1. (7 points) Compute the following two improper integrals, or explain why they do not converge. **Simplify your answer as much as possible.**

(a)

$$\int_{-\infty}^{-1} \frac{e^x}{(1-e^x)^2}\,dx.$$

$$= \lim_{t\to-\infty} \int_t^{-1} \frac{e^x}{(1-e^x)^2}\,dx$$

$$= \lim_{t\to-\infty} \left[\frac{-1}{-1(1-e^x)}\right]_t^{-1}$$

substitution
$u = 1-e^x$
$du = -e^x\,dx$

$$= \lim_{t\to-\infty} \left(\frac{1}{1-e^{-1}} - \frac{1}{1-e^t}\right)$$

$$= \frac{1}{1-e^{-1}} - 1$$

as $t\to-\infty$,
$e^t \to 0$
$1-e^t \to 1$

$$= \frac{e}{e-1} - \frac{e-1}{e-1} = \frac{1}{e-1}.$$

(b)

$$\int_0^1 \frac{e^x}{(1-e^x)^2}\,dx.$$

$$= \lim_{t\to0^+} \int_t^1 \frac{e^x}{(1-e^x)^2}\,dx$$

$$= \lim_{t\to0^+} \left[\frac{1}{1-e^x}\right]_t^1$$

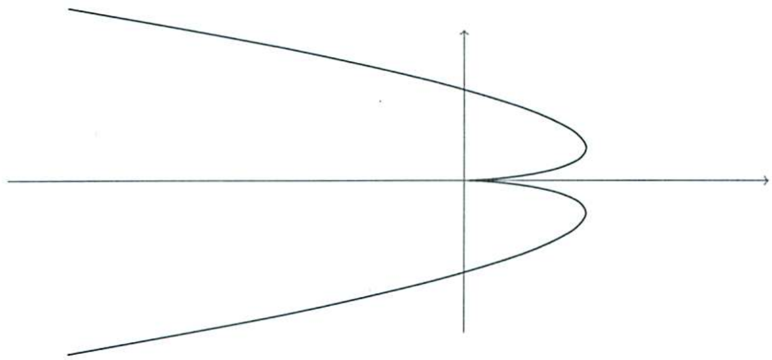
$$= \lim_{t\to0^+} \left(\frac{1}{1-e} - \frac{1}{1-e^t}\right)$$

This is divergent: as $t\to0^+$,
$e^t \to 1^+$
$1-e^t \to 0^-$
so $\frac{1}{2(1-e^t)} \to -\infty$

2. (14 points) Let C be the parametrised curve with equation

$$x = 4t^2 - \frac{t^4}{2}, \quad y = \frac{8}{3}t^3,$$

as shown in the diagram below.



(a) Find the point(s) where C has a vertical tangent. **Simplify your answer as much as possible.**

C has a vertical tangent when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

$$8t - 2t^3 = 0$$

$$2t(4 - t^2) = 0$$

$$t = 0 \quad \text{or} \quad t = 2 \quad \text{or} \quad t = -2.$$

When $t=0$: $\frac{dy}{dt} = 8t^2 = 0$ but $t=0$ corresponds to the point $(0,0)$, and from the picture we see there is no

When $t=2$: $\frac{dy}{dt} = 8(2)^2 \neq 0$

$t=-2$: $\frac{dy}{dt} = 8(-2)^2 \neq 0$

$\therefore$ indeed tangents are vertical here.

$t=2 \Rightarrow x = 4(2)^2 - \frac{2^4}{2} = 8$; $y = \frac{8}{3}(2)^3 = \frac{64}{3}$

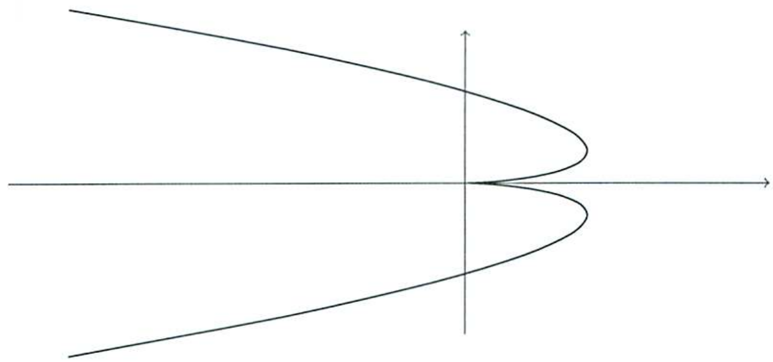
$t=-2 \Rightarrow x = 4(-2)^2 - \frac{(-2)^4}{2} = 8$ $y = \frac{8}{3}(-2)^3 = \frac{-64}{3}$

$\therefore C$ has vertical tangents at $(8, \frac{64}{3})$ and $(8, \frac{-64}{3})$

(Question 2 continues on the next page.)

(b) For your convenience, here again is the information about the parametrised curve C:

$$x = 4t^2 - \frac{t^4}{2}, \quad y = \frac{8}{3}t^3.$$



Find the length of the part of C with $-3 \le t \le -1$. **Simplify your answer as much as possible.**

$$\text{length} = \int_{-3}^{-1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_{-3}^{-1} \sqrt{(8t - 2t^3)^2 + (8t^2)^2}\, dt$$

$$= \int_{-3}^{-1} \sqrt{64t^2 - 32t^4 + 4t^6 + 64t^4}\, dt$$

$$= \int_{-3}^{-1} \sqrt{4t^6 + 32t^4 + 64t^2}\, dt$$

$$= \int_{-3}^{-1} \sqrt{4t^2(t^2+4)^2}\, dt$$

$$= \int_{-3}^{-1} |2t||t^2+4|\, dt$$

$t^2+4>0$ always, and $2t<0$ for $-3 \le t \le -1$.

$$= \int_{-3}^{-1} -2t(t^2+4)\, dt$$

$$= \left[-2\frac{t^4}{4} - 8\frac{t^2}{2}\right]_{-3}^{-1} = \frac{-2}{4} - \frac{8}{2} - \left(\frac{-2}{4}81 - \frac{8}{2}\cdot 9\right) = 72.$$