

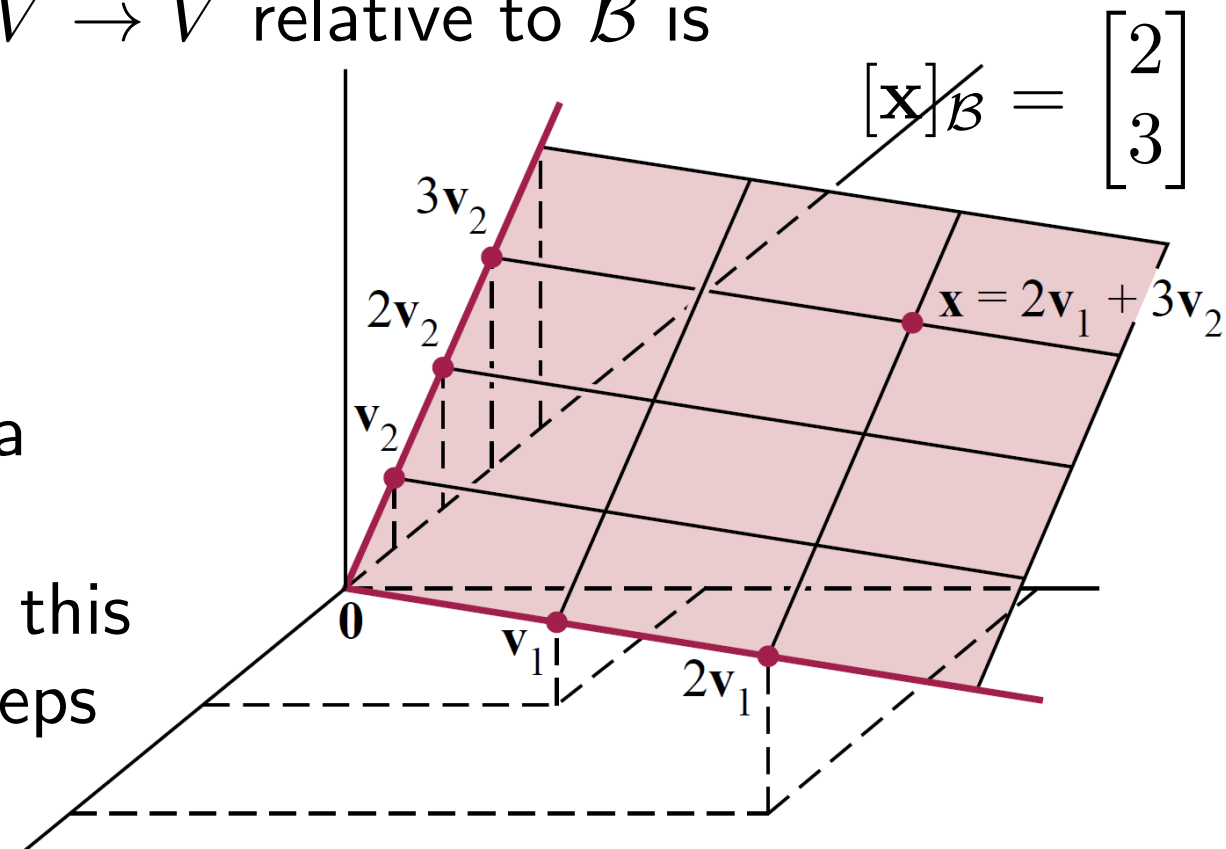
§4.4, 4.7, 5.4: Change of Basis

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for V . Remember:

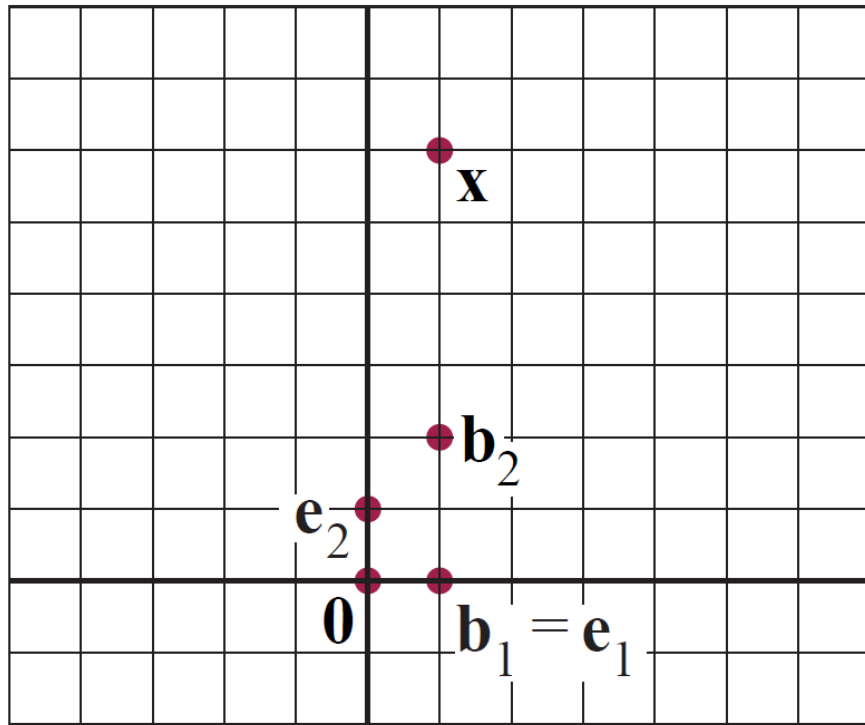
- The \mathcal{B} -coordinate vector of \mathbf{x} is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ where $\mathbf{x} = c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n$.
- The matrix for a linear transformation $T : V \rightarrow V$ relative to \mathcal{B} is

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & \dots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix}.$$

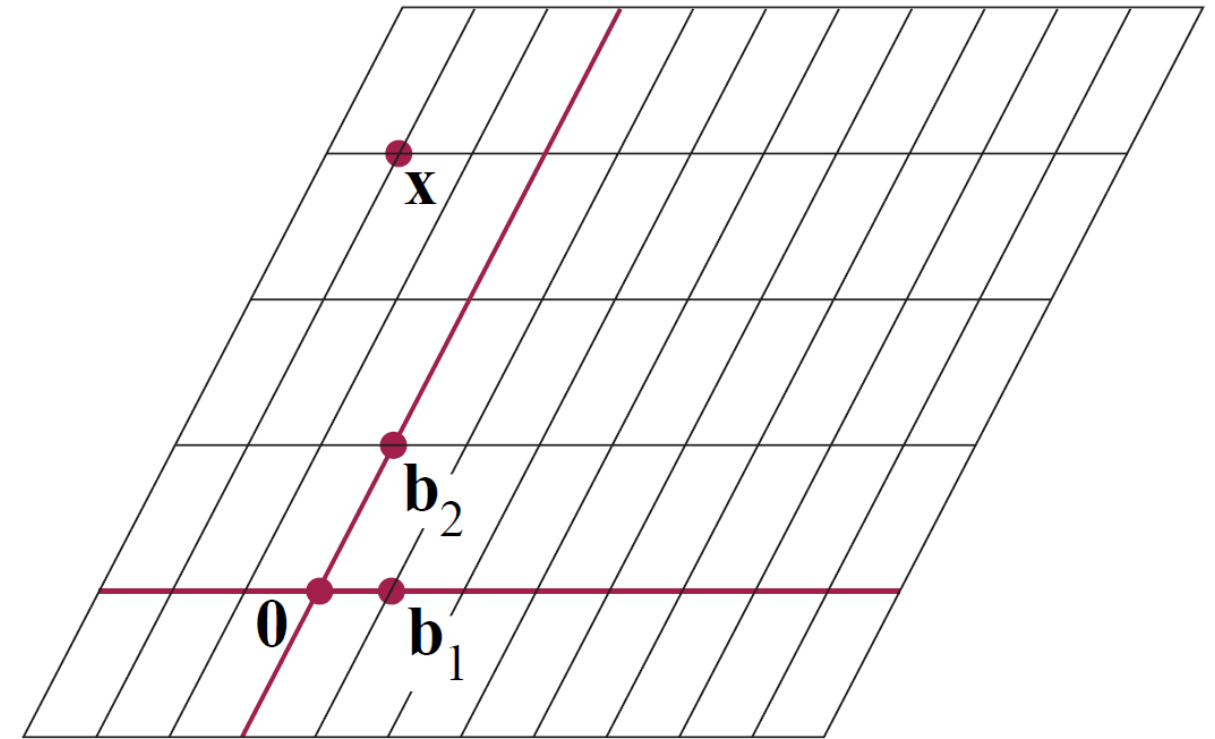
A basis for this plane in \mathbb{R}^3 allows us to draw a coordinate grid on the plane. The coordinate vectors then describe the location of points on this plane relative to this coordinate grid (e.g. 2 steps in \mathbf{v}_1 direction, 3 steps in \mathbf{v}_2 direction.)



Although we already have the standard coordinate grid on \mathbb{R}^n , some computations are much faster and more accurate in a different basis i.e. using a different coordinate grid (later, p17-19).



standard coordinate grid



\mathcal{B} -coordinate grid

Important questions:

- i how are x and $[x]_{\mathcal{B}}$ related (p3-7, §4.4 in textbook);
- ii how are $[x]_{\mathcal{B}}$ and $[x]_{\mathcal{F}}$ related for two bases \mathcal{B} and \mathcal{F} (p8-11, §4.7);
- iii how are the standard matrix of T and the matrix $[T]_{\mathcal{B}}$ related (p12-15, §5.4).

i: Changing between the standard basis and another basis in \mathbb{R}^n :

Example: Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis of \mathbb{R}^2 .

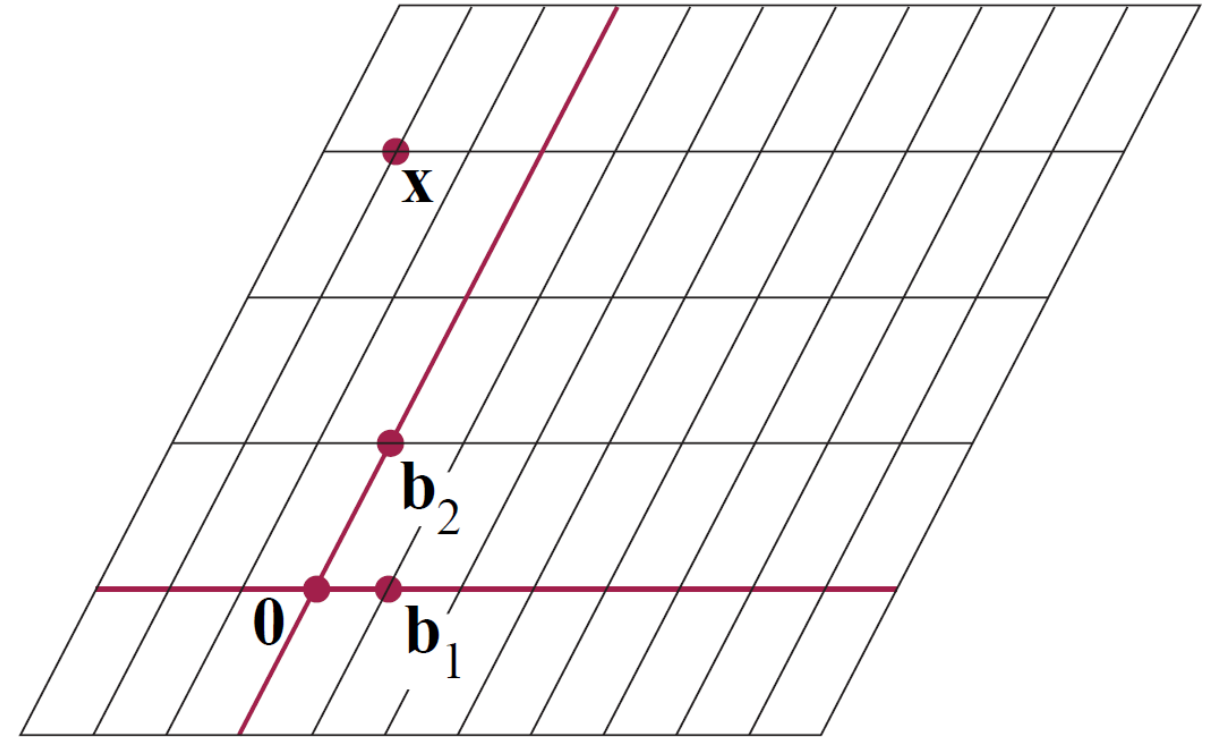
If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, then what is \mathbf{x} ?

Answer:

$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ means that

$$\mathbf{x} = -2\mathbf{b}_1 + 3\mathbf{b}_2 = -2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

(You can check this from the picture on the previous page.)



$$\text{So, if } [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \text{ then } \mathbf{x} = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 = \begin{bmatrix} | & | \\ \mathbf{b}_1 & \mathbf{b}_2 \\ | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

In general, if $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for \mathbb{R}^n , and $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, then

$$\mathbf{x} = c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n = \begin{bmatrix} | & | & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_n \\ | & | & | \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}.$$

Notation: write $\mathcal{P}_{\mathcal{B}}$ for the matrix $\begin{bmatrix} | & | & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_n \\ | & | & | \end{bmatrix}$. This is the
change-of-coordinates matrix from \mathcal{B} to the standard basis.

Going the other direction:

Example: Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis of \mathbb{R}^2 .

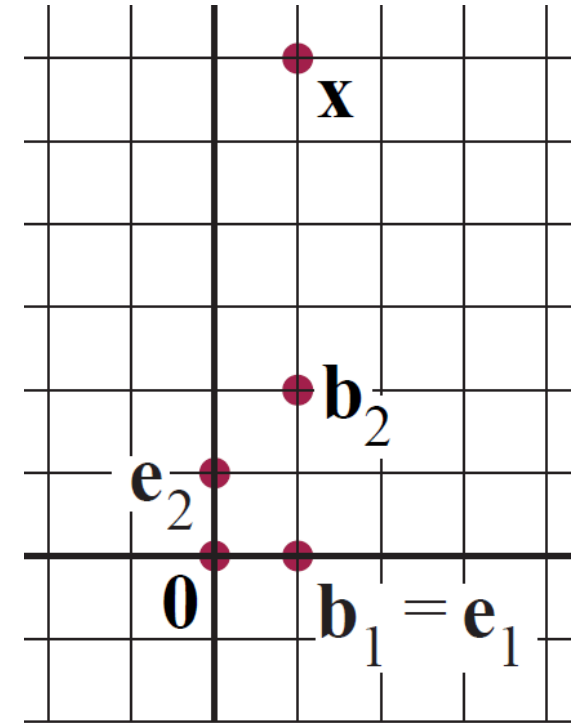
If $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$, then what are its \mathcal{B} -coordinates $[\mathbf{x}]_{\mathcal{B}}$?

Answer:

$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ where $\begin{bmatrix} 1 \\ 6 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ i.e. (c_1, c_2) is the solution to the linear system $\begin{bmatrix} 1 & 1 & | & 1 \\ 0 & 2 & | & 6 \end{bmatrix}$. (Exercise: do the row-reduction and show that $c_1 = -2, c_2 = 3$ as on the last pages.)

So the \mathcal{B} -coordinate vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ of \mathbf{v} satisfies $\mathbf{v} = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 = \begin{bmatrix} | & | \\ \mathbf{b}_1 & \mathbf{b}_2 \\ | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

So $[\mathbf{v}]_{\mathcal{B}}$ is the solution to $\begin{bmatrix} | & | \\ \mathbf{b}_1 & \mathbf{b}_2 \\ | & | \end{bmatrix} \mathbf{x} = \mathbf{v}$.



In general, if $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for \mathbb{R}^n , and \mathbf{v} is any vector in \mathbb{R}^n , then $[\mathbf{v}]_{\mathcal{B}}$ is a solution to

$$\begin{array}{c} \left[\begin{array}{ccc} | & & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_n \\ | & & | \end{array} \right] \mathbf{x} = \mathbf{v}. \\ \begin{array}{c} | \\ | \\ \mathcal{P}_{\mathcal{B}} \end{array} \end{array}$$

Because \mathcal{B} is a basis, the columns of $\mathcal{P}_{\mathcal{B}}$ are linearly independent, so by the Invertible Matrix Theorem, $\mathcal{P}_{\mathcal{B}}$ is invertible, and the unique solution to $\mathcal{P}_{\mathcal{B}}\mathbf{x} = \mathbf{v}$ is

$$[\mathbf{v}]_{\mathcal{B}} = \left[\begin{array}{ccc} | & & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_n \\ | & & | \end{array} \right]^{-1} \mathbf{v}.$$

In other words, the change-of-coordinates matrix from the standard basis to \mathcal{B} is $\mathcal{P}_{\mathcal{B}}^{-1}$.

Indeed, in the previous example, $\mathcal{P}_{\mathcal{B}}^{-1}\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & -1 \\ -0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$

A very common mistake is to get the direction wrong:

Does multiplication by $\mathcal{P}_{\mathcal{B}}$ change from standard coordinates to \mathcal{B} -coordinates, or from \mathcal{B} -coordinates to standard coordinates?

Don't memorise the formulas. Instead, remember the **definition** of coordinates:

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \text{ means } \mathbf{x} = c_1 \mathbf{b}_1 + \cdots + c_n \mathbf{b}_n = \begin{bmatrix} | & | & | \\ \mathbf{b}_1 & \cdots & \mathbf{b}_n \\ | & | & | \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}$$

and you won't go wrong.

ii: Changing between two non-standard bases:

Example: As before, $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$.

Another basis: $\mathbf{f}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{f}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2\}$.

If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, then what are its \mathcal{F} -coordinates $[\mathbf{x}]_{\mathcal{F}}$?

Answer 1: \mathcal{B} to standard to \mathcal{F} - works only in \mathbb{R}^n , in general easiest to calculate.

$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ means $\mathbf{x} = -2\mathbf{b}_1 + 3\mathbf{b}_2 = -2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$.

So if $[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, then $d_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$.

Row-reducing $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 6 \end{array} \right]$ shows $d_1 = 1, d_2 = 5$ so $[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

In other words, $\mathbf{x} = \mathcal{P}_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{F}} = \mathcal{P}_{\mathcal{F}}^{-1}\mathbf{x}$, so $[\mathbf{x}]_{\mathcal{F}} = \mathcal{P}_{\mathcal{F}}^{-1}\mathcal{P}_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$.

Answer 2: A different view that works for abstract vector spaces (without reference to a standard basis) - important theoretically, but may be hard to calculate for general examples in \mathbb{R}^n .

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \text{ means } \mathbf{x} = -2\mathbf{b}_1 + 3\mathbf{b}_2.$$

$$\text{So } [\mathbf{x}]_{\mathcal{F}} = [-2\mathbf{b}_1 + 3\mathbf{b}_2]_{\mathcal{F}} = -2[\mathbf{b}_1]_{\mathcal{F}} + 3[\mathbf{b}_2]_{\mathcal{F}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{F}} & [\mathbf{b}_2]_{\mathcal{F}} \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

because $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{F}}$ is an isomorphism, so every vector space calculation is accurately reproduced using coordinates.

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{f}_1 - \mathbf{f}_2 \text{ so } [\mathbf{b}_1]_{\mathcal{F}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{f}_1 + \mathbf{f}_2 \text{ so } [\mathbf{b}_2]_{\mathcal{F}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\text{So } [\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

This step can be hard to calculate if the \mathbf{b}_i are not “easy” linear combinations of the \mathbf{f}_i . But if you need to change bases in a practical application, the bases are probably “nicely” related.

Theorem 15: Change of Basis: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{F} = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ be two bases of a vector space V . Then, for all \mathbf{x} in V ,

$$[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix} | & | & | \\ [\mathbf{b}_1]_{\mathcal{F}} & \dots & [\mathbf{b}_n]_{\mathcal{F}} \\ | & | & | \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}.$$

Notation: write $\mathcal{P}_{\mathcal{F} \leftarrow \mathcal{B}}$ for the matrix $\begin{bmatrix} | & | & | \\ [\mathbf{b}_1]_{\mathcal{F}} & \dots & [\mathbf{b}_n]_{\mathcal{F}} \\ | & | & | \end{bmatrix}$, the
change-of-coordinates matrix from \mathcal{B} to \mathcal{F} .

A tip to get the direction correct:

$$[\mathbf{x}]_{\mathcal{F}} = \underbrace{\mathcal{P}_{\mathcal{F} \leftarrow \mathcal{B}}}_{\text{a linear combination of columns of } \mathcal{P}_{\mathcal{F} \leftarrow \mathcal{B}}, \text{ so these columns should be } \mathcal{F}\text{-coordinate vectors}} [\mathbf{x}]_{\mathcal{B}}$$

A \mathcal{F} -coordinate vector

Theorem 15: Change of Basis: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{F} = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ be two bases of a vector space V . Then, for all \mathbf{x} in V ,

$$[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix} | & & | \\ [\mathbf{b}_1]_{\mathcal{F}} & \dots & [\mathbf{b}_n]_{\mathcal{F}} \\ | & & | \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}.$$

Properties of the change-of-coordinates matrix $\mathcal{P}_{\mathcal{F} \leftarrow \mathcal{B}} = \begin{bmatrix} | & & | \\ [\mathbf{b}_1]_{\mathcal{F}} & \dots & [\mathbf{b}_n]_{\mathcal{F}} \\ | & & | \end{bmatrix}$:

- $\mathcal{P}_{\mathcal{B} \leftarrow \mathcal{F}}^{-1} = \mathcal{P}_{\mathcal{F} \leftarrow \mathcal{B}}$.
- If V is \mathbb{R}^n and \mathcal{E} is the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, then

$$\mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} = \mathcal{P}_{\mathcal{B}} = \begin{bmatrix} | & & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_n \\ | & & | \end{bmatrix}, \text{ because } [\mathbf{b}_i]_{\mathcal{E}} = \mathbf{b}_i. \text{ Also } \mathcal{P}_{\mathcal{B} \leftarrow \mathcal{E}} = \mathcal{P}_{\mathcal{B}}^{-1}.$$

- If V is \mathbb{R}^n , then $\mathcal{P}_{\mathcal{F} \leftarrow \mathcal{B}} = \mathcal{P}_{\mathcal{F}}^{-1} \mathcal{P}_{\mathcal{B}}$ (see p8).

iii: Change of coordinates and linear transformations:

Remember that the matrix for a linear transformation $T : V \rightarrow V$ relative to \mathcal{B} is

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix}, \text{ and this matrix is useful because}$$

$$[T(\mathbf{x})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} \quad (*)$$

(i.e. if you're working in \mathcal{B} -coordinates, then applying the function T is the same as multiplying by the matrix $[T]_{\mathcal{B}}$).

Often, it is easier to find the matrix for T relative to one basis than to another (later, p14). So it's important to know how to find $[T(\mathbf{x})]_{\mathcal{F}}$ if we know $[T(\mathbf{x})]_{\mathcal{B}}$.

In \mathbb{R}^n , the following is true:

$$T(\mathbf{x}) = \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} [T(\mathbf{x})]_{\mathcal{B}} \stackrel{(*)}{=} \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} [T]_{\mathcal{B}} [\mathbf{x}]_{\mathcal{B}} = \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} [T]_{\mathcal{B}} \mathcal{P}_{\mathcal{B} \leftarrow \mathcal{E}} \mathbf{x}.$$

Let $[T]_{\mathcal{E}}$ be the standard matrix of T . Then the equation above shows that

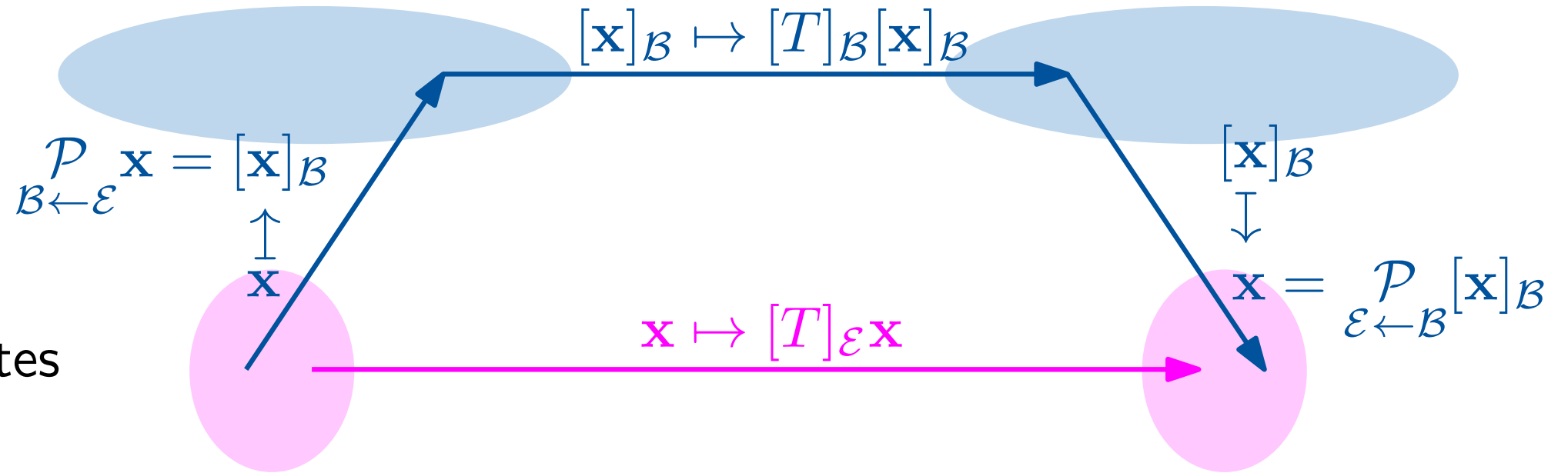
$[T]_{\mathcal{E}} \mathbf{x} = \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} [T]_{\mathcal{B}} \mathcal{P}_{\mathcal{B} \leftarrow \mathcal{E}} \mathbf{x}$ for all \mathbf{x} . So

$$[T]_{\mathcal{E}} = \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} [T]_{\mathcal{B}} \mathcal{P}_{\mathcal{B} \leftarrow \mathcal{E}}.$$

A picture to illustrate $[T]_{\mathcal{E}} = \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} [T]_{\mathcal{B}} \mathcal{P}_{\mathcal{B} \leftarrow \mathcal{E}}$:

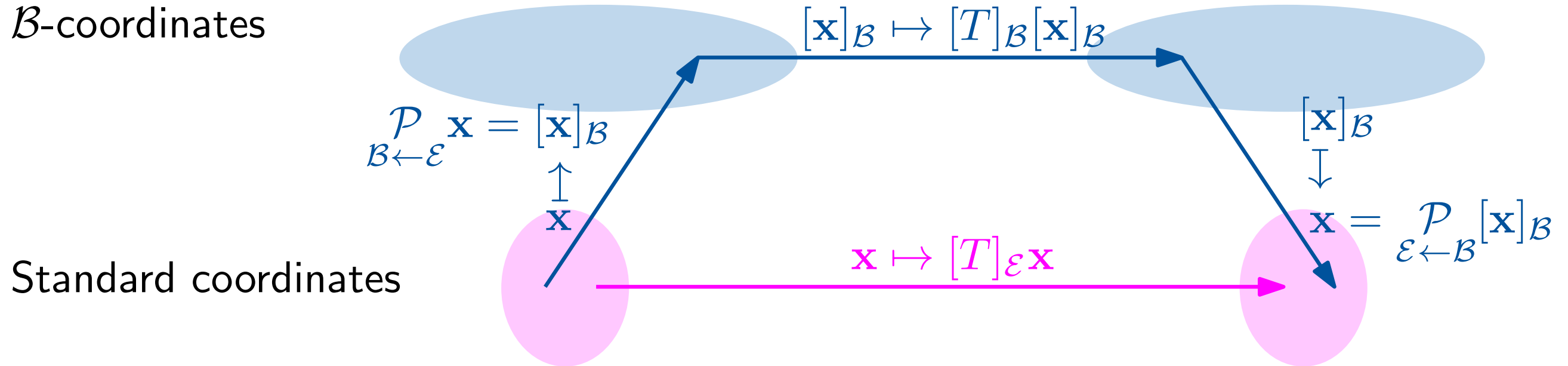
\mathcal{B} -coordinates

Standard coordinates



A picture to illustrate $[T]_{\mathcal{E}} = \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} [T]_{\mathcal{B}} \mathcal{P}_{\mathcal{B} \leftarrow \mathcal{E}}$:

\mathcal{B} -coordinates



Standard coordinates

Because $\mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} = \mathcal{P}_{\mathcal{B}}$ and $\mathcal{P}_{\mathcal{B} \leftarrow \mathcal{E}} = \mathcal{P}_{\mathcal{B}}^{-1}$:

$$[T]_{\mathcal{E}} = \mathcal{P}_{\mathcal{B}} [T]_{\mathcal{B}} \mathcal{P}_{\mathcal{B}}^{-1}.$$

Multiply both sides by $\mathcal{P}_{\mathcal{B}}^{-1}$ on the left and by $\mathcal{P}_{\mathcal{B}}$ on the right:

$$\mathcal{P}_{\mathcal{B}}^{-1} [T]_{\mathcal{E}} \mathcal{P}_{\mathcal{B}} = [T]_{\mathcal{B}}$$

These two equations are hard to remember (“where does the inverse go?”). Instead, remember $[T]_{\mathcal{E}} = \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} [T]_{\mathcal{B}} \mathcal{P}_{\mathcal{B} \leftarrow \mathcal{E}}$ (which works for all vector spaces, not just \mathbb{R}^n).

Change of basis is useful for finding matrices of geometric linear transformations:

Example: Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be reflection through the line $y = 2x$.

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is on the line $y = 2x$, so it is unchanged by the reflection: $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
 $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is perpendicular to $y = 2x$, so its image is its negative: $T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

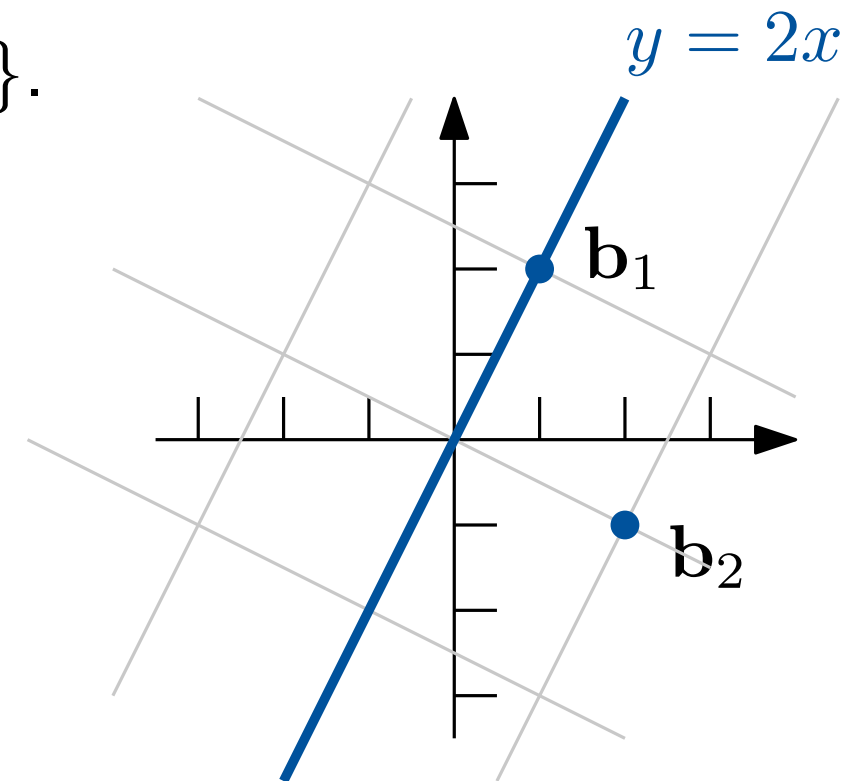
Find the standard matrix of T .

Answer: Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$.

By above, $T(\mathbf{b}_1) = 1\mathbf{b}_1 + 0\mathbf{b}_2$ so $[T(\mathbf{b}_1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

And $T(\mathbf{b}_2) = 0\mathbf{b}_1 + (-1)\mathbf{b}_2$ so $[T(\mathbf{b}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

So $[T]_{\mathcal{B}} = \begin{bmatrix} | & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.



Answer: (continued) To get the standard matrix $[T]_{\mathcal{E}}$ from $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$:

$$[T]_{\mathcal{E}} = {}_{\mathcal{E} \leftarrow \mathcal{B}} \mathcal{P} [T]_{\mathcal{B}} \mathcal{P}_{\mathcal{B} \leftarrow \mathcal{E}}$$

and

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \text{ means } \mathbf{x} = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 = \begin{bmatrix} | & | \\ \mathbf{b}_1 & \mathbf{b}_2 \\ | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}$$

so

$$[T]_{\mathcal{E}} = {}_{\mathcal{E} \leftarrow \mathcal{B}} \mathcal{P} [T]_{\mathcal{B}} \mathcal{P}_{\mathcal{B} \leftarrow \mathcal{E}} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}.$$

(To check your arithmetic: check that $[T]_{\mathcal{E}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, as given in the question).

Many alternative methods exist: e.g. see Homework 3 Q3 solution.

Remember

$$[T]_{\mathcal{E}} = {}_{\mathcal{E} \leftarrow \mathcal{B}} \mathcal{P} [T]_{\mathcal{B}} {}_{\mathcal{B} \leftarrow \mathcal{E}} \mathcal{P} = {}_{\mathcal{E} \leftarrow \mathcal{B}} \mathcal{P} [T]_{\mathcal{B}} {}_{\mathcal{E} \leftarrow \mathcal{B}} \mathcal{P}^{-1}.$$

This motivates the following definition:

Definition: Two square matrices A and D are *similar* if there is an invertible matrix P such that $A = PDP^{-1}$.

Similar matrices represent the *same linear transformation in different bases*.

Similar matrices have the *same determinant* and the *same rank*, because the signed volume scaling factor and the dimension of the image are coordinate-independent properties of the linear transformation. (You can also prove $\det D = \det(PDP^{-1})$ from the multiplicative property of determinants.)

Why is change of basis important?

Example: If x, y are the prices of two stocks on a particular day, then their prices the next day are respectively $\frac{1}{2}y$ and $-x + \frac{3}{2}y$. How are the prices after many days related to the prices today?

Answer: Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the function representing the changes in stock prices from one day to the next, i.e. $T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} \frac{1}{2}y \\ -x + \frac{3}{2}y \end{bmatrix}$. We are interested in T^k for large k . (You will NOT be required to do this step.)

T is a linear transformation; its standard matrix is $[T]_{\mathcal{E}} = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix}$. Calculating

$\begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix}^k$ by direct matrix multiplication will take a long time.

Answer: (continued) Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$.

$$T(\mathbf{b}_1) = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2}\mathbf{b}_1, \quad T(\mathbf{b}_2) = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathbf{b}_2,$$

$$\text{so } [T]_{\mathcal{B}} = \begin{bmatrix} | & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} \\ | & | \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}. \text{ Use } [T]_{\mathcal{E}} = \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} [T]_{\mathcal{B}} \mathcal{P}_{\mathcal{B} \leftarrow \mathcal{E}} = \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} [T]_{\mathcal{B}} \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}}^{-1}:$$

$$\begin{aligned} [T]_{\mathcal{E}}^k &= \left(\mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} [T]_{\mathcal{B}} \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}}^{-1} \right)^k \\ &= \left(\mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} [T]_{\mathcal{B}} \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}}^{-1} \right) \left(\mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} [T]_{\mathcal{B}} \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}}^{-1} \right) \cdots \left(\mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} [T]_{\mathcal{B}} \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}}^{-1} \right) \\ &= \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}} [T]_{\mathcal{B}}^k \mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}}^{-1} \end{aligned}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2}^k & 0 \\ 0 & 1^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -1 + \frac{1}{2}^{k-1} & 1 - \frac{1}{2}^k \\ -2 + \frac{1}{2}^{k-1} & 2 - \frac{1}{2}^k \end{bmatrix}.$$

So $[T]_{\mathcal{E}}^k = \begin{bmatrix} -1 + \frac{1}{2}^{k-1} & 1 - \frac{1}{2}^k \\ -2 + \frac{1}{2}^{k-1} & 2 - \frac{1}{2}^k \end{bmatrix}$. When k is very large, this is very close to $\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$.

So essentially the stock prices after many days is $-x + y$ and $-2x + 2y$, where x, y are the prices today. (In particular, the prices stabilise, which was not clear from $[T]_{\mathcal{E}}$.)

The **important points** in this example:

- We have a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and we want to find T^k for large k .
- We find a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ where $T(\mathbf{b}_1) = \lambda_1 \mathbf{b}_1$ and $T(\mathbf{b}_2) = \lambda_2 \mathbf{b}_2$ for some scalars λ_1, λ_2 . (In the example, $\lambda_1 = \frac{1}{2}, \lambda_2 = 1$.)
- Relative to the basis \mathcal{B} , the matrix for T is a **diagonal matrix** $[T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.
- It is easy to compute with $[T]_{\mathcal{B}}$, and we can then use change of coordinates to transfer the result to the standard matrix $[T]_{\mathcal{E}}$.

Next week (§5): does a “magic” basis like this always exist, and how to find it?

(Don’t worry: you can do many of the computations in §5 without fully understanding change of coordinates.)