§13.1-3: Extreme Values

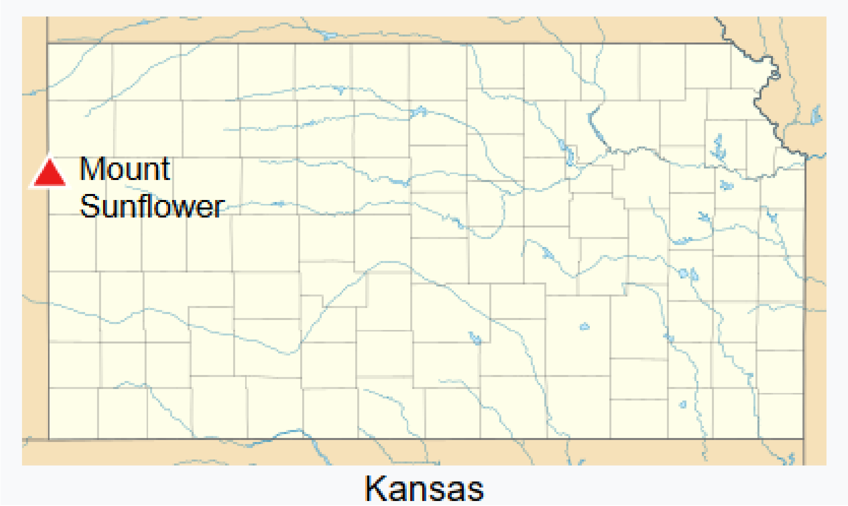
In this week's notes, we develop techniques for finding (absolute) maxima and minima of multivariate functions, i.e. points $\mathbf{a}$ where $f(\mathbf{a}) \geq f(\mathbf{x})$ for all $\mathbf{x}$ in the domain of f (or $f(\mathbf{a}) \leq f(\mathbf{x})$, for minima). The value of f at the maxima and minima are called *extreme values*, or extrema

As a start, imagine we wish to find the tallest person in the class. We can measure the heights of all 100 students and compare the measurements to find the tallest student.

Now imagine we wish to find the highest point in Kansas (a state in USA). Now the domain of the height function is "Kansas", which contains infinitely many points, so it's impractical to measure the height at every point and compare. So instead, let's measure the height of all the mountain tops (all the local maxima) in Kansas and compare their heights.

But this is "Mount Sunflower", the highest point in Kansas. It does not look like a mountain.

As Wikipedia explains, "the state of Kansas gradually increases in [height] from the east to the west". The height continues to increase to the west of Mount Sunflower, but that area is in Colorado, not Kansas.



(pictures from Wikipedia and AtlasObscura)

Another way to explain this: let f be the height function. If $\nabla f(a,b) \neq \mathbf{0}$, then I can walk from (a,b) in the direction of $\nabla f(a,b)$ to increase my height. So, if (a,b) is the highest point in Kansas, one of three things must happen:

- $\nabla f(a,b) = \mathbf{0}$ so there is no direction I can walk in to climb higher;
- f is not differentiable at (a,b) (i.e. (a,b) is a *singular* point), so I don't know which direction to walk in to climb higher;
- (a,b) is on the boundary, and $\nabla f(a,b)$ "points out of the domain", so I must walk outside Kansas to climb higher.

This idea is true for functions of any number of variables:

Theorem 1: Necessary conditions for extreme values: If $f : \mathcal{D} \to \mathbb{R}$ achieves a maximum or minimum value at $\mathbf{a}$, then $\mathbf{a}$ must be a critical point of f, or a singular point of f, or a boundary point of D.

For single-variable functions, we handle the possibility of extreme values on the boundary by checking the endpoints of an interval (Homework 4 last question). For higher-dimensional domains, the boundaries are more complicated, so we cannot check every boundary point.

As stated in the theorem, if f achieves its extreme value at a non-boundary point $\mathbf{a}$, then either $\nabla f(\mathbf{a}) = \mathbf{0}$, or f is not differentiable at $\mathbf{a}$.

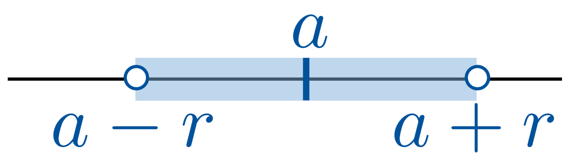
The main topic in this week's notes is to derive similar conditions that a boundary point must satisfy if f achieves its extreme value there. We will give two different set of conditions, which are useful for differently-shaped boundaries. Then we only need to check the shorter list of boundary points that satisfy these conditions.

In this week's notes:

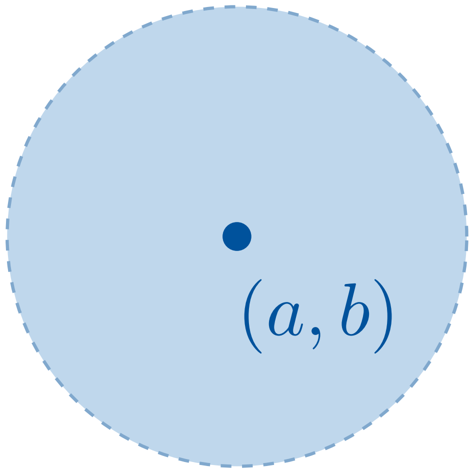
- What is the boundary? (p5-9, §10.1)
- General strategy for finding extreme values (p10, §13.1)
- Two ways to check for extreme values on the boundary:
  - Parametrisation: for straight lines and simple boundaries (p11-13, §13.2)
  - Lagrange multipliers: for complicated boundaries (p14-?, §13.3)
- Do extreme values exist? (p?-?, §13.1)

Definition: Let P be a point in $\mathbb{R}^n$. The *ball* of radius r around P is the set of points whose distance from P is less than r.

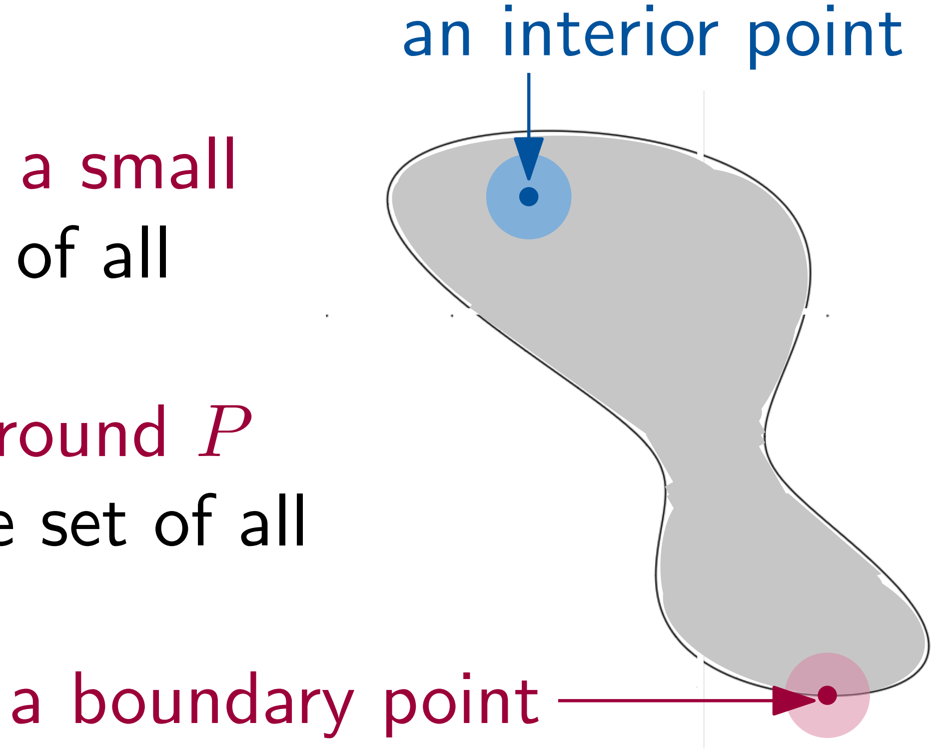
Example: In $\mathbb{R}^1$, the ball of radius r around a number a is the open interval $(a-r, a+r)$.



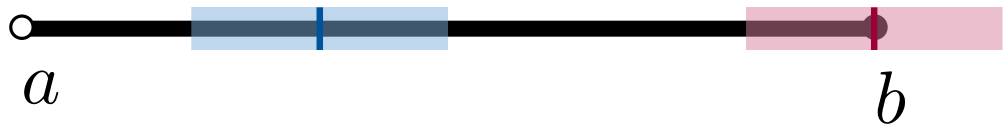
Example: In $\mathbb{R}^2$, the ball of radius r around a point (a,b) is the region inside the circle of radius r and centre (a,b):
$\{(x,y) \in \mathbb{R}^2 \mid (x-a)^2 + (y-b)^2 < r^2\}$.



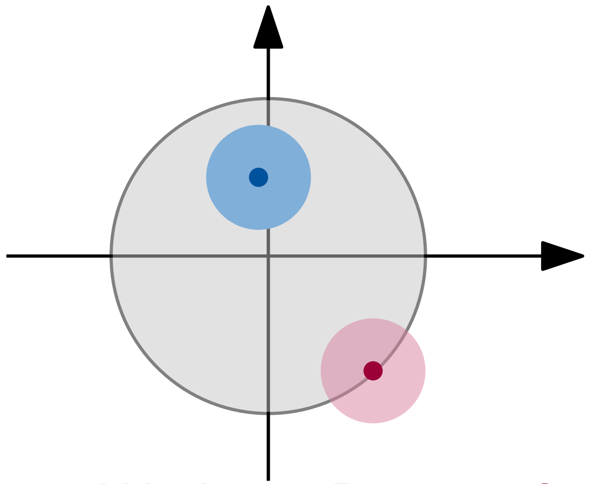
Definition: Let S be a subset of $\mathbb{R}^n$.
A point P in S is an *interior point* of S if there is a small ball around P that is completely inside S. The set of all interior points of S is the *interior of S*.
A point P is a *boundary point* of S if every ball around P contains both points in S and points not in S. The set of all boundary points of S is the *boundary of S*.



Example: Consider $S = (a, b]$.
The interior is (a, b).
The boundary are the two points $\{a\} \cup \{b\}$.
Note that a boundary point of S may be in S or not in S.



Example: Consider the unit disk $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.
The interior is $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$.
The boundary is $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

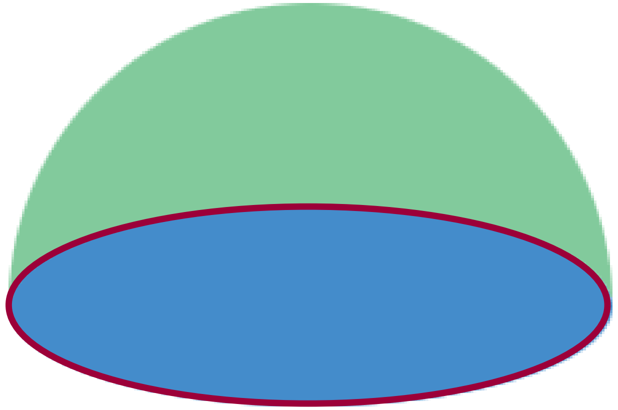


As these examples show, if S is defined by inequalities, then its interior is usually the part with a $<$ or $>$ sign, and its boundary is the part with a $=$ sign.

It is possible for a set to have no interior points e.g. the unit sphere $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, or any surface in $\mathbb{R}^3$ defined by one equation.

If S is defined by more than 1 inequality, then it is often convenient to write its boundary as the union of several sets ("boundary pieces") which are each defined by $<$, $>$ or $=$ signs (i.e. no $\leq$ or $\geq$ signs, because it is hard to search for extrema on this type of set).

Example: Consider the upper half ball $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1, \ z \geq 0\}$.
Its interior is $\{x^2 + y^2 + z^2 < 1, \ z > 0\}$.
Its boundary is in three pieces:
the "curved part on top": $\{x^2 + y^2 + z^2 = 1, \ z > 0\}$;
the "flat part on the bottom": $\{x^2 + y^2 + z^2 < 1, \ z = 0\}$;
the "circle on the edge": $\{x^2 + y^2 + z^2 = 1, \ z = 0\}$.



(picture from URegina MathCentral)

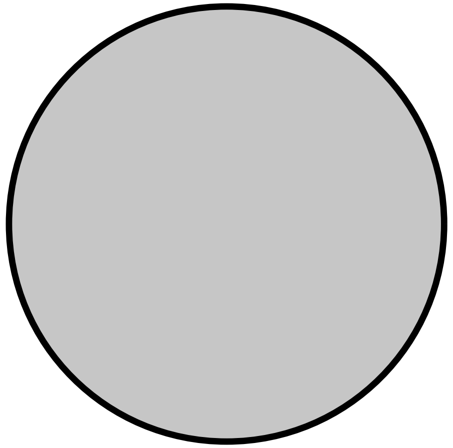
Two more definitions are useful:

Definition: A set is *closed* if it contains all its boundary points.
So sets that are defined by $\leq$, $\geq$ or $=$ signs are often closed.

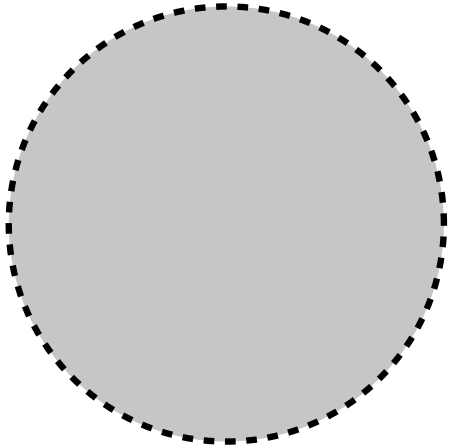
Definition: A set is *bounded* if it is contained in a large ball around the origin
Informally, a set is bounded if it doesn't "go to infinity".

Note that being bounded is unrelated to the boundary.

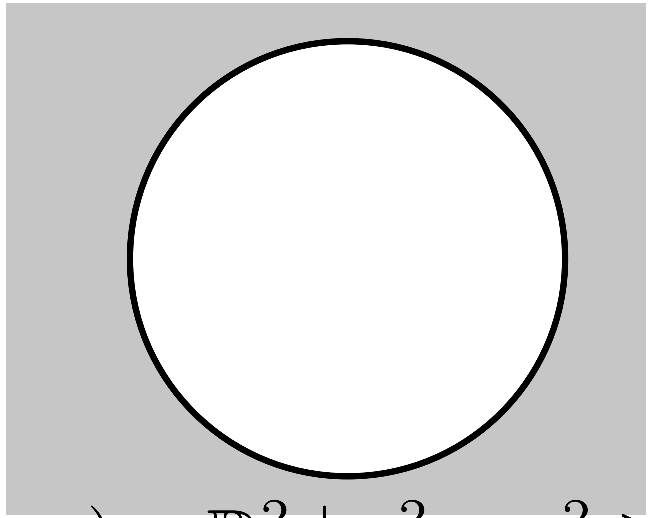
Examples:



$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$
is closed and bounded



$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$
is bounded but not closed



$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq 1\}$
is closed but not bounded

A closed and bounded set in $\mathbb{R}^n$ behaves like a closed interval $[a, b]$ in $\mathbb{R}^1$, in the following sense:

Theorem 2: Continuous functions on closed and bounded sets have extremum: If f is a continuous function whose domain $\mathcal{D}$ is a closed and bounded set in $\mathbb{R}^n$, then there are points in $\mathcal{D}$ where f achieves its maximum and minimum values.

Note that f may have maximum and minimum values even if the domain is not closed or not bounded, see ex sheet #19 Q1d. P?-? has some tips for determining whether extreme values exist when the domain is not closed or not bounded.

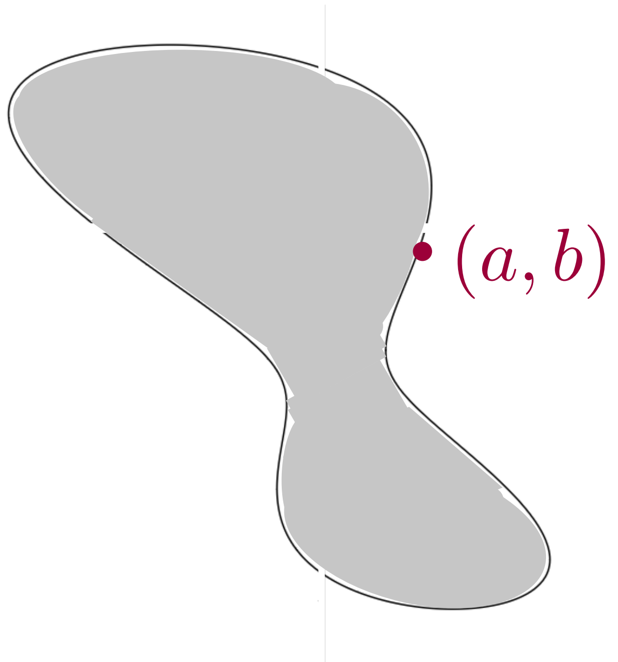
Remember our toy example of finding the tallest student by measuring all students and comparing these heights. This is the idea in the following algorithm.

Strategy for finding extreme values:

1. Draw the domain, and separate it into the interior (usually $<$ or $>$) and the boundary pieces (usually $=$).
2. Explain why the extremum exists, e.g. "f is a continuous function on a closed and bounded domain, so it has a maximum and a minimum."
3. Make a list of candidate extrema in the interior:
   - critical points: where $\nabla f = \mathbf{0}$.
   - singular points: where f is not differentiable.
4. Make a list of candidate extrema on each boundary piece: 2 possible methods
   - parametrise the boundary: for lines, boundaries of the form $y = p(x)$ or $z = p(x, y)$ where you can isolate one variable; circles and ellipses if f is simple.
   - Lagrange multipliers - circles and ellipses if f is complicated, other complicated boundaries
5. Compare the values of f at the candidate extrema from Steps 3 and 4.

Method 1 to find extrema on the boundary: boundary parametrisation

The main idea of this method is: suppose f achieves its maximum at (a,b), a point on the boundary. Then, if I were to walk along the boundary, I would experience a maximum of f when I pass through (a,b). In other words, if $t \to (x(t), y(t))$ is a parametrisation of a boundary piece, with $x(c) = a, y(c) = b$, then the single-variable function $f(x(t), y(t))$ must have a maximum at $t = c$ - i.e. c must be a critical point or singular point of $f(x(t), y(t))$.



This technique also works for boundary surfaces in $\mathbb{R}^3$: these are parametrised by $(s,t) \to (x(s,t), y(s,t), z(s,t))$, and we then look for critical and singular points of the 2-variable function $f(x(s,t), y(s,t), z(s,t))$ (see Homework 6 questions).

Example: Find the maximum value of $f(x, y) = x^2 + xy - 2y$ on the closed triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$, and the point(s) where this maximum value is achieved.

Redo example: (ex. sheet #19 Q2) Find the maximum and minimum values of $f(x, y) = x^2 + y$ on the unit circle $x^2 + y^2 = 1$, and the point(s) where these extreme values are achieved.