More examples of implicit differentiation:

Example: Suppose $2x + 2\ln(2y) = 5 - z^2 + xz$. Find $\dfrac{\partial z}{\partial x}$ when $(x, y, z) = (4, \frac{1}{2}, 1)$.

product rule

Differentiate both sides with respect to x

$$2 + 0 = 0 - 2z\frac{\partial z}{\partial x} + \frac{\partial x}{\partial x} \cdot z + x\frac{\partial z}{\partial x}$$

$$= -2z\frac{\partial z}{\partial x} + z + x\frac{\partial z}{\partial x}$$

At $(4, \frac{1}{2}, 1)$:

$$2 = -2\frac{\partial z}{\partial x} + 1 + 4\frac{\partial z}{\partial x}$$

$$\frac{1}{2} = \frac{\partial z}{\partial x}$$

While we're talking about implicit differentiation: we can use implicit differentiation to obtain second-order partial derivatives.

Example: Suppose $2x + 2\ln(2y) = 5 - z^2 + xz$. Find $\dfrac{\partial^2 z}{\partial x^2}$ when $(x, y, z) = (4, \frac{1}{2}, 1)$.

$$2 = 0 - 2z\frac{\partial z}{\partial x} + \left(z + x\frac{\partial z}{\partial x}\right)$$

$$2 - z = (x - 2z)\frac{\partial z}{\partial x} \longrightarrow \text{At } (4, \tfrac{1}{2}, 1),\ \frac{\partial z}{\partial x} = \frac{1}{2}$$

$$2 - z = (x - 2z) z_x$$

$\therefore$ differentiate with respect to x again

$$0 - \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x - 2z) \cdot z_x + (x - 2z)\frac{\partial}{\partial x} z_x$$

$$-\frac{\partial z}{\partial x} = \left(1 - 2\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial x} + (x - 2z)\frac{\partial^2 z}{\partial x^2}$$

At $(4, \frac{1}{2}, 1)$

$$-\frac{1}{2} = \left(1 - 2\cdot\frac{1}{2}\right)\frac{1}{2} + (4 - 2\cdot 1)\left.\frac{\partial^2 z}{\partial x^2}\right|_{(4,\frac{1}{2},1)}$$

$$\left.\frac{\partial^2 z}{\partial x^2}\right|_{(4,\frac{1}{2},1)} = -\frac{1}{4}.$$

Example: Suppose $xy + y^2z + zw = 3$ and $w^2x + 3yz = 4$. Calculate $\left(\frac{\partial z}{\partial x}\right)_y$ and $\left(\frac{\partial w}{\partial x}\right)_y$ at the point P where $(x, y, z, w) = (1, 1, 1, 1)$.

differentiating both sides with respect to x, fixing y

$$xy + y^2z + zw = 3 \qquad w^2x + 3yz = 4$$

$$y + y^2\left(\frac{\partial z}{\partial x}\right)_y + z\left(\frac{\partial w}{\partial x}\right)_y + w\left(\frac{\partial z}{\partial x}\right)_y = 0 \qquad w^2 + 2wx\left(\frac{\partial w}{\partial x}\right)_y + 3y\left(\frac{\partial z}{\partial x}\right)_y = 0$$

At $(1,1,1,1)$: $1 + 2\left(\frac{\partial z}{\partial x}\right)_y + \left(\frac{\partial w}{\partial x}\right)_y = 0 \qquad 1 + 2\left(\frac{\partial w}{\partial x}\right)_y + 3\left(\frac{\partial z}{\partial x}\right)_y = 0$

many ways to solve this. e.g. combine into a matrix equation

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} \left(\partial z/\partial x\right)_y \\ \left(\frac{\partial w}{\partial x}\right)_y \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

can solve by row-reduction

$$\begin{pmatrix} \left(\frac{\partial z}{\partial x}\right)_y \\ \left(\frac{\partial w}{\partial x}\right)_y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so $\left(\frac{\partial z}{\partial x}\right)_y = 1$, $\left(\frac{\partial w}{\partial x}\right)_y = -1$.

Example: Suppose
$$\begin{aligned} 3x + 2y + u - v^2 &= 0 \\ 4x + 3y + u^2 + vw &= 2 \\ xu^2 + w &= 1 \end{aligned}$$
Show that u, v, w can be written as functions of x, y when $u = 0$ and $v = 1$.

By Implicit Function Theorem, we need to show $\dfrac{\partial(F,G,H)}{\partial(u,v,w)} \neq 0$

where
$$\begin{aligned} F(x,y,u,v,w) &= 3x+2y+u-v^2 \\ G(x,y,u,v,w) &= 4x+3y+u^2+vw-2 \\ H(x,y,u,v,w) &= xu^2+w-1 \end{aligned}$$

$$\frac{\partial(F,G,H)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} & \frac{\partial F}{\partial w} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} & \frac{\partial G}{\partial w} \\ \frac{\partial H}{\partial u} & \frac{\partial H}{\partial v} & \frac{\partial H}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & -2v & 0 \\ 2u & w & v \\ 2xu & 0 & 1 \end{vmatrix}$$

when $u=0$ and $v=1$, this is $\begin{vmatrix} 1 & -2 & 0 \\ 0 & w & 1 \\ 0 & 0 & 1 \end{vmatrix} = w$

$xu^2+w=1$ so when $u=0$, $w=1$, so $\dfrac{\partial(F,G,H)}{\partial(u,v,w)} = 1 \neq 0$