

## §1.1: Systems of Linear Equations

Example of a linear equation:  $y = 5x + 2$

Can be rearranged to give:  $(-5)x + (1)y = 2$

Another example:  $3(x_1 + 2x_2) + 1 = x_1 + 1$

Rearrangement:  $(2)x_1 + (6)x_2 = 0$

Another example:  $x_2 = \sqrt{2}(\sqrt{6} - x_1) + x_3$

Rearrangement:  $\sqrt{2}x_1 + (1)x_2 + (-1)x_3 = 2\sqrt{3}$

Not linear:  $x_2 = 2\sqrt{x_1}$

$$xy + x = e^5$$

**Definition:** A linear system is *consistent* if it has a solution,

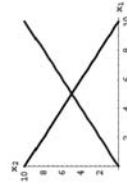
and *inconsistent* if it does not have a solution.

**Fact:** A linear system has either

- exactly one solution
  - infinitely many solutions
  - no solutions
- consistent  
consistent  
inconsistent

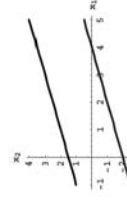
**EXAMPLE** Two equations in two variables:

$$\begin{aligned} x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0 \end{aligned}$$



one unique solution  
consistent

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8 \end{aligned}$$



no solution  
inconsistent

$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$



infinitely many solutions  
consistent

In general, a linear equation is an equation of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b.$$

$x_1, x_2, \dots, x_n$  are the variables.

$a_1, a_2, \dots, a_n$  are the coefficients.

**Definition:** A *system of linear equations* (or a *linear system*) is a collection of linear equations involving the same set of variables.

Example:

$$\begin{aligned} x + y &= 3 \\ 3x + 2z &= -2 \end{aligned}$$

A solution is:

$$(x, y, z) = (2, 1, -4).$$

This is a system of 2 equations in 3 variables,  $x, y, z$ .

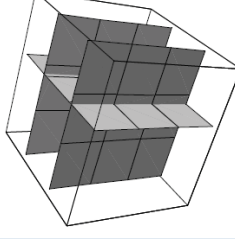
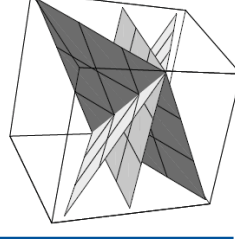
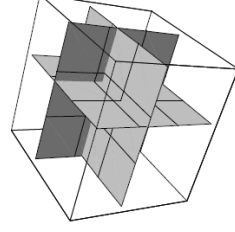
**Definition:** A *solution* of a linear system is a list  $(s_1, s_2, \dots, s_n)$  of numbers that makes each equation a true statement when the values  $s_1, s_2, \dots, s_n$  are substituted for  $x_1, x_2, \dots, x_n$  respectively.

**Definition:** The *solution set* of a linear system is the set of all possible solutions.

$$ax + by + cz = d, \text{ or } z = d'x + b'y + d''$$

**EXAMPLE:** Three equations in three variables. Each equation determines a plane in 3-space.

- The planes intersect in one point. (*one solution*)
- The planes intersect in one line. (*infinitely many solutions*)
- There is no point in common to all three planes. (*no solution*)



Which of these cases are consistent?



**Warning:** Do not do multiple elementary row operations at the same time, **except** adding multiples of **the same** row to several rows.

$$\begin{array}{l} x_1 - 2x_2 = 1 \\ -x_1 + 3x_2 = 3 \end{array} \quad \begin{array}{l} x_2 = 2 \\ x_2 = 2 \end{array}$$

$$\left[ \begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 0 & 1 & 2 \\ 0 & 1 & 2 \end{array} \right]$$

These are NOT equivalent systems.

$$\begin{array}{l} \leftarrow R_1 + R_2 \\ \leftarrow R_2 + R_1 \end{array}$$

$$\begin{array}{l} x_1 - 2x_2 = -3 \\ x_2 = 16 \\ x_3 = 3 \end{array} \quad \begin{array}{l} \leftarrow R_1 - R_3 \\ \leftarrow R_2 + 4R_3 \end{array}$$

$$\left[ \begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 1 & 16 \\ 0 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{array} \quad \left[ \begin{array}{cc|c} 1 & 0 & 29 \\ 0 & 1 & 16 \\ 0 & 0 & 1 \end{array} \right]$$

Two fundamental questions:

1. Existence of solutions: is the system consistent?
2. Uniqueness of solutions: if a solution exists, is it the only one?

Answering this requires less work than finding the solution.

**Example:**

$$\begin{array}{l} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{array} \quad \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right]$$

$$\begin{array}{l} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{array} \quad \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

$$\begin{array}{l} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{array} \quad \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

$$\begin{array}{l} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{array} \quad \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

We can stop here:  
back-substitution shows  
that we can find a unique  
solution.

**EXAMPLE:** Is this system consistent?

$$\begin{aligned}x_1 - 2x_2 + 3x_3 &= -1 \\5x_1 - 7x_2 + 9x_3 &= 0 \\3x_2 - 6x_3 &= 8\end{aligned}$$

**EXAMPLE:** For what values of  $h$  will the following system be consistent?

$$\begin{aligned}x_1 - 3x_2 &= 4 \\-2x_1 + 6x_2 &= h\end{aligned}$$

## Section 1.2: Row Reduction and Echelon Forms

Motivation: it is easy to solve a linear system whose augmented matrix is in reduced echelon form

**Echelon form (or row echelon form):**

1. All nonzero rows are above any rows of all zeros.
2. Each *leading entry* (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

**EXAMPLE:** Echelon forms

$$\begin{aligned}\text{(a)} \quad & \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \text{(b)} \quad & \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix} \\ \text{(c)} \quad & \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}\end{aligned}$$

**Reduced echelon form:** Add the following conditions to conditions 1, 2, and 3 above:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

**EXAMPLE** (continued):

Reduced echelon form :

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

**EXAMPLE:** Are these matrices in echelon form, reduced echelon form, or neither?

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{l}
 x_1 - 2x_2 + x_3 = 0 \\
 2x_2 - 8x_3 = 8 \\
 -4x_1 + 5x_2 + 9x_3 = -9 \\
 \\
 x_1 - 2x_2 + x_3 = 0 \\
 2x_2 - 8x_3 = 8 \\
 -3x_2 + 13x_3 = -9 \\
 \\
 x_1 - 2x_2 + x_3 = 0 \\
 x_2 - 4x_3 = 4 \\
 -3x_2 + 13x_3 = -9 \\
 \\
 x_1 - 2x_2 + x_3 = 0 \\
 x_2 - 4x_3 = 4 \\
 x_3 = 3 \\
 \\
 x_1 - 2x_2 = -3 \\
 x_2 = 16 \\
 x_3 = 3 \\
 \\
 x_1 = 29 \\
 x_2 = 16 \\
 x_3 = 3
 \end{array}$$

echelon form

reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

**Theorem:** Any matrix  $A$  is row-equivalent to exactly one reduced echelon matrix, which is called its **reduced echelon form** and written  $\text{rref}(A)$ .

General strategy for solving a linear system: apply row operations to its augmented matrix to obtain its  $\text{rref}$ .

General strategy for determining existence/uniqueness of solutions: apply row operations to its augmented matrix to obtain an **echelon form**, i.e. a row-equivalent echelon matrix.

Warning: an echelon form is not unique. Its entries depend on the row operations we used. But its pattern of  $\blacksquare$  and  $*$  is unique.

These processes of row operations (to get to echelon or reduced echelon form) are called **row reduction**.

Row reduction:

augmented matrix of linear system

→ echelon form

→ reduced echelon form

existence and uniqueness of solutions

solution set

The rest of this section:

- The row reduction algorithm
- Getting the solution, existence/uniqueness from the (reduced) echelon form

Important terms in the row reduction algorithm:

- **pivot position:** the position of a leading entry in a row-equivalent echelon matrix.
- **pivot:** a nonzero entry of the matrix that is used in a pivot position to create zeroes below it.
- **pivot column:** a column containing a pivot position.

The black squares are the pivot positions.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Row reduction algorithm:**

**EXAMPLE:**

$$\left[ \begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{array} \right]$$

1. The top of the leftmost nonzero column is a pivot position.
2. Put a pivot in this position, by scaling or interchanging rows.

$$\left[ \begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array} \right] \quad \begin{array}{l} R_3 \\ R_1 \end{array}$$

3. Create zeroes in all positions below the pivot, by adding multiples of the top row to each row.

$$\left[ \begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array} \right] \quad R_2 - 3R_1$$

4. Ignore this row and all rows above, and repeat steps 1-3.

$$\left[ \begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array} \right]$$

1. The top of the leftmost nonzero column is a pivot position.
2. Put a pivot in this position, by scaling or interchanging rows.

$$\left[ \begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array} \right] \quad {}^{1/2}R_2$$

3. Create zeroes in all positions below the pivot, by adding multiples of the top row to each row.

$$\left[ \begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right] \quad R_3 - 3R_2$$

4. Ignore this row and all rows above, and repeat steps 1-3.

$$\left[ \begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right]$$

1. The top of the leftmost nonzero column is a pivot position.
2. Put a pivot in this position, by scaling or interchanging rows.
3. Create zeroes in all positions below the pivot, by adding multiples of the top row to each row.

We are at the bottom row, so we don't need to repeat anymore. We have arrived at an echelon form.

5. To get from echelon to reduced echelon form:

Starting from the bottom row: for each pivot, add multiples of the row with the pivot to the other rows to create zeroes above the pivot.

$$\left[ \begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right] \quad \begin{array}{l} R_1 - 2R_3 \\ R_2 - R_3 \end{array}$$

$$\left[ \begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right] \quad R_1 + 3R_2$$

Check your answer: [www.wolframalpha.com](http://www.wolframalpha.com)

WolframAlpha computational knowledge engine

Input:  $\text{ref}\{\{0, 3, -6, 6, 4, -5\}, \{3, -7, 8, -5, 8, 9\}, \{1, -3, 4, -3, 2, 5\}\}$

Result:

row reduce  $\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 & 4 \end{pmatrix}$

Getting the solution set from the reduced echelon form:

A **basic variable** is a variable corresponding to a pivot column.  
All other variables are **free variables**.

**Example:**

$$\left[ \begin{array}{cccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 1 & 4 & 4 \end{array} \right] \quad \begin{array}{l} x_1 - 2x_3 + 3x_4 = -24 \\ x_2 - 2x_3 + 2x_4 = -7 \\ x_5 = 4 \end{array}$$

basic variables:  $x_1, x_2, x_5$ , free variables:  $x_3, x_4$ .

The free variables can take any value. These values then uniquely determine the basic variables.

**Example:**

$$\begin{array}{l} x_1 = -24 + 2x_3 - 3x_4 \\ x_2 = -7 + 2x_3 - 2x_4 \\ x_3 = x_3 \\ x_4 = x_4 \\ x_5 = 4 \end{array} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -24 + 2s - 3t \\ -7 + 2s - 2t \\ s \\ t \\ 4 \end{pmatrix}$$

Getting the solution set from the reduced echelon form:

Another example: reduced echelon form is  $\left[ \begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array} \right]$

$0x_1 + 0x_2 + 0x_3 = 15 \longrightarrow$  the system is inconsistent

## Theorem 2: Existence and Uniqueness:

A linear system is consistent if and only if an echelon form of its augmented matrix has **no** row of the form  $[0 \dots 0 | *]$  with  $* \neq 0$ .

If a linear system is consistent, then:

- it has a unique solution if there are no free variables;
- it has infinitely many solutions if there are free variables.

Next week: we talk about this:

