

§1.8-1.9: Linear Transformations

Goal: think of the equation $A\mathbf{x} = \mathbf{b}$ in terms of the “multiplication by A ” function: its input is \mathbf{x} and its output is \mathbf{b} .

$$\begin{array}{lcl}
 2^2 = 4 & & \\
 3^2 = 9 & \xrightarrow{\text{squaring}} & \begin{array}{c} 4 \\ 9 \end{array} \\
 \\
 \begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \end{bmatrix} & \xrightarrow{\text{multiply by } A} & \begin{bmatrix} 10 \\ 9 \end{bmatrix} \\
 \text{Think of this as: } & & \\
 \\
 \begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix} & \xrightarrow{\text{multiply by } A} & \begin{bmatrix} 4 \\ 7 \end{bmatrix} \\
 \text{Think of this as: } & &
 \end{array}$$

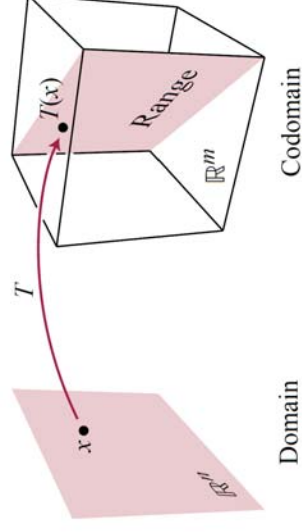
Examples:

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^3, \text{ defined by } f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^3 x_2 \\ 2x_2 \\ 0 \end{bmatrix}.$$

The range of f is the plane $z = 0$.

$$h: \mathbb{R}^3 \rightarrow \mathbb{R}^2, \text{ given by the matrix transformation } h(\mathbf{x}) = \begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \mathbf{x}.$$

Definition: A *function* f from \mathbb{R}^n to \mathbb{R}^m is a rule that assigns to each vector \mathbf{x} in \mathbb{R}^n a vector $f(\mathbf{x})$ in \mathbb{R}^m . We write $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$.



\mathbb{R}^n is the *domain* of f .
 \mathbb{R}^m is the *codomain* of f .
 $f(\mathbf{x})$ is the *image* of \mathbf{x} under f .
The *range* is the set of all images. It is a subset of the codomain.

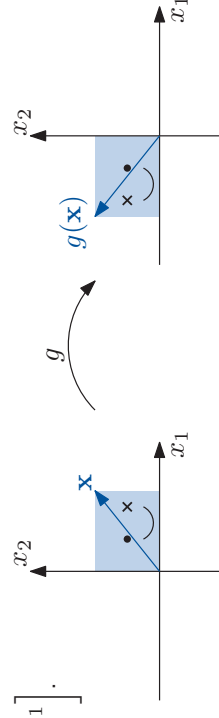
Example: $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$.

Its domain = codomain = \mathbb{R} , its range = $\{\text{zero and positive numbers}\}$.

Examples:

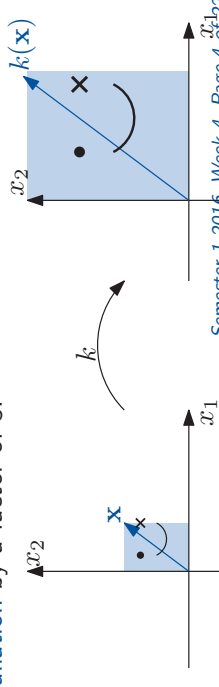
$g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, given by reflection through the x_2 -axis.

$$g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}.$$



$k: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, given by *dilation* by a factor of 3.

$$k(\mathbf{x}) = 3\mathbf{x}.$$



In this class, we will concentrate on functions that are **linear**.

Definition: A function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a **linear transformation** if:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all \mathbf{u}, \mathbf{v} in the domain of T ;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and for all \mathbf{u} in the domain of T .

For your intuition: the name “linear” is because these functions preserve lines:

A line through the point \mathbf{p} in the direction \mathbf{v} is the set $\mathbf{p} + s\mathbf{v}$, where s is any number. If T is linear, then the image of this set is

$$T(\mathbf{p} + s\mathbf{v}) \stackrel{1}{=} T(\mathbf{p}) + T(s\mathbf{v}) \stackrel{2}{=} T(\mathbf{p}) + sT(\mathbf{v}),$$

the line through the point $T(\mathbf{p})$ in the direction $T(\mathbf{v})$.

(If $T(\mathbf{v}) = \mathbf{0}$, then the image is just the point $T(\mathbf{p})$.)

Fact: A linear transformation T must satisfy $T(\mathbf{0}) = \mathbf{0}$.

Proof: Put $c = 0$ in condition 2.

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Example: $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^3 x_2 \\ 2x_2 \\ 0 \end{bmatrix}$ is not linear:

Take $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $c = 2$:

$$f\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 16 \\ 4 \\ 0 \end{bmatrix}.$$

$$2f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 16 \\ 4 \\ 0 \end{bmatrix}.$$

So condition 2 is false for f .

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Example: $g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$ (reflection through the x_2 -axis) is linear:

$$1. \ g\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right) = \begin{bmatrix} -u_1 - v_1 \\ u_2 + v_2 \end{bmatrix} = \begin{bmatrix} -u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix} = g\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + g\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right).$$

$$2. \ g\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) = \begin{bmatrix} -cu_1 \\ cu_2 \end{bmatrix} = c\begin{bmatrix} -u_1 \\ u_2 \end{bmatrix} = cg\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right).$$

Alternatively, we can combine the two conditions at the same time, and check just one statement: $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$, for all scalars c, d and all vectors \mathbf{u}, \mathbf{v} .

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Example: $k(\mathbf{x}) = 3\mathbf{x}$ (dilation by a factor of 3) is linear:

$$k(c\mathbf{u} + d\mathbf{v}) = 3(c\mathbf{u} + d\mathbf{v}) = 3c\mathbf{u} + 3d\mathbf{v} = k(c\mathbf{u}) + k(d\mathbf{v}).$$

Important Example: All matrix transformations $T(\mathbf{x}) = A\mathbf{x}$ are linear:

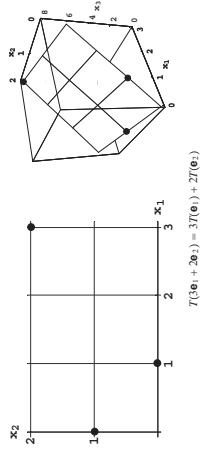
$$T(c\mathbf{u} + d\mathbf{v}) = A(c\mathbf{u} + d\mathbf{v}) = A(c\mathbf{u}) + A(d\mathbf{v}) = cA\mathbf{u} + dA\mathbf{v} = cT(\mathbf{u}) + dT(\mathbf{v}).$$

EXAMPLE: Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation with

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Find the image of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Solution:



Theorem 10: The matrix of a linear transformation: Every linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation: $T(\mathbf{x}) = A\mathbf{x}$ where A is the *standard matrix for T* , the $m \times n$ matrix given by

$$A = \begin{bmatrix} | & | & | & | \\ T(\mathbf{e}_1) & \dots & T(\mathbf{e}_n) \\ | & | & | & | \end{bmatrix}.$$

Example: $k: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, given by **dilation** by a factor of 3, $k(\mathbf{x}) = 3\mathbf{x}$.

$$k(\mathbf{e}_1) = k\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 3\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad k(\mathbf{e}_2) = k\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 3\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

So the standard matrix of k is $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, i.e. $k(\mathbf{x}) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{x}$.

In general:

Write \mathbf{e}_i for the vector with 1 in row i and 0 in all other rows.

$$\text{For example, in } \mathbb{R}^3, \text{ we have } \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \text{ span } \mathbb{R}^n, \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{e}_1 + \dots + x_n\mathbf{e}_n.$$

So, if $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation, then

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + \dots + x_n\mathbf{e}_n) = x_1T(\mathbf{e}_1) + \dots + x_nT(\mathbf{e}_n) = \begin{bmatrix} | & | & | & | \\ T(\mathbf{e}_1) & \dots & T(\mathbf{e}_n) \\ | & | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

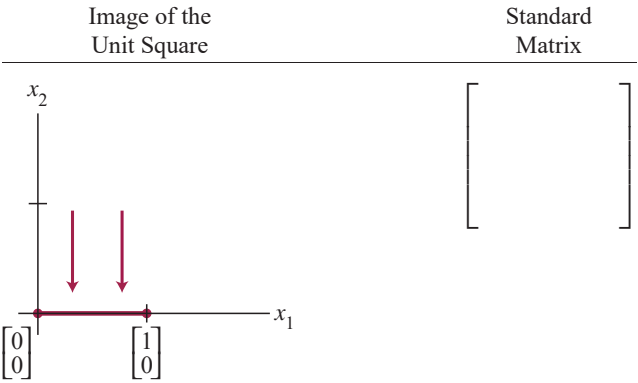
Example: $g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$ (reflection through the x_2 -axis):



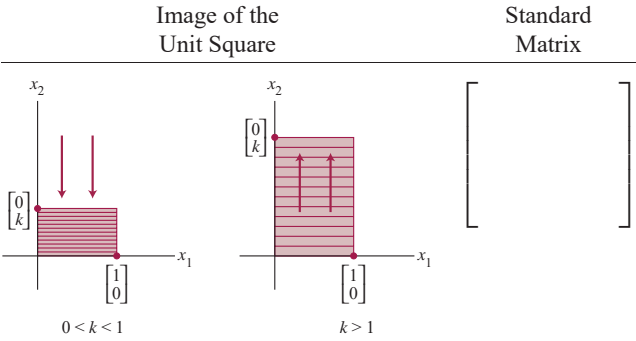
The standard matrix of g is $\begin{bmatrix} | & | \\ g(\mathbf{e}_1) & g(\mathbf{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\text{Indeed, } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}.$$

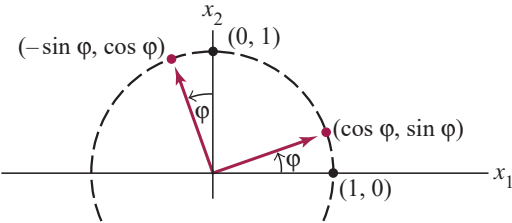
Projection onto the x_1 -axis



Vertical Contraction and Expansion



EXAMPLE: $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by rotation counterclockwise about the origin through an angle φ :



Definition: A function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *onto* (surjective) if each y in \mathbb{R}^m is the image of at least one x in \mathbb{R}^n .

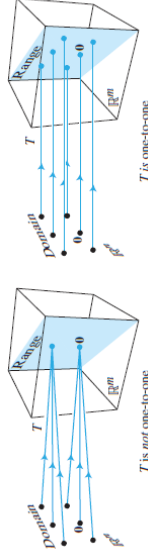
Other ways of saying this:

- The range is all of the codomain \mathbb{R}^m ,
- The equation $f(x) = y$ always has a solution.

Definition: A function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *one-to-one* (injective) if each y in \mathbb{R}^m is the image of at most one x in \mathbb{R}^n .

Other ways of saying this:

- ??? (something that only works for linear transformations),
- The equation $f(x) = y$ has no solutions or a unique solution.



Definition: A function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *onto* (surjective) if each y in \mathbb{R}^m is the image of at least one x in \mathbb{R}^n .

Definition: A function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is *one-to-one* (injective) if each y in \mathbb{R}^m is the image of at most one x in \mathbb{R}^n .

Example: $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$, defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^3 x_2 \\ 2x_2 \\ 0 \end{bmatrix}$.

f is not onto, because $f(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ does not have a solution. Indeed, the range of f is the plane $z = 0$.

f is one-to-one: the solution to $f(x) = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}$ is $x_2 = \frac{1}{2}y_2$, $x_1 = \sqrt[3]{\frac{2y_1}{y_2}}$.

There is an easier way to check if a linear transformation is one-to-one:

Definition: The *kernel* of a linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the set of solutions to $T(x) = 0$.

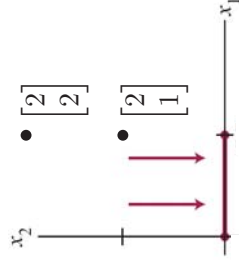
Fact: If $T(v_1) = T(v_2)$, then $v_1 - v_2$ is in the kernel of T .

Example: Let T be projection onto the x_1 -axis.

The kernel of T is the x_2 -axis.

$$T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ which is in the kernel.}$$



Proof of Fact: If $T(v_1) = T(v_2) = y$, then $T(v_1 - v_2) = T(v_1) - T(v_2) = y - y = 0$.

There is an easier way to check if a linear transformation is one-to-one:

Definition: The *kernel* of a linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the set of solutions to $T(x) = 0$.

Fact: If $T(v_1) = T(v_2)$, then $v_1 - v_2$ is in the kernel of T .

Theorem: A linear transformation is *one-to-one* if and only if its kernel is $\{0\}$.

Warning: this only works for linear transformations. For other functions, the solution sets of $f(x) = y$ and $f(x) = 0$ are not related.

Proof:

Suppose T is one-to-one. So $T(x) = 0$ has at most one solution. Since 0 is a solution, it must be the only one. So its kernel is $\{0\}$.

Suppose the kernel of T is $\{0\}$. Then, from the Fact, if there are vectors v_1, v_2 with $T(v_1) = T(v_2) = y$, then $v_1 - v_2 = 0$, i.e. $v_1 = v_2$.

For a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ whose standard matrix is A , the following are equivalent:

- a. $Ax = 0$ has no non-trivial solution.
- b. If $Ax = b$ is consistent, then it has a unique solution.
- c. The columns of A are linearly independent.
- d. $\text{rref}(A)$ has a pivot in every column (i.e. all variables are basic).
- e. T is a one-to-one function.

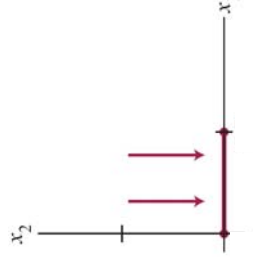
The range of a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the set of images, i.e. the set of y in \mathbb{R}^m with $T(x) = y$ for some x .

So, if A is the standard matrix of T , then the range of T is the set of b for which $Ax = b$ has a solution.

So the range of T is the span of the columns of A .

Example: The standard matrix of projection onto the x_1 -axis is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Its range is the x_1 -axis, which is also $\text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.



So the range of T is the span of the columns of A .

For a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ whose standard matrix is A , the following statements are logically equivalent (i.e. for any particular matrix A , they are all true or all false):

- a. For each b in \mathbb{R}^m , the equation $Ax = b$ has a solution.
- b. Each b in \mathbb{R}^m is a linear combination of the columns of A .
- c. The columns of A span \mathbb{R}^m .
- d. $\text{rref}(A)$ has a pivot in every row.
- e. T is onto.