Hello 141-ites,

THANK YOU once again for a fun semester, I've really enjoyed being your lecturer. Well done on finishing the class, have a great winter break, and I'll see you around.

Dr. Pang

p.s. If you want to look at your graded final paper, email me to make an appointment for early January: cy.amy.pang@mcgill.ca, or if you hear no reply in 24 hours, amypang@lacim.ca.

PART A: ANSWER ONLY

Enter **only your final answer** in the boxes provided; if there is any ambiguity, you will not receive credit.

1. (a) (2 points) Calculate the indefinite integral

$$\int (x+1)^2 dx.$$

$$= \int x^2+2x+1 \; dx$$

$$= \frac{x^3}{3} + x^2 + x + C$$

Answer only

$$\frac{x^3}{3} + x^2 + x + C$$

(b) (2 points) Calculate the indefinite integral

$$\int e^x(e^x + 3)dx.$$

Simplify your answer as much as possible.

$$= \int e^{2x} + 3e^x \, dx$$

$$= \frac{e^{2x}}{2} + 3e^x + C$$

Answer only

$$\frac{e^{2x}}{2} + 3e^x + C$$

(c) (2 points) Calculate the indefinite integral

$$\int \frac{\cos x}{3 + \sin x} dx.$$

top is derivative of bottom

Answer only

$$\ln|3 + \sin x| + C$$

(Question 1 continues on the next page.)

(d) (4 points) Calculate the definite integral

$$\int_0^{\frac{\pi}{4}} 16\cos^3 x \sin^5 x dx.$$

Simplify your answer as much as possible.

$$= \int_0^{\frac{\pi}{4}} 16\,(1-\sin^2 x)\sin^5 x\,(\cos x\,dx)$$

$$= 16\int_0^{\frac{\pi}{4}} (\sin^5 x - \sin^7 x)\cos x\,dx$$

substitution
$u = \sin x$

$$= 16\left[\frac{\sin^6 x}{6} - \frac{\sin^8 x}{8}\right]_0^{\frac{\pi}{4}}$$

$u = \cos x$
also works,
but takes
a little longer.

$$= 16\left(\left(\frac{1}{6}\left(\frac{1}{\sqrt{2}}\right)^6 - \frac{1}{8}\left(\frac{1}{\sqrt{2}}\right)^8\right) - 0\right)$$

$$= \frac{1}{3} - \frac{1}{8}$$

$$= \frac{8-3}{24}$$

$$= \frac{5}{24}$$

Answer only

$\frac{5}{24}$

(e) (4 points) Approximate the integral

$$\int_1^{10} e^{t^2}\,dt$$

by a left Riemann sum with 3 subintervals.

$\Delta x = \frac{b-a}{n} = \frac{10-1}{3} = 3$

$x_0 = 1$
$x_1 = 4$
$x_2 = 7$
$x_3 = 10$

Answer only

$3e^{1^2} + 3e^{4^2} + 3e^{7^2}$

(f) (4 points) Consider the function

$$g(x) = \frac{d}{dx}\int_{2x}^{x^2} \arctan(t^2)\,dt.$$

Give an expression for $g(x)$ without integral signs.

Let $F(x)$ be an antiderivative of $\arctan(x^2)$.

Then $\int_{2x}^{x^2} \arctan(t^2)\,dt = F(x^2) - F(2x)$.

So its derivative is $F'(x^2)\frac{d}{dx}(x^2) - F'(2x)\frac{d}{dx}(2x)$
$= \arctan(x^4)2x - \arctan((2x)^2)2$

Answer only

$2x\arctan(x^4) - 2\arctan(4x^2)$

(g) (4 points) The velocity of a particle at time t is given by

$$v(t) = 2(t-1).$$

Calculate the **total distance** travelled by the particle in the time interval [-1,4]. (You may wish to double-check your arithmetic, since there is no partial credit available on this question.)

Distance travelled $= \int_{-1}^{4} |v(t)|\, dt$

$v(t) > 0$ when $t > 1$
< 0 when $t < 1$

$$= \int_{-1}^{1} -2(t-1)\, dt + \int_{1}^{4} 2(t-1)\, dt$$

$$= \left[-t^2+2t\right]_{-1}^{1} + \left[t^2-2t\right]_{1}^{4}$$

$$= (-1+2)-(-1-2)+(16-8)-(1-2) = 13$$

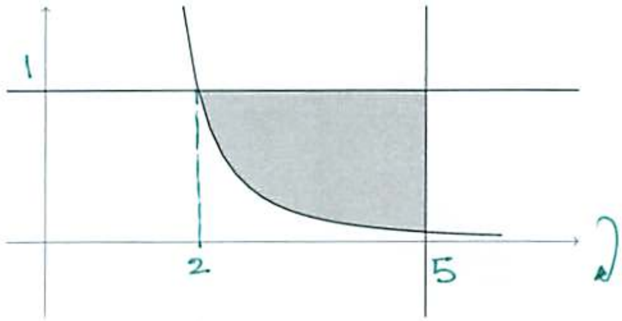
Answer only
13

PART B: FULL SOLUTIONS REQUIRED

In order to receive full credit, please show all of your work and justify your answers.

2. (14 points) The shaded region in the diagram below is bounded by

$$x = 5, \quad y = 1, \quad \text{and } y = \frac{1}{(x-1)^2}.$$



When $y=1$,

$\frac{1}{(x-1)^2} = 1$

$1 = (x-1)^2$

$1 = x-1$

$2 = x$

(a) Calculate the volume of the solid formed by rotating this region about the x-**axis**.

Use slices: $\text{volume} = \int_2^5 \pi \left(1^2 - \left(\frac{1}{(x-1)^2}\right)^2\right) dx$

$= \int_2^5 \pi - \frac{\pi}{(x-1)^4}\, dx$

substitution

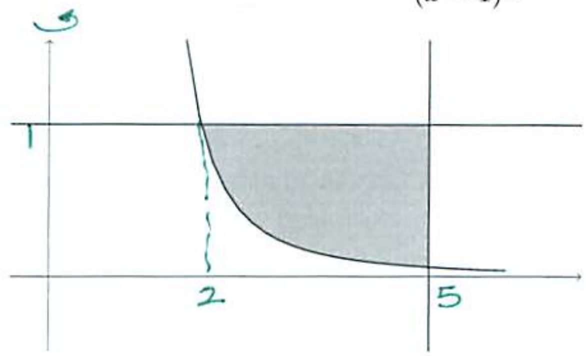
$= \left[\pi x + \frac{\pi}{3(x-1)^3}\right]_2^5$

$= \left(5\pi + \frac{\pi}{3(4)^3}\right) - \left(2\pi + \frac{\pi}{3}\right)$

$= \frac{171\pi}{64}$

(b) For your convenience, here again is the diagram of the region bounded by

$$x = 5, \quad y = 1, \quad \text{and } y = \frac{1}{(x-1)^2}.$$



Calculate the volume of the solid formed by rotating this region about the y-**axis**.

Use shells: $\text{volume} = \int_2^5 2\pi x \left(1 - \frac{1}{(x-1)^2}\right) dx$

$$= \int_2^5 2\pi x - \frac{2\pi x}{(x-1)^2}\, dx$$

$$= \Big[\pi x^2\Big]_2^5 - 2\pi \int_2^5 \frac{x}{(x-1)^2}\, dx$$

$$= (25\pi - 4\pi) - 2\pi \int_2^5 \frac{x}{(x-1)^2}\, dx$$

$u = x-1$

$$= 21\pi - 2\pi \int_1^4 \frac{u+1}{u^2}\, du$$

$$= 21\pi - 2\pi \int_1^4 u^{-1} + u^{-2}\, du$$

$$= 21\pi - 2\pi \left[\ln u - \frac{1}{u}\right]_1^4$$

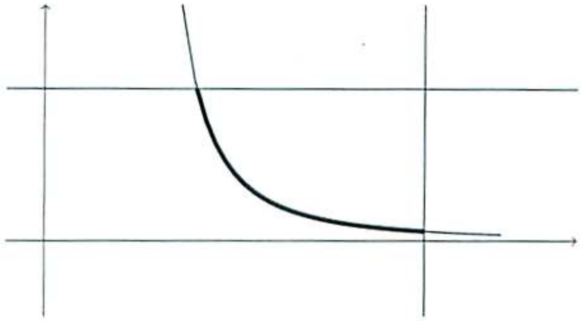
$$= 21\pi - 2\pi \left(\ln 4 - \frac{1}{4} - (\ln 1 - 1)\right)$$

$$= 21\pi - 2\pi\left(\ln 4 + \frac{3}{4}\right)$$

$$= \frac{39\pi}{2} - 2\pi \ln 4$$

(c) For your convenience, here again is the diagram of the lines

$$x = 5, \quad y = 1, \quad \text{and } y = \frac{1}{(x-1)^2}.$$



A surface is created by rotating the curve segment in bold **about the y-axis**. Write down a definite integral, in a single variable, which computes the area of this surface. You do **not** need to evaluate this integral.

$$\frac{dy}{dx} = \frac{-2}{(x-1)^3}$$

$$\text{Surface area} = \int_2^5 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_2^5 2\pi x \sqrt{1 + \frac{4}{(x-1)^6}}\, dx$$

3. (6 points)

Compute the following improper integral, or explain why it diverges:

$$\int_0^e x^{-\frac{5}{2}} \ln x dx. \quad = \lim_{t\to 0} \int_t^e x^{-5/2} \ln x \, dx$$

Simplify your answer as much as possible.

Integration by parts:

$$\int x^{-\frac{5}{2}} \ln x \, dx$$

$u = \ln x \qquad dv = x^{-5/2}$

$du = \frac{1}{x} \qquad v = \frac{x^{-3/2}}{-3/2}$

$$= \ln x \frac{x^{-3/2}}{(-3/2)} - \int \frac{x^{-3/2}}{-3/2} \frac{1}{x} \, dx$$

$$= -\frac{2}{3} x^{-3/2} \ln x + \frac{2}{3} \int x^{-5/2} \, dx$$

$$= -\frac{2}{3} x^{-3/2} \ln x + \frac{2}{3} \frac{x^{-3/2}}{-3/2} + C$$

$$= -\frac{2}{3} x^{-3/2} \ln x - \frac{4}{9} x^{-3/2} + C$$

So the improper integral is

$$\lim_{t\to 0} \left[-\frac{2}{3} x^{-3/2} \ln x - \frac{4}{9} x^{-3/2} \right]_t^e$$

$$= \frac{-2}{3} e^{-3/2} - \frac{4}{9} e^{-3/2} - \lim_{t\to 0} \left(-\frac{2}{3} t^{-3/2} \ln t - \frac{4}{9} t^{-3/2} \right)$$

as $t \to 0$, $\ln t \to -\infty$,
$t^{-3/2} \to \infty$

so this limit doesn't exist

$\therefore$ the integral diverges.

4. (12 points)

(a) Compute the following improper integral, or explain why it diverges:

$$\int_{-\infty}^{0} \frac{x^2}{\left(\sqrt{-x^3+4}\right)^5}\,dx.$$

Simplify your answer as much as possible.

$$\int \frac{x^2}{\sqrt{-x^3+4}^5}\,dx = \int \frac{1}{-3} u^{-5/2}\,du$$

$$= \frac{1}{-3}\,\frac{u^{-3/2}}{-3/2} + C$$

$$= \frac{2}{9}\left(-x^3+4\right)^{-3/2} + C$$

$u = -x^3+4$
$du = -3x^2\,dx$

So $\displaystyle\int_{-\infty}^{0} \frac{x^2}{\sqrt{-x^3+4}^5}\,dx = \lim_{t\to-\infty} \int_t^0 \frac{x^2}{\sqrt{-x^3+4}^5}\,dx$

(integrand is defined whenever $-x^3+4>0$
$4 > x^3$
$\sqrt[3]{4} > x$)

$$= \lim_{t\to-\infty} \left[\frac{2}{9}\left(-x^3+4\right)^{-3/2}\right]_t^0$$

$$= \lim_{t\to-\infty} \left(\frac{2}{9}4^{-3/2} - \frac{2}{9}\left(-t^3+4\right)^{-3/2}\right)$$

$$= \frac{2}{9}\left(\frac{1}{8}\right) + 0$$

because, as $t\to-\infty$,
$-t^3+4 \to \infty$
$(-t^3+4)^{-3/2} \to 0$

$$= \frac{1}{36}$$

(Question 4 continues on the next page.)

(b) Compute the following indefinite integral:

$$\int \frac{x^2}{\left(\sqrt{-x^2+4}\right)^5}dx.$$

(Hint: You may find it easier to calculate this **without** using part a.)

$$= \int \frac{4\sin^2\theta}{2^5\cos^5\theta}\, 2\cos\theta\, d\theta$$

$x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta\, d\theta$

$-x^2+4 = -4\sin^2\theta + 4$

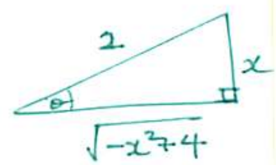
$= 4\cos^2\theta$

$$= \frac{1}{4}\int \frac{\sin^2\theta}{\cos^4\theta}\, d\theta$$

$$= \frac{1}{4}\int \tan^2\theta \sec^2\theta\, d\theta$$

substitution

$$= \frac{1}{4}\,\frac{\tan^3\theta}{3} + C$$



$$= \frac{1}{12}\left(\frac{x}{\sqrt{-x^2+4}}\right)^3 + C$$

5. (8 points)

(a) Compute the following indefinite integral:

$$\int x^2 e^{x^3}\,dx\,.$$

Simplify your answer as much as possible.

$= \dfrac{e^{x^3}}{3} + C$

substitution $u = x^3$

$du = 3x^2$.

(Question 5 continues on the next page.)

(b) Compute the following indefinite integral:

$$\int x^3 e^{x^2}\, dx$$

Simplify your answer as much as possible.

$= \int \frac{x^2}{2} e^{x^2} 2x \, dx$

substitution $t = x^2$

$dt = 2x\, dx$

$= \int \frac{t}{2} e^t \, dt$

$= \frac{t}{2} e^t - \int \frac{1}{2} e^t \, dt$

integration by parts:

$u = \frac{t}{2}$ $\quad dv = e^t \, dt$

$du = \frac{1}{2} dt$ $\quad v = e^t$

$= \frac{t}{2} e^t - \frac{1}{2} e^t + C$

$= \frac{x^2}{2} e^{x^2} - \frac{1}{2} e^{x^2} + C$

6. (6 points) Compute the following indefinite integral:

$$\int \frac{8}{x(x-2)^2} dx.$$

Partial fractions:

$$\frac{8}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$8 = A(x-2)^2 + Bx(x-2) + Cx$$

$x=0$: $8 = 4A \Rightarrow A=2$

$x=2$: $8 = C2 \Rightarrow C=4$

coeff of x^2: $0 = A+B \Rightarrow B=-2$

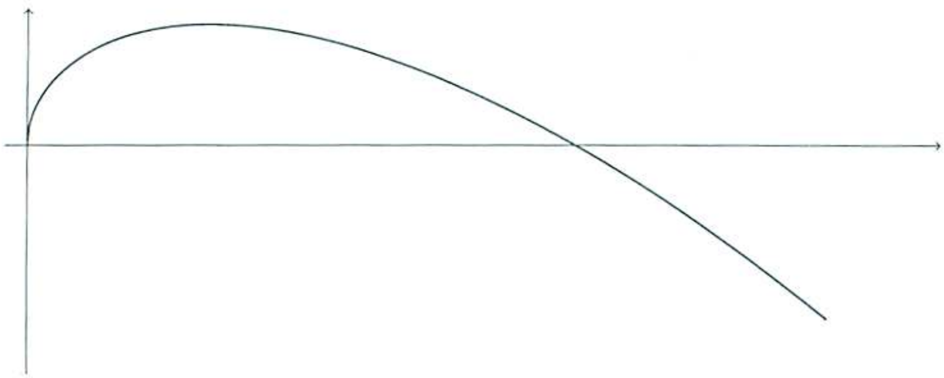
$$\therefore \int \frac{8}{x(x-2)^2} dx = \int \frac{2}{x} - \frac{2}{x-2} + \frac{4}{(x-2)^2} dx$$

$$= 2\ln|x| - 2\ln|x-2| - \frac{4}{x-2} + C$$

7. (12 points) Let C be the parametrised curve with equation

$$x = 2t^2, \quad y = -\frac{t^3}{3} + 4t, \quad t \geq 0,$$

as shown in the diagram below.



(a) Find the equation of the tangent line at the point $(18, 3)$.

The value of t corresponding to $(18,3)$ is $2t^2 = 18$, $\frac{-t^3}{3} + 4t = 3$

$t^2 = 9$

$t = 3$ because $t \geqslant 0$.

The gradient when $t=3$ is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

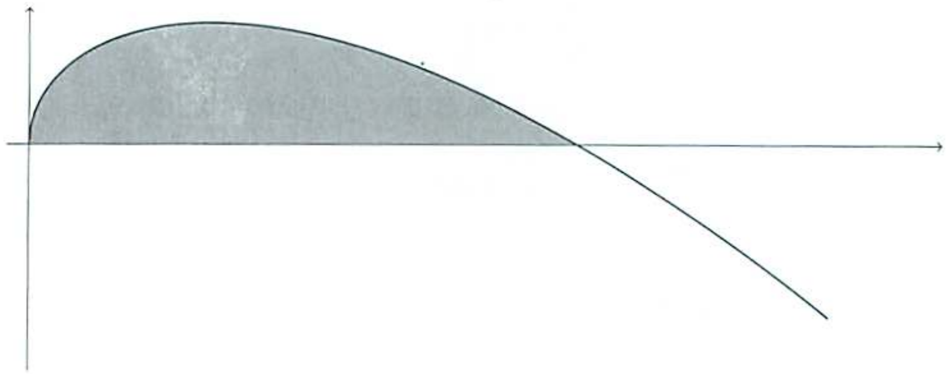
$= \frac{-t^2+4}{4t}$

$= \frac{-9+4}{12} = \frac{-5}{12}$

$\therefore$ equation of tangent line is $y - 3 = \frac{-5}{12}(x-18)$.

(b) For your convenience, here again is the information about the parametrised curve C:

$$x = 2t^2, \quad y = -\frac{t^3}{3} + 4t, \quad t \geq 0.$$



Find the **perimeter** of the shaded region. (Hint: if you cannot determine the limits of the associated integral, you can earn partial credit by computing the indefinite integral.)

The corners of the shaded region are where $y=0$.

$$-\frac{t^3}{3}+4t=0$$

$$-t^3+12t=0$$

$$t(-t^2+12)=0$$

so $t=0$ and $t=\sqrt{12}$

($t=-\sqrt{12}$ isn't relevant because the curve is just for $t \geq 0$)

$t=\sqrt{12}$ corresponds to $x=2(12)=24$, $y=0$.

So perimeter $= 24 + \int_0^{\sqrt{12}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= 24 + \int_0^{\sqrt{12}} \sqrt{(4t)^2 + (-t^2+4)^2}\, dt$$

$$= 24 + \int_0^{\sqrt{12}} \sqrt{16t^2 + t^4 - 8t^2 + 16}\, dt$$

$$= 24 + \int_0^{\sqrt{12}} \sqrt{t^4 + 8t^2 + 16}\, dt$$

$$= 24 + \int_0^{\sqrt{12}} \sqrt{(t^2+4)^2}\, dt$$

$$= 24 + \int_0^{\sqrt{12}} t^2+4 \; dt$$ because $t^2+4>0$ always

$$= 24 + \left[\frac{t^3}{3} + 4t \right]_0^{\sqrt{12}}$$

$$= 24 + \frac{12\sqrt{12}}{3} + 4\sqrt{12}$$

$$= 24 + 8\sqrt{12}.$$

8. (12 points)

(a) Determine whether the following series is convergent or divergent. Clearly state which tests you are using.

$$\sum_{n=1}^{\infty} \frac{2n}{5n^3 - n + 6}.$$

(Note that, for $n \geq 1$, $2n > 0$, $5n^3 - n + 6 = 5n(n^2-1) + 6 > 0$, so this is a positive series.)

Use limit comparison test: let $b_n = \frac{2n}{5n^3} = \frac{2}{5n^2}$

$\frac{a_n}{b_n} = \frac{2n}{5n^3-n+6} \frac{5n^3}{2n} = \frac{5}{5-n^{-2}+6n^{-3}}$ so $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{5}{5+0+0} = 1$.

$\sum_{n=1}^{\infty} \frac{2}{5n^2}$ converges (p-series with $p=2>1$)

so $\sum_{n=1}^{\infty} \frac{2n}{5n^3-n+6}$ also converges.

(Question 8 continues on the next page.)

(b) Determine whether the following series is convergent or divergent. Clearly state which tests you are using.

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n}.$$

(This is clearly a positive series.)

Use ratio test: $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^2}{3^{n+1}} \dfrac{3^n}{n^2} = \dfrac{1}{3} \dfrac{(n+1)^2}{n^2} = \dfrac{1}{3}(1 + 2n^{-1} + n^{-2}).$

so $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{3}(1+2n^{-1}+n^{-2}) = \frac{1}{3} < 1.$

So $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ converges.

(Question 8 continues on the next page.)

(c) Determine whether the following series is convergent or divergent. Clearly state which tests you are using.

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}.$$

(This is a positive series.)

Use comparison test: $b_n = \frac{1}{n}$.

Because $\frac{1}{n} > 0$ and e^x is an increasing function, $e^{1/n} > e^0 = 1$.

So $\frac{e^{1/n}}{n} > \frac{1}{n}$.

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series with $p=1$)

So $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ diverges.

9. (8 points) Let D be the region bounded by the polar curve

$$r = \frac{1}{1+\cos\theta},$$

and the lines $\theta = 0$ and $\theta = \frac{\pi}{2}$. Calculate the area of D. **Simplify your answer as much as possible.**

(Hint: you may wish to use the identity $1 + \cos(2x) = 2\cos^2(x)$.)

$$\text{Area} = \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} \frac{1}{(1+\cos\theta)^2} \, d\theta$$

$\theta = 2x$ $\quad \theta = 0 \to x = 0$

$d\theta = 2\,dx$ $\quad \theta = \frac{\pi}{2} \to x = \frac{\pi}{4}$.

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2} \frac{1}{(1+\cos(2x))^2} 2\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{(2\cos^2 x)^2} \, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{4} \sec^4 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{4} \sec^2 x (1+\tan^2 x) \, dx$$

substitution $u = \tan x$

$$= \left[\frac{1}{4}\left(\tan x + \frac{\tan^3 x}{3}\right) \right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left(1+\frac{1}{3}\right) - \frac{1}{4}(0+0)$$

$$= \frac{1}{3}$$