You must justify your answers to receive full credit.

1. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis of $\mathbb{R}^3$, and suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation with
$$T(\mathbf{v_1}) = \mathbf{v_2}, \quad T(\mathbf{v_2}) = \mathbf{v_1}, \quad T(\mathbf{v_3}) = \mathbf{v_1} + \mathbf{v_3}.$$

   a) Show that $\mathbf{v_1} + \mathbf{v_2}$ is an eigenvector of T, and find its corresponding eigenvalue.
   b) Find the matrix for T relative to $\mathcal{B}$.

   Now let
$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ -5 \\ 2 \end{bmatrix}.$$

   c) Find the change-of-coordinates matrix from the standard basis in $\mathbb{R}^3$ to $\mathcal{B}$.
   d) Find the standard matrix for T.

2. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be bases for a vector space V, and suppose
$$\mathbf{f}_1 = 2\mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3, \qquad \mathbf{f}_2 = 3\mathbf{b}_2 + \mathbf{b}_3, \qquad \mathbf{f}_3 = -3\mathbf{b}_1 + 2\mathbf{b}_3$$

   a) What is the dimension of V?
   b) Find the change-of-coordinates matrix from $\mathcal{F}$ to $\mathcal{B}$.
   c) Now suppose $V = \mathbb{P}_2$, the set of polynomials of degree at most 2, and $\mathcal{B}$ is the standard basis $\{1, t, t^2\}$. If $[\mathbf{p}]_{\mathcal{F}} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, find $\mathbf{p}$. (Hint: $\mathbf{p}$ is a polynomial, not a vector.)

3. Let A be the matrix
$$A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$$

   a) Diagonalise A, i.e find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.
   b) Using your answer to part a), find a matrix B such that $B^2 = A$. **You may give your answer as a product of matrices and/or their inverses.** (Hint: first find a matrix C such that $C^2 = D$.)

4. Are the following matrices diagonalisable? Calculate as little as possible, and explain your answers.

a) $A = \begin{bmatrix} 0 & 3 & -2 \\ -3 & 10 & -6 \\ -4 & 12 & -7 \end{bmatrix}$ b) $B = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$ c) $C = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

5. Let A be a 2 x 2 matrix.

   a) Explain why there is a polynomial p of degree at most 4 (i.e. $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$) such that $p(A) = 0$. (Hint: think about linear independence of the set $\{I, A, A^2, \dots\}$ in the vector space of 2 x 2 matrices.)

   b) **Optional**: Show that, if $a_0 \neq 0$, then A is invertible and A^{-1} is a polynomial in A.

6. State whether each of the following statements is (always) true or (sometimes) false. If it is true, give a brief justification; if it is false, give a counterexample with an explanation.

   a) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ spans a vector space V, then any 7 vectors in V are linearly dependent.

   b) If A is 4×7 matrix and $\text{rank}A = 4$, then $\text{Col}A = \mathbb{R}^4$.

   c) If A is 4×7 matrix and $\text{rank}A = 4$, then $\text{Nul}A = \mathbb{R}^3$.

   d) The sum of the dimensions of the row space and the null space of A equals the number of rows in A.

   e) If a square matrix A satisfies $A^2 = A$, then the only possibly eigenvalues of A are 0 and 1.

   f) Let A and B be square matrices. If A is similar to B, and B is diagonalisable, then A is diagonalisable.

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