


From the beginning of last week:

Remember from calculus the addition and scalar multiplication of polynomials:

$$\text{e.g. } (2t^2 + 1) + (-t^2 + 3t + 2) = t^2 + 3t + 3.$$

$$\text{e.g. } (-3)(-t^2 + 3t + 2) = 3t^2 - 9t - 6.$$



We want to represent abstract vectors as column vectors so we can do calculations (e.g. row-reduction) to study linear systems (e.g. week 7 p20) and linear transformations (e.g. week 7 p43).

Is this really different from \mathbb{R}^3 ?

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}.$$

$$(-3) \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \\ 3 \end{bmatrix}.$$

← coefficient of 1
← coefficient of t
← coefficient of t^2

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$$\text{In } \mathbb{R}^n, \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \mathbf{e}_1 + \cdots + x_n \mathbf{e}_n.$$

We can copy this idea: in V , pick a special set of vectors $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, write each \mathbf{v} in V as $c_1 \mathbf{b}_1 + \cdots + c_n \mathbf{b}_n$ and represent \mathbf{v} by the column vector $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

Example: In \mathbb{P}_2 , let $\mathbf{b}_1 = 1, \mathbf{b}_2 = t, \mathbf{b}_3 = t^2$.

Then we represent $a_0 + a_1 t + a_2 t^2$ by $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$ (see previous page).

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$\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$
must span V



We can copy this idea: in V , pick a special set of vectors $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, write **each**

\mathbf{x} in V **uniquely** as $c_1 \mathbf{b}_1 + \cdots + c_n \mathbf{b}_n$ and represent \mathbf{x} by the column vector $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

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§4.3: Bases

Definition: Let W be a subspace of a vector space V . An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a *basis for W* if

- i \mathcal{B} is a linearly independent set, and
- ii $\text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\} = W$.

↑
The order matters:
 $\{\mathbf{b}_1, \mathbf{b}_2\}$ and $\{\mathbf{b}_2, \mathbf{b}_1\}$
are different bases.

i means: The only solution to $x_1\mathbf{b}_1 + \dots + x_p\mathbf{b}_p = \mathbf{0}$ is $x_1 = \dots = x_p = 0$.

ii means: W is the set of vectors of the form $c_1\mathbf{b}_1 + \dots + c_p\mathbf{b}_p$ where c_1, \dots, c_p can take any value.

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Condition ii implies that $\mathbf{b}_1, \dots, \mathbf{b}_p$ must be in W , because $\text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ contains each of $\mathbf{b}_1, \dots, \mathbf{b}_p$.

Every vector space V is a subspace of itself, so we can take $W = V$ in the definition and talk about bases for V .

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Example: The *standard basis* for \mathbb{R}^3 is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

To check that this is a basis: $\begin{bmatrix} | & | & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is in reduced echelon form.

The matrix has a pivot in every column, so its columns are linearly independent.

The matrix has a pivot in every row, so its columns span \mathbb{R}^3 .

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A basis for W is **not** unique: (different bases are useful in different situations, see week 9).

Let's look for a different basis for \mathbb{R}^3 .

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for \mathbb{R}^3 ?

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Answer: No, because two vectors cannot span \mathbb{R}^3 : $\begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$ cannot have a pivot in every row.

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Answer: Form the matrix $A = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. Because

$\det A = 1 \neq 0$, the matrix A is invertible, so (by Invertible Matrix Theorem) its columns are linearly independent and its columns span \mathbb{R}^3 .

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Answer: No, because four vectors in \mathbb{R}^3 must be linearly dependent:

$\begin{bmatrix} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ cannot have a pivot in every column.

By the same logic as in the above examples:

Fact: $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis for \mathbb{R}^n if and only if:

- $p = n$ (i.e. the set has exactly n vectors), and

- $\det \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_n \\ | & & | \end{bmatrix} \neq 0.$

Fewer than n vectors: not enough vectors, can't span \mathbb{R}^n .
More than n vectors: too many vectors, linearly dependent.

Example: The **standard basis** for \mathbb{P}_n is $\mathcal{B} = \{1, t, t^2, \dots, t^n\}$.

To check that this is a basis:

- ii By definition of \mathbb{P}_n , every element of \mathbb{P}_n has the form $a_0 + a_1t + a_2t^2 + \dots + a_nt^n$, so \mathcal{B} spans \mathbb{P}_n .
- i To see that \mathcal{B} is linearly independent, we show that $c_0 = c_1 = \dots = c_n = 0$ is the only solution to

$$c_0 + c_1t + c_2t^2 + \dots + c_nt^n = 0. \text{ (the zero function)}$$

Substitute $t = 0$: we find $c_0 = 0$.

Differentiate, then substitute $t = 0$: we find $c_1 = 0$.

Differentiate again, then substitute $t = 0$: we find $c_2 = 0$.

Repeating many times, we find $c_0 = c_1 = \dots = c_n = 0$.

Once we have the standard basis of \mathbb{P}_n , it will be easier to check if other sets are bases of \mathbb{P}_n , using **coordinates** (later, p14).

Advanced exercise: what do you think is the standard basis for $M_{m \times n}$?

One way to make a basis for V is to start with a set that spans V .

Theorem 5: Spanning Set Theorem: If $V = \text{Span} \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, then some subset of $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis for V .

Proof: basically, the idea of the casting-out algorithm (week 7 p28-34) works in abstract vector spaces too.

- If $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly independent, it is a basis for V .
- If $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly dependent, then one of the \mathbf{v}_i s is a linear combination of the others. Removing this \mathbf{v}_i from the set still gives a set that spans V . Continue removing vectors in this way until the remaining vectors are linearly independent.

Example: $\mathbb{P}_2 = \text{Span} \{5, 3 + t, 1 + 2t^2, 4 + 2t - 4t^2\}$, but this set is not linearly independent because $4 + 2t - 4t^2$ is a linear combination of the other polynomials: $4 + 2t - 4t^2 = 2(3 + t) - 2(1 + 2t^2)$. So remove $4 + 2t - 4t^2$ to get the set $\{5, 3 + t, 1 + 2t^2\}$, which is in fact a basis (we can show this with coordinates, p14-15).

PP 234, 238-240 (§4.4), 307-308 (§5.4): Coordinates

Recall (p2) that our motivation for finding a basis is because we want to write each vector \mathbf{x} as $c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$ in a unique way. Let's show that this is indeed possible

Theorem 7: Unique Representation Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V . Then for each \mathbf{x} in V , there exists a unique set of scalars c_1, \dots, c_n such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

Proof:

Since \mathcal{B} spans V , there exists scalars c_1, \dots, c_n such that the above equation holds.

Suppose \mathbf{x} has another representation

$$\mathbf{x} = d_1\mathbf{b}_1 + \cdots + d_n\mathbf{b}_n.$$

for some scalars d_1, \dots, d_n . Then

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \cdots + (c_n - d_n)\mathbf{b}_n.$$

Because \mathcal{B} is linearly independent, all the weights in this equation must be zero, i.e. $(c_1 - d_1) = \cdots = (c_n - d_n) = 0$. So $c_1 = d_1, \dots, c_n = d_n$.

Because of the Unique Representation Theorem, we can make the following definition:

Definition: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for V . Then, for any \mathbf{x} in V , the *coordinates of \mathbf{x} relative to \mathcal{B}* , or the *\mathcal{B} -coordinates of \mathbf{x}* , are the unique weights c_1, \dots, c_n such that

$$\mathbf{x} = c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n.$$

The vector in \mathbb{R}^n

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the *coordinate vector of \mathbf{x} relative to \mathcal{B}* , or the *\mathcal{B} -coordinate vector of \mathbf{x}* .

Example: Let $\mathcal{B} = \{1, t, t^2, t^3\}$ be the standard basis for \mathbb{P}_3 . Then the coordinate

vector of an arbitrary polynomial is $[a_0 + a_1 t + a_2 t^2 + a_3 t^3]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

Why are coordinate vectors useful?

Because of the Unique Representation Theorem, the function V to \mathbb{R}^n given by $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ (e.g. $a_0 + a_1t + a_2t^2 + a_3t^3 \mapsto \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$) is linear, one-to-one and onto.

Definition: A linear transformation $T : V \rightarrow W$ that is both one-to-one and onto is called an *isomorphism*. We say V and W are *isomorphic*.

This means that, although the notation and terminology for V and W are different, the two spaces behave the same as vector spaces. Every vector space calculation in V is accurately reproduced in W , and vice versa.

Important consequence: if V has a basis of n vectors, then V and \mathbb{R}^n are isomorphic, so we can solve problems about V (e.g. span, linear independence) by working in \mathbb{R}^n .

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Example: Is the set of polynomials $\{1, 2 - t, (2 - t)^2, (2 - t)^3\}$ linearly independent?

Answer: The coordinates of these polynomials relative to the standard basis of \mathbb{P}_3 are

$$\begin{aligned} [1]_{\mathcal{B}} &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & [(2 - t)^2]_{\mathcal{B}} = [4 - 4t + t^2]_{\mathcal{B}} &= \begin{bmatrix} 4 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \\ [2 - t]_{\mathcal{B}} &= \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, & [(2 - t)^3]_{\mathcal{B}} = [8 - 12t + 6t^2 - t^3]_{\mathcal{B}} &= \begin{bmatrix} 8 \\ -12 \\ 6 \\ -1 \end{bmatrix} \end{aligned}$$

The set of polynomials is linearly independent if and only if their coordinate vectors are linearly independent (continued on next page).

Example: Is the set of polynomials $\{1, 2 - t, (2 - t)^2, (2 - t)^3\}$ linearly independent?

Answer: (continued). The matrix

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & -1 & -4 & -12 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

has determinant $1 \neq 0$ (it is upper triangular so its determinant is the product of the diagonal entries), so it is invertible, and by the Invertible Matrix Theorem, its columns are linearly independent in \mathbb{R}^4 . So the polynomials are linearly independent. (In fact they form a basis, because IMT says they also span \mathbb{R}^4 .)

Advanced exercise: if \mathbf{p}_i has degree exactly i , then $\{\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n\}$ is a basis for \mathbb{P}_n . (This idea is how I usually prove that a set is a basis in my research work.)

Harder example: (in preparation for week 9, change of coordinates) Let $\mathcal{F} = \{1, 2 - t, (2 - t)^2, (2 - t)^3\}$. We just showed that \mathcal{F} is a basis. So if the

\mathcal{F} -coordinates of a polynomial \mathbf{p} is $[\mathbf{p}]_{\mathcal{F}} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}$, then what is \mathbf{p} ?

Answer:

$$[\mathbf{p}]_{\mathcal{F}} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix} \text{ means } \mathbf{p} = 2 + 4(2 - t) + 0(2 - t)^2 - 1(2 - t)^3 = 2 + 8t - 6t^2 + t^3.$$

If V has a basis of n vectors, then V and \mathbb{R}^n are isomorphic, so we can solve problems about V (e.g. span, linear independence) by working in \mathbb{R}^n .

What about problems concerning linear transformations $T : V \rightarrow W$?

Remember from week 4 §1.9: Every linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} | & & | \\ T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \\ | & & | \end{bmatrix}$$

apply T to i th basis vector, put the coordinates of the result into column i
(standard matrix of T).

The standard matrix is useful because we can solve $T(\mathbf{x}) = \mathbf{y}$ by row-reducing $[A|\mathbf{y}]$.

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Definition: If V is a vector space with basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $T : V \rightarrow V$ is a linear transformation, then the *matrix for T relative to \mathcal{B}* is

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix}.$$

(so the standard matrix of T is the matrix for T relative to the standard basis of \mathbb{R}^n .)

The matrix $[T]_{\mathcal{B}}$ is useful because

$$[T(\mathbf{x})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}, \quad (*)$$

so we can solve $T(\mathbf{x}) = \mathbf{y}$ by row-reducing $\left[[T]_{\mathcal{B}} \mid [\mathbf{y}]_{\mathcal{B}} \right]$.

Example: Let $T : \mathbb{P}_2 \rightarrow \mathbb{P}_2$ be the differentiation function $T(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$ as on the previous page. Here is an example of equation $(*)$ for $\mathbf{x} = 2 + 3t - t^2$.

$$T(2 + 3t - t^2) = \frac{d}{dt}(2 + 3t - t^2) = 3 - 2t$$

$$[T]_{\mathcal{B}} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}.$$

Some other things about T that we can learn from the matrix $[T]_{\mathcal{B}}$:

- We can solve the differential equation $\frac{d}{dt}\mathbf{p} = 1 - 3t$ by row-reducing $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right]$.
- $[T]_{\mathcal{B}}$ is in echelon form, and it does not have a pivot in every column, so T is not one-to-one (which you know from calculus - this is why indefinite integrals have $+C$).

Basis and coordinates for subspaces:

Example: Let W be the set of vectors of the form $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$, where a, b can take any value.

We showed (week 7 p14) that W is a subspace of \mathbb{R}^3 because $W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Since $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is furthermore linearly independent, it is a basis for W .

Because $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, the coordinate vector of $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$, relative to the basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, is $\begin{bmatrix} a \\ b \end{bmatrix}$. So $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \mapsto \begin{bmatrix} a \\ b \end{bmatrix}$ is an isomorphism from W to \mathbb{R}^2 .

Coordinates for subspaces (e.g. planes in \mathbb{R}^3) are useful as they allow us to represent points in the subspace with fewer numbers (e.g. with 2 numbers instead of 3 numbers).

In this picture (p239 of textbook, example 7 in §4.4), $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for a plane. The basis allows us to draw a coordinate grid on the plane.

The \mathcal{B} -coordinate vector of \mathbf{x} is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. This coordinate vector describes the location of \mathbf{x} relative to this coordinate grid.

