How to find a basis 2: by extending a linearly independent set.

Related theory:

Th. 6.4.6 Steinitz replacement theorem (exchange lemma)

If $\{\alpha_1, \cdots, \alpha_n\}$ spans V and $\{\beta_1, \cdots, \beta_m\} \subseteq V$ is linearly independent, then, after relabelling the α_i, $\{\beta_1, \cdots, \beta_m, \alpha_{m+1}, \cdots, \alpha_n\}$ spans V.

In particular, $m \leq n$.

Proof: replace α_i by β_i one at a time (induction on m)

First: replace α_1 by β_1 (base case $m=1$)

$\because \{\alpha_1, \cdots, \alpha_n\}$ spans V, so

① $\beta_1 = a_1\alpha_1 + \cdots + a_n\alpha_n$.

$\because \beta_1$ is in a linearly independent set, $\beta_1 \neq \vec{0}$ so not all a_i are 0.

Relabel α_i so that $a_1 \neq 0$.

② Then $\alpha_1 = \frac{1}{a_1}\beta_1 - \frac{a_2}{a_1}\alpha_2 - \cdots - \frac{a_n}{a_1}\alpha_n$

$\therefore V = \operatorname{span}\{\alpha_1, \cdots, \alpha_n\}$

$= \operatorname{span}\{\beta_1, \alpha_1, \cdots, \alpha_n\}$ 6.3.11, ①

$= \operatorname{span}\{\beta_1, \alpha_2, \cdots, \alpha_n\}$ 6.3.11, ②.

Next step: replace α_2 by β_2:

$\beta_2 \in V = \operatorname{span}\{\beta_1, \alpha_2, \cdots, \alpha_n\}$

③ $\beta_2 = b_1\beta_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$.

Not all c_i are 0, $\because \beta_2 \neq b_1\beta_1$, $\because \{\beta_1, \beta_2, \cdots \beta_m\}$ is linearly independent.

Relabel α_i so $c_2 \neq 0$.

④ Then $\alpha_2 = \frac{1}{c_2}\beta_2 - \frac{b_1}{c_2}\beta_1 - \frac{a_3}{c_2}\alpha_3 - \cdots - \frac{a_n}{c_2}\alpha_n$

$V = \operatorname{span}\{\beta_1, \alpha_2, \cdots, \alpha_n\}$

$= \operatorname{span}\{\beta_1, \beta_2, \alpha_2, \cdots, \alpha_n\}$ 6.3.11 ③

$= \operatorname{span}\{\beta_1, \beta_2, \alpha_3, \cdots \alpha_n\}$ 6.3.11 ④

After k steps: $V = \operatorname{span}\{\beta_1, \cdots, \beta_k, \alpha_{k+1}, \cdots, \alpha_n\}$

⑤ $\beta_{k+1} = b_1'\beta_1 + \cdots + b_k'\beta_k + c_{k+1}'\alpha_{k+1} + \cdots + c_n'\alpha_n$.

Not all c_i' are 0 $\because \{\beta_1, \cdots, \beta_m\}$ is linearly independent, so there can be no linear dependence relation with $\{\beta_1, \cdots \beta_{k+1}\}$

$\therefore$ relabel α_i so $c_{k+1}' \neq 0$

Then rearranging ⑤ shows that α_{k+1} is a linear combination of $\beta_1, \cdots, \beta_{k+1}, \alpha_{k+2}, \cdots, \alpha_n$. So $V = \operatorname{span}\{\beta_1, \cdots, \beta_{k+1}, \alpha_{k+2}, \cdots, \alpha_n\}$

In practice: replace ALL α_i at the same time, using casting-out algorithm: row reduce $\begin{pmatrix} | & & | & | & & | \\ \beta_1 & \cdots & \beta_m & \alpha_1 & \cdots & \alpha_n \\ | & & | & | & & | \end{pmatrix}$

take all β_i, and α_j whose columns have pivots.

(All β_i columns will have pivots.)

Ex: Find a basis for $\mathbb{R}^4$ including $\beta_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$, $\beta_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}$.

$\mathrm{Span}\{e_1, e_2, e_3, e_4\} = \mathbb{R}^4$ $\therefore$ we can let $\alpha_i = e_i$.

$$\begin{array}{c} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \\ \begin{array}{cccccc} \beta_1 & \beta_2 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{array} \end{array} \longrightarrow \underbrace{\begin{pmatrix} \boxed{1} & 0 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 \end{pmatrix}}_{\text{echelon form}}$$

$\therefore$ basis is $\{\beta_1, \beta_2, \underset{\alpha_1}{\underset{\parallel}{e_1}}, \underset{\alpha_3}{\underset{\parallel}{e_3}}\}$