__Ex:__ $A = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 2 \end{pmatrix}$, $m_A = ?$

$\chi_A(x) = (1-x)(1-x)(2-x)$

so, by above, m_A can be

$(x-1)(x-2)$ or $(x-1)^2(x-2)$

this has lower degree:
if A satisfies $(x-1)(x-2)$, then
$m_A = (x-1)(x-2)$.
otherwise, $m_A = (x-1)^2(x-2)$

$(A-I)(A-2I)$

$= \begin{pmatrix} 0 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & 0 \end{pmatrix}$

$\therefore m_A(x) = (x-1)(x-2)$

In general: __Th. 5.3.10__

$\sigma \in L(V,V)$ is diagonalisable

$\Leftrightarrow m_\sigma(x) = (x-\lambda_1)\cdots(x-\lambda_k)$,

all λ_i distinct.

Proof (very sketch):

$\Rightarrow$: similar to example above.

$\Leftarrow$: either: consequence of Jordan form

or: Lemma:

$\ker((\sigma-\lambda_1\iota)\cdots(\sigma-\lambda_j\iota)) = \ker(\sigma-\lambda_1\iota) \oplus \cdots \oplus \ker(\sigma-\lambda_j\iota)$

by induction on j.

$\therefore$ if $(\sigma-\lambda_1\iota)\cdots(\sigma-\lambda_k\iota) =$ zero function,

then $V = \ker(\downarrow) \overset{\text{Lemma}}{=} \ker(\sigma-\lambda_1\iota) \oplus \cdots \oplus \ker(\sigma-\lambda_k\iota)$

$= \mathcal{E}_{\lambda_1} \oplus \cdots \oplus \mathcal{E}_{\lambda_k}$

Choose a basis for each $\mathcal{E}_{\lambda_i}$ – their union is a basis for V consisting of eigenvectors

§8.2 Jordan form:

Recall: not all matrices are diagonalisable

all matrices over $\mathbb{C}$ are triangularisable i.e. $P^{-1}AP$ is triangular

BUT: an arbitrary triangular matrix is complicated.

Can we make $P^{-1}AP$ into a particularly simple triangular matrix?

Def: A λ-Jordan block of size m $(\lambda \in \mathbb{F}, m \in \mathbb{N})$ is the $m \times m$ matrix with λ on diagonal

1 immediately above the diagonal

0 elsewhere

e.g. size 3: $\begin{pmatrix} \lambda & 1 & \cdot \\ \cdot & \lambda & 1 \\ \cdot & \cdot & \lambda \end{pmatrix}$

size 4: $\begin{pmatrix} \lambda & 1 & \cdot & \cdot \\ \cdot & \lambda & 1 & \cdot \\ \cdot & \cdot & \lambda & 1 \\ \cdot & \cdot & \cdot & \lambda \end{pmatrix}$

size 1: (λ)

A matrix is <u>in Jordan form</u> if it is

$$\left(\begin{array}{c|c|c|c} J_1 & & & \\ \hline & J_2 & & \\ \hline & & \ddots & \\ \hline & & & J_p \end{array}\right)$$

where each J_i is a Jordan block.

(different J_i may have same eigenvalues, or same sizes)

Ex: $\left(\begin{array}{cc|cc} 2 & 1 & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \\ \hline \cdot & \cdot & 3 & 1 \\ \cdot & \cdot & \cdot & 3 \end{array}\right)$ 2 3

Ex: $\left(\begin{array}{ccc|c|c} 3 & 1 & \cdot & \cdot & \cdot \\ \cdot & 3 & 1 & \cdot & \cdot \\ \cdot & \cdot & 3 & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \boxed{7} & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & 7 \end{array}\right)$ 3 7 7

Any diagonal matrix is in Jordan form, where all blocks have size 1, i.e. $\lambda_1, \lambda_2, \dots, \lambda_n$.

Th. Let V be a finite-dimensional vector space over $\mathbb{C}$, $\sigma \in L(V,V)$. Then $\exists$ basis $\mathcal{B}$ of V such that $[\sigma]_{\mathcal{B}}$ is in Jordan form, and the Jordan form is unique up to reordering of blocks (but many choices of $\mathcal{B}$) — i.e. the number of blocks of each size and each eigenvalue is unique.

(Works for any field where all polynomials have solutions.)

Th. in terms of matrices:

for each $A \in M_{n,n}(\mathbb{C})$, $\exists$ invertible P, and $\exists$ J in Jordan form, such that $A = PJP^{-1}$.

$$\left(P = [\,\iota\,]_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{pmatrix} | & & | \\ \beta_1 & \cdots & \beta_n \\ | & & | \end{pmatrix} \right.$$

$$\left. \text{where } \mathcal{B} = \{\beta_1, \beta_2, \cdots, \beta_n\} \right)$$

$\boxed{!?}$ how to find P (i.e. find $\mathcal{B}$) and J?