

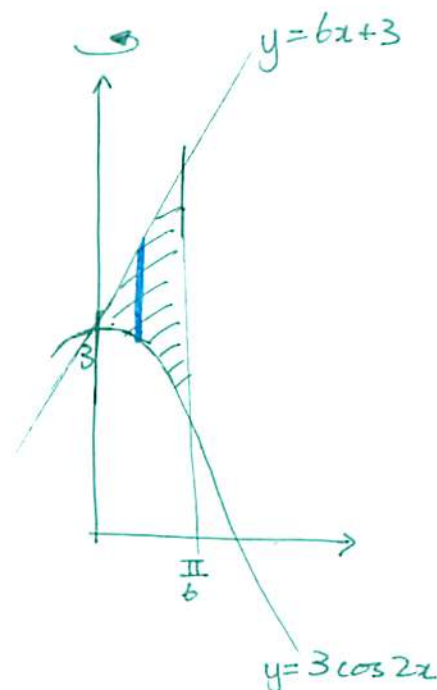
1. (7 points) Let R be the region bounded by the curves

$$y = 3 \cos 2x, \quad y = 6x + 3, \quad x = \frac{\pi}{6}.$$

Find the volume of the solid obtained by rotating R about the y -axis. Simplify your answer as much as possible.

Use cylindrical shells:

$$\begin{aligned} \text{volume} &= \int_0^{\frac{\pi}{6}} 2\pi x (6x+3 - 3\cos 2x) dx \\ &= \int_0^{\frac{\pi}{6}} 12\pi x^2 + 6\pi x - 6\pi x \cos 2x dx \\ &= \left[12\pi \frac{x^3}{3} + \frac{6\pi x^2}{2} \right]_0^{\frac{\pi}{6}} - 6\pi \int_0^{\frac{\pi}{6}} x \cos 2x dx \\ &= \frac{12\pi}{3} \left(\frac{\pi}{6} \right)^3 + \frac{6\pi}{2} \left(\frac{\pi}{6} \right)^2 - 6\pi \int_0^{\frac{\pi}{6}} x \cos 2x dx \end{aligned}$$



Integration by parts:

$$\begin{aligned} \int_0^{\frac{\pi}{6}} x \cos 2x dx &= \left[x \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \frac{\sin 2x}{2} dx & u=x & dv = \cos 2x dx \\ & & du=dx & v = \frac{\sin 2x}{2} \\ &= \left(\frac{\pi}{6} \frac{\sin \frac{\pi}{3}}{2} - 0 \right) - \left[-\frac{\cos 2x}{4} \right]_0^{\frac{\pi}{6}} \\ &= \frac{\pi}{6} \frac{\sin \frac{\pi}{3}}{2} - \left(-\frac{\cos \frac{\pi}{3}}{4} + \frac{\cos 0}{4} \right) \\ &= \frac{\pi}{6} \frac{\sqrt{3}}{4} + \frac{1}{8} - \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \therefore \text{volume} &= \frac{12\pi}{3} \left(\frac{\pi}{6} \right)^3 + \frac{6\pi}{2} \left(\frac{\pi}{6} \right)^2 - 6\pi \left(\frac{\pi}{6} \frac{\sqrt{3}}{4} + \frac{1}{8} - \frac{1}{4} \right) \\ &= \frac{\pi^4}{54} + \frac{\pi^3}{12} - \frac{\sqrt{3}\pi^2}{4} + \frac{3\pi}{4} \end{aligned}$$

2. (7 points) Compute the following integral:

$$\int \frac{2x^2 - 7}{(x-1)(x^2+4)} dx.$$

Partial fractions: $\frac{2x^2-7}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$

$$2x^2-7 = A(x^2+4) + (Bx+C)(x-1)$$

$x=1$: $2-7 = A(1+4) \Rightarrow -5 = 5A \Rightarrow A=-1$

coefficients of x^2 : $2 = A+B \Rightarrow 2 = -1+B \Rightarrow B=3$

coefficients of x : $0 = -B+C \Rightarrow C=B=3$

So $\int \frac{2x^2-7}{(x-1)(x^2+4)} dx = \int \frac{-1}{x-1} + \frac{3x}{x^2+4} + \frac{3}{x^2+4} dx$

$$= -\ln|x-1| + \frac{3}{2} \ln|x^2+4| + \frac{3}{2} \arctan\left(\frac{x}{2}\right) + C$$

↑
substitution
 $u=x-1$

↑
substitution
 $u=x^2+4$

↑
substitution
 $u=x/2$

3. (7 points) Compute the following integral:

$$\int x^2 (7 - x^2)^{-\frac{7}{2}} dx.$$

substitution:

$$x = \sqrt{7} \sin \theta$$

$$dx = \sqrt{7} \cos \theta d\theta$$

$$= \int (\sqrt{7} \sin \theta)^2 (\sqrt{7} \cos \theta)^{-7} (\sqrt{7} \cos \theta d\theta)$$

$$= \int \frac{1}{49} \frac{\sin^2 \theta}{\cos^6 \theta} d\theta$$

$$= \int \frac{1}{49} \tan^2 \theta \sec^4 \theta d\theta$$

$$= \int \frac{1}{49} \tan^2 \theta (1 + \tan^2 \theta) \sec^2 \theta d\theta$$

$$= \int \frac{1}{49} (\tan^2 \theta + \tan^4 \theta) \sec^2 \theta d\theta$$

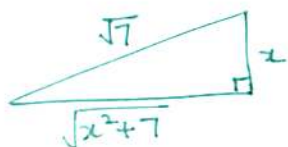
substitution

$$u = \tan \theta$$



$$= \frac{1}{49} \left(\frac{\tan^3 \theta}{3} + \frac{\tan^5 \theta}{5} \right) + C$$

$$= \frac{1}{49} \left(\frac{1}{3} \left(\frac{x}{\sqrt{x^2+7}} \right)^3 + \frac{1}{5} \left(\frac{x}{\sqrt{x^2+7}} \right)^5 \right) + C$$



$$\frac{x}{\sqrt{7}} = \sin \theta = \frac{\text{opp}}{\text{hyp}}$$

$$\therefore \tan \theta = \frac{x}{\sqrt{x^2+7}}$$