Example: Find the rate of change of $f(x,y) = x^2 - y^2$ at $(3,-1)$ in the direction $2\mathbf{i} + \mathbf{j}$. Is f increasing or decreasing in this direction?

$$\nabla f(3,-1) = \frac{\partial f}{\partial x}\Big|_{(3,-1)} \vec{i} + \frac{\partial f}{\partial y}\Big|_{(3,-1)} \vec{j}$$

$$= 2x\Big|_{(3,-1)} \vec{i} + (-2y)\Big|_{(3,-1)} \vec{j} = 6\vec{i} + 2\vec{j}.$$

Unit vector in the direction of $2\vec{i}+\vec{j}$ is $\frac{2}{\sqrt{2^2+1^2}}\vec{i} + \frac{1}{\sqrt{2^2+1^2}}\vec{j}$

$$\vec{u} = \frac{2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{j}$$

Rate of change is $D_{\vec{u}} f(3,-1) = \nabla f(3,-1) \cdot \vec{u}$

$$= (6\vec{i} + 2\vec{j}) \cdot \left(\frac{2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{j}\right)$$

$$= \frac{12}{\sqrt{5}} + \frac{2}{\sqrt{5}} = \frac{14}{\sqrt{5}}.$$

f is increasing in this direction because $D_{\vec{u}} f(3,-1)$ is positive.

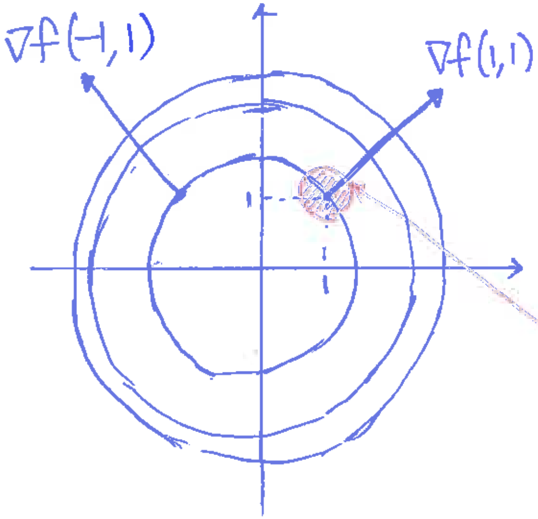
The following example explains why it is useful to put the partial derivatives into a gradient vector.

Example: Let $f(x, y) = x^2 + y^2$.

a. Draw the level curves of f.

b. Draw on the same diagram $\nabla f(1, 1)$ and $\nabla f(-1, 1)$.

c. By considering the value of f at points close to $(1, 1)$, estimate the direction at $(1, 1)$ in which f increases most quickly.



$$\nabla f(1,1) = \frac{\partial f}{\partial x}\Big|_{(1,1)} \vec{i} + \frac{\partial f}{\partial y}\Big|_{(1,1)} \vec{j}$$

$$= 2x\Big|_{(1,1)} \vec{i} + (2y)\Big|_{(1,1)} \vec{j} = 2\vec{i} + 2\vec{j}$$

$$\nabla f(-1,1) = 2x\Big|_{(-1,1)} \vec{i} + 2y\Big|_{(-1,1)} \vec{j} = -2\vec{i} + 2\vec{j}$$

f is largest at this point because
f is distance-squared from the origin
so the direction in which f increases most quickly
is $\vec{i} + \vec{j}$.

Because we can express any surface defined by an equation as the level set of a function (see week 2 p14), we can use this technique to find the normal line and tangent plane to any surface.

Example: Find the normal line and tangent plane to the surface $2x + 2\ln(2y) = 9 - z^2$ at the point $(x, y, z) = (4, \frac{1}{2}, 1)$.

This surface is the level set of $F(x,y,z) = 2x + 2\ln(2y) - 9 + z^2$ $(c=0)$.

Normal vector to the surface at $(4, \frac{1}{2}, 1)$ is $\nabla F(4, \frac{1}{2}, 1)$

$$= \left(2\vec{i} + 2\frac{1}{2y}\cdot 2\vec{j} + 2z\vec{k}\right)\Big|_{(4,\frac{1}{2},1)}$$

$$= 2\vec{i} + 4\vec{j} + 2\vec{k}$$

Normal line: $\vec{r} = 4\vec{i} + \frac{1}{2}\vec{j} + \vec{k} + t(2\vec{i} + 4\vec{j} + 2\vec{k})$

Tangent plane: $\left((x-4)\vec{i} + (y-\frac{1}{2})\vec{j} + (z-1)\vec{k}\right)\cdot(2\vec{i} + 4\vec{j} + 2\vec{k}) = 0$

$$2(x-4) + 4(y-\tfrac{1}{2}) + 2(z-1) = 0$$

We can use this technique to find tangent planes to graphs:

Example: Find an equation in standard form for the tangent plane to the graph of $f(x,y) = 3ye^{-x}$ when $x = 0$ and $y = 2$.

The graph of $f(x,y)$ is $z = f(x,y)$ i.e. $z = 3ye^{-x}$.

when $x=0$, $y=2$, then $z = 3 \cdot 2 \cdot e^{-0} = 6$.

Now forget about f: we want the tangent plane to $z=3ye^{-x}$ at $(0,2,6)$.

$z = 3ye^{-x}$ is the level set $(c=0)$ of $F(x,y,z) = z - 3ye^{-x}$

Normal vector to the surface at $(0,2,6)$ is $\nabla F(0,2,6) = \left(3ye^{-x}\vec{i} - 3e^{-x}\vec{j} + 1\vec{k}\right)\Big|_{(0,2,6)}$

$$= 6\vec{i} - 3\vec{j} + \vec{k}$$

So tangent plane is $6(x-0) - 3(y-2) + 1(z-6) = 0$

$$6x - 3y + z = 0$$

Alternative method: (week 7 p27) Tangent plane to a graph is the graph of the linearisation.

$$z = f(0,2) + f_x(0,2)(x-0) + f_y(0,2)(y-2)$$

$$= 6 + \left(-3ye^{-x}\right)\Big|_{(0,2)}(x-0) + \left(3e^{-x}\right)\Big|_{(0,2)}(y-2)$$

$$= 6 - 6(x-0) + 3(y-2) = -6x + 3y$$

standard form: $6x - 3y + z = 0$