§14.5-6: Triple Integrals

We can define the integral of a 3-variable function f just like we did for 2-variable functions last week:

$$\iiint_D f(x,y,z)\,dV = \lim_{m,n,p\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} \hat{f}(x_i^*, y_j^*, z_k^*)\Delta V_{ijk},$$

where $\hat{f}$ is the extension of f to a rectangular box containing D, and x_i, y_j, z_k divide the sides of the box equally, and ΔV_{ijk} is the volume of the smaller boxes.

In practice, we calculate triple integrals by iterated integrals (see p9, 16, 21, 24)

$$\int_a^b \int_{c(x)}^{d(x)} \int_{p(x,y)}^{q(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

Important skills:

- Why calculate triple integrals? (p2-5)
- Finding the limits of triple integrals (p9-16, 21-25)
- Cylindrical and spherical coordinates (p18-20, 26-33 §10.6, 14.6)

$$\iiint_D f(x,y,z)dV = \lim_{m,n,p\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} \hat{f}(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}, \quad \hat{f} = \begin{cases} f & \text{on } D \\ 0 & \text{outside } D. \end{cases}$$

The graph of a 3-variable function is in $\mathbb{R}^4$, so the "hypervolume" under such a graph is not a naturally interesting quantity. However, there are many good reasons to consider Riemann sums (and therefore integrals) of a 3-variable function. (The applications on these four pages also apply to 1D and 2D integrals.)

$$\iiint_D f(x,y,z)dV = \lim_{m,n,p\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} \hat{f}(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}, \quad \hat{f} = \begin{cases} f & \text{on } D \\ 0 & \text{outside } D. \end{cases}$$

The graph of a 3-variable function is in $\mathbb{R}^4$, so the "hypervolume" under such a graph is not a naturally interesting quantity. However, there are many good reasons to consider Riemann sums (and therefore integrals) of a 3-variable function. (The applications on these four pages also apply to 1D and 2D integrals.)

1. (Geometry) The volume of D is $\iiint_D 1\, dV$.

This is the "primary school method" of calculating areas/volumes: put D in a rectangular grid, count the number of rectangles inside D, and multiply this number by the area/volume of each rectangle. (Remember that $\hat{1}(x_i^*, y_j^*, z_k^*) = 1$ if (x_i^*, y_j^*, z_k^*) is in D, and 0 otherwise.)

$$\iiint_D f(x,y,z)dV = \lim_{m,n,p\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} \hat{f}(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}, \quad \hat{f} = \begin{cases} f & \text{on } D \\ 0 & \text{outside } D. \end{cases}$$

There are many good reasons to consider Riemann sums of a 3-variable function:

2. (Probability) The average value of f on D is $\bar{f} = \dfrac{\iiint_D f(x,y,z)\,dV}{\iiint_D 1\,dV}$.

   (Don't be confused by the notation: $\bar{f}$ is a number, not a function.)

To understand where this formula comes from: to approximate the average value of f, we can take the value of f at many points throughout D and take the average of those values - this gives us $\dfrac{1}{N}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} \hat{f}(x_i^*, y_j^*, z_k^*)$, where N is the number of points (x_i^*, y_j^*, z_k^*) lying inside D. Now multiply the numerator and denominator by ΔV_{ijk} and take the limit.

$$\iiint_D f(x,y,z)dV = \lim_{m,n,p\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} \hat{f}(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}, \quad \hat{f} = \begin{cases} f & \text{on } D \\ 0 & \text{outside } D. \end{cases}$$

There are many good reasons to consider Riemann sums of a 3-variable function:

3. (Physics) The mass of an object occupying the region D is $\iiint_D \delta(x,y,z)\, dV$, where $\delta(x,y,z)$ is the density function.

To understand where this formula comes from: the density of an object is its mass per unit volume. If an object has constant density, then its mass is this constant multiplied by its volume. Hence the mass of the small rectangular box $x_{i-1} < x < x_i, y_{j-1} < y < y_j, z_{k-1} < z < z_k$ is approximately $\delta(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}$.

(This formula also works for 2D objects, if $\delta(x,y)$ is a 2D-density function, i.e. mass per unit area.)

$$\iiint_D f(x,y,z)dV = \lim_{m,n,p\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} \hat{f}(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}, \quad \hat{f} = \begin{cases} f & \text{on } D \\ 0 & \text{outside } D. \end{cases}$$

There are many good reasons to consider Riemann sums of a 3-variable function:

4. (Physics) The centre of mass of an object occupying the region D, with density function $\delta(x,y,z)$ is the point with coordinates
$$\left(\frac{\iiint_D x\delta(x,y,z)\,dV}{\iiint_D \delta(x,y,z)\,dV}, \frac{\iiint_D y\delta(x,y,z)\,dV}{\iiint_D \delta(x,y,z)\,dV}, \frac{\iiint_D z\delta(x,y,z)\,dV}{\iiint_D \delta(x,y,z)\,dV} \right).$$
(This formula will be given to you in exams if necessary.)

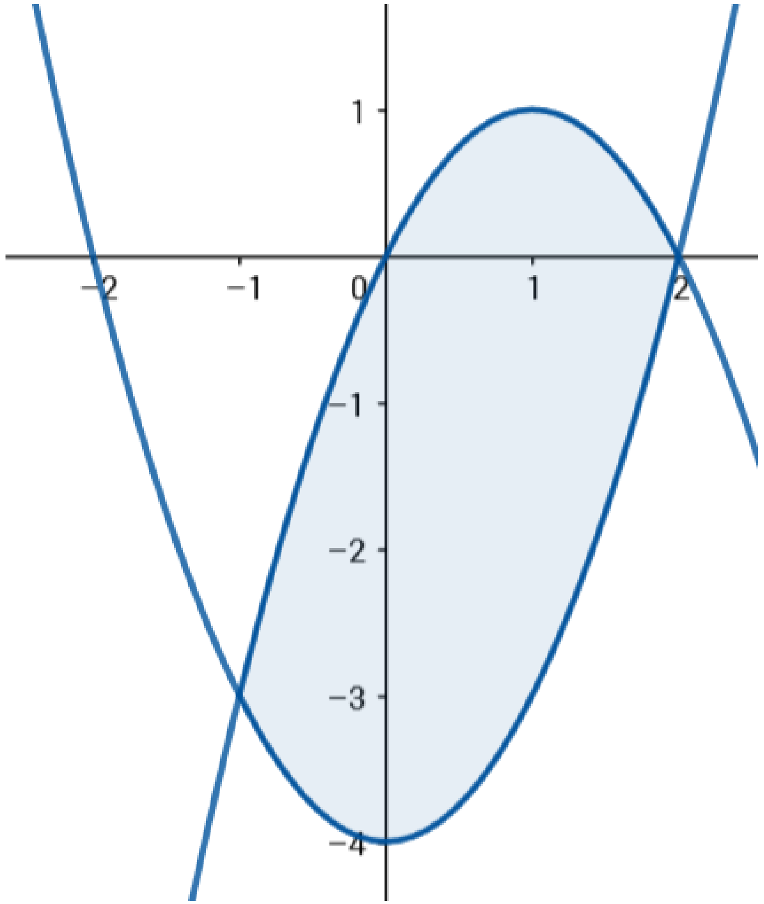
If the density is constant, then the centre of mass is the "average position" $(\bar{x}, \bar{y}, \bar{z})$, and is called the centroid.

The main difficulty with triple integrals is in visualising the region and finding the correct limits.

Let's first think more closely about how this works in two dimensions.

Example: (Week 5 p3): The area of the region R bounded by $y = x^2 - 4$ and $y = -x^2 + 2x$ is

$$\int_{-1}^{2} \quad (-x^2 + 2x) \quad - \quad (x^2 - 4) \quad dx.$$



The main difficulty with triple integrals is in visualising the region and finding the correct limits.

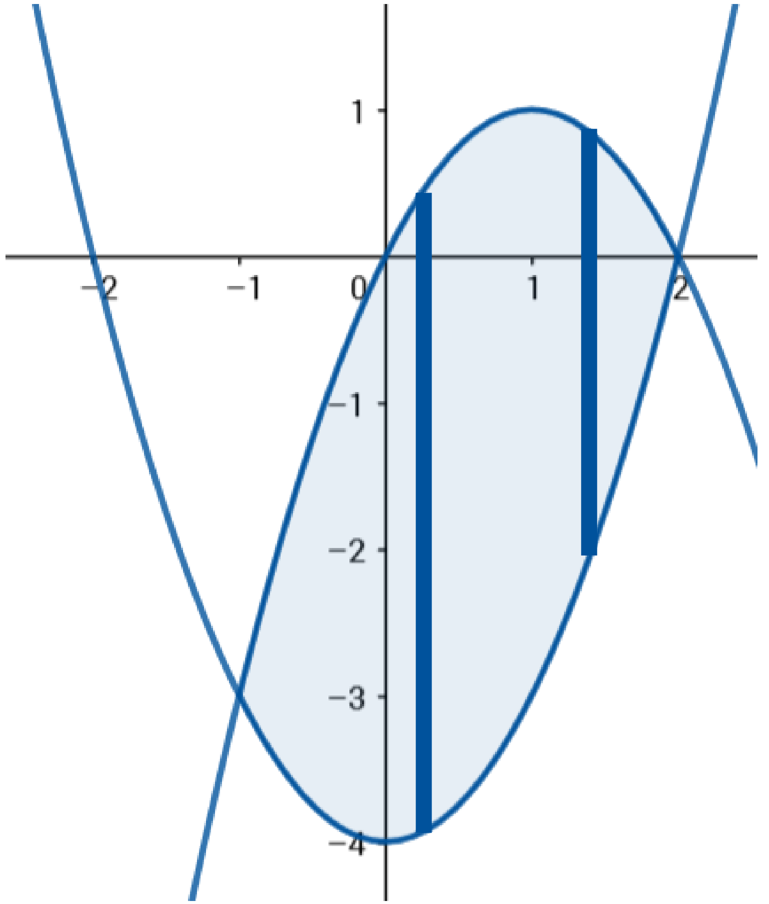
Let's first think more closely about how this works in two dimensions.

Example: (Week 5 p3): The area of the region R bounded by $y = x^2 - 4$ and $y = -x^2 + 2x$ is

$$\int_{-1}^{2} (-x^2 + 2x) - (x^2 - 4)\, dx.$$

top value of y bottom value of y

This difference is the area of the blue strip.



The main difficulty with triple integrals is in visualising the region and finding the correct limits.

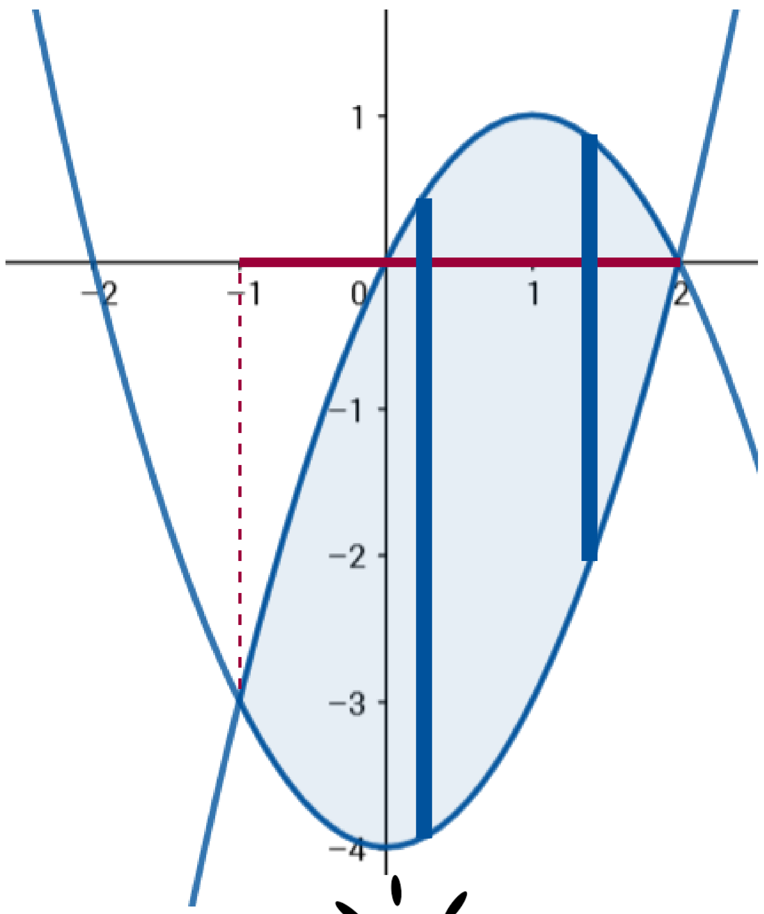
Let's first think more closely about how this works in two dimensions.

Example: (Week 5 p3): The area of the region R bounded by $y = x^2 - 4$ and $y = -x^2 + 2x$ is

$$\int_{-1}^{2} \underbrace{(-x^2 + 2x)}_{\text{top value of } y} - \underbrace{(x^2 - 4)}_{\text{bottom value of } y} \, dx.$$

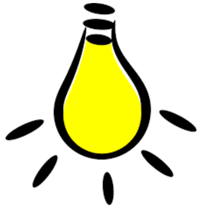
This difference is the area of the blue strip.

The domain of integration is the "shadow" that R makes on the x-axis. We find it by setting y equal in the given boundary equations (i.e. solving $x^2 - 4 = -x^2 + 2x$ in this example). This finds the shadows of the intersection points.



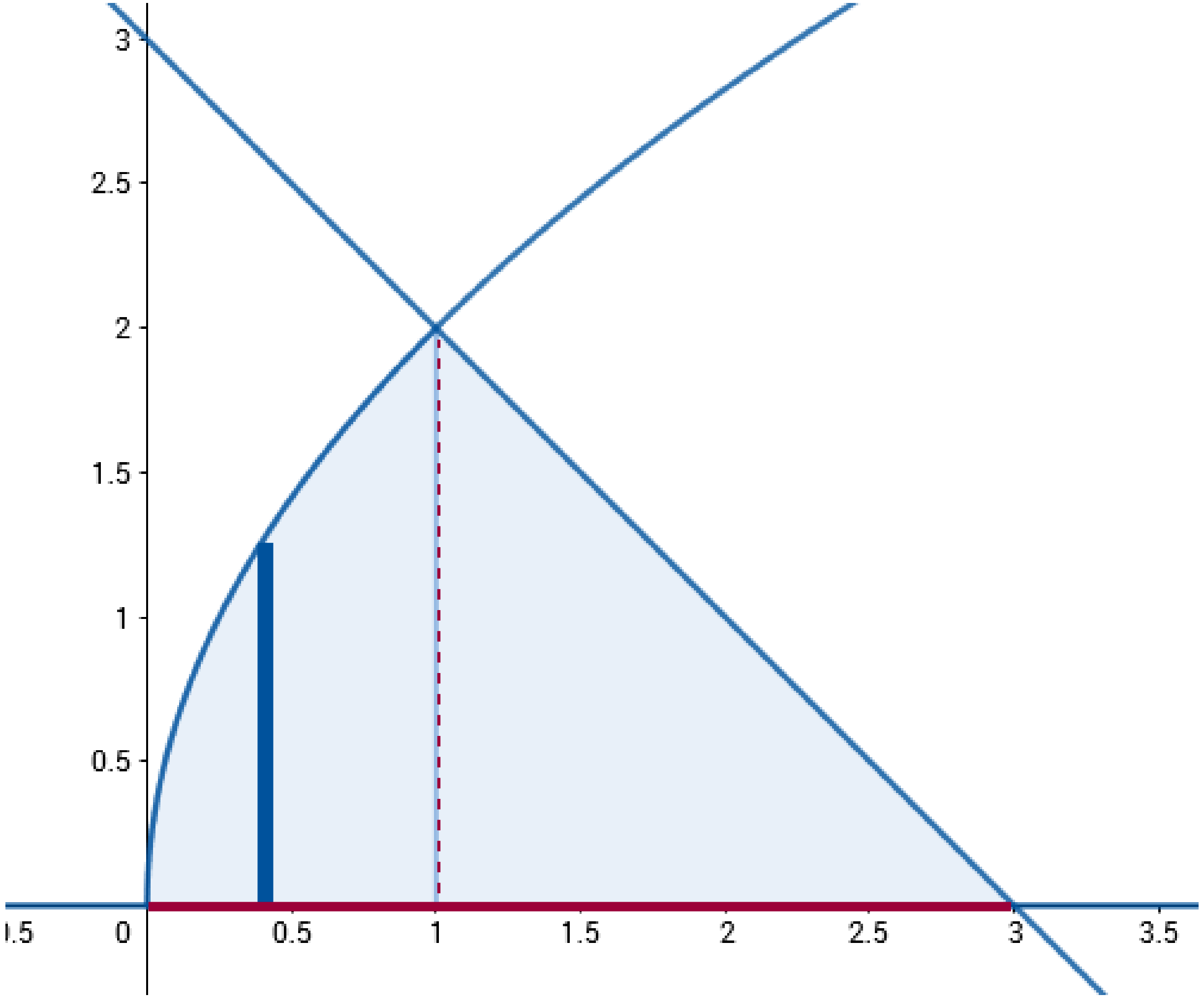
Example: (Week 5 p4): The area of the region R bounded by $y = 2\sqrt{x}$, $y = 3 - x$ and $y = 0$ is

$$\int_0^1 2\sqrt{x} - 0\,dx + \int_1^3 (3-x) - 0\,dx.$$



Note that we find all the "important x values" of the shadow by setting y equal in each pair of boundary equations:

$y = 2\sqrt{x}$ and $y = 0$	$x = 0$
$y = 2\sqrt{x}$ and $y = 3 - x$	$x = 1$
$y = 3 - x$ and $y = 0$	$x = 3$



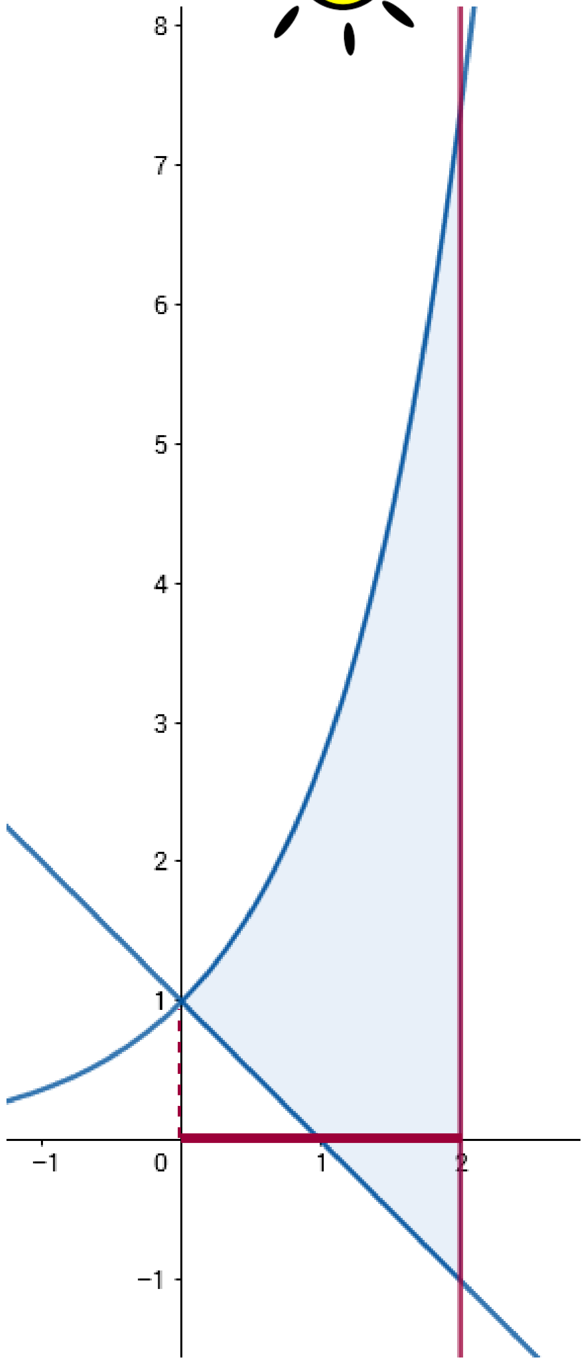
Example: (ex. sheet #8 q2): The area of the region R bounded by $y = e^x$, $y = 1 - x$ and $x = 2$ is

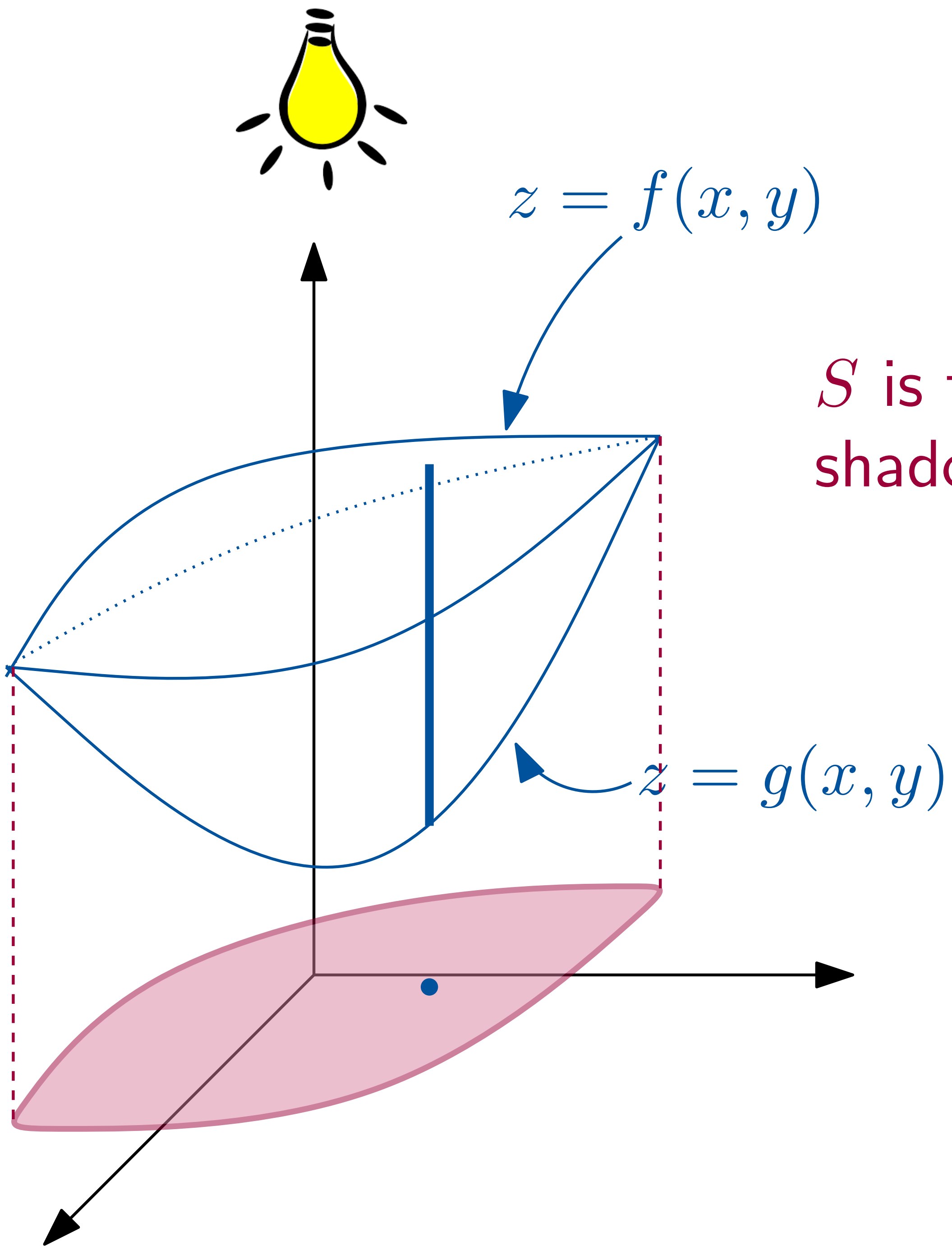
$$\int_0^2 (e^x) - (1 - x)\, dx.$$

Here, there are two "important shadow points":

- $x = 0$, which comes from setting the y-values equal in the equations that involve y (i.e. from solving $e^x = 1 - x$ - this is a difficult equation to solve, so it's important to know the shapes of your curves).
- $x = 2$, which is one of the given boundary equations.

Note that we do not combine $x = 2$ with the other equations.



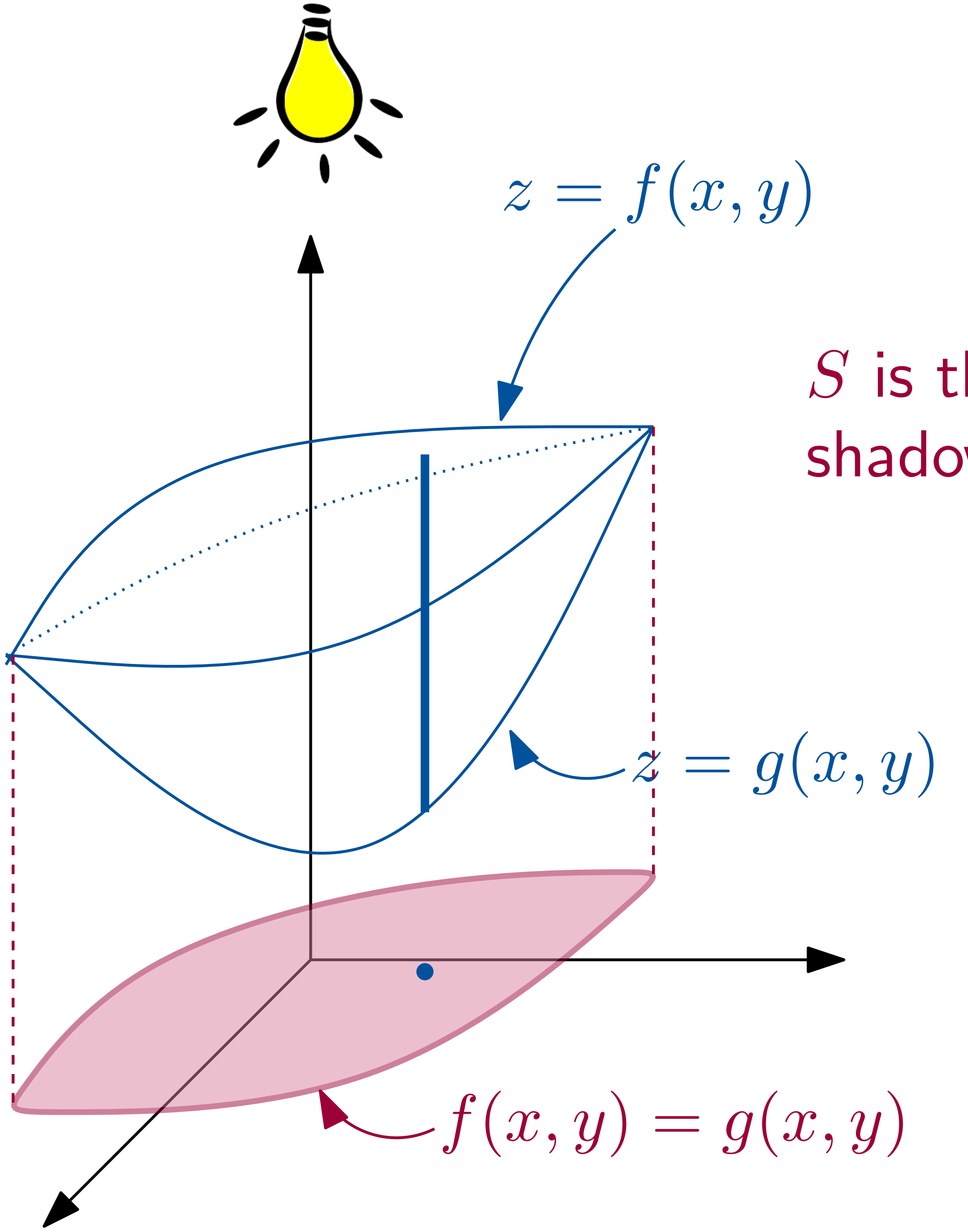


We can compute the volume of a 3D region R in a similar way: integrate the height of each "blue stick" over the 2D shadow of R. (This is sometimes called the shadow method.)

S is the shadow of R

$$\iint_S \underbrace{f(x, y)}_{\text{top value of } z} - \underbrace{g(x, y)}_{\text{bottom value of } z} \, dA.$$

This difference is the volume of the blue stick.



We can compute the volume of a 3D region R in a similar way: integrate the height of each "blue stick" over the 2D shadow of R. (This is sometimes called the shadow method.)

S is the shadow of R

$$\iint_S \underbrace{f(x, y)}_{\text{top value of } z} - \underbrace{g(x, y)}_{\text{bottom value of } z} \, dA.$$

This difference is the volume of the blue stick.

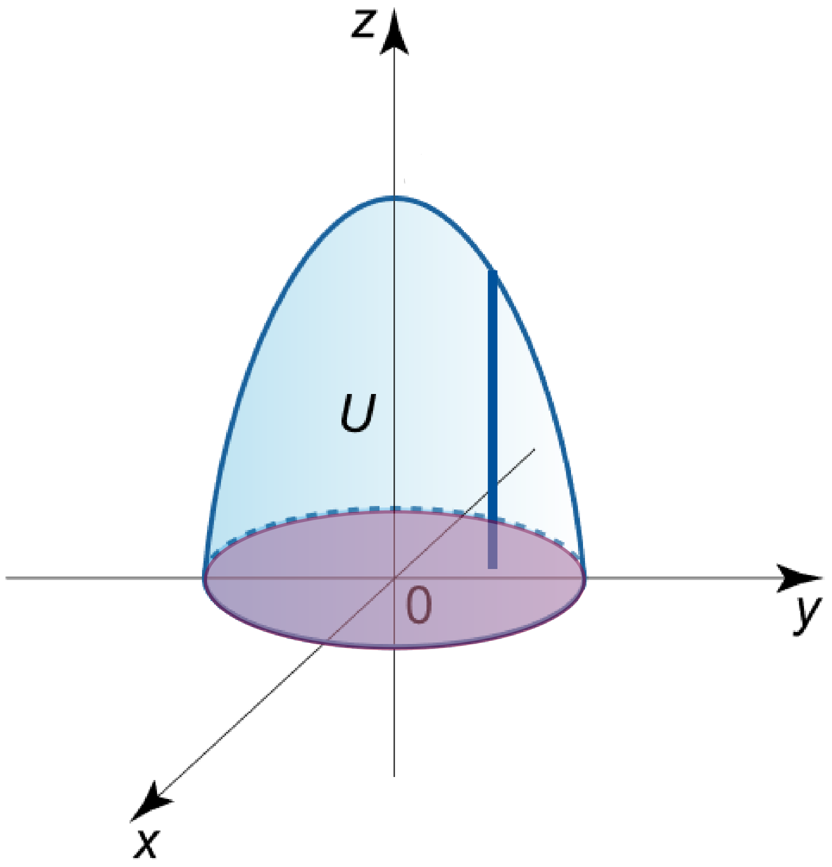
We find the shadow S by sketching the region, and using "important shadow curves", which we usually find by

- setting the z-coordinate equal for each pair of equations which involve z (p10-11, ex. sheet #11 q2);
- the equations involving only x and y, no z (p13, 24).

Example: (Week 5 p33) The volume of the region bounded by $z = 1 - x^2 - y^2$ and $z = 0$ is

$$\int_S \underbrace{(1 - x^2 - y^2)}_{\text{top value of } z} - \underbrace{0}_{\text{bottom value of } z} \, dA,$$

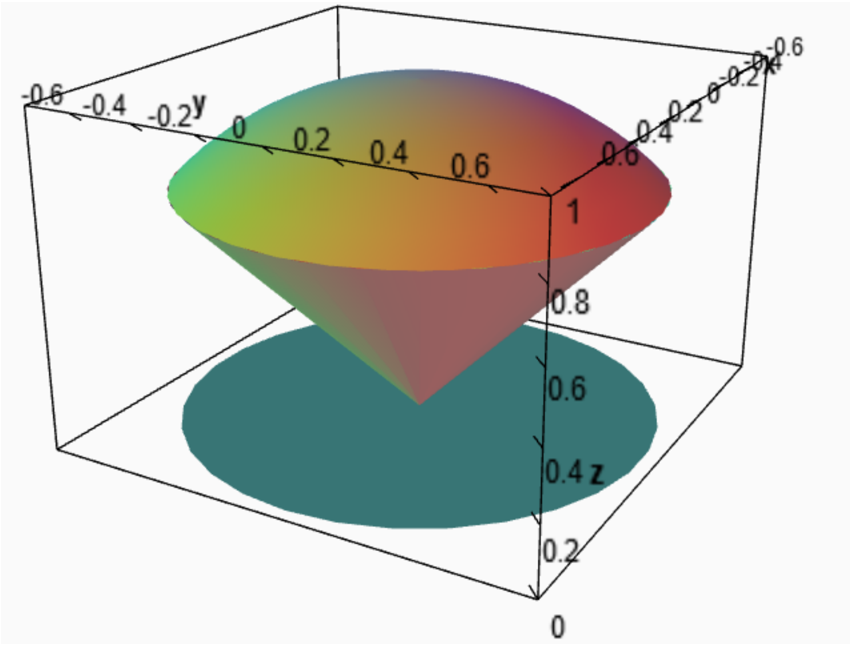
where the shadow S is $\left\{(x, y) \in \mathbb{R}^2 \middle| x^2 + y^2 \leq 1\right\}$.



Notice that the boundary of the shadow comes from setting z equal in the two equations: $1 - x^2 - y^2 = 0$. This describes the x, y values of the intersection of the two surfaces.

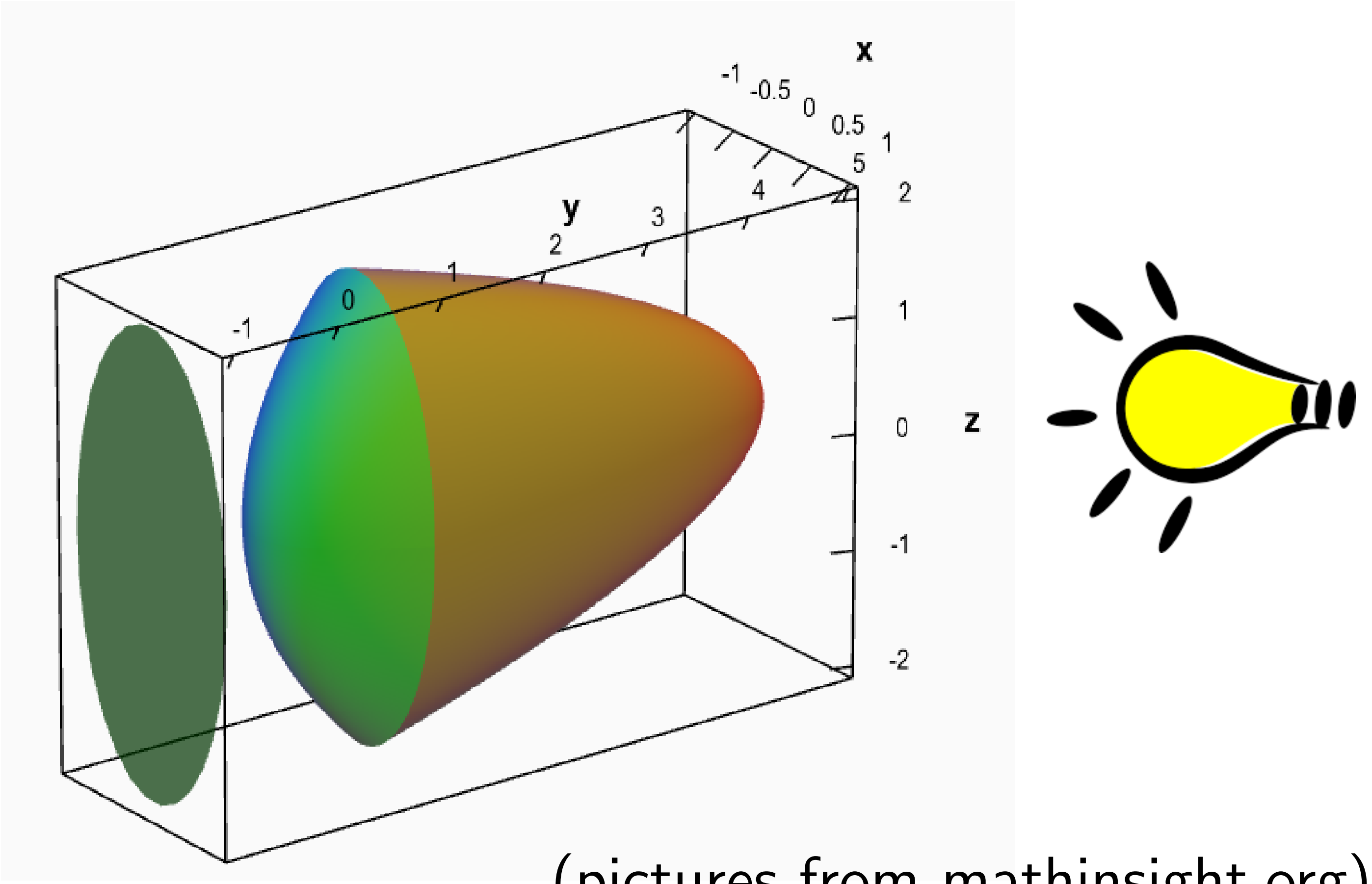
When the shadow is complicated, it may be useful to draw a separate 2D picture of the shadow.

(picture from math24.net)



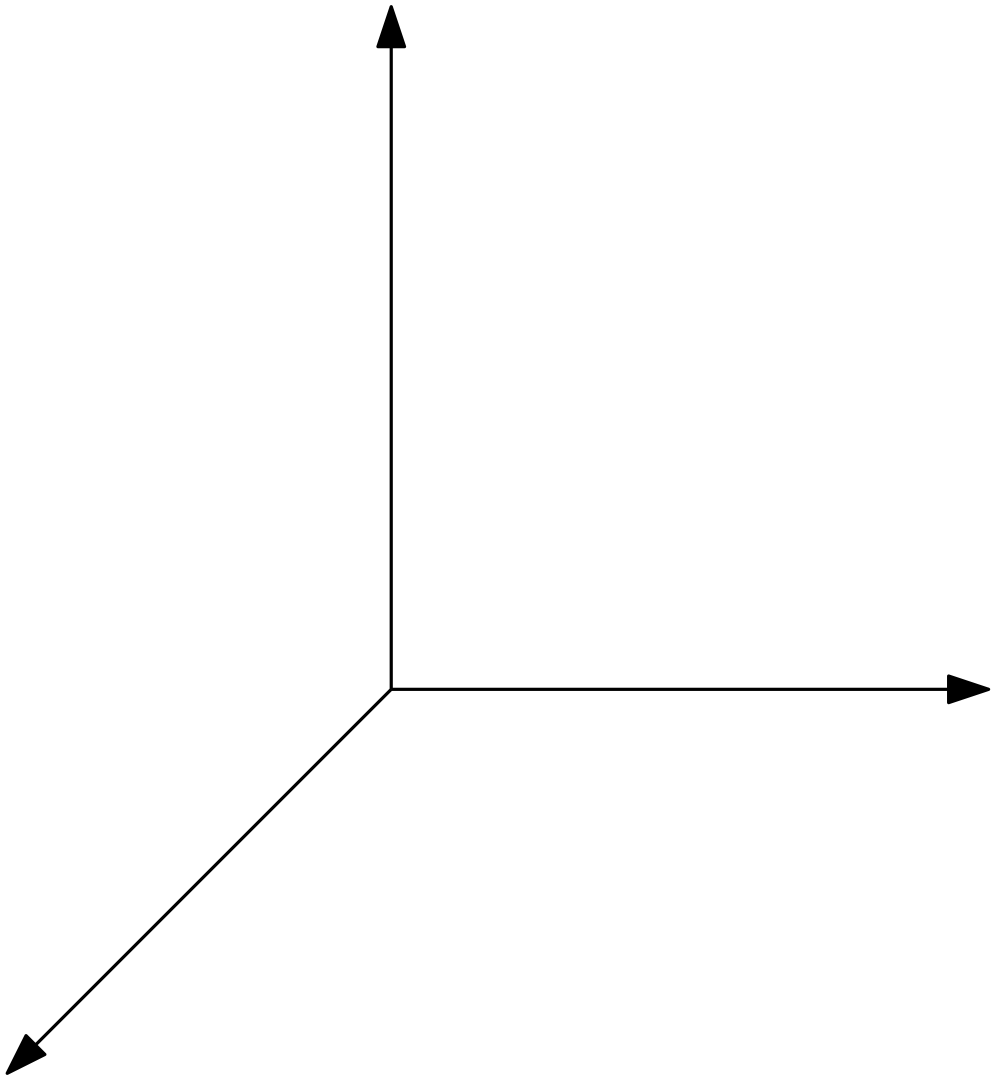
The mathematical word for the "shadow" is the projection to the xy-plane.

For some regions (or some integrands, see week 5 p20), it might be useful to consider shining our light in a different direction. For example, in this picture the light comes from the y direction, and the shadow is the projection to the xz-plane.

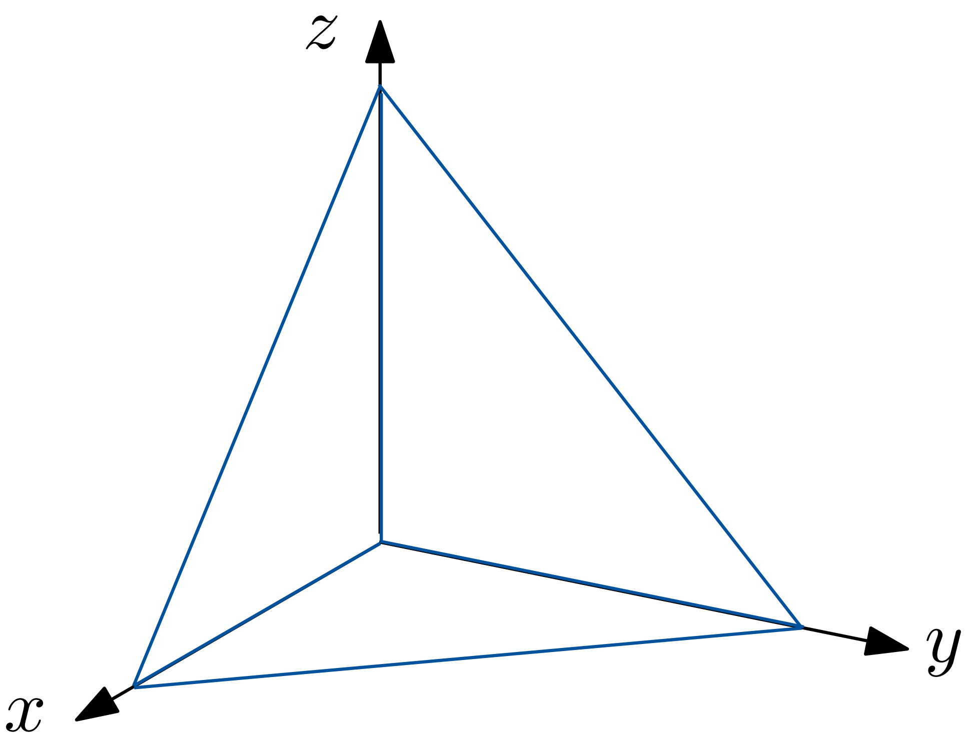


(pictures from mathinsight.org)

Example: Express as an iterated integral the volume of the smaller region bounded by $z = \sqrt{3x^2 + 3y^2}$ and $x^2 + y^2 + z^2 = 1$.



Example: The picture shows the tetrahedron T, bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. Express its volume in terms of iterated integrals.



From areas/volumes to mass, average value, etc.

Again, let's look at the 2D case first.

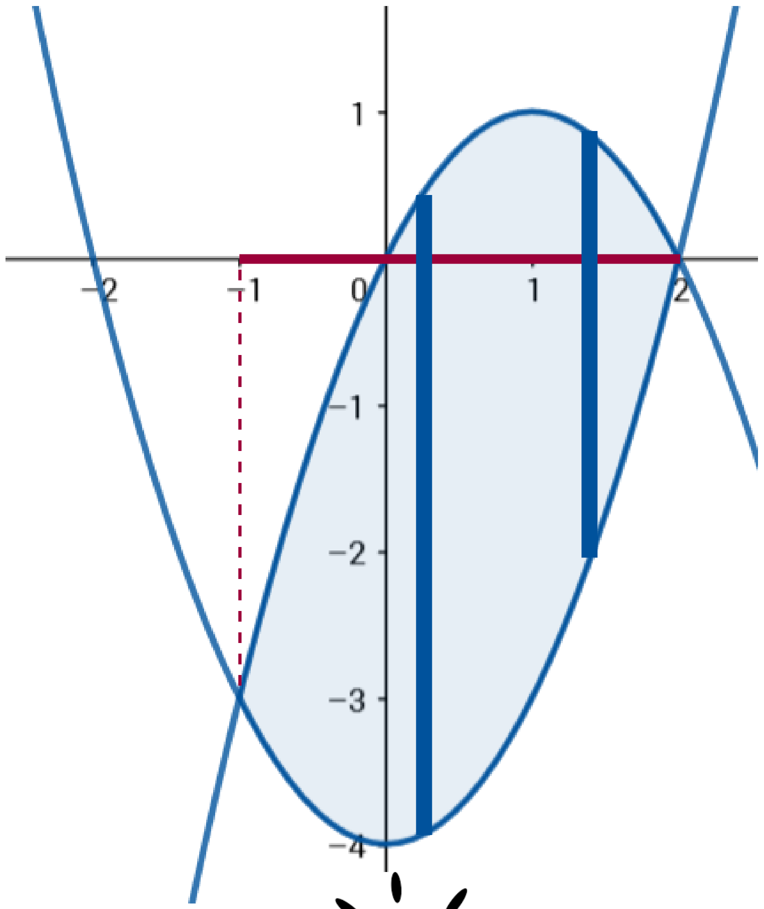
Example: Let R be the region bounded by $y = x^2 - 4$ and $y = -x^2 + 2x$. Recall from p6 that its area is

$$\int_{-1}^{2} (-x^2 + 2x) - (x^2 - 4)\, dx.$$

The shadow of R

top value of y bottom value of y

This difference is the area of the blue strip.



If its density function is $\delta(x, y)$, then the mass of R is

$$\int_{-1}^{2} \boxed{\int_{x^2-4}^{-x^2+2x} \delta(x, y)\, dy}\, dx.$$

top value of y

bottom value of y

The limits for the outer integral is the shadow of R.

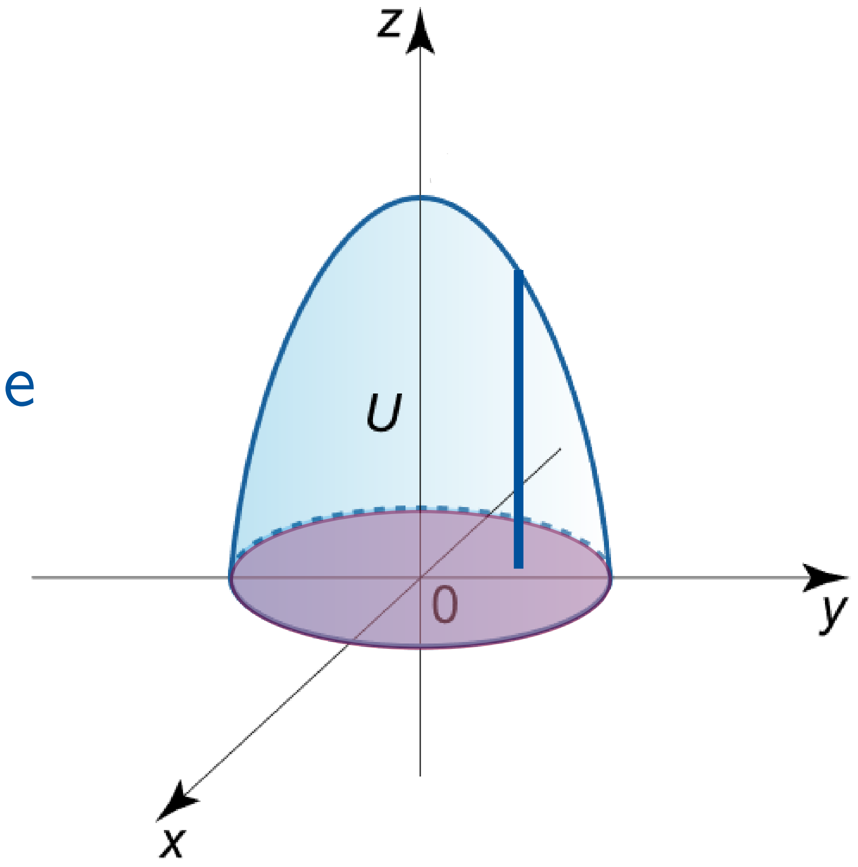
The inner integral is the mass of the blue strip.

The 3D case is similar: the top and bottom z values now appear as limits of the innermost integral, instead of inside the integrand.

Example: (see p10) The mass of the region bounded by $z = 1 - x^2 - y^2$ and $z = 0$ is

$$\int_S \boxed{\overset{\text{top value of } z}{\underset{\text{bottom value of } z}{\int_0^{1-x^2-y^2}}} \delta(x,y,z)\, dz} \, dA,$$

The inner integral is the mass of the blue stick above the point (x, y).



where the shadow S is $\left\{(x,y) \in \mathbb{R}^2 \middle| x^2 + y^2 \leq 1\right\}$.

(picture from math24.net)

The 3D case is similar: the top and bottom z values now appear as limits of the innermost integral, instead of inside the integrand.

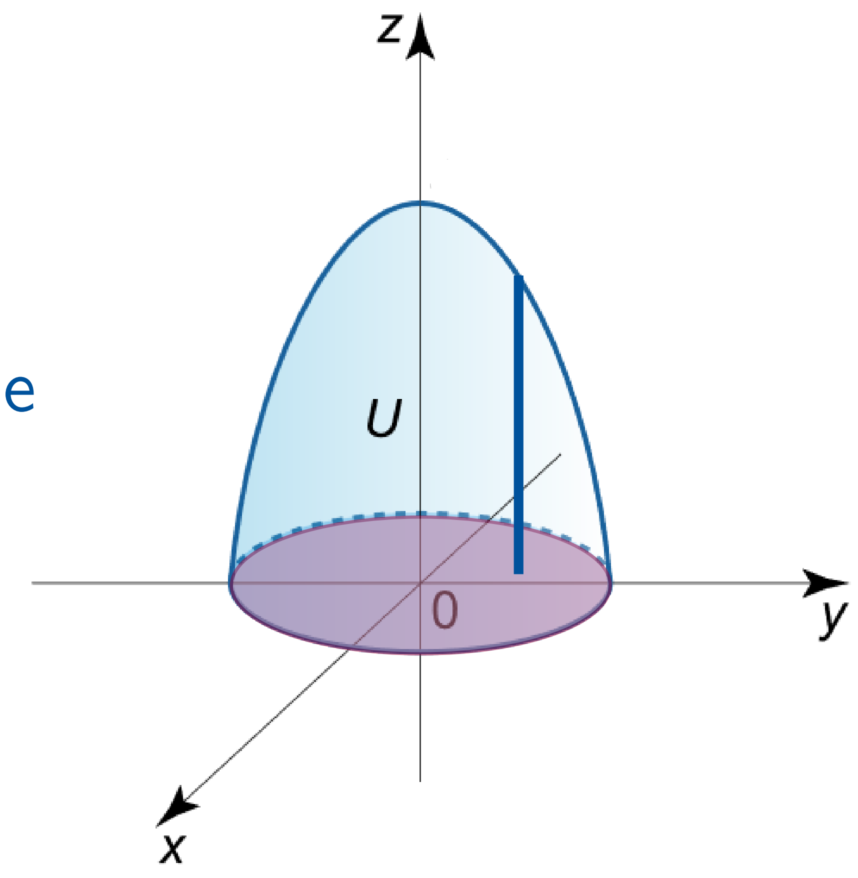
Example: (see p10) The mass of the region bounded by $z = 1 - x^2 - y^2$ and $z = 0$ is

top value of z

$$\int_S \left[\int_0^{1-x^2-y^2} \delta(x,y,z)\,dz \right] dA,$$

bottom value of z

The inner integral is the mass of the blue stick above the point (x, y).

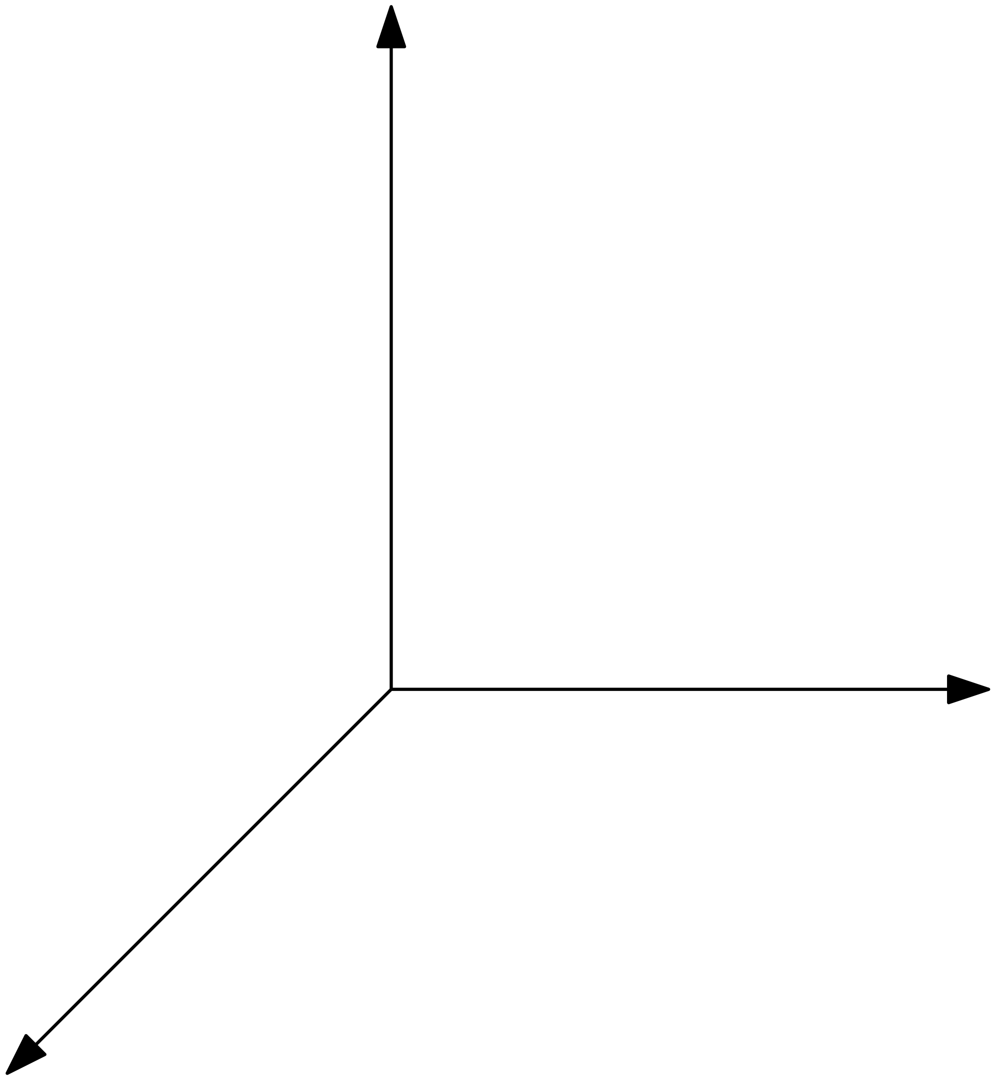


where the shadow S is $\left\{(x,y) \in \mathbb{R}^2 \middle| x^2 + y^2 \leq 1\right\}$.

Warning: The volume of a 3D region R is either $\iiint_R 1\,dV$ or $\iint_S f(x,y) - g(x,y)\,dA$.

(picture from math24.net)

Example: (see p12) Find the mass of the smaller region bounded by $z = \sqrt{3x^2 + 3y^2}$ and $x^2 + y^2 + z^2 = 1$, with density function $\delta(x, y, z) = x^2 z$.



On the previous two pages, the shadows were circular, so we used 2D polar coordinates on the outer, shadow integral. The notation is slightly easier if we use cylindrical coordinates.

Recall (p1) that the general triple integral formula $\int_a^b \int_{c(x)}^{d(x)} \int_{p(x,y)}^{q(x,y)} f(x,y,z)\,dz\,dy\,dx$ comes from dividing the region into rectangular subdomains. Cylindrical coordinates correspond to a different subdivision method.

The *cylindrical coordinates* $[r, \theta, z]$ of a point P are:

- r is the distance from P to the z-axis (i.e. horizontal distance);
- θ is the counterclockwise angle from the x, z-plane to the plane containing P and the z-axis;
- z is the distance from P to the x, y-plane (i.e. vertical distance).

So the distance from P to the origin is $\sqrt{r^2 + z^2}$.

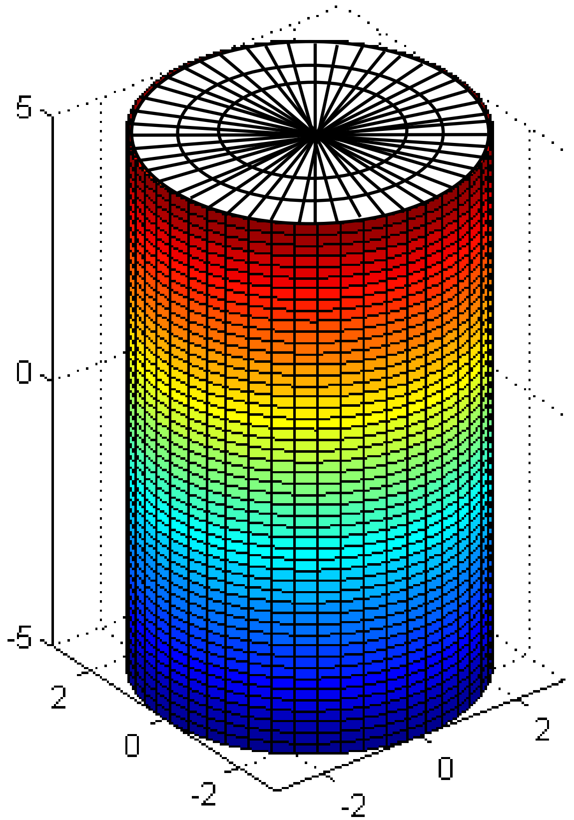
Informally, in terms of dividing our domain, we are slicing horizontally along the planes $z = z_k$, and then dividing each slice according to 2D polar coordinates.

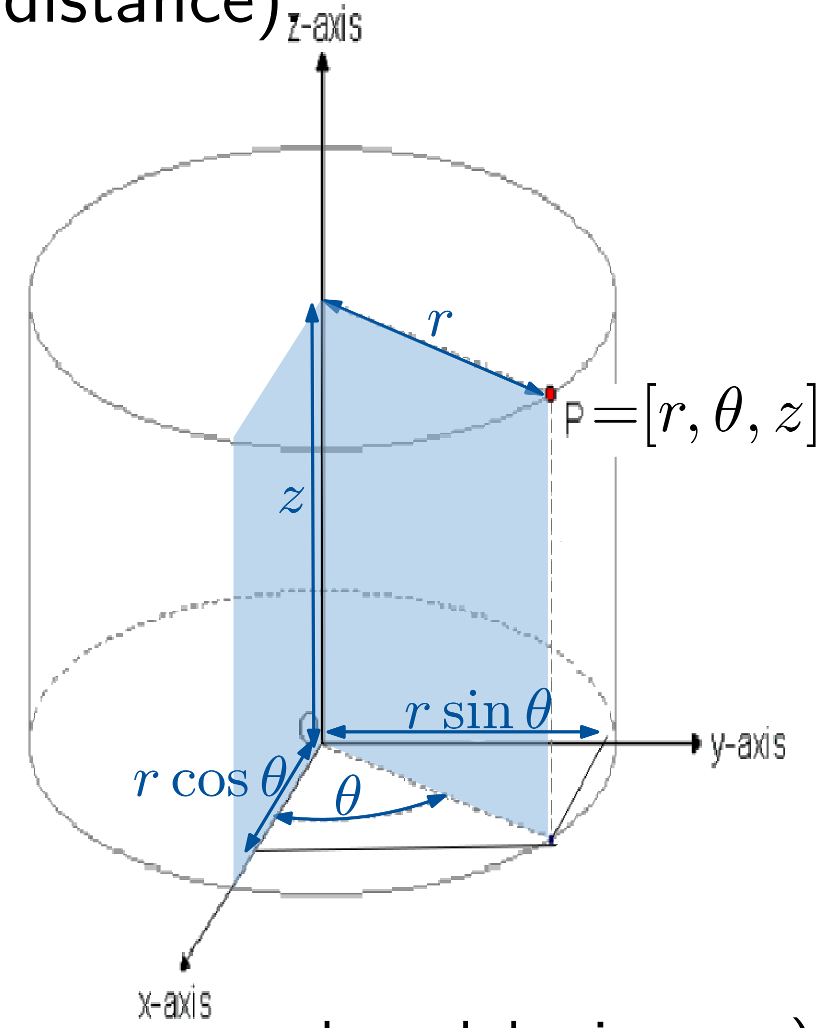
To change to Cartesian coordinates:

$x = r\cos\theta$,

$y = r\sin\theta$,

$z = z$.





(pictures from santoshlinkha.wordpress.com, astarmathsandphysics.com)

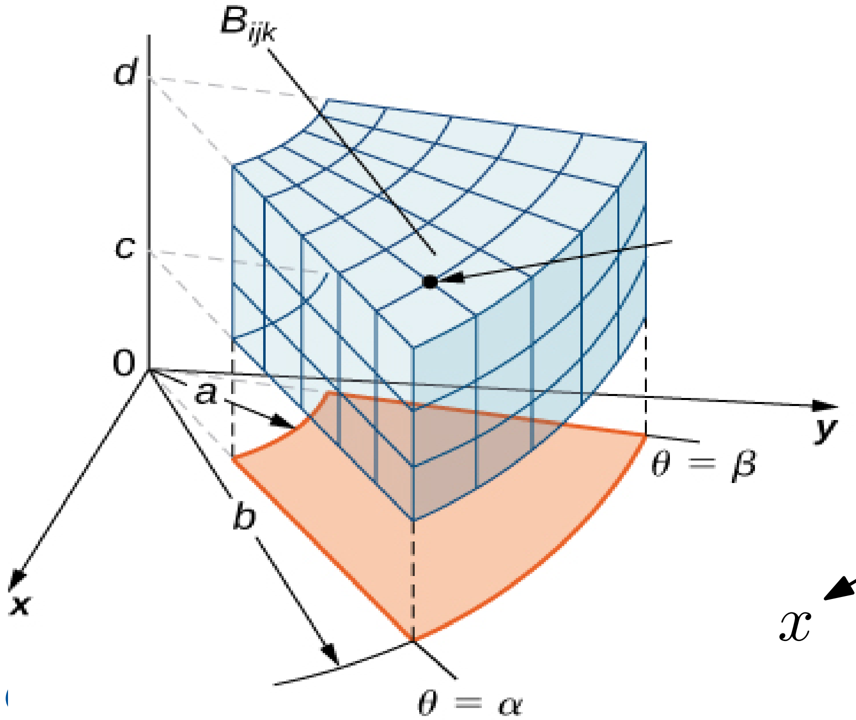
To compute iterated integrals using cylindrical coordinates, we need to know the volume ΔV_{ijk} of each small piece B_{ijk} in the cylindrical coordinate grid.

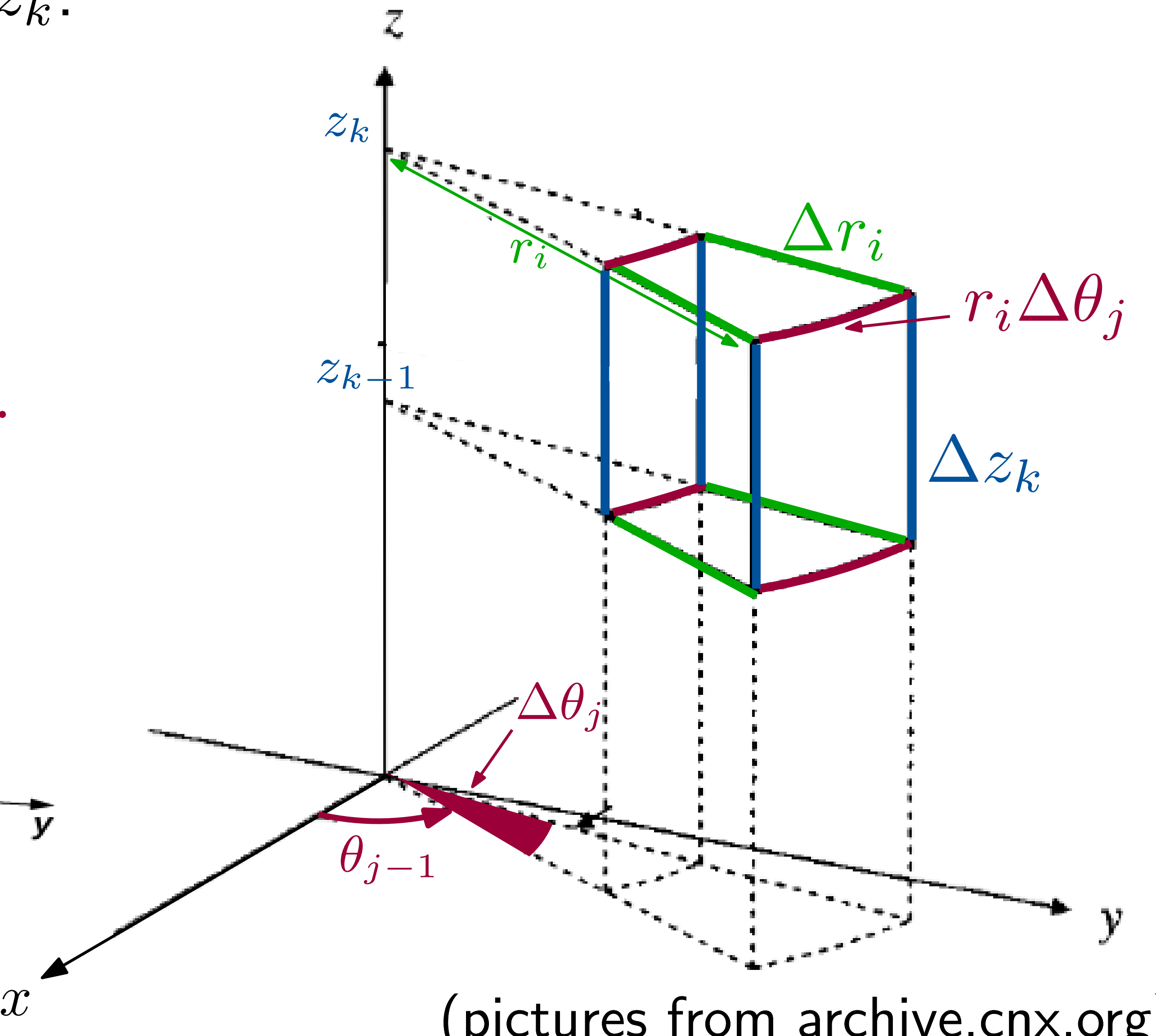
The base of B_{ijk} is a piece of a 2D polar grid, so its area is $\Delta A_{ij} \approx r_i \Delta r_i \Delta\theta_j$ (week 5 p32). And the height of B_{ijk} is Δz_k.

So $\Delta V_{ijk} \approx r_i \Delta r_i \Delta \theta_j \Delta z_k$.

So $\displaystyle\iiint_D f(x,y,z)dV$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz.$$





(pictures from archive.cnx.org)

The notation is easier with cylindrical coordinates:

Example: (see p16) Express as an iterated integral the mass of the smaller region bounded by $z = \sqrt{3x^2 + 3y^2}$ and $x^2 + y^2 + z^2 = 1$, with density function $\delta(x, y, z) = x^2 z$.

We know from p12 that the region satisfies $\sqrt{3x^2 + 3y^2} \leq z \leq \sqrt{1 - x^2 - y^2}$, which gives us the limits for the z-integral in terms of r. We also know that the shadow is the disk $r \leq \frac{1}{2}$. So the mass is

$$\boxed{\int_0^{2\pi} \int_0^{1/2}} \underbrace{\boxed{\int_{\sqrt{3}r}^{\sqrt{1-r^2}} (r\cos\theta)^2 z \, dz}}_{\text{mass of a stick}} \boxed{r \, dr \, d\theta} = \int_0^{2\pi} \int_0^{1/2} \frac{r^3 \cos^2\theta}{2} z^2 \Big|_{\sqrt{3}r}^{\sqrt{1-r^2}} dr \, d\theta$$

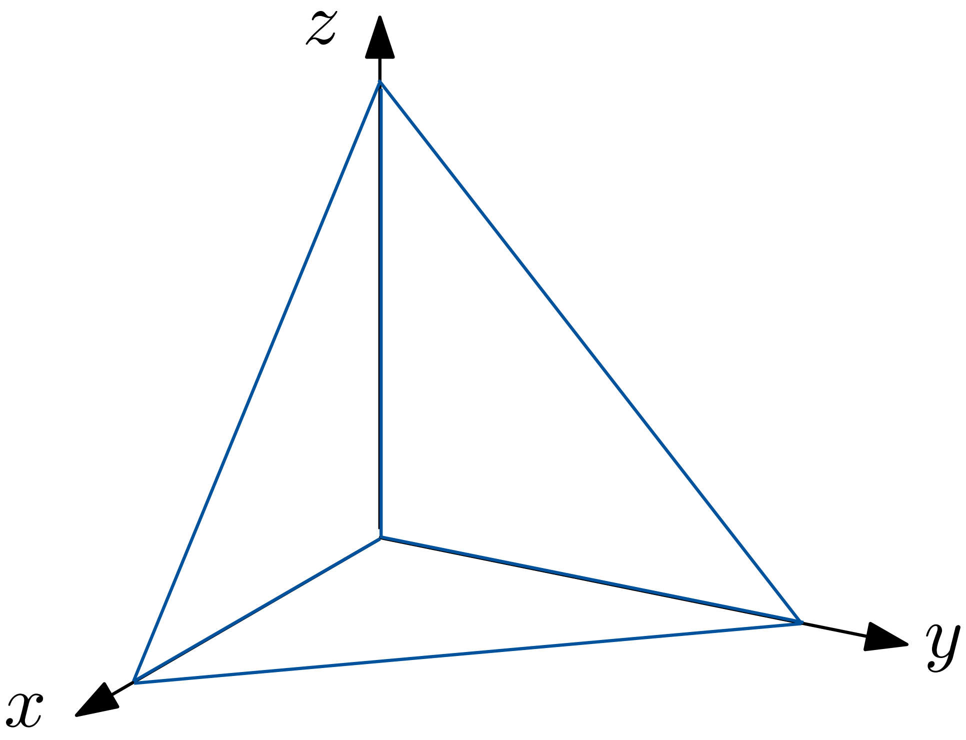
$$= \int_0^{2\pi} \int_0^{1/2} \frac{r^3 \cos^2\theta}{2} \left(1 - r^2 - 3r^2\right) dr \, d\theta$$

$= \ldots$ (as on p16)

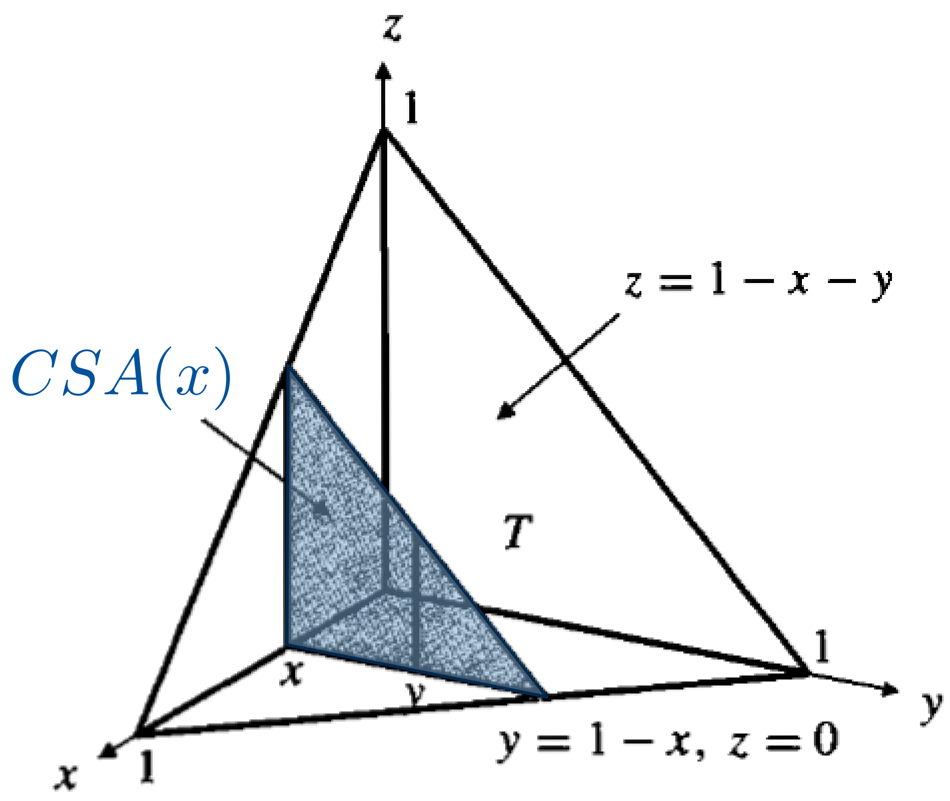
integrate over the shadow

Back to triple integration in general:

Example: (see p13) The picture shows the tetrahedron T, bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. If its density is $\delta(x, y, z) = y$, find its mass.



As an alternative to the shadow method (1D integral followed by 2D integral), Example 3 of §14.5 in the textbook computes this mass using the cross-section method (2D integral followed by 1D integral):



$$\int_0^1 \boxed{\iint_{CSA(x)} y\,dA(y,z)} \, dx.$$

$CSA(x)$ is the cross-sectional area, i.e. the area of the blue slice.

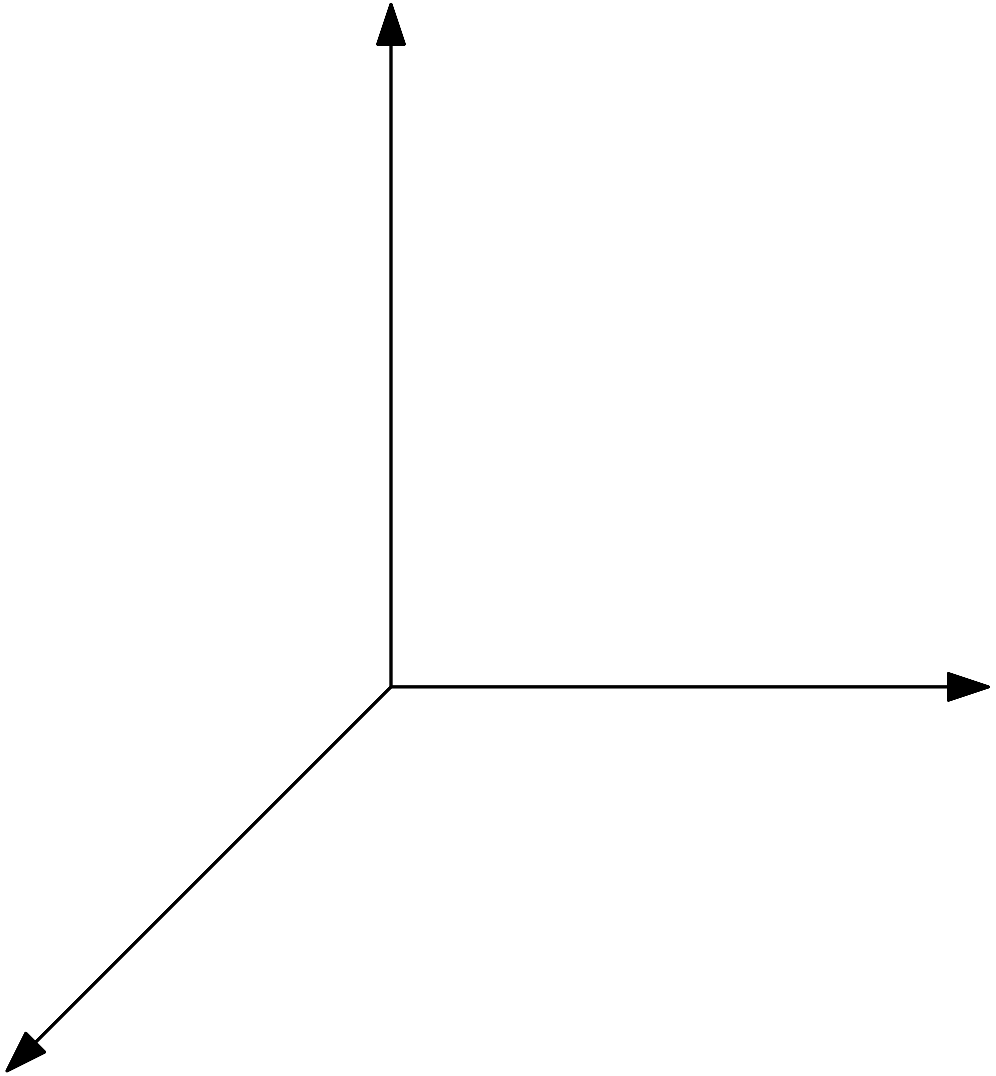
This is the mass of the blue slice.

I personally prefer the shadow method over the cross-section method because I find it hard to work out the limits of a 2D integral that depend on a third variable. But for some situations the cross-section method may be better.

Situations with cylinders and other surfaces are more complicated. For example, with one cylinder and two planes, there are two possibilities:

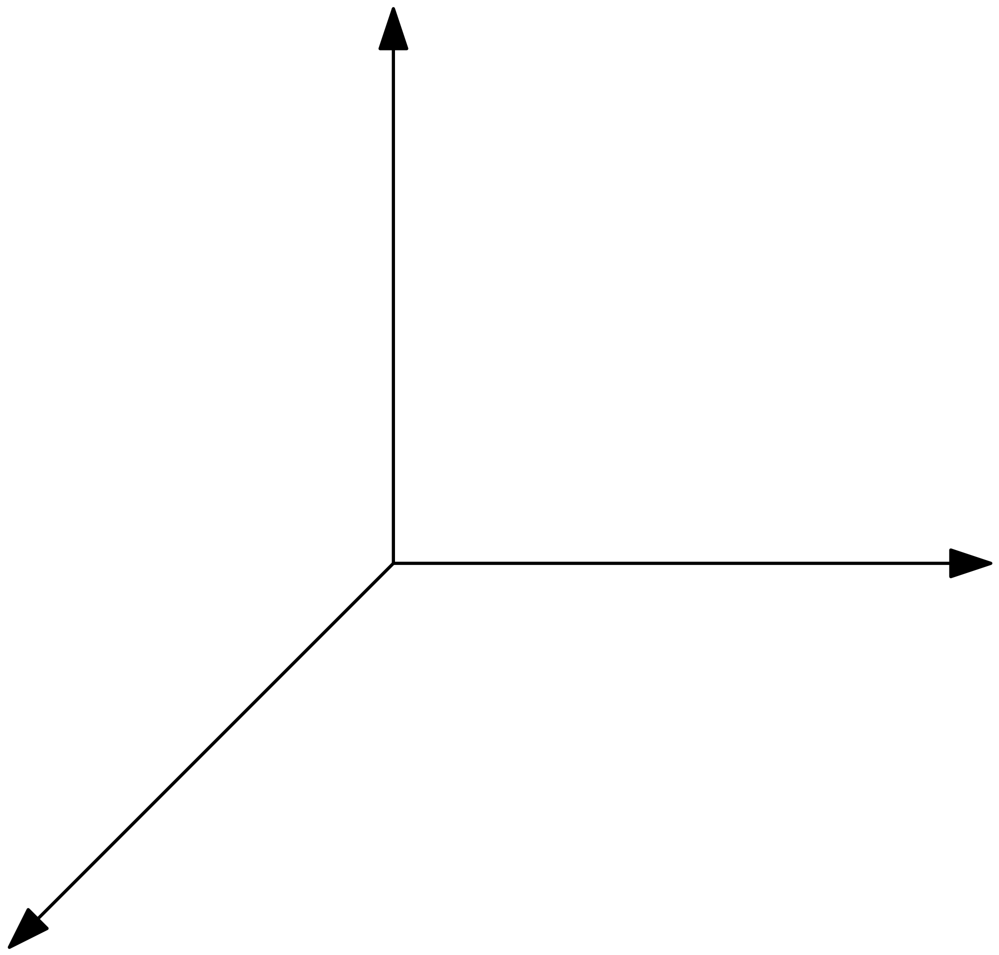
To find out which case we are in, we can find the x, y coordinates of the intersection of the two planes, and compare it to the equation of the cylinder (these are the two " important shadow curves").

Example: Let R be the region bounded by $9x^2 + y^2 = 9$, $y + z = 1$ and $z = 0$, with $z \geq 0$. Describe and sketch R, and express its volume in terms of iterated integrals.



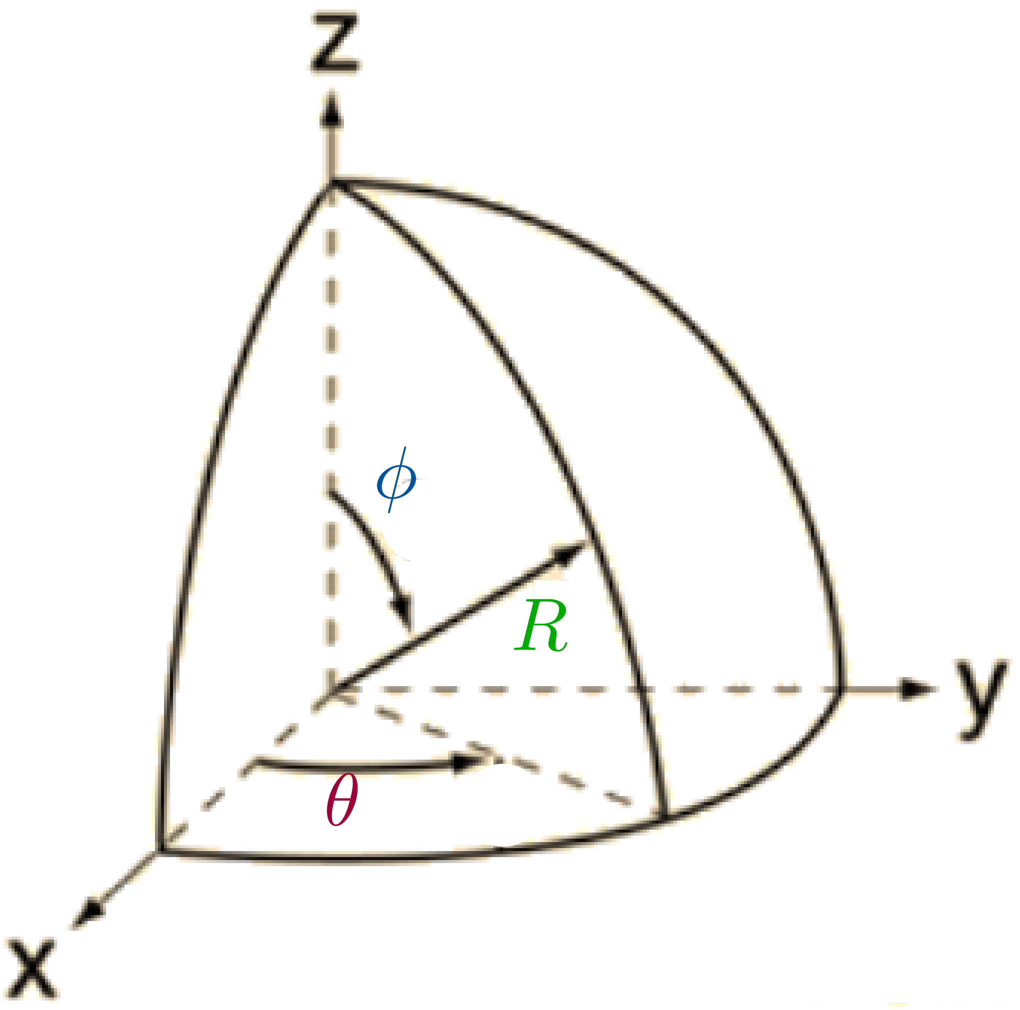
One example where the "important shadow curves" do not come from equating z values, nor from equations without zs:

Example: (see ex. sheet #12 and p32) A chocolate occupies the region between $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$. Its density function is $\delta(x, y, z) = x^2$. Find the mass of the chocolate.



The previous example showed that cylindrical coordinates may be inconvenient in problems involving spheres.
A possibly better choice is *spherical coordinates* $[R, \theta, \phi]$ (see next page for the coordinate grid):

- R is the distance from P to the origin;
- θ is the counterclockwise angle from the x, z-plane to the plane containing P and the z-axis;
- ϕ is the angle from the positive z-axis to the vector $\overrightarrow{OP}$.

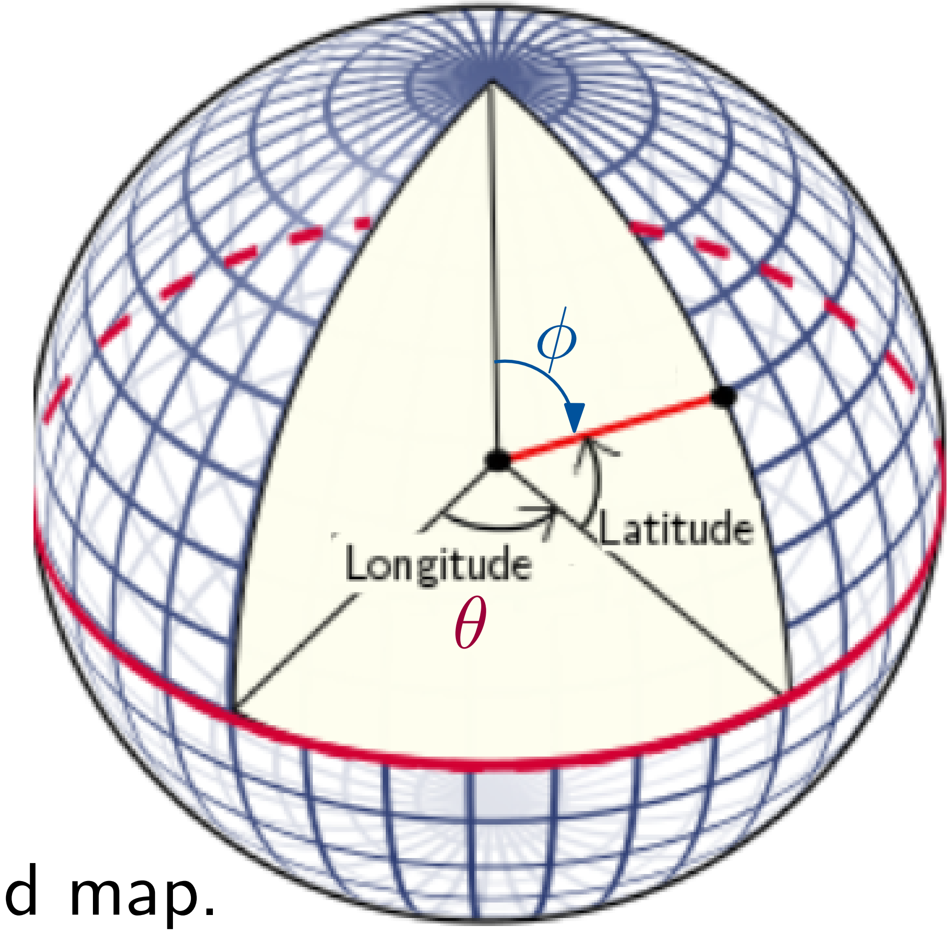


Warnings:

- R in spherical coordinates is not the same as r in cylindrical coordinates (but θ is the same in both coordinates);
- θ goes from 0 to 2π, but ϕ goes from 0 to π.

(picture from Hyperphysics)

- R is the distance from P to the origin;
- θ is the counterclockwise angle from the x, z-plane to the plane containing P and the z-axis;
- ϕ is the angle from the positive z-axis to the vector $\overrightarrow{OP}$.



Informally, in terms of dividing our domain, the spherical coordinate grid first separates the domain into spheres of different radii centred at the origin (R), then slices the sphere along the "horizontal", latitude lines (ϕ) and vertical, longitude lines (θ), like on a world map.

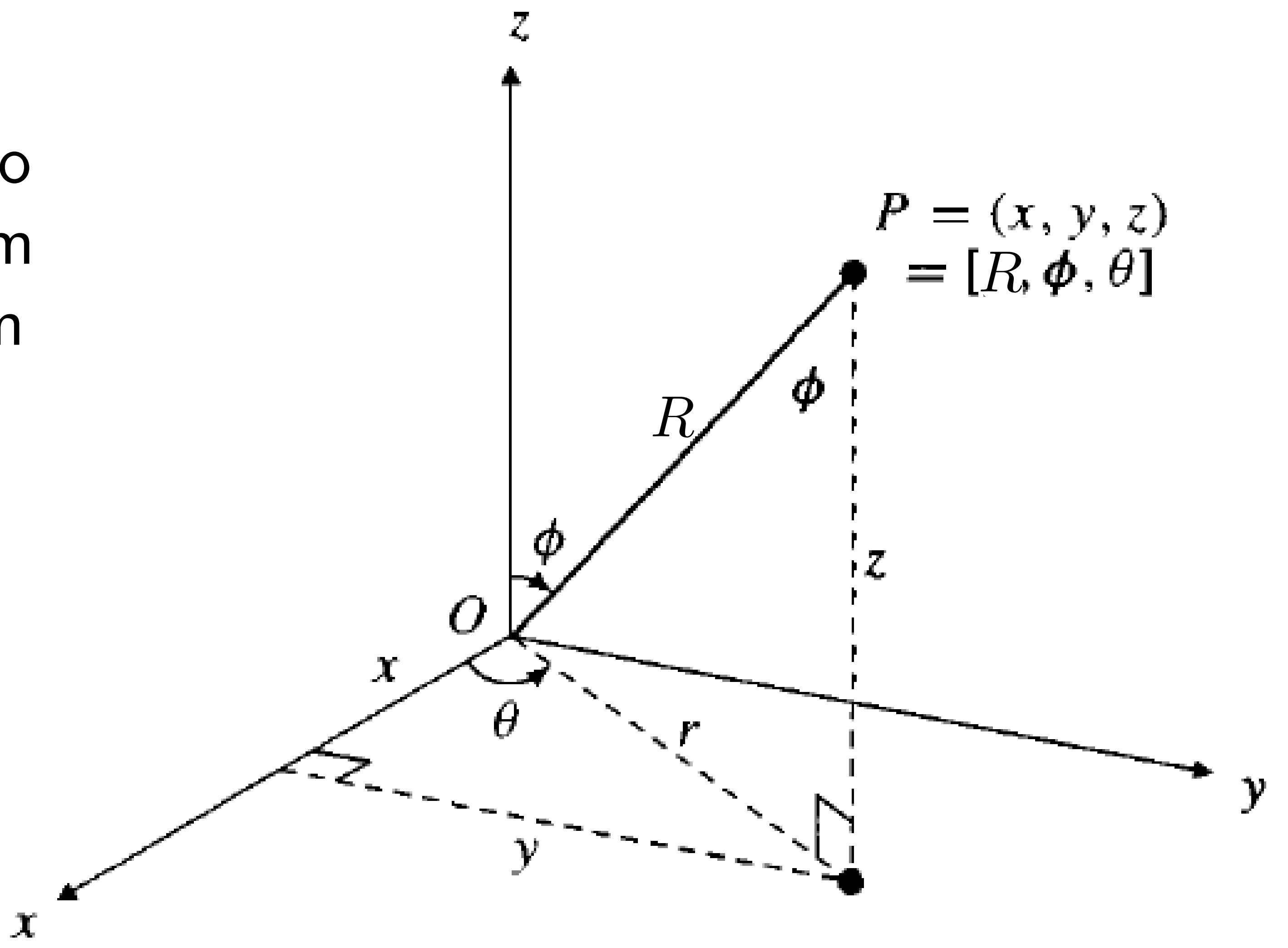
(A small difference between geographic and mathematical conventions: latitude is measured from the equator (so the equator is 0, the north pole is $\frac{\pi}{2}$, the south pole is $\frac{-\pi}{2}$); ϕ is measured from the north pole (so the north pole is $\phi = 0$, the equator is $\phi = \frac{\pi}{2}$, the south pole is $\phi = \pi$. So ϕ is called the "colatitude".)

Different authors use different symbols for the angles in spherical coordinates - outside of this class, you should say "colatitude" or "longitude".

(picture from vvvv.org)

- R is the distance from P to the origin;
- θ is the counterclockwise angle from the x, z-plane to the plane containing P and the z-axis;
- ϕ is the angle from the positive z-axis to the vector $\overrightarrow{OP}$.

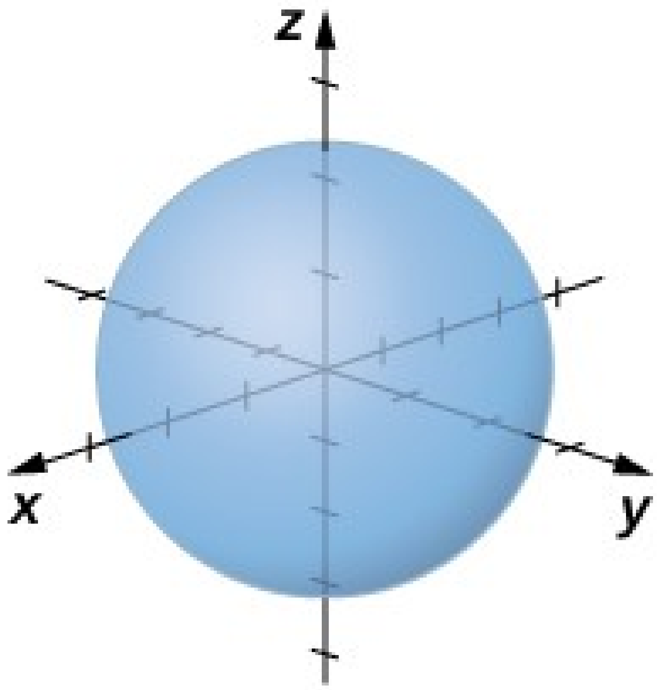
To change from spherical coordinates to Cartesian coordinates, first observe from the right-angled triangle in this diagram that $z = R\cos\phi$ and $r = R\sin\phi$. So
$x = r\cos\theta = R\sin\phi\cos\theta,$
$y = r\sin\theta = R\sin\phi\sin\theta,$
$z = R\cos\phi.$



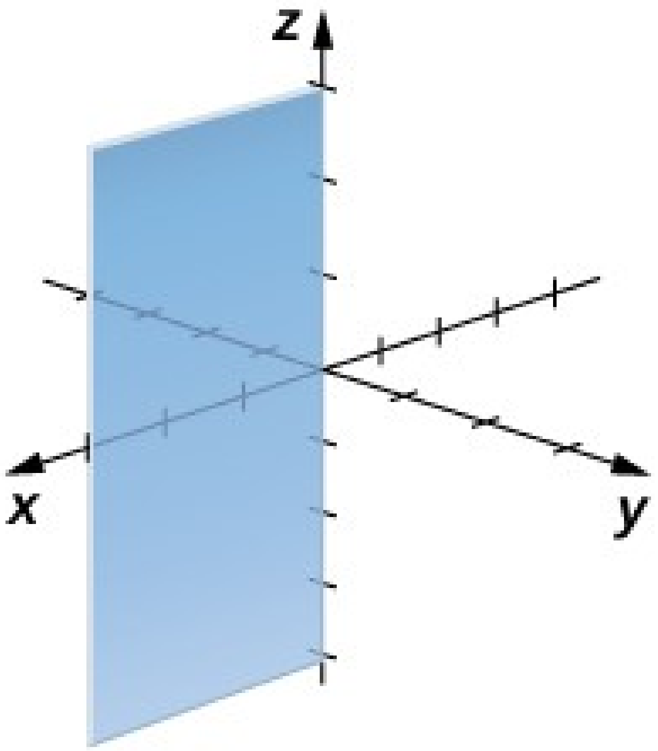
To understand spherical coordinates, it may help to consider the surfaces where one coordinate is fixed and the other two change.

$$r = R\sin\phi,$$
$$x = r\cos\theta = R\sin\phi\cos\theta,$$
$$y = r\sin\theta = R\sin\phi\sin\theta,$$
$$z = R\cos\phi.$$

The surfaces $R = R_i$ are spheres centred at the origin.



The surfaces $\theta = \theta_j$ are half-planes which include the z-axis.

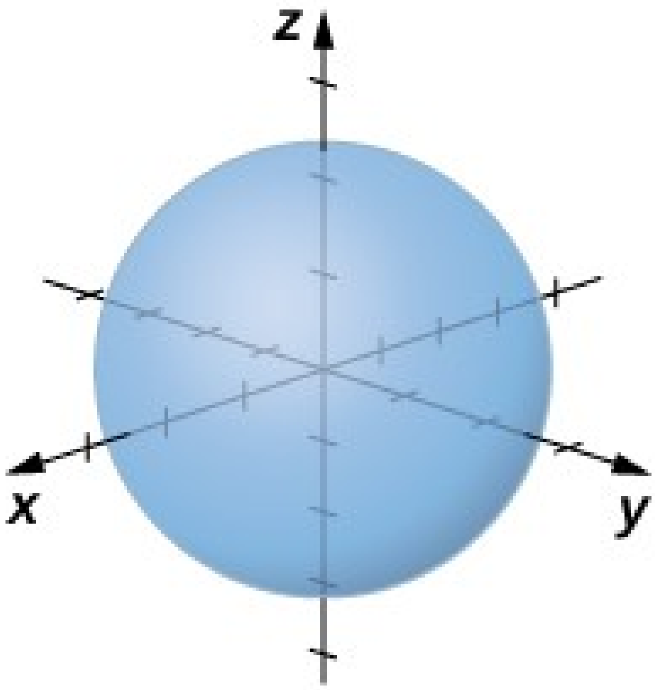


(picture from archive.cnx.org)

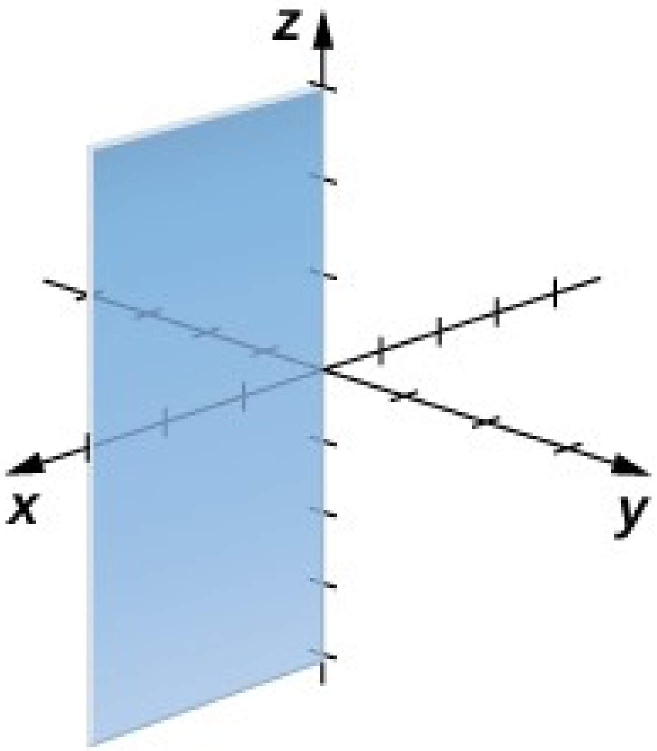
To understand spherical coordinates, it may help to consider the surfaces where one coordinate is fixed and the other two change.

$$r = R\sin\phi,$$
$$x = r\cos\theta = R\sin\phi\cos\theta,$$
$$y = r\sin\theta = R\sin\phi\sin\theta,$$
$$z = R\cos\phi.$$

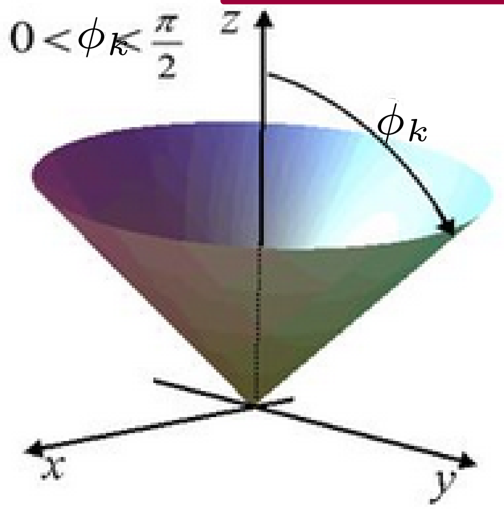
The surfaces $R = R_i$ are spheres centred at the origin.

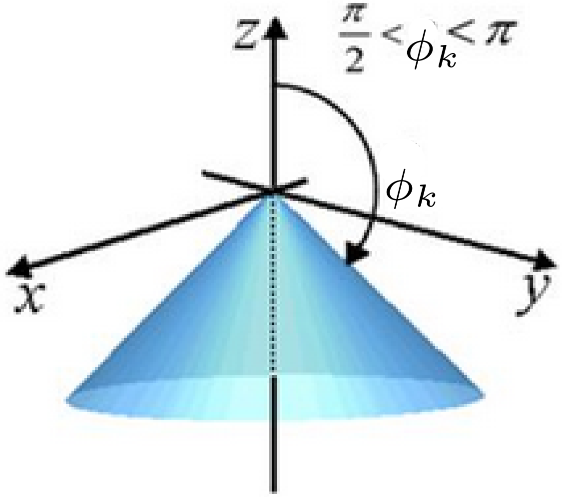


The surfaces $\theta = \theta_j$ are half-planes which include the z-axis.



(picture from archive.cnx.org)





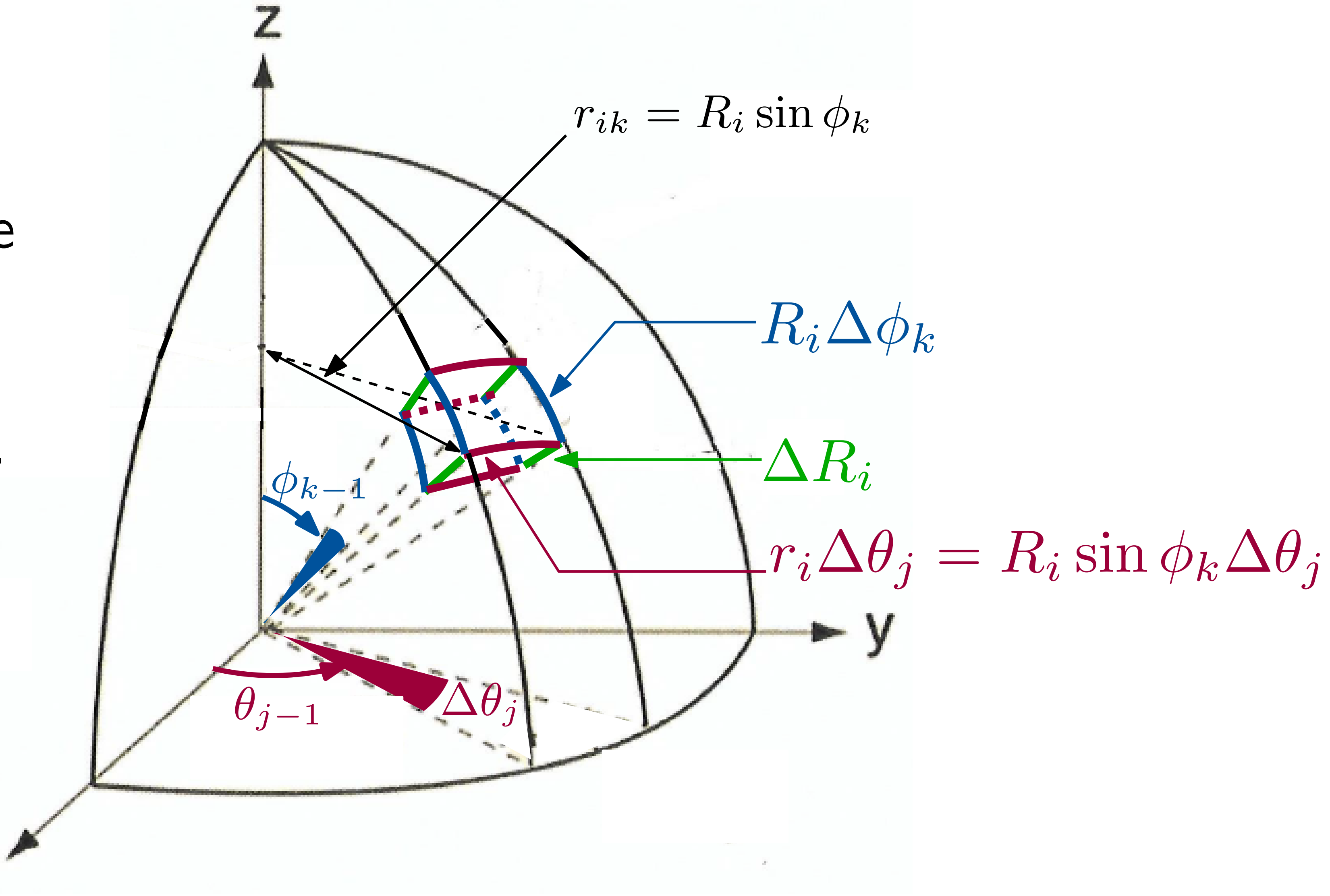
The surfaces $\phi = \phi_k$ are half-cones opening in the z-direction. To see this, remember that a half-cone has equation

$$z = c\sqrt{x^2+y^2} = cr,$$

so $\dfrac{1}{c} = \dfrac{r}{z} = \tan\phi.$

To compute iterated integrals using spherical coordinates, we need to know the volume ΔV_{ijk} of each small piece in the spherical coordinate grid.

We approximate each small piece by a rectangular box. The diagram shows the lengths of the sides of this box. So its volume is the product of these lengths: $\Delta V_{ijk} = R_i^2 \sin\phi_k \Delta R_i \Delta\theta_j \Delta\phi_k$.



$$\iiint_D f(x,y,z)dV = \int_\gamma^\delta \int_\alpha^\beta \int_a^b f(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)R^2\sin\phi\, dR\, d\theta\, d\phi$$

(picture from Hyperphysics)

$$\int_{\gamma}^{\delta}\int_{\alpha}^{\beta}\int_{a}^{b} f(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)R^2\sin\phi\,dR\,d\theta\,d\phi$$

Redo Example: (p16) Find the mass of the smaller region bounded by $z = \sqrt{3x^2+3y^2}$ and $x^2+y^2+z^2=1$, with density function $\delta(x,y,z) = x^2z$.

$$\int_{\gamma}^{\delta}\int_{\alpha}^{\beta}\int_{a}^{b} f(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)R^2\sin\phi\,dR\,d\theta\,d\phi$$

Redo Example: (ex. sheet #12) A chocolate occupies the region between $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$. Its density function is $\delta(x,y,z)=x^2$. Find the mass of the chocolate.

Given a integral over a 3D domain that is "circular" in some way, it may not be obvious whether to use cylindrical or spherical coordinates. In many examples, both methods work, and even for the same situation different people may have different opinions about which method is easier. The only way to find out what's easiest for you is to do many examples.

Here are my preferences (you may disagree):

- If the region involves cylinders or paraboloids (which are hard to describe in spherical coordinates), I try cylindrical coordinates first. This applies even if spheres, cones and other shapes are involved.
- If the region involves spheres and cones only, I look at the integrand:
  - If the integrand involves complicated functions of $x^2 + y^2$, I try cylindrical coordinates first.
  - Otherwise, I try spherical coordinates first.

If, after changing into my chosen coordinates, the integral looks very complicated, I will rewrite it in the other coordinates before trying to evaluate it.