Homework 3, due 12:00 Wednesday, 20 March 2019 to Dr. Pang's mailbox

You must justify your answers to receive full credit.

1. Let V be a vector space over $\mathbb{F}$ and $\sigma \in L(V, V)$. Let $\alpha \in V$ be an eigenvector of σ with eigenvalue λ.

   a) Show that α is an eigenvector of σ^n with eigenvalue λ^n (for $n = 1, 2, \dots$).

   Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, for some $a_i \in \mathbb{F}$.

   b) Show that α is an eigenvector of $f(\sigma)$ with eigenvalue $f(\lambda)$.

   c) Show that, if $f(\sigma) = 0$ (zero function), then $f(\lambda) = 0$ (zero number).

2. Let $C^0(\mathbb{R})$ denote the vector space of continuous functions on $\mathbb{R}$. Consider $\sigma : C^0(\mathbb{R}) \to C^0(\mathbb{R})$ given by
$$\sigma(f)(x) = f''(x) + f(x) + f(0).$$

   a) Show that, for each n, the function $f(x) = \sin(nx)$ is an eigenvector of σ, and find its corresponding eigenvalue.

   b) Let $W = P_{<3}(\mathbb{R})$ be the subspace of polynomials with degree less than 3. Find three linearly independent eigenvectors of the restriction $\sigma|_W$. (Click here for a hint)

3. Consider
$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & -2 \\ -4 & 3 & 3 & -2 \\ -2 & 2 & 1 & 0 \end{pmatrix}.$$
You are given that the characteristic polynomial of A is $\chi_A(x) = (x-2)^4$. Find the Jordan form J of A and find a matrix P such that $P^{-1}AP = J$. (You do **not** need to find P^{-1}.) (You may use an online RREF calculator, but remember you only have an ordinary calculator in the exams.)

4. Consider
$$B = \begin{pmatrix} 4 & 1 & -3 & -6 & 3 \\ -9 & -2 & 9 & 18 & -9 \\ -12 & -4 & 15 & 28 & -16 \\ 6 & 2 & -7 & -13 & 8 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \operatorname{rref}(B - I) = \begin{pmatrix} 1 & \frac{1}{3} & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
You are given that $(B-I)^2 = 0$. (Hint: so what are the eigenvalues of B?) Find the Jordan form J of B and find a matrix P such that $B = PJP^{-1}$. (You do **not** need to find P^{-1}.)

5. Consider $n \times n$ complex matrices A and B.

   a) Show that A, B are similar if and only if they have the same Jordan form. (Click here for a hint)

   b) Let $n = 3$. Show that A, B are similar if and only if they have the same characteristic polynomial and the same minimal polynomial. (Click here for a hint)

   c) Let $n = 4$. Give a counterexample to show that the second sentence of part b) is false.

6. Find, with explanation, all the possible Jordan forms of the following matrices:

   a) The characteristic polynomial of A is $-(x-2)^3(x+7)^2$, and the minimal polynomial of A has degree 3.

   b) The characteristic polynomial of B is $-(x-3)^5$, and $\ker(B-3I)$ is 3-dimensional.

   c) The characteristic polynomial of C is $(x+2)^4(x-1)^2$, and $\ker(C+2I)^2$ is 3-dimensional.

The following two questions are to prepare you for upcoming classes, and is unrelated to the material from recent classes.

7. Consider the subspace W of $\mathbb{R}^3$:

$$W = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

   a) Find an orthogonal basis for W.

   b) Find a basis for $W^\perp$, the orthogonal complement of W.

8. Consider

$$\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}.$$

Note that $\{\alpha_1, \alpha_2, \alpha_3\}$ is an orthogonal basis of $\mathbb{R}^3$.

   a) Find the length of β.

   b) By computing dot products, express β as a linear combination of $\alpha_1, \alpha_2, \alpha_3$.

Optional question If you attempted seriously all the above questions, then your scores for the following question may replace any lower scores for one of the above questions.

9. Let V be a finite-dimensional vector space, and $\sigma, \tau \in L(V, V)$.

   a) Suppose σ and τ are *simultaneously diagonalisable*, i.e. there is a basis of V that are eigenvectors for both σ and τ (with maybe different eigenvalues). Show that $\sigma \circ \tau = \tau \circ \sigma$. (Hint: consider the image of the eigenvectors under $\sigma \circ \tau$.)

   b) Suppose σ, τ are diagonalisable, and $\sigma \circ \tau = \tau \circ \sigma$. Show that σ and τ are simultaneously diagonalisable. (Hint: first show that each eigenspace of σ is invariant under τ. Then use that the restriction of τ to these eigenspaces is diagonalisable.)

10. Let $C^0(\mathbb{R})$ be the vector space of continuous functions on $\mathbb{R}$. Show that $\sigma : C^0(\mathbb{R}) \to C^0(\mathbb{R})$ given by

$$\sigma(f)(x) = \int_0^x f(t)\,dt$$

does not have any eigenvalues.

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