

## §4.5: Dimension

From last week:

- Given a vector space  $V$ , a basis for  $V$  is a linearly independent set that spans  $V$ .
- If  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  is a basis for  $V$ , then the  $\mathcal{B}$ -coordinates of  $\mathbf{x}$  are the weights  $c_i$  in the linear combination  $\mathbf{x} = c_1 \mathbf{b}_1 + \dots + c_p \mathbf{b}_p$ .
- Coordinate vectors allow us to test for spanning / linear independence, to solve linear systems, and to test for one-to-one / onto by working in  $\mathbb{R}^n$ .

Another example of this idea:

**Theorem:** Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  be a basis for a vector space  $V$ .

- Any set in  $V$  containing more than  $n$  vectors must be linearly dependent (theorem 9 in textbook).
- Any set in  $V$  containing fewer than  $n$  vectors cannot span  $V$ .

We prove this (next page) using coordinate vectors, and the fact that we already know it is true for  $V = \mathbb{R}^n$ .

**Theorem:** Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  be a basis for a vector space  $V$ .

- i Any set in  $V$  containing more than  $n$  vectors must be linearly dependent.
- ii Any set in  $V$  containing fewer than  $n$  vectors cannot span  $V$ .

**Proof:** Let our set of vectors in  $V$  be  $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ , and consider the matrix

$$A = \left[ \begin{array}{c|c|c} | & | & | \\ [\mathbf{u}_1]_{\mathcal{B}} & \cdots & [\mathbf{u}_p]_{\mathcal{B}} \\ | & | & | \end{array} \right],$$

which has  $p$  columns and  $n$  rows.

- i If  $p > n$ , then  $\text{rref}(A)$  cannot have a pivot in every column, so  $\{[\mathbf{u}_1]_{\mathcal{B}}, \dots, [\mathbf{u}_p]_{\mathcal{B}}\}$  is linearly dependent in  $\mathbb{R}^n$ , so  $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$  is linearly dependent in  $V$ .
- ii If  $p < n$ , then  $\text{rref}(A)$  cannot have a pivot in every row, so the set of coordinate vectors  $\{[\mathbf{u}_1]_{\mathcal{B}}, \dots, [\mathbf{u}_p]_{\mathcal{B}}\}$  cannot span  $\mathbb{R}^n$ , so  $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$  cannot span  $V$ .

**Theorem:** Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  be a basis for a vector space  $V$ .

- i Any set in  $V$  containing more than  $n$  vectors must be linearly dependent.
- ii Any set in  $V$  containing fewer than  $n$  vectors cannot span  $V$ .

Warning: the theorem does **not** say that “any set of more than  $n$  vectors must span  $V$ ” - this is false, e.g.  $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}$  is a set of 3 vectors in  $\mathbb{R}^2$  that does not span  $\mathbb{R}^2$ . What the theorem says is:

- Fewer than  $n$  vectors: cannot span  $V$ .
- $n$  or more vectors: has a chance of spanning  $V$ , depending on the set.

Similarly, any set of fewer than  $n$  vectors may be linearly independent or dependent (think about  $\mathbf{0}$ ).

**Theorem:** Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  be a basis for a vector space  $V$ .

- i Any set in  $V$  containing more than  $n$  vectors must be linearly dependent.
- ii Any set in  $V$  containing fewer than  $n$  vectors cannot span  $V$ .

As a consequence:

**Theorem 10: Every basis has the same size:** If a vector space  $V$  has a basis of  $n$  vectors, then every basis of  $V$  must consist of exactly  $n$  vectors.

So the following definition makes sense:

**Definition:** Let  $V$  be a vector space.

- If  $V$  is spanned by a finite set, then  $V$  is *finite-dimensional*.  
The *dimension* of  $V$ , written  $\dim V$ , is the number of vectors in a basis for  $V$ .  
(This number is finite because of the spanning set theorem.)
- If  $V$  is not spanned by a finite set, then  $V$  is *infinite-dimensional*.

Note that the definition does not involve “infinite sets”.

**Definition:** (or convention) The dimension of the zero vector space  $\{\mathbf{0}\}$  is 0.

**Definition:** The *dimension* of  $V$  is the number of vectors in a basis for  $V$ .

**Examples:**

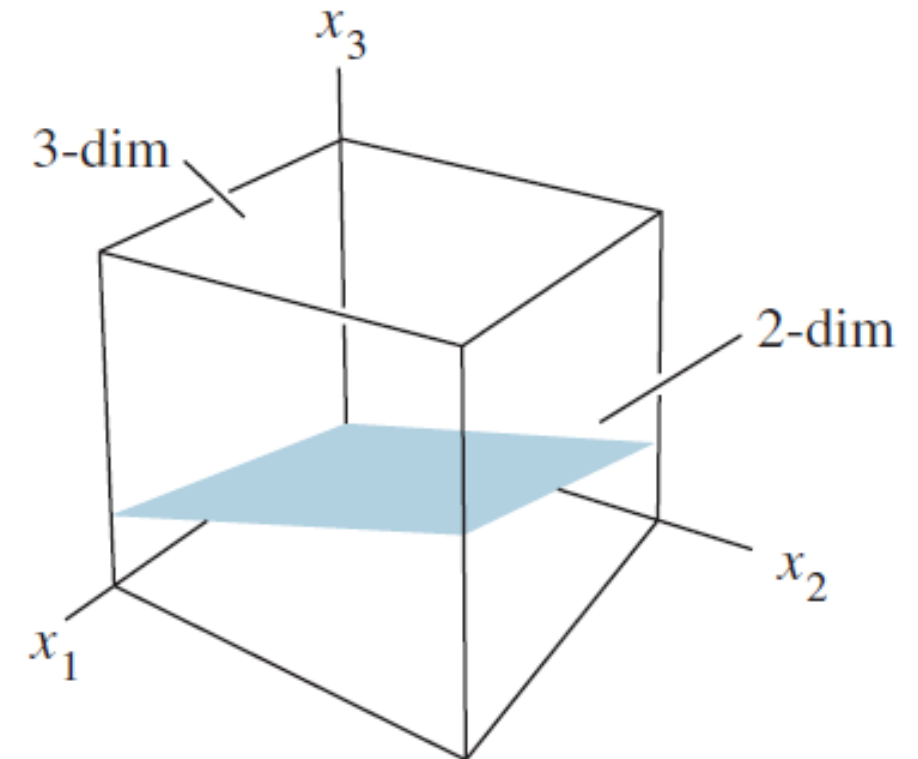
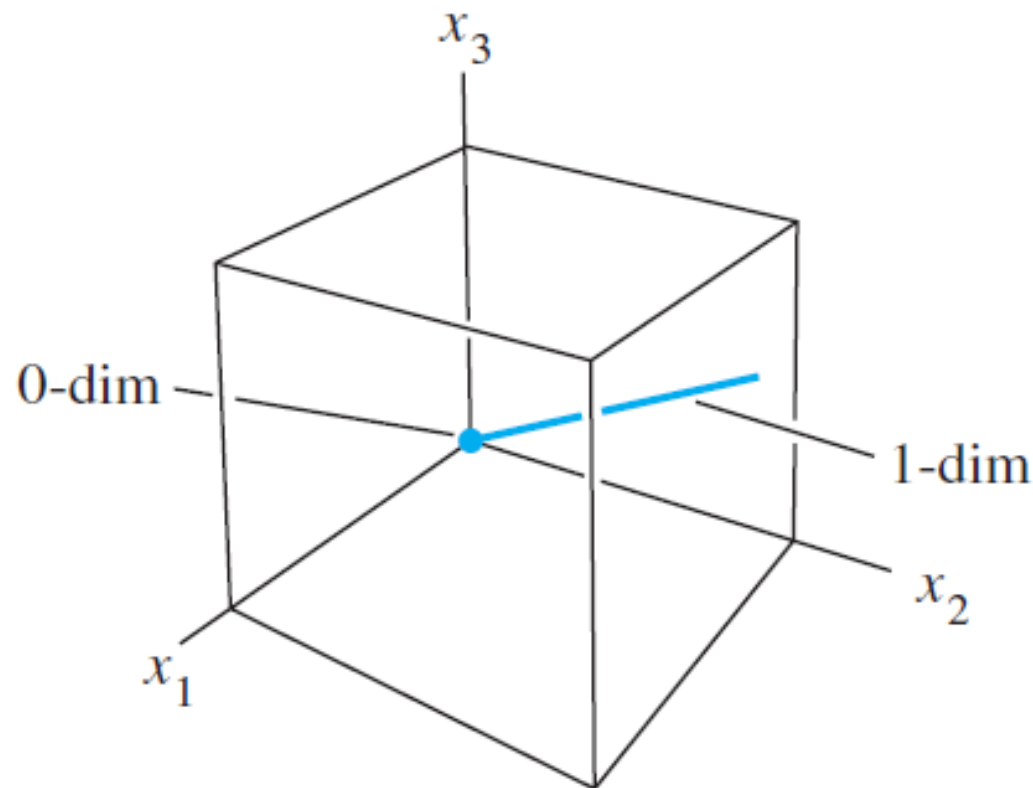
- The standard basis for  $\mathbb{R}^n$  is  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ , so  $\dim \mathbb{R}^n = n$ .
- The standard basis for  $\mathbb{P}^n$  is  $\{1, t, \dots, t^n\}$ , so  $\dim \mathbb{P}^n = n + 1$ .
- Exercise: Show that  $\dim M_{m \times n} = mn$ .

**Example:** Let  $W$  be the set of vectors of the form  $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$ , where  $a, b$  can take any value. We showed (week 8 p19) that a basis for  $W$  is  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ . So  $\dim W = 2$ .

From the theorem on p2, we know that any set of 3 vectors in  $W$  must be linearly dependent, because  $3 > \dim W$ .

**Example:** We classify the subspaces of  $\mathbb{R}^3$  by dimension:

- 0-dimensional: only the zero subspace  $\{\mathbf{0}\}$ .
- 1-dimensional, i.e.  $\text{Span}\{\mathbf{v}\}$ : lines through the origin.
- 2-dimensional, i.e.  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$  where  $\{\mathbf{u}, \mathbf{v}\}$  is linearly independent: planes through the origin.
- 3-dimensional: by Invertible Matrix Theorem, 3 linearly independent vectors in  $\mathbb{R}^3$  spans  $\mathbb{R}^3$ , so the only 3-dimensional subspace of  $\mathbb{R}^3$  is  $\mathbb{R}^3$  itself.



Here is a counterpart to the spanning set theorem (week 8 p10):

**Theorem 11: Linearly Independent Set Theorem:** Let  $W$  be a subspace of a finite-dimensional vector space  $V$ . If  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is a linearly independent set in  $W$ , we can find  $\mathbf{v}_{p+1}, \dots, \mathbf{v}_n$  so that  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is a basis for  $W$ .

**Proof:**

- If  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = W$ , then  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is a basis for  $W$ .
- Otherwise  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  does not span  $W$ , so there is a vector  $\mathbf{v}_{p+1}$  in  $W$  that is not in  $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ . Adding  $\mathbf{v}_{p+1}$  to the set still gives a linearly independent set. Continue adding vectors in this way until the set spans  $W$ . This process must stop after at most  $\dim V - p$  additions, because a set of more than  $\dim V$  elements must be linearly dependent.

The above logic proves something stronger:

**Theorem 11 part 2: Subspaces of Finite-Dimensional Spaces:** If  $W$  is a subspace of a finite-dimensional vector space  $V$ , then  $W$  is also finite-dimensional and  $\dim W \leq \dim V$ .

Because of the spanning set theorem and linearly independent set theorem:

**Theorem 12: Basis Theorem:** If  $V$  is a  $p$ -dimensional vector space, then

- i Any linearly independent set of exactly  $p$  elements in  $V$  is a basis for  $V$ .
- ii Any set of exactly  $p$  elements that span  $V$  is a basis for  $V$ .

In other words, to prove that  $\mathcal{B}$  is a basis of a  $p$ -dimensional vector space  $V$ , we only need to show **two of the following three** things (the third will be automatic):

- $\mathcal{B}$  contains exactly  $p$  vectors;
  - $\mathcal{B}$  is linearly independent;
  - $\text{Span}\mathcal{B} = V$ .
- } If  $V$  is a subspace of  $U$ , these two statements are usually easier to check because we can work in the big space  $U$  (see p10 and p14).

**Proof:**

- i By the linearly independent set theorem, we can add elements to any linearly independent set to obtain a basis for  $V$ . But that larger set must contain exactly  $\dim V = p$  elements. So our starting set must already be a basis.
- ii By the spanning set theorem, we can remove elements from any set that spans  $V$  to obtain a basis for  $V$ . But that smaller set must contain exactly  $\dim V = p$  elements. So our starting set must already be a basis.



## Summary:

- If  $V$  is spanned by a finite set, then  $V$  is finite-dimensional and  $\dim V$  is the number of vectors in any basis for  $V$ .
- If  $V$  is not spanned by a finite set, then  $V$  is infinite-dimensional.
- If  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  spans  $V$ , then some subset is a basis for  $V$  (week 8 p10).
- If  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is linearly independent and  $V$  is finite-dimensional, then it can be expanded to a basis for  $V$  (p5).

If  $\dim V = p$  (so  $V$  and  $\mathbb{R}^p$  are isomorphic):

- Any set of more than  $p$  vectors in  $V$  is linearly dependent (p2).
- Any set of fewer than  $p$  vectors in  $V$  cannot span  $V$  (p2).
- Any linearly independent set of exactly  $p$  elements in  $V$  is a basis for  $V$  (p6).
- Any set of exactly  $p$  elements that span  $V$  is a basis for  $V$  (p6).

To prove that  $\mathcal{B}$  is a basis of  $V$ , show two of the following three things:

- $\mathcal{B}$  contains exactly  $p$  vectors;
- $\mathcal{B}$  is linearly independent;
- $\text{Span}\mathcal{B} = V$ .

The basis theorem is useful for finding bases of subspaces:

**Example:**

Let  $W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ . Is  $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$  a basis for  $W$ ?

**Answer:** We are given that  $W = \text{Span} \{ \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4 \}$  and  $\{ \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4 \}$  is a linearly independent set, so  $\{ \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4 \}$  is a basis for  $W$ , and so  $\dim W = 3$ .

The vectors in  $\mathcal{B}$  are all in  $W$ , and  $\mathcal{B}$  consists of exactly 3 vectors, so it's enough to check whether  $\mathcal{B}$  is linearly independent.

Row reduction:  $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 3 & 5 & 2 \end{bmatrix} \xrightarrow{R_4 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -7 \end{bmatrix} \xrightarrow{R_4} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  has a pivot

in each column, so  $\mathcal{B}$  is linearly independent, and is therefore a basis.

Note that we never had to work in  $W$ , only in  $\mathbb{R}^4$ .

## §4.6: Rank

Next we look at how the idea of dimension can help us answer questions about existence and uniqueness of solutions to linear systems.

**Definition:** The *rank* of a matrix  $A$  is the dimension of its column space.  
The *nullity* of a matrix  $A$  is the dimension of its null space.

**Example:** Let  $A = \begin{bmatrix} 5 & -3 & 10 \\ 7 & 2 & 14 \end{bmatrix}$ ,  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$ .

A basis for  $\text{Col}A$  is  $\left\{ \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$   $\longleftarrow$  one vector per pivot

A basis for  $\text{Nul}A$  is  $\left\{ \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$   $\longleftarrow$  one vector per free variable

A basis for  $\text{Row}A$  is  $\{(1, 0, 1/2), (0, 1, 0)\}$ .  $\longleftarrow$  one vector per pivot

So  $\text{rank}A = 2$ ,  $\text{nullity}A = 1$ .

So  $\text{rank}A + \text{nullity}A = ?$

## Theorem 14:

**Rank Theorem:**  $\text{rank}A = \dim \text{Col}A = \dim \text{Row}A = \text{number of pivots in } \text{rref}(A).$

**Rank-Nullity Theorem:** For an  $m \times n$  matrix  $A$ ,

$$\text{rank}A + \text{nullity}A = n.$$

**Proof:** From our algorithms for bases of  $\text{Col}A$  and  $\text{Nul}A$  (see week 7 slides):  
 $\text{rank}A = \text{number of pivots in } \text{rref}(A) = \text{number of basic variables},$   
 $\text{nullity}A = \text{number of free variables}.$

Each variable is either basic or free, and the total number of variables is  $n$ , the number of columns.

An application of the Rank-Nullity theorem:

**Example:** Suppose a homogeneous system of 10 equations in 12 variables has a solution set that is spanned by two linearly independent vectors. Then the nullity of this system is 2, so the rank is  $12 - 2 = 10$ . So this system has 10 pivots. Since there are ten equations, there must be a pivot in every row, so any nonhomogeneous system with the same coefficients always has a solution.

## Theorem 8 (The Invertible Matrix Theorem)

Let  $A$  be a square  $n \times n$  matrix. The the following statements are equivalent (i.e., for a given  $A$ , they are either all true or all false).

a.  $A$  is an invertible matrix.

b.  $A$  is row equivalent to  $I_n$ .

c.  $A$  has  $n$  pivot positions.

d. The equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.

e. The columns of  $A$  form a linearly independent set.

f. The linear transformation  $\mathbf{x} \rightarrow A\mathbf{x}$  is one-to-one.

g. The equation  $A\mathbf{x} = \mathbf{b}$  has at least one solution for each  $\mathbf{b}$  in  $\mathbf{R}^n$ .

h. The columns of  $A$  span  $\mathbf{R}^n$ .

i. The linear transformation  $\mathbf{x} \rightarrow A\mathbf{x}$  maps  $\mathbf{R}^n$  onto  $\mathbf{R}^n$ .

j. There is an  $n \times n$  matrix  $C$  such that  $CA = I_n$ .

k. There is an  $n \times n$  matrix  $D$  such that  $AD = I_n$ .

l.  $A^T$  is an invertible matrix.

m. The columns of  $A$  form a basis for  $\mathbf{R}^n$

n.  $\text{Col } A = \mathbf{R}^n$

o.  $\dim \text{Col } A = n$

p.  $\text{rank } A = n$

q.  $\text{Nul } A = \{\mathbf{0}\}$

r.  $\dim \text{Nul } A = 0$

↑  
**new** (rephrasing  
the previous  
statements in  
the new  
terminology)

## Advanced application of the Rank-Nullity Theorem and the Basis Theorem:

**Redo Example:** (p8) Let  $A = \begin{bmatrix} 5 & -3 & 10 \\ 7 & 2 & 14 \end{bmatrix}$ . Find a basis for  $\text{Nul}A$  and  $\text{Col}A$ .

**Answer:** (a clever trick without any row-reduction)

- Observe that  $2 \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$ , so  $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$  is a solution to  $A\mathbf{x} = \mathbf{0}$ . So  $\text{nullity}A \geq 1$ .
- The first two columns of  $A$  are linearly independent (not multiples of each other), so  $\left\{ \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$  is a linearly independent set in  $\text{Col}A$ , so  $\text{rank}A \geq 2$ .
- But  $\text{rank}A + \text{nullity}A = 3$ , so in fact  $\text{rank}A = 2$  and  $\text{nullity}A = 1$ , and, by the Basis Theorem, the linearly independent sets we found above are bases:  
so  $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \right\}$  is a basis for  $\text{Nul}A$ ,  $\left\{ \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$  is a basis for  $\text{Col}A$ .

So for a general  $m \times n$  matrix, it's enough to find  $k$  linearly independent vectors in  $\text{Nul}A$  and  $n - k$  linearly independent vectors in  $\text{Col}A$ .

The Rank-Nullity theorem also holds for linear transformations  $T : V \rightarrow W$  whenever  $V$  is finite-dimensional (to prove it yourself, work through q8 of homework 5 from 2015):

$$\dim \text{range of } T + \dim \text{kernel of } T = \dim V.$$

Advanced application:

**Example:** Find a basis for  $Q$ , the set of polynomials  $\mathbf{p}(t)$  of degree at most 3 satisfying  $\mathbf{p}(2) = 0$ .

**Answer:** Remember (week 7 p28) that  $Q$  is the kernel of the evaluation-at-2 function  $E_2 : \mathbb{P}_3 \rightarrow \mathbb{R}$  given by  $E_2(\mathbf{p}) = \mathbf{p}(2)$ ,

$$E_2(a_0 + a_1t + a_2t^2 + a_3t^3) = a_0 + a_12 + a_22^2 + a_32^3.$$

$E_2$  is onto, so its range has dimension 1. So  $\dim Q = \dim \mathbb{P}_3 - 1 = 4 - 1 = 3$ .

Now  $\mathcal{B} = \{(2 - t), (2 - t)^2, (2 - t)^3\}$  is a subset of  $Q$ , and is linearly independent (check with coordinate vectors, or because (week 8 p14-15)

$\{1, (2 - t), (2 - t)^2, (2 - t)^3\}$  is a basis and any subset of a basis is linearly independent). Since  $\mathcal{B}$  contains exactly 3 vectors, it is a basis for  $Q$ .