We can describe a bilinear form using a matrix:

Suppose $\mathcal{A}=\{\alpha_1,\alpha_2,\alpha_3\}$ is a basis of V.

(order is important)

and $\alpha = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3$ i.e. $[\alpha]_{\mathcal{A}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\beta = y_1\alpha_1 + y_2\alpha_2 + y_3\alpha_3$ $\quad [\beta]_{\mathcal{A}} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Then
$$\begin{aligned} f(\alpha,\beta) &= f(x_1\alpha_1+x_2\alpha_2+x_3\alpha_3,\ y_1\alpha_1+y_2\alpha_2+y_3\alpha_3) \\ &= x_1 f(\alpha_1, y_1\alpha_1+y_2\alpha_2+y_3\alpha_3) \\ &\quad + x_2 f(\alpha_2, y_1\alpha_1+y_2\alpha_2+y_3\alpha_3) \\ &\quad + x_3 f(\alpha_3, y_1\alpha_1+y_2\alpha_2+y_3\alpha_3) \end{aligned}$$

$$\begin{aligned} &= x_1[y_1 f(\alpha_1,\alpha_1) + y_2 f(\alpha_1,\alpha_2) + y_3 f(\alpha_1,\alpha_3)] \\ &\quad + x_2[y_1 f(\alpha_2,\alpha_1) + \cdots \\ &\quad + x_3 \cdots \\ &= \sum_{i,j} x_i y_j f(\alpha_i,\alpha_j) \quad (*) \end{aligned}$$

we can compute $f(\alpha,\beta)$ for any α,β using only these numbers.

Define $\{f\}_{\mathcal{A}}$ to be the matrix with $f(\alpha_i,\alpha_j)$ in row i, column j.

This is the matrix representing f relative to $\mathcal{A}$.

Then $f(\alpha,\beta) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \{f\}_{\mathcal{A}} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = [\alpha]_{\mathcal{A}}^T \{f\}_{\mathcal{A}} [\beta]_{\mathcal{A}}$

from (*)

Ex: $f(\alpha,\beta) = \alpha\cdot\beta$ on $\mathbb{R}^3$, $\mathcal{A}=\{e_1,e_2,e_3\}$.

$$\{f\}_{\mathcal{A}} = \begin{pmatrix} e_1\cdot e_1 & e_1\cdot e_2 & e_1\cdot e_3 \\ e_2\cdot e_1 & e_2\cdot e_2 & e_2\cdot e_3 \\ e_3\cdot e_1 & e_3\cdot e_2 & e_3\cdot e_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Check: $f(\alpha,\beta) = \alpha^T \{f\}_{\mathcal{A}} \beta = \alpha^T I \beta = \alpha^T\beta$ ✓

Ex: $g\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = x_1 y_1 + 2x_1 y_2 + x_3 y_1$. on $\mathbb{R}^3$, $A = \{e_1, e_2, e_3\}$.

$$\{g\}_A = \begin{pmatrix} g(e_1,e_1) & g(e_1,e_2) & g(e_1,e_3) \\ g(e_2,e_1) & g(e_2,e_2) & g(e_2,e_3) \\ g(e_3,e_1) & g(e_3,e_2) & g(e_3,e_3) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

row i, column j is the coefficient of $x_i y_j$

Ex $V = P_{<2}(\mathbb{R})$ $\mathcal{A} = \{1, x\}$

$$k(F,G) = \int_0^1 F(x)\,G(x)\,dx$$

$$\{k\}_{\mathcal{A}} = \begin{pmatrix} k(1,1) & k(1,x) \\ k(x,1) & k(x,x) \end{pmatrix}$$

$$k(1,1) = \int_0^1 1\cdot 1\,dx = x\Big|_0^1 = 1$$

$$k(1,x) = \int_0^1 1\cdot x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$k(x,1) = \int_0^1 x\cdot 1\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$k(x,x) = \int_0^1 x\cdot x\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\therefore \{k\}_{\mathcal{A}} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

To calculate e.g. $k(-1+x, 1+x)$:

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{-2}{3}$$

Check: $\int_0^1 (-1+x)(1+x)\,dx$

$$= \int_0^1 -1+x^2\,dx = -x + \frac{x^3}{3}\Big|_0^1 = \frac{-2}{3}.$$

A bilinear form f is <u>symmetric</u> if $f(\alpha,\beta)=f(\beta,\alpha)$,

equivalently, $[f]_{\mathcal{A}}$ is a symmetric matrix.

Ex: in the previous examples, f, k are symmetric,

g is not.

Def 9.4.11: A function $q: V \to \mathbb{F}$ is a <u>quadratic form</u> if $q(\alpha)=f(\alpha,\alpha)$ for some bilinear form f.

Ex: from $f(\alpha,\beta)=\alpha\cdot\beta$:

$$q(\alpha) = \alpha\cdot\alpha = \|\alpha\|^2$$

i.e. $$q\left(\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right) = x_1^2+x_2^2+x_3^2$$

Ex: from $k(F,G)$: $q(F)=\int_0^1 F(x)^2\,dx$.

Rem 9.4.12: Given any bilinear form f, define

$$f_{sym}(\alpha,\beta)=\frac{1}{2}\big(f(\alpha,\beta)+f(\beta,\alpha)\big).$$

this is symmetric.

and $f(\alpha,\alpha)=f_{sym}(\alpha,\alpha)$

i.e. f, f_{sym} make the same q

$\therefore$ we can assume that q comes from a symmetric f.