Last time: Given $\sigma \in L(U,V)$, its dual $\hat{\sigma} \in L(\hat{V},\hat{U})$ is given by $\hat{\sigma}(\phi) = \phi \circ \sigma$.

If $\mathcal{A}, \mathcal{B}$ are bases of U, V, then

Th. 9.3.3 $${}_{\hat{\mathcal{A}} \leftarrow \hat{\mathcal{B}}}[\hat{\sigma}] = \left({}_{\mathcal{B} \leftarrow \mathcal{A}}[\sigma] \right)^T.$$

§ 9.2 Change of coordinates for linear forms

? Given $\mathcal{A}, \mathcal{B}$ bases of V, what is the matrix relating $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$, for $\hat{V}$?

i.e. what is $${}_{\hat{\mathcal{A}} \leftarrow \hat{\mathcal{B}}}[\iota_{\hat{V}}] \;?$$

Note: $\iota_{\hat{V}} = \widehat{\iota_V}$ $\because$ $\widehat{\iota_V}(\phi) = \phi \circ \iota_V = \phi$

$\therefore$ From Th 9.3.3:

$$ {}_{\hat{\mathcal{A}} \leftarrow \hat{\mathcal{B}}}[\iota_{\hat{V}}] = {}_{\hat{\mathcal{A}} \leftarrow \hat{\mathcal{B}}}[\widehat{\iota_V}] = \left({}_{\mathcal{B} \leftarrow \mathcal{A}}[\iota_V] \right)^T $$

equivalently

$$ {}_{\hat{\mathcal{A}} \leftarrow \hat{\mathcal{B}}}[\iota_{\hat{V}}] = \left({}_{\mathcal{A} \leftarrow \mathcal{B}}[\iota_V] \right)^{T^{-1}} $$

A way to understand this when

$\mathcal{A}$ = standard basis of $\mathbb{F}^n$,

$\mathcal{B} = \{\beta_1, \cdots, \beta_n\}$

Let $\widehat{\mathcal{B}} = \{\psi_1, \cdots, \psi_n\} \subseteq \hat{V}$,

so $\psi_i : V \to \mathbb{F}$

Consider

$$\underbrace{\begin{pmatrix} — [\psi_1] — \\ \vdots \\ — [\psi_n] — \end{pmatrix}}_{\substack{\text{standard matrix of } \psi_i \\ (1\times n \text{ matrix})}} \begin{pmatrix} | & & | \\ \beta_1 & \cdots & \beta_n \\ | & & | \end{pmatrix} = \begin{pmatrix} \psi_1(\beta_1) & \psi_1(\beta_2) & \cdots & \psi_1(\beta_n) \\ \vdots & & & \vdots \\ \psi_n(\beta_1) & \cdots & \cdots & \psi_n(\beta_n) \end{pmatrix}$$

i,j entry of ↑ is $\psi_i(\beta_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$\therefore$ matrix on RHS is I.

$$\text{So}\begin{pmatrix} — [\psi_1] — \\ \vdots \\ — [\psi_n] — \end{pmatrix} = \begin{pmatrix} | & & | \\ \beta_1 & \cdots & \beta_n \\ | & & | \end{pmatrix}^{-1} = \left([\iota_V]_{\mathcal{A} \leftarrow \mathbb{B}} \right)^{-1}$$

$$\| $$

$$[\iota_{\hat{V}}]^T_{\hat{\mathcal{A}} \leftarrow \hat{\mathbb{B}}}$$

$\therefore$ the ith column of $\begin{pmatrix} — [\psi_1] — \\ \vdots \\ — [\psi_n] — \end{pmatrix}^T$ is $[\psi_i]^T = [\psi_i]_{\hat{\mathcal{A}}}$

Ex. 9.2.3 If $\mathbb{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^3 = V$

find $\hat{\mathbb{B}}$.

Method 1 : use change of coordinates matrix:

let $\mathcal{A} = \{e_1, e_2, e_3\}$, and let $\hat{\mathbb{B}} = \{\psi_1, \psi_2, \psi_3\}$.

The columns of ${}_{\hat{A}\leftarrow\hat{B}}[\iota_{\hat{V}}]$ are $[\psi_1]_{\hat{A}}, [\psi_2]_{\hat{A}}, [\psi_3]_{\hat{A}}$.

$$ {}_{\hat{A}\leftarrow\hat{B}}[\iota_{\hat{V}}] = {}_{B\leftarrow A}[\iota_V]^T $$

columns of this are $[e_i]_B$.

$$ = \left({}_{A\leftarrow B}[\iota_V]^{-1} \right)^T $$

$$ = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}^{-1\,T} $$

row reduction to find inverse

$$ = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}^T $$

$$ = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix} $$

read the columns:

$[\psi_1]_{\hat{A}} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. i.e. $\psi_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2}x + \frac{1}{2}y - \frac{1}{2}z$

$\psi_1 = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2 - \frac{1}{2}\phi_3$ if $\hat{A} = \{\phi_1, \phi_2, \phi_3\}$

$$ \psi_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{2}x + \frac{1}{2}y - \frac{1}{2}z $$

$$ \psi_3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z. $$

Check: $\psi_i(\beta_i) = 1$ and $\psi_i(\beta_j) = 0$ if $i \neq j$.

e.g. $\psi_1(\beta_1) = \psi_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{2}1 + \frac{1}{2}0 - \frac{1}{2}(-1) = 1$ ✓

$\psi_2(\beta_3) = \psi_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = -\frac{1}{2}0 + \frac{1}{2}1 - \frac{1}{2}1 = 0$ ✓

Method 2: use definition of $\hat{B}$.

let $\psi_1\begin{pmatrix}x\\y\\z\end{pmatrix} = ax+by+cz$.

We need $\psi_1(\beta_1)=1$, $\psi_1(\beta_2)=0$, $\psi_1(\beta_3)=0$

$$\psi_1(\beta_1) = \psi_1\begin{pmatrix}1\\0\\-1\end{pmatrix} = a\cdot 1 + b\cdot 0 + c(-1) = 1$$

$$\underline{a - c = 1}$$

$$\psi_1(\beta_2) = \psi_1\begin{pmatrix}-1\\1\\0\end{pmatrix} = -a+b = 0$$

$$\psi_1(\beta_3) = \psi_1\begin{pmatrix}0\\1\\1\end{pmatrix} = b+c = 0$$

$$a=b=-c=\frac{1}{2}.$$

so $\psi_1\begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{1}{2}x+\frac{1}{2}y-\frac{1}{2}z$

similarly solve for ψ_2, ψ_3.