Example: Find the volume lying under the surface $z = 2x^2y + 3y^2$ and above the region $0 \le x \le 3$, $1 \le y \le 2$.

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$\int_0^3 \left(\int_1^2 2x^2y + 3y^2\,dy \right) dx$$

$$= \int_0^3 \left(x^2y^2 + y^3 \Big|_{y=1}^{y=2} \right) dx$$

$$= \int_0^3 4x^2 + 8 - (x^2+1)\,dx = \int_0^3 3x^2 + 7\,dx = x^3 + 7x \Big|_0^3$$

$$= 27 + 21 = 48$$

Redo Example: (p8) Find, by first integrating in x, the volume lying under the surface $z = 2x^2y + 3y^2$ and above the region $0 \le x \le 3$, $1 \le y \le 2$.

$$\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

$$\int_1^2 \left(\int_0^3 2x^2y + 3y^2\,dx \right) dy \quad \because \{y \text{ is a constant for this step}\}$$

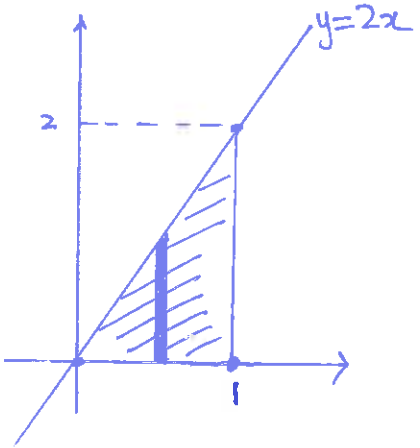
$$= \int_1^2 \left(\frac{2x^3y}{3} + 3xy^2 \Big|_{x=0}^{x=3} \right) dy = \int_1^2 18y + 9y^2 - 0\,dy$$

$$= 9y^2 + 3y^3 \Big|_1^2$$

$$= 36 + 24 - (9+3) = 48$$

Example: Find the volume lying under the surface $z = x^2 + y^2$ and above the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$.

first step: draw a picture of the domain to find the limits



region is
$0 \le y \le 2x$
$0 \le x \le 1$.

$$\int_0^1 \int_0^{2x} x^2+y^2 \, dy \, dx$$

$$= \int_0^1 x^2 y + \frac{y^3}{3} \Big|_{y=0}^{y=2x} dx$$

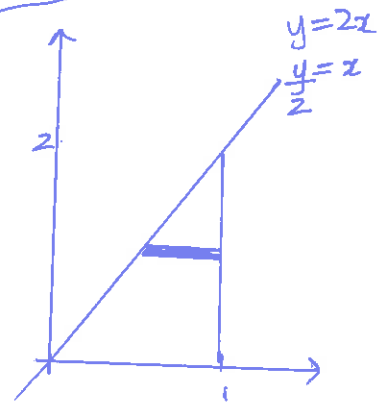
$$= \int_0^1 2x^3 + \frac{8x^3}{3} \, dx$$

$$= \int_0^1 \frac{14x^3}{3} \, dx = \frac{14x^4}{12} \Big|_0^1 = \frac{7}{6}.$$

check: first/inner integral is in y, its limits are functions of x.

check: no more y's

Alternative solution: integrate in x first.



$$\int_0^2 \int_{y/2}^1 x^2+y^2 \, dx \, dy = \int_0^2 \frac{x^3}{3} + xy^2 \Big|_{x=y/2}^{x=1} dy$$

$$= \int_0^2 \frac{1}{3} + y^2 - \left(\frac{y^3}{24} + \frac{y^3}{2}\right) dy = \int_0^2 \frac{1}{3} + y^2 - \frac{13y^3}{24} \, dy$$

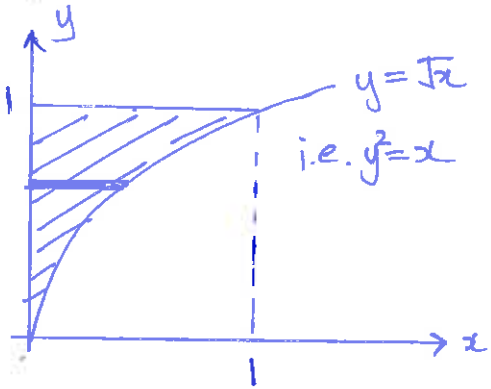
$$= \frac{y}{3} + \frac{y^3}{3} - \frac{13y^4}{24 \cdot 4} \Big|_0^2 = \frac{2}{3} + \frac{8}{3} - \frac{13}{6} = \frac{7}{6}.$$

As we saw above, we get the same answer whether we integrate first in x and then in y, or first in y and then in x. Sometimes one order is much easier than the other - changing the order is called reiterating the integral.

Example: Evaluate $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 e^{-y^3}\, dy\, dx$.

region is
$\sqrt{x} \le y \le 1$
$0 \le x \le 1$

we can rewrite this as
$0 \le x \le y^2$
$0 \le y \le 1$



We don't know any antiderivative of e^{-y^3}.
– so try integrating with respect to x first

$$\int_?^? \int_?^? e^{-y^3}\, dx\, dy$$

$$\int_0^1 \int_0^{y^2} e^{-y^3}\, dx\, dy = \int_0^1 x e^{-y^3} \Big|_{x=0}^{x=y^2} dy$$

$$= \int_0^1 y^2 e^{-y^3}\, dy$$

substitution:
$u = y^3$
$du = 3y^2\, dy$
$y=0 \to u=0$ $\quad y=1 \to u=1$

$$= \int_0^1 \frac{e^{-u}}{3}\, du$$

$$= -\frac{e^{-u}}{3}\Big|_0^1 = -\left(\frac{e^{-1}}{3} - \frac{1}{3}\right) = \frac{1-e^{-1}}{3}$$