**Def 6.5.11** If $W_1, W_2 \subseteq V$ and $W_1 \oplus W_2 = V$,

then $W_2$ is a <u>complement of $W_1$ (in $V$)</u>

and <u>$\operatorname{codim}(W_1)$</u> $= \dim W_2 = \dim V - \dim W_1$

**Th. 6.5.12** If $V$ is finite-dimensional,

then every subspace $W \subseteq V$ has a complement.

**Proof:** Take $\{\alpha_1, \cdots, \alpha_m\}$ a basis of $W$.

This is a linearly independent set, $\therefore$ can extend

to $\{\alpha_1, \cdots, \alpha_m, \alpha_{m+1}, \cdots, \alpha_n\}$ a basis of $V$.

Then $W' = \operatorname{span}\{\alpha_{m+1}, \cdots, \alpha_n\}$ is a complement of $W$, because:

- $W \cap W' = \{\vec{0}\}$: if $\alpha \in W \cap W'$ then

$\alpha \in W \Rightarrow \alpha = a_1\alpha_1 + \cdots + a_m\alpha_m$ (*)

$\alpha \in W' \Rightarrow \alpha = a_{m+1}\alpha_{m+1} + \cdots + a_n\alpha_n$.

$a_1\alpha_1 + \cdots + a_m\alpha_m = a_{m+1}\alpha_{m+1} + \cdots + a_n\alpha_n$.

$a_1\alpha_1 + \cdots + a_m\alpha_m - a_{m+1}\alpha_{m+1} - \cdots - a_n\alpha_n = \vec{0}$

$\because \{\alpha_1, \cdots, \alpha_n\}$ is a basis of $V$, it's linearly independent $\therefore a_1 = \cdots = a_n = 0$

Substitute into (*): $\alpha = \vec{0}$.

6.5.5

- $W + W' = V$: $W + W' = \operatorname{Span}\{\alpha_1, \cdots, \alpha_m, \alpha_{m+1}, \cdots, \alpha_n\}$
$= V$.

Note: A complement is NOT unique

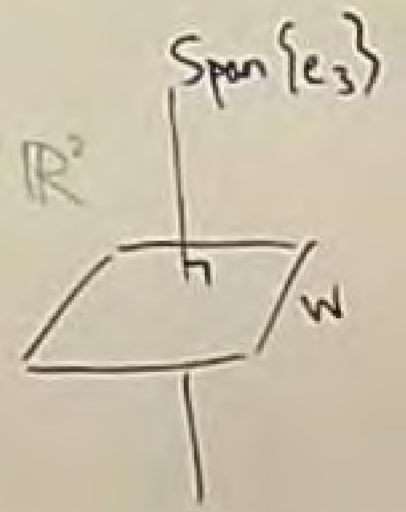
(e.g. many choices of bases in the above proof.)

e.g. $W = \operatorname{Span}\{e_1, e_2\} \leq \mathbb{R}^3$

$\operatorname{Span}\{e_3\}$ is a complement

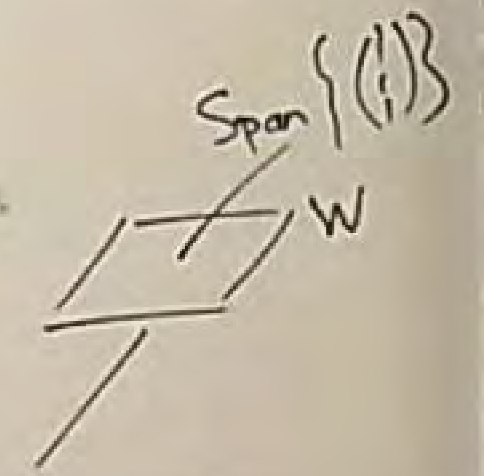
using the basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ in the above proof

(the orthogonal complement)




$\operatorname{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ is another complement

$\because \{e_1, e_2, e_3 + e_1 + e_2\}$ is also a basis of $\mathbb{R}^3$




## Direct sum for many subspaces:

**Def 6.5.13:** $W_1 + \cdots + W_k$ is a direct sum

— i.e. $W_1 \oplus \cdots \oplus W_k$, $\bigoplus_{i=1}^{k} W_i$ —

if $\forall i$, $W_i \cap \sum_{j \neq i} W_j = \{\vec{0}\}$.

e.g. $k=3$: need $W_1 \cap (W_2 + W_3) = \{\vec{0}\}$

$W_2 \cap (W_1 + W_3) = \{\vec{0}\}$

$W_3 \cap (W_1 + W_2) = \{\vec{0}\}$.

Prop 6.5.14 / Th. 6.5.15:

$W_1 + \cdots + W_k$ is direct

$\iff$ $\alpha \in W_1 + \cdots + W_k$ can be written uniquely as $\alpha = \alpha_1 + \cdots + \alpha_k$ with $\alpha_i \in W_i$,

$\iff$ $\dim(W_1 + \cdots + W_k) = \dim W_1 + \cdots + \dim W_k$.

## 7.1 Linear transformations

Def 7.1.1 A function $\sigma : U \to V$ is a linear transformation

if $\forall \alpha, \beta \in U$, $a \in \mathbb{F}$

$$\sigma(a\alpha + \beta) = a\sigma(\alpha) + \sigma(\beta)$$

(Equivalent to $\sigma(a\alpha + b\beta) = a\sigma(\alpha) + b\sigma(\beta)$)

Important consequences:

$$\sigma(a_1\alpha_1 + \cdots + a_n\alpha_n) = a_1\sigma(\alpha_1) + \cdots + a_n\sigma(\alpha_n)$$

$$\sigma(\vec{0}) = \vec{0}$$

Ex: $\sigma : \mathbb{F}^n \to \mathbb{F}^m$ given by multiplication by a fixed $A \in M_{m,n}(\mathbb{F})$

i.e. $\sigma(\alpha) = A\alpha$

e.g. $\sigma : \mathbb{R}^3 \to \mathbb{R}^2$

$$\sigma(\alpha) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \end{pmatrix} \alpha$$

$$\sigma\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 4x + 5z \end{pmatrix} \in \mathbb{R}^2$$

more general:

- $\sigma : M_{n,p}(\mathbb{F}) \longrightarrow M_{m,p}(\mathbb{F})$ given by "left-multiplication by $A \in M_{m,n}(\mathbb{F})$"

$$\sigma(X) = AX$$

similarly: "right-multiplication": $\sigma(X) = XA$

exercise: find the domain and codomain.

- $\sigma : \mathbb{F}[x] \longrightarrow \mathbb{F}[x]$ differentiation.

  i.e. $[\sigma(f)](x) = \frac{df}{dx}$ or $\sigma(f) = f'$

  e.g. $\sigma(a_0 + a_1 x + a_2 x^2 + \cdots) = a_1 + 2a_2 x + \cdots$

How to check this is linear:

$$[\sigma(af+g)](x)$$

input of $\sigma$ is $f$, not $x$

$$= \frac{d}{dx}(af+g)(x)$$

$$= \left(a\frac{df}{dx} + \frac{dg}{dx}\right)(x)$$

$$= \left(a\sigma(f) + \sigma(g)\right)(x).$$