Given a symmetric bilinear form $f: V \times V \to \mathbb{F}$ (symmetric matrix A)

find a basis $\mathcal{B} = \{\beta_1, \dots, \beta_n\}$ (invertible matrix P)

such that $\{f\}_{\mathcal{B}}$ is diagonal ($P^T A P$ is diagonal)

$$\left(P = [\,\iota\,]_{\mathcal{A} \leftarrow \mathcal{B}} = \begin{pmatrix} | & & | \\ \beta_1 & \cdots & \beta_n \\ | & & | \end{pmatrix} \right)$$

Th. 9.4.16 This diagonalisation is possible if $1+1 \neq 0$ in $\mathbb{F}$.

$\{f\}_{\mathcal{B}}$ is diagonal means the i,j entries are zero if $i \neq j$

i.e. $f(\beta_i, \beta_j) = 0$ if $i \neq j$.

Main idea of algorithm:

choose β_j one-by-one so that β_j satisfies $f(\beta_i, \beta_j) = 0$ for all previously chosen β_i, i.e. for $i < j$

proof by induction on $\dim V$

Proof:

(base case: if $\dim V = 1$ then $\{f\}_{\mathcal{B}}$ is 1×1 matrix $\therefore$ diagonal)

Let $q(\alpha) = f(\alpha, \alpha)$.

Important: if $q(\alpha) = 0 \ \forall \alpha$, then $f(\alpha, \beta) = 0 \ \forall \alpha, \beta$, by polarisation identity.

(then $\{f\}_{\mathcal{B}}$ = zero matrix for all $\mathcal{B}$

$\therefore$ done)

So, if $f \neq$ zero function, then $\exists \beta_1$ with $q(\beta_1) \neq 0$.

Define a function $\phi_1 = f(\beta_1, -)$ i.e. $\phi_1(\alpha) = f(\beta_1, \alpha)$

and a subspace $W_1 = \ker \phi_1$.

Note: $\phi_1 : V \to \mathbb{F}$ is linear and $\phi_1(\beta_1) = q(\beta_1) \neq 0$

$\therefore$ rank $\phi_1 \neq 0$ $\therefore$ rank $\phi_1 = 1$

RNT: nullity $\phi_1 = n-1$ $\therefore \dim W_1 = n-1$

$\therefore$ apply the inductive hypothesis to $f|_{W_1}$ to $\{\beta_2, \cdots, \beta_n\}$ a basis of W_1 so that $f|_{W_1}(\beta_i, \beta_j) = 0$

$\Rightarrow f(\beta_i, \beta_j) = 0 \ \forall i \neq j,\ i, j \geq 2$.

By definition of W_1, we also have $f(\beta_1, \beta_j) = 0 \ \forall j > 1$.

$\therefore$ done QED.

How to do the induction part in the algorithm:

after finding $\beta_1, \phi_1\ (W_1)$:

find $\beta_2 \in W_1$ with $q(\beta_2) \neq 0$

(if there is no such β_2, i.e. $q(\beta_2) = 0\ \forall \beta_2 \in W_1$,
then $f|_{W_1}$ is zero function, so choose any
$\beta_2, \beta_3, \cdots \in W_1$).

Let $\phi_2 = f(\beta_2, -)$ i.e. $\phi_2(\alpha) = f(\beta_2, \alpha)$

$$W_2 = \ker(\phi_2|_{W_1})$$
$$= \ker \phi_2 \cap W_1 = \ker \phi_2 \cap \ker \phi_1.$$

repeat with $\beta_3, \beta_4, \cdots$

Ex: (same as before) $A=\begin{pmatrix}0&1&2\\1&0&1\\2&1&0\end{pmatrix}$

- Choose β_1 with $q(\beta_1)\neq 0$:

  tip: if diagonal entry $a_{ii}\neq 0$, then we can choose $\beta_1=e_i$ ($\because a_{ii}=q(e_i)$).

  any simple linear combination

  here, all diagonal entries are zero, so try e_1+e_2?

$$q\begin{pmatrix}1\\1\\0\end{pmatrix}=(1\ 1\ 0)\begin{pmatrix}0&1&2\\1&0&1\\2&1&0\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}=(1\ 1\ 3)\begin{pmatrix}1\\1\\0\end{pmatrix}=2\neq 0$$

$\therefore$ let $\beta_1=\begin{pmatrix}1\\1\\0\end{pmatrix}$.

- $$\phi_1\begin{pmatrix}x\\y\\z\end{pmatrix}=f\left(\beta_1,\begin{pmatrix}x\\y\\z\end{pmatrix}\right)=(1\ 1\ 0)\begin{pmatrix}0&1&2\\1&0&1\\2&1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=(1\ 1\ 3)\begin{pmatrix}x\\y\\z\end{pmatrix}=x+y+3z.$$

$$W_1 = \ker \phi_1 = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 3 \end{pmatrix} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, x+y+3z=0 \right\}$$

· Choose any $\beta_2 \in W_1$, with $q(\beta_2) \neq 0$. Try $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$?

$$q\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -2 \neq 0$$

$\therefore \beta_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is ok.

· $$\phi_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -x+y+z.$$

$$W_2 = \ker \phi_1 \cap \ker \phi_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, \begin{array}{l} x+y+3z=0 \\ -x+y+z=0 \end{array} \right\} = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 3 \\ -1 & 1 & 1 \end{pmatrix} \begin{array}{l} [\phi_1] \\ [\phi_2] \end{array}$$

$$\begin{pmatrix} 1 & 1 & 3 \\ -1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{row reduction}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \end{pmatrix}$$

so $W_2 = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\}$

$\therefore \beta_3 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

$q(\beta_3) = \cdots = -4$

$$\therefore D = \begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & -2 & \cdot \\ \cdot & \cdot & -4 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$