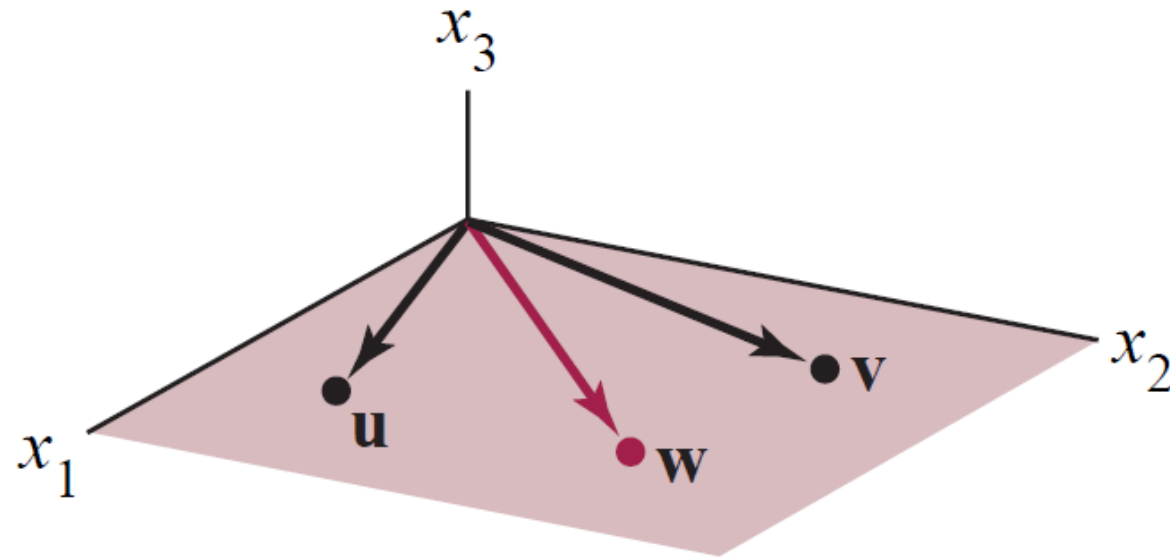


## §1.7: Linear Independence



In this picture, the plane is  $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \text{Span}\{\mathbf{u}, \mathbf{v}\}$ , so we do not need to include  $\mathbf{w}$  to describe this plane.

We can think that  $\mathbf{w}$  is “too similar” to  $\mathbf{u}$  and  $\mathbf{v}$  - and linear dependence is the way to make this idea precise.

**Definition:** A set of vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is *linearly independent* if the **only solution** to the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

is the **trivial solution** ( $x_1 = \cdots = x_p = 0$ ).

The opposite of linearly independent is linearly dependent:

**Definition:** A set of vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is *linearly dependent* if there are weights  $c_1, \dots, c_p$ , **not all zero**, such that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$

The equation  $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$  is a **linear dependence relation**.

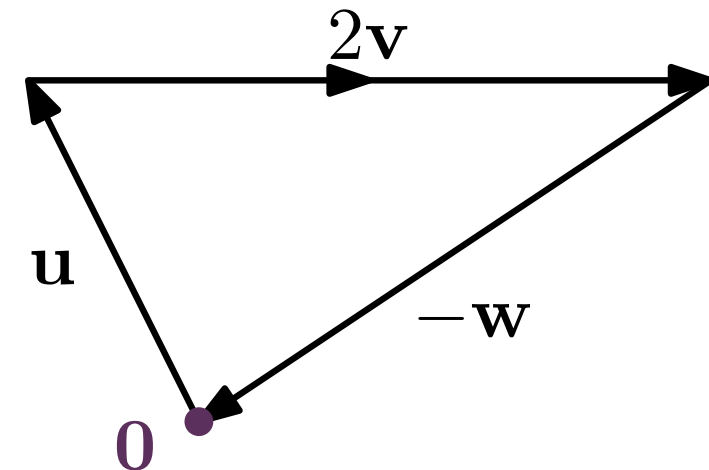
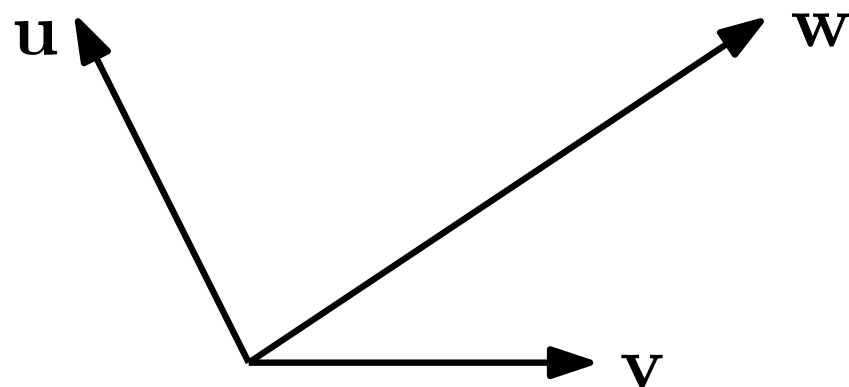
**Definition:** A set of vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is *linearly dependent* if there are weights  $c_1, \dots, c_p$ , **not all zero**, such that

$$c_1 \mathbf{v}_1 + \dots + c_p \mathbf{v}_p = \mathbf{0}.$$

The equation  $c_1 \mathbf{v}_1 + \dots + c_p \mathbf{v}_p = \mathbf{0}$  is a *linear dependence relation*.

A picture of a linear dependence relation: “you can use the given directions to move in a circle”.

$$\mathbf{u} + 2\mathbf{v} - \mathbf{w} = \mathbf{0}$$



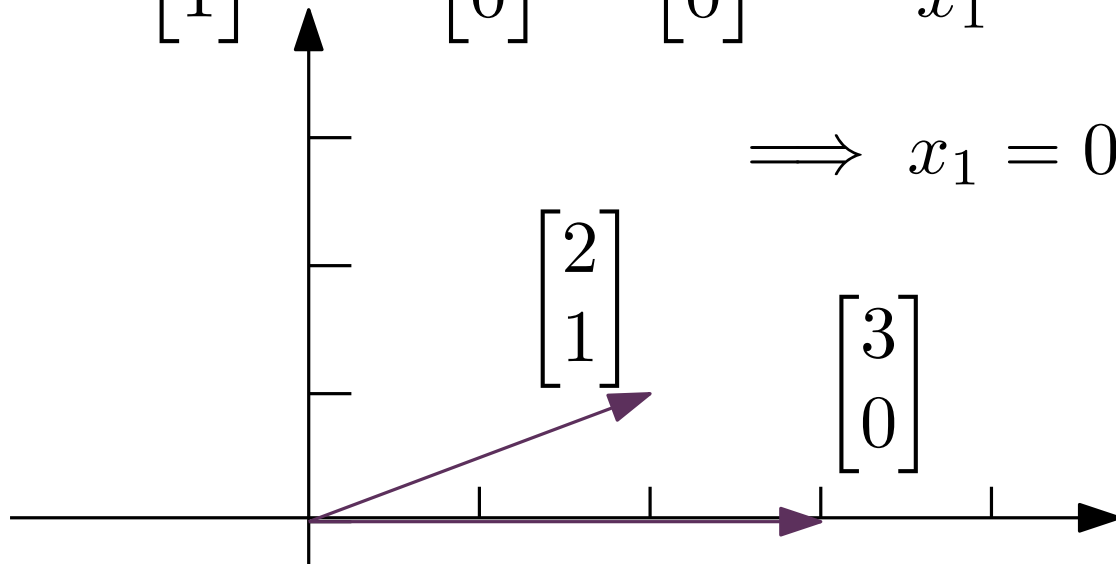
$$x_1 \mathbf{v}_1 + \cdots + x_p \mathbf{v}_p = \mathbf{0}$$

The only solution is  $x_1 = \cdots = x_p = 0$   
 $\rightarrow$  linearly independent

**Example:**  $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}$  is linearly independent because

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} 2x_1 + 3x_2 &= 0 \\ x_1 &= 0 \end{aligned}$$

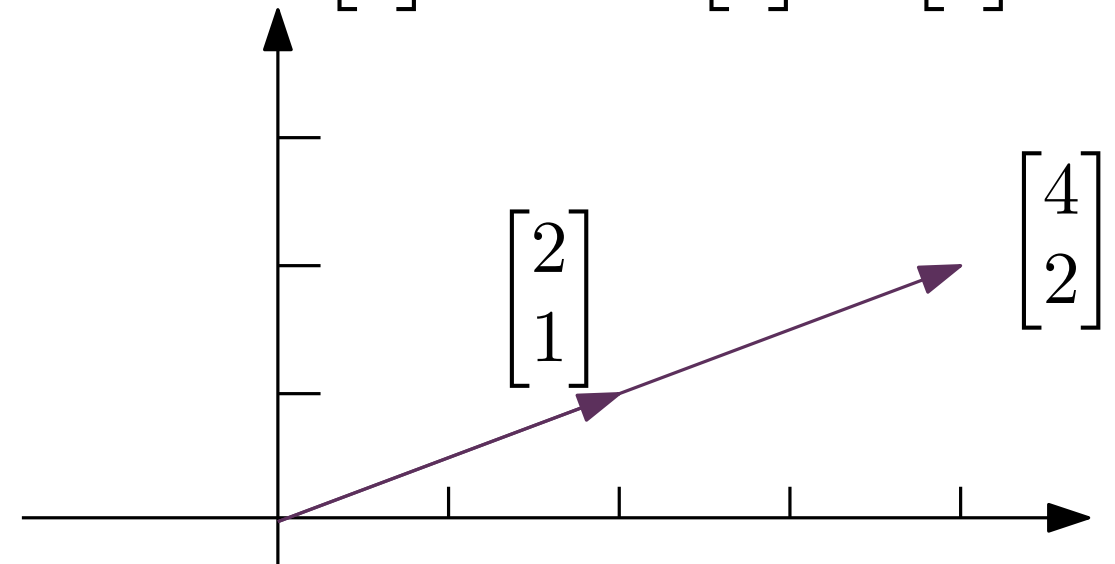
$$\Rightarrow x_1 = 0, x_2 = 0.$$



There is a solution with some  $x_i \neq 0$   
 $\rightarrow$  linearly dependent

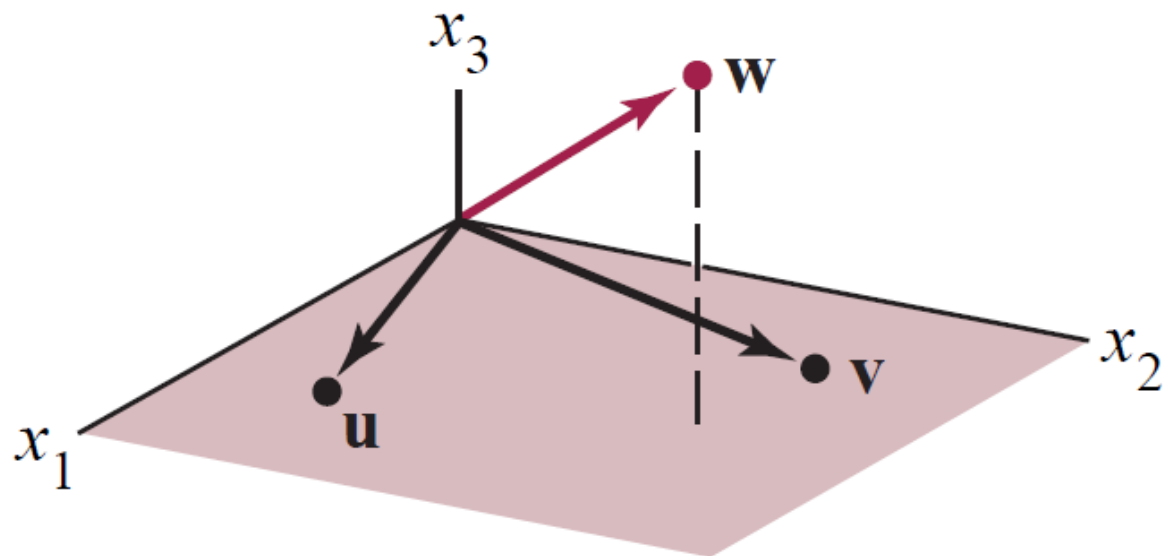
**Example:**  $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$  is linearly dependent because

$$2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$



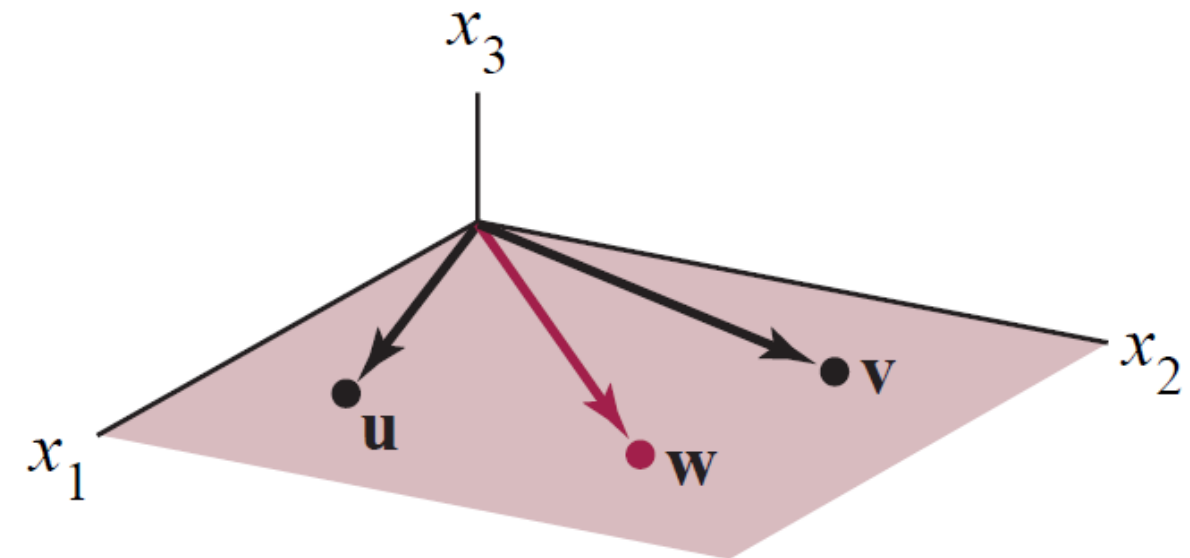
$$x_1 \mathbf{v}_1 + \cdots + x_p \mathbf{v}_p = \mathbf{0}$$

The only solution is  $x_1 = \cdots = x_p = 0$   
 (i.e. unique solution)  
 → linearly independent



**Informally:**  $\mathbf{v}_1, \dots, \mathbf{v}_p$  are in “totally different directions”; there is “no relationship” between  $\mathbf{v}_1, \dots, \mathbf{v}_p$ .

There is a solution with some  $x_i \neq 0$   
 (i.e. infinitely many solutions)  
 → linearly dependent



**Informally:**  $\mathbf{v}_1, \dots, \mathbf{v}_p$  are in “similar directions”

Some easy cases:

- Sets containing the zero vector  $\{\mathbf{0}, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ : then the linear dependence equation is

$$x_1 \mathbf{0} + x_2 \mathbf{v}_2 + \dots + x_p \mathbf{v}_p = \mathbf{0}.$$

A non-trivial solution is

$$(1)\mathbf{0} + (0)\mathbf{v}_2 + \dots + (0)\mathbf{v}_p = \mathbf{0},$$

so such a set is **linearly dependent** (it doesn't matter what  $\mathbf{v}_2, \dots, \mathbf{v}_p$  are).

- Sets containing one vector  $\{\mathbf{v}\}$ : then the linear dependence equation is

$$x\mathbf{v} = \mathbf{0} \quad \text{i.e.} \quad \begin{bmatrix} xv_1 \\ \vdots \\ xv_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If some  $v_i \neq 0$ , then  $x = 0$  is the only solution. So  $\{\mathbf{v}\}$  is **linearly independent** if  $\mathbf{v} \neq \mathbf{0}$ .

Some easy cases:

- Sets containing two vectors  $\{\mathbf{u}, \mathbf{v}\}$ : then the linear dependence equation is

$$x_1 \mathbf{u} + x_2 \mathbf{v} = \mathbf{0}.$$

Using the same argument as in the example on p4, we can show that, if  $\mathbf{v} = c\mathbf{u}$  for any  $c$ , then  $\mathbf{u}$  and  $\mathbf{v}$  are linearly dependent:

$$\mathbf{v} = c\mathbf{u} \quad \text{means} \quad c\mathbf{u} + (-1)\mathbf{v} = \mathbf{0}.$$

The same argument applies if  $\mathbf{u} = d\mathbf{v}$  for any  $d$ .

Is this the only way in which two vectors can be linearly dependent?

Suppose we have  $x_1 \mathbf{u} + x_2 \mathbf{v} = \mathbf{0}$  and  $x_1, x_2$  are not both zero.

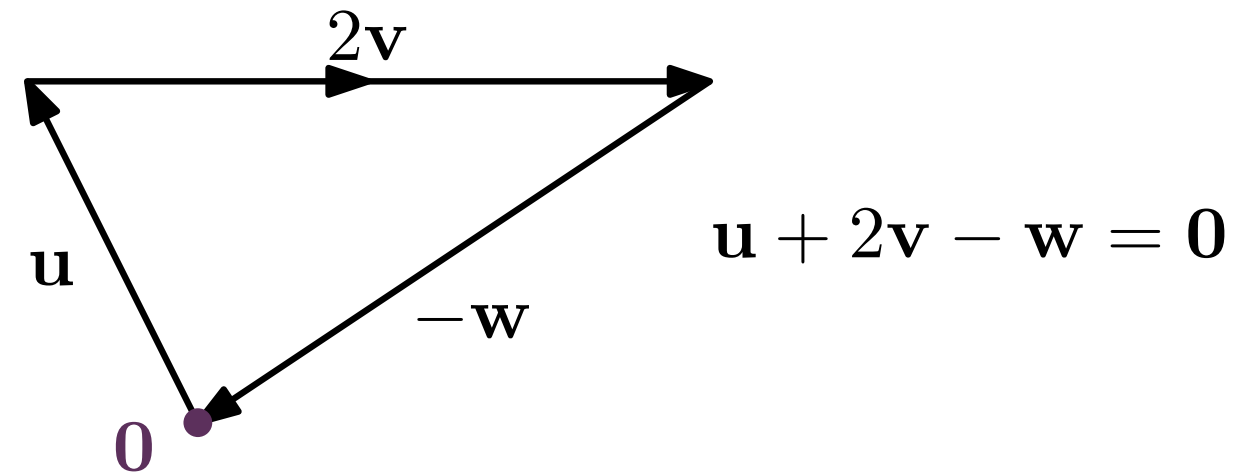
If  $x_1 \neq 0$ , then we can divide by it:  $\mathbf{u} = \frac{-x_2}{x_1} \mathbf{v}$ .

Similarly, if  $x_2 \neq 0$ , then  $\mathbf{v} = \frac{-x_1}{x_2} \mathbf{u}$ .

So  $\{\mathbf{u}, \mathbf{v}\}$  is linearly dependent if and only if one of the vectors is a multiple of the other, i.e.  $\mathbf{u}, \mathbf{v}$  are in the same or opposite direction.

When there are more vectors, it is hard to tell quickly if a set is linearly independent or dependent.

As shown in this example from p3, three vectors can be linearly dependent without any of them being a multiple of any other vector.



The correct generalisation of the two-vector case is the following: a set of vectors is **linearly dependent** if and only if **one of the vectors is a linear combination of the others**. (More specifically: if the weight  $x_i$  in the linear dependency relation  $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$  is non-zero, then  $\mathbf{v}_i$  is a linear combination of the other  $\mathbf{v}$ s, by the same argument as in the case of two vectors.)



How to determine if  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$  is linearly independent:

**EXAMPLE** Let  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$ ,  $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$ .

- Determine if  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly independent.
- If possible, find a linear dependence relation among  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ .

*Solution:* (a)  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly independent if \_\_\_\_\_

Augmented matrix:

$$\left[ \begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{array} \right] \quad \text{row reduces to} \quad \left[ \begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$x_3$  is a free variable  $\Rightarrow$  there are nontrivial solutions.

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is \_\_\_\_\_

(b) Reduced echelon form:  $\left[ \begin{array}{ccc|c} 1 & 0 & 33 & 0 \\ 0 & 1 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$

Let  $x_3 = \underline{\hspace{1cm}}$  (any nonzero number). Then  $x_1 = \underline{\hspace{1cm}}$  and  $x_2 = \underline{\hspace{1cm}}$ .

$$\underline{\hspace{1cm}} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \underline{\hspace{1cm}} \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + \underline{\hspace{1cm}} \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\underline{\hspace{1cm}} \mathbf{v}_1 + \underline{\hspace{1cm}} \mathbf{v}_2 + \underline{\hspace{1cm}} \mathbf{v}_3 = \mathbf{0}$$

(one possible linear dependence relation)

A non-trivial solution to  $A\mathbf{x} = \mathbf{0}$  is a linear dependence relation between the columns of  $A$ :  $A\mathbf{x} = \mathbf{0}$  means  $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$ .

**Theorem: Uniqueness of solutions for linear systems:** For a matrix  $A$ , the following are equivalent:

- a.  $A\mathbf{x} = \mathbf{0}$  has no non-trivial solution.
- b. If  $A\mathbf{x} = \mathbf{b}$  is consistent, then it has a unique solution.
- c. The columns of  $A$  are linearly independent.
- d.  $\text{rref}(A)$  has a pivot in every column (i.e. all variables are basic).

In particular: the row reduction algorithm produces at most one pivot in each row of  $\text{rref}(A)$ . So, if  $A$  has more columns than rows (a “fat” matrix), then  $\text{rref}(A)$  cannot have a pivot in every column.

So a set of **more than  $n$  vectors in  $\mathbb{R}^n$**  is always **linearly dependent**.

Exercise: Combine this with the Theorem of Existence of Solutions (Week 2 p23) to show that a set of  $n$  linearly independent vectors span  $\mathbb{R}^n$ .

**Theorem: Uniqueness of solutions for linear systems:** For a matrix  $A$ , the following are equivalent:

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- d.  $\text{rref}(A)$  has a pivot in every column (i.e. all variables are basic).

Study tip: now that we're working with different types of mathematical objects (matrices, vectors, equations, numbers), you should be careful which properties apply to which objects: e.g. linear independence applies to a set of vectors, not to

a matrix (at least not until Chapter 4). Do **not** say “ $\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix}$  is linearly

independent” when you mean “ $\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} \right\}$  are linearly dependent”.

Abstract proofs of linear dependence and independence:

To prove that  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is linearly dependent, we need to find **one** choice of non-zero weights  $c_1, \dots, c_p$  such that  $c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p = \mathbf{0}$ . The technique that we saw in Week 2 applies here: express the information in the question as mathematical formulae, then reorganise the equations until we have something of the form  $c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p = \mathbf{0}$ .

A proof that  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is linearly independent has a different structure. Now we need to show that the **only** choice of weights  $c_1, \dots, c_p$  such that  $c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p = \mathbf{0}$  is  $c_1 = \dots = c_p = 0$ . So we need to start with the equation  $c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p = \mathbf{0}$ , and solve it using the information given in the question.

(See p4 for this difference in a numerical example.)

**EXAMPLE:** Suppose  $\{\mathbf{u}, \mathbf{v}\}$  is linearly independent. Show that  $\{\mathbf{u}, \mathbf{u} + \mathbf{v}\}$  is linearly independent.

What we know:

What we want to show:

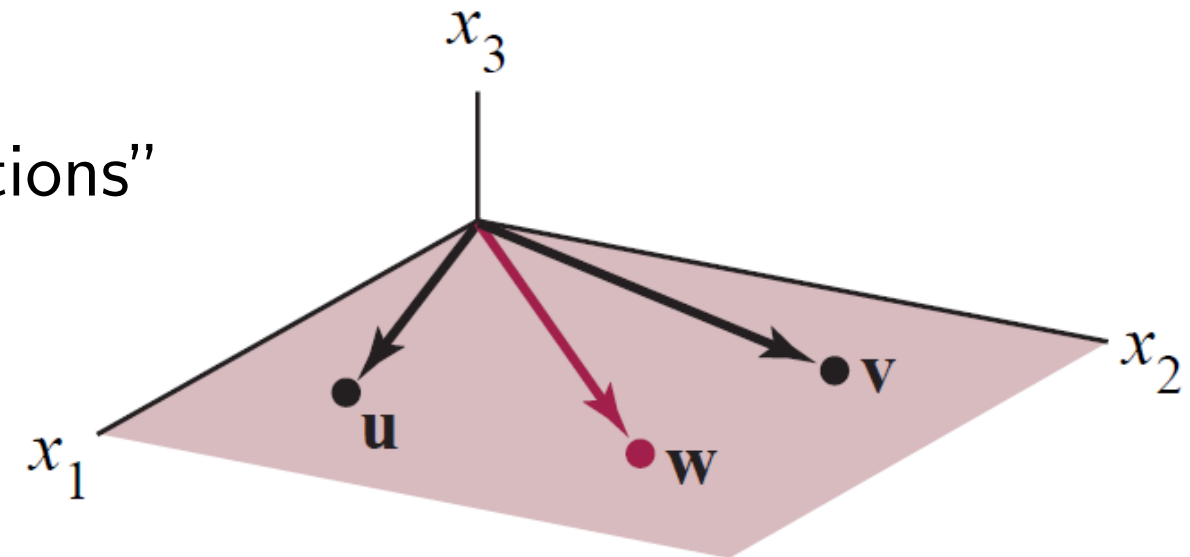
## Partial summary of linear dependence:

The definition:  $x_1 \mathbf{v}_1 + \cdots + x_p \mathbf{v}_p = \mathbf{0}$  has a non-trivial solution (not all  $x_i$  are zero); equivalently, it has infinitely many solutions.

Equivalently: **one** of the vectors is a linear combination of the others (see p8, also Theorem 7 in textbook). But it might not be the case that every vector in the set is a linear combination of the others (see ex. sheet #5 q2c).

Computation:  $\text{rref} \left( \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_p \\ | & & | \end{bmatrix} \right)$  has at least one free variable.

Informal idea: the vectors are in “similar directions”



## Partial summary of linear dependence (continued):

Easy examples:

- Sets containing the zero vector;
- Sets containing “too many” vectors (more than  $n$  vectors in  $\mathbb{R}^n$ );
- Multiples of vectors: e.g.  $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$  (this is the only possibility if the set has two vectors);
- Other examples: e.g.  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ . Make your own examples!

Adding vectors to a linearly dependent set still makes a linearly dependent set (see ex. sheet #5 Q2d).

Equivalent: removing vectors from a linearly independent set still makes a linearly independent set (because  $P$  implies  $Q$  is equivalent to  $(\text{not } Q)$  implies  $(\text{not } P)$  - this is the **contrapositive**).

## Study tips:

- Linear independence will appear again in many topics throughout the class, so I suggest you add to this summary throughout the semester, so you can see the connections between linear independence and the other topics.
- Topic summaries like this one is useful for exam revision, but even more useful is [making these summaries yourself](#). I encourage you to use my summary as a template for your own summaries of the other topics.
- Examples can be useful for solving true/false questions: if a true/false question is about a linear dependent set, try it on the examples on the previous page. Try to make a counterexample, and if you can't, it will give you some idea of why the statement is true.