

Remember the addition and scalar multiplication of matrices:

$$(A + B)_{ij} = a_{ij} + b_{ij},$$

$$\text{e.g. } \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}.$$

$$(cA)_{ij} = ca_{ij},$$

$$\text{e.g. } (-3) \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -12 & 0 & -15 \\ 3 & -9 & -6 \end{bmatrix}.$$

Is this really different from  $\mathbb{R}^6$ ?

$$\begin{bmatrix} 4 \\ 0 \\ 5 \\ -1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 6 \\ 2 \\ 8 \\ 9 \end{bmatrix}.$$

$$(-3) \begin{bmatrix} 4 \\ 0 \\ 5 \\ -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ 0 \\ -15 \\ 3 \\ -9 \\ -6 \end{bmatrix}.$$

Remember from calculus the addition and scalar multiplication of polynomials:

$$\text{e.g } (2t^2 + 1) + (-t^2 + 3t + 2) = t^2 + 3t + 3.$$

$$\text{e.g } (-3)(-t^2 + 3t + 2) = 3t^2 - 9t - 6.$$

Is this really different from  $\mathbb{R}^3$ ?

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}.$$

$$(-3) \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \\ 3 \end{bmatrix}.$$

← coefficient of 1  
← coefficient of  $t$   
← coefficient of  $t^2$

## §4.1, pp217-218: Abstract Vector Spaces

As the examples above showed, there are many objects in mathematics that “looks” and “feels” like  $\mathbb{R}^n$ . We will also call these **vectors**.

The real power of linear algebra is that everything we learned in Chapters 1-3 can be applied to all these abstract vectors, not just to column vectors.

You should think of abstract vectors as objects which can be added and multiplied by scalars - i.e. where the concept of “linear combination” makes sense. This addition and scalar multiplication must obey some “sensible rules” called **axioms** (see next page).

The axioms guarantee that the proof of every result and theorem from Chapters 1-3 will work for our new definition of vectors.

A **vector space** is a nonempty set  $V$  of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms below. The axioms must hold for all  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  in  $V$  and for all scalars  $c$  and  $d$ .

1.  $\mathbf{u} + \mathbf{v}$  is in  $V$ .
2.  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ .
3.  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a vector (called the zero vector)  $\mathbf{0}$  in  $V$  such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$ .
5. For each  $\mathbf{u}$  in  $V$ , there is vector  $-\mathbf{u}$  in  $V$  satisfying  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ .
6.  $c\mathbf{u}$  is in  $V$ .
7.  $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ .
8.  $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ .
9.  $(cd)\mathbf{u} = c(d\mathbf{u})$ .
10.  $1\mathbf{u} = \mathbf{u}$ .

Examples of vector spaces:

$M_{2 \times 3}$ , the set of  $2 \times 3$  matrices.

Let's check axiom 4. There is a vector (called the zero vector)  $\mathbf{0}$  in  $V$  such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$ .

The zero vector for  $M_{2 \times 3}$  is  $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

You can check the other 9 axioms by using the properties of matrix addition and scalar multiplication (page 5 of week 4 slides, theorem 2.1 in textbook).

Similarly,  $M_{m \times n}$ , the set of all  $m \times n$  matrices, is a vector space.

Is the set of all matrices (of all sizes) a vector space?

No, because we cannot add two matrices of different sizes, so axiom 1 does not hold.

Examples of vector spaces:

$\mathbb{P}_n$ , the set of polynomials of degree **at most**  $n$ .

Each of these polynomials has the form

$$a_0 + a_1t + a_2t^2 + \cdots + a_nt^n,$$

for some numbers  $a_0, a_1, \dots, a_n$ .

Let's check axiom 4. There is a vector (called the zero vector)  $\mathbf{0}$  in  $V$  such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$ .

The zero vector for  $\mathbb{P}_n$  is  $0 + 0t + 0t^2 + \cdots + 0t^n$ .

Let's check axiom 1.  $\mathbf{u} + \mathbf{v}$  is in  $V$ .

$$\begin{aligned} & (a_0 + a_1t + a_2t^2 + \cdots + a_nt^n) + (b_0 + b_1t + b_2t^2 + \cdots + b_nt^n) \\ &= (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \cdots + (a_n + b_n)t^n, \text{ which also has degree} \\ & \text{at most } n. \end{aligned}$$

Exercise: convince yourself that the other axioms are true.

Examples of vector spaces:

Warning: the set of polynomials of degree **exactly**  $n$  is **not** a vector space, because axiom 1 does not hold:

$$\underbrace{(t^3 + t^2)}_{\text{degree 3}} + \underbrace{(-t^3 + t^2)}_{\text{degree 3}} = \underbrace{2t^2}_{\text{degree 2}}$$

$\mathbb{P}$  , the set of all polynomials (no restriction on the degree) is a vector space.

$C(\mathbb{R})$ , the set of all continuous functions is a vector space (because the sum of two continuous functions is continuous, the zero function is continuous, etc.)

These last two examples are a bit different from  $M_{m \times n}$  and  $\mathbb{P}_n$  because they are infinite-dimensional (more later, see week 8 §4.5).

(You do **not** have to remember the notation  $M_{m \times n}$ ,  $\mathbb{P}_n$ , etc. for the vector spaces.)

Let  $W$  be the set of symmetric  $2 \times 2$  matrices. Is  $W$  a vector space?

1.  $\mathbf{u} + \mathbf{v}$  is in  $V$ .

2.  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ .

3.  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

4. There is a vector (called the zero vector)  $\mathbf{0}$  in  $V$  such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$ .

5. For each  $\mathbf{u}$  in  $V$ , there is vector  $-\mathbf{u}$  in  $V$  satisfying  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ .


6.  $c\mathbf{u}$  is in  $V$ .

7.  $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ .

8.  $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ .

9.  $(cd)\mathbf{u} = c(d\mathbf{u})$ .

10.  $1\mathbf{u} = \mathbf{u}$ .


$$A = A^T, \text{ i.e. } A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \text{ for some } a, b, d$$

$W$  is a subset of  $M_{2 \times 2}$ .

Axioms 2, 3, 5, 7, 8, 9, 10 hold for  $W$  because they hold for  $M_{2 \times 2}$ .

So we only need to check axioms 1, 4, 6.



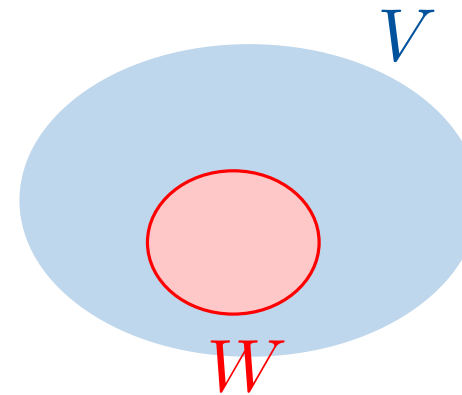
**Definition:** A subset  $W$  of a vector space  $V$  is a *subspace* of  $V$  if the closure axioms 1,4,6 hold:

- 4. The zero vector is in  $W$ .
- 1. If  $\mathbf{u}, \mathbf{v}$  are in  $W$ , then their sum  $\mathbf{u} + \mathbf{v}$  is in  $W$ . (closed under addition)
- 6. If  $\mathbf{u}$  is in  $W$  and  $c$  is any scalar, the scalar multiple  $c\mathbf{u}$  is in  $W$ . (closed under scalar multiplication)

**Fact:**  $W$  is itself a vector space (with the same addition and scalar multiplication as  $V$ ) if and only if  $W$  is a subspace of  $V$ .

To show that  $W$  is a subspace, check **all** three axioms directly, for all  $\mathbf{u}, \mathbf{v}, c$  (i.e. use variables).

To show that  $W$  is not a subspace, show that **one** of the axioms is false, for a particular value of  $\mathbf{u}, \mathbf{v}, c$ .



**Definition:** A subset  $W$  of a vector space  $V$  is a *subspace* of  $V$  if the *closure axioms* 1,4,6 hold:

4. The zero vector is in  $W$ .

1. If  $\mathbf{u}, \mathbf{v}$  are in  $W$ , then their sum  $\mathbf{u} + \mathbf{v}$  is in  $W$ . (*closed under addition*)

6. If  $\mathbf{u}$  is in  $W$  and  $c$  is any scalar, the scalar multiple  $c\mathbf{u}$  is in  $W$ . (*closed under scalar multiplication*)

**Example:** Let  $W$  be the set of vectors of the form  $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$ , where  $a, b$  can take any value.

( $W$  is the  $x_1x_3$ -plane.) We show that  $W$  is a subspace of  $\mathbb{R}^3$ :

4. The zero vector is in  $W$  because it is the vector with  $a = 0, b = 0$ .

$$1. \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} x \\ 0 \\ y \end{bmatrix} = \begin{bmatrix} a+x \\ 0 \\ b+y \end{bmatrix} \text{ is in } W.$$

$$6. c \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} ca \\ 0 \\ cb \end{bmatrix} \text{ is in } W.$$

Although  $W$  “feels like”  $\mathbb{R}^2$ , note that  $\mathbb{R}^2$  is **not** a subspace of  $\mathbb{R}^3$  - vectors in  $\mathbb{R}^2$  have two entries, so they are not in  $\mathbb{R}^3$ .

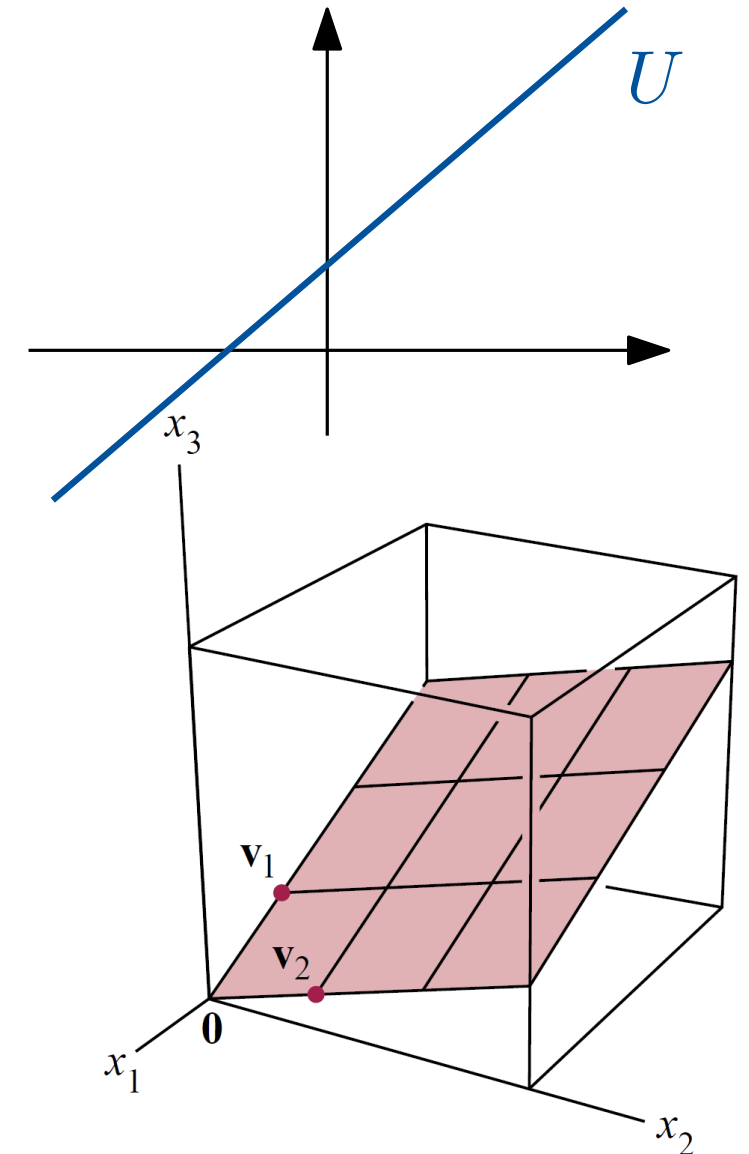
**Example:** Let  $U$  be the set of vectors of the form  $\begin{bmatrix} x \\ x+1 \end{bmatrix}$ , where  $x$  can take any value. To show that  $U$  is not a subspace of  $\mathbb{R}^2$ , we need to find one counterexample to one of the axioms, e.g.

4. The zero vector is not in  $U$ , because there is no value of  $x$  with  $\begin{bmatrix} x \\ x+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .

An alternative answer:

1.  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  are in  $U$ , but  $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$  is not of the form  $\begin{bmatrix} x \\ x+1 \end{bmatrix}$ , so  $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$  is not in  $U$ . So  $U$  is not closed under addition.

Best examples of a subspace: **lines and planes containing the origin** in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ .



**Example:** Let  $Q$  be the set of polynomials  $\mathbf{p}(t)$  of degree at most 3 with  $\mathbf{p}(2) = 0$ . We show that  $Q$  is a subspace of  $\mathbb{P}_3$ :

- 4. The zero polynomial is in  $Q$  because  $\mathbf{0}(2) = 0 + 0 \cdot 2 + 0 \cdot 2^2 + 0 \cdot 2^3 = 0$ .
- 1. For  $\mathbf{p}, \mathbf{q}$  in  $Q$ , we have  $(\mathbf{p} + \mathbf{q})(2) = \mathbf{p}(2) + \mathbf{q}(2) = 0 + 0 = 0$ , so  $\mathbf{p} + \mathbf{q}$  is in  $Q$ .
- 6. For  $\mathbf{p}$  in  $Q$  and any scalar  $c$ , we have  $(c\mathbf{p})(2) = c(\mathbf{p}(2)) = c0 = 0$ , so  $c\mathbf{p}$  is in  $Q$ .

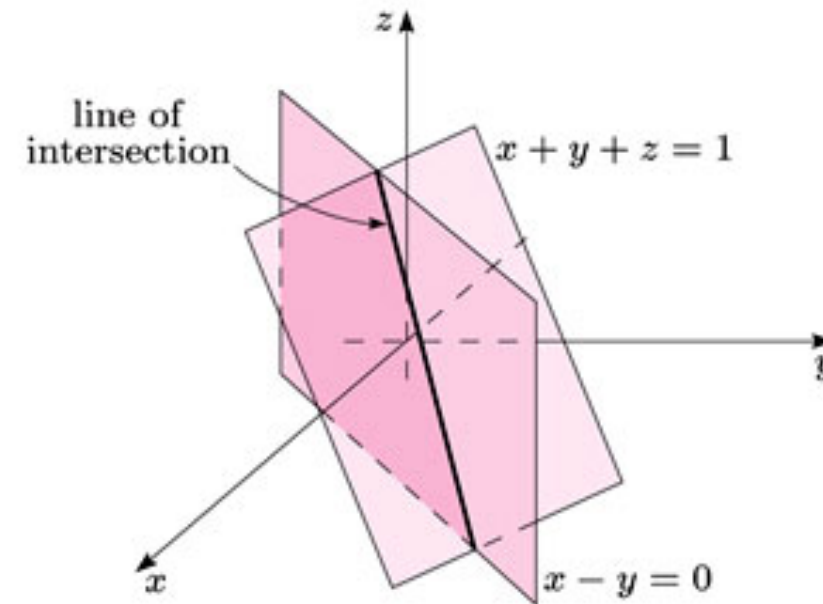
**Example:** In every vector space  $V$ , the set  $\{\mathbf{0}\}$  containing only the zero vector is a subspace:

- 4.  $\mathbf{0}$  is clearly in the subspace.
  - 1.  $\mathbf{0} + \mathbf{0} = \mathbf{0}$  (use axiom 4:  $\mathbf{0} + \mathbf{u} = \mathbf{u}$  for all  $\mathbf{u}$  in  $V$ ).
  - 6.  $c\mathbf{0} = \mathbf{0}$  (use axiom 7:  $c(\mathbf{0} + \mathbf{0}) = c\mathbf{0} + c\mathbf{0}$ ; and left hand side is  $c\mathbf{0}$ .)
- $\{\mathbf{0}\}$  called the **zero subspace**.

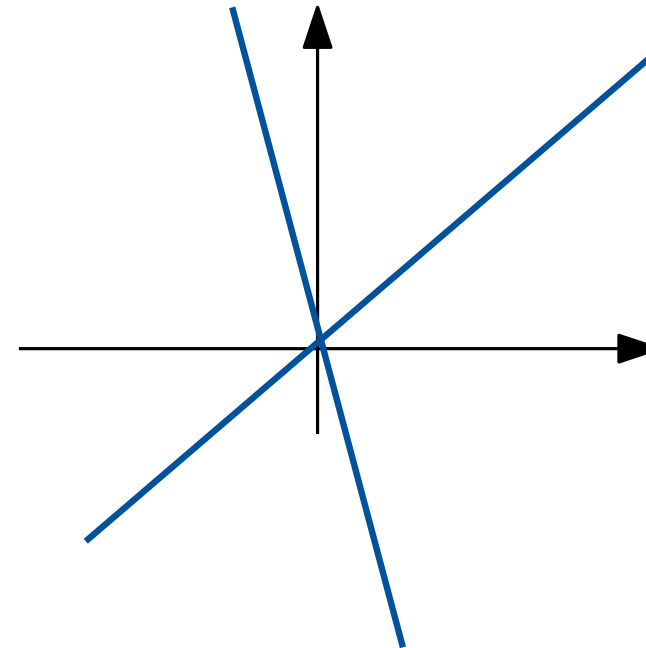
**Example:** For every vector space  $V$ , the whole space  $V$  is a subspace.

Non-examinable:

**Fact:** The intersection of two subspaces is a subspace (e.g. the intersection of two planes through the origin is a line through the origin).  
Exercise: prove this from the axioms.



But the union of two subspaces is in general not a subspace (e.g. the union of two lines through the origin is not a line nor a plane through the origin, see also ex sheet Q1b).



The first of two shortcuts to show that a set is a subspace:

**Theorem 1: Spans are subspaces:** If  $\mathbf{v}_1, \dots, \mathbf{v}_p$  are vectors in a vector space  $V$ , then  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is a subspace of  $V$ .

**Redo Example:** (p10) Let  $W$  be the set of vectors of the form  $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$ , where

$a, b$  can take any value. We can rewrite such a vector as

$\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ . Because  $a$  and  $b$  can take any value, this shows that

$W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ . So  $W$  is a subspace of  $\mathbb{R}^3$ .

The first of two shortcuts to show that a set is a subspace:

**Theorem 1: Spans are subspaces:** If  $\mathbf{v}_1, \dots, \mathbf{v}_p$  are vectors in a vector space  $V$ , then  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is a subspace of  $V$ .

**Redo Example:** (p8) Let  $\text{Sym}_{2 \times 2}$  be the set of symmetric  $2 \times 2$  matrices, i.e. the set of matrices of the form  $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$  where  $a, b, d$  can take any value. We can rewrite such a matrix as  $\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ . So  $\text{Sym}_{2 \times 2} = \text{Span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$  and is therefore a subspace of  $M_{2 \times 2}$ .

Warning: Theorem 1 does not help us show that a set is **not** a subspace.

### THEOREM 1: Spans are subspaces

If  $\mathbf{v}_1, \dots, \mathbf{v}_p$  are vectors in a vector space  $V$ , then  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is a subspace of  $V$ .

**Proof:** We check axioms 4, 1 and 6 in the definition of a subspace.

4.  $\mathbf{0}$  is in  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  since

$$\mathbf{0} = \text{---} \mathbf{v}_1 + \text{---} \mathbf{v}_2 + \cdots + \text{---} \mathbf{v}_p$$

1. To show that  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is closed under addition, we choose two arbitrary vectors in  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  :

$$\mathbf{u} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_p \mathbf{v}_p$$

and

$$\mathbf{v} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \cdots + b_p \mathbf{v}_p.$$

Then

$$\mathbf{u} + \mathbf{v} = (a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_p \mathbf{v}_p) + (b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \cdots + b_p \mathbf{v}_p)$$

$$= \text{---} \mathbf{v}_1 + \text{---} \mathbf{v}_2 + \cdots + \text{---} \mathbf{v}_p$$

So  $\mathbf{u} + \mathbf{v}$  is in  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ .

6. To show that  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is closed under scalar multiplication, choose an arbitrary number  $c$  and an arbitrary vector in  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  :

$$\mathbf{v} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \cdots + b_p \mathbf{v}_p.$$

Then

$$c\mathbf{v} = c(b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \cdots + b_p \mathbf{v}_p)$$

$$= \text{---} \mathbf{v}_1 + \text{---} \mathbf{v}_2 + \cdots + \text{---} \mathbf{v}_p$$

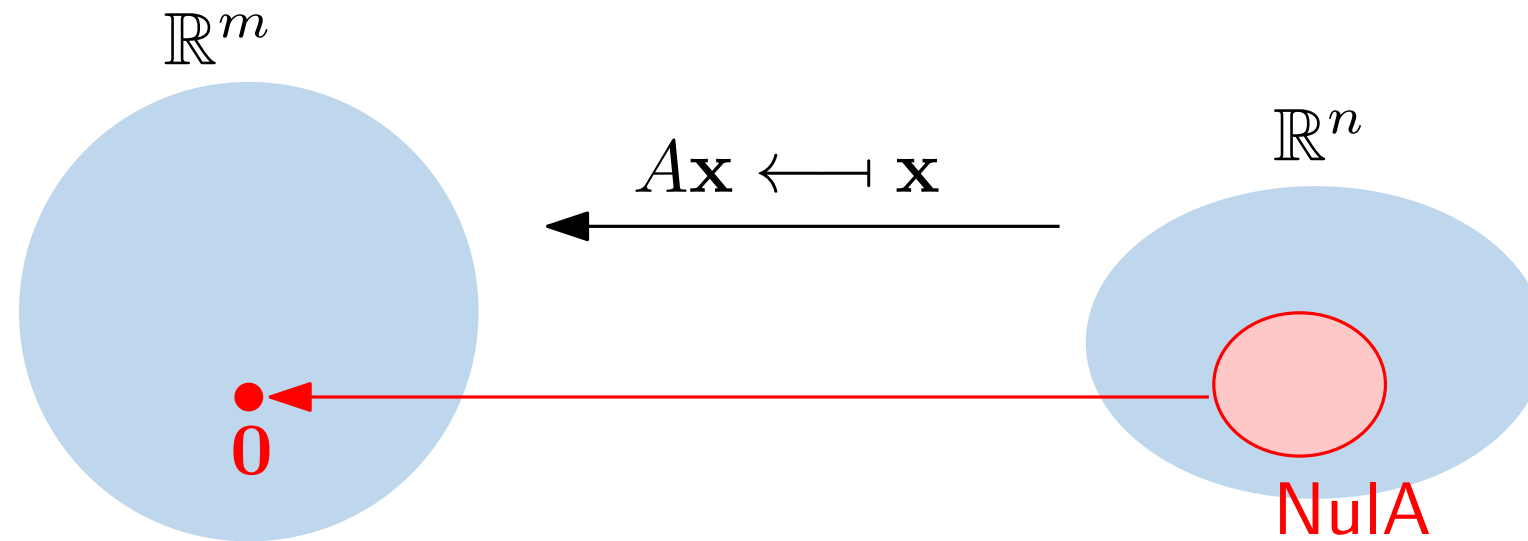
So  $c\mathbf{v}$  is in  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ .

Since 4, 1, 6 hold,  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is a subspace of  $V$ .



The second of two shortcuts to show that a set is a subspace:

**Definition:** The **null space** of a  $m \times n$  matrix  $A$ , written  $\text{Nul}A$ , is the **solution set** to the homogeneous equation  $A\mathbf{x} = \mathbf{0}$ .



**Theorem 2: Null Spaces are Subspaces:** The null space of an  $m \times n$  matrix  $A$  is a subspace of  $\mathbb{R}^n$ .

This theorem is useful for showing that a set defined by conditions is a subspace.

**Example:** Show that the line  $y = 2x$  is a subspace of  $\mathbb{R}^2$ .

**Answer:**  $y = 2x$  is the solution set to  $2x - y = 0$ , which in matrix form is  $\begin{bmatrix} 2 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0}$ . So this solution set is the null space of  $\begin{bmatrix} 2 & -1 \end{bmatrix}$ .

The **null space** of an  $m \times n$  matrix  $A$ , written as  $\text{Nul } A$ , is the set of all solutions to the homogeneous equation  $A\mathbf{x} = \mathbf{0}$ .

## **THEOREM 2: Null spaces are subspaces**

The null space of an  $m \times n$  matrix  $A$  is a subspace of  $\mathbf{R}^n$ .

**Proof:**  $\text{Nul } A$  is a subset of  $\mathbf{R}^n$  since  $A$  has  $n$  columns. We check axioms 4,1,6 in the definition of a subspace.

4.  $\mathbf{0}$  is in  $\text{Nul } A$  because

1. If  $\mathbf{u}$  and  $\mathbf{v}$  are in  $\text{Nul } A$ , we show that  $\mathbf{u} + \mathbf{v}$  is in  $\text{Nul } A$ . Because  $\mathbf{u}$  and  $\mathbf{v}$  are in  $\text{Nul } A$

Therefore

$$A(\mathbf{u} + \mathbf{v}) =$$

6. If  $\mathbf{u}$  is in  $\text{Nul } A$  and  $c$  is a scalar, we show that  $c\mathbf{u}$  is in  $\text{Nul } A$ :

$$A(c\mathbf{u}) =$$

Since axioms 4,1,6 hold,  $\text{Nul } A$  is a subspace of  $\mathbf{R}^n$ .

## Summary:

### Axioms for a subspace:

4. The zero vector is in  $W$ .
1. If  $\mathbf{u}, \mathbf{v}$  are in  $W$ , then  $\mathbf{u} + \mathbf{v}$  is in  $W$ . (closed under addition)
6. If  $\mathbf{u}$  is in  $W$  and  $c$  is a scalar, then  $c\mathbf{u}$  is in  $W$ . (closed under scalar multiplication)

### Ways to show that a set $W$ is a subspace:

- Show that  $W = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  for some  $\mathbf{v}_1, \dots, \mathbf{v}_p$  (if  $W$  is explicitly defined - i.e. its description has variables that can take any value).
- Show that  $W$  is  $\text{Nul}A$  for some matrix  $A$  (if  $W$  is implicitly defined - i.e. by conditions that vectors must satisfy).
- Show that  $W$  is the kernel or range of a linear transformation (later, p42-43).
- Check all three axioms directly, for all  $\mathbf{u}, \mathbf{v}, c$ .

### To show that a set is not a subspace:

- Show that one of the axioms is false, for a particular value of  $\mathbf{u}, \mathbf{v}, c$ .

Best examples of a subspace: lines and planes containing the origin in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ .

One example of the power of abstract vector spaces - solving differential equations:

**Question:** What are all the polynomials  $\mathbf{p}$  of degree at most 5 that satisfy

$$\frac{d^2}{dt^2}\mathbf{p}(t) - 4\frac{d}{dt}\mathbf{p}(t) + 3\mathbf{p}(t) = 3t - 1?$$

**Answer:** The differentiation function  $D : \mathbb{P}_5 \rightarrow \mathbb{P}_5$  given by  $D(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$  is a linear transformation (later, p39).

The function  $T : \mathbb{P}_5 \rightarrow \mathbb{P}_5$  given by  $T(\mathbf{p}) = \frac{d^2}{dt^2}\mathbf{p}(t) - 4\frac{d}{dt}\mathbf{p}(t) + 3\mathbf{p}(t)$  is a sum of compositions of linear transformations, so  $T$  is also linear.

We can check that the polynomial  $t + 1$  is a solution.

So, by the Solutions and Homogeneous Equations Theorem, the solution set to the above differential equation is all polynomials of the form  $t + 1 + \mathbf{q}(t)$  where  $T(\mathbf{q}) = 0$ .

**Extra:**  $\mathbb{P}_5$  is both the domain and codomain of  $T$ , so the Invertible Matrix Theorem applies. So, if the above equation has more than one solution, then there is a polynomial  $\mathbf{g}$  such that  $\frac{d^2}{dx^2}\mathbf{p}(t) - 4\frac{d}{dt}\mathbf{p}(t) + 3\mathbf{p}(t) = \mathbf{g}(t)$  has no solutions.

## §4.2, pp229-230, pp249-250: Subspaces and Matrices

Each linear transformation has two vector subspaces associated to it.

For each of these two subspaces, we are interested in two problems:

- given a vector  $\mathbf{v}$ , is it in the subspace?
- can we write this subspace as  $\text{Span} \{ \mathbf{v}_1, \dots, \mathbf{v}_p \}$  for some vectors  $\mathbf{v}_1, \dots, \mathbf{v}_p$ ?

The set  $\{ \mathbf{v}_1, \dots, \mathbf{v}_p \}$  is then called a **spanning set** of the subspace.

- $\mathbf{v}_1, \dots, \mathbf{v}_p$ ? The set  $\{ \mathbf{v}_1, \dots, \mathbf{v}_p \}$  is then called a **basis** of the subspace.

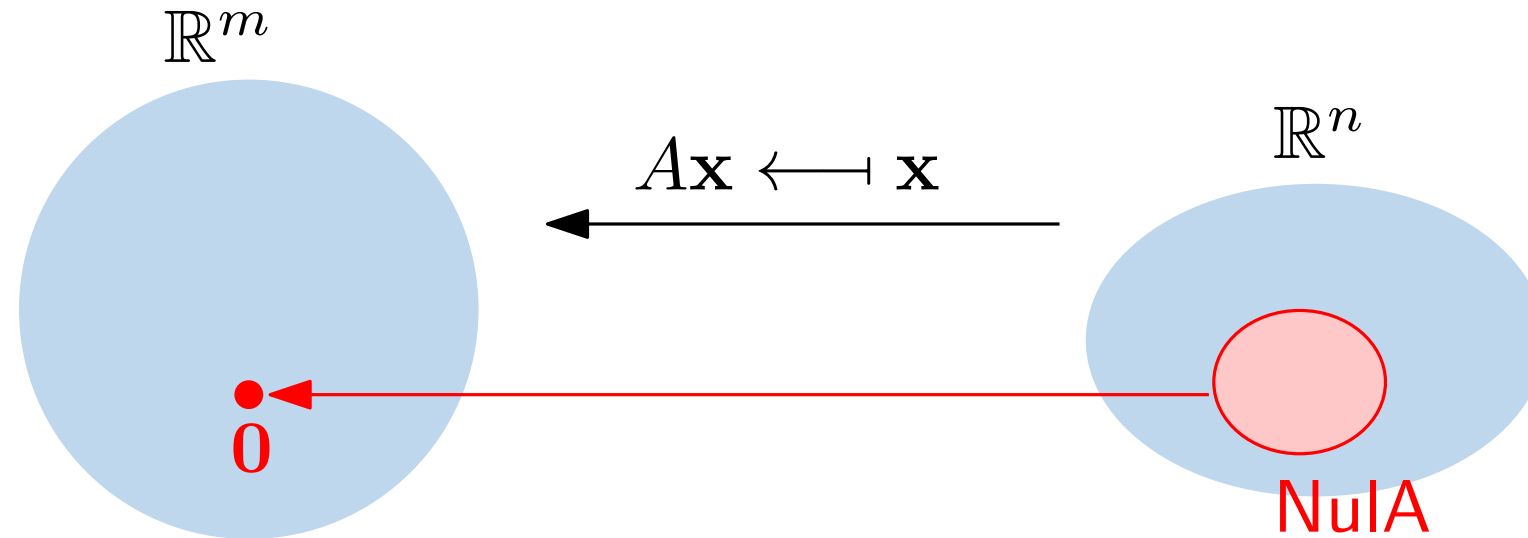
Problem b is important because it means every vector in the subspace can be written as  $c_1 \mathbf{v}_1 + \dots + c_p \mathbf{v}_p$ . This allows us to calculate with and prove statements about arbitrary vectors in the subspace.

Problem  $\mathbf{b}^*$  is important because it means every vector in the subspace can be written **uniquely** as  $c_1 \mathbf{v}_1 + \dots + c_p \mathbf{v}_p$  (proof next week, §4.4).

We turn a spanning set into a basis by removing some vectors - this is the **Spanning Set Theorem / casting-out algorithm** (p27, also week 7 p10).

Remember from p17:

**Definition:** The null space of a  $m \times n$  matrix  $A$ , written  $\text{Nul}A$ , is the solution set to the homogeneous equation  $A\mathbf{x} = \mathbf{0}$ .



$\text{Nul}A$  is **implicitly** defined (i.e. defined by conditions) - problem a is easy, problem b takes more work.

**Example:** Let  $A = \begin{bmatrix} 1 & -3 & 4 & -3 \\ 3 & -7 & 8 & -5 \end{bmatrix}$ . a. Is  $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$  in  $\text{Nul}A$ ?

b. Find vectors  $\mathbf{v}_1, \dots, \mathbf{v}_p$  which span  $\text{Nul}A$ .

**Answer:**

a.  $A\mathbf{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq \mathbf{0}$ , so  $\mathbf{v}$  is not in  $\text{Nul}A$ .

b.  $[A|\mathbf{0}] \xrightarrow{\text{row reduction}} \left[ \begin{array}{cccc|c} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \end{array} \right] \longrightarrow \begin{array}{l} x_1 = 2x_3 - 3x_4 \\ x_2 = 2x_3 - 2x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{array}$

So the solution set is  $s \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$  where  $s, t$  can take any value. So  $\text{Nul}A = \text{Span} \left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ .  
linearly independent

In general: the solution to  $A\mathbf{x} = \mathbf{0}$  in parametric form looks like  $s_i\mathbf{w}_i + s_j\mathbf{w}_j + \dots$ , where  $x_i, x_j, \dots$  are the free variables (one vector for each free variable). The vector  $\mathbf{w}_i$  has a 1 in row  $i$  and a 0 in row  $j$  for every other free variable  $x_j$ , so  $\{\mathbf{w}_i, \mathbf{w}_j, \dots\}$  are automatically linearly independent.

b.  $[A|\mathbf{0}] \xrightarrow{\text{row reduction}} \left[ \begin{array}{cccc|c} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \end{array} \right] \longrightarrow \begin{array}{l} x_1 = 2x_3 - 3x_4 \\ x_2 = 2x_3 - 2x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{array}$

So the solution set is  $s \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$  where  $s, t$  can take any value. So  $\text{Nul}A = \text{Span} \left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ .

linearly independent

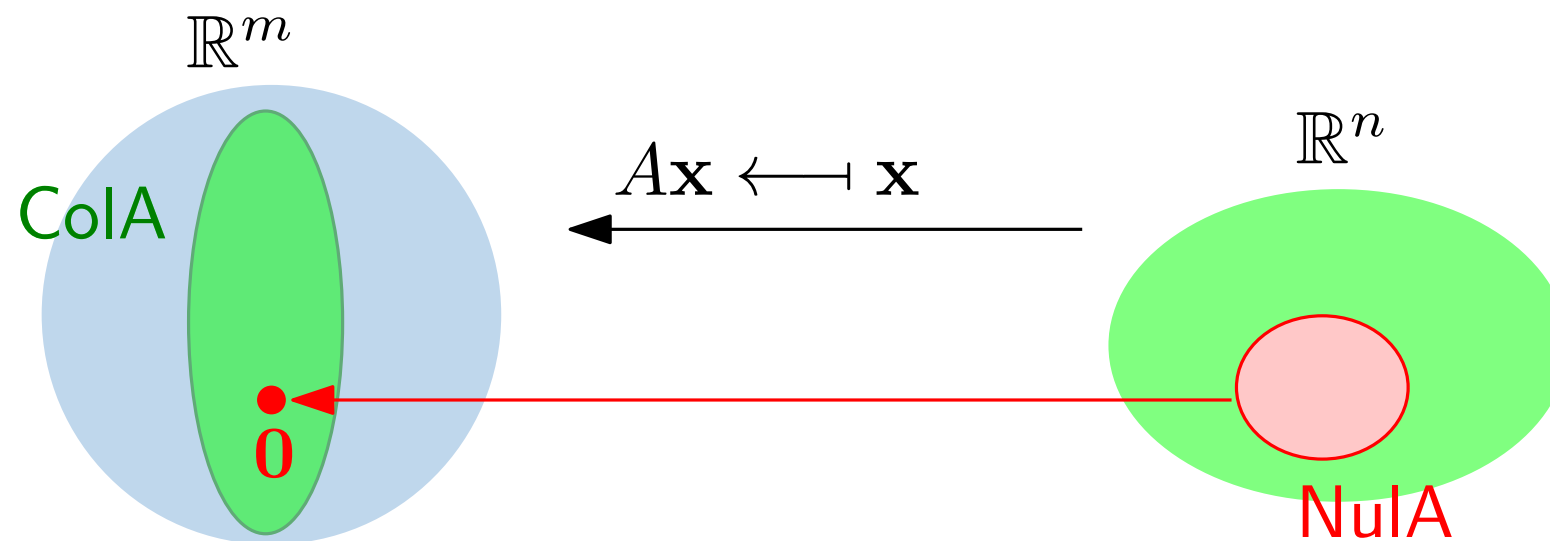
$\mathbf{w}_3$   $\mathbf{w}_4$



**Definition:** The column space of a  $m \times n$  matrix  $A$ , written  $\text{Col}A$ , is the span of the columns of  $A$ .

Because spans are subspaces, it is obvious that  $\text{Col}A$  is a subspace of  $\mathbb{R}^m$ .

It follows from §1.3-1.4 that  $\text{Col}A$  is the set of  $\mathbf{b}$  for which  $A\mathbf{x} = \mathbf{b}$  has solutions.



ColA is explicitly defined - problem a takes work, problem b is easy.

**Example:** Let  $A = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix}$ . a. Is  $\mathbf{v} = \begin{bmatrix} -5 \\ 9 \\ 5 \end{bmatrix}$  in ColA?

b. Find vectors  $\mathbf{v}_1, \dots, \mathbf{v}_p$  which span ColA.

**Answer:**

$$\text{a. } \left[ \begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{array} \right] \xrightarrow[\text{to echelon form}]{\text{row reduction}} \left[ \begin{array}{ccccc|c} 1 & -3 & 4 & 3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right]$$

There is no row  $[0 \dots 0 | *]$  with  $* \neq 0$ , so  $\mathbf{v}$  is in ColA.

b. By definition, ColA is the span of the columns of  $A$ , so

$$\text{ColA} = \text{Span} \left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}, \begin{bmatrix} -6 \\ 8 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} \right\}.$$

Note that this spanning set is not linearly independent (more than 3 vectors in  $\mathbb{R}^3$ ).

## Contrast Between Nul $A$ and Col $A$ for an $m \times n$ Matrix $A$

p.222 of  
textbook

Nul $A$	Col $A$
1. Nul $A$ is a subspace of $\mathbb{R}^n$ .	1. Col $A$ is a subspace of $\mathbb{R}^m$ .
2. Nul $A$ is implicitly defined; that is, you are given only a condition ( $A\mathbf{x} = \mathbf{0}$ ) that vectors in Nul $A$ must satisfy.	2. Col $A$ is explicitly defined; that is, you are told how to build vectors in Col $A$ .
3. It takes time to find vectors in Nul $A$ . Row operations on $[A \mid \mathbf{0}]$ are required.	3. It is easy to find vectors in Col $A$ . The columns of $A$ are displayed; others are formed from them.
4. There is no obvious relation between Nul $A$ and the entries in $A$ .	4. There is an obvious relation between Col $A$ and the entries in $A$ , since each column of $A$ is in Col $A$ .
5. A typical vector $\mathbf{v}$ in Nul $A$ has the property that $A\mathbf{v} = \mathbf{0}$ .	5. A typical vector $\mathbf{v}$ in Col $A$ has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$ , it is easy to tell if $\mathbf{v}$ is in Nul $A$ . Just compute $A\mathbf{v}$ .	6. Given a specific vector $\mathbf{v}$ , it may take time to tell if $\mathbf{v}$ is in Col $A$ . Row operations on $[A \mid \mathbf{v}]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	7. Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$ .
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	8. Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ .

← problem b

← problem a

As we saw on p26, it is easy to obtain a spanning set for  $\text{Col}A$  (just take all the columns of  $A$ ), but usually this spanning set is not linearly independent.

To obtain a linearly independent set that spans  $\text{Col}A$ , take the pivot columns of  $A$  - this is called the casting-out algorithm.

**Example:** Let  $A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix}$ .

Find a linearly independent set that spans  $\text{Col}A$ .

**Answer:**  $\begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to echelon form}]{\text{row reduction}} \begin{bmatrix} 1 & -3 & 4 & 3 & 2 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

The pivot columns are 1, 2 and 5, so  $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\} = \left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} \right\}$  is one answer.

(The answer from the casting-out algorithm is not the only answer - see p34.)

Casting-out algorithm: the **pivot columns** of  $A$  is a **linearly independent set** that **spans  $\text{Col}A$** .

Why does the casting-out algorithm work part 1: why the pivot columns are linearly independent:

**Example:**

$$A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to echelon form}]{\text{row reduction}} \begin{bmatrix} 1 & -3 & 4 & 3 & 2 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

So  $\begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_5 \\ | & | & | \end{bmatrix}$  is row-equivalent to  $\begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ , which has no free variables.

So  $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$  is linearly independent.

Casting-out algorithm: the **pivot columns** of  $A$  is a **linearly independent set** that **spans ColA**.

Why does the casting-out algorithm work part 2: why the pivot columns span ColA:

To explain this, we need to look at the solutions to  $A\mathbf{x} = \mathbf{0}$ :

**Example:**

$$A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to rref}]{\text{row reduction}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So the solution set to  $A\mathbf{x} = \mathbf{0}$  is

$$\begin{matrix} x_3 = 1 \\ x_4 = 0 \end{matrix} s \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{where } s, t \text{ can take any value.}$$

$$\begin{matrix} x_3 = 0 \\ x_4 = 1 \end{matrix}$$

These correspond respectively to the linear dependence relations  $2\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$  and  $-3\mathbf{a}_1 - 2\mathbf{a}_2 + \mathbf{a}_4 = \mathbf{0}$ .

Rearranging:  $\mathbf{a}_3 = -2\mathbf{a}_1 - 2\mathbf{a}_2$  and  $\mathbf{a}_4 = 3\mathbf{a}_1 + 2\mathbf{a}_2$ .

$$A(2, 2, 1, 0, 0) = \mathbf{0} \longrightarrow 2\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0} \longrightarrow \mathbf{a}_3 = -2\mathbf{a}_1 - 2\mathbf{a}_2.$$

$$A(-3, -2, 0, 1, 0) = \mathbf{0} \longrightarrow -3\mathbf{a}_1 - 2\mathbf{a}_2 + \mathbf{a}_4 = \mathbf{0} \longrightarrow \mathbf{a}_4 = 3\mathbf{a}_1 + 2\mathbf{a}_2.$$

In other words: consider the solution to  $A\mathbf{x} = \mathbf{0}$  where one free variable  $x_i$  is 1, and all other free variables are 0. This corresponds to a linear dependence relation among the columns of  $A$ , which can be rearranged to express the column  $\mathbf{a}_i$  as a linear combination of the pivot columns.

Why this is useful: any vector  $\mathbf{v}$  in  $\text{Col}A$  has the form

$$\mathbf{v} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 + c_4\mathbf{a}_4 + c_5\mathbf{a}_5,$$

which we can rewrite as

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3(-2\mathbf{a}_1 - 2\mathbf{a}_2) + c_4(3\mathbf{a}_1 + 2\mathbf{a}_2) + c_5\mathbf{a}_5$$

$$= (c_1 - 2c_3 + 3c_4)\mathbf{a}_1 + (c_2 - 2c_3 + 2c_4)\mathbf{a}_2 + c_5\mathbf{a}_5,$$

a linear combination of the pivot columns  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5$ . So  $\mathbf{v}$  is in  $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$ , and so  $\text{Col}A = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$ .

Another view: the casting-out algorithm as a greedy algorithm:

**Example:**

$$\begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to rref}]{\text{row reduction}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rref} \left( \begin{bmatrix} | \\ \mathbf{a}_1 \\ | \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ has a pivot in every column, so } \{\mathbf{a}_1\} \text{ is linearly independent, so we keep } \mathbf{a}_1.$$

$$\text{rref} \left( \begin{bmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ has a pivot in every column, so } \{\mathbf{a}_1, \mathbf{a}_2\} \text{ is linearly independent, so we keep } \mathbf{a}_2.$$

$$\text{rref} \left( \begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ | & | & | \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \text{ does not have a pivot in every column, so } \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \text{ is linearly dependent, so we remove } \mathbf{a}_3.$$



Another view: the casting-out algorithm as a greedy algorithm (continued):

**Example:**

$$\begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to rref}]{\text{row reduction}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rref} \left( \begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_4 \\ | & | & | \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ does not have a pivot in every column, so } \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\} \text{ is linearly dependent, so we remove } \mathbf{a}_4.$$

$$\text{rref} \left( \begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_5 \\ | & | & | \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ has a pivot in every column, so } \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\} \text{ is linearly independent, so we keep } \mathbf{a}_5.$$

So the casting-out algorithm is a greedy algorithm in that it prefers vectors that are earlier in the set.

**Example:** Let  $A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix}$ .

Find a linearly independent set **containing  $\mathbf{a}_3$**  that spans  $\text{Col}A$ .

**Answer:** To ensure that the set contains  $\mathbf{a}_3$ , we should make it the leftmost column - e.g. we row-reduce  $\begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_3 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix}$  and take the pivot columns.

**Warning:** the example on the previous two pages is a little misleading: a subset of the columns of  $\text{rref}(A)$  is **not** always the reduced echelon form of those columns of

$A$ , e.g.  $\text{rref} \left( \begin{bmatrix} | & | \\ \mathbf{a}_2 & \mathbf{a}_3 \\ | & | \end{bmatrix} \right) \neq \begin{bmatrix} 0 & -2 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}$  (because this isn't in reduced echelon form).

The correct statement is that a subset of the columns of  $\text{rref}(A)$  is **row equivalent** to those columns of  $A$ .

**Definition:** The **row space** of a  $m \times n$  matrix  $A$ , written  $\text{Row}A$ , is the **span** of the rows of  $A$ . It is a subspace of  $\mathbb{R}^n$ .

**Example:**  $A = \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 2 & 0 & 8 \\ 1 & 2 & -3 & 6 \end{bmatrix} \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\text{Row}A = \text{Span} \{(0, 1, 0, 4), (0, 2, 0, 8), (1, 2, -3, 6)\}.$$

$\text{Row}A$  is explicitly defined - indeed, it is equivalent to  $\text{Col}A^T$ .

So, to see if a vector  $\mathbf{v}$  is in  $\text{Row}A$ , row-reduce  $[A^T | \mathbf{v}^T]$ .

To find a linear independent set that spans  $\text{Row}A$ , take the pivot columns of  $A^T$ , or..

**Theorem 13:** Row operations do not change the row space. In particular, **the nonzero rows of  $\text{rref}(A)$**  is a linearly independent set whose span is  $\text{Row}A$ .

E.g. for the above example,  $\text{Row}A = \text{Span} \{(1, 0, -3, -2), (0, 1, 0, 4)\}$ .

Warning: the “pivot rows” of  $A$  do not usually span  $\text{Row}A$ :

e.g. here  $(1, 2, -3, 6)$  is in  $\text{Row}A$  but not in  $\text{Span} \{(0, 1, 0, 4), (0, 2, 0, 8)\}$ .

**Theorem 13:** Row operations do not change the row space. In particular, the nonzero rows of  $\text{rref}(A)$  is a linearly independent set whose span is  $\text{Row}A$ .

An example to explain why row operations do not change the row space:

$$A = \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 2 & 0 & 8 \\ 1 & 2 & -3 & 6 \end{bmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & -2 \end{bmatrix} \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(1, 4, -3, 14) = R_2 + R_3 = (R_2 - 2R_1) + (R_3 - R_1) + 3R_1$$

Similarly, any linear combination of  $R_1, R_2, R_3$  can be written as a linear combination of  $R_1, R_2 - 2R_1, R_3 - R_1$ .

Proof of the second sentence in Theorem 13:

From the first sentence,  $\text{Row}(A) = \text{Row}(\text{rref}(A)) = \text{Span of the nonzero rows of } \text{rref}(A)$ . Because each nonzero row has a 1 in one pivot column (different column for each row) and 0s in all other pivot columns, these rows are linearly independent.

## Summary:

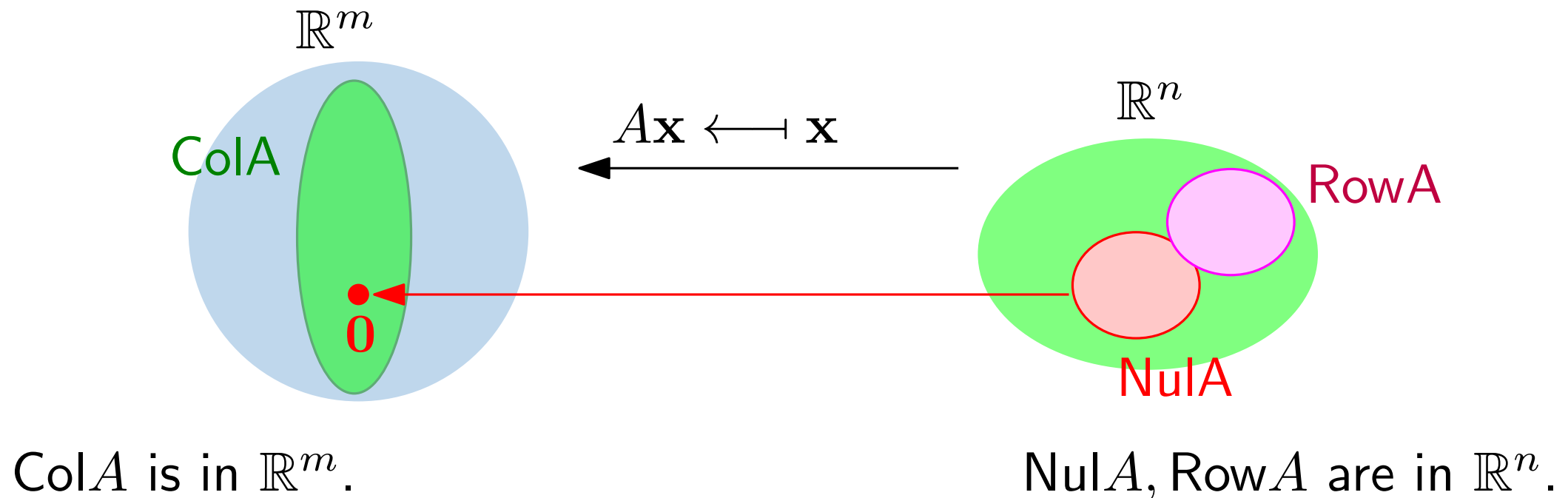
A basis for  $W$  is a linearly independent set that spans  $W$  (more later).

- $\text{Nul}A$ =solutions to  $A\mathbf{x} = \mathbf{0}$ ,
- $\text{Col}A$ =span of columns of  $A$ ,
- $\text{Row}A$ =span of rows of  $A$ .

basis for  $\text{Nul}A$ : solve  $A\mathbf{x} = \mathbf{0}$  via the rref.

basis for  $\text{Col}A$ : pivot columns of  $A$ .

basis for  $\text{Row}A$ : nonzero rows of  $\text{rref}(A)$ .



In general,  $\text{Col}A \neq \text{Col}(\text{rref}(A))$ .

$\text{Nul}A = \text{Nul}(\text{rref}(A))$ ,  $\text{Row}A = \text{Row}(\text{rref}(A))$ .

# PP222-223: Linear Transformations for Vector Spaces

Recall (week 3 §1.8) the definition of a linear transformation:

**Definition:** A function  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a *linear transformation* if:

1.  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$  for all  $\mathbf{u}, \mathbf{v}$  in the domain of  $T$ ;
2.  $T(c\mathbf{u}) = cT(\mathbf{u})$  for all scalars  $c$  and for all  $\mathbf{u}$  in the domain of  $T$ .

Now consider a function  $T : V \rightarrow W$ , where  $V, W$  are abstract vector spaces. Because we can add and scalar-multiply in  $V$ , the left hand sides of the equations in 1,2 make sense.

Because we can add and scalar-multiply in  $W$ , the right hand sides of the equations in 1,2 make sense.

So we can ask if functions between abstract vector spaces are linear:

**Definition:** A function  $T : V \rightarrow W$  is a *linear transformation* if:

1.  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$  for all  $\mathbf{u}, \mathbf{v}$  in  $V$ ;
2.  $T(c\mathbf{u}) = cT(\mathbf{u})$  for all scalars  $c$  and for all  $\mathbf{u}$  in  $V$ .

Hard exercise: show that the set of all linear transformations  $V \rightarrow W$  is a vector space.

**Definition:** A function  $T : V \rightarrow W$  is a *linear transformation* if:

1.  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$  for all  $\mathbf{u}, \mathbf{v}$  in  $V$ ;
2.  $T(c\mathbf{u}) = cT(\mathbf{u})$  for all scalars  $c$  and for all  $\mathbf{u}$  in  $V$ .

**Example:** The differentiation function  $D : \mathbb{P}_n \rightarrow \mathbb{P}_{n-1}$  given by  $D(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$ ,

$D(a_0 + a_1t + a_2t^2 + \cdots + a_nt^n) = a_1 + 2a_2t + \cdots + na_nt^{n-1}$ ,  
is linear.

If you've taken a calculus class, then you already know this:

When you calculate  $\frac{d}{dt}(3t + 2t^2) = 3 + 2 \cdot 2t$   
you're really thinking  $3\frac{d}{dt}t + 2\frac{d}{dt}t^2$

Method A to show that  $D$  is linear:

$$D(\mathbf{p} + \mathbf{q}) = \frac{d}{dt}(\mathbf{p} + \mathbf{q}) = \frac{d}{dt}\mathbf{p} + \frac{d}{dt}\mathbf{q} = D(\mathbf{p}) + D(\mathbf{q}); \text{ and}$$

$$D(c\mathbf{p}) = \frac{d}{dt}(c\mathbf{p}) = c\frac{d}{dt}\mathbf{p} = cD(\mathbf{p})$$

**Definition:** A function  $T : V \rightarrow W$  is a *linear transformation* if:

1.  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$  for all  $\mathbf{u}, \mathbf{v}$  in  $V$ ;
2.  $T(c\mathbf{u}) = cT(\mathbf{u})$  for all scalars  $c$  and for all  $\mathbf{u}$  in  $V$ .

**Example:** The differentiation function  $D : \mathbb{P}_n \rightarrow \mathbb{P}_{n-1}$  given by  $D(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$ ,

$$D(a_0 + a_1t + a_2t^2 + \cdots + a_nt^n) = a_1 + 2a_2t + \cdots + na_nt^{n-1},$$

is linear.

Method B to show that  $D$  is linear - use the formula:

$$\begin{aligned} & D((a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \cdots + (a_n + b_n)t^n) \\ &= (a_1 + b_1) + 2(a_2 + b_2)t + \cdots + n(a_n + b_n)t^{n-1} \\ &= a_1 + 2a_2t + \cdots + na_nt^{n-1} + b_1 + 2b_2t + \cdots + nb_nt^{n-1} \\ &= D(a_0 + a_1t + a_2t^2 + \cdots + a_nt^n) + D(b_0 + b_1t + b_2t^2 + \cdots + b_nt^n); \text{ and} \\ & D((ca_0) + (ca_1)t + (ca_2)t^2 + \cdots + (ca_n)t^n) = (ca_1) + 2(ca_2)t + \cdots + n(ca_n)t^{n-1} \\ &= c(a_1 + 2a_2t + \cdots + na_nt^{n-1}) \\ &= cD(a_0 + a_1t + a_2t^2 + \cdots + a_nt^n). \end{aligned}$$



**Example:** The “multiplication by  $t$ ” function  $M : \mathbb{P}_n \rightarrow \mathbb{P}_{n+1}$  given by  $M(\mathbf{p}(t)) = t\mathbf{p}(t)$ ,

$$M(a_0 + a_1t + \cdots + a_nt^n) = t(a_0 + a_1t + \cdots + a_nt^n),$$

is linear:

Method A:  $M(\mathbf{p} + \mathbf{q}) = t[(\mathbf{p} + \mathbf{q})(t)] = t\mathbf{p}(t) + t\mathbf{q}(t) = M(\mathbf{p}) + M(\mathbf{q});$  and

$$M(c\mathbf{p}) = t[(c\mathbf{p})(t)] = c[t(\mathbf{p}(t))] = cM(\mathbf{p})$$

Method B:  $M((a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n)$   
 $= t(a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n)$   
 $= t(a_0 + a_1t + \cdots + a_nt^n) + t(a_0 + a_1t + \cdots + a_nt^n)$   
 $= M(a_0 + a_1t + \cdots + a_nt^n) + M(b_0 + b_1t + \cdots + b_nt^n);$  and

$$\begin{aligned} M((ca_0) + (ca_1)t + \cdots + (ca_n)t^n) &= t((ca_0) + (ca_1)t + \cdots + (ca_n)t^n) \\ &= ct(a_0 + a_1t + \cdots + a_nt^n) \\ &= cM(a_0 + a_1t + \cdots + a_nt^n). \end{aligned}$$

**Definition:** A function  $T : V \rightarrow W$  is a *linear transformation* if:

1.  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$  for all  $\mathbf{u}, \mathbf{v}$  in  $V$ ;
2.  $T(c\mathbf{u}) = cT(\mathbf{u})$  for all scalars  $c$  and for all  $\mathbf{u}$  in  $V$ .

The concepts of kernel and range (week 3, §1.9) make sense for linear transformations between abstract vector spaces:

**Definition:** The *kernel* of  $T$  is the set of  $\mathbf{v}$  in  $V$  satisfying  $T(\mathbf{v}) = \mathbf{0}$ .

**Definition:** The *range* of  $T$  is the set of  $\mathbf{w}$  in  $W$  such that  $\mathbf{w} = T(\mathbf{v})$  for some  $\mathbf{v}$  in  $V$ .

**Example:** The kernel of the differentiation function  $D : \mathbb{P}_n \rightarrow \mathbb{P}_{n-1}$ , given by  $D(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$ , is the set of constant polynomials  $\mathbf{p}(t) = a_0$  for any number  $a_0$ . The range of  $D$  is all of  $\mathbb{P}_{n-1}$ .

Our proof that null spaces are subspaces (p18) shows that the kernel of a linear transformation is a subspace.

Exercise: show that the range of a linear transformation is a subspace.

**Redo Example:** (p12) Let  $Q$  be the set of polynomials  $\mathbf{p}(t)$  of degree at most 3 with  $\mathbf{p}(2) = 0$ . We show that  $Q$  is a subspace of  $\mathbb{P}_3$  by showing that it is the kernel of a linear transformation. (This argument is hard; if you prefer the axiom-checking on p12 that is fine.)

The evaluation-at-2 function  $E_2 : \mathbb{P}_3 \rightarrow \mathbb{R}$  given by  $E_2(\mathbf{p}) = \mathbf{p}(2)$ ,

$$E_2(a_0 + a_1t + a_2t^2 + a_3t^3) = a_0 + a_12 + a_22^2 + a_32^3$$

is a linear transformation because

1. For  $\mathbf{p}, \mathbf{q}$  in  $\mathbb{P}_3$ , we have

$$E_2(\mathbf{p} + \mathbf{q}) = (\mathbf{p} + \mathbf{q})(2) = \mathbf{p}(2) + \mathbf{q}(2) = E_2(\mathbf{p}) + E_2(\mathbf{q}).$$

2. For  $\mathbf{p}$  in  $\mathbb{P}_3$  and any scalar  $c$ , we have  $E_2(c\mathbf{p}) = (c\mathbf{p})(2) = c(\mathbf{p}(2)) = cE_2(\mathbf{p})$ .

So  $E_2$  is a linear transformation.  $Q$  is the kernel of  $E_2$ , so  $Q$  is a subspace.

Can we write  $Q$  as  $\text{Span}\{\mathbf{p}_1, \dots, \mathbf{p}_p\}$  for some linearly independent polynomials  $\mathbf{p}_1, \dots, \mathbf{p}_p$ ?

One idea: associate a matrix  $A$  to  $E_2$  and take a basis of  $\text{Nul}A$  using the rref.

To do computations like this, we need **coordinates**.