

You must justify your answers to receive full credit.

1. Consider  $\mathbb{C}^2$  with the standard dot product. Let

$$\alpha = \begin{pmatrix} 2 + 3i \\ -1 - i \end{pmatrix}, \beta = \begin{pmatrix} -2i \\ 2 \end{pmatrix}, \gamma = \begin{pmatrix} -1 + 4i \\ 2 - i \end{pmatrix}.$$

Calculate the following quantities:

- a)  $\alpha \cdot \beta$
- b)  $\beta \cdot \alpha$
- c) the length of  $\alpha$
- d) the distance between  $\beta$  and  $\gamma$ .

2. Prove the following “parallelogram law”:

$$\|\alpha + \beta\|^2 + \|\alpha - \beta\|^2 = 2\|\alpha\|^2 + 2\|\beta\|^2.$$

(You should use a general inner product  $\langle \cdot, \cdot \rangle$ .)

3. Consider  $P_{<3}(\mathbb{R})$ , the vector space of polynomials over  $\mathbb{R}$  of degree less than 3, with inner product

$$\langle f, g \rangle = \int_{-1}^1 (1 + 3x) f(x) g(x) dx.$$

Construct an orthogonal basis for  $P_{<3}(\mathbb{R})$ . You may use the following:

$$\int_{-1}^1 x^n dx = \begin{cases} 0 & \text{if } n \text{ odd} \\ \frac{2}{n+1} & \text{if } n \text{ even.} \end{cases}$$

(Click here for a hint)

4. Consider  $M_{2,2}(\mathbb{R})$  with the inner product

$$\langle A, B \rangle = \text{Tr}(A^T B).$$

Consider the subspace  $W$  with orthogonal basis  $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\}$ .

- a) Calculate the orthogonal projection of  $\begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}$  onto  $W$ .
- b) Find the closest point in  $W$  to  $\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$ .

5. Recall that, for a square matrix  $X$ , its trace  $\text{Tr}(X)$  is the sum of the diagonal entries of  $X$ .
- a) Show that  $\text{Tr}(BC) = \text{Tr}(CB)$  for all  $B, C \in M_{2,2}(\mathbb{F})$ .
  - b) It is true that  $\text{Tr}(BC) = \text{Tr}(CB)$  for all  $B, C \in M_{n,n}(\mathbb{F})$  (you do not need to prove this). Show that, if  $A, J \in M_{n,n}(\mathbb{F})$  are similar matrices, then  $\text{Tr } A = \text{Tr } J$ .
  - c) Now suppose  $\mathbb{F} = \mathbb{C}$ , i.e.  **$A$  is an  $n \times n$  complex matrix**. Using part b or otherwise, explain why  $\text{Tr } A$  is the sum (with multiplicity) of the eigenvalues of  $A$ .

6. Prove the (simplified) Riesz Representation Theorem: let  $\Phi : V \rightarrow \hat{V}$  be given by  $\Phi(\gamma) = \langle \gamma, - \rangle$ . Show that:
- a)  $\Phi$  is conjugate-linear, i.e.  $\Phi(a\gamma + \gamma') = \bar{a}\Phi(\gamma) + \Phi(\gamma')$ .
  - b)  $\Phi$  is injective, i.e.  $\Phi(\gamma) = \Phi(\gamma')$  means  $\gamma = \gamma'$ . (Please give full details.)

7. Let  $V = P_{<2}(\mathbb{R})$ , the vector space of polynomials over  $\mathbb{R}$  of degree less than 2, with inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x) dx.$$

Define  $\phi \in \hat{V}$  by  $\phi(g) = g(-1)$ .

- a) By direct calculation, find  $f \in V$  such that  $\langle f, g \rangle = \phi(g)$ .  
You are given that  $\mathcal{A} = \{1, \sqrt{3} - 2\sqrt{3}x\}$  is an orthonormal basis for  $V$  (you do **not** need to check this).
  - b) Find the same  $f$  as in part a, using the formula for  $\Lambda(\phi)$  from class.
8. Let  $U_1, U_2, W_1, W_2$  be subspaces of an inner product space  $V$ .
- a) Prove that, if  $U_1 \subseteq U_2$ , then  $U_2^\perp \subseteq U_1^\perp$ .
  - b) Prove that  $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$ .

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**Optional questions** If you attempted seriously all the above questions, then your scores for the following questions may replace any lower scores for two of the above questions.

9. (Sturm-Liouville theory for PDEs) Let  $V$  be the subspace of continuous functions on  $[0, 1]$  defined by  $V = \{f \in C^0([0, 1]) \mid f'' \text{ is continuous and } f(0) = f(1) = 0\}$ . Let  $\sigma : V \rightarrow V$  be the double-derivative operator, i.e.  $\sigma(f) = f''$ .

- a) Let  $V$  have inner product given by

$$\langle f, g \rangle = \int_0^1 f(x)g(x) dx.$$

By integration by parts twice, or otherwise, show that  $\sigma$  is self-adjoint, i.e.  $\langle f, \sigma(g) \rangle = \langle \sigma(f), g \rangle$ .

- b) Now let  $V$  have inner product given by

$$\langle f, g \rangle = \int_0^1 e^x f(x)g(x) dx.$$

By integration by parts twice, or otherwise, find the function  $\tau : V \rightarrow V$  such that  $\langle f, \sigma(g) \rangle = \langle \tau(f), g \rangle$ .

10. Let  $V$  be an inner product space, and let  $f : V \rightarrow V$  be a self-adjoint function, i.e. for all  $\alpha, \beta \in V$ ,

$$\langle f(\alpha), \beta \rangle = \langle \alpha, f(\beta) \rangle.$$

Show that  $f$  is linear.

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