Homework 3, due 12:00 Wednesday, 20 March 2019 to Dr. Pang's mailbox

You must justify your answers to receive full credit.

1. Let V be a vector space over $\mathbb{F}$ and $\sigma \in L(V, V)$. Let $\alpha \in V$ be an eigenvector of σ with eigenvalue λ.

   a) Show that α is an eigenvector of σ^n with eigenvalue λ^n (for $n = 1, 2, \dots$).

   Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, for some $a_i \in \mathbb{F}$.

   b) Show that α is an eigenvector of $f(\sigma)$ with eigenvalue $f(\lambda)$.

   c) Show that, if $f(\sigma) = 0$ (zero function), then $f(\lambda) = 0$ (zero number).

2. Let $C^0(\mathbb{R})$ denote the vector space of continuous functions on $\mathbb{R}$. Consider $\sigma : C^0(\mathbb{R}) \to C^0(\mathbb{R})$ given by
$$\sigma(f)(x) = f''(x) + f(x) + f(0).$$

   a) Show that, for each n, the function $f(x) = \sin(nx)$ is an eigenvector of σ, and find its corresponding eigenvalue.

   b) Let $W = P_{<3}(\mathbb{R})$ be the subspace of polynomials with degree less than 3. Find three linearly independent eigenvectors of the restriction $\sigma|_W$. (Click here for a hint)

3. Consider
$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & -2 \\ -4 & 3 & 3 & -2 \\ -2 & 2 & 1 & 0 \end{pmatrix}.$$
   You are given that the characteristic polynomial of A is $\chi_A(x) = (x-2)^4$. Find the Jordan form J of A and find a matrix P such that $P^{-1}AP = J$. (You do **not** need to find P^{-1}.) (You may use an online RREF calculator, but remember you only have an ordinary calculator in the exams.)

4. Consider
$$B = \begin{pmatrix} 4 & 1 & -3 & -6 & 3 \\ -9 & -2 & 9 & 18 & -9 \\ -12 & -4 & 15 & 28 & -16 \\ 6 & 2 & -7 & -13 & 8 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \operatorname{rref}(B - I) = \begin{pmatrix} 1 & \frac{1}{3} & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
   You are given that $(B - I)^2 = 0$. (Hint: so what are the eigenvalues of B?) Find the Jordan form J of B and find a matrix P such that $B = PJP^{-1}$. (You do **not** need to find P^{-1}.)

5. Consider $n \times n$ complex matrices A and B.

   a) Show that A, B are similar if and only if they have the same Jordan form. (Click here for a hint)

   b) Let $n = 3$. Show that A, B are similar if and only if they have the same characteristic polynomial and the same minimal polynomial. (Click here for a hint)

   c) Let $n = 4$. Give a counterexample to show that the second sentence of part b) is false.

6. Find, with explanation, all the possible Jordan forms of the following matrices:

   a) The characteristic polynomial of A is $-(x-2)^3(x+7)^2$, and the minimal polynomial of A has degree 3.

   b) The characteristic polynomial of B is $-(x-3)^5$, and $\ker(B-3I)$ is 3-dimensional.

   c) The characteristic polynomial of C is $(x+2)^4(x-1)^2$, and $\ker(C+2I)^2$ is 3-dimensional.

The following two questions are to prepare you for upcoming classes, and is unrelated to the material from recent classes.

7. Consider the subspace W of $\mathbb{R}^3$:

$$W = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

   a) Find an orthogonal basis for W.

   b) Find a basis for $W^\perp$, the orthogonal complement of W.

8. Consider

$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}.$$

Note that $\{\alpha_1, \alpha_2, \alpha_3\}$ is an orthogonal basis of $\mathbb{R}^3$.

   a) Find the length of β.

   b) By computing dot products, express β as a linear combination of $\alpha_1, \alpha_2, \alpha_3$.

Optional question If you attempted seriously all the above questions, then your scores for the following question may replace any lower scores for one of the above questions.

9. to be released later

10. to be released later

— END —