4.4 Coordinate Systems

In general, people are more comfortable working with the vector space $\mathbf{R}^n$ and its subspaces than with other types of vectors spaces and subspaces. The goal here is to *impose* coordinate systems on vector spaces, even if they are not in $\mathbf{R}^n$.

THEOREM 7 The Unique Representation Theorem

Let $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

DEFINITION

Suppose $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis β** (or the **β – coordinates of x**) are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

In this case, the vector in $\mathbf{R}^n$

$$[\mathbf{x}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

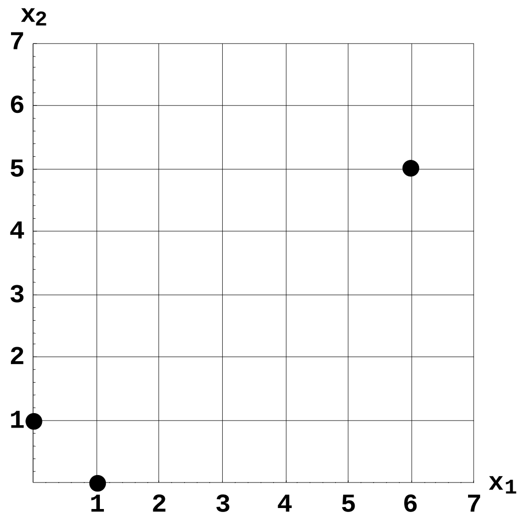
is called the **coordinate vector of x** (**relative to β**), or the **β – coordinate vector of x**.

EXAMPLE: Let $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$ where $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and let $E = \{\mathbf{e}_1, \mathbf{e}_2\}$ where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

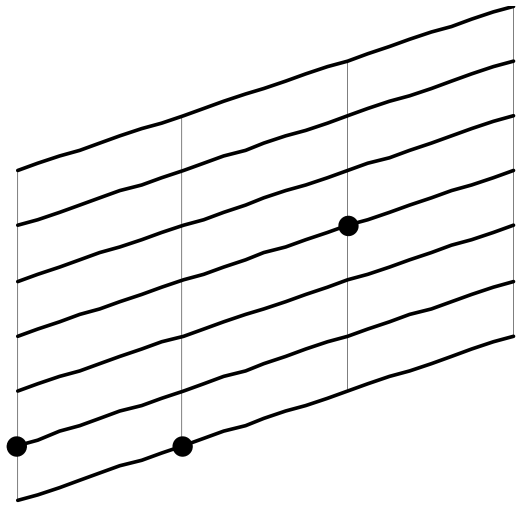
Solution:

If $[\mathbf{x}]_\beta = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then $\mathbf{x} = \_\_\_\_ \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$.

If $[\mathbf{x}]_E = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$, then $\mathbf{x} = \_\_\_\_ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$.



Standard graph paper



β – graph paper

From the last example,

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

For a basis $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, let

$$P_\beta = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{bmatrix} \text{ and } [\mathbf{x}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Then

$$\mathbf{x} = P_\beta [\mathbf{x}]_\beta.$$

We call P_β the **change-of-coordinates matrix** from β to the standard basis in $\mathbf{R}^n$. Then

$$[\mathbf{x}]_\beta = P_\beta^{-1} \mathbf{x}$$

and therefore P_β^{-1} is a **change-of-coordinates matrix** from the standard basis in $\mathbf{R}^n$ to the basis β.

EXAMPLE: Let $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathbf{x} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$. Find the change-of-coordinates matrix P_β from β to the standard basis in $\mathbf{R}^2$ and change-of-coordinates matrix P_β^{-1} from the standard basis in $\mathbf{R}^2$ to β.

Solution $\quad P_\beta = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$ and so $P_\beta^{-1} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}$

(b) If $\mathbf{x} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$, then use P_β^{-1} to find $[\mathbf{x}]_\beta = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$.

Solution: $\quad [\mathbf{x}]_\beta = P_\beta^{-1}\mathbf{x} = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$

Change between two bases, matrix for a linear transformation relative to a basis: see 4.7, 5.4 in textbook.

Coordinate mappings allow us to introduce coordinate systems for unfamiliar vector spaces.

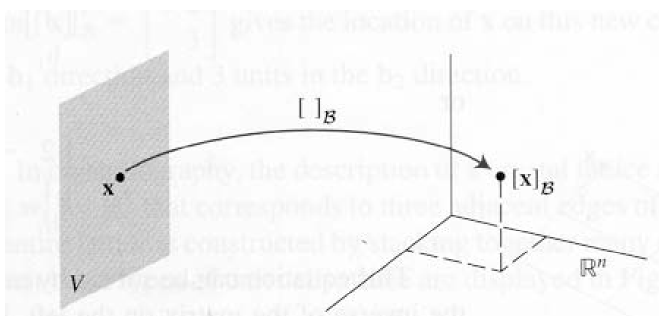


FIGURE 5 The coordinate mapping from V onto $\mathbb{R}^n$.

Standard basis for $\mathbf{P}_2$: $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} = \{1, t, t^2\}$

Polynomials in $\mathbf{P}_2$ behave like vectors in $\mathbf{R}^3$. Since $a + bt + ct^2 =$ ____$\mathbf{p}_1 +$ ____$\mathbf{p}_2 +$ ____$\mathbf{p}_3$,

$$[a + bt + ct^2]_\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

We say that the vector space $\mathbf{R}^3$ is *isomorphic* to $\mathbf{P}_2$.

EXAMPLE: Parallel Worlds of $\mathbf{R}^3$ and $\mathbf{P}_2$.

Vector Space $\mathbf{R}^3$	**Vector Space $\mathbf{P}_2$**
Vector Form: $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$	Vector Form: $a + bt + ct^2$
Vector Addition Example	*Vector Addition Example*
$\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$	$(-1 + 2t - 3t^2) + (2 + 3t + 5t^2)$ $= 1 + 5t + 2t^2$

Informally, we say that vector space V is **isomorphic** to W if *every vector space calculation in V is accurately reproduced in W*, and vice versa.

Assume β is a basis set for vector space V. Exercise 25 (page 254) shows that a set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ in V is linearly independent if and only if $\left\{[\mathbf{u}_1]_\beta, [\mathbf{u}_2]_\beta, \ldots, [\mathbf{u}_p]_\beta\right\}$ is linearly independent in $\mathbf{R}^n$.

EXAMPLE: Use coordinate vectors to determine if $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a linearly independent set, where $\mathbf{p}_1 = 1 - t$, $\mathbf{p}_2 = 2 - t + t^2$, and $\mathbf{p}_3 = 2t + 3t^2$.

Solution: The standard basis set for $\mathbf{P}_2$ is $\beta = \{1, t, t^2\}$. So

$$[\mathbf{p}_1]_\beta = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}, \ [\mathbf{p}_2]_\beta = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}, \ [\mathbf{p}_3]_\beta = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

By the IMT, $\left\{[\mathbf{p}_1]_\beta, [\mathbf{p}_2]_\beta, [\mathbf{p}_3]_\beta\right\}$ is linearly ____________________ and therefore

$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly ____________________.

Coordinate vectors also allow us to associate vector spaces with subspaces of other vectors spaces.

EXAMPLE Let $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$ where $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ and let $H =$span$\{\mathbf{b}_1, \mathbf{b}_2\}$.

Find $[\mathbf{x}]_\beta$, if $\mathbf{x} = \begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix}$.

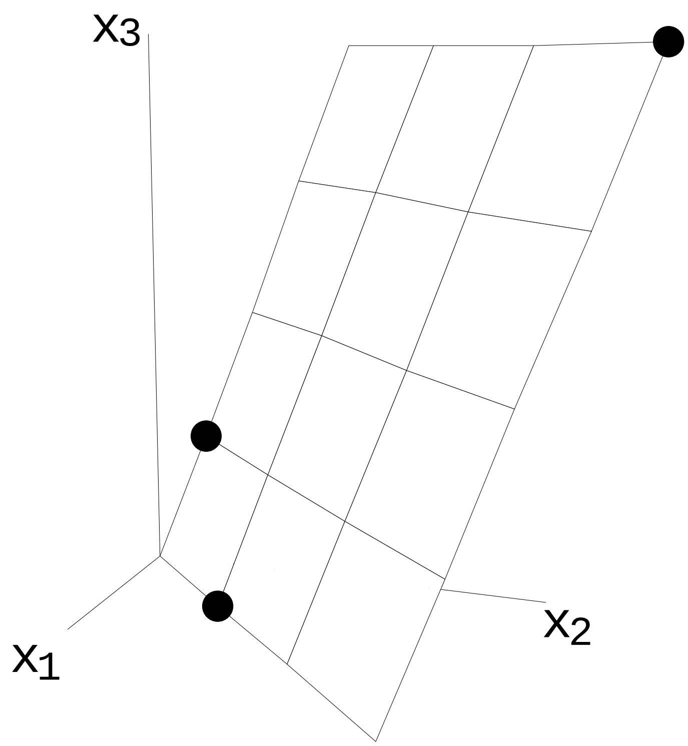
Solution: (a) Find c_1 and c_2 such that

$$c_1 \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix}$$

Corresponding augmented matrix:

$$\begin{bmatrix} 3 & 0 & 9 \\ 3 & 1 & 13 \\ 1 & 3 & 15 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $c_1 =$ ____ and $c_2 =$ _____ and so $[\mathbf{x}]_\beta = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$.



$$\begin{bmatrix} 9 \\ 13 \\ 15 \end{bmatrix} \text{ in } \mathbf{R}^3 \text{ is associated with the vector } \begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ in } \mathbf{R}^2$$

H is isomorphic to $\mathbf{R}^2$

4.5 The Dimension of a Vector Space

THEOREM 9

If a vector space V has a basis $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Proof: Suppose $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is a set of vectors in V where $p > n$. Then the coordinate vectors $\left\{[\mathbf{u}_1]_\beta, \cdots, [\mathbf{u}_p]_\beta\right\}$ are in $\mathbf{R}^n$. Since $p > n$, $\left\{[\mathbf{u}_1]_\beta, \cdots, [\mathbf{u}_p]_\beta\right\}$ are linearly dependent and therefore $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$are linearly dependent. ■

THEOREM 10

If a vector space V has a basis of n vectors, then every basis of V must consist of n vectors.

Proof: Suppose β_1 is a basis for V consisting of exactly n vectors. Now suppose β_2 is any other basis for V. By the definition of a basis, we know that β_1 and β_2 are both linearly independent sets.

By Theorem 9, if β_1 has more vectors than β_2, then ______ is a linearly dependent set (which cannot be the case).

Again by Theorem 9, if β_2 has more vectors than β_1, then ______ is a linearly dependent set (which cannot be the case).

Therefore β_2 has exactly n vectors also. ■

DEFINITION

If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be 0. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

EXAMPLE: The standard basis for $\mathbf{P}_3$ is $\{ \qquad \qquad \qquad \}$. So $\dim \mathbf{P}_3 =$ ______.

$$\boxed{\text{In general, } \dim \mathbf{P}_n = n + 1.}$$

EXAMPLE: The standard basis for $\mathbf{R}^n$ is $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the columns of I_n. So, for example, dim $\mathbf{R}^3 = 3$.

EXAMPLE: Find a basis and the dimension of the subspace

$$W = \left\{ \begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} : a,b,c,d \text{ are real} \right\}.$$

Solution: Since $\begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$,

$W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

- Note that $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, so by the Spanning Set Theorem, we may discard $\mathbf{v}_3$.
- $\mathbf{v}_4$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis for W.
- Also, dim $W =$____.

EXAMPLE: ***Dimensions of subspaces of $\mathbf{R}^3$***

0-dimensional subspace contains only the zero vector $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

1-dimensional subspaces. Span$\{\mathbf{v}\}$ where $\mathbf{v} \neq \mathbf{0}$ is in $\mathbf{R}^3$.

These subspaces are ______________ through the origin.

2-dimensional subspaces. Span$\{\mathbf{u}, \mathbf{v}\}$ where $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbf{R}^3$ and are not multiples of each other.

These subspaces are ______________ through the origin.

3-dimensional subspaces. Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent vectors in $\mathbf{R}^3$. This subspace is $\mathbf{R}^3$ itself because the columns of $A = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$ span $\mathbf{R}^3$ according to the IMT.

THEOREM 11

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V.$$

EXAMPLE: Let $H = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Then H is a subspace of $\mathbf{R}^3$ and $\dim H < \dim \mathbf{R}^3$.

We could expand the spanning set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ to $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ to form a basis for $\mathbf{R}^3$.

THEOREM 12 THE BASIS THEOREM

Let V be a p – dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p vectors in V is automatically a basis for V. Any set of exactly p vectors that spans V is automatically a basis for V.

EXAMPLE: Show that $\{t, 1-t, 1+t-t^2\}$ is a basis for $\mathbf{P}_2$.

Solution: Let $\mathbf{v}_1 = t, \mathbf{v}_2 = 1-t, \mathbf{v}_3 = 1+t-t^2$ and $\beta = \{1, t, t^2\}$.

Corresponding coordinate vectors

$$[\mathbf{v}_1]_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, [\mathbf{v}_2]_\beta = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, [\mathbf{v}_3]_\beta = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$[\mathbf{v}_2]_\beta$ is not a multiple of $[\mathbf{v}_1]_\beta$

$[\mathbf{v}_3]_\beta$ is not a linear combination of $[\mathbf{v}_1]_\beta$ and $[\mathbf{v}_2]_\beta$

$\Rightarrow$ $\{[\mathbf{v}_1]_\beta, [\mathbf{v}_2]_\beta, [\mathbf{v}_3]_\beta\}$ is linearly independent and therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent.

Since dim $\mathbf{P}_2 = 3$, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbf{P}_2$ according to The Basis Theorem.

Dimensions of Col A and Nul A

Recall our techniques to find basis sets for column spaces and null spaces.

EXAMPLE: Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix}$. Find dim Col A and dim Nul A.

Solution

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} \quad \end{bmatrix}, \begin{bmatrix} \quad \end{bmatrix} \right\}$ is a basis for Col A and dim Col $A = 2$.

Now solve $A\mathbf{x} = \mathbf{0}$ by row-reducing the corresponding augmented matrix. Then we arrive at

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 2 & 4 & 7 & 8 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= -2x_2 - 4x_4 \\ x_3 &= 0 \end{aligned}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul A and dim Nul $A = 2$.

Note

$$\boxed{\textit{dim Col } A = \textit{number of pivot columns of } A}$$

$$\boxed{\textit{dim Nul } A = \textit{number of free variables of } A}.$$

Section 4.6 Rank

The set of all linear combinations of the row vectors of a matrix A is called the **row space** of A and is denoted by Row A.

EXAMPLE: Let

$$A = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix} \quad \text{and} \quad \begin{aligned} \mathbf{r}_1 &= (-1, 2, 3, 6) \\ \mathbf{r}_2 &= (2, -5, -6, -12) \\ \mathbf{r}_3 &= (1, -3, -3, -6) \end{aligned}.$$

Row A = Span$\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ (a subspace of $\mathbf{R}^4$)

While it is natural to express row vectors horizontally, they can also be written as column vectors if it is more convenient. Therefore

$$\boxed{\text{Col } A^T = \text{Row } A}.$$

When we use row operations to reduce matrix A to matrix B, we are taking linear combinations of the rows of A to come up with B. We could reverse this process and use row operations on B to get back to A. Because of this, the row space of A equals the row space of B.

THEOREM 13

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as B.

EXAMPLE: The matrices

$$A = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are row equivalent. Find a basis for row space, column space and null space of A. Also state the dimension of each.

Basis for Row A : $\{ \qquad\qquad\qquad\qquad\qquad\qquad \}$

dim Row A :_____

Basis for Col A : $\left\{ \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \\ \\ \end{bmatrix} \right\}$

dim Col A :_____

To find Nul A, solve $A\mathbf{x} = \mathbf{0}$ first:

$$\begin{bmatrix} -1 & 2 & 3 & 6 & 0 \\ 2 & -5 & -6 & -12 & 0 \\ 1 & -3 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & 6 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -6 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_3 + 6x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis for Nul A : $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and dim Nul $A =$ ______

Note the following:

$\dim$ Col A = # of pivots of A = # of nonzero rows in B = $\dim$ Row A.

$\dim$ Nul A = # of free variables = # of nonpivot columns of A.

DEFINITION

The **rank** of A is the dimension of the column space of A.

$$\boxed{\text{rank } A = \dim \text{Col } A = \text{\# of pivot columns of } A = \dim \text{Row } A}.$$

$$\underbrace{\text{rank } A}_{\updownarrow \atop \left\{\begin{array}{c}\text{\# of pivot}\\ \text{columns}\\ \text{of } A\end{array}\right\}} + \underbrace{\dim \text{Nul } A}_{\updownarrow \atop \left\{\begin{array}{c}\text{\# of nonpivot}\\ \text{columns}\\ \text{of } A\end{array}\right\}} = \underbrace{n}_{\updownarrow \atop \left\{\begin{array}{c}\text{\# of}\\ \text{columns}\\ \text{of } A\end{array}\right\}}$$

THEOREM 14 THE RANK THEOREM

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n.$$

Since Row A = Col A^T,

$$\boxed{\text{rank } A = \text{rank } A^T}.$$

EXAMPLE: Suppose that a 5×8 matrix A has rank 5. Find $\dim$ Nul A, $\dim$ Row A and rank A^T. Is Col $A = \mathbf{R}^5$?

Solution:

$$\underbrace{\text{rank } A}_{\updownarrow \atop 5} + \underbrace{\dim \text{ Nul } A}_{\downarrow \atop ?} = \underbrace{n}_{\updownarrow \atop 8}$$

$$5 + \dim \text{ Nul } A = 8 \qquad \Rightarrow \qquad \dim \text{ Nul } A = \underline{\qquad}$$

$$\dim \text{ Row } A = \text{rank } A = \underline{\qquad} \qquad \Rightarrow \qquad \text{rank } A^T = \text{rank } \underline{\qquad} = \underline{\qquad}$$

Since rank A = # of pivots in $A = 5$, there is a pivot in every row. So the columns of A span $\mathbf{R}^5$ (by Theorem 4, page 43). Hence Col $A = \mathbf{R}^5$.

EXAMPLE: For a 9×12 matrix A, find the smallest possible value of $\dim$ Nul A.

Solution:

$$\text{rank } A + \dim \text{ Nul } A = 12$$

$$\dim \text{ Nul } A = 12 - \underbrace{\text{rank } A}_{\text{largest possible value}=\underline{\qquad}}$$

$$\text{smallest possible value of } \dim \text{ Nul } A = \underline{\qquad}$$

Visualizing Row A and Nul A

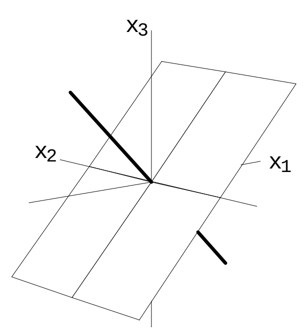
EXAMPLE: Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 2 \end{bmatrix}$. One can easily verify the following:

Basis for Nul $A = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and therefore Nul A is a plane in $\mathbf{R}^3$.

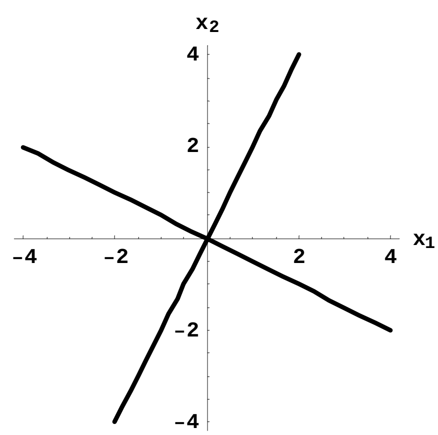
Basis for Row $A = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ and therefore Row A is a line in $\mathbf{R}^3$.

Basis for Col $A = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and therefore Col A is a line in $\mathbf{R}^2$.

Basis for Nul $A^T = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ and therefore Nul A^T is a line in $\mathbf{R}^2$.



Subspaces Nul A and Row A



Subspaces Nul A^T and Col A

The Rank Theorem provides us with a powerful tool for determining information about a system of equations.

EXAMPLE: A scientist solves a homogeneous system of 50 equations in 54 variables and finds that exactly 4 of the unknowns are free variables. Can the scientist be *certain* that any associated nonhomogeneous system (with the same coefficients) has a solution?

Solution: Recall that

rank $A = \dim$ Col A = # of pivot columns of A

$\dim$ Nul A = # of free variables

In this case $A\mathbf{x} = \mathbf{0}$ of where A is 50×54.

By the rank theorem,

$$\text{rank } A + \_\_\_\_\_\_\_ = \_\_\_\_\_\_\_$$

or

$$\text{rank } A = \_\_\_\_\_\_\_.$$

So any nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ has a solution because there is a pivot in every row.

THE INVERTIBLE MATRIX THEOREM (continued)

Let A be a square $n \times n$ matrix. The the following statements are equivalent:

m. The columns of A form a basis for $\mathbf{R}^n$

n. Col $A = \mathbf{R}^n$

o. dim Col $A = n$

p. rank $A = n$

q. Nul $A = \{\mathbf{0}\}$

r. dim Nul $A = 0$