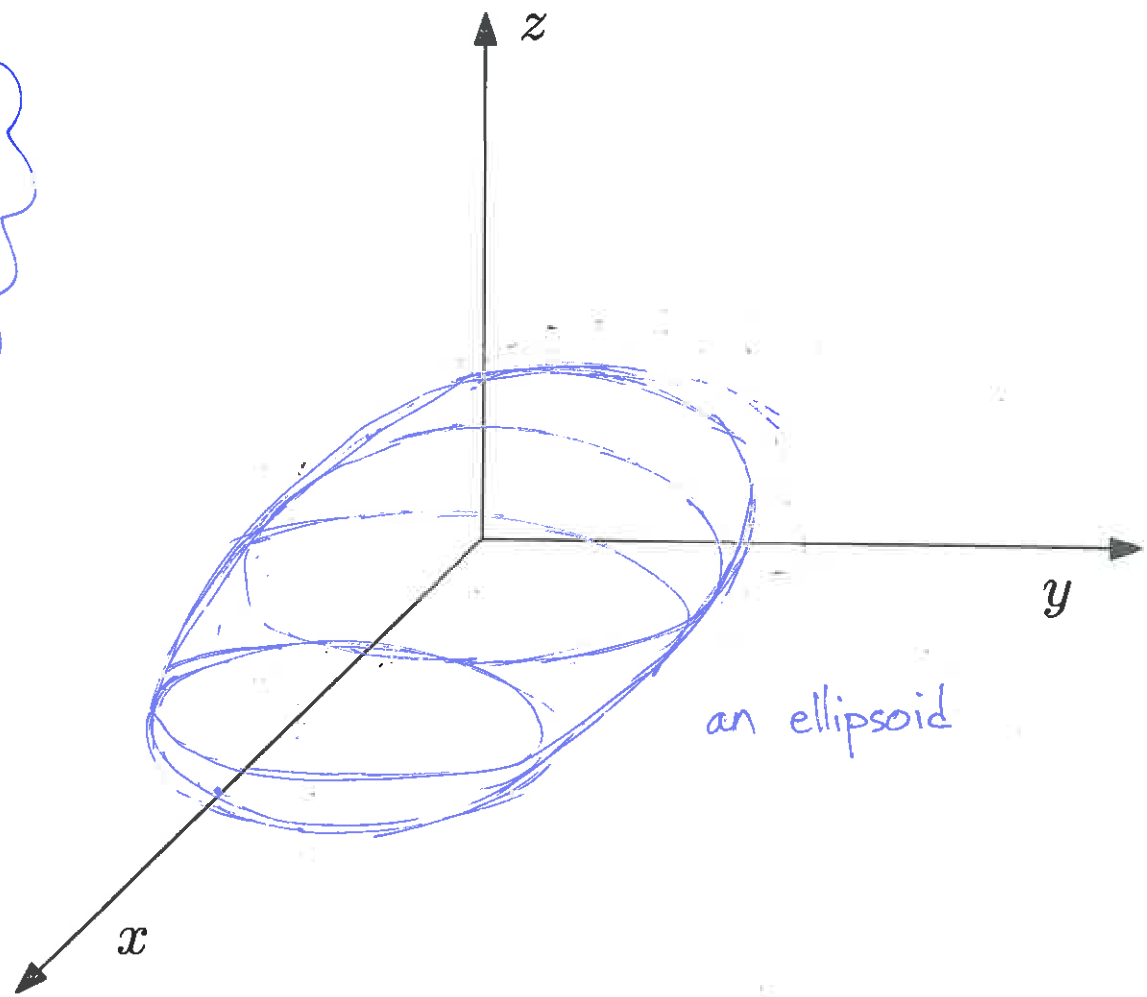
sphere of radius 2,

transformation in z-direction

points: $(\pm 2, 0, 0)$

$(0, \pm 2, 0)$

$(0, 0, \pm 1)$



Example: Describe and sketch the set satisfying $z^2 = x^2 + y^2$ (i.e. $J = 0$).

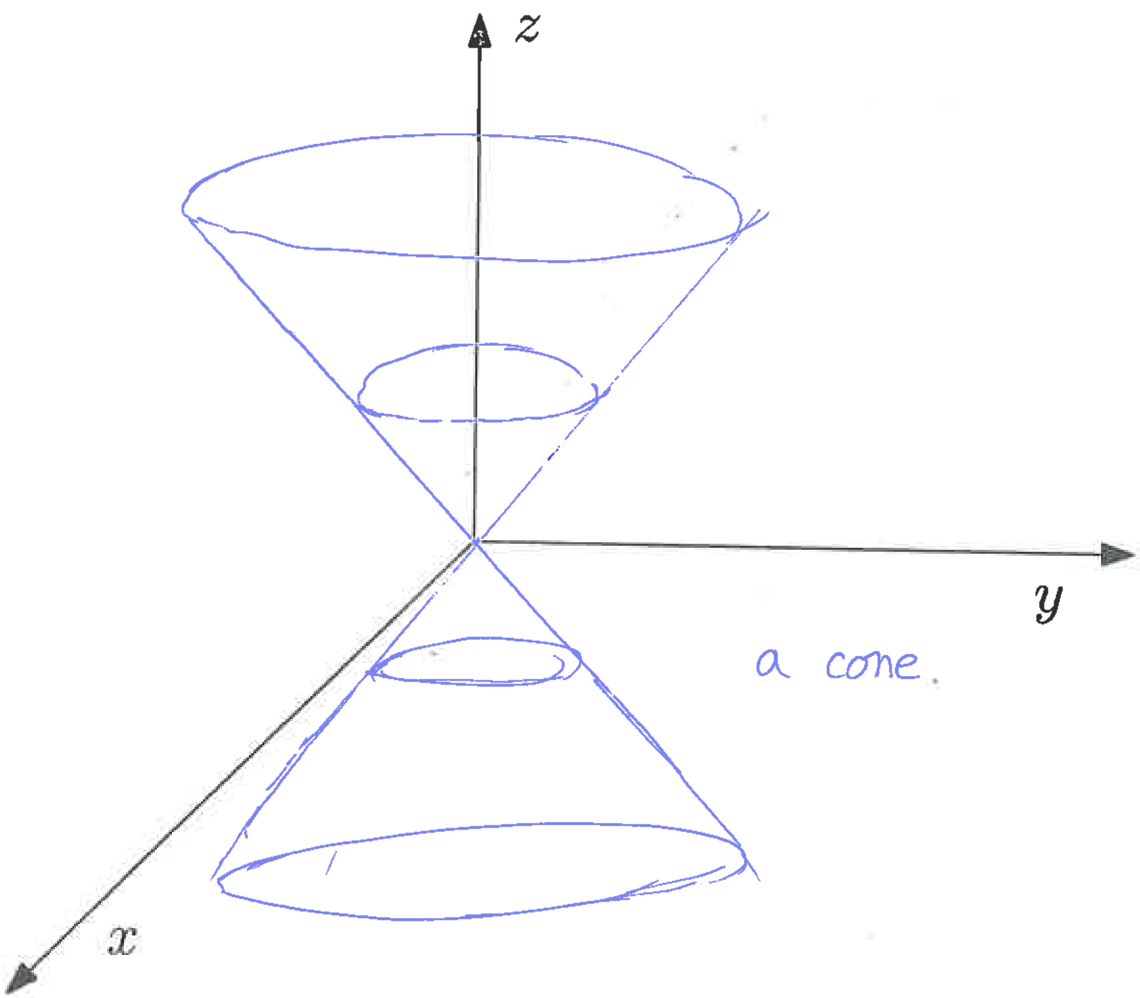
cross-sections:

$x=0$: $z^2 = y^2$

$z = y$ or $z = -y$.

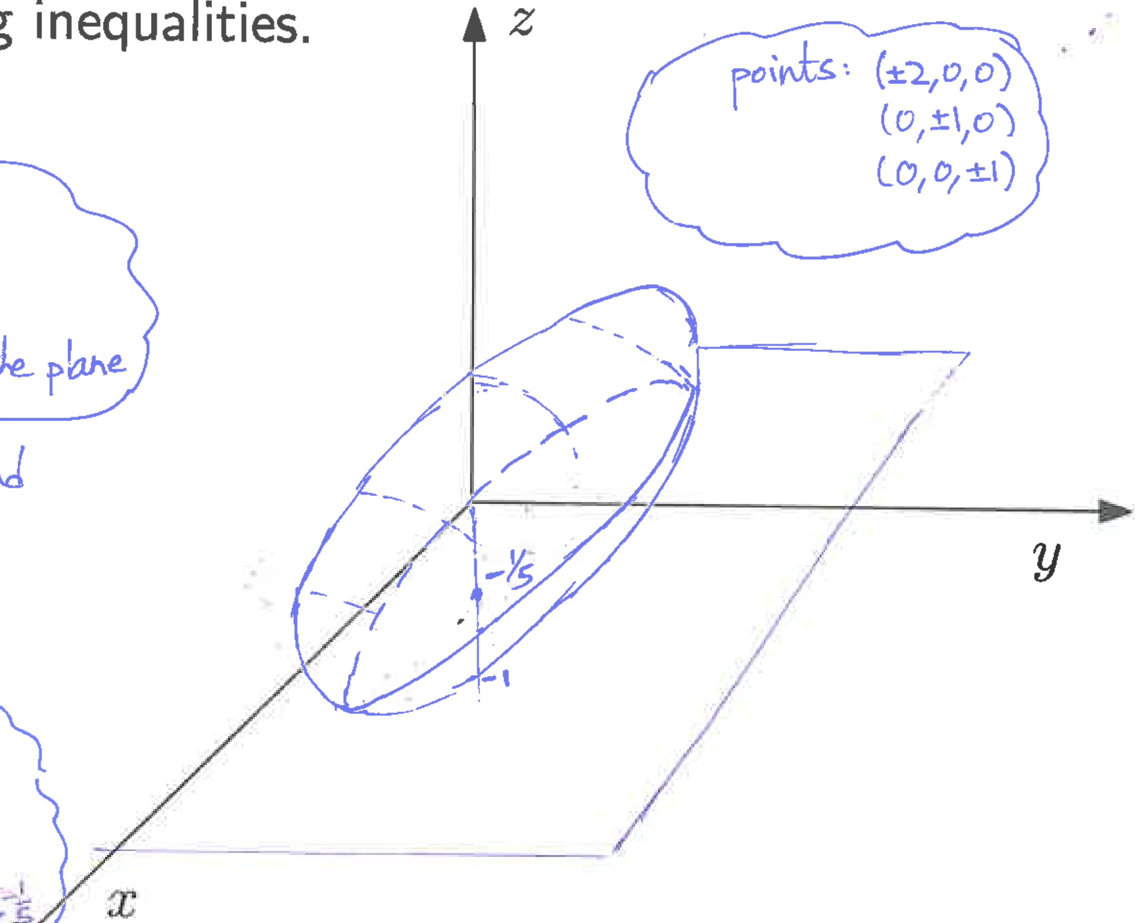
$z = C$: $C^2 = x^2 + y^2$

circle of radius C



Regions bounded by surfaces and inequalities

Example: Describe and sketch the larger region bounded by $\frac{1}{4}x^2 + y^2 + z^2 = 1$ (ellipsoid) and $z = -\frac{1}{5}$ (plane), and describe it using inequalities.



points: $(\pm 2,0,0)$, $(0,\pm 1,0)$, $(0,0,\pm 1)$

There are 2 regions bounded by the ellipsoid and the plane: inside the ellipsoid and above/below the plane

The larger region is $\underbrace{\text{inside the ellipsoid}}$ and $\underbrace{\text{above the plane}}$.

$z > -\frac{1}{5}$. $\frac{1}{4}x^2 + y^2 + z^2 < 1$.

Another way to tell the signs of the inequalities ($<$ or $>$): we know from the sketch that $(0,0,0)$ is in our region. At $(0,0,0)$: $\frac{1}{4}x^2+y^2+z^2 = 0 < 1$. and $z = 0 > -\frac{1}{5}$

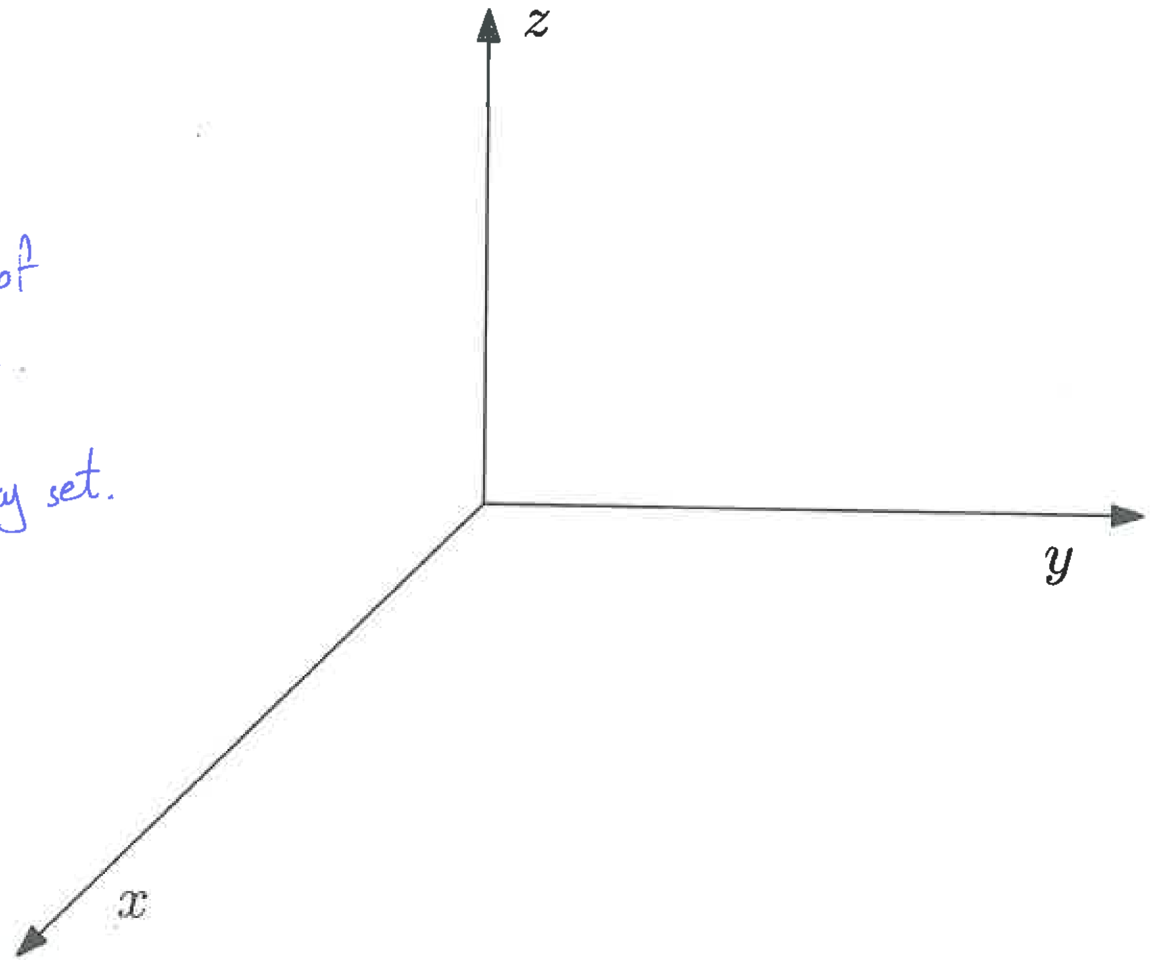
Degenerate cases

Example: Describe and sketch the set satisfying $x^2 + y^2 + z^2 + 1 = 0$.

$x^2+y^2+z^2=-1$

This is never satisfied, because the left hand side is a sum of squares, so it cannot be negative.

So this equation describes the empty set.



Example: Describe and sketch the set in $\mathbb{R}^3$ satisfying $x^2 - y^2 = 0$.

$(x+y)(x-y)=0$

$x+y=0$ or $x-y=0$

- union of two planes through the origin, with normals $(1,1,0)$ and $(1,-1,0)$.