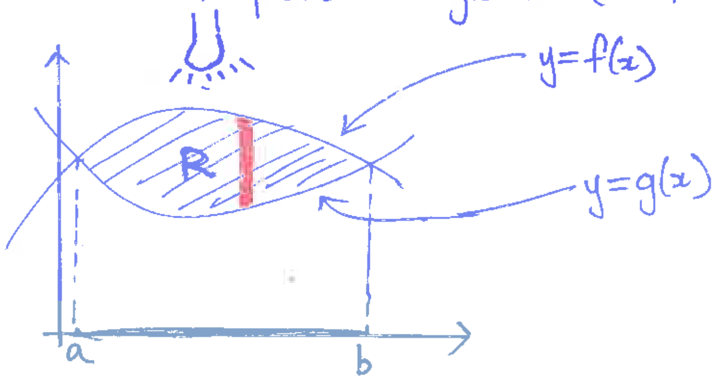
* Areas of planar regions (Week 4 p23-25, sheet #7 Q2)



$$\text{Area of } R = \int_a^b \underbrace{\overset{\text{top}}{f(x)} - \overset{\text{bottom}}{g(x)}}_{\text{area of red rectangle}} \, dx$$

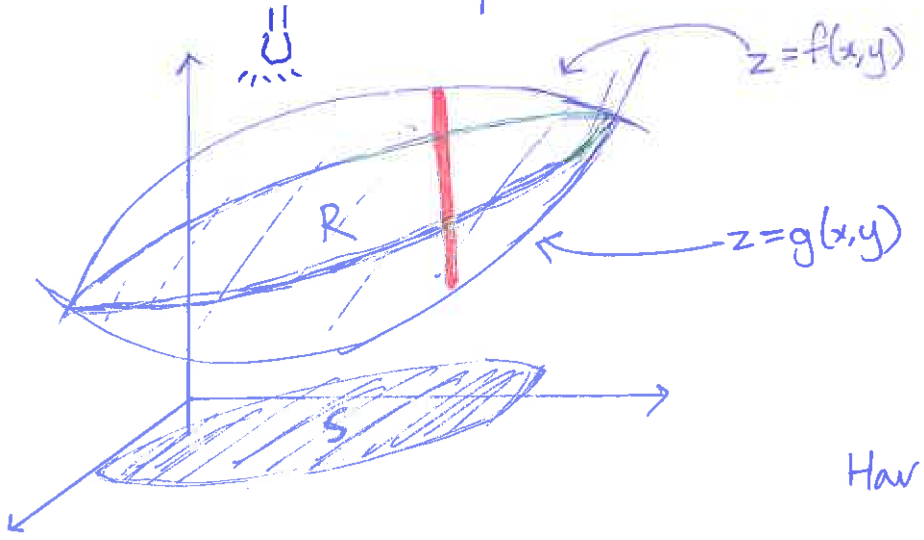
a, b are where $y=f(x)$ and $y=g(x)$ intersect.
i.e. $f(x)=g(x)$

The domain of integration is $[a,b]$,
i.e. the "shadow" of R.

* Integrating over a 2D domain (Week 5 p17, sheet #8 Q2)

$$\text{e.g. mass of } R = \iint_R \underset{\substack{\uparrow \\ \text{density function}}}{\delta(x,y)} \, dA = \int_a^b \underbrace{\int_{g(x)}^{f(x)} \delta(x,y) \, dy}_{\text{mass of red rectangle}} \, dx$$

* Volumes of 3D regions (Week 5 p31,32, sheet #9Q2)
(in Cartesian or polar coordinates)



$$\text{Volume of } R = \iint_S \underbrace{\overset{\text{top}}{f(x,y)} - \overset{\text{bottom}}{g(x,y)}}_{\text{volume of red stick}} \, dA$$

S: "shadow" of R on the xy-plane (projection)

How to find the shadow? We solve $f(x,y) = g(x,y)$ to find the boundary of S.

(Cases with ≥ 3 surfaces are more complicated (Week 5 p24.5, sheet #9Q1):
usually, parts of the boundary of S: • cylinders (no z's)
• intersections of any surfaces with z's.

* Integrating over a 3D region

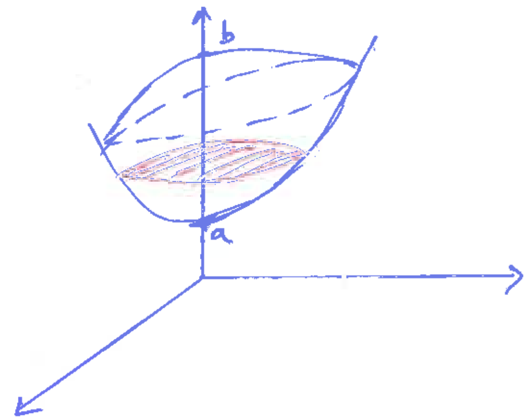
e.g. $\text{mass of } R = \iint_S \underbrace{\int_{g(x,y)}^{f(x,y)} \delta(x,y,z)\, dz}_{\text{mass of red stick}} \, dA$

Shadow method

Or: mass of $R = \int_a^b \underbrace{\iint_{CS(z)} \delta(x,y,z)\, dA}_{\text{mass of red plane}}\, dz$

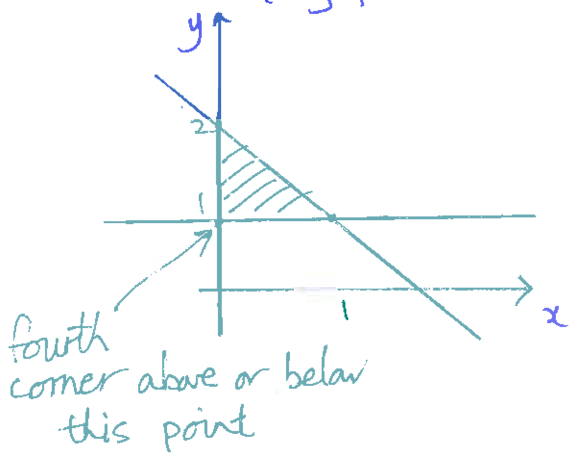
cross-section method

e.g. textbook §14.5 Ex.3



Tetrahedron: Sheet #10 Q1 $\underbrace{x=0,\ y=1}_{\text{no } z\text{'s: sides}},\ \underbrace{z=0,\ 2x+2y+z=4}_{\text{have } z\text{'s: tops/bottoms}}$.

Shadow: (xy-plane)



boundary of shadow comes from

$x=0$

$y=1$

intersection of $z=0$ and $2x+2y+z=4$

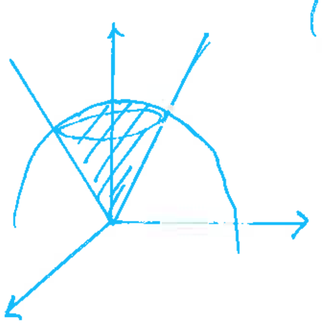
i.e. $2x+2y=4$

To build the tetrahedron from the shadow, find the fourth corner: $(0,1,z)$ on $2x+2y+z=4$. $\rightarrow z=2$.

Example: Find the mass of the smaller region bounded by $z=\sqrt{3x^2+3y^2}$ and $x^2+y^2+z^2=1$, with density function $\delta(x,y,z)=x^2z$.

$$\int_c^d\int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta, z)r\,dr\,d\theta\,dz$$

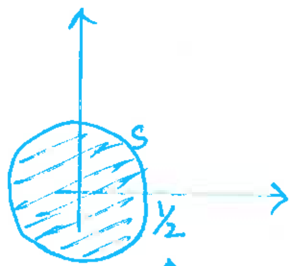
(from p32)



region is above the cone $z \geq \sqrt{3}r$

below the sphere $z \leq \sqrt{1-r^2}$

Projection to the xy plane:



from $\sqrt{3}r=\sqrt{1-r^2}$, this has solution $r=\frac{1}{2}$.

$$\text{mass} = \iint_S \int_{\sqrt{3}r}^{\sqrt{1-r^2}} (r\cos\theta)^2 z \, dz \, dA$$

$$= \int_0^{2\pi}\int_0^{1/2} (r\cos\theta)^2 \frac{z^2}{2}\Big|_{z=\sqrt{3}r}^{z=\sqrt{1-r^2}} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1/2} (r\cos\theta)^2 \left(\frac{1-r^2-3r^2}{2}\right) r\,dr\,d\theta$$

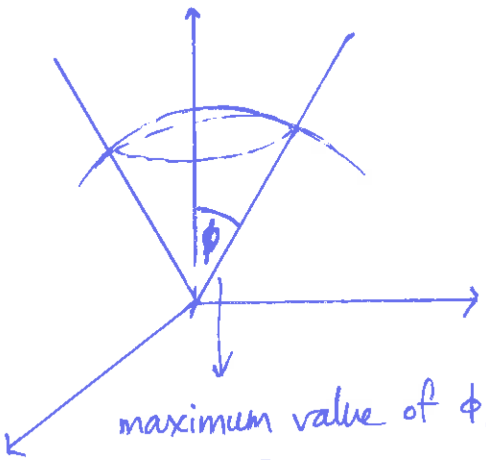
$$= \int_0^{2\pi}\int_0^{1/2} \cos^2\theta\left(\frac{r^3}{2}-2r^5\right)dr\,d\theta$$

$$= \int_0^{2\pi} \cos^2\theta\left(\frac{r^4}{8}-\frac{2r^6}{6}\right)\Big|_{r=0}^{r=1/2} d\theta = \int_0^{2\pi}\cos^2\theta\,\frac{1}{384}\,d\theta$$

$$= \left(\frac{1}{2}\theta+\frac{1}{2}\sin\theta\cos\theta\right)\frac{1}{384}\Big|_0^{2\pi} = \frac{\pi}{384}$$

$$\int_\gamma^\delta \int_\alpha^\beta \int_a^b f(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi) R^2 \sin\phi \, dR\, d\theta\, d\phi$$

Redo Example: (p46) Find the mass of the smaller region bounded by $z = \sqrt{3x^2+3y^2}$ and $x^2+y^2+z^2=1$, with density function $\delta(x,y,z) = x^2 z$.



maximum value of ϕ:

$z = \sqrt{3}\, r$

$\frac{1}{\sqrt{3}} = \frac{r}{z} = \tan\phi$

$\phi = \frac{\pi}{6}$.

$$\int_0^{\frac{\pi}{6}} \int_0^{2\pi} \int_0^1 (R\sin\phi\cos\theta)^2 (R\cos\phi) R^2 \sin\phi \, dR\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{6}} \int_0^{2\pi} \frac{R^6}{6} \sin^3\phi \cos\phi \cos^2\theta \Big|_{R=0}^{R=1} d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{6}} \int_0^{2\pi} \frac{1}{6} \sin^3\phi \cos\phi \cos^2\theta \, d\theta\, d\phi$$

table

$$= \int_0^{\frac{\pi}{6}} \frac{1}{6} \sin^3\phi \cos\phi \, \frac{1}{2}(\theta + \sin\theta\cos\theta) \Big|_{\theta=0}^{\theta=2\pi} d\phi$$

$$= \frac{\pi}{6} \int_0^{\frac{\pi}{6}} \sin^3\phi \cos\phi \, d\phi$$

substitution $u = \sin\phi$
$du = \cos\phi$

$$= \frac{\pi}{6} \frac{\sin^4\phi}{4} \Big|_0^{\frac{\pi}{6}} = \frac{\pi}{6} \frac{1}{4} \left(\frac{1}{2}\right)^4 = \frac{\pi}{384}$$

$$\int_{\gamma}^{\delta}\int_{\alpha}^{\beta}\int_{a}^{b} f(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)R^2\sin\phi\, dR\, d\theta\, d\phi$$

Example: A chocolate occupies the region between $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$. Its density function is $\delta(x,y,z)=x^2$. Find the mass of the chocolate.

Let $D=\{(x,y,z)\in\mathbb{R}^3 \mid 4\le x^2+y^2+z^2\le 9\}$

D is symmetric in x, y and z. so $\iiint_D x^2\,dV=\iiint_D y^2\,dV=\iiint_D z^2\,dV$

So mass $=\iiint_D x^2\,dV$

$$=\frac{1}{3}\iiint_D x^2+y^2+z^2\,dV$$

$$=\frac{1}{3}\int_0^{\pi}\int_0^{2\pi}\int_2^3 R^2(R^2\sin\phi)\,dR\,d\theta\,d\phi$$

$$=\frac{1}{3}\int_0^{\pi}\int_0^{2\pi}\frac{R^5}{5}\sin\phi\Big|_{R=2}^{R=3}d\theta\,d\phi=\frac{1}{3}\int_0^{\pi}\int_0^{2\pi}\frac{3^5-2^5}{5}\sin\phi\,d\theta\,d\phi$$

$$=\frac{1}{3}2\pi\int_0^{\pi}\frac{3^5-2^5}{5}\sin\phi\,d\phi$$

Exercise: what if the chocolate was only the part with $z\geq 0$?

$$=\frac{2\pi(3^5-2^5)}{15}(-\cos\phi)\Big|_0^{\pi}=\frac{2\pi(3^5-2^5)}{15}(-1-(-1))$$

$$=\frac{844}{15}\pi.$$