Research Statement: Chung Yin Amy Pang

My favourite research avenues concern the interaction of combinatorics, algebra and probability. In my current work, I study, in a unified way, Markov chains on various combinatorial objects (partitions, graphs, trees) whose transition probabilities arise from the same linear maps on different graded Hopf algebras. In my thesis, this linear map was the composition of the coproduct and product operators, and the resulting chains model the breaking-then-recombining of the combinatorial objects. Over the past year, I generalised this defining linear map to the "descent operators" of Patras, which allowed me to deform the breaking step of these chains. Applying this theory to the shuffle algebra and the algebra of symmetric functions respectively analyses the card-shuffling models of Diaconis, Fill and Pitman, and Fulman's restriction-then-induction chains on the representations of the symmetric group.

By combining the Poincare-Birkhoff-Witt theorem on the structure of Hopf algebras and the Perron-Frobenius theorem on stochastic matrices, I derived a single expression for the eigenvalues and multiplicities of the diverse range of chains in this extended framework. This gives information about the long term behaviour of the chains, such as their stationary distributions. From maps between Hopf algebras, I can detect Markov statistics of these chains - that is, functions θ on the Markov chain $\{X_t\}$ which themselves are Markov chains, so $\theta(X_{t+1})$ depends only of $\theta(X_t)$ and not on X_t or X_{t+1}. This is useful as $\{\theta(X_t)\}$ is often simpler to study than $\{X_t\}$.

In the coming years, I plan to extend this analysis to linear maps beyond the descent operators, as well as to investigate some algebraic questions that this model has inspired, such as determining the minimal polynomial of the descent operators, and analysing the structure of the algebras generated by interesting subsets of these operators. I'm also keen to explore unrelated problems involving symmetric functions - recently, I've been learning about the Shareshian-Wachs conjectures regarding chromatic symmetric functions and representations on Hessenberg varieties. All these projects are heavily guided by computational data, an aspect which I enjoy.

Background

Two models of card-shuffling best illustrate the notion of descent-operator-chains. Firstly, the *riffle-shuffle* (as studied in [BD92]) takes a binomially-chosen number of cards off the top of the deck, then drops the cards one-by-one from the bottom of the resulting two piles, where the probability of dropping from each pile is proportional to the pile size. An equivalent description of this second step is that all interleavings of the two piles of cards that keep cards from the same pile in the same relative order are equally likely. The second model is the *top-to-random* shuffle (as studied in [AD87]), which removes the top card off the deck, then re-inserts it at a uniformly chosen position.

The Hopf algebra associated with these shuffling schemes is Ree's shuffle algebra $\mathcal{S}$ [Ree58]. It is a *combinatorial Hopf algebra* in the sense of Joni and Rota [JR79], as its product and coproduct describe respectively the combining and breaking of objects. Its basis is the set of all words or strings in the letters $\{1, 2, \ldots, N\}$, graded by length. The product $m : \mathcal{S} \otimes \mathcal{S} \to \mathcal{S}$ of two words is the sum of their interleavings (with multiplicity), and the coproduct $\Delta : \mathcal{S} \to \mathcal{S} \otimes \mathcal{S}$ of a word is the sum of its deconcatenations. (m and Δ then extend linearly to the rest of the Hopf algebra.)

For example,

$$m([\![15]\!] \otimes [\![52]\!]) = [\![15]\!][\![52]\!] = 2[\![1552]\!] + [\![1525]\!] + [\![5152]\!] + [\![5125]\!] + [\![5215]\!];$$

$$\Delta([\![316]\!]) = \epsilon \otimes [\![316]\!] + [\![3]\!] \otimes [\![16]\!] + [\![31]\!] \otimes [\![6]\!] + [\![316]\!] \otimes \epsilon.$$

Here, ϵ denotes the unit element of $\mathcal{S}$. Since ϵ spans the subspace of degree zero, $\mathcal{S}$ is *connected.*

The shuffle algebra $\mathcal{S}$ and the riffle-shuffle are related in the following way: the probability of riffle-shuffling a deck in the order $x_1, x_2, \dots, x_n$ (read from top to bottom) and obtaining $y_1, y_2, \dots, y_n$ is the coefficient of the word $y_1 y_2 \dots y_n$ in $2^{-n} m \circ \Delta(x_1 \dots x_n)$. Similarly, the transition probabilities of top-to-random are the coefficients in the linear map $\frac{1}{n} m \circ \Delta_{1,n-1}$, where $\Delta_{1,n-1}$ denotes the coproduct followed by projection to the graded subspace $\mathcal{S}_1 \otimes \mathcal{S}_{n-1}$ (e.g. $\Delta_{1,2}([\![316]\!]) = [\![3]\!] \otimes [\![16]\!]$). Indeed, any shuffle from [DFP92], where the deck is cut into multiple piles according to an arbitrary distribution, then interleaved, admits an analogous description in terms of the descent operators of [Pat94]. Specifically, fix the number n of cards in the deck, and for each *weak-composition* D of n (that is, $D := (d_1, \dots, d_a)$ with $\sum d_i = n$ and $d_i \geq 0$), let $P(D)$ denote the probability that the cutting of the deck places d_1 cards in the first pile, d_2 cards in the second pile, and so on. So, for the riffle-shuffle, P is the binomial distribution: $P((i, n-i)) = 2^{-n}\binom{n}{i}$, and, for top-to-random, $P((1, n-1)) = 1$ and $P(D) = 0$ for all other D. In this notation, the descent operator encoding such a shuffle is

$$\mathbf{T}_P := \sum_D \frac{P(D)}{\binom{n}{d_1 \dots d_a}} m \circ \Delta_D$$

where $\Delta_D : \mathcal{S} \to \mathcal{S}_{d_1} \otimes \cdots \otimes \mathcal{S}_{d_a}$ is a projection of an iterated coproduct. (Here, we abuse notation and write m for the multiplication of any number of factors.)

Current Work

Markov chains from descent operators

My first main result is a definition of analogous Markov chains for other combinatorial Hopf algebras to model the breaking and recombining of other combinatorial objects. That is, given any (graded) basis $\mathcal{B} = \amalg \mathcal{B}_n$ of an arbitrary graded Hopf algebra, and any $x, y \in \mathcal{B}_n$, I would like the probability of moving from x to y to be the coefficient of y when $\mathbf{T}(x)$ is expanded in the basis $\mathcal{B}$, for some fixed descent operator $\mathbf{T}$. A frequent obstacle to this naive generalisation is that, for each fixed origin state x, the coefficients of the destination states y in $\mathbf{T}(x)$ need not sum to 1. For many bases $\mathcal{B}$, I solved this problem with an atypical application of the *Doob h-transform*, which essentially rescales the basis. To refer to bases that admit such a rescaling solution, I coined the term *state space basis*; its precise definition is detailed in [Pan15b, Def. 2.2; Pan14, Def. 4.3.3].

Theorem: Construction of P-chains. *[Pan15b, Def. 3.1] Let $\mathcal{H}$ be a graded, connected Hopf algebra over $\mathbb{R}$ with state space basis $\mathcal{B}$. Define a linear function $\eta : \mathcal{S} \to \mathbb{R}$ by setting*

$$\eta(x) := \textit{sum of coefficients (in the } \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_1 \textit{ basis) of } \Delta_{1,\dots,1}(x).$$

For each $x \in \mathcal{B}$, set $\check{x} := \frac{x}{\eta(x)}$, and work in this rescaled basis. Then, for each descent operator $\mathbf{T}_P$ defined above, setting

$$\textit{Prob}(x \to y) = \textit{coefficient of } \check{y} \textit{ in } \mathbf{T}_P(\check{x})$$

defines a Markov chain.

I call this the *P-Markov chain on $\mathcal{B}_n$*. Note that the rescaling weight $\eta(x)$ is independent of the descent operator $\mathbf{T}_P$, and has an interpretation as the number of ways to break x into pieces of size 1. As an example, when $\mathcal{B}$ is the Schur basis of symmetric functions, the P-chains are precisely those of [Ful04], on the irreducible representations of the symmetric group $\mathfrak{S}_n$:

1. Choose a Young subgroup $\mathfrak{S}_{d_1} \times \cdots \times \mathfrak{S}_{d_a}$ of $\mathfrak{S}_n$ with probability $P(D)$.
2. Restrict the current irreducible representation to the chosen subgroup.
3. Induce it back up to $\mathfrak{S}_n$, then pick an irreducible constituent with probability proportional to the dimension of its isotypic component.

Other processes described by these chains include a rock-breaking model [DPR12, Chap. 4; Pan14, Sec. 5.2], the removal of edges from graphs [DPR12, Sec. 3.2], and the pruning of trees [Pan14, Sec. 5.3], which [Pan15b, Ex 5.3] rephrases as changes to an organisational structure. These arise respectively from the elementary basis of the symmetric functions [Sta99], an algebra of graphs [Sch93; Fis10], the Connes-Kreimer algebra of rooted forests [CK98].

Probabilistic conclusions from Hopf-algebraic techniques

The main advantage of this Hopf-algebraic viewpoint is that abstract theorems regarding descent operators and Hopf algebras translate into interesting information about the Markov chains. For instance, I identified all eigenvectors of eigenvalue 1 for the descent operators on any graded connected Hopf algebra, which yields the following expression for the stationary distributions (i.e. limiting distributions) of the associated chains.

Theorem: Stationary Distributions. *[Pan15b, Th. 4.5] For a fixed state space basis $\mathcal{B}$, all P-Markov chains on $\mathcal{B}_n$ have the same set of stationary distributions. These can be uniquely written as a linear combination of the functions*

$$\pi_{c_1,\dots,c_n}(x) := \frac{\eta(x)}{n!^2} \sum_{\sigma \in \mathfrak{S}_n} \textit{coefficient of } x \textit{ in the product } c_{\sigma(1)} \dots c_{\sigma(n)}$$

for each multiset $\{c_1, \dots, c_n\}$ in $\mathcal{B}_1$.

For the restriction-then-induction chain on the irreducible representations of symmetric groups, the unique stationary distribution is the ubiquitous Plancherel measure $\pi(x) := \frac{(\dim x)^2}{n!}$, related to growth processes and the eigenvalues of random matrices [Ker99; Oko00].

[Pan15b, Th. 4.2] gives the eigenvalues and multiplicities of an arbitrary descent operator on any graded connected Hopf algebra. Furthermore, for two interesting families of descent operators (of which the riffle-shuffle and top-to-random are representative), I have explicit algorithms for a full basis of eigenvectors [Pan14, Th. 2.5.1; Pan15b, Th. 5.1], from which follows many interesting probability estimates. For instance, starting from a deck of n distinct cards in ascending order, [Pan15b, Prop. 5.2] discovers that, after t top-to-random shuffles,

$$\text{Prob(second bottommost card} < \text{bottommost card)} = \left(1 - \left(\frac{n-2}{n}\right)^t\right)\frac{1}{2};$$

$$\text{Prob(third bottommost card} < \text{second bottommost card} > \text{bottommost card)} = \left(1 - \left(\frac{n-3}{n}\right)^t\right)\frac{1}{3}.$$

In a different flavour, the theorem below gives two situations where a function θ on $\mathcal{B}_n$ is a Markov statistic, simultaneously for all the P-chains.

Theorem: Markov Statistics. *[Pan15a, Th. 7.2, Th. 7.3] Let $\mathcal{H}$, $\bar{\mathcal{H}}$, $\mathcal{H}'$ be graded, connected Hopf algebras with state space bases $\mathcal{B}$, $\bar{\mathcal{B}}$, $\mathcal{B}'$ respectively. Fix an integer n, and a distribution P on the weak-compositions of n.*

i) If $\theta : \mathcal{H} \to \bar{\mathcal{H}}$ is a Hopf-morphism such that $\theta(\mathcal{B}_j) = \bar{\mathcal{B}}_j$ for all j, then the image under θ of the P-Markov chain on $\mathcal{B}_n$ is the P-Markov chain on $\bar{\mathcal{B}}_n$.

ii) Suppose $\theta : \mathcal{B}_n \to \mathcal{B}'_n$ is such that the "preimage sum" map $\theta^* : \mathcal{B}'_n \to \mathcal{H}$, defined by $\theta^*(x') := \sum_{x \in \mathcal{B}, \theta(x) = x'} x$, extends linearly to a Hopf-morphism. Then the image under θ of the P-Markov chain on $\mathcal{B}_n$ is the P-Markov chain on $\mathcal{B}'_n$, provided the starting distribution X_0 satisfies $\frac{X_0(x)}{\eta(x)} = \frac{X_0(y)}{\eta(y)}$ whenever $\theta(x) = \theta(y)$.

In [Pan13], I constructed a Hopf-morphism from the shuffle algebra to the algebra of quasisymmetric functions, to conclude that the positions of descents under any P-shuffling chain of a deck of distinct cards is a Markov statistic. (A *descent* occurs where a card has greater value than the card immediately below it.) [Pan15a] applied both parts of the above theorem to two classic Hopf-morphisms to exhibit the restriction-then-induction chain above as the image of a chain on permutations under the Robinsten-Schensted-Knuth shape map [Sta99, Sec. 7.11].

Research Plan

Many card-shuffling schemes can potentially be expressed in terms of Hopf algebras and extended to other combinatorial objects, though I expect this will require extra ingredients on top of the product, coproduct and degree. Two examples particularly fascinate me. The *a-handed ruffle* of [DR08, Sec. 5.4] cuts the deck multinomially into a piles and turns over every other pile before interleaving them. I wonder if turning the piles over can be generalised to an involution τ that is an algebra morphism and a coalgebra antimorphism (i.e. $\tau(xy) = \tau(x)\tau(y)$, and if $\Delta(x) = \sum_i w_i \otimes z_i$, then $\Delta(\tau(x)) = \sum_i \tau(z_i) \otimes \tau(w_i)$). The *weighted-random-to-top* shuffle of [Pha91] chooses a card to remove from the deck, with probability depending on the card's face value, and places it on the top of the deck. An analogue for other combinatorial objects will likely involve the Hopf monoid theory of Aguiar-Mahajan [AM10], a finer structure than the Hopf algebra. If these chain constructions are successful, I would like to determine the eigenvectors of the chains, and an analogous theorem regarding Markov statistics should come easily.

The diagonalisation of a Markov chain from a linear map $\mathbf{T}$ is closely related to a purely algebraic problem, of finding orthogonal idempotents for the algebra generated by $\mathbf{T}$. For example, the multi-handed generalisation of the riffle-shuffle corresponds to the Eulerian algebra of the symmetric group algebra, whose basis consists of sums of permutations having the same number of descents. [DFP92, Sec. 4.2] finds the idempotents for a new commutative subalgebra of the descent algebra by considering the top-to-random shuffle. I've begun, with Matthieu Josuat-Verges, this analysis for the algebra of the above ruffles, and our strategies should apply to a wide range of algebras.

Furthermore, implementing the existing mechanisms on the plethora of new combinatorial Hopf algebras [NT14; And14; CP14] would allow analysis of more Markov chains within this unified framework. For example, the "chipping gems out of rocks" chain preliminarily described in [BC15, App. B] is the only chain to date with multiple stationary distributions, and we do not yet have an interpretation for the transition probabilities. Calculating and interpreting the transition probabilities, stationary distribution and eigenfunctions on these new chains would be suitable projects for advanced undergraduate research or masters theses.

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