SUBALGEBRAS OF SOLOMON'S DESCENT ALGEBRA BASED ON ALTERNATING RUNS

MATTHIEU JOSUAT-VERGÈS AND C.Y. AMY PANG

Abstract. The number of alternating runs is a natural permutation statistic. We show it can be used to define some commutative subalgebras of the symmetric group algebra, and more precisely of the descent algebra. The Eulerian peak algebras naturally appear as subalgebras of our run algebras. We also calculate the orthogonal idempotents for run algebras in terms of noncommutative symmetric functions.

1. Introduction

An *alternating run* of a permutation $\sigma \in \mathfrak{S}_n$ is a maximal monotone sequence of consecutive elements in $\sigma_1 \dots \sigma_n$. For example, the alternating runs of $\sigma = 14523687$ are 145, 52, 2368, and 87. We denote by $\mathrm{run}(\sigma)$ the number of alternating runs of σ. The enumeration of permutations refined by the number of alternating runs have been the subject of various previous works, beginning with André [3] as early as the late 19th century, then Carlitz in the 70's [9, 10, 11, 12]. See also [6, 8, 13, 15, 20, 21, 30] for more recent references. It is worth mentioning that the work [15] was motivated by computations in quantum field theory. Also, alternating runs play a role in some algorithms such as pattern matching [7].

However, we take here a different perspective, our goal being to define and study various algebras based on this notion of alternating runs. Such an algebra first arose in Doyle and Rockmore's study of "ruffle" card-shuffling: [14, Section 5.5] shows that the elements

$$W_k = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{run}(\sigma)=k}} \sigma,$$

for $1 \le k \le n-1$, linearly span a commutative subalgebra of $\mathbb{Z}[\mathfrak{S}_n]$, called the *reduced turning algebra*. Doyle and Rockmore also considered the number of alternating runs when we add an initial 0 in front of the permutation and obtain in this way another algebra, called the *turning algebra* [14, Section 5.4]. We will in particular obtain new proofs of these results in the present article.

In fact, it is easily seen that $\mathrm{run}(\sigma)$ only depends on the descent set of σ, so that the elements W_k lie in the descent algebra $\mathcal{D}_n \subset \mathbb{Z}[\mathfrak{S}_n]$ (see [27] or the next section for a definition). This widely studied algebra provides other examples of combinatorial statistics such that there is an algebra linearly spanned by sums of permutations having

---

2000 *Mathematics Subject Classification.* 05A18, 05E18, 11B68, 20F55.

The first author was supported by the Laboratoire International Franco-Québecois de Recherche en Combinatoire (LIRCO), and by ANR CARMA (ANR-12-BS01-0017). The second author was supported by a CRM-ISM postdoctoral fellowship, and an NSERC individual research grant of François Bergeron.

the same value. Consider for example the descent statistic, then the n elements

$$E_k = \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \operatorname{des}(\sigma)=k}} \sigma,$$

for $0 \leq k \leq n-1$ linearly span a subalgebra of $\mathcal{D}_n$ called the *Eulerian algebra* $\mathcal{E}_n$ (see [16, 19]). But those being particularly relevant here are the Eulerian peak algebra and its left peak analogue [2, 23, 24, 25], because peaks and alternating runs are tightly connected.

Our main goal is to introduce a new algebra based on alternating runs, as the linear span of the $2n-2$ elements:

$$W_k^+ := \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \operatorname{run}(\sigma)=k,\, \sigma_1<\sigma_2}} \sigma, \qquad W_k^- := \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \operatorname{run}(\sigma)=k,\, \sigma_1>\sigma_2}} \sigma,$$

for $1 \leq k \leq n-1$. This algebra is noncommutative, but it contains several commutative subalgebras, in particular the two considered by Doyle and Rockmore and the two Eulerian peak subalgebras.

We give in Section 3 a bijective proof of its existence, based on Atkinson's proof of the existence of the descent algebra [4]. Then we show that once we know the existence of this $2n-2$ dimensional algebra, the existence of the various subalgebras easily follows.

Then in Section 4, we provide a Hopf algebraic proof of the existence of run algebras, using the formalism of noncommutative symmetric functions [16]. Indeed, whereas each homogeneous component has an internal product which is essentially that of the descent algebra, the external Hopf structure permits to build new bases. Also, this algebraic approach easily gives the commutativity of the subalgebras, a property that seems very difficult to get by bijective proofs.

We go on in Section 5 by computing primitive orthogonal idempotents for each of the various run algebras, as generating functions in noncommutative symmetric functions. Part of these calculations appear to be equivalent to Petersen's computation of peak idempotents via order polynomials of enriched P-partitions [24]. We also give some properties of the related characters of the symmetric groups, in the same way as idempotents of the Eulerian subalgebra are related with Foulkes characters [17].

Acknowledgements. We thank Jean-Yves Thibon for suggesting an exploration of algebras based on alternating runs. We thank Kyle Petersen for helpful discussions, and for pointing out to us the reference by Doyle and Rockmore. We are also grateful to François Bergeron, Christophe Hohlweg and Franco Saliola for their help and comments. Sage computer software [29] was very useful throughout this project. We also thank the Laboratoire International Franco-Québécois de Recherche en Combinatoire (LIRCO), who supported the French author in his visits to Montréal.

2. Preliminaries

2.1. **Notations.** A permutation $\sigma \in \mathfrak{S}_n$ is denoted as the sequence $\sigma_1, \ldots, \sigma_n$ where $\sigma_i = \sigma(i)$. We denote by $\operatorname{Des}(\sigma)$ the set of *descents* of σ, i.e. integers i such that $1 \leq i \leq n-1$ and $\sigma(i) > \sigma(i+1)$. Let also $\operatorname{des}(\sigma) = \#\operatorname{Des}(\sigma)$. Let $[n-1]$ denote $\{1, \ldots, n-1\}$. The *descent class* $\mathfrak{D}_I$ for $I \subset [n-1]$ is the set of $\sigma \in \mathfrak{S}_n$ with $\operatorname{Des}(\sigma) = I$.

A *partition* λ of n is a decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ called *parts*, such that $\sum_{i=1}^k \lambda_i = n$. We also denote by $|\lambda|$ the sum of parts of λ. The integer k is the *length* of λ, and is denoted $\ell(\lambda)$. Also, we denote $\ell_e(\lambda)$ (respectively, $\ell_o(\lambda)$) the even length (respectively, the odd length) of λ, i.e. the number of even parts (respectively, odd parts). Write z_λ for $1^{i_1} i_1! 2^{i_2} i_2! \ldots$, where i_r is the number of parts of size r in λ. We denote $\lambda \vdash n$ to say that λ is a partition of n.

A *composition* of n is a sequence $I = (i_1, \ldots, i_r)$ of positive integers such that $\sum_{j=1}^r i_j = n$. We also denote by $|I|$ the sum of parts of I. The integer r is called the *length* of I, and is denoted $\ell(I)$. We write $I \vDash n$ to say that I is a composition of n.

In a partition or a composition, we write 1^n to mean that 1 is repeated n times.

The *descent set* of a composition $I = (i_1, \ldots, i_r)$, denoted $\mathrm{Des}(I)$, is

$$\left\{ i_1, i_1 + i_2, \ldots, \sum_{j=1}^{r-1} i_j \right\}.$$

The map $I \mapsto \mathrm{Des}(I)$ is a bijection from compositions of n to subsets of $[n-1]$. When $I, J \vDash n$, we say that J is a refinement of I, denoted $I \preccurlyeq J$, if $\mathrm{Des}(I) \subset \mathrm{Des}(J)$. When we write $I \preccurlyeq J$, we always understand that $|I| = |J|$.

2.2. **The descent algebra.** For $I \vDash n$, let

$$R_I = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{Des}(\sigma) = \mathrm{Des}(I)}} \sigma, \qquad S^I = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{Des}(\sigma) \subset \mathrm{Des}(I)}} \sigma.$$

Note that by an immediate inclusion-exclusion, we have:

$$S^I = \sum_{J \preccurlyeq I} R_J, \qquad R_I = \sum_{J \preccurlyeq I} (-1)^{\ell(J) - \ell(I)} S^J.$$

The linear span of the elements R_I (or equivalently, S^I) for $I \vDash n$ is denoted $\mathcal{D}_n$ and called the *descent algebra*. It was indeed shown by Solomon [27] that $\mathcal{D}_n$ is a subalgebra of the group algebra $\mathbb{Z}[\mathfrak{S}_n]$.

2.3. **The Eulerian peak algebras.** Two previously identified subalgebras of the descent algebra will play a key role in the analysis of our new subalgebras. These are based on peaks:

Definition 2.1. A *peak* of a permutation $\sigma \in \mathfrak{S}_n$ is an integer i with $2 \leq i \leq n-1$ and $\sigma(i-1) < \sigma(i) > \sigma(i+1)$. We denote by $\mathrm{pk}(\sigma)$ the number of peaks of σ.

Definition 2.2. A *left peak* of a permutation $\sigma \in \mathfrak{S}_n$ is an integer i with $1 \leq i \leq n-1$ and $\sigma(i-1) < \sigma(i) > \sigma(i+1)$ (with the convention $\sigma(0) = 0$). We denote by $\mathrm{pk}^\circ(\sigma)$ the number of left peaks of σ.

Proposition 2.3. [25, Sec. 9] *The elements*

$$P_k = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{pk}(\sigma) = k}} \sigma$$

for $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ *span the* Eulerian peak algebra $\mathcal{P}_n$, *a commutative subalgebra of* $\mathcal{D}_n$. *Note that* $\dim(\mathcal{P}_n) = \lceil \frac{n}{2} \rceil$.

The Eulerian peak algebra first appeared in Schocker's analysis [25] of a larger, noncommutative, peak algebra, spanned by sums of permutations with the same peak positions. This work determined the idempotents of $\mathcal{P}_n$, which we will recover in the present Theorem 5.3.

Aguiar, Bergeron and Nyman [1] also studied this larger peak algebra, by viewing it as an ideal within the image of projecting the type B descent algebra to $\mathcal{D}_n$. This viewpoint uncovered a left peak variant of $\mathcal{P}_n$:

Proposition 2.4. [1, Sec. 6.3] *The elements*

$$P_k^\circ = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{pk}^\circ(\sigma) = k}} \sigma$$

for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ *span the* Eulerian left peak algebra $\mathcal{P}_n^\circ$, *a commutative subalgebra of* $\mathcal{D}_n$. *Note that* $\dim(\mathcal{P}_n^\circ) = \lfloor \frac{n}{2} \rfloor + 1$.

Petersen [24] computed the idempotents of both $\mathcal{P}_n$ and $\mathcal{P}_n^\circ$, in terms of enriched P-partitions. Our re-computation in the present Theorem 5.3 is uncannily similar, despite being in the different language of noncommutative symmetric functions.

Both $\mathcal{P}_n$ and $\mathcal{P}_n^\circ$ are subalgebras of our new algebra $\mathcal{W}_n^\pm$.

2.4. **Noncommutative symmetric functions.** To find and express the idempotents of the run algebras, it is useful to view the descent algebra from another perspective, that contains extra algebraic structure.

Let **Sym** denote the Hopf algebra of noncommutative symmetric functions, which is the direct sum of $\mathbf{Sym}_n$, its degree n homogeneous components (see [16] as a general reference on this subject). Its product $\star$ and coproduct Δ are defined below. It will be convenient to also work in $\hat{\mathbf{Sym}}$, the completion, which allows infinite sums of the elements of **Sym**, of unbounded degree.

As a vector space, we have the identification $\mathbf{Sym}_n = \mathcal{D}_n$. Under this identification, the image of S^I are the *complete noncommutative symmetric functions*, and the image of R_I are the *ribbon noncommutative symmetric functions*. We write S^I, R_I for these images also.

The product on $\mathcal{D}_n$ is extended to **Sym** by the rule $xy = 0$ if $x \in \mathbf{Sym}_m$, $y \in \mathbf{Sym}_n$, and $m \neq n$. We call it the *internal product* of **Sym**, to distinguish from the external product. However, it is to be noted that the usual internal product of noncommutative symmetric function is taken in the opposite order (see [16]).

Sym also admits an *external product* $\star$, for which degree is additive: $\mathbf{Sym}_n \star \mathbf{Sym}_{n'} \subseteq \mathbf{Sym}_{n+n'}$. In the complete basis, this is given by concatenation of compositions:

$$S^{(i_1,\dots,i_{\ell(I)})} \star S^{(j_1,\dots,j_{\ell(J)})} = S^{(i_1,\dots,i_{\ell(I)},j_1,\dots,j_{\ell(J)})}.$$

Hence, for any composition I, we have $S^I = S_{i_1} \star \cdots \star S_{i_r}$, so **Sym** is a free algebra for the external product, generated by the S_i, the complete noncommutative symmetric functions indexed by compositions with a single part.

In the ribbon basis, the external product results in "near-concatenation":

$$R_{(i_1,\dots,i_{\ell(I)})} \star R_{(j_1,\dots,j_{\ell(J)})} = R_{(i_1,\dots,i_{\ell(I)},j_1,\dots,j_{\ell(J)})} + R_{(i_1,\dots,i_{\ell(I)-1},i_{\ell(I)}+j_1,j_2,\dots,j_{\ell(J)})}.$$

For example, $S^{(2,3)} \star S^{(5,1,1)} = S^{(2,3,5,1,1)}$ and $R_{(2,3)} \star R_{(5,1,1)} = R_{(2,3,5,1,1)} + R_{(2,8,1,1)}$. We write $F^{\star k}$ for powers under the external product, to distinguish from the internal product power F^k.

The two products are related by the *splitting formula* of [16, Prop. 5.2]; this involves a third algebraic operation, the coproduct. This is the (external) algebra homomorphism $\Delta : \mathbf{Sym} \to \mathbf{Sym} \otimes \mathbf{Sym}$ with $\Delta(S_n) = \sum_{i=0}^{n} S_i \otimes S_{n-i}$. (The convention is that $S_0 = 1$.) It follows that $\sum_{n\geq 0} R_n$ and $\sum_{n\geq 0} R_{(1^n)}$ are *grouplike* - that is, they satisfy $\Delta(G) = G \otimes G$. It is a standard fact of Hopf algebras that (external) products of grouplike elements are also grouplike. For grouplike G, the splitting formula simplifies to:

$$G(F_1 \star \cdots \star F_k) = (GF_1) \star \cdots \star (GF_k).$$

This simplified version is the only case of the splitting formula that we will need.

Remark 2.5. Since we are interested in subalgebras of the descent algebra, it is natural to use a special symbol for the external product of (noncommutative) symmetric functions, the product without a special symbol meaning the internal product. However this differs from the usual notation.

2.5. **Symmetric functions.** Let Sym denote the algebra of symmetric functions and Sym_n its degree n homogeneous component. We follow the notation of [22] so that h_λ, e_λ, p_λ, s_λ are respectively the homogeneous, elementary, power sum, and Schur symmetric functions associated with a partition λ. If $I = (i_1, \ldots, i_{\ell(I)})$ is a composition, there is a unique ribbon (i.e. a skew shape which is connected and contains no 2×2 square) whose row lengths from top to bottom in French notation are $i_1, \ldots, i_{\ell(I)}$, and the skew Schur function associated with it is denoted r_I. To be coherent with the notation on noncommutative symmetric functions, the usual product of Sym is denoted $\star$, whereas the *internal product* on Sym_n has no particular symbol.

There is a well-known map $\Gamma : \mathbf{Sym} \to Sym$, defined by:

$$\Gamma(R_I) = r_I.$$

In particular, we have:

$$\Gamma(S_n) = h_n, \qquad \Gamma(R_{(1^n)}) = e_n.$$

This map is an algebra morphism for both the internal and external product of symmetric functions:

$$\Gamma(xy) = \Gamma(x)\Gamma(y), \qquad \Gamma(x \star y) = \Gamma(x) \star \Gamma(y).$$

One fact about Sym_n that will be particularly useful for us is that the rescaled power sums $\frac{1}{z_\lambda} p_\lambda$ form a basis of orthogonal idempotents (under the internal product). It follows that, if $\Lambda_1, \Lambda_2, \ldots, \Lambda_K$ are disjoint subsets of partitions of n, then $\sum_{\lambda \in \Lambda_k} \frac{1}{z_\lambda} p_\lambda$ form a coarser set of orthogonal idempotents. In fact, every subalgebra of Sym_n has a basis of orthogonal idempotents of this form, and identifying the Λ_k is particularly interesting if we view Sym_n as the character ring of $\mathfrak{S}_n$ [28, Sec. 7.18]. (In this light, Γ is the homomorphism of Solomon from the descent algebra of a finite Coxeter group to its character ring [27].) For example, the idempotents of $\Gamma(\mathcal{E}_n)$, the symmetric function image of the Eulerian algebra, are $\sum_{\ell(\lambda)=k} \frac{1}{z_\lambda} p_\lambda$ [16, Sec. 5.3], corresponding to the indicator functions on permutations with k cycles. By definition, $\sum_{\ell(I)=k} r_I$ is also a basis of $\Gamma(\mathcal{E}_n)$. So, writing χ_I for the character corresponding to r_I, this slickly recovers a result of Foulkes [17, Th. 3.1], [18]:

$$\left\{ \sum_{\ell(I)=k} \chi_I : 1 \leq k \leq n \right\}$$

gives a basis for the subspace of functions $\mathfrak{S}_n \to \mathbb{R}$ depending only on the number of cycles. Applying this idea to our new commutative run algebras yields similar characters that depend only on the numbers of odd cycles and of even cycles, see Corollary 5.4.

3. Bijective proof for the existence of run algebras

Definition 3.1. Let

$$\begin{aligned}
\mathfrak{W}_k^+ &:= \{\sigma \in \mathfrak{S}_n \,:\, \mathrm{run}(\sigma) = k \text{ and } \sigma_1 < \sigma_2\}, && (1 \leq k \leq n-1),\\
\mathfrak{W}_k^- &:= \{\sigma \in \mathfrak{S}_n \,:\, \mathrm{run}(\sigma) = k \text{ and } \sigma_1 > \sigma_2\}, && (1 \leq k \leq n-1),\\
\mathfrak{W}_k &:= \{\sigma \in \mathfrak{S}_n \,:\, \mathrm{run}(\sigma) = k\} = \mathfrak{W}_k^+ \cup \mathfrak{W}_k^-, && (1 \leq k \leq n-1),\\
\mathfrak{W}_k^\circ &:= \{\sigma \in \mathfrak{S}_n \,:\, \mathrm{run}^\circ(\sigma) = k\} = \mathfrak{W}_k^+ \cup \mathfrak{W}_{k-1}^-, && (1 \leq k \leq n).
\end{aligned}$$

The given range for k indicates when these sets are nonempty. Note that in the definition of $\mathfrak{W}_k^\circ$ we need to specify that $\mathfrak{W}_n^+ = \mathfrak{W}_0^- = \emptyset$. And let

$$W_k^- := \sum_{\sigma \in \mathfrak{W}_k^-} \sigma, \qquad W_k^+ := \sum_{\sigma \in \mathfrak{W}_k^+} \sigma, \qquad W_k := \sum_{\sigma \in \mathfrak{W}_k} \sigma, \qquad W_k^\circ := \sum_{\sigma \in \mathfrak{W}_k^\circ} \sigma.$$

Note that $W_k = W_k^- + W_k^+$ and $W_k^\circ = W_{k-1}^- + W_k^+$. Then we define subspaces of the descent algebra $\mathcal{D}_n$ as follows:

$$\begin{aligned}
\mathcal{W}^\pm &:= \mathrm{Vect}(W_k^+, W_k^- \,:\, 1 \leq k \leq n-1\},\\
\mathcal{W} &:= \mathrm{Vect}(W_k \,:\, 1 \leq k \leq n-1\},\\
\mathcal{W}^\circ &:= \mathrm{Vect}(W_k^\circ \,:\, 1 \leq k \leq n\},\\
\mathcal{C} &:= \mathrm{Vect}(W_{2k} \,:\, 1 \leq k \leq \lfloor \tfrac{n-1}{2} \rfloor\} \oplus \mathrm{Vect}(W_{2k-1}^+, W_{2k-1}^- \,:\, 1 \leq k \leq \lfloor \tfrac{n}{2} \rfloor\}.
\end{aligned}$$

Note that in each definition, the given generators are linearly independent, since they are sums of permutations over disjoint nonempty subsets. It follows that:

$$\dim(\mathcal{W}^\pm) = 2n-2, \qquad \dim(\mathcal{W}) = n-1, \qquad \dim(\mathcal{W}^\circ) = n, \qquad \dim(\mathcal{C}) = \lfloor \tfrac{3n-2}{2} \rfloor.$$

Theorem 3.2. *The space $\mathcal{W}^\pm$ is a subalgebra of $\mathcal{D}_n$, and the spaces $\mathcal{W}$, $\mathcal{W}^\circ$, $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{P}^\circ$ are subalgebras of $\mathcal{W}^\pm$.*

Note that $\mathcal{W}$ and $\mathcal{W}^\circ$ are respectively the reduced turning algebra and the turning algebra of Doyle and Rockmore [14]. And $\mathcal{P}$ and $\mathcal{P}^\circ$ are the Eulerian peak and Eulerian left peak algebra, as described in Subsection 2.3.

The rest of this section contains the proof. In Subsections 3.1 to 3.3, we prove the result for $\mathcal{W}^\pm$. Then, in Subsection 3.4, we show how the other parts of the theorem follow.

3.1. **Alternating runs and the left weak order.** This subsection contains a preliminary result about the sets $\mathfrak{W}_j^+$ and $\mathfrak{W}_j^-$. Let s_i denote the simple transposition $(i, i+1) \in \mathfrak{S}_n$. Let us recall that the left weak order on $\mathfrak{S}_n$ is defined by the covering relations $\sigma \lessdot s_i\sigma$ if i is to the left of $i+1$ in σ, for all $\sigma \in \mathfrak{S}_n$ and $1 \leq i \leq n-1$. It is known that descent classes are intervals for this order [5]. In particular, each descent class is a connected subset in $\mathfrak{S}_n$ (seen as an undirected graph whose edges are given by the cover relation of the left weak order).

Lemma 3.3. *Let $I, J \subset [n-1]$ with $I \neq J$. The descent classes $\mathfrak{D}_I$ and $\mathfrak{D}_J$ are connected to each other (in the sense that there are $u \in \mathfrak{D}_I$, $v \in \mathfrak{D}_J$ such that $u \lessdot v$ or $v \gtrdot u$) if and only if one of the following conditions is true:*

- *$I \subset J$ and $\#I = \#J - 1$, or*
- *$J \subset I$ and $\#J = \#I - 1$.*

Proof. We compare the descents sets of σ and $s_i\sigma$. If i and $i+1$ are not adjacent in σ, $\mathrm{Des}(\sigma) = \mathrm{Des}(s_i\sigma)$. Otherwise, either $\mathrm{Des}(\sigma) = \mathrm{Des}(s_i\sigma) \cup \{i\}$ or $\mathrm{Des}(s_i\sigma) = \mathrm{Des}(\sigma) \cup \{i\}$. It follows that if $\mathfrak{D}_I$ and $\mathfrak{D}_J$ are connected, I and J satisfy the given conditions.

As for the other implication, suppose I and J satisfy the given conditions, for example $I = J \cup \{i\}$. It is easy to find $\sigma \in \mathcal{D}_I$ such that $\sigma_i = \sigma_{i+1} + 1$. Then the two permutations σ and $s_i\sigma \in \mathcal{D}_J$ show that $\mathfrak{D}_I$ and $\mathfrak{D}_J$ are connected. $\square$

Lemma 3.4. *Each set $\mathfrak{W}_j^+$ or $\mathfrak{W}_j^-$ is connected in the left weak order.*

Proof. By symmetry, we only need to consider the case of $\mathfrak{W}_j^+$. This set is a union of descent classes. Since each descent class is connected, it remains to show that these descent classes are connected to each other, using the previous lemma.

The descent classes included in $\mathfrak{W}_j^+$ are those indexed by subsets $X \subset \{1, \ldots, n-1\}$ such that:

- X is a union of $\lfloor j/2 \rfloor$ intervals and no less (i.e. the number of descending runs),
- $1 \notin X$ (since the elements in $\mathfrak{W}_j^+$ begin with an ascending run),
- $n-1 \in X$ if and only if j is even (since the first run is ascending, the last run is descending if and only if j is even).

By adding or removing elements in these subsets as in the previous lemma, (and staying in the same class of subsets) we can reach a particular chosen subset, for example $\{2, 4, 6, \ldots, 2\lfloor j/2 \rfloor\}$ if j is odd and $\{2, 4, 6, \ldots, 2\lfloor j/2 \rfloor - 1, n-1\}$ if j is even. This fact is rather straightforward so we omit the full formal proof. Connectedness easily follows. $\square$

3.2. **The scheme of proof.** To prove that $\mathcal{W}^\pm$ is an algebra, we show $W_j^+ W_k^+ \in \mathcal{W}^\pm$. The other products, i.e. $W_j^+ W_k^-$, $W_j^- W_k^+$, and $W_j^- W_k^-$, can be done similarly (or more formally, one can use symmetry properties described in Subsection 3.4).

For each permutation $\sigma \in \mathfrak{S}_n$, let

$$\mathfrak{C}_{j,k}^\sigma = \{(\alpha, \beta) \in \mathfrak{W}_j^+ \times \mathfrak{W}_k^+ \,:\, \alpha\beta = \sigma\}.$$

Then $c_{j,k}^\sigma = \#\mathfrak{C}_{j,k}^\sigma$ is the coefficient of σ in the product $W_j^+ W_k^+$. Our goal is to prove that $c_{j,k}^\sigma = c_{j,k}^\tau$ if σ, τ are both in $\mathfrak{W}_m^+$ or both in $\mathfrak{W}_m^-$.

We only consider the case where σ and τ are neighbours in the left weak order, i.e., either $\sigma \lessdot \tau$ or $\tau \lessdot \sigma$. Indeed, using the connectedness shown in Lemma 3.4, this implies the general case. So we assume $s_i\sigma = \tau$ for some i.

Let us first consider the case where i and $i+1$ are not adjacent in $\sigma = \sigma_1 \dots \sigma_n$ (and hence in $\tau = s_i\sigma$). It implies that σ and τ are in the same descent class. But since $W_j^+ W_k^+$ is in the descent algebra (as W_j^+ and W_k^+ both are), the coefficients of σ and τ are equal, i.e. $c_{j,k}^{\sigma} = c_{j,k}^{\tau}$ as we wanted.

It remains to consider the case where i and $i+1$ are adjacent in σ (and hence in τ). We will give a bijection $\Phi : \mathfrak{C}_{j,k}^{\sigma} \to \mathfrak{C}_{j,k}^{\tau}$. This bijection is rather long to define as we need to consider many cases, and it is described in the next subsection. Note that σ and τ have symmetric roles here, since we did not specify which of $\sigma \lessdot \tau$ or $\tau \lessdot \sigma$ is true. So by exchanging the role of σ and τ, the definition in the next subsection also describes another map $\Psi : \mathfrak{C}_{j,k}^{\tau} \to \mathfrak{C}_{j,k}^{\sigma}$. At each step of the definition, we can check that Ψ is in the inverse map of Φ, so that these maps are indeed bijections.

3.3. **The bijection Φ.** We are now in the case where i and $i+1$ are adjacent in σ. So there is an integer h such that $\tau = s_i\sigma = \sigma s_h$, and also $\sigma(\{h, h+1\}) = \tau(\{h, h+1\}) = \{i, i+1\}$.

Note that for any $u \in \mathfrak{S}_n$, u and us_1 cannot be both in $\mathfrak{W}_m^+$ or both in $\mathfrak{W}_m^-$. The same is true for u and us_{n-1}. So we have $2 \le h \le n-2$.

3.3.1. *First case.* In the first case, we assume that $s_i\alpha \in \mathfrak{W}_j^+$. Thus we can define $\Phi((\alpha, \beta)) = (s_i\alpha, \beta)$, this pair being clearly in $\mathfrak{C}_{j,k}^{\tau}$.

Note that the pair $(s_i\alpha, \beta) \in \mathfrak{C}_{j,k}^{\tau}$ also leads to the first case when we apply the map Ψ, so that $\Psi \circ \Phi$ is the identity when applied on (α, β) falling in the first case.

From now on, assume $s_i\alpha \notin \mathfrak{W}_j^+$. It follows that $s_i\alpha$ is not in the same descent class as α, so i and $i+1$ are adjacent in α. So there is an integer g such that $s_i\alpha = \alpha s_g$. We also have $s_g\beta = \beta s_h$ since $\alpha s_g \beta = s_i \alpha\beta = \alpha\beta s_h$. Also, $\alpha(\{g, g+1\}) = \{i, i+1\}$ and $\beta(\{h, h+1\}) = \{g, g+1\}$.

3.3.2. *Second case.* In the second case, we assume that $s_g\beta \in \mathfrak{W}_k^+$. Thus we can define $\Phi((\alpha, \beta)) = (\alpha, s_g\beta)$, which is in $\mathfrak{C}_{j,k}^{\tau}$ (since $\alpha s_g \beta = s_i\alpha\beta = s_i\sigma = \tau$).

Note that the pair $(\alpha, s_g\beta) \in \mathfrak{C}_{j,k}^{\tau}$ also leads to this second case when we apply the map Ψ, so that $\Psi \circ \Phi$ is the identity when applied on (α, β) falling in the second case.

So from now on, assume that $s_g\beta \notin \mathfrak{W}_k^+$. This is our last case.

3.3.3. *Third case.* We will extensively use the following fact. The proof is clear upon inspection of a few cases.

Lemma 3.5. *Let $u \in \mathfrak{S}_n$ and a, b such that $s_a u = u s_b$ (so that $u(\{b, b+1\}) = \{a, a+1\}$). Then u and $s_a u$ are both in $\mathfrak{W}_m^+$ or both in $\mathfrak{W}_m^-$ for some m, if and only if:*

- $2 \le b \le n-2$, *and*
- $u(b-1)$ *and* $u(b+2)$ *are both* $< a$ *or both* $> a+1$.

To begin, let us apply this lemma to $\beta \in \mathfrak{W}_k^+$ and $s_g\beta = \beta s_h \notin \mathfrak{W}_k^+$. Since we already know $2 \le h \le n-2$, we obtain:

- either $\beta(h-1) < g$ and $\beta(h+2) > g+1$, (A)
- or $\beta(h-1) > g+1$ and $\beta(h+2) < g$. (B)

Note that this implies $2 \leq g \leq n-2$.

Next, we apply the lemma to $\alpha \in \mathfrak{W}_j^+$ and $s_i\alpha = \alpha s_g \notin \mathfrak{W}_j^+$. Since we just got $2 \leq g \leq n-2$, it follows:

- either $\alpha(g-1) < i$ and $\alpha(g+2) > i+1$, (C)
- or $\alpha(g-1) > i+1$ and $\alpha(g+2) < i$. (D)

Next, we apply the lemma to σ and $s_i\sigma = \sigma s_h$, which are both in $\mathfrak{W}_m^+$. Since $2 \leq h \leq n-1$, we get (writing $\alpha\beta$ instead of σ):

- either $\alpha(\beta(h-1)) < i$ and $\alpha(\beta(h+2)) < i$, (E)
- or $\alpha(\beta(h-1)) > i+1$ and $\alpha(\beta(h+2)) > i+1$. (F)

To define Φ, we only need to distinguish between cases C or D, and E or F. Suppose first that we are in case C-E.

Lemma 3.6. *In case C-E as above, there is a unique (word) factorization $\alpha_1 \dots \alpha_n = X_1X_2X_3X_4$ where the four factors are nonempty and such that: $X_2 = i, i+1$ or $X_2 = i+1, i$, X_3 contains only letters $> i+1$, the first letter of X_4 is $< i$.*

Proof. To begin, X_2 is defined as the factor containing i and $i+1$ (we have $\alpha(\{g, g+1\}) = \{i, i+1\}$). Since $g \geq 2$, we deduce that X_1 is nonempty. Since $\alpha(g+2) > i+1$, i.e. the first letter to the right of X_2 is $> i+1$, we can define X_3 as the largest factor following X_2 and containing letters $> i+1$.

Then, we have $\alpha(\beta(h-1)) < i$ and $\alpha(\beta(h+2)) < i$. Since at least one of $\beta(h-1)$ and $\beta(h+2)$ is $> g+1$, it follows that there is at least one letter $< i$ to the right of X_2. So X_4 is nonempty. $\square$

Then we define $\Phi((\alpha,\beta)) = (\alpha', \beta')$ as follows:

- $\alpha' = X_1X_3\overline{X_2}X_4$ where $\overline{X_2}$ is X_2 in reversed order.
- β' is such that $\beta'(h) = \beta(h) + |X_3|$, $\beta'(h+1) = \beta(h+1) + |X_3|$, if $\alpha(\beta(u))$ appears in X_3 then $\beta'(u) = \beta(u) - 2$, otherwise (i.e. $\alpha(\beta(u))$ appears in X_1 or X_4) then $\beta'(u) = \beta(u)$. Here $|X_3|$ is the length of X_3.

Lemma 3.7. *We have $(\alpha', \beta') \in \mathfrak{C}_{j,k}^{\tau}$.*

Proof. Since the last letter of X_1 (i.e. $\alpha(g-1)$) and the first of X_4 are both $< i$, and X_3 contains only letters $> i+1$, we easily get that $\alpha' \in \mathcal{W}_j^+$ (knowing $\alpha \in \mathcal{W}_j^+$).

Among the two letters $\alpha(\beta(h-1))$ and $\alpha(\beta(h+2))$, one appears in X_1 and the other in X_4. One can deduce that four letters $\beta'(h-1), \dots, \beta'(h+2)$ are in the same relative order as $\beta(h-1), \dots, \beta(h+2)$. Then, suppose we are in the case $\beta'(u) = \beta(u) - 2$, i.e. $\alpha(\beta(u))$ appears in X_3. Let $v = u \pm 1$, then $\alpha(\beta(v))$ can appear in X_1, X_3, or X_4. In each case, we check that $\beta'(u)$ and $\beta'(v)$ are in the same relative order as $\beta(u)$ and $\beta(v)$. So β' is in the same descent class as β and in paticular $\beta' \in \mathfrak{W}_k^+$.

By distinguishing which of the factors $X_1, \dots, X_4$ contains $\alpha(\beta(u))$, from the definition of β' we can check that $\alpha'(\beta'(u)) = s_i(\alpha(\beta(u)))$. So $\alpha'\beta' = s_i\alpha\beta$. $\square$

Let $g' = g + |X_3|$. We have $s_i\alpha' = \alpha' s_{g'}$ and $s_{g'}\beta' = \beta' s_h$ by definition of α' and β'. So (α', β') falls in the third case when we apply Ψ. Moreover:

- $\alpha'(g'-1) > i+1$ (since this number is the last letter of X_3) , so (α', β') falls in case D.
- $\alpha'(\beta'(h-1)) = \alpha(\beta(h-1)) < i$ (since $\alpha'\beta' = s_i\alpha\beta$) , so (α', β') falls in case E.

So (α', β') falls in case D-E. Consequently, there is only one way to define Φ in case D-E in order to get that $\Psi \circ \Phi$ is the identity. Namely, if (α, β) falls in case D-E, we factor $\alpha = X_1 X_2 X_3 X_4$ where X_3 is $i, i+1$ or $i+1, i$, X_2 contains only letters $> i+1$, the last letter of X_1 is $< i$. Once again the factorization exists and is unique. Then we define $\Phi((\alpha, \beta)) = (\alpha', \beta')$ as follows:

- $\alpha' = X_1 \overline{X_3} X_2 X_4$.
- β' is such that $\beta'(h) = \beta(h) - |X_2|$, $\beta'(h+1) = \beta(h+1) - |X_2|$, if $\alpha(\beta(u))$ appears in X_2 then $\beta'(u) = \beta(u) + 2$, otherwise (i.e. $\alpha(\beta(u))$ appears in X_1 or X_4) then $\beta'(u) = \beta(u)$.

From the definitions, we check that $\Psi \circ \Phi$ is the identity on all (α, β) that falls in case C-E or D-E.

It remains to define Φ and Ψ in a similar manner to exchange the case C-F with the case D-F. This is completely similar except that we exchange the conditions "$< i$" and "$> i+1$".

This ends the definition of the bijection, hence of the proof of $W_j^+ W_k^+ \in \mathcal{W}^{\pm}$.

3.4. **The case of the other subalgebras.** Knowing that $\mathcal{W}^{\pm}$ is an algebra, we can finish the proof of Theorem 3.2.

Proposition 3.8. *$\mathcal{W}^{\circ}$ is a subalgebra of $\mathcal{W}^{\pm}$.*

Proof. For each permutation σ, we define:

$$\mathfrak{D}_{j,k}^{\sigma} = \{(\alpha, \beta) \in \mathfrak{W}_j^{\circ} \times \mathfrak{W}_k^{\circ} \, : \, \alpha\beta = \sigma\},$$

and $d_{j,k}^{\sigma} = \#\mathfrak{D}_{j,k}^{\sigma}$. If $\sigma \in \mathfrak{W}_m^+$ or $\sigma \in \mathfrak{W}_m^-$, then $d_{j,k}^{\sigma}$ is the coefficient of W_m^+ or W_m^-, respectively, in $W_j^{\circ} W_k^{\circ}$. We will show that for each m such that $2 \leq m \leq n-1$, we have $d_{j,k}^{\sigma} = d_{j,k}^{\tau}$ for some well-chosen $\sigma \in \mathfrak{W}_m^+$ and $\tau \in \mathfrak{W}_{m-1}^-$. Indeed this implies $W_j^{\circ} W_k^{\circ} \in \mathcal{W}^{\circ}$.

It is easy to find an element $\sigma \in \mathfrak{W}_m^+$ such that $\sigma_1 = n-1$ and $\sigma_2 = n$. We deduce that $\tau = s_{n-1}\sigma = \sigma s_1$ is in $\mathfrak{W}_{m-1}^-$. Then, it remains only to find a bijection beween $\mathfrak{D}_{j,k}^{\sigma}$ and $\mathfrak{D}_{j,k}^{\tau}$.

This bijection Δ is defined as follows. Let $(\alpha, \beta) \in \mathfrak{D}_{j,k}^{\sigma}$. In the first case, if $s_{n-1}\alpha \in \mathfrak{W}_j^{\circ}$, we set $\Delta(\alpha, \beta) = (s_{n-1}\alpha, \beta)$.

Otherwise, it easily follows $s_{n-1}\alpha = \alpha s_{n-1}$. In the second case, if $s_{n-1}\beta \in \mathfrak{W}_k^{\circ}$, we set $\Delta(\alpha, \beta) = (\alpha, s_{n-1}\beta)$.

There is in fact no other case to consider: if we are not in the first case, we have necessarily $s_{n-1}\beta \in \mathfrak{W}_k^{\circ}$. Otherwise, we would have $s_{n-1}\beta = \beta s_{n-1}$, and it would follow that $s_{n-1}\sigma = \sigma s_{n-1}$ (this is a contradiction to $s_{n-1}\sigma = \sigma s_1$). $\square$

As for the other subalgebras, we don't need to do a similar bijection, as it is more convenient to use some symmetry properties. Let $\omega := W_1^- = n \dots 21$. We clearly have $\mathrm{run}(\omega\sigma) = \mathrm{run}(\sigma\omega) = \mathrm{run}(\sigma)$ for any $\sigma \in \mathfrak{S}_n$. Moreover, $1 \in \mathrm{Des}(\sigma)$ iff $1 \notin \mathrm{Des}(\omega\sigma)$, so that:

$$\omega W_k^+ = W_k^-, \qquad \omega W_k^- = W_k^+.$$

Note that if $\sigma \in \mathfrak{S}_n$ has an odd number of alternating runs, the first and last runs are of similar nature (both ascending or both descending), but if σ has an even number of runs, they are of different nature (one is ascending, the other descending). It is then easy to read the alternating runs of $\sigma\omega = \sigma_n \dots \sigma_1$ from those of $\sigma = \sigma_1 \dots \sigma_n$. It follows:

$$W_k^+\omega = \begin{cases} W_k^- & \text{if } k \text{ is odd,} \\ W_k^+ & \text{if } k \text{ is even,} \end{cases} \qquad W_k^-\omega = \begin{cases} W_k^+ & \text{if } k \text{ is odd,} \\ W_k^- & \text{if } k \text{ is even,} \end{cases} \tag{1}$$

Note that this implies

$$\begin{aligned} W_j^- W_k^+ &= \omega W_j^+ W_k^+ \\ W_j^+ W_k^- &= \begin{cases} W_j^+ W_k^+ & \text{if } j \text{ is even,} \\ \omega W_j^+ W_k^+ & \text{if } j \text{ is odd,} \end{cases} \\ W_j^- W_k^- &= \begin{cases} W_j^+ W_k^+ & \text{if } j \text{ is odd,} \\ \omega W_j^+ W_k^+ & \text{if } j \text{ is even.} \end{cases} \end{aligned}$$

As a consequence, to show that $\mathcal{W}^\pm$ is an algebra it suffices to show that $W_j^+ W_k^+ \in \mathcal{W}^\pm$, as was mentioned earlier.

Proposition 3.9. *$\mathcal{W}$ is a subalgebra of $\mathcal{W}^\pm$.*

Proof. From the elementary properties of $\omega = W_1^-$ (and W_1^+, the unit element), we have $W_1 W_k^+ = W_1 W_k^- = W_k$. It follows that $\mathcal{W}$ is the right ideal $W_1\mathcal{W}^\pm$ and *a fortiori* a subalgebra. □

Remark 3.10. The previous proposition should be understood in the sense of nonunital algebras, since $\mathcal{W}$ does not contain contain the unit of $\mathcal{W}^\pm$. Still, $\mathcal{W}$ has a unit element as an abstract algebra, namely $\frac{1}{2}W_1$ (at the condition of allowing rational coefficients).

Proposition 3.11. *$\mathcal{C}$ is a subalgebra of $\mathcal{W}^\pm$.*

Proof. From the elementary properties of $\omega = W_1^-$, we have:

$$\omega W_k^+\omega = \begin{cases} W_k^+ & \text{if } k \text{ is odd,} \\ W_k^- & \text{if } k \text{ is even,} \end{cases} \qquad \omega W_k^-\omega = \begin{cases} W_k^- & \text{if } k \text{ is odd,} \\ W_k^+ & \text{if } k \text{ is even.} \end{cases}$$

Using that $x \mapsto \omega x \omega$ is a linear symmetry, it follows that

$$\{x \in \mathcal{W}^\pm : \omega x \omega = x\} = \mathrm{Vect}(W_{2k+1}^+, W_{2k+1}^-, W_{2k}) = \mathcal{C}.$$

If $x, y \in \mathcal{C}$, we have $\omega xy\omega = \omega x\omega^2 y\omega = xy$, so $\mathcal{C}$ is closed under product. □

We turn to the case of peak algebras.

Lemma 3.12. *We have:*

$$P_k = W_{2k}^+ + W_{2k+1}^- + W_{2k+1}^+ + W_{2k+2}^-, \tag{2}$$

$$P_k^\circ = W_{2k-1}^- + W_{2k}^+ + W_{2k}^- + W_{2k+1}^+. \tag{3}$$

Proof. For the first equality, consider a permutation σ with $\mathrm{pk}(\sigma) = k$. It has at least $2k$ runs, since each peak follows an ascending run, and precedes a descending run. Besides these $2k$ runs, whose first is ascending, there is possibly an additional descending run at the beginning, and possibly an additional ascending run at the end. The four possibilities give the four terms in the sum.

As for the second equality, consider a permutation $\mathrm{pk}^\circ(\sigma) = k$. It has at least $2k-1$ runs, since now the first peak might not follow an ascending run (if it is equal to 1). Besides these $2k-1$ runs, whose first is descending, there is possibly an additional ascending run at the beginning, and possibly an additional ascending run at the end. Once again, the four possibilities give the four terms in the sum. □

Proposition 3.13. *$\mathcal{P}$ and $\mathcal{P}^\circ$ are subalgebras of $\mathcal{W}^\pm$.*

Proof. The previous lemma shows that $\mathcal{P}$ and $\mathcal{P}^\circ$ are subspaces of $\mathcal{W}^\pm$, so it remains to show that they are closed under product.

We begin with the case of $\mathcal{P}^\circ$. From the previous lemma, we have $P_k^\circ = W_{2k}^\circ + W_{2k+1}^\circ$. Now let $x \in \mathcal{W}^\circ$, then it is in $\mathcal{P}^\circ$ iff the coefficients of W_{2k}° and W_{2k+1}° are the same for all k. This is also equivalent to the equality of the coefficients of W_{2k}^+ and W_{2k}^- in x, i.e. to the condition $x \in \mathcal{C}$. So we have $\mathcal{P}^\circ = \mathcal{W}^\circ \cap \mathcal{C}$, and this is a subalgebra as an intersection of subalgebras.

In the case of $\mathcal{P}$, we first note that it is closed under product iff $\omega\mathcal{P}\omega$ is also closed under product. We have:

$$\omega P_k \omega = W_{2k}^- + W_{2k+1}^- + W_{2k+1}^+ + W_{2k+2}^+ = W_{2k+1}^\circ + W_{2k+2}^\circ,$$

so $\omega\mathcal{P}\omega \subset \mathcal{W}^\circ$. Now let $x \in \mathcal{W}^\circ$, then it is in $\omega\mathcal{P}\omega$ iff the coefficients of W_{2k+1}° and W_{2k+2}° are the same for all k. This is also equivalent to the equality of the coefficients of W_{2k+1}^+ and W_{2k+1}^- in x, and also to the condition $x\omega = x$. So we have $\omega\mathcal{P}\omega = \{x \in \mathcal{W}^\circ : x\omega = x\}$, which is clearly closed under product. □

4. Algebraic proof for the existence of run algebras

This section gives a second proof, using noncommutative symmetric functions, that grouping permutations by the number of runs gives an algebra. This will require working in all degrees n at the same time, so from now on add a second index to all previous quantities indicating the degree, if these quantities involve one degree only. For example, we now write $\mathcal{W}_n$ for the commutative run algebra in degree n, and $W_{k,n}$ for its basis.

First define

$$V_1^+ := \sum_{i \geq 0} R_i = \sum_{n \geq 0} W_{1,n}^+, \qquad V_1^- := \sum_{i \geq 0} R_{(1^i)} = \sum_{n \geq 0} W_{1,n}^-.$$

The intuition is that V_1^+ is an ascending run and V_1^- is a descending run. To obtain multiple runs, one concatenates ascending runs and descending runs alternately. Since external product roughly expresses concatenation, it is no surprise that we define

$$V_{2k}^+ := (V_1^+ \star V_1^-)^{\star k}, \qquad V_{2k}^- := (V_1^- \star V_1^+)^{\star k},$$

$$V_{2k+1}^+ := V_1^+ \star (V_1^- \star V_1^+)^{\star k}, \qquad V_{2k+1}^- := V_1^- \star (V_1^+ \star V_1^-)^{\star k} \qquad V_k := V_k^+ + V_k^-.$$

Write $V_{k,n}^+$ for the degree n part of V_k^+, and similarly for $V_{k,n}^-, V_{k,n}$, so

$$V_k^+ = \sum_{n \geq 0} V_{k,n}^+, \qquad V_k^- = \sum_{n \geq 0} V_{k,n}^-, \qquad V_k = \sum_{n \geq 0} V_{k,n}.$$

Note that $V_{k,n}^+, V_{k,n}^-$ are non-zero even when $k > n$.

Recall from Section 2.4 that V_1^+, V_1^- are grouplike. Being a product of grouplike elements, V_k^+, V_k^- are also grouplike. Thus the simplified splitting formula in Section 2.4

holds, and this allows a straightforward proof that the Vs are "multiplicative" (under the internal product) in the following way:

Proposition 4.1. *We have:*

$$V_k^+ V_l^+ = V_{kl}^+, \qquad V_k^- V_l^+ = V_{kl}^-.$$

If k is even, then

$$V_k^+ V_l^- = V_{kl}^+, \qquad V_k^- V_l^- = V_{kl}^-.$$

If k is odd, then

$$V_k^+ V_l^- = V_{kl}^-, \qquad V_k^- V_l^- = V_{kl}^+.$$

Proof. By the splitting formula,

$$V_k^+(F_1 \star \cdots \star F_l) = (V_k^+ F_1) \star \cdots \star (V_k^+ F_l),$$

and similarly for V_k^-. So in particular:

$$V_k^+ V_l^+ = (V_k^+ V_1^+) \star (V_k^+ V_1^-) \star \cdots \star (V_k^+ V_1^\pm)$$

where the $\pm$ is $+$ or $-$ depending on the parity of l. For even k, once we know that $V_k^+ V_1^+ = V_k^+ V_1^- = V_k^+$ (the case $l = 1$), this simplifies to

$$V_k^+ V_l^+ = V_k^+ \star V_k^+ \star \cdots \star V_k^+ = V_{kl}^+.$$

For odd k, once we know that $V_k^+ V_1^+ = V_k^+$ and $V_k^+ V_1^- = V_k^-$ (the case $l = 1$), this simplifies to

$$V_k^+ V_l^+ = V_k^+ \star V_k^- \star \cdots \star V_k^\pm = V_{kl}^+.$$

The other products are treated similarly, and we are left with the case $l = 1$.

Recall $V_1^+ = \sum_i R_i$ and R_i is the identity for the internal product, so $V_k^+ V_1^+ = V_k^+$, and $V_k^- V_1^+ = V_k^-$.

Now $V_1^- = \sum_{i \geq 0} R_{1^i}$ and R_{1^i} is the maximal permutation $\omega_i = i, i-1, \ldots, 3, 2, 1$. Internal product to the right by ω_i sends a permutation σ to its reversal $\sigma_i, \ldots, \sigma_1$, so, for any $G_1, \ldots, G_l \in \hat{\mathbf{Sym}}$, we have $(G_1 \star \cdots \star G_l) V_1^- = (G_l V_1^-) \star \cdots \star (G_1 V_1^-)$ (it can be checked on the basis (S^I) and extended by multilinearity). Also it is easy to see that $V_1^- V_1^- = V_1^+$. As a result, if k is even, then

$$V_k^+ V_1^- = V_k^+, \qquad V_k^- V_1^- = V_k^-.$$

And if k is odd, then

$$V_k^+ V_1^- = V_k^-, \qquad V_k^- V_1^- = V_k^+.$$

(Note that the sign rule here is the same as in Equation (1). This will be explained by the relation between Vs and Ws given below.) $\square$

Now it is clear that $\mathrm{Vect}(V_{k,n}^+, V_{k,n}^- : k \in \mathbb{N})$ is closed under the internal product, and it is easy to identify subspaces, such as $\mathrm{Vect}(V_{k,n}^+ + V_{k,n}^- : k \in \mathbb{N})$, which form subalgebras. It remains to equate these to the run algebras $\mathcal{W}_n^\pm$ and $\mathcal{W}_n$, and see how the Vs give them new bases. We will do this by relating V_{2k}^- and V_{2k+1}^+ to the Eulerian peak and left peak algebras respectively, via a triangular change of basis.

Proposition 4.2. *For all k, we have $V^-_{2k,n} \in \mathcal{P}_n$, the Eulerian peak algebra. Moreover, if $k \leq \frac{n+1}{2}$, then*

$$V^-_{2k,n} = 2^{2k-1} P_{k-1,n} + \sum_{l<k-1} a_{k,l,n} P_{l,n}$$

for some constants $a_{k,l,n}$. Hence $\{V^-_{2k,n} : 1 \leq k \leq \lfloor \frac{n+1}{2} \rfloor\}$ is a basis for $\mathcal{P}_n$.

Proof. For the first assertion, it suffices to show that, for each composition I, the coefficient of R_I in $V^-_{2k,n}$ depends only on the number of peaks in I.

First observe that $V^-_{2k} = (V^-_2)^{\star k}$ by definition, and that $V^-_2 = 1 + 2\sum R_I$ over all compositions I with no peaks.

Recall that

$$R_{(i_1,\ldots,i_{\ell(I)})} \star R_{(j_1,\ldots,j_{\ell(J)})} = R_{(i_1,\ldots,i_{\ell(I)},j_1,\ldots,j_{\ell(J)})} + R_{(i_1,\ldots,i_{\ell(I)-1},i_{\ell(I)}+j_1,j_2,\ldots,j_{\ell(J)})}.$$

So the coefficient of R_I in any external product $F \star G$ of noncommutative symmetric functions is the sum, over all $J := (j_1, \ldots, j_{\ell(J)})$, $J' := (j'_1, \ldots, j'_{\ell(J')})$ with $I = (j_1, \ldots, j_{\ell(J)}, j'_1, \ldots, j'_{\ell(J')})$ or $I = (j_1, \ldots, j_{\ell(J)-1}, j_{\ell(J)} + j'_1, j'_2, \ldots, j'_{\ell(J')})$, of the products of the coeffficient of J in F with the coefficient of J' in G. Hence the coefficient of R_I in $V^-_{2k,n}$ is the weighted number of ways to "cut" I into k pieces, possibly trivial, so that each piece contains no peak. The weight is $2^{k'}$, where k' is the number of non-trivial pieces (because of the coefficient 2 in $V^-_2 = 1 + 2\sum R_I$).

Since each piece must be peakless, there must be a cut immediately to the left or immediately to the right of any peak; other than this the positions of cuts are unconstrained. These pair of positions, for any set of peaks, are disjoint, so the number of ways to cut I into k' non-trivial peakless pieces depends only on the number of peaks in I.

If I has more than $k-1$ peaks, then I cannot be cut into k peakless pieces, so R_I does not appear in V^-_{2k}.

If I has exactly $k-1$ peaks, then the ways to cut I into k peakless pieces are precisely when there is one cut either immediately to the left or immediately to the right of each peak. There are 2^{k-1} such ways, and the weight of this cut is 2^k, since no piece is trivial. Hence the coefficient of $P_{k-1,n}$ in $V^-_{2k,n}$ is 2^{2k-1}. □

Next, we apply a similar argument to $V^+_{2k+1,n}$.

Proposition 4.3. *For all k, we have $V^+_{2k+1,n} \in \mathcal{P}^\circ_n$, the Eulerian left peak algebra. Moreover, if $k \leq \frac{n}{2}$, then*

$$V^+_{2k+1,n} = 2^{2k} P^\circ_{k,n} + \sum_{l<k} a^\circ_{k,l,n} P^\circ_{l,n}$$

for some constants $a^\circ_{k,l,n}$. Hence $\{V^+_{2k+1,n} : 0 \leq k \leq \lfloor \frac{n}{2} \rfloor\}$ is a basis for $\mathcal{P}^\circ_n$.

Proof. By the same logic as above, the coefficient of R_I in $V^+_{2k+1,n}$ is the weighted number of ways to "cut" I into $k+1$ pieces, possibly trivial, so that the leftmost piece has no descents, and the k other pieces each contain no peak. The weight is $2^{k'}$, where k' is the number of non-trivial pieces among the k peakless pieces.

Since the first piece has no descents, the first cut must be immediately to the left or immediately to the right of the first left peak. The only other constraint is that there must be a cut immediately to the left or immediately to the right of all other peaks. Again, these pair of positions, for any set of left peaks, are disjoint, so the number of ways to cut into I into a descentless first piece and k' further non-trivial peakless pieces depends only on the number of left peaks in I.

If I has more than k left peaks, then I cannot be cut into a descentless first piece and k further peakless pieces, so R_I does not appear in V^+_{2k+1}.

If I has exactly k left peaks, then the ways to cut I into a descentless first piece and k further peakless pieces are precisely when there is one cut either immediately to the left or immediately to the right of each left peak. There are 2^k such ways, and the weight of this cut is 2^k, since none of the peakless pieces are trivial. Hence the coefficient of $P^\circ_{k,n}$ in $V^+_{2k+1,n}$ is 2^{2k}. □

From the above two propositions, and Equations (2) and (3) relating $P_{k,n}, P^\circ_{k,n}$ to $W^+_{k,n}, W^-_{k,n}$, we see that:

$$V^-_{2k,n} = 2^{2k-1}W^-_{2k,n} + \sum_{l<k} a_{k,l,n}W_{2l+1,n} + a_{k,l,n}W^+_{2l,n} + a_{k,l-1,n}W^-_{2l,n};$$
$$V^+_{2k+1,n} = 2^{2k}W^+_{2k+1,n} + \sum_{l\le k} a^\circ_{k,l,n}W_{2l,n} + a^\circ_{k,l-1,n}W^+_{2l-1,n} + a^\circ_{k,l,n}W^-_{2l-1,n}.$$

By exchanging ascending and descending runs, it is also true that

$$V^+_{2k,n} = 2^{2k-1}W^+_{2k,n} + \sum_{l<k} a_{k,l,n}W_{2l+1,n} + a_{k,l,n}W^-_{2l,n} + a_{k,l-1,n}W^+_{2l,n};$$
$$V^-_{2k+1,n} = 2^{2k}W^-_{2k+1,n} + \sum_{l\le k} a^\circ_{k,l,n}W_{2l,n} + a^\circ_{k,l-1,n}W^-_{2l-1,n} + a^\circ_{k,l,n}W^+_{2l-1,n}.$$

Also, because $P^\circ_k = W^\circ_{2k} + W^\circ_{2k+1}$, and

$$P_k = W^+_{2k} + W^-_{2k+1} + W^+_{2k+1} + W^-_{2k+2},$$
$$W^\circ_{2k+1} + W^\circ_{2k+2} = W^-_{2k} + W^+_{2k+1} + W^-_{2k+1} + W^+_{2k+2},$$

we have

$$V^+_{2k+1,n} = 2^{2k}W^\circ_{2k+1} + 2^{2k}W^\circ_{2k} + \sum_{l<k} a^\circ_{k,l,n}(W^\circ_{2l+1,n} + W^\circ_{2l,n});$$
$$V^+_{2k,n} = 2^{2k-1}W^\circ_{2k} + 2^{2k-1}W^\circ_{2k-1} + \sum_{l<k} a^\circ_{k,l-1,n}(W^\circ_{2l,n} + W^\circ_{2l-1,n}).$$

Combining all this with the multiplicativity of the V_k's in Proposition 4.1, we can conclude:

Theorem 4.4. *For each n,*

$$\mathrm{Vect}(V^+_{k,n}, V^-_{k,n} : k \in \mathbb{N}) = \mathrm{Vect}(V^+_{k,n}, V^-_{k,n} : 1 \le k < n) = \mathcal{W}^\pm_n$$

is an algebra under the internal product. Five commutative subalgebras of $\mathcal{W}_n^{\pm}$ are:

$$\begin{aligned}
\mathrm{Vect}(V_{k,n} : k \in \mathbb{N}) &= \mathrm{Vect}(V_{k,n} : 1 \leq k < n) &&= \mathcal{W}_n,\\
\mathrm{Vect}(V^+_{k,n} : k \in \mathbb{N}) &= \mathrm{Vect}(V^+_{k,n} : 1 \leq k \leq n) &&= \mathcal{W}^\circ_n,\\
\mathrm{Vect}(V^+_{2k+1,n}, V^-_{2k+1,n}, V_{2k,n} : k \in \mathbb{N}) &= \mathrm{Vect}(V^+_{2k+1,n}, V^-_{2k+1,n}, V_{2l,n} : 1 \leq 2k+1, 2l < n) &&= \mathcal{C}_n,\\
\mathrm{Vect}(V^-_{2k,n} : k \in \mathbb{N}) &= \mathrm{Vect}(V^-_{2k,n} : 1 \leq 2k \leq n+1) &&= \mathcal{P}_n,\\
\mathrm{Vect}(V^+_{2k+1,n} : k \in \mathbb{N}) &= \mathrm{Vect}(V^+_{2k+1,n} : 1 \leq 2k+1 \leq n+1) &&= \mathcal{P}^\circ_n.
\end{aligned}$$

Furthermore, $\mathcal{P}^\circ_n$ is central in $\mathcal{W}_n^{\pm}$.

Remark 4.5. The quantities $V^+_{k,n}, V^-_{k,n}$ previously appeared in [24], as order polynomials of enriched P-partitions. The translation between his notation and ours is

$$\begin{aligned}
\rho(x) &= V^-_{2x,n};\\
\bar{\rho}(x) &= V^+_{2x,n};\\
\rho^{(l)}(x) &= V^+_{2x+1,n};\\
\rho^{(r)}(x) &= V^-_{2x+1,n}.
\end{aligned}$$

Our Proposition 4.1, the multiplication rule, is his Theorems 3.1, 3.3 and 3.5.

5. Computation of the orthogonal idempotents

In this section, we compute a complete set of primitive idempotents for $\mathcal{W}_n^{\pm}$ and for its five commutative subalgebras in Theorem 4.4.

5.1. **Orthogonal idempotents of the commutative subalgebras.** The starting point is the orthogonal idempotents of the Eulerian peak and left peak algebras. These were computed in [24, 25], but we rederive them here so this paper is self-contained. The argument here mirrors the approach of Loday (see [16, Sec. 5.3]) for the idempotents of the Eulerian (descent) algebra. The key is the following observation:

Lemma 5.1. *For some multiplicatively-closed subset $I \subseteq \mathbb{N}$, suppose $\{X_i : i \in I\}$ generate a subalgebra of $\hat{\mathbf{Sym}}$ with $X_i X_j = X_{ij}$ (under the internal product). Assume also that there are elements J_k in this subalgebra such that, for each i, we have $X_i = \sum_{k\geq 0} i^k J_k$ (as an identity of formal power series within $\hat{\mathbf{Sym}}$). Then the non-zero J_k form a basis of orthogonal idempotents.*

Proof. For all $i, j \in I$, we have

$$\sum_k (ij)^k J_k = X_{ij} = X_i X_j = \sum_{r,s} i^r j^s J_r J_s.$$

Equating coefficients of $(ij)^k$ then shows that $J_r J_s = 0$ if $r \neq s$, and $J_k^2 = J_k$. □

Recall from Theorem 4.4 that the Eulerian peak algebra is generated by $\{V^-_{i,n} : i \text{ even}\}$ and the Eulerian left peak algebra by $\{V^+_{i,n} : i \text{ odd}\}$. By Proposition 4.1, these generating sets satisfy the multiplicative condition of Lemma 5.1. Thus, to calculate the orthogonal idempotents, it suffices to find $I^-_{k,n}, J^+_{k,n}$ such that $V^-_{2i,n} = \sum_{k\geq 0}(2i)^k I^-_{k,n}$

and $V^+_{2i+1,n} = \sum_{k\geq 0}(2i+1)^k J^+_{k,n}$. Such expressions come from the formal exponential and logarithm operations in $\hat{\mathbf{Sym}}$, defined via their familiar power series expansions:

$$\log_\star(1+F) := F - \frac{1}{2}F^{\star 2} + \frac{1}{3}F^{\star 3} - \dots,$$
$$\exp_\star F := 1 + F + \frac{1}{2!}F^{\star 2} + \frac{1}{3!}F^{\star 3} + \dots,$$

for $F \in \hat{\mathbf{Sym}}$ with no term in degree 0. It can be checked that $\exp_\star(F+G) = (\exp_\star F)\star(\exp_\star G)$ whenever F and G commute. Also, $(1+F)^{\star k} = \exp_\star(k\log_\star(1+F))$ for all positive integers k (and all $F \in \hat{\mathbf{Sym}}$ with no term in degree 0), and this can be used to define $(1+F)^{\star k}$ when k is not a positive integer.

Now we have

$$\begin{aligned} V^-_{2i} = (V^-_2)^{\star i} &= \exp_\star(i\log_\star V^-_2) \\ &= \sum_{k\geq 0}\frac{1}{k!}(i\log_\star V^-_2)^{\star k} \\ &= \sum_{k\geq 0}(2i)^k\left(\frac{1}{2^k k!}(\log_\star V^-_2)^{\star k}\right), \end{aligned}$$

and

$$\begin{aligned} V^+_{2i+1} &= V^+_1 \star (V^-_2)^{\star\left(-\frac{1}{2}\right)}(V^-_2)^{\star\left(i+\frac{1}{2}\right)} \\ &= V^+_1 \star (V^-_2)^{\star\left(-\frac{1}{2}\right)} \star \exp_\star\left(\left(i+\frac{1}{2}\right)\log_\star V^-_2\right) \\ &= \sum_{k\geq 0}(2i+1)^k\left(\tfrac{1}{2^k k!}V^+_1 \star (V^-_2)^{\star\left(-\frac{1}{2}\right)} \star (\log_\star V^-_2)^{\star k}\right). \end{aligned} \tag{4}$$

This motivates the definitions:

$$\begin{aligned} I^-_1 &:= \frac{1}{2}\log_\star V^-_2, & I^-_k &:= \frac{1}{k!}I^{-\star k}_1, \\ I^+_1 &:= \frac{1}{2}\log_\star V^+_2, & I^+_k &:= \frac{1}{k!}I^{+\star k}_1, \\ J^+_0 &:= V^+_1 \star (V^-_2)^{\star\left(-\frac{1}{2}\right)}, & J^+_k &:= J^+_0 \star I^-_k, \\ J^-_0 &:= V^-_1 \star (V^+_2)^{\star\left(-\frac{1}{2}\right)}, & J^-_k &:= J^-_0 \star I^+_k. \end{aligned}$$

Also, make the convention $I^-_0 = I^+_0 = 1$. As with the Vs, write $I^+_{k,n}, I^-_{k,n}, J^+_{k,n}, J^-_{k,n}$ for the degree n part of $I^+_k, I^-_k, J^+_k, J^-_k$. The above calculations show that $I^-_{k,n}, J^+_{k,n}$ are orthogonal idempotents for the Eulerian peak and left peak algebras respectively. However, we will see that $I^-_{k,n}$ is not orthogonal to $J^+_{k,n}$. A first attempt to rectify this might be to calculate products such as $I^-_{k,n}J^+_{k,n}$, and guess simple linear combinations of these idempotents that are orthogonal. These product calculations are considerably easier in the symmetric functions - that is, we calculate the images of $I^+_{k,n}, I^-_{k,n}, J^+_{k,n}, J^-_{k,n}$ under the homomorphism $\Gamma : \mathbf{Sym} \to Sym$, then find simple linear combinations whose images have the form $\sum_{\lambda\in\Lambda_i}\frac{1}{z_\lambda}p_\lambda$ over disjoint subsets Λ_i of partitions of n (see the last paragraph of Subsection 2.5).

Remark 5.2. Note that $J^-_{k,n}$ is not a system of orthogonal idempotents. Indeed, the proof below will show that the symmetric function images $\Gamma(J^-_{k,n})$ are not of the form $\sum_{\lambda\in\Lambda_k}\frac{1}{z_\lambda}p_\lambda$. The problem is that $\{V^-_{2i+1}\}$ is not closed under the internal product, so the analogue of Equation (4) cannot be applied in Proposition 5.1.

Theorem 5.3. *With notation as above:*

i) A basis of orthogonal idempotents of $\mathcal{W}_n$ are given by

$$\frac{1}{2}(I^+_{k,n}+I^-_{k,n}),\quad 1\le k\le n \text{ and } k\equiv n \mod 2$$

and

$$\frac{1}{2}(J^+_{l,n}+J^-_{l,n}-I^+_{l,n}-I^-_{l,n}),\quad 0\le l\le n-4 \text{ and } l\equiv n \mod 2.$$

Their images under $\Gamma:\mathbf{Sym}\to Sym$ are

$$\sum_{\substack{\lambda\vdash n\\ \ell_o(\lambda)=k,\,\ell_e(\lambda)=0}}\frac{1}{z_\lambda}p_\lambda,\quad 1\le k\le n \text{ and } k\equiv n \mod 2$$

and

$$\sum_{\substack{\lambda\vdash n\\ \ell_o(\lambda)=l,\,\ell_e(\lambda)>0 \text{ even}}}\frac{1}{z_\lambda}p_\lambda,\quad 0\le l\le n-4 \text{ and } l\equiv n \mod 2.$$

ii) A basis of orthogonal idempotents of $\mathcal{W}^\circ_n$ are given by

$$I^+_{k,n},\quad 1\le k\le n \text{ and } k\equiv n \mod 2$$

and

$$\frac{1}{2}(J^+_{l,n}-I^+_{l,n}),\quad 0\le l\le n-2 \text{ and } l\equiv n \mod 2.$$

Their images under $\Gamma:\mathbf{Sym}\to Sym$ are

$$\sum_{\substack{\lambda\vdash n\\ \ell_o(\lambda)=k,\,\ell_e(\lambda)=0}}\frac{1}{z_\lambda}p_\lambda,\quad 1\le k\le n \text{ and } k\equiv n \mod 2$$

and

$$\sum_{\substack{\lambda\vdash n\\ \ell_o(\lambda)=l,\,\ell_e(\lambda)>0}}\frac{1}{z_\lambda}p_\lambda,\quad 0\le l\le n-2 \text{ and } l\equiv n \mod 2.$$

iii) A basis of orthogonal idempotents of $\mathcal{C}_n$ are given by

$$\frac{1}{2}(I^+_{k,n}+I^-_{k,n}),\quad 1\le k\le n \text{ and } k\equiv n \mod 2$$

and

$$\frac{1}{2}(J^+_{l,n}+J^-_{l,n}-I^+_{l,n}-I^-_{l,n}),\quad 0\le l\le n-4 \text{ and } l\equiv n \mod 2$$

and

$$\frac{1}{2}(J^+_{m,n}-J^-_{m,n}),\quad 0\le m\le n-2 \text{ and } m\equiv n \mod 2.$$

Their images under $\Gamma : \mathbf{Sym} \to Sym$ *are*

$$\sum_{\substack{\lambda \vdash n \\ \ell_o(\lambda)=k,\ \ell_e(\lambda)=0}} \frac{1}{z_\lambda} p_\lambda, \quad 1 \leq k \leq n \text{ and } k \equiv n \mod 2$$

and

$$\sum_{\substack{\lambda \vdash n \\ \ell_o(\lambda)=l,\ \ell_e(\lambda) \text{ even}}} \frac{1}{z_\lambda} p_\lambda, \quad 0 \leq l \leq n-4 \text{ and } l \equiv n \mod 2$$

and

$$\sum_{\substack{\lambda \vdash n \\ \ell_o(\lambda)=m,\ \ell_e(\lambda)>0 \text{ odd}}} \frac{1}{z_\lambda} p_\lambda, \quad 0 \leq m \leq n-2 \text{ and } m \equiv n \mod 2.$$

iv) A basis of orthogonal idempotents of $\mathcal{P}_n$ *are given by*

$$I^-_{k,n}, \quad 1 \leq k \leq n \text{ and } k \equiv n \mod 2.$$

Their images under $\Gamma : \mathbf{Sym} \to Sym$ *are*

$$\sum_{\substack{\lambda \vdash n \\ \ell_o(\lambda)=k, \ell_e(\lambda)=0}} \frac{1}{z_\lambda} p_\lambda, \quad 1 \leq k \leq n \text{ and } k \equiv n \mod 2.$$

v) A basis of orthogonal idempotents of $\mathcal{P}^\circ_n$ *are given by*

$$J^+_{k,n}, \quad 0 \leq k \leq n \text{ and } k \equiv n \mod 2.$$

Their images under $\Gamma : \mathbf{Sym} \to Sym$ *are*

$$\sum_{\substack{\lambda \vdash n \\ \ell_o(\lambda)=k}} \frac{1}{z_\lambda} p_\lambda, \quad 0 \leq k \leq n \text{ and } k \equiv n \mod 2.$$

Proof. Observe that each set of claimed images under Γ have cardinality equal to the dimension of the corresponding subalgebra. Thus, if these images are correct, $\Gamma : \mathbf{Sym} \to Sym$ must restrict to an algebra isomorphism on each of these subalgebras. Hence the claimed sets are orthogonal and idempotent if and only if their images under Γ are orthogonal and idempotent.

Each claimed symmetric function image is a sum of $\frac{p_\lambda}{z_\lambda}$, the orthogonal idempotents of Sym, hence their sums are also idempotent. Furthermore, for each subalgebra, no partition λ appears in more than one sum, so the sums are orthogonal.

So it suffices to check that the claimed idempotents are indeed in the correct subalgebras, and have the claimed images under Γ. Note that I^-_1 is a series in V^-_2, and I^+_1 is a series in V^+_2, so

$$\begin{aligned} I^-_{k,n} &\in \mathrm{Vect}(V^-_{2k,n}) = \mathcal{P}_n, \\ I^+_{k,n} &\in \mathrm{Vect}(V^+_{2k,n}), \\ J^+_{k,n} &\in \mathrm{Vect}(V^+_{2k+1,n}) = \mathcal{P}^\circ_n, \\ J^-_{k,n} &\in \mathrm{Vect}(V^-_{2k+1,n}). \end{aligned}$$

By symmetry in the definitions of $I_k^+, I_k^-, J_k^+, J_k^-$, it is clear that

$$\begin{aligned} I_{k,n}^+ + I_{k,n}^- &\in \mathrm{Vect}(V_{2k,n}),\\ J_{k,n}^+ + J_{k,n}^- &\in \mathrm{Vect}(V_{2k+1,n}). \end{aligned}$$

Thus all claimed idempotents are in the correct subalgebras.

To calculate the symmetric function images, first recall that

$$\begin{aligned} \Gamma(V_1^+) &= \sum_{n\geq 0} h_n = \exp_\star\left(\sum_{i\geq 1}\frac{p_i}{i}\right),\\ \Gamma(V_1^-) &= \sum_{n\geq 0} e_n = \exp_\star\left(\sum_{i\geq 1}\frac{(-1)^{i-1}p_i}{i}\right). \end{aligned}$$

Hence

$$\begin{aligned} \Gamma(I_1^-) &= \frac{1}{2}\Gamma(\log_\star(V_1^- \star V_1^+))\\ &= \frac{1}{2}(\log_\star(\Gamma(V_1^-)) + \log_\star(\Gamma(V_1^+)))\\ &= \frac{1}{2}\left(\sum_{i\geq 1}\frac{p_i}{i} + \frac{(-1)^{i-1}p_i}{i}\right)\\ &= \sum_{i \text{ odd}}\frac{p_i}{i}, \end{aligned}$$

and similarly

$$\begin{aligned} \Gamma(I_1^+) &= \frac{1}{2}\Gamma(\log_\star(V_1^+ \star V_1^-))\\ &= \frac{1}{2}(\log_\star(\Gamma(V_1^+)) + \log_\star(\Gamma(V_1^-)))\\ &= \sum_{i \text{ odd}}\frac{p_i}{i}. \end{aligned}$$

Consequently,

$$\begin{aligned} \Gamma(I_k^-) = \Gamma(I_k^+) &= \frac{1}{k!}\left(\sum_{i \text{ odd}}\frac{p_i}{i}\right)^{\star k}\\ &= \sum_{\ell_o(\lambda)=k,\ \ell_e(\lambda)=0}\frac{1}{z_\lambda}p_\lambda. \end{aligned}$$

Also,

$$\begin{aligned}\Gamma(J_0^+) &= \Gamma(V_1^+) \star \Gamma(V_1^-)^{\star(-\frac{1}{2})} \star \Gamma(V_1^+)^{\star(-\frac{1}{2})} \\ &= \Gamma(V_1^+)^{\star(\frac{1}{2})} \star \Gamma(V_1^-)^{\star(-\frac{1}{2})} \\ &= \exp_\star\left(\frac{1}{2}\left(\sum_{i\geq 1}\frac{p_i}{i} - \frac{(-1)^{i-1}p_i}{i}\right)\right) \\ &= \exp_\star\left(\sum_{i>0 \text{ even}}\frac{p_i}{i}\right) = \sum_{\ell_o(\lambda)=0}\frac{1}{z_\lambda}p_\lambda,\end{aligned}$$

and

$$\begin{aligned}\Gamma(J_0^-) &= \Gamma(V_1^-) \star \Gamma(V_1^+)^{\star(-\frac{1}{2})} \star \Gamma(V_1^-)^{\star(-\frac{1}{2})} \\ &= \Gamma(V_1^+)^{\star(-\frac{1}{2})} \star \Gamma(V_1^-)^{\star(\frac{1}{2})} \\ &= \exp_\star\left(\frac{1}{2}\left(\sum_{i\geq 1}-\frac{p_i}{i} + \frac{(-1)^{i-1}p_i}{i}\right)\right) \\ &= \exp_\star\left(\sum_{i>0 \text{ even}}-\frac{p_i}{i}\right) = \sum_{\ell_o(\lambda)=0}\frac{(-1)^{\ell(\lambda)}}{z_\lambda}p_\lambda.\end{aligned}$$

So

$$\begin{aligned}\Gamma(J_k^+) &= \Gamma(J_0^+) \star \Gamma(I_k^+) \\ &= \left(\sum_{\ell_o(\lambda)=0}\frac{1}{z_\lambda}p_\lambda\right) \star \left(\sum_{\ell_o(\lambda)=k,\ \ell_e(\lambda)=0}\frac{1}{z_\lambda}p_\lambda\right) \\ &= \sum_{\ell_o(\lambda)=k}\frac{1}{z_\lambda}p_\lambda,\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\Gamma(J_k^+ + J_k^-) &= \frac{1}{2}\Gamma(J_0^+ + J_0^-) \star \Gamma(I_k^-) \\ &= \left(\sum_{\ell_o(\lambda)=0,\ \ell_e(\lambda) \text{ even}}\frac{1}{z_\lambda}p_\lambda\right) \star \left(\sum_{\ell_o(\lambda)=k,\ \ell_e(\lambda)=0}\frac{1}{z_\lambda}p_\lambda\right) \\ &= \sum_{\ell_o(\lambda)=k,\ \ell_e(\lambda) \text{ even}}\frac{1}{z_\lambda}p_\lambda,\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}\Gamma(J_k^+ - J_k^-) &= \frac{1}{2}\Gamma(J_0^+ - J_0^-) \star \Gamma(I_k^-) \\
&= \left(\sum_{\ell_o(\lambda)=0,\, \ell_e(\lambda) \text{ odd}} \frac{1}{z_\lambda} p_\lambda \right) \star \left(\sum_{\ell_o(\lambda)=k,\, \ell_e(\lambda)=0} \frac{1}{z_\lambda} p_\lambda \right) \\
&= \sum_{\ell_o(\lambda)=k,\, \ell_e(\lambda) \text{ odd}} \frac{1}{z_\lambda} p_\lambda .
\end{aligned}$$

□

As remarked at the end of Section 2.5, the identification of the commutative images of the idempotents leads to analogues of the characters of Foulkes [17], that only depend on certain features of the cycle type:

Corollary 5.4. *Write χ_I for the symmetric group character corresponding to the ribbon skew shape for I. Let $\mathrm{run}(I), \mathrm{pk}(I)$ etc. denote $\mathrm{run}(\sigma), \mathrm{pk}(\sigma)$ for any permutation σ with $\mathrm{Des}(\sigma) = \mathrm{Des}(I)$. Then*

i) The set of characters

$$\left\{ \sum_{\mathrm{run}(I)=k} \chi_I : 1 \leq k < n \right\}$$

is a basis for functions $\mathfrak{S}_n \to \mathbb{R}$ depending only on the number of odd cycles and whether there are even cycles, and are zero on permutations with an odd number of even cycles.

ii) The set of characters

$$\left\{ \sum_{\mathrm{run}^\circ(I)=k} \chi_I : 1 \leq k \leq n \right\}$$

is a basis for functions $\mathfrak{S}_n \to \mathbb{R}$ depending only on the number of odd cycles and whether there are even cycles.

iii) The set of characters

$$\left\{ \sum_{\mathrm{run}(I)=2l} \chi_I : 1 \leq 2l < n \right\} \amalg \left\{ \sum_{\substack{\mathrm{run}(I)=2k+1 \\ i_1=1}} \chi_I : 1 \leq 2k+1 < n \right\} \amalg \left\{ \sum_{\substack{\mathrm{run}(I)=2k+1 \\ i_1>1}} \chi_I : 1 \leq 2k+1 < n \right\}$$

is a basis for functions $\mathfrak{S}_n \to \mathbb{R}$ depending only on the number of odd cycles and whether the number of even cycles is zero, even positive, or odd.

iv) The set of characters

$$\left\{ \sum_{\mathrm{pk}(I)=k} \chi_I : 1 \leq 2k \leq n+1 \right\}$$

is a basis for functions $\mathfrak{S}_n \to \mathbb{R}$ depending only on the number of odd cycles and are zero on permutations with an odd number of even cycles.

v) The set of characters

$$\left\{ \sum_{\mathrm{pk}^\circ(I)=k} \chi_I : 1 \leq 2k+1 \leq n+1 \right\}$$

is a basis for functions $\mathfrak{S}_n \to \mathbb{R}$ depending only on the number of odd cycles.

Proof. If $\left\{ \sum_{I \in \mathfrak{I}_k} R_I \right\}$ span a commutative subalgebra of the descent algebra (for some disjoint sets $\mathfrak{I}_k$ of compositions), and the symmetric function image of this subalgebra has a basis of orthogonal idempotents of the form $\left\{ \sum_{\lambda \in \Lambda_k} \frac{p_\lambda}{z_\lambda} \right\}$, then $\left\{ \sum_{I \in \mathfrak{I}_k} \chi_I \right\}$ is a basis for functions $\mathfrak{S}_n \to \mathbb{R}$ that are constant on permutations with cycle type within each Λ_k, and are zero on permutations with cycle type not in any Λ_k. □

5.2. **Orthogonal idempotents for the noncommutative run algebra $\mathcal{W}_n^{\pm}$.** We now show that the idempotents of $\mathcal{C}_n$ constructed in Theorem 5.3 are in fact a complete set of primitive orthogonal idempotents for $\mathcal{W}_n^{\pm}$.

Theorem 5.5. *A complete set of primitive orthogonal idempotents for $\mathcal{W}_n^{\pm}$ are given by*

$$\frac{1}{2}(I_{k,n}^+ + I_{k,n}^-), \quad 1 \leq k \leq n \text{ and } k \equiv n \mod 2$$

and

$$\frac{1}{2}(J_{l,n}^+ + J_{l,n}^- - I_{l,n}^+ - I_{l,n}^-), \quad 0 \leq l \leq n-4 \text{ and } l \equiv n \mod 2$$

and

$$\frac{1}{2}(J_{m,n}^+ - J_{m,n}^-), \quad 0 \leq m \leq n-2 \text{ and } m \equiv n \mod 2.$$

Proof. The key is to show that $\mathcal{C}_n$ is a complement to $\mathcal{K}_n := \ker \Gamma \cap \mathcal{W}_n^{\pm}$. As noted in the previous section, Γ restricted to $\mathcal{C}_n$ is an isomorphism, so $\mathcal{C}_n$ and $\mathcal{K}_n$ have trivial intersection. Since $\dim \mathcal{C}_n = n - 1 + \lfloor \frac{n}{2} \rfloor$, it must be that $\dim \mathcal{K}_n \leq \lceil \frac{n}{2} \rceil - 1$. Now $W_{2k,n}^+ = \omega W_{2k,n}^- \omega$ for $1 \leq k \leq \lceil \frac{n}{2} \rceil - 1$, so $W_{2k,n}^+ - W_{2k,n}^- \in \mathcal{K}_n$, and these elements are linearly independent, so $\dim \mathcal{K}_n = \lceil \frac{n}{2} \rceil - 1$, and $\mathcal{W}_n^{\pm} = \mathcal{C}_n \oplus \mathcal{K}_n$.

By [26, Cor 2.2], the Jacobson radical of any subalgebra $\mathcal{A}$ of $\mathbf{Sym}_n$ (under the internal product) is equal to $\ker \Gamma \cap \mathcal{A}$. Hence $\mathcal{K}_n$ is the Jacobson radical of $\mathcal{W}_n^{\pm}$, so any complete set of primitive orthogonal idempotents for the complement $\mathcal{C}_n$ is a complete set of primitive orthogonal idempotents for $\mathcal{W}_n^{\pm}$. □

References

[1] M. Aguiar, N. Bergeron, and K. Nyman: The peak algebra and the descent algebras of types B and D. Trans. Amer. Math. Soc. 356 (2004), no. 7, 2781–2824.
[2] M. Aguiar, K. Nyman, and R. Orellana: New results on the peak algebra. J. of Alg. Comb. 23(2) (2006), 149–188.
[3] D. André: Étude sur les maxima, minima et séquences des permutations. Ann. Sci. École Norm. Sup. 3(1) (1884), 121–135.
[4] M.D. Atkinson: A new proof of a theorem of Solomon. Bull. London Math. Soc. 18(4) (1986), 351–354.
[5] A. Björner and M. Wachs: Generalized quotients in Coxeter groups Trans. Amer. Math. Soc. 308 (1988), 1–37
[6] M. Bóna and R. Ehrenborg: A combinatorial proof of the log-concavity of the numbers of permutations with k runs. J. Combin. Theory Ser. A 90 (2000), 293–303.
[7] M.-L. Bruner and M. Lackner: A fast algorithm for permutation pattern matching based on alternating runs. Algorithmica 75(1) (2016), 84–117.
[8] E.R. Canfield and H.S. Wilf: Counting permutations by their alternating runs. J. of Combin. Theory A 115 (2008), 213–225.
[9] L. Carlitz: Enumeration of permutations by sequences. Adv. Math. 14 (1974), 92–120.
[10] L. Carlitz: Enumeration of permutations by sequences. Fibonacci Quart. 16(3) (1978) 259–268.
[11] L. Carlitz: The number of permutations with a given number of sequences. Fibonacci Quart. 18(4) (1980) 347–352.
[12] L. Carlitz: Enumeration of permutations by sequences, II. Fibonacci Quart. 19(5) (1981) 398–406.
[13] C.-O. Chow and S.-M. Ma: Counting signed permutations by their alternating runs. Discrete Math. 323 (2014), 49–57.
[14] P.G. Doyle and D. Rockmore: Riffles, ruffles, and the turning algebra. Preprint arXiv:0804.0157.
[15] C.J. Fewster and D. Siemssen: Enumerating permutations by their run structure. El. J. Combin 21(4), article #P4.18.
[16] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh, and J.-Y. Thibon: Noncommutative symmetric functions. Adv. Math. 112 (1995), 218–348.
[17] H.O. Foulkes Eulerian numbers, Newcomb's problem and representations of symmetric groups. Discrete Math. 30(1) (1980), 3–49.
[18] A. Kerber and K.-J. Thürlings: Eulerian numbers, Foulkes characters and Lefschetz characters of S_n. Publ. I.R.M.A. Strasbourg 229/S-08 (1984), 31–36.
[19] J.-L. Loday: Opérations sur l'homologie cyclique des algèbres commutatives Inven. Math. 96 (1989), 205–230.
[20] S.-M. Ma: An explicit formula for the number of permutations with a given number of alternating runs. J. Combin. Theory Ser. A 119 (2012) 1660–1664.
[21] S.-M. Ma: Enumeration of permutations by number of alternating runs. Discrete Math. 313 (2013) 1816–1822.
[22] I.G. Macdonald: *Symmetric functions and Hall polynomials.* Second edition. Oxford University Press, New York, 1995.
[23] J.-C. Novelli, F. Saliola, and J.-Y. Thibon: Representation theory of the higher-order peak algebras. J. of Alg. Comb. 32(4) (2010), 465–495.
[24] T.K. Petersen: Enriched P-partitions and peak algebras. Adv. Math. 209 (2007), no. 2, 561–610.
[25] M. Schocker: The peak algebra of the symmetric group revisited. Adv. Math. 192(2) (2005), 259–309.
[26] M. Schocker: The module structure of the Solomon–Tits algebra of the symmetric group. J. Algebra. 301(2), (2006), 554–586
[27] L. Solomon: A Mackey formula in the group ring of a finite Coxeter group. J. Algebra 41 (1976), 255–264.
[28] R.P. Stanley: *Enumerative combinatorics Vol.2* Cambridge University Press, Cambridge 1999.
[29] W. A. Stein et al.: Sage Mathematics Software (Version 6.1.1) The Sage Development Team, 2015, `http://www.sagemath.org`.

[30] Y. Zhuang, Counting permutations by runs. J. Combin. Theory A 142 (2016), 147–176.

CNRS and Institut Gaspard Monge, Université Paris-Est Marne-la-Vallée, 5 Boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France
E-mail address: `matthieu.josuat-verges@u-pem.fr`

Laboratoire de Combinatoire et d'Informatique Mathématique, Université du Québec à Montréal, CP 8888, Succ. Centre-ville, Montréal, Québec, H3C 3P8, Canada
E-mail address: `amypang@lacim.ca`