Ex: $B = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 \\ -4 & 1 & -3 & 2 & 1 \\ -2 & -1 & 0 & 1 & 1 \\ -3 & -1 & -3 & 4 & 1 \\ -8 & -2 & -7 & 5 & 4 \end{pmatrix}$ $\quad \chi_B(x) = -(x-2)^5$

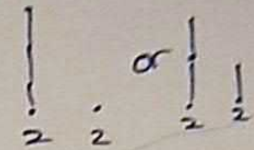
Step 1: number of vectors in level i of eigenstring diagram

$= \dim \mathrm{Nul}(B-2I)^i - \dim \mathrm{Nul}(B-2I)^{i+1}$

$$B-2I = \begin{pmatrix} -1 & 0 & -1 & 1 & 0 \\ -4 & -1 & -3 & 2 & 1 \\ -2 & -1 & -2 & 1 & 1 \\ -3 & -1 & -3 & 2 & 1 \\ -8 & -2 & -7 & 5 & 2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} \boxed{-1} & 0 & -1 & 1 & 0 \\ 0 & \boxed{-1} & 0 & -1 & 1 \\ 0 & 0 & \boxed{1} & -1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$\therefore \dim \mathrm{Nul}(B-2I) = 2$

$\therefore$ possibilities =

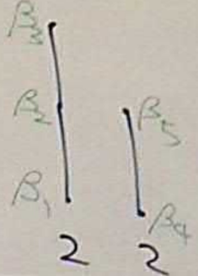


$$(B-2I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \end{pmatrix}$$

clearly only one pivot.

$\therefore \dim \mathrm{Nul}(B-2I)^2 = 5-1 = 4$

$\therefore$ diagram



(check that $\dim \mathrm{Nul}(B-2I)^3 = \dim \mathrm{Nul}(B-2I)^4$)

Step 2: Find longest eigenstrings i.e. find β_3.

· Basis $\{\alpha_i\}$ of $\mathrm{Null}(B-2I)^3$: $\{\alpha_1 = e_1, \alpha_2 = e_2, \alpha_3 = e_3, \alpha_4 = e_4, \alpha_5 = e_5\}$

$\because$ B has only one eigenvalue, so all of $\mathbb{C}^5$ is in K_2.

· Consider $\{(B-2I)^2(\alpha_i)\}$ = columns of $(B-2I)^2$.

· Take linearly independent subset

e.g. columns with pivot $\Rightarrow$ column 1 $\Rightarrow \beta_3 = e_1$

<u>or</u> we need linearly independent subset of one vector,

i.e. one nonzero vector $\therefore \beta_3$ can be any e_i such that

$(B-2I)^2 e_i \neq \vec{0}$. e.g. $\beta_3 = e_1$ or e_3 or e_4.

if choose e_1: $\beta_3 = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$, $\beta_2 = (B-2I)\beta_3 = \begin{pmatrix}-1\\-4\\-2\\-3\\-8\end{pmatrix}$, $\beta_1 = (B-2I)\beta_2 = (B-2I)^2\beta_3 = \begin{pmatrix}0\\0\\-1\\-1\\-1\end{pmatrix}$.

Step 3: Find the next maximal eigenstring length, i.e. the next level with new eigenstring tops – call this m'.

(in example B, $m'=2$)

Step 4: Find the length m' eigenstring tops $\alpha_1', \alpha_2', \ldots$

Repeat steps 3, 4 as necessary

⋮

$\alpha_i' \in \mathrm{Ker}(\sigma-\lambda_c)^{m'}$

BUT: α_i' cannot be in the span of the existing eigenstrings.

· new eigenstring bottoms must be linearly independent, and independent from existing eigenstring bottoms

---

· Let $\{\beta_{i_1}, \beta_{i_2}, \cdots\}$ be the vectors in the bottom m' levels of existing eigenstrings.

Extend to a basis of spanning set is ok $\mathrm{Ker}(\sigma-\lambda_c)^{m'}$:

$\{\beta_{i_1}, \beta_{i_2}, \cdots, \alpha_1', \alpha_2', \ldots\}$.

· Consider $\{(\sigma-\lambda_c)^{m'-1}(\beta_{i_1}), (\sigma-\lambda_c)^{m'-1}(\beta_{i_2}), \cdots, (\sigma-\lambda_c)^{m'-1}(\alpha_1'), \ldots\}$

· Take a linearly independent subset containing existing eigenstring bottoms, and some $(\sigma-\lambda_c)^{m'-1}(\alpha_i')$

– these α_i' are the new eigenstring tops.

Ex: Continue with B (Ex. 8.3.7), $m' = 2$

Bottom m' levels of existing eigenstrings: $\{\beta_1, \beta_2\}$

A basis of $\mathrm{Nul}(B-2I)^2$:

from matrix earlier: $-x_1 - x_3 + x_4 = 0$

$$
\begin{aligned}
x_1 &= -x_3 + x_4 \\
x_2 &= x_2 \\
x_3 &= \quad x_3 \\
x_4 &= \qquad x_4 \\
x_5 &= \qquad\quad x_5
\end{aligned}
\quad \longmapsto \quad
\left\{
\begin{array}{cccc}
\alpha_1' & \alpha_2' & \alpha_3' & \alpha_4' \\
\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, &
\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, &
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, &
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\end{array}
\right\}
$$