Remember the addition and scalar multiplication of matrices:

$$(A+B)_{ij} = a_{ij} + b_{ij},$$

e.g $\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}.$

$$(cA)_{ij} = ca_{ij},$$

e.g. $(-3)\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -12 & 0 & -15 \\ 3 & -9 & -6 \end{bmatrix}.$

Is this really different from $\mathbb{R}^6$?

$$\begin{bmatrix} 4 \\ 0 \\ 5 \\ -1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 6 \\ 2 \\ 8 \\ 9 \end{bmatrix}. \qquad (-3)\begin{bmatrix} 4 \\ 0 \\ 5 \\ -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ 0 \\ -15 \\ 3 \\ -9 \\ -6 \end{bmatrix}.$$

Remember from calculus the addition and scalar multiplication of polynomials:

e.g $(2t^2 + 1) + (-t^2 + 3t + 2) = t^2 + 3t + 3.$

e.g $(-3)(-t^2 + 3t + 2) = 3t^2 - 9t - 6.$

Is this really different from $\mathbb{R}^3$?

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}. \qquad (-3)\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \\ 3 \end{bmatrix}. \qquad \begin{matrix} \leftarrow \text{coefficient of } 1 \\ \leftarrow \text{coefficient of } t \\ \leftarrow \text{coefficient of } t^2 \end{matrix}$$

§4.1, pp217-218: Abstract Vector Spaces

As the examples above showed, there are many objects in mathematics that "looks" and "feels" like $\mathbb{R}^n$. We will also call these vectors.

The real power of linear algebra is that everything we learned in Chapters 1-3 can be applied to all these abstract vectors, not just to column vectors.

You should think of abstract vectors as objects which can be added and multiplied by scalars - i.e. where the concept of "linear combination" makes sense. This addition and scalar multiplication must obey some "sensible rules" called axioms (see next page).

The axioms guarantee that the proof of every result and theorem from Chapters 1-3 will work for our new definition of vectors.

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms below. The axioms must hold for all **u**, **v** and **w** in V and for all scalars c and d.

1. $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $(cd)\mathbf{u} = c(d\mathbf{u})$.
10. $1\mathbf{u} = \mathbf{u}$.

Examples of vector spaces:

$M_{2\times 3}$, the set of 2×3 matrices.

Let's check axiom 4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

The zero vector for $M_{2\times 3}$ is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

You can check the other 9 axioms by using the properties of matrix addition and scalar multiplication (page 5 of week 4 slides, theorem 2.1 in textbook).

Similarly, $M_{m\times n}$, the set of all $m \times n$ matrices, is a vector space.

Is the set of all matrices (of all sizes) a vector space?

No, because we cannot add two matrices of different sizes, so axiom 1 does not hold.

Examples of vector spaces:

$\mathbb{P}_n$, the set of polynomials of degree at most n.

Each of these polynomials has the form

$$a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n,$$

for some numbers $a_0, a_1, \ldots, a_n$.

Let's check axiom 4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

The zero vector for $\mathbb{P}_n$ is $0 + 0t + 0t^2 + \cdots + 0t^n$.

Let's check axiom 1. $\mathbf{u} + \mathbf{v}$ is in V.

$(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) + (b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n)$
$= (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \cdots + (a_n + b_n)t^n$, which also has degree at most n.

Exercise: convince yourself that the other axioms are true.

Examples of vector spaces:

Warning: the set of polynomials of degree exactly n is not a vector space, because axiom 1 does not hold:

$$\underbrace{(t^3+t^2)}_{\text{degree 3}} + \underbrace{(-t^3+t^2)}_{\text{degree 3}} = \underbrace{2t^2}_{\text{degree2}}$$

$\mathbb{P}$, the set of all polynomials (no restriction on the degree) is a vector space.

$C(\mathbb{R})$, the set of all continuous functions is a vector space (because the sum of two continuous functions is continuous, the zero function is continuous, etc.)

These last two examples are a bit different from $M_{m\times n}$ and $\mathbb{P}_n$ because they are infinite-dimensional (more later, see week 8.5 §4.5).

(You do not have to remember the notation $M_{m\times n}, \mathbb{P}_n$, etc. for the vector spaces.)

Let W be the set of symmetric 2×2 matrices. Is W a vector space?

$A = A^T$, i.e. $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ for some a, b, d

1. $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $(cd)\mathbf{u} = c(d\mathbf{u})$.
10. $1\mathbf{u} = \mathbf{u}$.

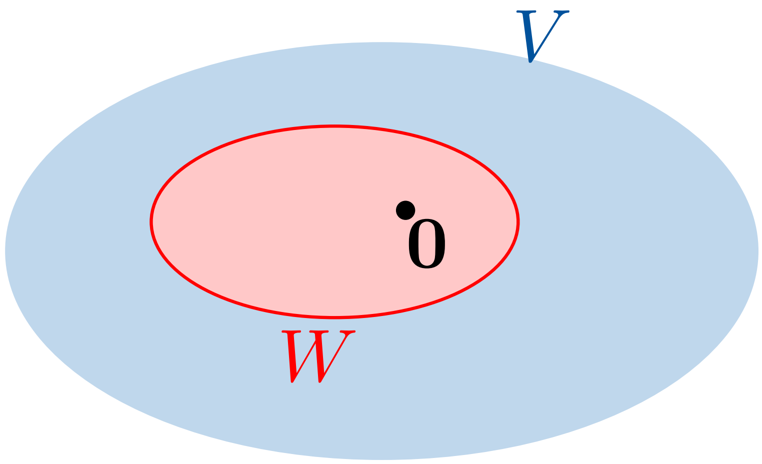
W is a subset of $M_{2\times 2}$.
Axioms 2, 3, 5, 7, 8, 9, 10 hold for W because they hold for $M_{2\times 2}$.
So we only need to check axioms 1, 4, 6.

Definition: A subset W of a vector space V is a *subspace* of V if the closure axioms 1,4,6 hold:

4. The zero vector is in W.
1. If $\mathbf{u}, \mathbf{v}$ are in W, then their sum $\mathbf{u} + \mathbf{v}$ is in W. (closed under addition)
6. If $\mathbf{u}$ is in W and c is any scalar, the scalar multiple $c\mathbf{u}$ is in W. (closed under scalar multiplication)

Fact: W is itself a vector space (with the same addition and scalar multiplication as V) if and only if W is a subspace of V.

To show that W is a subspace, check all three axioms directly, for all $\mathbf{u}, \mathbf{v}, c$ (i.e. use variables).
To show that W is not a subspace, show that one of the axioms is false, for a particular value of $\mathbf{u}, \mathbf{v}, c$.



Definition: A subset W of a vector space V is a *subspace* of V if the closure axioms 1,4,6 hold:

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6. If $\mathbf{u}$ is in W and c is any scalar, the scalar multiple $c\mathbf{u}$ is in W. (closed under scalar multiplication)

Example: Let $W = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$, i.e. the x_1x_3-plane. We show W is a subspace of $\mathbb{R}^3$:

4. The zero vector is in W because it is the vector with $a = 0$, $b = 0$.

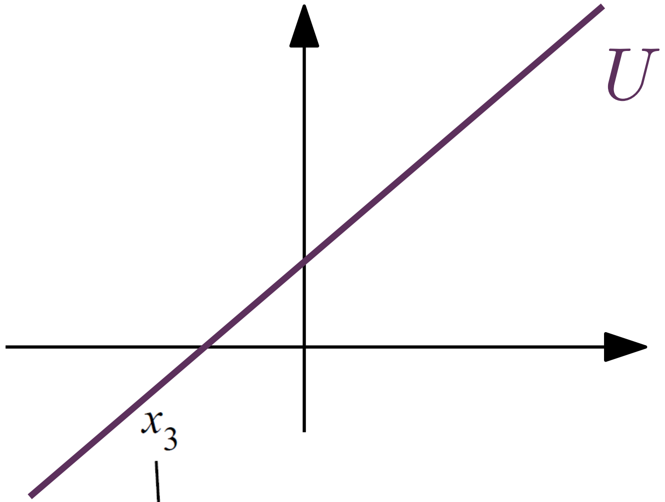
1. $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} x \\ 0 \\ y \end{bmatrix} = \begin{bmatrix} a+x \\ 0 \\ b+y \end{bmatrix}$ is in W.

6. $c \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} ca \\ 0 \\ cb \end{bmatrix}$ is in W.

Although W "feels like" $\mathbb{R}^2$, note that $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ - vectors in $\mathbb{R}^2$ have two entries, so they are not in $\mathbb{R}^3$.

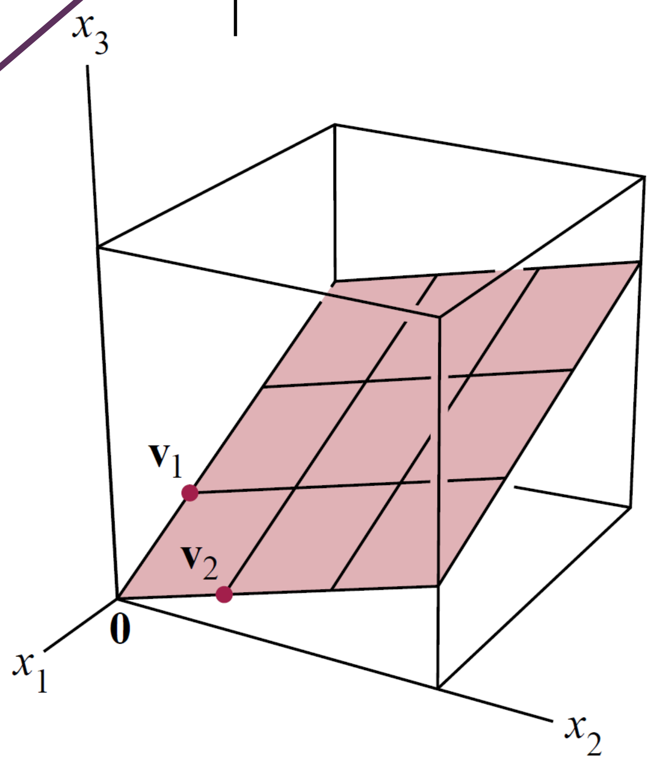
Example: Let $U = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} \middle| x \in \mathbb{R} \right\}$. To show that U is not a subspace of $\mathbb{R}^2$, we need to find one counterexample to one of the axioms, e.g.

4. The zero vector is not in U, because there is no value of x with $\begin{bmatrix} x \\ x+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.



An alternative answer:

1. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are in U, but $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is not of the form $\begin{bmatrix} x \\ x+1 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is not in U. So U is not closed under addition.



Best examples of a subspace: lines and planes containing the origin in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example: Let $Q = \{\mathbf{p} \in \mathbb{P}_3 | \mathbf{p}(2) = 0\}$, i.e. the polynomials $\mathbf{p}(t)$ of degree at most 3 with $\mathbf{p}(2) = 0$. We show that Q is a subspace of $\mathbb{P}_3$:

4. The zero polynomial is in Q because $\mathbf{0}(2) = 0 + 0 \cdot 2 + 0 \cdot 2^2 + 0 \cdot 2^3 = 0$.
1. For $\mathbf{p}, \mathbf{q}$ in Q, we have $(\mathbf{p} + \mathbf{q})(2) = \mathbf{p}(2) + \mathbf{q}(2) = 0 + 0 = 0$, so $\mathbf{p} + \mathbf{q}$ is in Q.
6. For $\mathbf{p}$ in Q and any scalar c, we have $(c\mathbf{p})(2) = c(\mathbf{p}(2)) = c0 = 0$, so $c\mathbf{p}$ is in Q.

Example: In every vector space V, the set $\{\mathbf{0}\}$ containing only the zero vector is a subspace:

4. $\mathbf{0}$ is clearly in the subspace.
1. $\mathbf{0} + \mathbf{0} = \mathbf{0}$ (use axiom 4: $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u}$ in V).
6. $c\mathbf{0} = \mathbf{0}$ (use axiom 7: $c(\mathbf{0} + \mathbf{0}) = c\mathbf{0} + c\mathbf{0}$; and left hand side is $c\mathbf{0}$.)

$\{\mathbf{0}\}$ called the zero subspace.

Example: For every vector space V, the whole space V is a subspace.

The first of two shortcuts to show that a set is a subspace:

Theorem 1: Spans are subspaces: If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are vectors in a vector space V, then Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of V.

Redo Example: (p10) Let $W = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$. We can rewrite W as

$$= \left\{ a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| a, b \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$ So W is a subspace of $\mathbb{R}^3$.

The first of two shortcuts to show that a set is a subspace:

Theorem 1: Spans are subspaces: If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in a vector space V, then Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Redo Example: (p8) Let $Sym_{2\times 2}$ be the set of symmetric 2×2 matrices. Then

$$\begin{aligned} Sym_{2\times 2} &= \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \in M_{2\times 2} \middle| a, b, d \in \mathbb{R} \right\} \\ &= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \middle| a, b, d \in \mathbb{R} \right\} \\ &= \text{Span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}, \end{aligned}$$

so $Sym_{2\times 2}$ is a subspace of $M_{2\times 2}$.

Warning: Theorem 1 does not help us show that a set is not a subspace.

THEOREM 1: Spans are subspaces

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Proof: We check axioms 4, 1 and 6 in the definition of a subspace.

4. $\mathbf{0}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ since

$$\mathbf{0} = \_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\_\mathbf{v}_p$$

1. To show that Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is closed under addition, we choose two arbitrary vectors in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$:

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p$$

and

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$\mathbf{u} + \mathbf{v} = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p) + (b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p)$$

$$= \_\_\_\_\ \_\_\_\_\mathbf{v}_1 + \_\_\_\_\ \_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\ \_\_\_\_\mathbf{v}_p$$

So $\mathbf{u} + \mathbf{v}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

6. To show that Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is closed under scalar multiplication, choose an arbitrary number c and an arbitrary vector in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$:

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$c\mathbf{v} = c(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p)$$

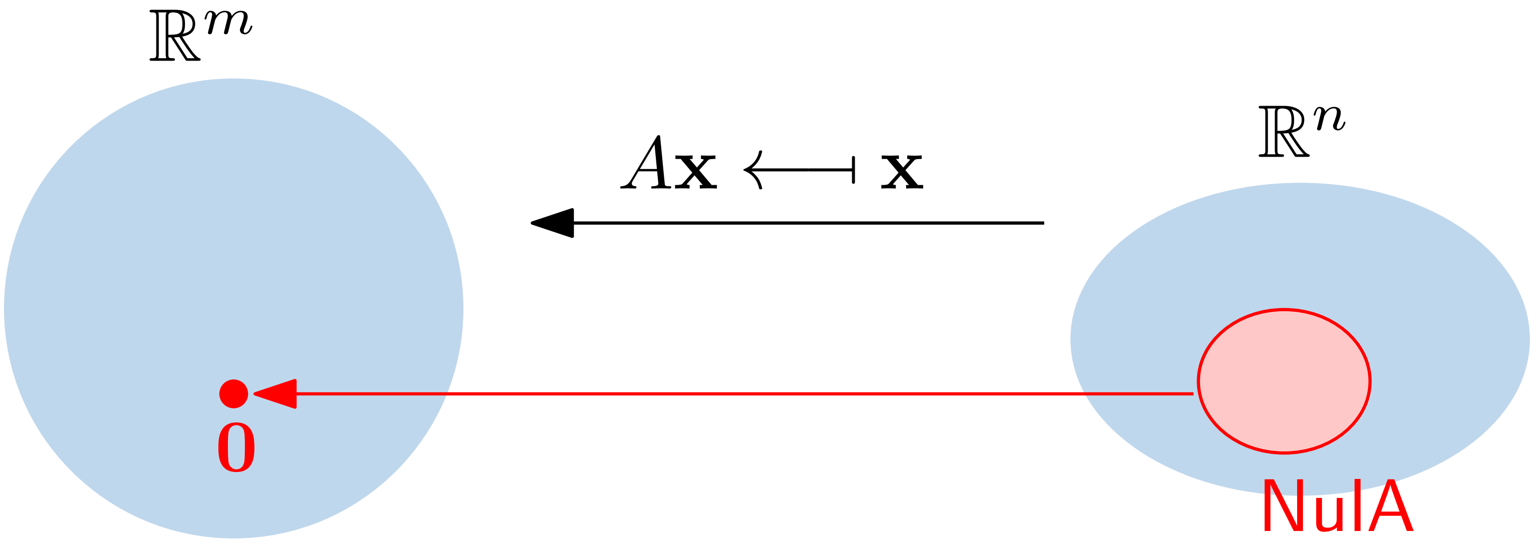
$$= \_\_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\_\_\mathbf{v}_p$$

So $c\mathbf{v}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Since 4,1,6 hold, Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

The second of two shortcuts to show that a set is a subspace:

Definition: The null space of a $m \times n$ matrix A, written Nul$A$, is the solution set to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.



Theorem 2: Null Spaces are Subspaces: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

This theorem is useful for showing that a set defined by conditions is a subspace.

Warning: If $\mathbf{b} \neq \mathbf{0}$, then the solution set of $A\mathbf{x} = \mathbf{b}$ is not a subspace, because it does not contain $\mathbf{0}$.

The second of two shortcuts to show that a set is a subspace:

Definition: The null space of a $m \times n$ matrix A, written Nul$A$, is the solution set to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Theorem 2: Null Spaces are Subspaces: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

Example: Show that the line $L = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \middle| y = 2x \right\}$ is a subspace of $\mathbb{R}^2$.

Here, we do not have "$x, y \in \mathbb{R}$": instead, x and y are related by the condition $y = 2x$. In these situations, it's often easier to show that the given set is a null space.

Answer: $y = 2x$ is the same as $2x - y = 0$, which in matrix form is $\begin{bmatrix} 2 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0}$. So L is the solution set to $\begin{bmatrix} 2 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0}$, which is the null space of the matrix $\begin{bmatrix} 2 & -1 \end{bmatrix}$. Because null spaces are subspaces, L is a subspace.

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

THEOREM 2: Null spaces are subspaces

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$.

Proof: Nul A is a subset of $\mathbf{R}^n$ since A has n columns. We check axioms 4,1,6 in the definition of a subspace.

4. $\mathbf{0}$ is in Nul A because

1. If $\mathbf{u}$ and $\mathbf{v}$ are in Nul A, we show that $\mathbf{u}+\mathbf{v}$ is in Nul A. Because $\mathbf{u}$ and $\mathbf{v}$ are in Nul A

Therefore

$$A(\mathbf{u}+\mathbf{v}) =$$

6. If $\mathbf{u}$ is in Nul A and c is a scalar, we show that $c\mathbf{u}$ iis in Nul A:

$$A(c\mathbf{u}) =$$

Since axioms 4,1,6 hold, Nul A is a subspace of $\mathbf{R}^n$.

Summary:

Axioms for a subspace:

4. The zero vector is in W.
1. If $\mathbf{u}, \mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W. (closed under addition)
6. If $\mathbf{u}$ is in W and c is a scalar, then $c\mathbf{u}$ is in W. (closed under scalar multiplication)

Ways to show that a set W is a subspace:

- Show that $W = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ for some $\mathbf{v}_1, \ldots, \mathbf{v}_p$ (if W is explicitly defined - i.e. its description has variables that can take any value).
- Show that W is Nul$A$ for some matrix A (if W is implicitly defined - i.e. by conditions that vectors must satisfy).
- Show that W is the kernel or range of a linear transformation (later, p42-43).
- Check all three axioms directly, for all $\mathbf{u}, \mathbf{v}, c$.

To show that a set is not a subspace:

- Show that one of the axioms is false, for a particular value of $\mathbf{u}, \mathbf{v}, c$.

Best examples of a subspace: lines and planes containing the origin in $\mathbb{R}^2$ and $\mathbb{R}^3$.

One example of the power of abstract vector spaces - solving differential equations:

Question: What are all the polynomials $\mathbf{p}$ of degree at most 5 that satisfy

$$\frac{d^2}{dt^2}\mathbf{p}(t) - 4\frac{d}{dt}\mathbf{p}(t) + 3\mathbf{p}(t) = 3t - 1?$$

Answer: The differentiation function $D : \mathbb{P}_5 \to \mathbb{P}_5$ given by $D(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$ is a linear transformation (later, p39).

The function $T : \mathbb{P}_5 \to \mathbb{P}_5$ given by $T(\mathbf{p}) = \frac{d^2}{dt^2}\mathbf{p}(t) - 4\frac{d}{dt}\mathbf{p}(t) + 3\mathbf{p}(t)$ is a sum of compositions of linear transformations, so T is also linear.

We can check that the polynomial $t + 1$ is a solution.

So, by the Solutions and Homogeneous Equations Theorem, the solution set to the above differential equation is all polynomials of the form $t + 1 + \mathbf{q}(t)$ where $T(\mathbf{q}) = 0$.

Extra: $\mathbb{P}_5$ is both the domain and codomain of T, so the Invertible Matrix Theorem applies. So, if the above equation has more than one solution, then there is a polynomial $\mathbf{g}$ such that $\frac{d^2}{dx^2}\mathbf{p}(t) - 4\frac{d}{dt}\mathbf{p}(t) + 3\mathbf{p}(t) = \mathbf{g}(t)$ has no solutions.

§4.2, pp229-230, pp249-250: Subspaces and Matrices

Each linear transformation has two vector subspaces associated to it.

For each of these two subspaces, we are interested in two problems:

a. given a vector $\mathbf{v}$, is it in the subspace?

b. can we write this subspace as Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ for some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$? The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is then called a spanning set of the subspace.

b*. can we write this subspace as Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ for linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$? The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is then called a basis of the subspace.

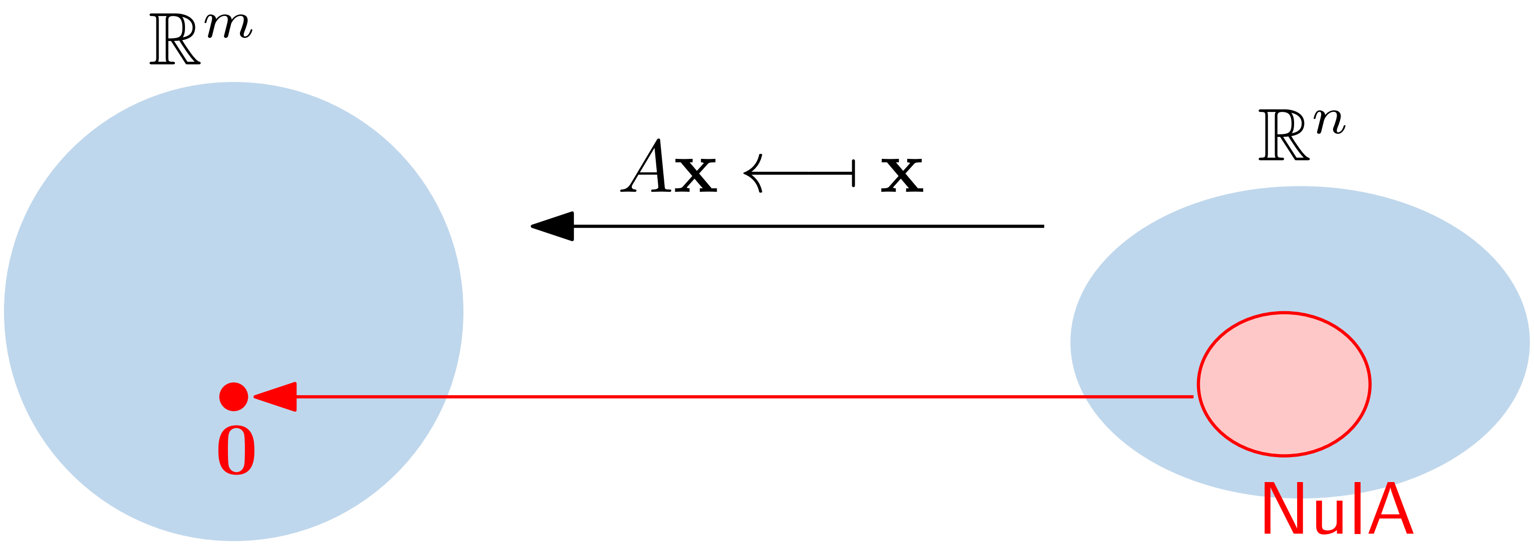
Problem b is important because it means every vector in the subspace can be written as $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$. This allows us to calculate with and prove statements about arbitrary vectors in the subspace.

Problem b* is important because it means every vector in the subspace can be written uniquely as $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$ (proof next week, §4.4).

We turn a spanning set into a basis by removing some vectors - this is the Spanning Set Theorem / casting-out algorithm (p28, also week 8 p10).

Remember from p17:

Definition: The null space of a $m \times n$ matrix A, written Nul$A$, is the solution set to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.



Nul$A$ is implicitly defined (i.e. defined by conditions) - problem a is easy, problem b takes more work.

Example: Let $A = \begin{bmatrix} 1 & -3 & 4 & -3 \\ 3 & -7 & 8 & -5 \end{bmatrix}$. a. Is $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in Nul$A$?

b. Find vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ which span Nul$A$.

Answer:

a. $A\mathbf{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq \mathbf{0}$, so $\mathbf{v}$ is not in Nul$A$.

b. $[A|\mathbf{0}] \xrightarrow{\text{row reduction}} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \end{array}\right] \longrightarrow$

$$\begin{aligned} x_1 &= 2x_3 - 3x_4 \\ x_2 &= 2x_3 - 2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

So the solution set is $s\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ where s, t can take any value.

So Nul$A$ = Span $\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

linearly independent

In general: the solution to $A\mathbf{x} = \mathbf{0}$ in parametric form looks like $s_i\mathbf{w_i} + s_j\mathbf{w_j} + \dots$, where $x_i, x_j, \dots$ are the free variables (one vector for each free variable).
The vector $\mathbf{w_i}$ has a 1 in row i and a 0 in row j for every other free variable x_j, so $\{\mathbf{w_i}, \mathbf{w_j}, \dots\}$ are automatically linearly independent.

b. $[A|\mathbf{0}] \xrightarrow{\text{row reduction}} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \end{array}\right] \longrightarrow$

$$\begin{aligned} x_1 &= 2x_3 - 3x_4 \\ x_2 &= 2x_3 - 2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

So the solution set is $s\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ where s, t can take any value.

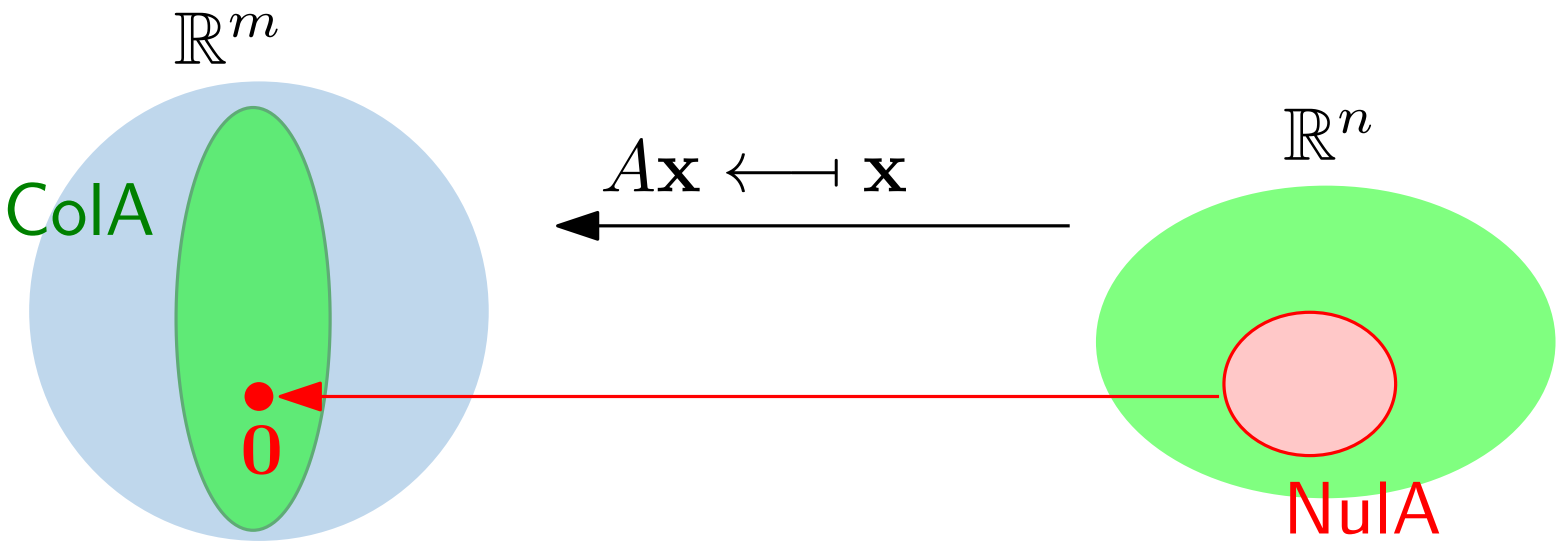
So $\text{Nul}A = \text{Span}\left\{ \underset{\mathbf{w_3}}{\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}}, \underset{\mathbf{w_4}}{\begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}} \right\}$.

linearly independent

Definition: The column space of a $m \times n$ matrix A, written Col$A$, is the span of the columns of A.

Because spans are subspaces, it is obvious that Col$A$ is a subspace of $\mathbb{R}^m$.

It follows from §1.3-1.4 that Col$A$ is the set of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has solutions.



Col$A$ is explicitly defined - problem a takes work, problem b is easy.

Example: Let $A=\begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix}$. a. Is $\mathbf{v} = \begin{bmatrix} -5 \\ 9 \\ 5 \end{bmatrix}$ in Col$A$?

b. Find vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ which span Col$A$.

Answer:

a. $\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{array}\right] \xrightarrow[\text{to echelon form}]{\text{row reduction}} \left[\begin{array}{ccccc|c} 1 & -3 & 4 & 3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$

There is no row $[0 \dots 0|*]$ with $* \neq 0$, so $\mathbf{v}$ is in Col$A$.

b. By definition, Col$A$ is the span of the columns of A, so

$$\text{Col}A = \text{Span}\left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ 8 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} \right\}.$$

Note that this spanning set is not linearly independent (more than 3 vectors in $\mathbb{R}^3$).

Contrast Between Nul A and Col A for an m x n Matrix A

p.222 of textbook

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	1. Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	2. Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	3. It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them. ← problem b
4. There is no obvious relation between Nul A and the entries in A.	4. There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	5. A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	6. Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \mid \mathbf{v}\,]$ are required. ← problem a
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	7. Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	8. Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

As we saw on p26, it is easy to obtain a spanning set for Col$A$ (just take all the columns of A), but usually this spanning set is not linearly independent.

To obtain a linearly independent set that spans Col$A$, take the pivot columns of A - this is called the casting-out algorithm.

Example: Let $A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix}$.

Find a linearly independent set that spans Col$A$.

Answer:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to echelon form}]{\text{row reduction}} \begin{bmatrix} 1 & -3 & 4 & 3 & 2 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The pivot columns are 1,2 and 5, so $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\} = \left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} \right\}$ is one answer.

(The answer from the casting-out algorithm is not the only answer - see p34.)

Casting-out algorithm: the pivot columns of A is a linearly independent set that spans Col$A$.

Why does the casting-out algorithm work part 1: why the pivot columns are linearly independent:

Example:

$$A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to echelon form}]{\text{row reduction}} \begin{bmatrix} 1 & -3 & 4 & 3 & 2 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

So $\begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_5 \\ | & | & | \end{bmatrix}$ is row-equivalent to $\begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, which has no free variables.

So $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$ is linearly independent.

Casting-out algorithm: the pivot columns of A is a linearly independent set that spans Col$A$.

Why does the casting-out algorithm work part 2: why the pivot columns span Col$A$:

To explain this, we need to look at the solutions to $A\mathbf{x} = \mathbf{0}$:

Example:

$$A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to rref}]{\text{row reduction}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So the solution set to $A\mathbf{x} = \mathbf{0}$ is $s\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ where s, t can take any value.

$x_3 = 1$ $x_4 = 0$ (for s); $x_3 = 0$ $x_4 = 1$ (for t)

These correspond respectively to the linear dependence relations $2\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0}$ and $-3\mathbf{a}_1 - 2\mathbf{a}_2 + \mathbf{a}_4 = \mathbf{0}$.

Rearranging: $\mathbf{a}_3 = -2\mathbf{a}_1 - 2\mathbf{a}_2$ and $\mathbf{a}_4 = 3\mathbf{a}_1 + 2\mathbf{a}_2$.

$$A(2,2,1,0,0)=\mathbf{0} \longrightarrow 2\mathbf{a}_1+2\mathbf{a}_2+\mathbf{a}_3=\mathbf{0} \longrightarrow \mathbf{a}_3=-2\mathbf{a}_1-2\mathbf{a}_2.$$

$$A(-3,-2,0,1,0)=\mathbf{0} \longrightarrow -3\mathbf{a}_1-2\mathbf{a}_2+\mathbf{a}_4=\mathbf{0} \longrightarrow \mathbf{a}_4=3\mathbf{a}_1+2\mathbf{a}_2.$$

In other words: consider the solution to $A\mathbf{x}=\mathbf{0}$ where one free variable x_i is 1, and all other free variables are 0. This corresponds to a linear dependence relation among the columns of A, which can be rearranged to express the column $\mathbf{a}_i$ as a linear combination of the pivot columns.

Why this is useful: any vector $\mathbf{v}$ in Col$A$ has the form

$$\mathbf{v} = c_1\mathbf{a}_1+c_2\mathbf{a}_2+c_3\mathbf{a}_3+c_4\mathbf{a}_4+c_5\mathbf{a}_5,$$

which we can rewrite as

$$\begin{aligned} &c_1\mathbf{a}_1+c_2\mathbf{a}_2+c_3(-2\mathbf{a}_1-2\mathbf{a}_2)+c_4(3\mathbf{a}_1+2\mathbf{a}_2)+c_5\mathbf{a}_5 \\ =&(c_1-2c_3+3c_4)\mathbf{a}_1+(c_2-2c_3+2c_4)\mathbf{a}_2+c_5\mathbf{a}_5, \end{aligned}$$

a linear combination of the pivot columns $\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_5$. So $\mathbf{v}$ is in Span $\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_5\}$, and so Col$A=$ Span $\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_5\}$.

Another view: the casting-out algorithm as a greedy algorithm:

Example:

$$\begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to rref}]{\text{row reduction}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rref}\left(\begin{bmatrix} | \\ \mathbf{a}_1 \\ | \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ has a pivot in every column, so $\{\mathbf{a}_1\}$ is linearly independent, so we keep $\mathbf{a}_1$.

$$\text{rref}\left(\begin{bmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$ has a pivot in every column, so $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent, so we keep $\mathbf{a}_2$.

$$\text{rref}\left(\begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ | & | & | \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$ does not have a pivot in every column, so $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is linearly dependent, so we remove $\mathbf{a}_3$.

Another view: the casting-out algorithm as a greedy algorithm (continued):

Example:

$$\begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to rref}]{\text{row reduction}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rref}\left(\begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_4 \\ | & | & | \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

does not have a pivot in every column, so $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is linearly dependent, so we remove $\mathbf{a}_4$.

$$\text{rref}\left(\begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_5 \\ | & | & | \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has a pivot in every column, so $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$ is linearly independent, so we keep $\mathbf{a}_5$.

So the casting-out algorithm is a greedy algorithm in that it prefers vectors that are earlier in the set.

Example: Let $A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix}$.

Find a linearly independent set containing $\mathbf{a}_3$ that spans Col$A$.

Answer: To ensure that the set contains $\mathbf{a}_3$, we should make it the leftmost column - e.g. we row-reduce $\begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_3 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix}$ and take the pivot columns.

Warning: the example on the previous two pages is a little misleading: a subset of the columns of rref(A) is not always the reduced echelon form of those columns of A, e.g.rref$\left(\begin{bmatrix} | & | \\ \mathbf{a}_2 & \mathbf{a}_3 \\ | & | \end{bmatrix}\right) \neq \begin{bmatrix} 0 & -2 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}$ (because this isn't in reduced echelon form).

The correct statement is that a subset of the columns of rref(A) is row equivalent to those columns of A.

Definition: The row space of a $m \times n$ matrix A, written Row$A$, is the span of the rows of A. It is a subspace of $\mathbb{R}^n$.

Example: $A = \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 2 & 0 & 8 \\ 1 & 2 & -3 & 6 \end{bmatrix}$ $\quad$ rref$(A) = \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\text{Row}A = \text{Span}\{(0,1,0,4),(0,2,0,8),(1,2,-3,6)\}.$$

Row$A$ is explicitly defined - indeed, it is equivalent to ColA^T.
So, to see if a vector $\mathbf{v}$ is in Row$A$, row-reduce $[A^T|\mathbf{v}^T]$.
To find a linear independent set that spans Row$A$, take the pivot columns of A^T, or...

Theorem 13: Row operations do not change the row space. In particular, the nonzero rows of rref(A) is a linearly independent set whose span is Row$A$.
E.g. for the above example, Row$A = \text{Span}\{(1,0,-3,-2),(0,1,0,4)\}$.

Warning: the "pivot rows" of A do not usually span Row$A$:
e.g. here $(1,2,-3,6)$ is in Row$A$ but not in Span$\{(0,1,0,4),(0,2,0,8)\}$.

Theorem 13: Row operations do not change the row space. In particular, the nonzero rows of $\text{rref}(A)$ is a linearly independent set whose span is Row$A$.

An example to explain why row operations do not change the row space:

$$A = \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 2 & 0 & 8 \\ 1 & 2 & -3 & 6 \end{bmatrix} \qquad \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & -2 \end{bmatrix} \qquad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(1, 4, -3, 14) = R_2 + R_3 = (R_2 - 2R_1) + (R_3 - R_1) + 3R_1$

Similarly, any linear combination of R_1, R_2, R_3 can be written as a linear combination of $R_1, R_2 - 2R_1, R_3 - R_1$.

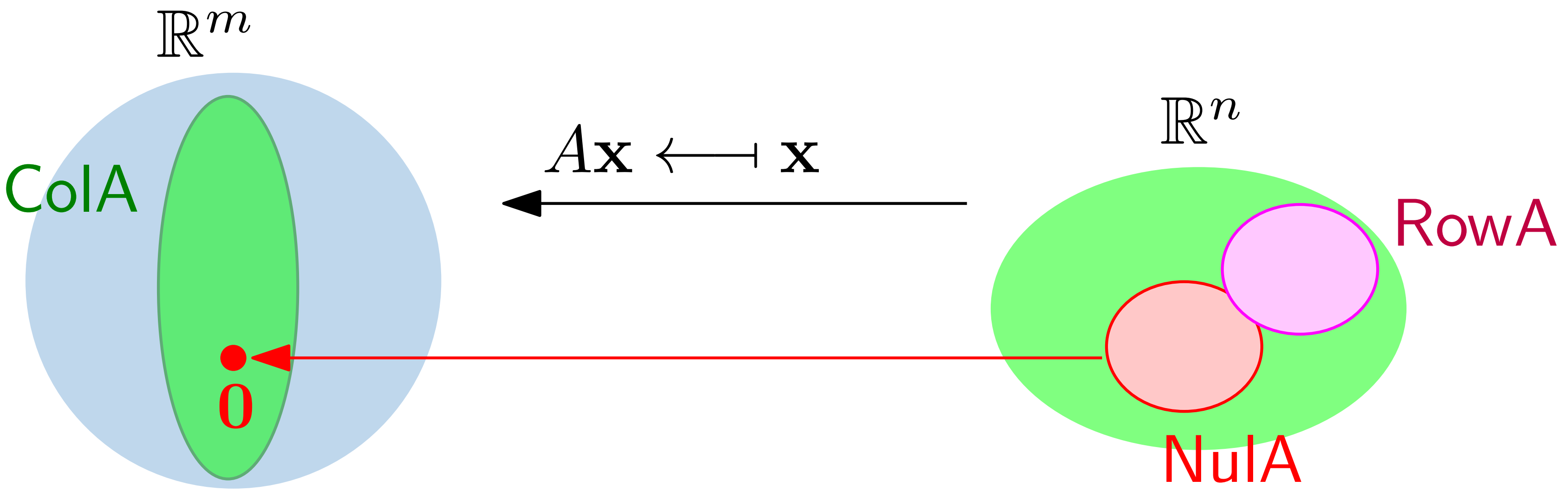
Proof of the second sentence in Theorem 13:
From the first sentence, $\text{Row}(A) = \text{Row}(\text{rref}(A)) =$ Span of the nonzero rows of $\text{rref}(A)$. Because each nonzero row has a 1 in one pivot column (different column for each row) and 0s in all other pivot columns, these rows are linearly independent.

Summary:

A basis for W is a linearly independent set that spans W (more later).

- Nul$A$=solutions to $A\mathbf{x} = \mathbf{0}$,
- Col$A$=span of columns of A,
- Row$A$=span of rows of A.

basis for Nul$A$: solve $A\mathbf{x} = \mathbf{0}$ via the rref.
basis for Col$A$: pivot columns of A.
basis for Row$A$: nonzero rows of rref(A).



Col$A$ is in $\mathbb{R}^m$.

Nul$A$, Row$A$ are in $\mathbb{R}^n$.

In general, Col$A \neq$ Col(rref(A)).

Nul$A$ = Nul(rref(A)), Row$A$ = Row(rref(A)).

PP222-223: Linear Transformations for Vector Spaces

Recall (week 4 §1.8) the definition of a linear transformation:

Definition: A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* if:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and for all $\mathbf{u}$ in the domain of T.

Now consider a function $T : V \to W$, where V, W are abstract vector spaces.
Because we can add and scalar-multiply in V, the left hand sides of the equations in 1,2 make sense.
Because we can add and scalar-multiply in W, the right hand sides of the equations in 1,2 make sense.
So we can ask if functions between abstract vector spaces are linear:

Definition: A function $T : V \to W$ is a *linear transformation* if:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and for all $\mathbf{u}$ in V.

Hard exercise: show that the set of all linear transformations $V \to W$ is a vector space.

Definition: A function $T : V \to W$ is a *linear transformation* if:

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and for all $\mathbf{u}$ in V.

Example: The differentiation function $D : \mathbb{P}_n \to \mathbb{P}_{n-1}$ given by $D(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$,

$$D(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) = a_1 + 2a_2 t + \cdots + n a_n t^{n-1},$$

is linear.

If you've taken a calculus class, then you already know this:

When you calculate $\frac{d}{dt}(3t + 2t^2) = 3 \quad + 2 \cdot 2t$

you're really thinking $3\frac{d}{dt}t + 2\frac{d}{dt}t^2$

Method A to show that D is linear:

$$D(\mathbf{p}+\mathbf{q}) = \frac{d}{dt}(\mathbf{p}+\mathbf{q}) = \frac{d}{dt}\mathbf{p} + \frac{d}{dt}\mathbf{q} = D(\mathbf{p}) + D(\mathbf{q}); \text{ and}$$

$$D(c\mathbf{p}) = \frac{d}{dt}(c\mathbf{p}) \qquad = c\frac{d}{dt}\mathbf{p} \qquad = cD(\mathbf{p})$$

Definition: A function $T : V \to W$ is a *linear transformation* if:

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and for all $\mathbf{u}$ in V.

Example: The differentiation function $D : \mathbb{P}_n \to \mathbb{P}_{n-1}$ given by $D(\mathbf{p}) = \dfrac{d}{dt}\mathbf{p}$,

$$D(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) = a_1 + 2a_2 t + \cdots + n a_n t^{n-1},$$

is linear.

Method B to show that D is linear - use the formula:

$$\begin{aligned}
&D((a_0+b_0) + (a_1+b_1)t + (a_2+b_2)t^2 + \cdots + (a_n+b_n)t^n) \\
&= (a_1+b_1) + 2(a_2+b_2)t + \cdots + n(a_n+b_n)t^{n-1} \\
&= a_1 + 2a_2 t + \cdots + n a_n t^{n-1} + b_1 + 2b_2 t + \cdots + n b_n t^{n-1} \\
&= D(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) + D(b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n);
\end{aligned}$$ and

$$\begin{aligned}
D((ca_0) + (ca_1)t + (ca_2)t^2 + \cdots + (ca_n)t^n) &= (ca_1) + 2(ca_2)t + \cdots + n(ca_n)t^{n-1} \\
&= c(a_1 + 2a_2 t + \cdots + n a_n t^{n-1}) \\
&= cD(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n).
\end{aligned}$$

Example: The "multiplication by t" function $M : \mathbb{P}_n \to \mathbb{P}_{n+1}$ given by $M(\mathbf{p}(t)) = t\mathbf{p}(t)$,

$$M(a_0 + a_1 t + \cdots + a_n t^n) = t(a_0 + a_1 t + \cdots + a_n t^n),$$

is linear:

Method A: $M(\mathbf{p}+\mathbf{q}) = t[(\mathbf{p}+\mathbf{q})(t)] = t\mathbf{p}(t) + t\mathbf{q}(t) = M(\mathbf{p}) + M(\mathbf{q})$; and

$$M(c\mathbf{p}) = t[(c\mathbf{p})(t)] \qquad = c[t(\mathbf{p}(t)] \qquad = cM(\mathbf{p})$$

Method B:

$$\begin{aligned} &M((a_0+b_0) + (a_1+b_1)t + \cdots + (a_n+b_n)t^n) \\ &= t((a_0+b_0) + (a_1+b_1)t + \cdots + (a_n+b_n)t^n)) \\ &= t(a_0 + a_1 t + \cdots + a_n t^n) + t(b_0 + b_1 t + \cdots + b_n t^n) \\ &= M(a_0 + a_1 t + \cdots + a_n t^n) + M(b_0 + b_1 t + \cdots + b_n t^n); \text{ and} \end{aligned}$$

$$\begin{aligned} M((ca_0) + (ca_1)t + \cdots + (ca_n)t^n) &= t((ca_0) + (ca_1)t + \cdots + (ca_n)t^n) \\ &= ct(a_0 + a_1 t + \cdots + a_n t^n) \\ &= cM(a_0 + a_1 t + \cdots + a_n t^n). \end{aligned}$$

Definition: A function $T : V \to W$ is a *linear transformation* if:

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and for all $\mathbf{u}$ in V.

The concepts of kernel and range (week 4, §1.9) make sense for linear transformations between abstract vector spaces:

Definition: The *kernel* of T is the set of $\mathbf{v}$ in V satisfying $T(\mathbf{v}) = \mathbf{0}$.

Definition: The *range* of T is the set of $\mathbf{w}$ in W such that $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V.

Example: The kernel of the differentiation function $D : \mathbb{P}_n \to \mathbb{P}_{n-1}$, given by $D(\mathbf{p}) = \dfrac{d}{dt}\mathbf{p}$, is the set of constant polynomials $\mathbf{p}(t) = a_0$ for any number a_0. The range of D is all of $\mathbb{P}_{n-1}$.

Our proof that null spaces are subspaces (p18) shows that the kernel of a linear transformation is a subspace.

Exercise: show that the range of a linear transformation is a subspace.

Redo Example: (p12) Let $Q = \{\mathbf{p} \in \mathbb{P}_3 | \mathbf{p}(2) = 0\}$, i.e. polynomials $\mathbf{p}(t)$ of degree at most 3 with $\mathbf{p}(2) = 0$. We show that Q is a subspace of $\mathbb{P}_3$ by showing that it is the kernel of a linear transformation. (This argument is hard; if you prefer the axiom-checking on p12 that is fine.)
The evaluation-at-2 function $E_2 : \mathbb{P}_3 \to \mathbb{R}$ given by $E_2(\mathbf{p}) = \mathbf{p}(2)$,

$$E_2(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = a_0 + a_1 2 + a_2 2^2 + a_3 2^3$$

is a linear transformation because

1. For $\mathbf{p}, \mathbf{q}$ in $\mathbb{P}_3$, we have
   $E_2(\mathbf{p} + \mathbf{q}) = (\mathbf{p} + \mathbf{q})(2) = \mathbf{p}(2) + \mathbf{q}(2) = E_2(\mathbf{p}) + E_2(\mathbf{q})$.
2. For $\mathbf{p}$ in $\mathbb{P}_3$ and any scalar c, we have $E_2(c\mathbf{p}) = (c\mathbf{p})(2) = c(\mathbf{p}(2)) = cE_2(\mathbf{p})$.

So E_2 is a linear transformation. Q is the kernel of E_2, so Q is a subspace.

Can we write Q as Span $\{\mathbf{p}_1, \ldots, \mathbf{p}_p\}$ for some linearly independent polynomials $\mathbf{p}_1, \ldots, \mathbf{p}_p$?

One idea: associate a matrix A to E_2 and take a basis of Nul$A$ using the rref.
To do computations like this, we need coordinates.