

How to find a basis 2: by

extending a linearly independent set.

Related theory:

Th. 6.4.6 Steinitz replacement theorem (exchange lemma)

If $\{\alpha_1, \dots, \alpha_n\}$ spans V and $\{\beta_1, \dots, \beta_m\} \subseteq V$ is linearly independent, then, after relabelling the α_i , $\{\beta_1, \dots, \beta_m, \alpha_{m+1}, \dots, \alpha_n\}$ spans V .

In particular, $m \leq n$.

Proof: replace α_i by β_i one at a time
(induction on m)

First: replace α_1 by β_1 (base case $m=1$)

$\because \{\alpha_1, \dots, \alpha_n\}$ spans V , so

$$\textcircled{1} \quad \beta_1 = a_1 \alpha_1 + \dots + a_n \alpha_n.$$

$\because \beta_1$ is in a linearly independent set,
 $\beta_1 \neq \vec{0}$ so not all a_i are 0.

Relabel α_i so that $a_1 \neq 0$.

$$\textcircled{2} \quad \text{Then } \alpha_1 = \frac{1}{a_1} \beta_1 - \frac{a_2}{a_1} \alpha_2 - \dots - \frac{a_n}{a_1} \alpha_n$$

$$\begin{aligned} \therefore V &= \text{span}\{\alpha_1, \dots, \alpha_n\} \\ &= \text{span}\{\beta_1, \alpha_2, \dots, \alpha_n\} \quad \text{6.3.11, } \textcircled{1} \\ &= \text{span}\{\beta_1, \alpha_2, \dots, \alpha_n\} \quad \text{6.3.11, } \textcircled{2}. \end{aligned}$$

Next step: replace α_2 by β_2 :

$$\beta_2 \in V = \text{span}\{\beta_1, \alpha_2, \dots, \alpha_n\}$$

$$\textcircled{3} \quad \beta_2 = b_1 \beta_1 + c_2 \alpha_2 + \dots + c_n \alpha_n.$$

Not all c_i are 0, $\because \beta_2 \neq b_1 \beta_1, \therefore \{\beta_1, \beta_2, \dots, \beta_m\}$
is linearly independent.

Relabel α_i so $c_2 \neq 0$.

$$\textcircled{4} \quad \text{Then } \alpha_2 = \frac{1}{c_2} \beta_2 - \frac{b_1}{c_2} \beta_1 - \frac{a_3}{c_2} \alpha_3 - \dots - \frac{a_n}{c_2} \alpha_n$$

$$\begin{aligned} V &= \text{span}\{\beta_1, \alpha_2, \dots, \alpha_n\} \\ &= \text{span}\{\beta_1, \beta_2, \alpha_2, \dots, \alpha_n\} \quad \text{6.3.11 } \textcircled{3} \\ &= \text{span}\{\beta_1, \beta_2, \alpha_3, \dots, \alpha_n\} \quad \text{6.3.11 } \textcircled{4} \end{aligned}$$

After k steps: $V = \text{span}\{\beta_1, \dots, \beta_k, \alpha_{k+1}, \dots, \alpha_n\}$

$$\textcircled{5} \quad \beta_{k+1} = b'_1 \beta_1 + \dots + b'_k \beta_k + c'_{k+1} \alpha_{k+1} + \dots + c'_n \alpha_n.$$

Not all c'_i are 0 $\because \{\beta_1, \dots, \beta_m\}$ is linearly independent,
so there can be no linear dependence relation with $\{\beta_1, \dots, \beta_{k+1}\}$
 \therefore relabel α_i so $c'_{k+1} \neq 0$

Then rearranging $\textcircled{5}$ shows that α_{k+1} is a linear combination of
 $\beta_1, \dots, \beta_{k+1}, \alpha_{k+2}, \dots, \alpha_n$. So $V = \text{span}\{\beta_1, \dots, \beta_{k+1}, \alpha_{k+2}, \dots, \alpha_n\}$

In practice: replace ALL α_i at the same time, using casting-out algorithm: row reduce $\left(\begin{array}{ccc|ccc} \beta_1 & \dots & \beta_m & \alpha_1 & \dots & \alpha_n \end{array} \right)$

take all β_i , and α_j whose columns have pivots.

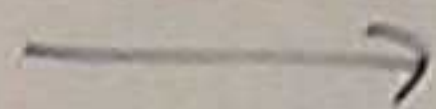
(All β_i columns will have pivots.)

Ex: Find a basis for \mathbb{R}^4 including $\beta_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}$, $\beta_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \end{pmatrix}$.

$\text{Span}\{e_1, e_2, e_3, e_4\} = \mathbb{R}^4$

\therefore we can let $\alpha_i = e_i$.

$$\left(\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ \beta_1 & \beta_2 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \right)$$



$$\left(\begin{array}{cccccc} \boxed{1} & 0 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 \end{array} \right)$$

echelon form

\therefore basis is

$\{\beta_1, \beta_2, \underset{\alpha_1}{e_1}, \underset{\alpha_3}{e_3}\}$