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Since FTC says that integration is antidifferentiation, we can derive from these differentiation rules two techniques of integration:

chain rule                       $\longrightarrow$     method of substitution (p2-15, §5.6)

product rule                    $\longrightarrow$     integration by parts (p16-22, §6.1)

These techniques are **not** rules. They do not give us the answer; they only **change our integral to a new integral**, which we hope will be easier to evaluate. There are no rules in integration: there is no guaranteed algorithm to integrate a function. Using the techniques require some creativity, and there are often multiple efficient ways to calculate the same integral.

## §5.6: The Method of Substitution

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Write  $u$  for  $g(x)$ :

$$F(u) + C = \int F'(u) \frac{du}{dx} dx$$

Write  $f$  for  $F'$ :

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Hence, if we can identify a function  $u(x)$  such that our integrand is a product, of the composition  $f(u(x))$  and the derivative  $\frac{du}{dx}$  then we can rewrite our integral as  $\int f(u) du$ .

$$\int f(u) \frac{du}{dx} dx = \int f(u) du.$$

(i.e. we can treat  $\frac{du}{dx}$  formally like a fraction)

**Example:** Evaluate  $\int \cos(x^3) 3x^2 dx$ .

$$\int f(u) \frac{du}{dx} dx = \int f(u) du.$$

**Example:** Evaluate  $\int e^{3x} dx$ .

$$\int f(u) \frac{du}{dx} dx = \int f(u) du.$$



**Example:** Evaluate  $\int x \sqrt{1 + x^2} \, dx$ .

$$\int f(u) \frac{du}{dx} \, dx = \int f(u) \, du.$$

There are two ways to calculate a definite integral by substitution:

1. Find the indefinite integral and then substitute in the limits for  $x$ ;
2. (Usually faster) Change the limits into limits for  $u$ .

**Example:** Evaluate  $\int_0^1 x \sqrt{1 + x^2} dx$ .

Two other correct ways to use method 1:

$$\begin{aligned} & \int x \sqrt{1+x^2} \, dx \\ &= \int \frac{1}{2} \sqrt{u} \, du \\ &= \frac{u^{3/2}}{2(3/2)} + C \\ &= \frac{1}{3} \sqrt{1+x^2}^3 + C, \end{aligned}$$

so

$$\begin{aligned} & \int_0^1 x \sqrt{1+x^2} \, dx \\ &= \left. \frac{1}{3} \sqrt{1+x^2}^3 \right|_1^0 = \frac{1}{3} (\sqrt{2}^3 - 1). \end{aligned}$$

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$$\begin{aligned} & \int_0^1 x \sqrt{1+x^2} dx \\ &= \int_{x=0}^{x=1} \frac{1}{2} \sqrt{u} du \\ &= \left. \frac{u^{3/2}}{2(3/2)} \right|_{x=0}^{x=1} \\ &= \left. \frac{1}{3} \sqrt{1+x^2}^3 \right|_1^0 = \frac{1}{3} (\sqrt{2}^3 - 1). \end{aligned}$$

Do **not** write  $\int_0^1 \frac{1}{2} \sqrt{u} du$  - that would mean you want to evaluate at  $u = 0, 1$ .

Note that the final two steps in method 1 are to change the indefinite integral from  $us$  to  $x$ , then substitute the limits of  $x$ . In method 2 below, we combine these two steps – simply substitute the corresponding limits for  $u$ .

**Example:** Evaluate  $\int_0^1 x \sqrt{1 + x^2} dx$ .

$$\int f(u) \frac{du}{dx} dx = \int f(u) du.$$

Tips for choosing a good  $u$ :

- If the integrand contains a composite function e.g.  $e^{g(x)}$ ,  $\cos(g(x))$ ,  $\sin(g(x))$ ,  $\sqrt{g(x)}$ ,  $\frac{1}{g(x)}$ , try  $u = g(x)$ .
- Choose a  $u$  for which  $\frac{du}{dx}$  appears in the integrand.

The best way to get better at choosing  $u$  is to do lots of problems, and **think about** why your chosen  $u$  was effective.

Very important: make sure your integrand is **entirely in terms of  $u$**  (no  $x$ s) before you start integrating.

**Harder example:** Evaluate  $\int_0^1 x^3 \sqrt{1-x^2} \, dx$ .

**Harder example:** Evaluate  $\int_0^1 \frac{x^2}{1+x^6} dx$ .



Using various trigonometric identities and the method of substitution, we can obtain the integrals of many trigonometric functions - these will be given to you on the exams.

**Examples:**

$$\begin{aligned}\int \cos^2 x \, dx &= \int \frac{1}{2}(1 + \cos(2x)) \, dx \\ &= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C \\ &= \frac{1}{2}x + \frac{1}{2}\sin x \cos x + C.\end{aligned}$$

by the identity  $\cos(2x) = 2\cos^2 x - 1$

substitution  $u = 2x$  in the second term

by the identity  $\sin(2x) = 2\sin x \cos x$

$$\begin{aligned}\int \cos^3 x \, dx &= \int \cos x(1 - \sin^2 x) \, dx \\ &= \int \cos x - \cos x \sin^2 x \, dx \\ &= \sin x + \frac{1}{3}\sin^3 x + C.\end{aligned}$$

by the identity  $\cos^2 x + \sin^2 x = 1$

substitution  $u = \sin x$  in the second term

The full list of trigonometric-power integrals you will be given in exams:

$$\int \sin^2 x \, dx = \frac{1}{2}(x - \sin x \cos x) + C,$$

$$\int \sin^3 x \, dx = -\cos x + \frac{1}{3} \cos^3 x + C.$$

$$\int \sin^4 x \, dx = \frac{1}{8}(3x - 3 \sin x \cos x - 2 \sin^3 x \cos x) + C,$$

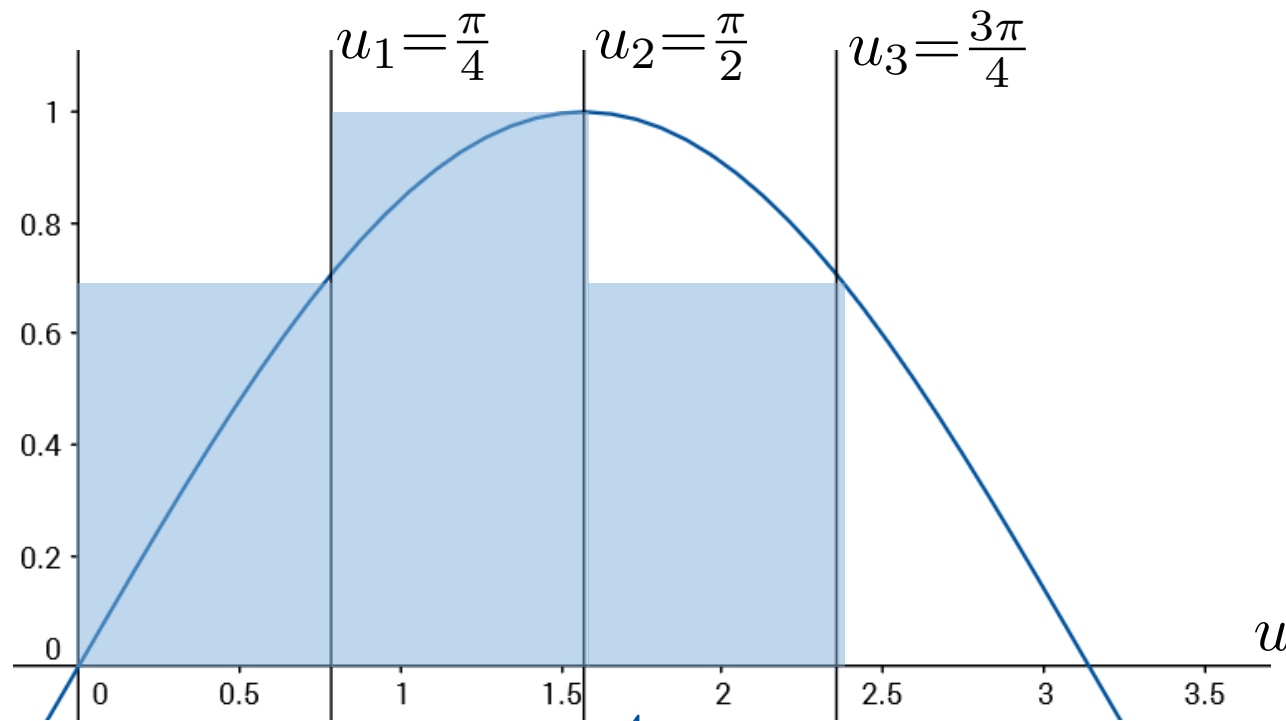
$$\int \cos^2 x \, dx = \frac{1}{2}(x + \sin x \cos x) + C,$$

$$\int \cos^3 x \, dx = \sin x - \frac{1}{3} \sin^3 x + C.$$

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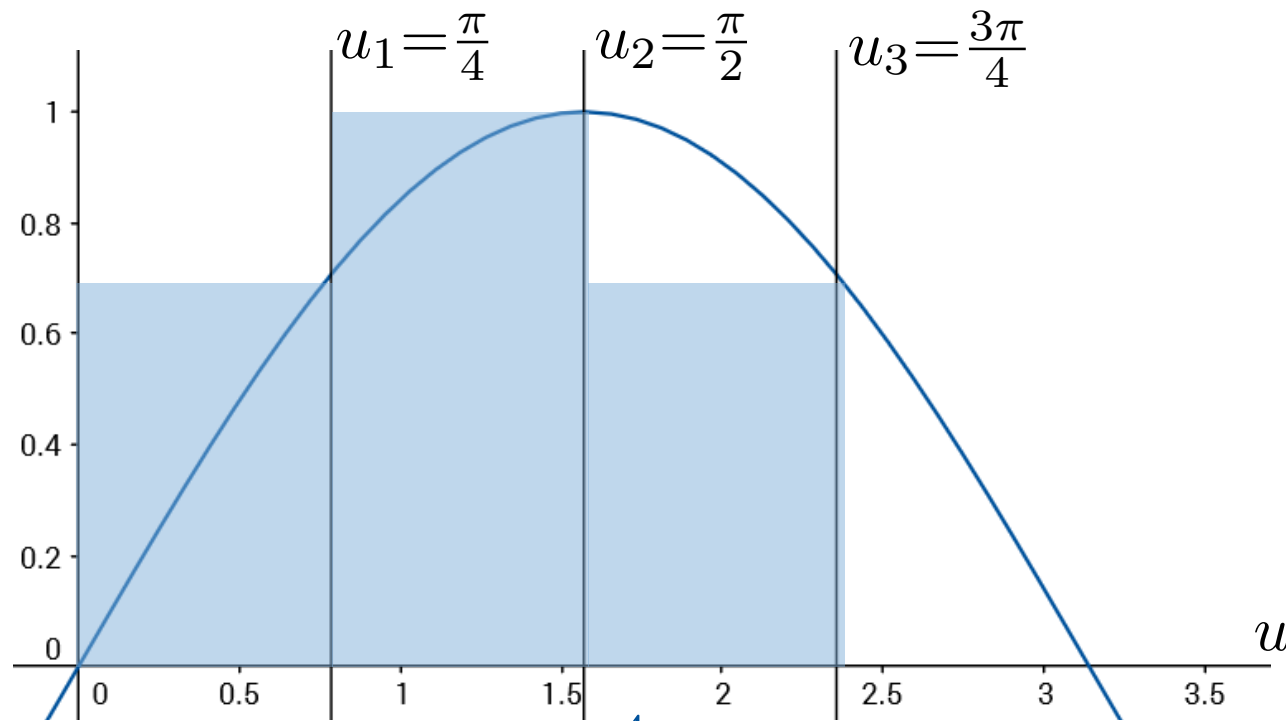
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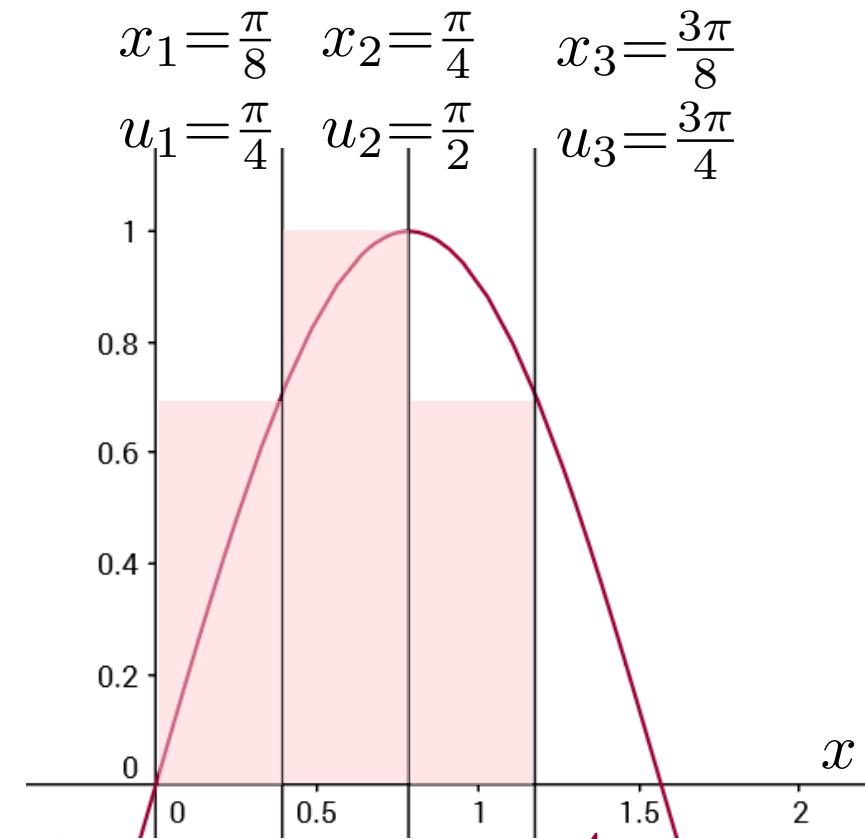


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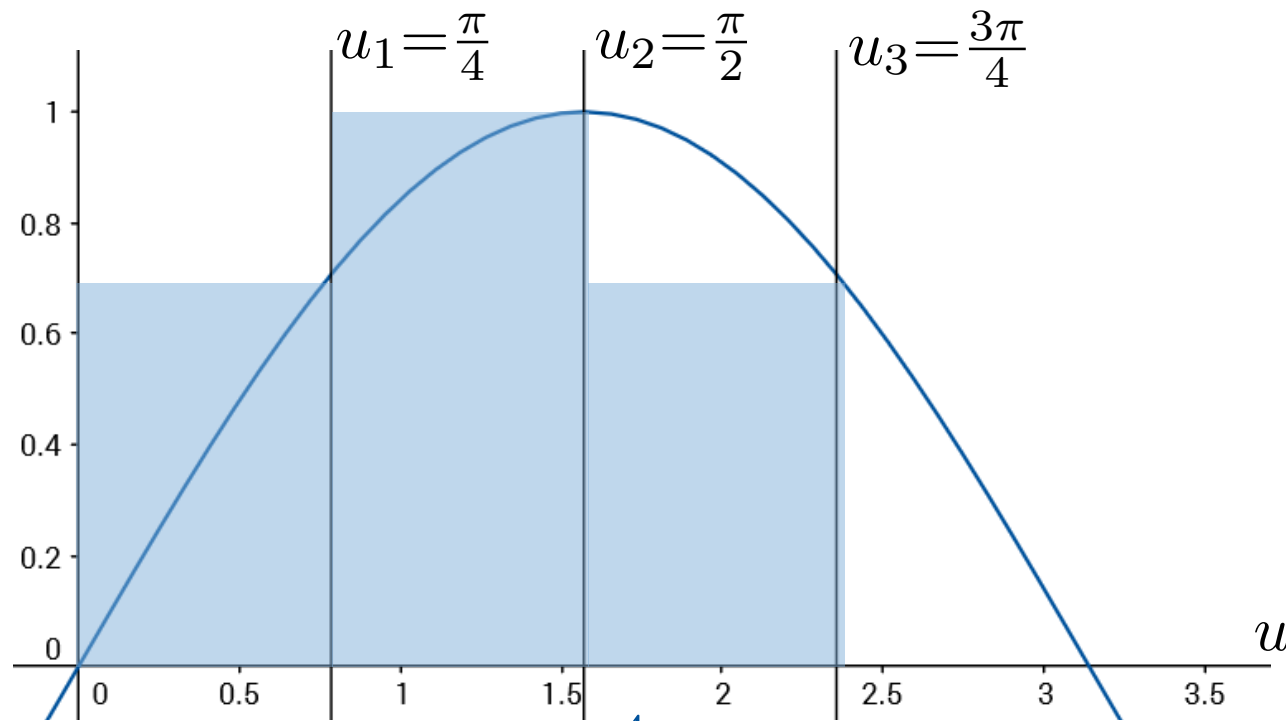


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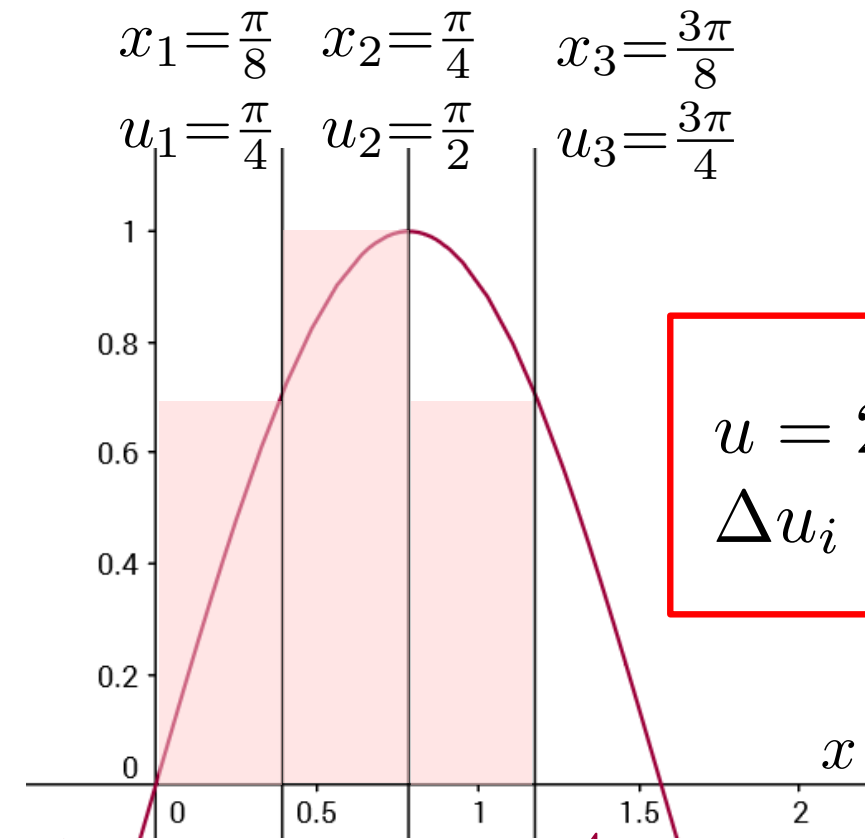
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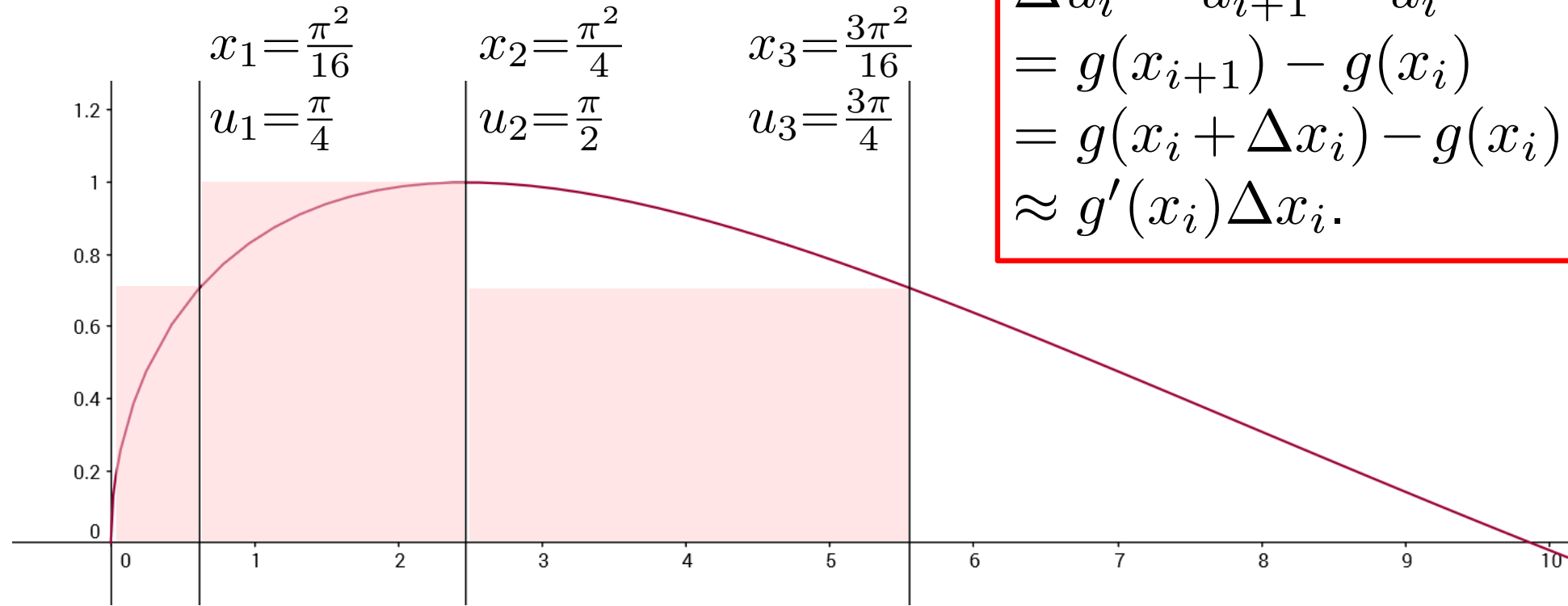
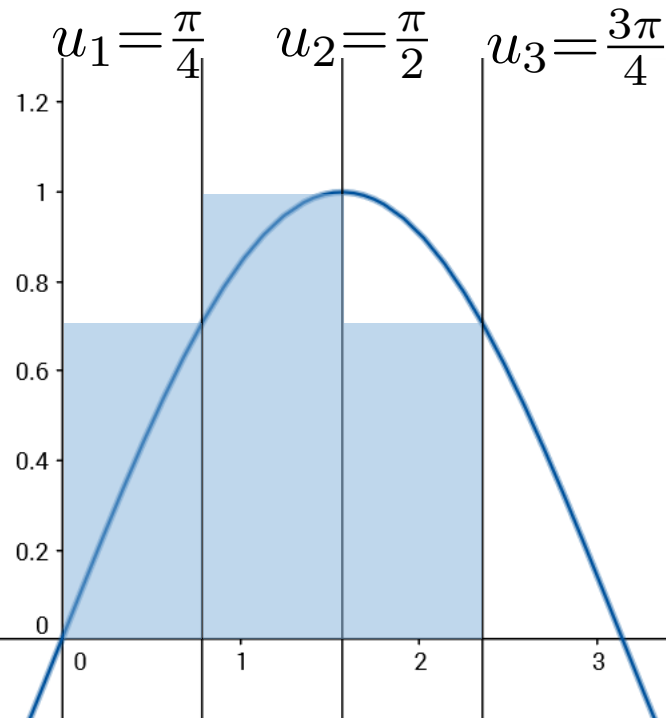
$$\int_0^{\pi} \sin u \, du \approx \sum_{i=1}^4 \sin(u_i) \Delta u_i.$$

The heights of the two sets of approximating rectangles are the same, but on the right the rectangles are half as wide.



$$\begin{aligned} \int_0^{\pi/2} \sin(2x) \, dx &\approx \sum_{i=1}^4 \sin(2x_i) \Delta x_i \\ &= \sum_{i=1}^4 \sin(u_i) \frac{1}{2} \Delta u_i \approx \int_0^{\pi} \sin u \, \frac{1}{2} du. \end{aligned}$$

When  $u$  is not a linear function of  $x$ , the widths of the rectangles stretch by different amounts.



When  $u = g(x)$ , then  
 $\Delta u_i = u_{i+1} - u_i$   
 $= g(x_{i+1}) - g(x_i)$   
 $= g(x_i + \Delta x_i) - g(x_i)$   
 $\approx g'(x_i) \Delta x_i.$

$$\int_0^\pi \sin u \, du \approx \sum_{i=1}^4 \sin(u_i) \Delta u_i.$$

In this example,  $u = \sqrt{x}$ , so

$$\Delta u_i \approx \frac{1}{2\sqrt{x_i}} \Delta x_i = \frac{1}{2u} \Delta x_i.$$

$$\int_0^{\pi^2} \sin \sqrt{x} \, dx \approx \sum_{i=1}^4 \sin \sqrt{x_i} \Delta x_i$$

$$\approx \sum_{i=1}^4 \sin(u_i) 2u \Delta u_i \approx \int_0^\pi \sin u \, 2u \, du.$$

## §6.1: Integration by Parts

Recall the product rule for differentiation:

$$\frac{d}{dx}(U(x)V(x)) = U(x)\frac{dV}{dx} + V\frac{dU}{dx}$$



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Take the antiderivative of both sides:

$$U(x)V(x) = \int U(x)\frac{dV}{dx} dx + \int V\frac{dU}{dx} dx$$

Re-arrange:

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Take the antiderivative of both sides:

$$U(x)V(x) = \int U(x)\frac{dV}{dx} dx + \int V\frac{dU}{dx} dx$$

Rearranging:

$$\int U(x)\frac{dV}{dx} dx = U(x)V(x) - \int V\frac{dU}{dx} dx$$

A shorthand that is easy to remember:

$$\int U dV = UV - \int V dU$$

**Standard example:** Evaluate  $\int x e^x dx$ .

$$\int U dV = UV - \int V dU$$

**Standard example:** Evaluate  $\int x \sin x \, dx$ .

$$\int U \, dV = UV - \int V \, dU$$

**Standard example:** Evaluate  $\int x \ln x \, dx$ .

$$\int U \, dV = UV - \int V \, dU$$

The technique of integration by parts relies on separating your integrand into two parts, a  $U$  and a  $\frac{dV}{dx}$ . Because we need to calculate  $\int V dU$ , we want  $U$  to be easy to differentiate and  $V$  to be easy to integrate. One good strategy to choose these parts is the DETAIL rule:

$$\int U dV = UV - \int V dU$$

**d** $V$  should be the part of the integrand that appears highest in this list:

**E**xponential:  $e^x$

**T**rigonometric:  $\sin x, \cos x$

**A**lgebraic:  $x^n$

**I**nverse trigonometric:  $\sin^{-1} x, \tan^{-1} x$

**L**ogarithmic:  $\ln x$

} nice to integrate

} hard to integrate

In our previous examples:

$xe^x$  (p17) is a product of an algebraic and an exponential function, and exponential is higher on the list, so  $dV = e^x dx$  and  $U = x$ .

$x \ln x$  (p19) is a product of an algebraic and a logarithmic function, and algebraic is higher on the list, so  $dV = x dx$  and  $U = \ln x$ .

Sometimes, after integration by parts, our new integral again requires integration by parts:

**Example:** Evaluate  $\int_0^2 (xe^x)^2 dx$ .

$$\int U dV = UV - \int V dU$$

Some integrals are best calculated using a substitution and then integration by parts. (It can also happen that, after integration by parts, the new integral requires a substitution.)

**Example:** Evaluate  $\int x^3 e^{x^2} dx$ .

$$\int U dV = UV - \int V dU$$



## §5.7: Areas of Plane Regions

This is a simple, first example where we calculate a geometric quantity by writing it as a limit of a Riemann sum, identifying it as an integral, and then using FTC2. (We will do more of this in higher dimensions).

Given functions  $f, g : [a, b] \rightarrow \mathbb{R}$  with  $f(x) \geq g(x)$  we wish to find the area bounded by  $y = f(x), y = g(x), x = a, x = b$ .

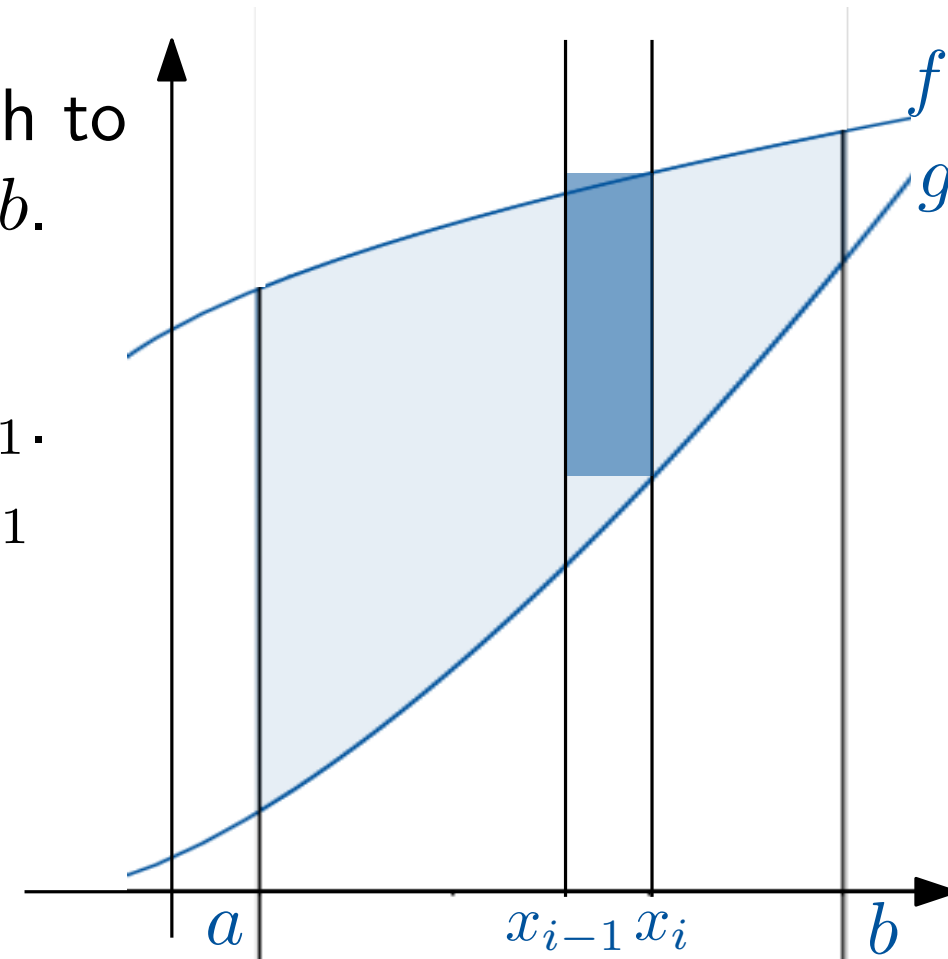
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1. Divide  $[a, b]$  into  $n$  subintervals by choosing  $x_i$  with  $a = x_0 < x_1 < \cdots < x_n = b$ , and let  $\Delta x_i = x_i - x_{i-1}$ .
2. Approximate the part of the desired area between  $x_{i-1}$  and  $x_i$  by a rectangle, whose width is  $\Delta x_i$  and whose height is  $f(x_i^*) - g(x_i^*)$ , for some  $x_i^* \in [x_{i-1}, x_i]$ .
3. So the area is

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (f(x_i^*) - g(x_i^*)) \Delta x_i =$$



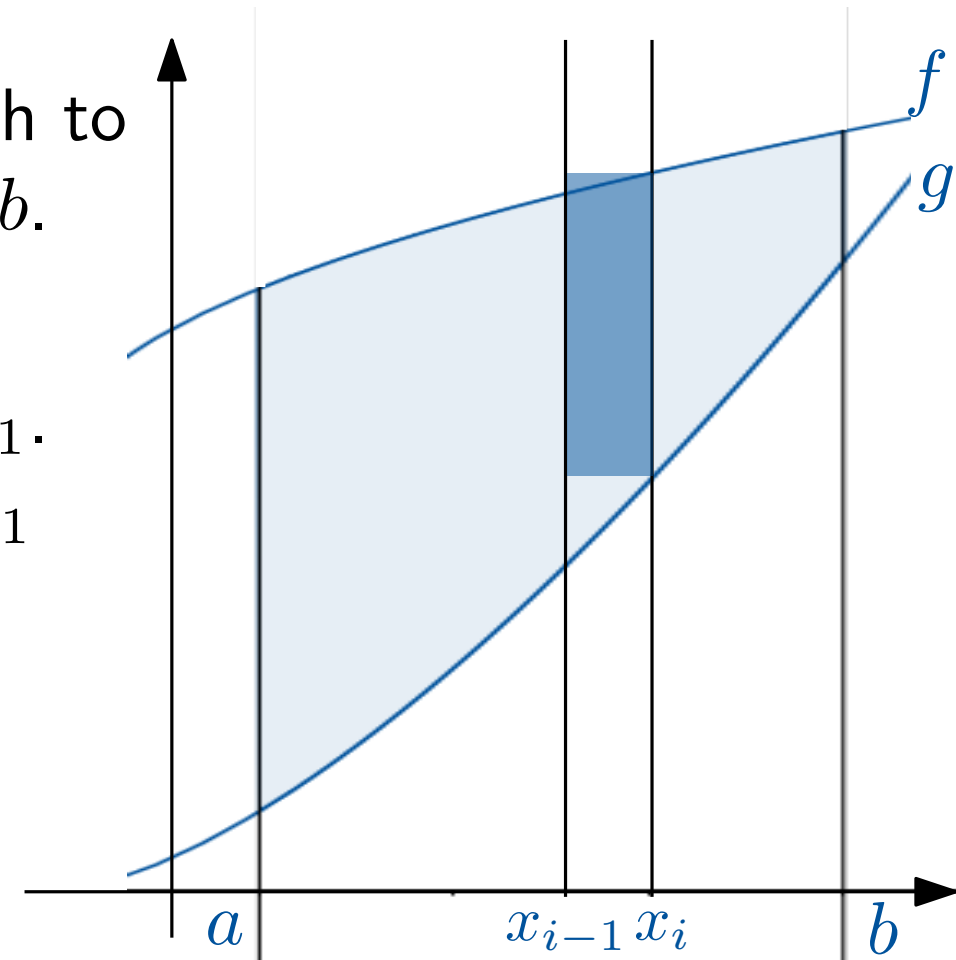
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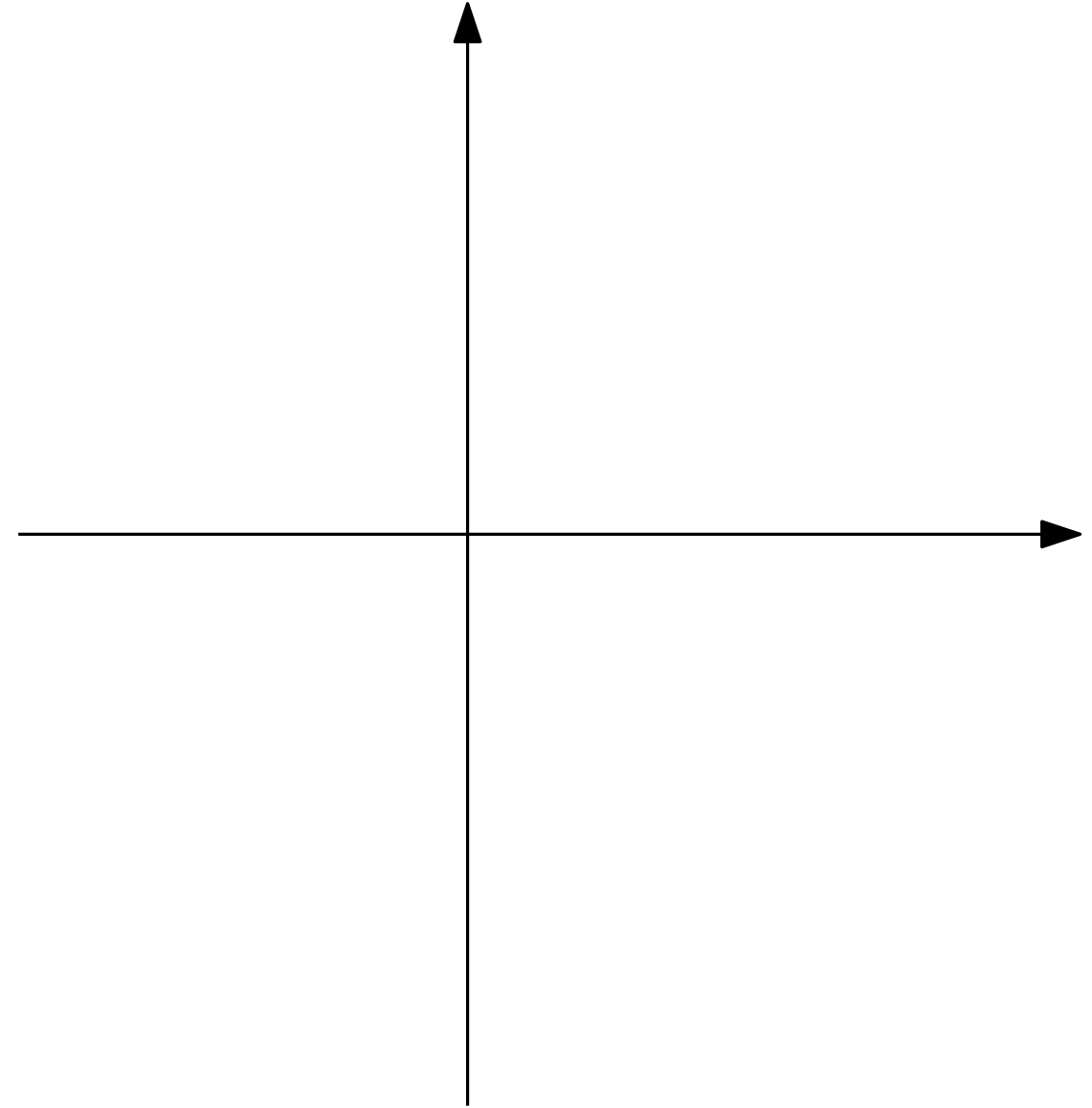
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$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (f(x_i^*) - g(x_i^*)) \Delta x_i = \int_a^b f(x) - g(x) dx.$$



**Example:** Find the area of the region bounded by  $y = x^2 - 4$  and  $y = -x^2 + 2x$ .



**Example:** Find the area of the region bounded by  $y = 2\sqrt{x}$ ,  $y = 3 - x$  and  $y = 0$ .

