Last time:

<u>The double dual</u>:

Recall $\hat{V} = L(V, \mathbb{F})$, so $\hat{\hat{V}} = L(\hat{V}, \mathbb{F})$.

i.e. each $f \in \hat{\hat{V}}$ is a function $\phi \mapsto$ number,

where the input ϕ is a function on V.

One example: f = evaluation at some fixed $\alpha \in V$,

i.e. f is $\phi \mapsto \phi(\alpha)$, i.e. $f(\phi) = \phi(\alpha)$.

Exercise: check this f is in $\hat{\hat{V}}$, i.e. f is linear (in its input in $\hat{V}$).

So, to every $\alpha \in V$, we can associate such an $f \in \hat{\hat{V}}$

Th. 9.2.1: The function $J: V \to \hat{\hat{V}}$ given by $J(\alpha)$ = evaluation at α, i.e. $[J(\alpha)](\phi) = \phi(\alpha)$, is an injective linear transformation.

Cor: If $\dim V < \infty$ then $\dim V = \dim \hat{V} = \dim \hat{\hat{V}}$

so the injection $J: V \to \hat{\hat{V}}$ is an isomorphism. (i.e. also a surjection — so every $f \in \hat{\hat{V}}$ is evaluation on some $\alpha \in V$)

Proof: linearity: we need to show $J(a\alpha+\beta) = aJ(\alpha) + J(\beta)$.

(an equation in $\hat{\hat{V}}$)

i.e. need to show that, $\forall \phi \in \hat{V}$,

(an equation in $\mathbb{F}$)

$$[J(a\alpha+\beta)](\phi) = [aJ(\alpha)+J(\beta)](\phi)$$
$$= a[J(\alpha)](\phi) + [J(\beta)](\phi) \quad \text{[definition of } af+g]$$

$$\text{LHS} = \phi(a\alpha+\beta) \quad \text{[definition of } J]$$
$$= a\phi(\alpha)+\phi(\beta) \quad [\because \phi \text{ is linear}]$$
$$= \text{RHS} \quad \text{[definition of } J]$$

injectivity: we need to show $\ker(J) = \{\vec{0}\}$ an equation in V

i.e. if $\underbrace{J(\alpha) = \vec{0}}$, then $\underbrace{\alpha = \vec{0}}_{\text{in } V}$

in $\hat{\hat{V}}$ zero function on $\hat{V}$.

if $J(\alpha) = \vec{0}$, then $\underbrace{[J(\alpha)](\phi) = 0}_{\text{an equation in } \mathbb{F}}$ for all $\phi \in \hat{V}$

so $\phi(\alpha) = 0$ for all $\phi \in \hat{V}$.

To show $\alpha = \vec{0}$: either use HW4 Q3a

or: let $\{\alpha_1, \cdots, \alpha_n\}$ be a basis of V,

and $\{\phi_1, \cdots, \phi_n\}$ be the dual basis of $\hat{V}$.

Then
$$\begin{aligned}
\alpha &= \phi_1(\alpha)\alpha_1 + \cdots + \phi_n(\alpha)\alpha_n && [\text{Prop. from before}] \\
&= 0\,\alpha_1 + \cdots + 0\,\alpha_n && [\because \phi(\alpha) = 0 \ \forall \phi] \\
&= \vec{0}
\end{aligned}$$

§9.4 Bilinear Forms and Quadratic Forms

Today: definitions

Monday: algorithm 1 for diagonalisation

Monday: " 2 for diagonalisation

(after: inner products:

§10.1 – redo things from 2207 in $P_n(\mathbb{R})$

§10.2 – inner products and ∇.)

Def 9.4.1: Let V be a vector space over $\mathbb{F}$

A <u>bilinear form on V</u> is a function $f: V \times V \to \mathbb{F}$ that is separately linear in each input.

i.e. $f(a\alpha_1 + \alpha_2, \beta) = af(\alpha_1, \beta) + f(\alpha_2, \beta)$

$f(\alpha, a\beta_1 + \beta_2) = af(\alpha, \beta_1) + f(\alpha, \beta_2)$

Ex 9.4.2: dot product in $\mathbb{R}^n$

$f(\alpha, \beta) = \alpha \cdot \beta = \alpha^T \beta$

e.g. on $\mathbb{R}^3$:

$$f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = x_1y_1 + x_2y_2 + x_3y_3$$

Ex: on $\mathbb{R}^3$:

$$g\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = x_1y_1 + 2x_1y_2 + x_3y_1$$

$$g\left(a\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = g\left(\begin{pmatrix} ax_1 + x_1' \\ ax_2 + x_2' \\ ax_3 + x_3' \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right)$$

$$= (ax_1 + x_1')y_1 + 2(ax_1 + x_1')y_2 + (ax_3 + x_3')y_1$$

$$= a(x_1y_1 + 2x_1y_2 + x_3y_1) + x_1'y_1 + 2x_1'y_2 + x_3'y_1$$

$$= a\,g\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) + g\left(\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right).$$

$$g\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, a\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix}\right) = \ldots$$ finish it as an exercise

Important Ex: on $\mathbb{R}^n$: $h(\alpha, \beta) = \alpha^T A \beta$ for some matrix A.

$= \alpha \cdot (A\beta)$

Exercise: check the axioms.

Ex 9.4.3 $V = C^0([0,1])$

$$k(F,G) = \int_0^1 F(x)\,G(x)\,dx$$

(can use any interval instead of $[0,1]$)

(More in §10.1)

Exercise: check this is bilinear.

Note:

$$af(\alpha,\beta) = f(a\alpha,\beta)$$

or: $af(\alpha,\beta) = f(\alpha,a\beta)$ $\neq f(a\alpha, a\beta)$.