How $[\sigma]_{\mathcal{B}\leftarrow\mathcal{A}}$ is useful:

($\alpha \in U$)

if $\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n$ i.e. $[\alpha]_{\mathcal{A}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

then $[\sigma(\alpha)]_{\mathcal{B}} = [\sigma(a_1\alpha_1 + \cdots + a_n\alpha_n)]_{\mathcal{B}}$

$= [a_1\sigma(\alpha_1) + \cdots + a_n\sigma(\alpha_n)]_{\mathcal{B}}$

$= a_1[\sigma(\alpha_1)]_{\mathcal{B}} + \cdots + a_n[\sigma(\alpha_n)]_{\mathcal{B}}$

$= \begin{pmatrix} | & & | \\ [\sigma(\alpha_1)]_{\mathcal{B}} & \cdots & [\sigma(\alpha_n)]_{\mathcal{B}} \\ | & & | \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

Th 7.2.11. $[\sigma(\alpha)]_{\mathcal{B}} = [\sigma]_{\mathcal{B}\leftarrow\mathcal{A}} [\alpha]_{\mathcal{A}}$

- To compute $\sigma(\alpha)$:

$\text{Decoord}_{\mathcal{B}}\left([\sigma]_{\mathcal{B}\leftarrow\mathcal{A}} \, [\alpha]_{\mathcal{A}} \right)$ (③ ② ①)

e.g. for previous example:

$\sigma(1+3x) =$

$= \text{Decoord}_{\mathcal{B}} \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$= \text{Decoord}_{\mathcal{B}} \begin{pmatrix} 2 \\ 6 \\ 1 \\ 3 \end{pmatrix}$

$= 2 + 6x + x^2 + 3x^3$

- To solve $\sigma(?) = y$:

same as $[\sigma]_{\mathcal{B}\leftarrow\mathcal{A}} [?]_{\mathcal{A}} = [y]_{\mathcal{B}}$

$\therefore$ solve $[\sigma]_{\mathcal{B}\leftarrow\mathcal{A}} X = [y]_{\mathcal{B}}$ for X, by e.g. row reduction

then apply $\text{Decoord}_{\mathcal{A}}$ to the solution set.

In particular, when $y = \vec{0}$:

- $\ker\sigma = \text{Decoord}_{\mathcal{B}}\left(\text{Nul} [\sigma]_{\mathcal{B}\leftarrow\mathcal{A}} \right)$ (③ ② ①)

- $\text{range}\,\sigma = \text{Decoord}_{\mathcal{B}}\left(\text{Col} [\sigma]_{\mathcal{B}\leftarrow\mathcal{A}} \right)$

Reason:

$\text{range}\,\sigma = \text{Span}\{\sigma(\alpha_1), \cdots, \sigma(\alpha_n)\}$ by Lemma.

$= \text{Span}\{ \text{Decoord}_{\mathcal{B}}([\sigma(\alpha_i)]_{\mathcal{B}}) \}$

$= \text{Decoord}_{\mathcal{B}}\left(\text{Span}\{ [\sigma(\alpha_i)]_{\mathcal{B}} \} \right)$

$= \text{Decoord}_{\mathcal{B}}$ (span of the columns of $[\sigma]_{\mathcal{B}\leftarrow\mathcal{A}}$)

- many other problems

Th 7.2.9 if $\sigma \in L(U,V)$, $\tau \in (V,W)$ and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are bases of U, V, W, then

$$\underset{C \leftarrow A}{[\tau \circ \sigma]_{\mathcal{A}}} = \underset{C \leftarrow B}{[\tau]_{\mathcal{B}}} \underset{B \leftarrow A}{[\sigma]_{\mathcal{A}}}$$

Proof: Given $\alpha \in U$, to find $(\tau \circ \sigma)(\alpha)$:

$$[(\tau \circ \sigma)(\alpha)]_C = \underset{C \leftarrow A}{[\tau \circ \sigma]_{\mathcal{A}}} [\alpha]_{\mathcal{A}}.$$

Also: let $\beta = \sigma(\alpha)$. Then $(\tau \circ \sigma)\alpha = \tau(\beta)$

$$[(\tau \circ \sigma)(\alpha)]_C = [\tau(\beta)]_C$$

$$= \underset{C \leftarrow B}{[\tau]_{\mathcal{B}}} [\beta]_{\mathcal{B}}$$

$$= \underset{C \leftarrow B}{[\tau]_{\mathcal{B}}} [\sigma(\alpha)]_{\mathcal{B}}$$

$$= \underset{C \leftarrow B}{[\tau]_{\mathcal{B}}} \underset{B \leftarrow A}{[\sigma]_{\mathcal{A}}} [\alpha]_{\mathcal{A}}$$

$$\therefore \underset{C \leftarrow A}{[\tau \circ \sigma]_{\mathcal{A}}} [\alpha]_{\mathcal{A}} = \underset{C \leftarrow B}{[\tau]_{\mathcal{B}}} \underset{B \leftarrow A}{[\sigma]_{\mathcal{A}}} [\alpha]_{\mathcal{A}}$$

for all $\alpha \in U$.

$$\therefore \underset{C \leftarrow A}{[\tau \circ \sigma]_{\mathcal{A}}} = \underset{C \leftarrow B}{[\tau]_{\mathcal{B}}} \underset{B \leftarrow A}{[\sigma]_{\mathcal{A}}}$$

(i.e. because matrix multiplication is defined to be associative.)