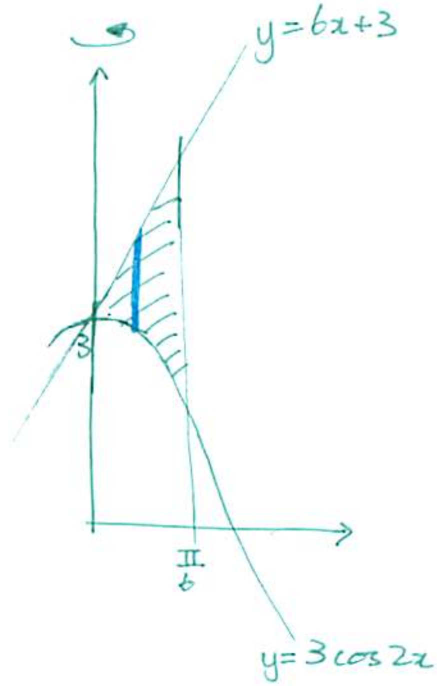
1. (7 points) Let R be the region bounded by the curves

$$y = 3\cos 2x, \quad y = 6x + 3, \quad x = \frac{\pi}{6}.$$

Find the volume of the solid obtained by rotating R about the y-axis. **Simplify your answer as much as possible.**



Use cylindrical shells:

$$\text{volume} = \int_0^{\frac{\pi}{6}} 2\pi x\left((6x+3) - 3\cos 2x\right) dx$$

$$= \int_0^{\frac{\pi}{6}} 12\pi x^2 + 6\pi x - 6\pi x\cos 2x \, dx$$

$$= \left[12\pi\frac{x^3}{3} + \frac{6\pi x^2}{2}\right]_0^{\frac{\pi}{6}} - 6\pi\int_0^{\frac{\pi}{6}} x\cos 2x\, dx$$

$$= \frac{12\pi}{3}\left(\frac{\pi}{6}\right)^3 + \frac{6\pi}{2}\left(\frac{\pi}{6}\right)^2 - 6\pi\int_0^{\frac{\pi}{6}} x\cos 2x\, dx$$

Integration by parts:

$$\int_0^{\frac{\pi}{6}} x\cos 2x\, dx = \left[x\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \frac{\sin 2x}{2}\, dx$$

$u = x \quad dv = \cos 2x\, dx$

$du = dx \quad v = \frac{\sin 2x}{2}$

$$= \left(\frac{\pi}{6}\frac{\sin\frac{\pi}{3}}{2} - 0\right) - \left[-\frac{\cos 2x}{4}\right]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{6}\frac{\sin\frac{\pi}{3}}{2} - \left(-\frac{\cos\frac{\pi}{3}}{4} + \frac{\cos 0}{4}\right)$$

$$= \frac{\pi}{6}\frac{\sqrt{3}}{4} + \frac{1}{8} - \frac{1}{4}$$

$$\therefore \text{volume} = \frac{12\pi}{3}\left(\frac{\pi}{6}\right)^3 + \frac{6\pi}{2}\left(\frac{\pi}{6}\right)^2 - 6\pi\left(\frac{\pi}{6}\frac{\sqrt{3}}{4} + \frac{1}{8} - \frac{1}{4}\right)$$

$$= \frac{\pi^4}{54} + \frac{\pi^3}{12} - \frac{\sqrt{3}\pi^2}{4} + \frac{3\pi}{4}.$$

2. (7 points) Compute the following integral:

$$\int \frac{2x^2-7}{(x-1)(x^2+4)}dx.$$

Partial fractions: $\dfrac{2x^2-7}{(x-1)(x^2+4)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+4}$

$2x^2-7 = A(x^2+4) + (Bx+C)(x-1)$

$x=1$: $2-7 = A(1+4) \Rightarrow -5 = 5A \Rightarrow A=-1$

coefficients of x^2: $2 = A+B \Rightarrow 2 = -1+B \Rightarrow B=3$

coefficients of x: $0 = -B+C \Rightarrow C=B=3$

So $\displaystyle\int \frac{2x^2-7}{(x-1)(x^2+4)}dx = \int \frac{-1}{x-1} + \frac{3x}{x^2+4} + \frac{3}{x^2+4}\,dx$

$$= \underbrace{-\ln|x-1|}_{\substack{\text{substitution}\\ u=x-1}} + \underbrace{\frac{3}{2}\ln|x^2+4|}_{\substack{\text{substitution}\\ u=x^2+4}} + \underbrace{\frac{3}{2}\arctan\left(\frac{x}{2}\right)}_{\substack{\text{substitution}\\ u=x/2.}} + C$$

3. (7 points) Compute the following integral:

$$\int x^2 \left(7 - x^2\right)^{-\frac{7}{2}} \, dx.$$

substitution:

$x = \sqrt{7} \sin\theta$

$dx = \sqrt{7} \cos\theta \, d\theta$

$$= \int (\sqrt{7}\sin\theta)^2 (\sqrt{7}\cos\theta)^{-7} (\sqrt{7}\cos\theta \, d\theta)$$

$$= \int \frac{1}{49} \frac{\sin^2\theta}{\cos^6\theta} \, d\theta$$

$$= \int \frac{1}{49} \tan^2\theta \sec^4\theta \, d\theta$$

$$= \int \frac{1}{49} \tan^2\theta \left(1 - \tan^2\theta\right) \sec^2\theta \, d\theta$$

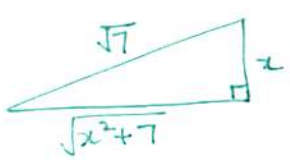
$$= \int \frac{1}{49} \left(\tan^2\theta - \tan^4\theta\right) \sec^2\theta \, d\theta$$

substitution

$u = \tan\theta$

$$= \frac{1}{49}\left(\frac{\tan^3\theta}{3} - \frac{\tan^5\theta}{5}\right) + C$$

$$= \frac{1}{49}\left(\frac{1}{3}\left(\frac{x}{\sqrt{x^2+7}}\right)^3 - \frac{1}{5}\left(\frac{x}{\sqrt{x^2+7}}\right)^5\right) + C$$



$$\frac{x}{\sqrt{7}} = \sin\theta = \frac{\text{opp}}{\text{hyp}}$$

$$\therefore \tan\theta = \frac{x}{\sqrt{x^2+7}}$$