MATH 2207: Linear Algebra
Homework 4, due 13:45 Friday 17 March 2017

You must justify your answers to receive full credit.

1. For each of the sets W_i below:
(i) determine, with explanation, whether it is a subspace of $\mathbb{R}^3$;
(ii) if it is a subspace, give a basis for it. (Hint: if you use the theorem "null spaces are subspaces" to show that W_i is a subspace, then you can use the algorithm for computing a basis of the null space to find a basis of W_i.)

   a) W_1 is the set of vectors of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x, y, z \geq 0$.

   b) W_2 is the set of vectors of the form $\begin{bmatrix} 5b + 2c \\ b \\ c \end{bmatrix}$ where b, c can take any value.

   c) W_3 is the set of vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfying $a - 2b = 4c$ and $2a = c + 3b$.

2. Let $\mathbb{P}_3$ denote the set of polynomials of degree at most 3. For each of the sets W_i below:
(i) determine, with explanation, whether it is a subspace of $\mathbb{P}_3$;
(ii) if it is a subspace, give a basis for it.

   a) W_4 is the subset of $\mathbb{P}_3$ satisfying $\mathbf{p}(1) = 1$.

   b) W_5 is the subset of $\mathbb{P}_3$ of the form $\mathbf{p}(t) = a + bt + at^2$, where a, b can take any value.

3. Suppose

$$A = \begin{bmatrix} | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} -2 & 2 & -2 & -4 \\ 2 & -3 & -3 & 1 \\ -3 & 4 & 2 & -3 \end{bmatrix}, \quad \operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   a) Find a basis for the null space of A.

   b) Find a basis for the column space of A.

   c) Find a basis for the row space of A.

   d) Is $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4\}$ a basis for the column space of A? Explain your answer.

4. Let $\mathbb{P}_3$ denote the set of polynomials of degree at most 3, with the standard basis $\mathcal{B} = \{1, t, t^2, t^3\}$.

a) Use coordinate vectors to determine if the set of polynomials

$$\{1 + 2t^3, 2 + t - 3t^2, -t + 3t^2 + 4t^3\}$$

is linearly independent.

Let $T : \mathbb{P}_3 \to \mathbb{P}_3$ be the function given by

$$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = t(a_0 + a_1 t + a_2 t^2) + a_3(t^2 + 2t - 1),$$

(i.e. multiply by t and then replace t^4 with $t^2 + 2t - 1$ - this type of function is common in abstract algebra).

b) Show that T is a linear transformation.

c) Find the matrix of T relative to the standard basis $\mathcal{B}$.

5. Let D be the determinant

$$D = \begin{vmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \\ 1 & x & x^2 & x^3 \end{vmatrix}$$

(so D is a function of x).

a) Without computation, find three solutions to $D(x) = 0$. Explain your answer.

b) **Optional**: Explain why these three are the only solutions to $D(x) = 0$. (Hint: $D(x)$ is a polynomial in x, of what degree?)
This is an example of a "Vandermonde determinant".

6. State whether each of the following statements is (always) true or (sometimes) false. If it is true, give a brief justification; if it is false, give a counterexample with an explanation.

a) The set of vectors of the form $\begin{bmatrix} a \\ 0 \\ b+1 \end{bmatrix}$, where a, b can take any value, is a subspace of $\mathbb{R}^3$.

b) Let $M_{2\times 2}$ denote the vector space of 2×2 matrices. The determinant function $\det : M_{2\times 2} \to \mathbb{R}$ is a linear transformation.

c) The differentiation function $D : \mathbb{P}_3 \to \mathbb{P}_2$ is onto.

d) The differentiation function $D : \mathbb{P}_3 \to \mathbb{P}_3$ is onto.

e) If V is a 6-dimensional vector space, then any set of 6 vectors in V is a basis for V.

f) Let $\mathbb{P}_3$ be the set of polynomials of degree at most 3. Then $\mathbb{R}^3$ and $\mathbb{P}_3$ have the same dimension.

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