You must justify your answers to receive full credit.

1. Let A be the matrix
$$A = \begin{bmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{bmatrix}.$$

   a) Find, in parametric form, the solutions to $A\mathbf{x} = \mathbf{0}$.

   b) Find, in parametric form, the solutions to $A\mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

   c) Find two particular solutions to $A\mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

2. Without doing any row-reduction, determine whether the following sets are linearly independent, and explain why:

   a) $\left\{ \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \end{bmatrix} \right\}$

   b) $\left\{ \begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}$

   c) $\left\{ \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

3. Let S be the set of vectors
$$S = \left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

   a) Find a linearly independent subset R of S such that $\mathrm{Span}R = \mathrm{Span}S$.

   b) Is $\mathrm{Span}S$ equal to $\mathbb{R}^3$? Explain your answer.

4. For each matrix A in parts a), b), c) below, determine:
   (i) does the equation $A\mathbf{x} = 0$ have a nontrivial solution; and,
   (ii) does the equation $A\mathbf{x} = \mathbf{b}$ have at least one solution for every possible $\mathbf{b}$?

   a) A is a 3×3 matrix with three pivot positions.

b) A is a 3×3 matrix with two pivot positions.

c) A is a 3×2 matrix with two pivot positions.

5. Suppose $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a linearly dependent subset of $\mathbb{R}^n$, and $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Show that $\{T(\mathbf{v_1}), T(\mathbf{v_2}), T(\mathbf{v_3})\}$ is linearly dependent.

6. State whether each of the following statements is (always) true or (sometimes) false. If it is true, give a brief justification; if it is false, give a counterexample with an explanation.

   a) $A\mathbf{x} = \mathbf{b}$ is a homogeneous equation if and only if $\mathbf{x} = \mathbf{0}$ is a solution.

   b) If $A\mathbf{x} = \mathbf{b}$ has a unique solution, then $A\mathbf{x} = \mathbf{c}$ cannot have infinitely many solutions.

   c) If $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ is linearly dependent, then $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly dependent.

   d) If $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ is linearly independent, then $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly independent.

   e) Let A be a 4 x 3 matrix with columns $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$, and suppose $\mathbf{b}$ is a vector in $\mathbb{R}^4$ such that $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{b}\}$ is linearly dependent. Then $A\mathbf{x} = \mathbf{b}$ has a solution.

   f) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\{\mathbf{v_1}, \mathbf{v_2}\}$ spans $\mathbb{R}^2$, then $\{T(\mathbf{v_1}), T(\mathbf{v_2})\}$ also spans $\mathbb{R}^2$.

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