Ex: $B = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 \\ -4 & 1 & -3 & 2 & 1 \\ -2 & -1 & 0 & 1 & 1 \\ -3 & -1 & -3 & 4 & 1 \\ -8 & -2 & -7 & 5 & 4 \end{pmatrix}$ $\quad \chi_B(x) = -(x-2)^5$

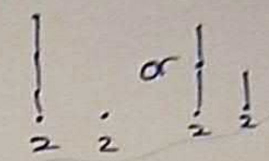
Step 1: number of vectors in level i of eigenstring diagram

$$= \dim \mathrm{Nul}(B-2I)^i - \dim \mathrm{Nul}(B-2I)^{i+1}$$

$$B-2I = \begin{pmatrix} -1 & 0 & -1 & 1 & 0 \\ -4 & -1 & -3 & 2 & 1 \\ -2 & -1 & -2 & 1 & 1 \\ -3 & -1 & -3 & 2 & 1 \\ -8 & -2 & -7 & 5 & 2 \end{pmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{pmatrix} \boxed{-1} & 0 & -1 & 1 & 0 \\ 0 & \boxed{-1} & 0 & -1 & 1 \\ 0 & 0 & \boxed{1} & -1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$\therefore \dim \mathrm{Nul}(B-2I) = 2$

$\therefore$ possibilities =

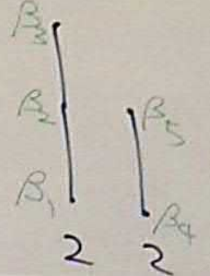


$$(B-2I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \end{pmatrix}$$

clearly only one pivot.

$\therefore \dim \mathrm{Nul}(B-2I)^2 = 5 - 1 = 4$

$\therefore$ diagram



(check that $\dim \mathrm{Nul}(B-2I)^3 = \dim \mathrm{Nul}(B-2I)^4$)

Step 2: Find longest eigenstrings i.e. find β_3.

- Basis $\{\alpha_i\}$ of $\mathrm{Nul}\,(B-2I)^3$: $\{\alpha_1 = e_1, \alpha_2 = e_2, \alpha_3 = e_3, \alpha_4 = e_4, \alpha_5 = e_5\}$

$\because$ B has only one eigenvalue, so all of $\mathbb{C}^5$ is in K_2.

- Consider $\{(B-2I)^2(\alpha_i)\}$ = columns of $(B-2I)^2$.
- Take linearly independent subset

e.g. columns with pivot $\Rightarrow$ column 1 $\Rightarrow \beta_3 = e_1$

<u>Or</u> we need linearly independent subset of one vector,

i.e. one nonzero vector $\therefore \beta_3$ can be any e_i such that

$(B-2I)^2 e_i \neq \vec{0}$. e.g. $\beta_3 = e_1$ or e_3 or e_4.

if choose e_1: $\beta_3 = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$, $\beta_2 = (B-2I)\beta_3 = \begin{pmatrix}-1\\-4\\-2\\-3\\-8\end{pmatrix}$, $\beta_1 = (B-2I)\beta_2 = (B-2I)^2\beta_3 = \begin{pmatrix}0\\0\\-1\\-1\\-1\end{pmatrix}$.

Step 3: Find the next maximal eigenstring length, i.e. the next level with new eigenstring tops — call this m'.

(in example B, $m'=2$)

Step 4: Find the length m' eigenstring tops $\alpha_1', \alpha_2', \ldots$

Repeat steps 3, 4 as necessary

$\therefore$ $\alpha_i' \in \mathrm{Ker}(\sigma-\lambda_c)^{m'}$

BUT $\therefore$ α_i' cannot be in the span of the existing eigenstrings.

- new eigenstring bottoms must be linearly independent, and independent from existing eigenstring bottoms

- Let $\{\beta_{i_1}, \beta_{i_2}, \cdots\}$ be the vectors in the bottom m' levels of existing eigenstrings.

Extend to a ~~basis of~~ spanning set of $\mathrm{Ker}(\sigma-\lambda_c)^{m'}$:

$\{\beta_{i_1}, \beta_{i_2}, \cdots, \alpha_1', \alpha_2', \ldots\}$.

- Consider $\{(\sigma-\lambda_c)^{m'-1}(\beta_{i_1}), (\sigma-\lambda_c)^{m'-1}(\beta_{i_2}), \cdots, (\sigma-\lambda_c)^{m'-1}(\alpha_1'), \ldots\}$

- Take a linearly independent subset containing existing eigenstring bottoms, and some $(\sigma-\lambda_c)^{m'-1}(\alpha_i')$

— these α_i' are the new eigenstring tops.

Ex: Continue with B (Ex. 8.3.7), $m'=2$

Bottom m' levels of existing eigenstrings: $\{\beta_1, \beta_2\}$

A basis of $\mathrm{Nul}(B-2I)^2$:

from matrix earlier: $-x_1 - x_3 + x_4 = 0$

$$\begin{aligned} x_1 &= -x_3 + x_4 \\ x_2 &= x_2 \\ x_3 &= x_3 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned} \quad \mapsto \quad \left\{ \begin{array}{cccc} \alpha_1' & \alpha_2' & \alpha_3' & \alpha_4' \\ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{array} \right\}$$

· Apply $B-2I$ to the spanning set of $\{\beta_1, \beta_2, \alpha_1', \alpha_2', \alpha_3', \alpha_4'\}$:

can stop here.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \\ 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & -1 & -1 & 0 & -1 & 1 \\ 0 & -1 & -1 & 0 & -1 & 1 \\ 0 & -1 & -2 & 1 & -3 & 2 \end{pmatrix} \xrightarrow{\text{row reduction}} \begin{pmatrix} 0 & \boxed{-1} & 1 & 0 & -1 & 1 \\ 0 & 0 & \boxed{-1} & 1 & -2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$(B-2I)(\beta_1)=\vec{0}$ $\quad(B-2I)(\beta_2)=\beta_1$ $\quad(B-2I)(\alpha_1')$ $\quad(B-2I)(\alpha_2')$ ...

∴ columns 2, 3 have pivots

∴ choose new top (β_5) to correspond to column 3

ie. $\beta_5 = \alpha_1' = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\beta_4 = (B-2I)\alpha_1'$

So: $P = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & -4 & 0 & -1 & 1 \\ -1 & -2 & 0 & -1 & 0 \\ -1 & -3 & 0 & -1 & 0 \\ -1 & 8 & 0 & -2 & 0 \end{pmatrix}$ $\quad J = \begin{pmatrix} 2 & 1 & \cdot & \cdot & \cdot \\ \cdot & 2 & 1 & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & 2 \end{pmatrix}$

Shortcut: we can stop at $\Big|$,

$\because$ we know from diagram we only need one new eigenstring top, i.e. only one α_i', so that $(B-2I)\alpha_i' \cup \{\text{previous eigenstring bottoms}\}$ is linearly independent, i.e. need an α_i' so that $(B-2I)\alpha_i'$ is not a multiple of β_1. And $\alpha_1' = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ satisfies this.