§6.3 Subspaces

Def: 6.3.1 A subset $W \subseteq V$ is <u>a subspace of V</u> which is itself a vector space, with the addition and scalar multiplication as in V.

Lem 6.3.2 / Prop 6.3.3: To check W is a subspace of V, it is enough to check <u>ONE</u> of the following equivalent things (you may use any in questions)

c) $W \ni \vec{0}$ and if $\alpha, \beta \in W$ and $a \in \mathbb{F}$, then $\alpha + \beta \in W$ and $a\alpha \in W$

closed under addition

closed under scalar multiplication

a) $W \ni \vec{0}$ and if $\alpha, \beta \in W$ and $a, b \in \mathbb{F}$, then $a\alpha + b\beta \in W$

b) $W \ni \vec{0}$ and if $\alpha, \beta \in W$ and $a \in \mathbb{F}$, then $a\alpha + \beta \in W$ mainly used in textbook / class

Furthermore, $W \ni \vec{0}$ in a,b,c above may be replaced by $W \neq \emptyset$.

Proof (outline): $a \Rightarrow b$ — special case $b=1$

$b \Rightarrow c$ — special case $a=1$ or $\beta = \vec{0}$

$c \Rightarrow a$ — $a\alpha \in W$, $b\beta \in W \Rightarrow a\alpha + b\beta \in W$

$c \Rightarrow W$ is a vector space:

$c \Rightarrow$ axioms V1, V4, V6

and other axioms don't mention W

so they are true in W $\because$ true in V

$W \ni \vec{0} \Rightarrow W \neq \phi$: clear

$W \neq \phi \Rightarrow W \ni \vec{0}$: for $\alpha \in W$,

$\vec{0} = 0\alpha \in W$.

Ex: (most subspaces are defined

either by a form (explicit) $\{ * \mid \text{no condition} \}$

or by a condition (implicit) $\{ \alpha \in V \mid \dagger \}$

or a combination see §7.1)

① $W = \left\{ \begin{pmatrix} a \\ 2a \\ 0 \\ b \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$

Check W is a subspace of $\mathbb{R}^4$:

$\vec{0} \in W$ $\because$ set $a=0$, $b=0$.

$$c\begin{pmatrix} a \\ 2a \\ 0 \\ b \end{pmatrix} + \begin{pmatrix} a' \\ 2a' \\ 0 \\ b' \end{pmatrix} = \begin{pmatrix} ca+a' \\ 2(ca+a') \\ 0 \\ cb+b' \end{pmatrix} \in W$$

② Let $C^0(\mathbb{R}) = \{f: \mathbb{R} \to \mathbb{R} \text{ continuous}\}$

$C^0_{even}(\mathbb{R}) = \{ f \in C^0(\mathbb{R}) \mid f(x) = f(-x)\ \forall x \in \mathbb{R}\}$

check this is subspace of $C^0(\mathbb{R})$:

$\vec{0} \in C^0_{even}(\mathbb{R}) \because \vec{0}(x) = 0 = \vec{0}(-x)\ \forall x.$

if $f, g \in C^0_{even}(\mathbb{R})$, then:

$$(af+g)(x) = af(x) + g(x)$$
$$= af(-x) + g(-x)$$
$$= (af+g)(-x).\ \forall x \in \mathbb{R}$$

So $af+g \in C^0_{even}(\mathbb{R})$.

③ $P_{<n}(\mathbb{R})$ is a subspace of $\mathbb{R}[x]$, and $\mathbb{R}[x]$ is a subspace of $C^0(\mathbb{R})$.

Th 6.3.5 The intersection of any collection of subspaces is a subspace

finitely or infinitely many

$P_{\leq 3}(\mathbb{R})$

Ex: $P_{<4}(\mathbb{R}) \cap C^0_{even}(\mathbb{R})$

$= \{\text{even polynomials of degree} < 4\}$

$= \{a_0 + a_2x^2 \mid a_0, a_2 \in \mathbb{R}\}$

Proof: Let Λ be some set, and for each $\lambda \in \Lambda$, let W_λ be a subspace of V.

Let $W = \bigcap_{\lambda \in \Lambda} W_\lambda \; \left(= \{ \alpha \in V \mid \alpha \in W_\lambda \text{ for each } \lambda \in \Lambda \} \right)$

$\vec{0} \in W \because \vec{0} \in W_\lambda$ for each λ
$\because$ each W_λ is a subspace.

if $\alpha, \beta \in W$, then $\alpha, \beta \in W_\lambda$ for each λ
$a\alpha + \beta \in W_\lambda$ for each λ,
$\because W_\lambda$ is a subspace.
$\therefore a\alpha + \beta \in W$.

Remark: The union of two subspaces is NOT a subspace — see §6.5.

We can use 6.3.5 to make a subspace from a set.

Def 6.3.6: Given a set $S \subseteq V$, the span of S, written $\mathrm{span}(S)$, is the intersection of all subspaces containing S.

Equivalently (Rem 6.3.8): $\mathrm{span}(S)$ is the subspace with this property: if any subspace $W \supseteq S$, then $W \supseteq \mathrm{span}(S)$.

(Reason: $\mathrm{span}(S) = W \cap \bigcap$ all other subspaces containing S)

i.e. $\mathrm{Span}(S)$ is the "smallest subspace containing S".

Equivalently (Th. 6.3.9)

$$\mathrm{span}(S) = \left\{ \sum_{i=1}^{n} a_i \alpha_i \;\middle|\; \begin{array}{l} \alpha_i \in S \\ a_i \in \mathbb{F} \end{array}, \text{ some } n \in \mathbb{N} \right\}$$

= "all linear combinations of finitely many elements from S"