There are two ways to calculate a definite integral by substitution:

1. Find the indefinite integral and then substitute in the limits for x;
2. (Usually faster) Change the limits into limits for u.

Example: Evaluate $\int_0^1 x\sqrt{1+x^2}\,dx$.

also an example of how to write down "easy" substitutions

$$= \int_0^1 \frac{\sqrt{1+x^2}}{2}\,(2x\,dx)$$

substitution
$u = 1+x^2$
$du = 2x\,dx$

$$= \left.\frac{(1+x^2)^{3/2}}{\cancel{2}\,3/\cancel{2}}\right|_0^1$$

$$= \frac{2^{3/2}}{3} - \frac{1}{3} = \frac{1}{3}\left(2\sqrt{2}-1\right)$$

Note that the final two steps in method 1 are to change the indefinite integral from us to x, then substitute the limits of x. In method 2 below, we combine these two steps – simply substitute the corresponding limits for u.

Example: Evaluate $\displaystyle\int_0^1 x\sqrt{1+x^2}\,dx$.

$$= \int_{x=0}^{x=1} x\sqrt{1+x^2}\ \frac{du}{2x}$$

$$= \int_1^2 \frac{u^{1/2}}{2}\,du$$

$$= \left.\frac{u^{3/2}}{2\cdot\frac{3}{2}}\right|_1^2 = \frac{2^{3/2}}{3} - \frac{1}{3}$$

substitution $u = 1+x^2$ ①

$du = 2x\ dx$ ②

$\frac{du}{2x} = dx$

$x=0 \to u = 1+0^2 = 1$ ③

$x=1 \to u = 1+1^2 = 2$ ④

Harder example: Evaluate $\int_0^1 x^3\sqrt{1-x^2}\,dx$.

$$= \int_0^1 \underline{\frac{x^2}{-2}} \sqrt{1-x^2}\,(-2x\,dx)$$

substitution: $u = 1-x^2 \longrightarrow x^2 = 1-u$

$du = -2x\,dx$

$$= \int_1^0 \frac{1-u}{-2}\, u^{1/2}\, du$$

$x=0 \to u = 1-0^2 = 1$

$x=1 \to u = 1-1^2 = 0$

$$= \int_1^0 \frac{-u^{1/2}}{2} + \frac{u^{3/2}}{2}\, du$$

$$= \frac{-u^{3/2}}{2\,3/2} + \frac{u^{5/2}}{5/2}\Bigg|_1^0 = 0 - \left(-\frac{1}{3} + \frac{1}{5}\right) = \frac{2}{15}.$$

This strategy will work for $\int x^n\sqrt{1-x^2}\,dx$ when n is odd.

$$= \int \underbrace{\frac{x^{n-1}}{-2}}\sqrt{1-x^2}\,(-2x\,dx)$$

even power of $x \to$ can write as a polynomial in u.

Harder example: Evaluate $\int_0^1 \frac{x^2}{1+x^6}\,dx$.

An example of "real problem solving"

$$= \int_0^1 \frac{\cancel{x^2}}{1+x^6}\frac{1}{6x^{\cancel{5}\,3}}(6x^5\,dx)$$

substitution
$u = 1+x^6$
$du = 6x^5\,dx$

$u-1 = x^6$
$\sqrt{u-1} = x^3$
positive square root, because $x \in [0,1]$.

$$= \int_1^2 \frac{1}{6u\sqrt{u-1}}\,du$$

$x=0 \to u=1$
$x=1 \to u=2$

Try again:
$$\int_0^1 \frac{x^2}{1+x^6}\frac{1}{2x}(2x\,dx)$$
$u = x^2$
$du = 2x\,dx$
$$= \int_0^1 \frac{u}{1+u^3}\frac{1}{2\sqrt{u}}\,du$$
does not seem useful

$$= \int_0^1 \frac{1}{6(v+1)\sqrt{v}}\,dv$$

$v = u-1$
$dv = du$

$u = v+1$

$u=1 \to v=0$
$u=2 \to v=1$

$\sqrt{\ }$ in denominator is bad

$$= \int_0^1 \frac{1}{6(w^2+1)\cancel{w}}\cancel{2w}\,dw$$

$w = \sqrt{u-1}$
$w^2 = u-1 \longrightarrow u = w^2+1$
$2w\,dw = du$

$u=1 \to w=0$
$u=2 \to w=\sqrt{2-1}=1$

$$= \int_0^1 \frac{1}{3(w^2+1)}\,dw$$

$$= \frac{1}{3}\tan^{-1}w\Big|_0^1 = \frac{1}{3}\tan^{-1}(1) = \frac{1}{3}\frac{\pi}{4} = \frac{\pi}{12}$$

To do this with one substitution: $w = \sqrt{u-1} = \sqrt{(1+x^6)-1} = \sqrt{x^6} = x^3$

What we learned: if denominator is $1+g(x)^2$, try $u = g(x)$.

Example: Find the area of the region bounded by $y = x^2 - 4$ and $y = -x^2 + 2x$.

$= -x(x-2)$

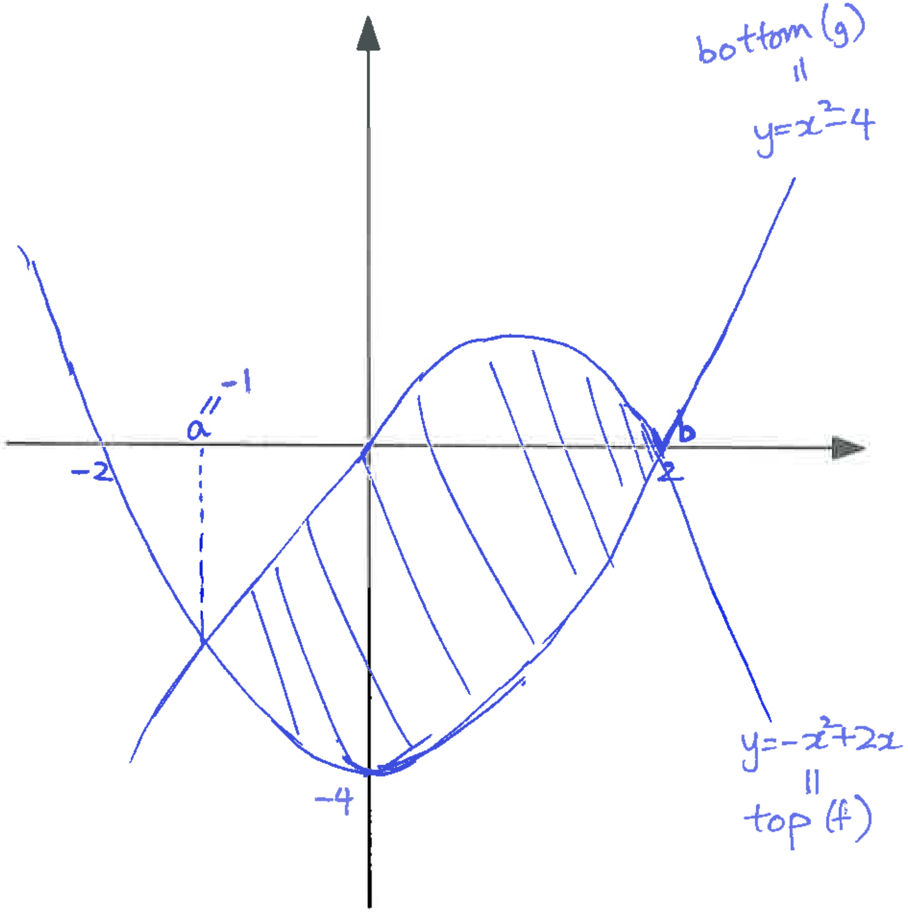
To find a:

Intersections: $x^2 - 4 = -x^2 + 2x$

$2x^2 - 2x - 4 = 0$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0 \longrightarrow x = 2$ and $x = -1$.



$$\text{Area} = \int_{-1}^{2} (-x^2 + 2x) - (x^2 - 4)\, dx$$

$$= \int_{-1}^{2} -2x^2 + 2x + 4\, dx$$

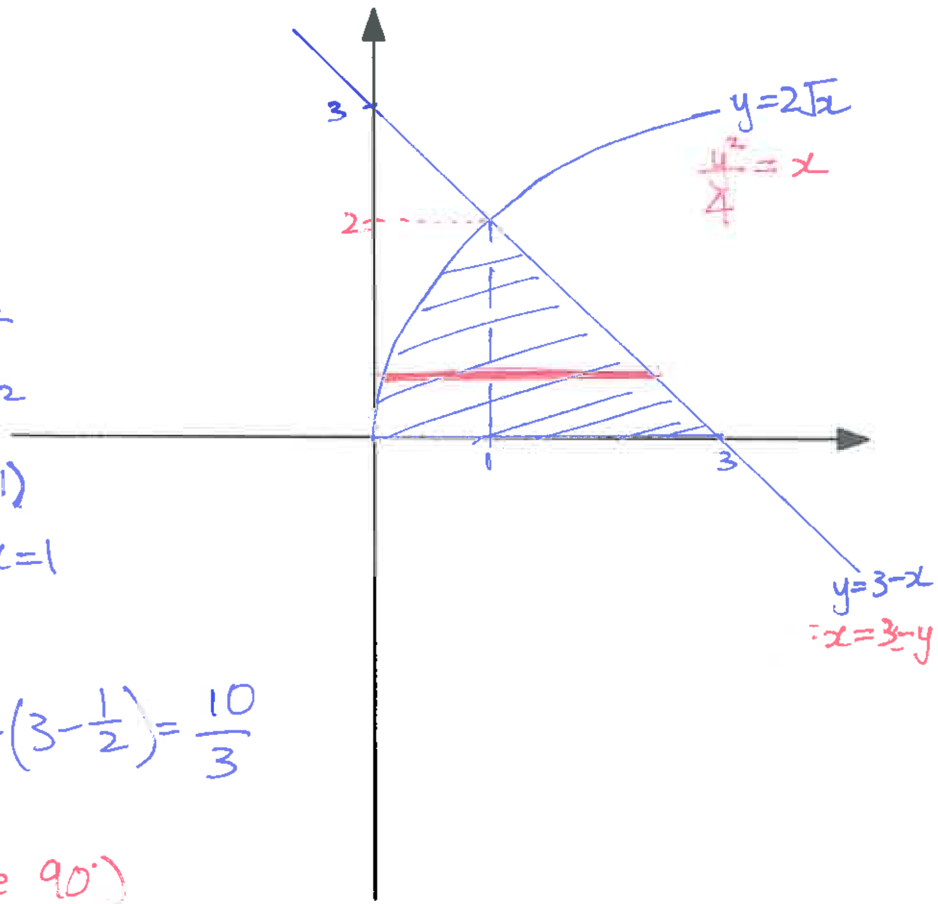
$$= \frac{-2x^3}{3} + x^2 + 4x \Big|_{-1}^{2}$$

$$= \left(\frac{-16}{3} + 4 + 4\right) - \left(\frac{+2}{3} + 1 - 4\right) = 9$$

Example: Find the area of the region bounded by $y = 2\sqrt{x}$, $y = 3 - x$ and $y = 0$.

Problem: the "top" of the region is sometimes given by one function, sometimes another.

Solution: divide the region



At the intersections:
$$2\sqrt{x} = 3-x$$
$$4x = (3-x)^2 = 9-6x+x^2$$
$$0 = 9-10x+x^2$$
$$0 = (x-9)(x-1)$$
$$x=1$$

$$\text{Area} = \int_0^1 2\sqrt{x}\,dx + \int_1^3 3-x\,dx$$
$$= \frac{2x^{3/2}}{3/2}\Big|_0^1 + \left(3x-\frac{x^2}{2}\right)\Big|_1^3 = \frac{4}{3} + \left(9-\frac{9}{2}\right)-\left(3-\frac{1}{2}\right) = \frac{10}{3}$$

Alternative solution: integrate in y (ie. rotate your picture 90°)

$$\text{Area} = \int_0^2 \underbrace{(3-y) - \frac{y^2}{4}}_{\text{right}-\text{left}}\,dy = \left(3y - \frac{y^2}{2} - \frac{y^3}{4\cdot 3}\right)\Big|_0^2 = 6 - \frac{4}{2} - \frac{8}{12} = \frac{10}{3}$$