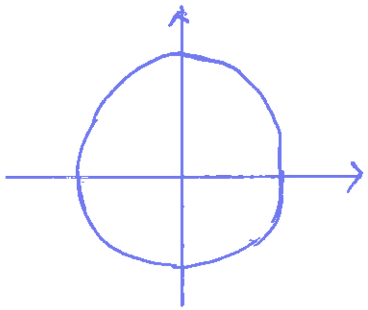
objective function. $x=\cos t$, $y=\sin t$

Redo example: (ex. sheet #19 Q2) Find the maximum and minimum values of $f(x,y) = x^2 + y$ on the unit circle $x^2 + y^2 = 1$, and the point(s) where these extreme values are achieved.

1.



No interior

boundary $\{x^2+y^2=1\}$

2. f is a continuous function, and the domain $\{x^2+y^2=1\}$ is closed and bounded, so f achieves a maximum and a minimum.

3. no interior

4. On the boundary $x^2+y^2=1$

$x^2=1-y^2$

so $f(x,y) = x^2+y$

$= (1-y^2)+y$ $\quad -1 \le y \le 1$

So at a candidate extrema

$\frac{d}{dy}(1-y^2+y)=0$

$-2y+1 = 0 \rightarrow y=\frac{1}{2}$.

and $x^2+y^2=1$

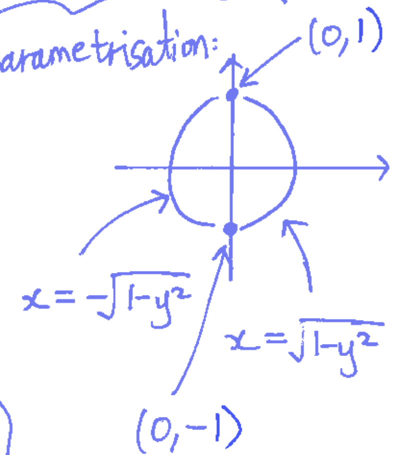
$x^2+\frac{1}{4}=1 \rightarrow x=\frac{\sqrt{3}}{2}$ or $\frac{-\sqrt{3}}{2}$.

also $y=-1$ and $y=1$ (endpoints)

$x^2+y^2=1$

$x=0$ $\quad x=0$

parametrisation:



5. $f\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right) = \frac{3}{4}+\frac{1}{2}=\frac{5}{4}$

$f\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right) = \frac{3}{4}+\frac{1}{2}=\frac{5}{4}$.

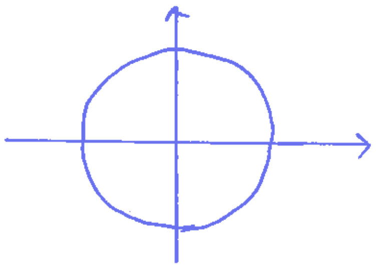
$f(0,1)=1$, $\quad f(0,-1)=-1$.

maximum is $\frac{5}{4}$, achieved at $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ and $\left(\frac{-\sqrt{3}}{2},\frac{1}{2}\right)$

minimum is -1, achieved at $(0,-1)$.

We will explain why Lagrange multipliers work after an example.

Redo example: (ex. sheet #19 Q2, p13) Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = x^2 + y$ on the unit circle $x^2 + y^2 = 1$, and the point(s) where these extreme values are achieved.



No interior

boundary $\{x^2+y^2=1\}$

f is continuous and $\{x^2+y^2=1\}$ is closed and bounded so f achieves a maximum and minimum.

The boundary $\{x^2+y^2=1\}$ is the level set of $g(x,y)=x^2+y^2-1$.

At a candidate extrema:

- f or g does not have continuous partial derivatives $\longrightarrow$ no points.
- $\nabla g = \vec{0}$ $\quad 2x\vec{i}+2y\vec{j}=\vec{0} \longrightarrow (x,y)=(0,0)$
  but $(0,0)$ is not in this boundary piece: doesn't satisfy $x^2+y^2=1$.
- $\nabla f = \lambda \nabla g$.

$2x\vec{i}+\vec{j}=\lambda(2x\vec{i}+2y\vec{j})$

i.e. $2x = \lambda 2x$ ①

$1 = \lambda 2y$ ②

and the point must be on the circle so $x^2+y^2=1$ ③

multiply up ①, ②
to eliminate λ

①$\times y$: $2xy = \lambda 2xy$

②$\times x$: $x = \lambda 2yx$

so $2xy = x$

$2xy - x = 0$

$(2y-1)x = 0$

$2y - 1 = 0$ or $x = 0$

$y = \frac{1}{2}$

substitute into ③: $x^2 + (\frac{1}{2})^2 = 1$ | $0^2 + y^2 = 1$

$x = \frac{\sqrt{3}}{2}$ or $\frac{-\sqrt{3}}{2}$ | $y = 1$ or -1.

$f(\frac{\sqrt{3}}{2}, \frac{1}{2}) = \frac{5}{4}$

$f(\frac{-\sqrt{3}}{2}, \frac{1}{2}) = \frac{5}{4}$

$f(0,1) = 1$

$f(0,-1) = -1$

so f achieves a maximum of $\frac{5}{4}$ at $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ and $(\frac{-\sqrt{3}}{2}, \frac{1}{2})$
and a minimum of -1 at $(0,-1)$.