MATH 2207: Linear Algebra
Homework 6, due 13:45 Friday, December 2

You must justify your answers to receive full credit.

1. Are the following matrices diagonalisable? Calculate as little as possible, and explain your answers.

a) $A = \begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$ b) $B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & -1 \\ 0 & 1 & 5 \end{bmatrix}$ c) $C = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

2. Let
$$\mathbf{y} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ -5 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix},$$
and let $W = \operatorname{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

a) Use the Gram-Schmidt process to find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for W.

b) Write $\mathbf{y}$ as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ from part a).

c) Let $\operatorname{proj}_W : \mathbb{R}^4 \to \mathbb{R}^4$ be the orthogonal projection onto W. Find the standard matrix of proj_W.

d) Let $\mathbf{x}_4$ be a nonzero vector in $W^\perp$, and let $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$. Find the matrix of proj_W relative to $\mathcal{B}$. (Hint: you do not need to do any calculation.)

3. Let
$$\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}, \quad \mathbf{v_1} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix},$$
and let $W = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

a) Find the distance from $\mathbf{y}$ to W, and the closest point in W to $\mathbf{y}$.

b) Find the distance from $\mathbf{y}$ to $W^\perp$, and the closest point in $W^\perp$ to $\mathbf{y}$.

4. Suppose radioactive substances A and B have decay constants of 0.02 and 0.07, respectively. If a mixture of these two substances at time $t = 0$ contains M_A grams of A and M_B grams of B, then a model for the total mass in grams y of the mixture at time t is
$$y = M_A e^{-0.02t} + M_B e^{-0.07t}.$$
Suppose the initial amounts M_A and M_B are unknown, but a scientist is able to measure the total mass at several times t_i to obtain the following data points:

t_i	10	11	12	14	15
y_i	21.34	20.68	20.05	18.87	18.30

Write down the observation vector, the design matrix and the parameter vector. (Do **not** evaluate the exponentials, just write $e^{-0.2}$ etc.) You do **not** need to solve for M_A and M_B.

5. Remember from class that the least-squares line through the data points $(x_1, y_1), \dots, (x_n, y_n)$ is given by $y = \hat{\beta}_0 + \hat{\beta}_1 x$, where $(\hat{\beta}_0, \hat{\beta}_1)$ is the least-squares solution to

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

The goal of this problem is to prove, from this linear algebra perspective, the formulas for the regression coefficients $\hat{\beta}_0$ and $\hat{\beta}_1$ from Probability and Statistics class:

$$\hat{\beta}_1 = \frac{\mathrm{SS}_{xy}}{\mathrm{SS}_{xx}}, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

where

$$\mathrm{SS}_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}, \quad \mathrm{SS}_{xx} = \Sigma xx - \frac{(\Sigma x)^2}{n}, \quad \bar{y} = \frac{(\Sigma y)}{n}, \quad \bar{x} = \frac{(\Sigma x)}{n}.$$

a) By considering the normal equations, find a matrix M such that $M \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \Sigma y \\ \Sigma xy \end{bmatrix}$. (The entries of M will be in terms of x and y.)

b) Assume that $x_1, \dots, x_n$ are not all the same. Using a theorem from class, explain why this means $\hat{\beta}_0$ and $\hat{\beta}_1$ are unique.

c) The uniqueness of $\hat{\beta}_0$ and $\hat{\beta}_1$ implies that the matrix M is invertible. By inverting M, show that $\hat{\beta}_1$ has the formula given above, then show that $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$.

6. State whether each of the following statements is (always) true or (sometimes) false. If it is true, give a brief justification; if it is false, give a counterexample with an explanation.

a) Let A and B be square matrices. If A is similar to B, and B is diagonalisable, then A is diagonalisable.

b) If a square matrix A is diagonalisable, then it is invertible.

c) If a square matrix A is invertible, then it is diagonalisable.

d) Let W be a subspace of $\mathbb{R}^n$. If $\{\mathbf{w}_1, \dots \mathbf{w}_p\}$ is an orthogonal set in W and $\{\mathbf{u}_1, \dots \mathbf{u}_q\}$ is an orthogonal set in $W^\perp$, then their union $\{\mathbf{w}_1, \dots \mathbf{w}_p, \mathbf{u}_1, \dots \mathbf{u}_q\}$ is an orthogonal set.

e) Let U be an $n \times n$ orthogonal matrix. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in $\mathbb{R}^n$, then $\{U\mathbf{v}_1, U\mathbf{v}_2, U\mathbf{v}_3\}$ is also an orthogonal set.

f) Let U and W be subspaces of $\mathbb{R}^n$, and suppose U is a subset of W (i.e. every vector in U is also in W). Then $U^\perp$ is a subset of $W^\perp$.

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