This week's notes is the last week on integration (until the final week of class).
We will define and calculate improper integrals, of which there are two types:

- Type I: integrals over an unbounded domain (i.e. domains that "go to infinity")
- Type II: integrals over a domain containing points where the integrand is not defined (e.g. the integrand "goes to infinity").

An integral can be of both types, see p12.

For simplicity, we will discuss these mainly in 1D and 2D, although the results and techniques work in any dimension.

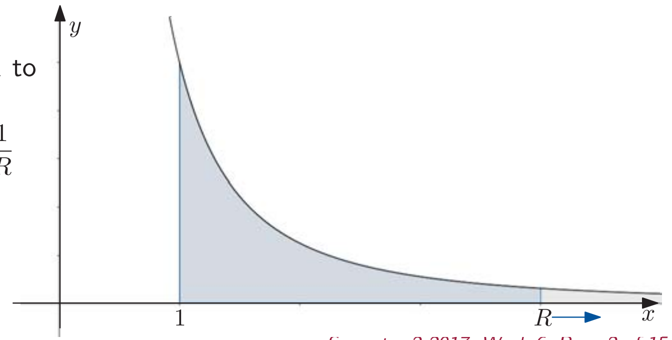
§6.5: Improper Integrals in Single Variable

Type I improper integrals

Consider the region under the graph of $\frac{1}{x^2}$, to the right of $x = 1$.
Although the region has infinite width horizontally, its height goes to 0 as $x \to \infty$.
So the area of the region might not be infinite.

Let's investigate: the area under the graph from $x = 1$ to $x = R$ is
$\int_1^R \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right|_1^R = 1 - \frac{1}{R}$

This area tends to 1 as $R \to \infty$, so it is reasonable to say that the area of the whole region is 1.



The previous example showed that there is a good meaning for an integral whose limit is infinity:

Definition: We define

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty}\int_a^R f(x)\,dx;$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{R\to-\infty}\int_R^b f(x)\,dx;$$

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{R\to-\infty}\int_R^c f(x)\,dx + \lim_{R\to\infty}\int_c^R f(x)\,dx \text{ for any value of } c.$$

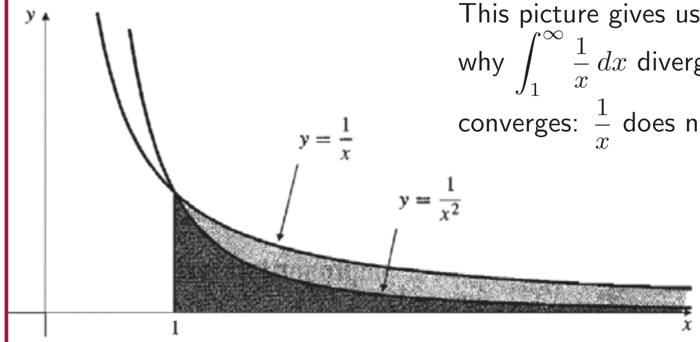
These are called *improper integrals of type I*.

- If the limit exists (i.e. is a finite number), then the improper integral *converges*. (In the third case where the limits are $-\infty$ and ∞, both limits must exist for the integral to converge)
- If the limit does not exist (which includes when the limit is ∞ or $-\infty$), then the improper integral *diverges*.

Rewriting our example on p2 in this terminology:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{R\to\infty}\int_1^R \frac{1}{x^2}\,dx = \lim_{R\to\infty}\left.\frac{-1}{x}\right|_1^R = \lim_{R\to\infty} 1 - \frac{1}{R} = 1.$$

Example: Evaluate $\int_1^\infty \frac{1}{x}\,dx$.



This picture gives us an informal explanation for why $\int_1^\infty \frac{1}{x}\,dx$ diverges whilst $\int_1^\infty \frac{1}{x^2}\,dx$ converges: $\frac{1}{x}$ does not "tend to 0 fast enough".

Note that an improper integral can diverge without "becoming infinite": e.g. $\int_0^\infty \cos x\,dx = \lim_{R\to\infty} \int_0^R \cos x\,dx = \lim_{R\to\infty} \sin x|_0^R = \lim_{R\to\infty} \sin R$ which doesn't exist. So $\int_0^\infty \cos x\,dx$ diverges.

Example: Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

In the previous examples, it looks like we just "substitute in ∞" for the upper limit: "$\frac{1}{\infty} = 0$", "$\ln\infty = \infty$", etc. Although this informal view may be helpful in your scratch work, it can be misleading, so in your final solution you should always use the limit notation.

Another warning: what is $\int_{-\infty}^{\infty} x^3\,dx$?

$\lim_{R\to\infty} \int_{-R}^{R} x^3\,dx = \lim_{R\to\infty} 0 = 0$, because x^3 is an odd function.

But it is misleading to say $\int_{-\infty}^{\infty} x^3\,dx = 0$, because that would require an infinite positive area to cancel out an infinite negative area. The property of additive dependence of domains would not hold.

This is why the definition of $\int_{-\infty}^{\infty} f(x)\,dx$ is the sum of two separate limits of two separate integrals. So the correct answer is that $\int_{-\infty}^{\infty} x^3\,dx$ diverges.

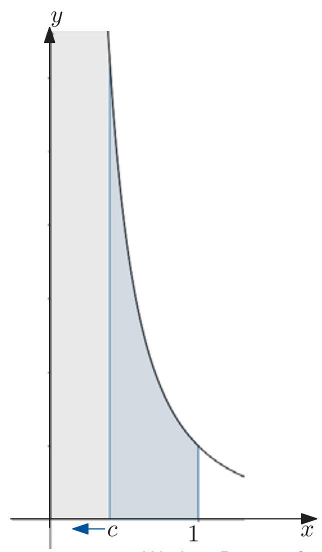
Type II improper integrals

Similarly, we can consider the region under the graph of $\frac{1}{x^2}$, to the left of $x = 1$.
This region has infinite height vertically, but its width tends to 0 as we move up, so again we can ask if its area is finite.

Now the relevant finite approximation is the area under the graph from $x = c$ to $x = 1$, for c close to 0.

This area is $\int_c^1 \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right|_c^1 = \frac{1}{c} - 1.$

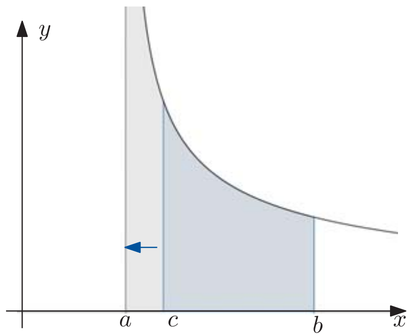
This area becomes larger and larger as $c \to 0$, so the area of the whole region is infinite.



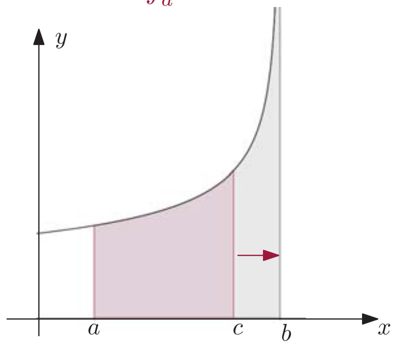
As the previous example showed, there is also a reasonable idea of an integral on a half-open interval $(a, b]$, by taking a limit:

Definition:

If f is not defined at a, then we define
$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

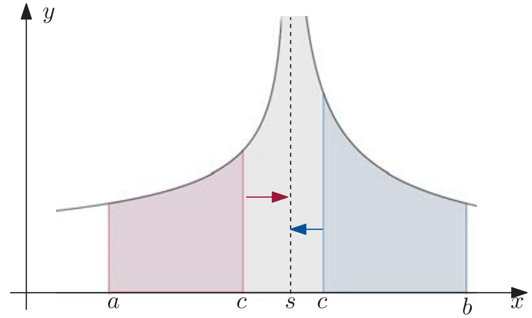


If f is not defined at b, then we define
$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$



Definition: (continued) If f is not defined at some point s in $[a, b]$, then we define
$$\int_a^b f(x)\,dx = \lim_{c\to s^-} \int_a^c f(x)\,dx + \lim_{c\to s^+} \int_c^b f(x)\,dx.$$

If f is not defined at multiple points within $[a, b]$, we similarly divide $[a, b]$ into more subintervals.



All three types (including those on the previous page) are called *improper integrals of type II*. As with type I improper integrals, these may converge (all limits involved exist and are finite) or diverge (some limit does not exist).

Since type II improper integrals may not be obvious, you should always check for points where the integrand is not defined.

Example: Evaluate $\displaystyle\int_2^6 (x-3)^{-1/3}\,dx.$

Here is an improper integral that is of both types.

Example: Evaluate $\displaystyle\int_{1/e}^{\infty} \frac{1}{x(\ln x)^2}\,dx.$

§14.3: Improper Multiple Integrals

The definition of an improper multiple integral is quite complicated, so we won't go into the details, but we remark that, if the integrand is non-negative, then the value of an improper multiple integral can be calculated using iterated integrals, and the involved 1D integrals may be improper or not.

Example: (type I) Evaluate $\iint_D e^{-x-y}\,dx$, where D is the region $0 \leq y \leq x$.

Example: (type II) Evaluate $\iint_D \frac{1}{(x+y)^2}\,dx$, where D is the region bounded by $y=0$, $x=1$ and $y=x$.