? Given $\psi \in \widehat{P_{<2}(\mathbb{R})}$, how to write ψ as a linear combination of ϕ_1, ϕ_2, ϕ_3?

Answer: $\psi = a'\phi_1 + b'\phi_2 + c'\phi_3$.

i.e. $\psi(p) = a'\phi_1(p) + b'\phi_2(p) + c'\phi_3(p) \quad \forall p \in P_{<2}(\mathbb{R})$.

in particular: $\psi(1) = a'\phi_1(1) + b'\phi_2(1) + c'\phi_3(1)$
$= a' \, 1 + b' \, 0 + c' \, 0 = a'$

$1 = \alpha_1$

$x = \alpha_2$

$\psi(x) = a'\phi_1(x) + b'\phi_2(x) + c'\phi_3(x)$
$= a' \, 0 + b' \, 1 + c' \, 0 = b'$

For same reason: $\psi(x^2) = c'$

For general V:

Prop: if $\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\}$ is a basis of V, and $\hat{\mathcal{A}} = \{\phi_1, \cdots, \phi_n\}$ is the dual basis of $\hat{V}$,

then $\beta = \phi_1(\beta)\alpha_1 + \cdots + \phi_n(\beta)\alpha_n$
$\psi = \psi(\alpha_1)\phi_1 + \cdots + \psi(\alpha_n)\phi_n$. (*)

Ex: (continue from before)

Let $\psi(f) = \int_0^1 f(t)\,dt$

Write ψ as a linear combination of $\hat{\mathcal{A}}$:

$\psi(1) = \int_0^1 1\,dt = 1 \qquad \psi(x^2) = \int_0^1 t^2\,dt = \frac{1}{3}$.

$\psi(x) = \int_0^1 t\,dt = \frac{1}{2} \qquad \therefore \psi = 1\phi_1 + \frac{1}{2}\phi_2 + \frac{1}{3}\phi_3$.

i.e. $\psi(a + bx + cx^2) = 1a + \frac{1}{2}b + \frac{1}{3}c$.

Note: ψ as a vector: $[\psi]_{\hat{\mathcal{A}}} = \begin{pmatrix} \psi(\alpha_1) \\ \vdots \\ \psi(\alpha_n) \end{pmatrix}$ from (*)

ψ as a linear transformation: $V \to \mathbb{F}$:

$[\psi]_{\{1\} \leftarrow \mathcal{A}} = \begin{pmatrix} \psi(\alpha_1) & \cdots & \psi(\alpha_n) \end{pmatrix}$

so $[\psi]_{\hat{\mathcal{A}}} = \left([\psi]_{\{1\} \leftarrow \mathcal{A}} \right)^T$

$\{1\}$ is the standard basis of $\mathbb{F} = \mathbb{F}^1$
e_1 in $\mathbb{F}^1$

(more §9.2 later)

§9.3 The dual of a linear transformation:

$$\begin{tikzcd} U \arrow[r, "\sigma"] \arrow[dr, "{\hat{\sigma}(\phi) = \phi\circ\sigma}"'] & V \arrow[d, "\phi"] \\ & \mathbb{F} \end{tikzcd}$$

Given $\sigma \in L(U,V)$ and $\phi \in \hat{V} \in L(V,\mathbb{F})$

we can define $\phi\circ\sigma \in L(U,\mathbb{F}) = \hat{U}$

$\because$ a composition of linear functions is linear.

This works for every $\phi \in \hat{V}$ — so we have a function $\hat{\sigma} : \hat{V} \longrightarrow \hat{U}$ given by

$\hat{\sigma}(\phi) = \phi\circ\sigma$ i.e. $\hat{\sigma}$ is "precomposition by σ".

Def 9.3.2. $\hat{\sigma}$ is the <u>dual</u> of σ.

Ex 1: $\sigma : P_{<3}(\mathbb{R}) \to P_{<2}(\mathbb{R})$ given by differentiation $\sigma(f) = f'$

$E_2 : P_{<2}(\mathbb{R}) \to \mathbb{R}$ given by

$E_2(g) = g(2)$.

$$\begin{tikzcd} P_{<3} \arrow[r, "\sigma"] \arrow[dr, "\hat{\sigma}(E_2)"'] & P_{<2} \arrow[d, "E_2"] \\ & \mathbb{R} \end{tikzcd}$$

Then $\hat{\sigma}(E_2) : P_{<3}(\mathbb{R}) \to \mathbb{R}$

$$\hat{\sigma}(E_2) = E_2 \circ \sigma$$

i.e. $$\begin{aligned} [\hat{\sigma}(E_2)](f) &= (E_2 \circ \sigma)(f) \\ &= E_2(\sigma(f)) \\ &= E_2(f') = f'(2). \end{aligned}$$

Ex 2: $\sigma : \mathbb{R}^3 \to \mathbb{R}^2$, $\sigma(\alpha) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \end{pmatrix} \alpha$

i.e. $\sigma\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 4x + 5z \end{pmatrix}$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "\sigma"] \arrow[dr, "\hat{\sigma}(\phi)"'] & \mathbb{R}^2 \arrow[d, "\phi"] \\ & \mathbb{R} \end{tikzcd}$$

$\phi \in \widehat{\mathbb{R}^2}$ i.e. $\phi : \mathbb{R}^2 \to \mathbb{R}$

$\phi\begin{pmatrix} u \\ v \end{pmatrix} = au + bv$

$\hat{\sigma}(\phi) \in \widehat{\mathbb{R}^3}$, i.e. $\hat{\sigma}(\phi) : \mathbb{R}^3 \to \mathbb{R}$

$$[\hat{\sigma}(\phi)]\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \phi\left(\sigma\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \phi\begin{pmatrix} x+2y+3z \\ 4x+5z \end{pmatrix}$$

$$= a(x+2y+3z) + b(4x+5z)$$

<u>Th 9.3.1</u> for $\sigma \in L(U,V)$, the dual $\hat{\sigma} : \hat{V} \longrightarrow \hat{U}$ is linear (i.e. $\hat{\sigma} \in L(\hat{V},\hat{U})$)

<u>Proof:</u> We need to show, for all $\phi, \psi \in \hat{V}$, and $a \in \mathbb{F}$

$$\hat{\sigma}(a\phi + \psi) = a\hat{\sigma}(\phi) + \hat{\sigma}(\psi).$$

$$\hat{\sigma}(a\phi + \psi) = (a\phi + \psi) \circ \sigma \qquad [\text{definition of } \hat{\sigma}]$$

So
$$\begin{aligned}
[\hat{\sigma}(a\phi+\psi)](\alpha) &= (a\phi+\psi)(\sigma(\alpha)) \\
&= a\phi(\sigma(\alpha)) + \psi(\sigma(\alpha)) \quad [\text{definition of } a\phi+\psi] \\
&= a(\phi\circ\sigma)(\alpha) + (\psi\circ\sigma)(\alpha) \\
&= (a\phi\circ\sigma + \psi\circ\sigma)(\alpha) \quad [\text{definition of } a\phi\circ\sigma + \psi\circ\sigma] \\
&= (a\hat{\sigma}(\phi) + \hat{\sigma}(\psi))(\alpha)
\end{aligned}$$

This holds $\forall \alpha \in U$, so $\hat{\sigma}(a\phi+\psi) = a\hat{\sigma}(\phi) + \hat{\sigma}(\psi)$.