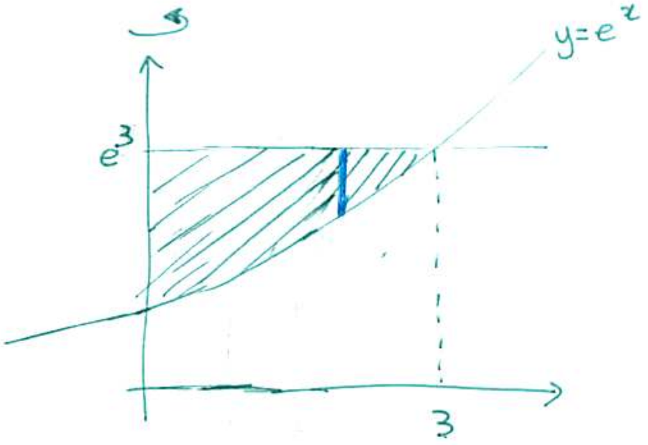
1. (7 points) Let R be the region bounded by the curves

$$y = e^x, \quad x = 0, \quad y = e^3.$$

Find the volume of the solid obtained by rotating R about the y-axis. **Simplify your answer as much as possible.**

Use cylindrical shells:



$$\text{Volume} = \int_0^3 2\pi x (e^3 - e^x)\,dx$$

$$= \int_0^3 2\pi e^3 x - 2\pi x e^x \,dx$$

$$= \left[2\pi e^3 \frac{x^2}{2}\right]_0^3 - 2\pi \int_0^3 x e^x\,dx$$

$$= (9\pi e^3 - 0) - 2\pi \int_0^3 x e^x\,dx$$

Integration by parts:

$$\int_0^3 x e^x\,dx = \left[x e^x\right]_0^3 - \int_0^3 e^x\,dx$$

$u = x \qquad dv = e^x\,dx$

$du = dx \qquad v = e^x$

$$= (3e^3 - 0) - \left[e^x\right]_0^3$$

$$= 3e^3 - (e^3 - 1) = 2e^3 + 1$$

$$\text{So Volume} = 9\pi e^3 - 2\pi(2e^3 + 1)$$

$$= 5\pi e^3 - 2\pi$$

2. (7 points) Compute the following integral:

$$\int (x^2+16)^{-\frac{7}{2}}\,dx.$$

substitution:

$x = 4\tan\theta$

$dx = 4\sec^2\theta\,d\theta$

$x^2+16 = 16\tan^2\theta + 16$

$= 16\sec^2\theta$

$$= \int (4\sec\theta)^{-7}(4\sec^2\theta\,d\theta)$$

$$= \int 4^{-6}(\sec\theta)^{-5}\,d\theta$$

$$= \int 4^{-6}\cos^5\theta\,d\theta$$

$$= \int 4^{-6}(1-\sin^2\theta)^2\cos\theta\,d\theta$$

$$= \int 4^{-6}(1-2\sin^2\theta+\sin^4\theta)\cos\theta\,d\theta$$

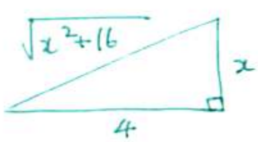
substitution

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= 4^{-6}\left(\sin\theta - \frac{2\sin^3\theta}{3} + \frac{\sin^5\theta}{5}\right) + C$$

$$= 4^{-6}\left(\frac{x}{\sqrt{x^2+16}} - \frac{2}{3}\left(\frac{x}{\sqrt{x^2+16}}\right)^3 + \frac{1}{5}\left(\frac{x}{\sqrt{x^2+16}}\right)^5\right) + C$$



$\frac{x}{4} = \tan\theta = \frac{\text{opp}}{\text{adj}}$

$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{x^2+16}}$

3. (7 points) Compute the following integral:

$$\int \frac{-5}{(x-2)(x^2+1)}dx.$$

Partial fractions:

$$\frac{-5}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$$

$$-5 = A(x^2+1) + (Bx+C)(x-2)$$

$x=2$: $\quad -5 = 5A \quad \Rightarrow A = -1$

coeff of x^2: $\quad 0 = A + B = -1 + B \Rightarrow B = 1$

coeff of x: $\quad 0 = C - 2B = C - 2 \Rightarrow C = 2$

So $$\int \frac{-5}{(x-2)(x^2+1)}\,dx = \int \frac{-1}{x-2} + \frac{x}{x^2+1} + \frac{2}{x^2+1}\,dx$$

$$= -\ln|x-2| + \frac{1}{2}\ln|x^2+1| + 2\arctan x + C$$