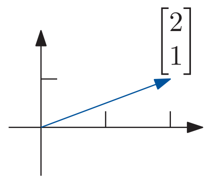
§1.3: Vector Equations

A column vector is a matrix with only one column.

Until Chapter 4, we will say "vector" to mean "column vector".

A vector $\mathbf{u}$ is in $\mathbb{R}^n$ if it has n rows, i.e. $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

Example: $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are vectors in $\mathbb{R}^2$.

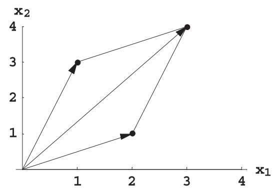


Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ have a geometric meaning: think of $\begin{bmatrix} x \\ y \end{bmatrix}$ as the point (x, y) in the plane.

Parallelogram rule for addition of two vectors:

If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\mathbf{0}$, $\mathbf{u}$ and $\mathbf{v}$. (Note that $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.)

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



There are two operations we can do on vectors:

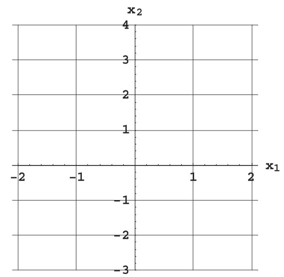
addition: if $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, then $\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}$.

scalar multiplication: if $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and c is a number (a scalar), then $c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$.

These satisfy the usual rules for arithmetic of numbers, e.g.

$$\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}, \quad c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}, \quad 0\mathbf{u} = \mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Express $\mathbf{u}$, $2\mathbf{u}$, and $\frac{-1}{2}\mathbf{u}$ on a graph.



Definition: Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p$ in $\mathbb{R}^n$ and scalars $c_1, c_2, \ldots c_p$, the vector

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

is a *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p$ with *weights* $c_1, c_2, \ldots c_p$.

Example: $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Some linear combinations of $\mathbf{u}$ and $\mathbf{v}$ are:

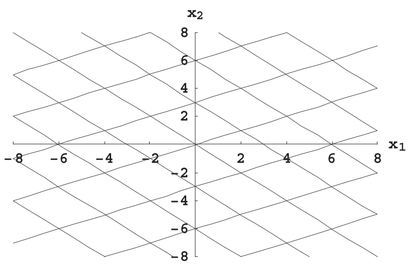
$$3\mathbf{u} + 2\mathbf{v} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}. \qquad \tfrac{1}{3}\mathbf{u} + 0\mathbf{v} = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}.$$

$$\mathbf{u} - 3\mathbf{v} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}. \qquad \mathbf{0} = 0\mathbf{u} + 0\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Geometric interpretation of linear combinations:

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Express each of the following as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \mathbf{d} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$



Definition: Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$. The *span* of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, written

$$\text{Span}\left\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\right\},$$

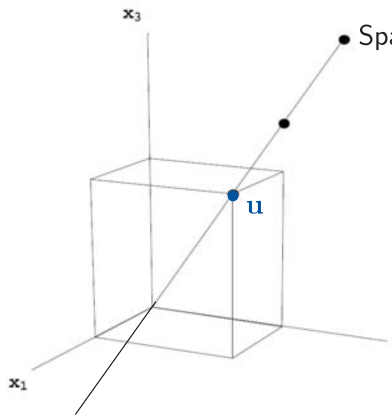
is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.

In other words, Span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is the set of all vectors which can be written as $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p$ for any choice of scalars $x_1, x_2, \ldots, x_p$.

Example: Span of one vector in $\mathbb{R}^3$

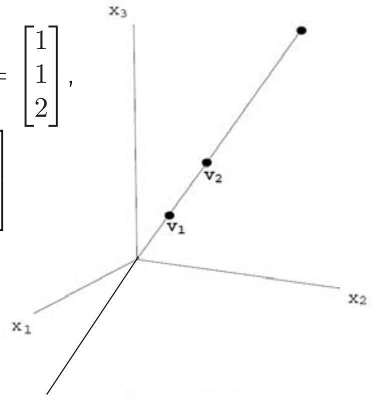
- Span $\{\mathbf{0}\} = \{\mathbf{0}\}$, because $c\mathbf{0} = \mathbf{0}$ for all scalars c.
- If $\mathbf{u}$ is not the zero vector, then Span $\{\mathbf{u}\}$ is a line through the origin in the direction $\mathbf{u}$.

We can also say "$\{\mathbf{u}\}$ spans a line through the origin".



Example: Span of two vectors in $\mathbb{R}^3$

e.g. $\mathbf{v}_1 = \begin{bmatrix}1\\1\\2\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}2\\2\\4\end{bmatrix}$

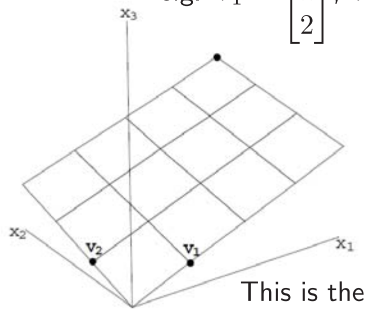


v$_2$ is a multiple of **v**$_1$

Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ =**Span**$\{\mathbf{v}_1\}$ =**Span**$\{\mathbf{v}_2\}$

(line through the origin)

e.g. $\mathbf{v}_1 = \begin{bmatrix}3\\1\\2\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}1\\4\\1\end{bmatrix}$



This is the plane spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$.

v$_2$ is **not** a multiple of **v**$_1$

Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ =plane through the origin

EXAMPLE: Let $\mathbf{a}_1 = \begin{bmatrix}4\\2\\14\end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix}3\\6\\10\end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix}-2\\8\\-8\end{bmatrix}$.

Determine if **b** is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

Solution: Vector **b** is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$ if we can find weights x_1, x_2 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}.$$

Vector equation:

Corresponding linear system:

Corresponding augmented matrix:

$$\left[\begin{array}{cc|c}4 & 3 & -2\\2 & 6 & 8\\14 & 10 & -8\end{array}\right]$$

Reduced echelon form:

$$\left[\begin{array}{cc|c}1 & 0 & -2\\0 & 1 & 2\\0 & 0 & 0\end{array}\right]$$

The vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_p\mathbf{a}_p = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\left[\begin{array}{cccc|c}| & | & & | & |\\\mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_p & \mathbf{b}\\| & | & & | & |\end{array}\right].$$

In particular, $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ (i.e. $\mathbf{b}$ is in Span $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p\}$) if and only if there is a solution to the linear system with augmented matrix

$$\left[\begin{array}{cccc|c}| & | & & | & |\\\mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_p & \mathbf{b}\\| & | & & | & |\end{array}\right].$$

§1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$

We can think of the weights $x_1, x_2, \dots, x_p$ as a vector.

The product of an $\underbrace{m \times p}_{m \text{ rows, } p \text{ columns}}$ matrix A and a vector $\mathbf{x}$ in $\mathbb{R}^p$ is the linear combination of the columns of A using the entries of $\mathbf{x}$ as weights:

$$A\mathbf{x} = \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_p \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_p\mathbf{a}_p.$$

Example: $\begin{bmatrix} 4 & 3 \\ 2 & 6 \\ 14 & 10 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ -8 \end{bmatrix}.$

Example: $\begin{bmatrix} 4 & 3 \\ 2 & 6 \\ 14 & 10 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ -8 \end{bmatrix}.$

There is another way to compute $A\mathbf{x}$, one row of A at a time:

Example: $\begin{bmatrix} 4 & 3 \\ 2 & 6 \\ 14 & 10 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4(-2) + 3(2) \\ 2(-2) + 6(2) \\ 14(-2) + 10(2) \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ -8 \end{bmatrix}.$

Warning: The product $A\mathbf{x}$ is only defined if the number of columns of A equals the number of rows of x. The number of rows of $A\mathbf{x}$ is the number of rows of A.

It is easy to check that $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ and $A(c\mathbf{u}) = cA\mathbf{u}$.

We have three ways of viewing the same problem:

1. The system of linear equations with augmented matrix $[A|\mathbf{b}]$,
2. The vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_p\mathbf{a}_p = \mathbf{b}$,
3. The matrix equation $A\mathbf{x} = \mathbf{b}$.

So these three things are the same:

1. The system of linear equations with augmented matrix $[A|\mathbf{b}]$ has a solution,
2. $\mathbf{b}$ is a linear combination of the columns of A (or $\mathbf{b}$ is in the span of the columns of A),
3. The matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(The three problems have the same solution set.)

Another way of saying this: The span of the columns of A is the set of vectors $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution.

Theorem 4: Existence of solutions to linear systems: For an $m \times n$ matrix A, the following statements are logically equivalent (i.e. for any particular matrix A, they are all true or all false):

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. rref(A) has a pivot in every row.

Warning: the theorem says nothing about the uniqueness of the solution.

Proof: (outline): By previous discussion, (a), (b) and (c) are logically equivalent. So, to finish the proof, we only need to show that (a) and (d) are logically equivalent, i.e. we need to show that,

- if (d) is true, then (a) is true;
- if (d) is false, then (a) is false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
d. rref(A) has a pivot in every row.

Proof: (continued)

Suppose (d) is true. Then, for every $\mathbf{b}$ in $\mathbb{R}^m$, the augmented matrix $[A|\mathbf{b}]$ row-reduces to [rref$(A)|\mathbf{d}$] for some $\mathbf{d}$ in $\mathbb{R}^m$. This does not have a row of the form $[0 \dots 0|*]$, so, by Theorem 2, $A\mathbf{x} = \mathbf{b}$ is consistent. So (a) is true.

Suppose (d) is false. We want to find a counterexample to (a): i.e. we want to find a vector $\mathbf{b}$ in $\mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ has no solution.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
d. rref(A) has a pivot in every row.

Proof: (continued) Suppose (d) is false. We want to find a counterexample to (a): i.e. we want to find a vector $\mathbf{b}$ in $\mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ has no solution.

rref(A) does not have a pivot in every row, so its last row is $[0 \dots 0]$.

Let $\mathbf{d} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. Then the linear system with augmented matrix [rref$(A)|\mathbf{d}$] is inconsistent.

Now we apply the row operations in reverse to get an equivalent linear system $[A|\mathbf{b}]$ that is inconsistent.

Example: $\left[\begin{array}{cc|c} 1 & -3 & -1 \\ -2 & 6 & -1 \end{array}\right] \underset{R_2 \to R_2 - 2R_1}{\overset{R_2 \to R_2 + 2R_1}{\rightleftharpoons}} \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 0 & 1 \end{array}\right]$

§1.5: Solution Sets of Linear Systems

Goal: use vector notation to give geometric descriptions of solution sets
to compare the solution sets of $A\mathbf{x} = \mathbf{b}$ and of $A\mathbf{x} = \mathbf{0}$.

Definition: A linear system is *homogeneous* if the right hand side is the zero vector, i.e.

$$A\mathbf{x} = \mathbf{0}.$$

When we row-reduce $[A|\mathbf{0}]$, the right hand side stays $\mathbf{0}$, so the reduced echelon form does not have a row of the form $[0 \dots 0|*]$ with $* \neq 0$.
So a homogeneous system is always consistent.

In fact, $\mathbf{x} = \mathbf{0}$ is always a solution, because $A\mathbf{0} = \mathbf{0}$. The solution $\mathbf{x} = \mathbf{0}$ called the trivial solution.

A non-trivial solution $\mathbf{x}$ is a solution where at least one x_i is non-zero.

If there are non-trivial solutions, what does the solution set look like?

EXAMPLE:

$$2x_1 + 4x_2 - 6x_3 = 0$$

$$4x_1 + 8x_2 - 10x_3 = 0$$

Corresponding augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 0 \end{array}\right]$$

Corresponding reduced echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Solution set:

Geometric representation:

EXAMPLE: Compare the solution sets of:

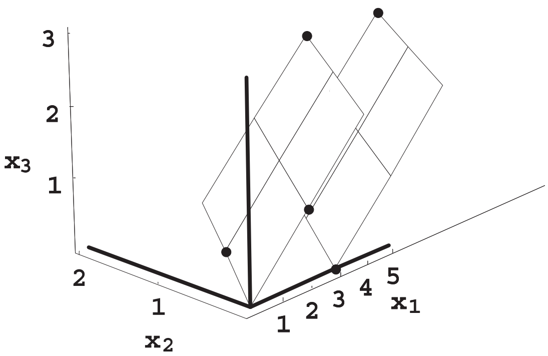
$x_1 - 2x_2 - 2x_3 = 0$ $\qquad$ $x_1 - 2x_2 - 2x_3 = 3$

Corresponding augmented matrices:

$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 0 \end{array}\right]$ $\qquad$ $\left[\begin{array}{ccc|c} 1 & -2 & -2 & 3 \end{array}\right]$

These are already in reduced echelon form.
Solution sets:

Geometric representation:



Parallel Solution Sets of $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$

EXAMPLE: (same left hand side as before)

$$2x_1 + 4x_2 - 6x_3 = 0$$

$$4x_1 + 8x_2 - 10x_3 = 4$$

Corresponding augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 4 \end{array}\right]$$

Corresponding reduced echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solution set:

Geometric representation:

In our first example:

- The solution set of $A\mathbf{x} = \mathbf{0}$ is a line through the origin parallel to $\mathbf{v}$.
- The solution set of $A\mathbf{x} = \mathbf{b}$ is a line through $\mathbf{p}$ parallel to $\mathbf{v}$.

In our second example:

- The solution set of $A\mathbf{x} = \mathbf{0}$ is a plane through the origin parallel to $\mathbf{u}$ and $\mathbf{v}$.
- The solution set of $A\mathbf{x} = \mathbf{b}$ is a plane through $\mathbf{p}$ parallel to $\mathbf{u}$ and $\mathbf{v}$.

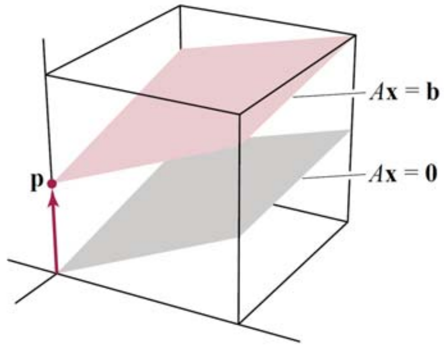
In both cases: to get the solution set of $A\mathbf{x} = \mathbf{b}$, start with the solution set of $A\mathbf{x} = \mathbf{0}$ and translate it by $\mathbf{p}$.

$\mathbf{p}$ is called a particular solution (one solution out of many).

In general:

Theorem 6: Solutions and homogeneous equations: Suppose $\mathbf{p}$ is a solution to $A\mathbf{x} = \mathbf{b}$. Then the solution set to $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v_h}$, where $\mathbf{v_h}$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Theorem 6: Solutions and homogeneous equations: Suppose $\mathbf{p}$ is a solution to $A\mathbf{x} = \mathbf{b}$. Then the solution set to $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v_h}$, where $\mathbf{v_h}$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.



Parallel solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$.

Theorem 6: Solutions and homogeneous equations: Suppose $\mathbf{p}$ is a solution to $A\mathbf{x} = \mathbf{b}$. Then the solution set to $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v_h}$, where $\mathbf{v_h}$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Proof: (outline)
We show that $\mathbf{w} = \mathbf{p} + \mathbf{v_h}$ is a solution:

$$\begin{aligned} & A(\mathbf{p} + \mathbf{v_h}) \\ =& A\mathbf{p} + A\mathbf{v_h} \\ =& \mathbf{b} + \mathbf{0} \\ =& \mathbf{b}. \end{aligned}$$

We also need to show that all solutions are of the form $\mathbf{w} = \mathbf{p} + \mathbf{v_h}$ - see q25 in Section 1.5 of the textbook.

Question:

Suppose A is a matrix with rref(A) $= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find the solution set to $A\mathbf{x} = A\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$.

Answer:

rref$(A) \rightarrow$ the solution set to $A\mathbf{x} = \mathbf{0}$ is Span $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$ (see earlier today).

$\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$ is a particular solution to $A\mathbf{x} = A\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$.

So the solution set to $A\mathbf{x} = A\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$ is $\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, where s can take any value.