How $[\sigma]_{B\leftarrow A}$ is useful:

if $\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n$ $(\alpha \in U)$ i.e. $[\alpha]_A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

then $[\sigma(\alpha)]_B = [\sigma(a_1\alpha_1 + \cdots + a_n\alpha_n)]_B$

$= [a_1\sigma(\alpha_1) + \cdots + a_n\sigma(\alpha_n)]_B$

$= a_1[\sigma(\alpha_1)]_B + \cdots + a_n[\sigma(\alpha_n)]_B$

$= \begin{pmatrix} | & & | \\ [\sigma(\alpha_1)]_B & \cdots & [\sigma(\alpha_n)]_B \\ | & & | \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

Th 7.2.11. $[\sigma(\alpha)]_B = [\sigma]_{B\leftarrow A} [\alpha]_A$

- To compute $\sigma(\alpha)$:

$\overset{③}{\mathrm{Decoord}_B}\left(\overset{②}{[\sigma]_{B\leftarrow A}} \overset{①}{[\alpha]_A}\right)$

e.g. for previous example:

$\sigma(1+3x) =$

$= \mathrm{Decoord}_B \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$= \mathrm{Decoord}_B \begin{pmatrix} 2 \\ 6 \\ 1 \\ 3 \end{pmatrix}$

$= 2 + 6x + x^2 + 3x^3$

- To solve $\sigma(?) = \gamma$:

Same as $[\sigma]_{B\leftarrow A} [?]_A = [\gamma]_B$

$\therefore$ solve $[\sigma]_{B\leftarrow A} X = [\gamma]_B$

for $X$, by e.g. row reduction

then apply $\mathrm{Decoord}_A$ to the solution set.

In particular, when $\gamma = \vec{0}$:

- $\ker\sigma = \overset{③}{\mathrm{Decoord}_B}\left(\overset{②}{\mathrm{Nul}} \overset{①}{[\sigma]_{B\leftarrow A}}\right)$

- range $\sigma = \mathrm{Decoord}_B\left(\mathrm{Col}\ [\sigma]_{B\leftarrow A}\right)$

Reason:

range $\sigma = \mathrm{Span}\{\sigma(\alpha_1), \ldots, \sigma(\alpha_n)\}$ by Lemma.

$= \mathrm{Span}\{\mathrm{Decoord}_B([\sigma(\alpha_i)]_B)\}$

$= \mathrm{Decoord}_B(\mathrm{Span}\{[\sigma(\alpha_i)]_B\})$

$= \mathrm{Decoord}_B($span of the columns of $[\sigma]_{B\leftarrow A})$

- many other problems

**Th 7.2.9** if $\sigma \in L(U,V)$, $\tau \in (V,W)$ and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are bases of $U, V, W$, then

$$[\tau\circ\sigma]_{\mathcal{C}\leftarrow\mathcal{A}} = [\tau]_{\mathcal{C}\leftarrow\mathcal{B}}[\sigma]_{\mathcal{B}\leftarrow\mathcal{A}}$$

Proof: Given $\alpha \in U$, to find $(\tau\circ\sigma)(\alpha)$:

$$[(\tau\circ\sigma)(\alpha)]_{\mathcal{C}} = [\tau\circ\sigma]_{\mathcal{C}\leftarrow\mathcal{A}}[\alpha]_{\mathcal{A}}.$$

Also: let $\beta = \sigma(\alpha)$. Then $(\tau\circ\sigma)\alpha = \tau(\beta)$

$$[(\tau\circ\sigma)(\alpha)]_{\mathcal{C}} = [\tau(\beta)]_{\mathcal{C}}$$
$$= [\tau]_{\mathcal{C}\leftarrow\mathcal{B}}[\beta]_{\mathcal{B}}$$
$$= [\tau]_{\mathcal{C}\leftarrow\mathcal{B}}[\sigma(\alpha)]_{\mathcal{B}}$$
$$= [\tau]_{\mathcal{C}\leftarrow\mathcal{B}}[\sigma]_{\mathcal{B}\leftarrow\mathcal{A}}[\alpha]_{\mathcal{A}}$$

$\therefore [\tau\circ\sigma]_{\mathcal{C}\leftarrow\mathcal{A}}[\alpha]_{\mathcal{A}} = [\tau]_{\mathcal{C}\leftarrow\mathcal{B}}[\sigma]_{\mathcal{B}\leftarrow\mathcal{A}}[\alpha]_{\mathcal{A}}$

<u>for all $\alpha \in U$</u>.

$\therefore [\tau\circ\sigma]_{\mathcal{C}\leftarrow\mathcal{A}} = [\tau]_{\mathcal{C}\leftarrow\mathcal{B}}[\sigma]_{\mathcal{B}\leftarrow\mathcal{A}}$

(i.e. because matrix multiplication is defined to be associative.)

## §7.3 Change of basis

$\boxed{?}$ If $\mathcal{A}, \mathcal{B}$ are bases of $U$
$\mathcal{C}, \mathcal{D}$ are bases of $V$
$\sigma \in L(U,V)$
then how is $[\sigma]_{\mathcal{C}\leftarrow\mathcal{A}}$ related to $[\sigma]_{\mathcal{D}\leftarrow\mathcal{B}}$?

Answer. $\sigma = \iota_V \circ \sigma \circ \iota_U$ ($\iota_U$ = identity transformation on $U$)

Th 7.3.1 $[\sigma]_{\mathcal{D}\leftarrow\mathcal{B}} = [\iota_V]_{\mathcal{D}\leftarrow\mathcal{C}}[\sigma]_{\mathcal{C}\leftarrow\mathcal{A}}[\iota_U]_{\mathcal{A}\leftarrow\mathcal{B}}$

what is $[\iota]_{\mathcal{A}\leftarrow\mathcal{B}}$

By 7.2.11: $[\iota]_{\mathcal{A}\leftarrow\mathcal{B}}[\alpha]_{\mathcal{B}} = [\iota(\alpha)]_{\mathcal{A}} = [\alpha]_{\mathcal{A}}$

$P_{\mathcal{A}\leftarrow\mathcal{B}}$ from 2207

$\therefore$ multiplying by $[\iota]_{\mathcal{A}\leftarrow\mathcal{B}}$ changes from $\mathcal{B}$-coordinates to $\mathcal{A}$-coordinates