1. (7 points) Compute the following two improper integrals, or explain why they do not converge. **Simplify your answer as much as possible.**

(a)

$$\int_e^\infty \frac{1}{x\sqrt{1+\ln x}}dx.$$

$$= \lim_{t\to\infty} \int_e^t \frac{1}{x\sqrt{1+\ln x}}\,dx$$

$u = 1+\ln x$

$$= \lim_{t\to\infty} \left[\frac{\sqrt{1+\ln x}}{1/2}\right]_e^t$$

$$= \lim_{t\to\infty} 2\sqrt{1+\ln t} - 2\sqrt{1+1}$$

This integral is divergent because, as $t\to\infty$, $\ln t\to\infty$ so $2\sqrt{1+\ln t}\to\infty$

(b)

$$\int_{\frac{1}{e}}^1 \frac{1}{x\sqrt{1+\ln x}}dx.$$

$$= \lim_{t\to 1/e^+} \int_t^1 \frac{1}{x\sqrt{1+\ln x}}\,dx$$

$$= \lim_{t\to 1/e^+} \left[2\sqrt{1+\ln x}\right]_t^1$$

$$= \lim_{t\to 1/e^+} 2 - 2\sqrt{1+\ln x}$$

$$= 2$$

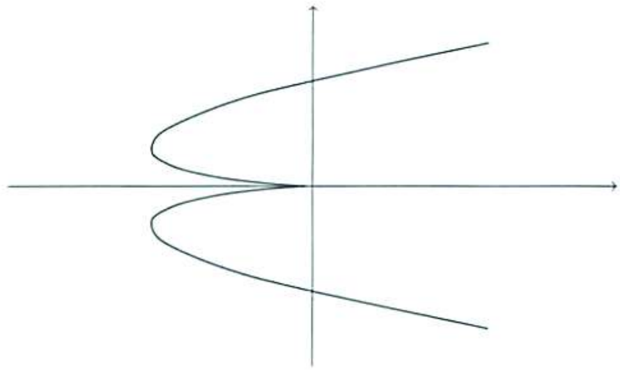
as $t\to 1/e^+$, $\ln t \to -1^+$

so $\sqrt{1+\ln t}\to 0^+$

2. (14 points) Let C be the parametrised curve with equation

$$x = \frac{t^6}{3} - 2t^4, \quad y = -\frac{8}{5}t^5,$$

as shown in the diagram below.



(a) Find the point(s) where C has a vertical tangent. **Simplify your answer as much as possible.**

C has a vertical tangent when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

$\frac{dx}{dt} = 0$ when $2t^5 - 8t^3 = 0$

$2t^3(t^2 - 4) = 0 \quad \therefore t = 0$ or $t = 2$ or $t = -2$.

When $t = 0$, $\frac{dy}{dt} = -8t^4 = 0$ – but this corresponds to the point $(0,0)$, and from the picture we see there is no vertical tangent here.

When $t = 2$, $\frac{dy}{dt} = -8(2)^4 \neq 0$
$t = -2$, $\frac{dy}{dt} = -8(-2)^4 \neq 0$
$\Big\}$ $\therefore$ these indeed give vertical tangents

$t = 2 \Rightarrow x = \frac{2^6}{3} - 2(2)^4 = \frac{-32}{3}$, $y = -\frac{8}{5}(2)^5 = \frac{-256}{5}$

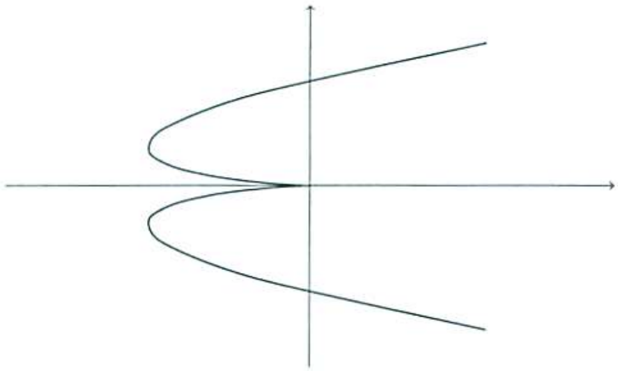
$t = -2 \Rightarrow x = \frac{(-2)^6}{3} - 2(-2)^4 = \frac{-32}{3}$, $y = -\frac{8}{5}(-2)^5 = \frac{256}{5}$

$\therefore C$ has vertical tangents at $\left(\frac{-32}{3}, \frac{-256}{5}\right)$ and $\left(\frac{-32}{3}, \frac{256}{5}\right)$

(Question 2 continues on the next page.)

(b) For your convenience, here again is the information about the parametrised curve C:

$$x = \frac{t^6}{3} - 2t^4, \quad y = -\frac{8}{5}t^5.$$



Find the length of the part of C with $-2 \le t \le -1$. **Simplify your answer as much as possible.**

$$\text{length} = \int_{-2}^{-1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_{-2}^{-1} \sqrt{(2t^5 - 8t^3)^2 + (-8t^4)^2}\, dt$$

$$= \int_{-2}^{-1} \sqrt{4t^{10} - 32t^8 + 64t^6 + 64t^8}\, dt$$

$$= \int_{-2}^{-1} \sqrt{4t^{10} + 32t^8 + 64t^6}\, dt$$

$$= \int_{-2}^{-1} \sqrt{4t^6(t^2+4)^2}\, dt$$

$$= \int_{-2}^{-1} |2t^3|\,|t^2+4|\, dt$$

$t^2+4>0$ always.
when $t \in [-2,-1]$, $2t^3<0$.

$$= \int_{-2}^{-1} -2t^3(t^2+4)\, dt$$

$$= \left[\frac{-2t^6}{6} - \frac{8t^4}{4}\right]_{-2}^{-1} = \left(\frac{-2}{6} - \frac{8}{4}\right) - \left(\frac{-2}{6}64 - \frac{8}{4}16\right) = 51.$$