

# §7.1: Diagonalisation of Symmetric Matrices

Symmetric matrices ( $A = A^T$ ) arise naturally in many contexts, when  $a_{ij}$  depends on  $i$  and  $j$  but not on their order (e.g. the friendship matrix from Homework 3 Q7, the Hessian matrix of second partial derivatives from Multivariate Calculus). The goal of this section is to observe some very nice properties about the eigenvectors of a symmetric matrix.

**Example:**  $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$  is a symmetric matrix.

$$\begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \text{ so } \begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ is a } -1\text{-eigenvector.}$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \text{ so } \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ is a } 4\text{-eigenvector.}$$

Notice that the eigenvectors are orthogonal:  $\begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$ . This is not a coincidence...

**Theorem 1: Eigenvectors of Symmetric Matrices:** If  $A$  is a symmetric matrix, then eigenvectors corresponding to **distinct eigenvalues** are **orthogonal**. Compare: for an arbitrary matrix, eigenvectors corresponding to distinct eigenvalues are linearly independent (week 10 p22).

**Proof:** Suppose  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are eigenvectors corresponding to distinct eigenvalues  $\lambda_1$  and  $\lambda_2$ . Then

$$(A\mathbf{v}_1) \cdot \mathbf{v}_2 = (\lambda_1 \mathbf{v}_1) \cdot \mathbf{v}_2 = \lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2),$$

and

$$\mathbf{v}_1 \cdot (A\mathbf{v}_2) = \mathbf{v}_1 \cdot (\lambda_2 \mathbf{v}_2) = \lambda_2(\mathbf{v}_1 \cdot \mathbf{v}_2).$$

But the two left hand sides above are equal, because (see also week 12 p25)

$$(A\mathbf{v}_1) \cdot \mathbf{v}_2 = (A\mathbf{v}_1)^T \mathbf{v}_2 = \mathbf{v}_1^T A^T \mathbf{v}_2 = \mathbf{v}_1^T A \mathbf{v}_2 = \mathbf{v}_1 \cdot (A\mathbf{v}_2).$$

$A$  is symmetric

So the two right hand sides are equal:  $\lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2) = \lambda_2(\mathbf{v}_1 \cdot \mathbf{v}_2)$ . Since  $\lambda_1 \neq \lambda_2$ , it must be that  $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ .

Remember from week 10 §5:

**Definition:** A square matrix  $A$  is *diagonalisable* if there is an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $A = PDP^{-1}$ .

**Diagonalisation Theorem:** An  $n \times n$  matrix  $A$  is *diagonalisable* if and only if  $A$  has  $n$  linearly independent eigenvectors. Those eigenvectors are the columns of  $P$ .

Given our previous observation, we are interested in when a matrix has  $n$  orthogonal eigenvectors. Because any scalar multiple of an eigenvector is also an eigenvector, this is the same as asking, when does a matrix have  $n$  orthonormal eigenvectors, i.e. when is the matrix  $P$  in the Diagonalisation Theorem an orthogonal matrix?

**Definition:** A square matrix  $A$  is *orthogonally diagonalisable* if there is an orthogonal matrix  $P$  and a diagonal matrix  $D$  such that  $A = PDP^{-1}$ , or equivalently,  $A = PDP^T$ .

We can extend the previous theorem (being careful about eigenvectors with the same eigenvalue) to show that any diagonalisable symmetric matrix is orthogonally diagonalisable, see the example on the next page.

**Example:** Orthogonally diagonalise  $B = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$ , i.e. find an orthogonal  $P$  and diagonal  $D$  with  $B = PDP^{-1}$ .

**Answer:**

**Step 1** Solve the characteristic equation  $\det(B - \lambda I) = 0$  to find the eigenvalues.

Eigenvalues are 2 and 5.

**Step 2** For each eigenvalue  $\lambda$ , solve  $(B - \lambda I)\mathbf{x} = \mathbf{0}$  to find a basis for the  $\lambda$ -eigenspace.

This gives  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$  as a basis for the 2-eigenspace, and  $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$  as a basis for the 5-eigenspace. Notice the 2-eigenvector is orthogonal to both the 5-eigenvectors, but the two 5-eigenvectors are not orthogonal.

**Step 2A** For each eigenspace of dimension  $> 1$ , find an orthogonal basis (e.g. by Gram-Schmidt) Applying Gram-Schmidt to the above basis for the 5-eigenspace

gives  $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$ . To avoid fractions, let's use  $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$ , which is still

an orthogonal set.

**Step 2B** Normalise all the eigenvectors

$\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$  is an orthonormal basis for the 2-eigenspace, and  $\left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}$

is an orthonormal basis for the 5-eigenspace.

**Step 3** Put the normalised eigenvectors from Step 2B as the columns of  $P$ .

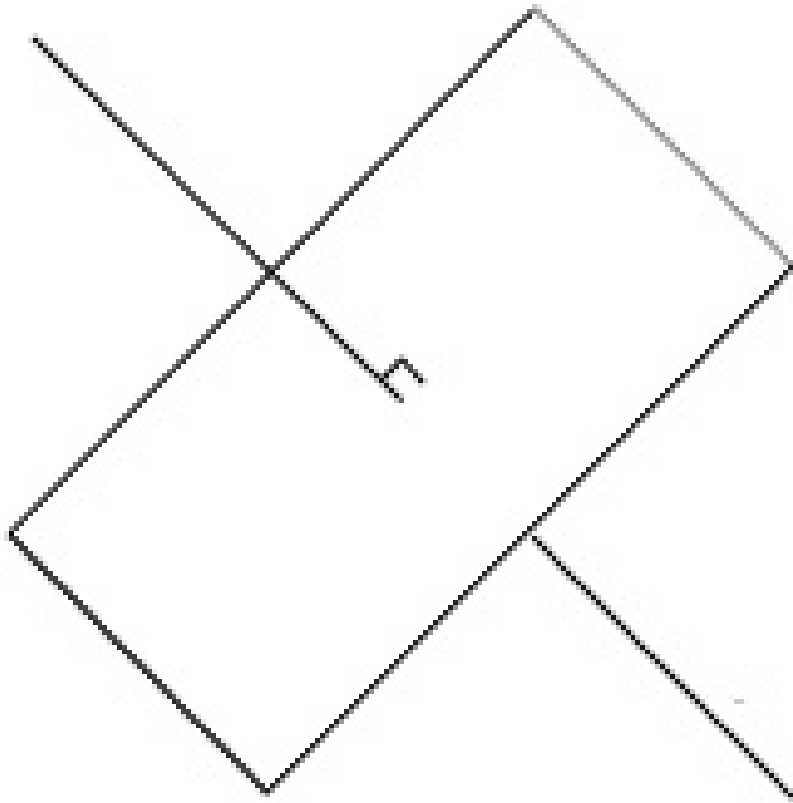
**Step 4** Put the corresponding eigenvalues as the diagonal entries of  $D$ .

$$P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Check our answer:

$$PDP^T = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}.$$

A geometric illustration of "orthonormalising" the eigenvectors:



This algorithm shows that any diagonalisable symmetric matrix is orthogonally diagonalisable.

Amazingly, every symmetric matrix is diagonalisable:

**Theorem 3: Spectral Theorem for Symmetric Matrices:** A symmetric matrix is **orthogonally diagonalisable**, i.e. it has a orthonormal basis of eigenvectors.

(The name of the theorem is because the **set** of eigenvalues and multiplicities of a matrix is called its **spectrum**. There are spectral theorems for many types of linear transformations.)

The reverse direction is also true, and easy:

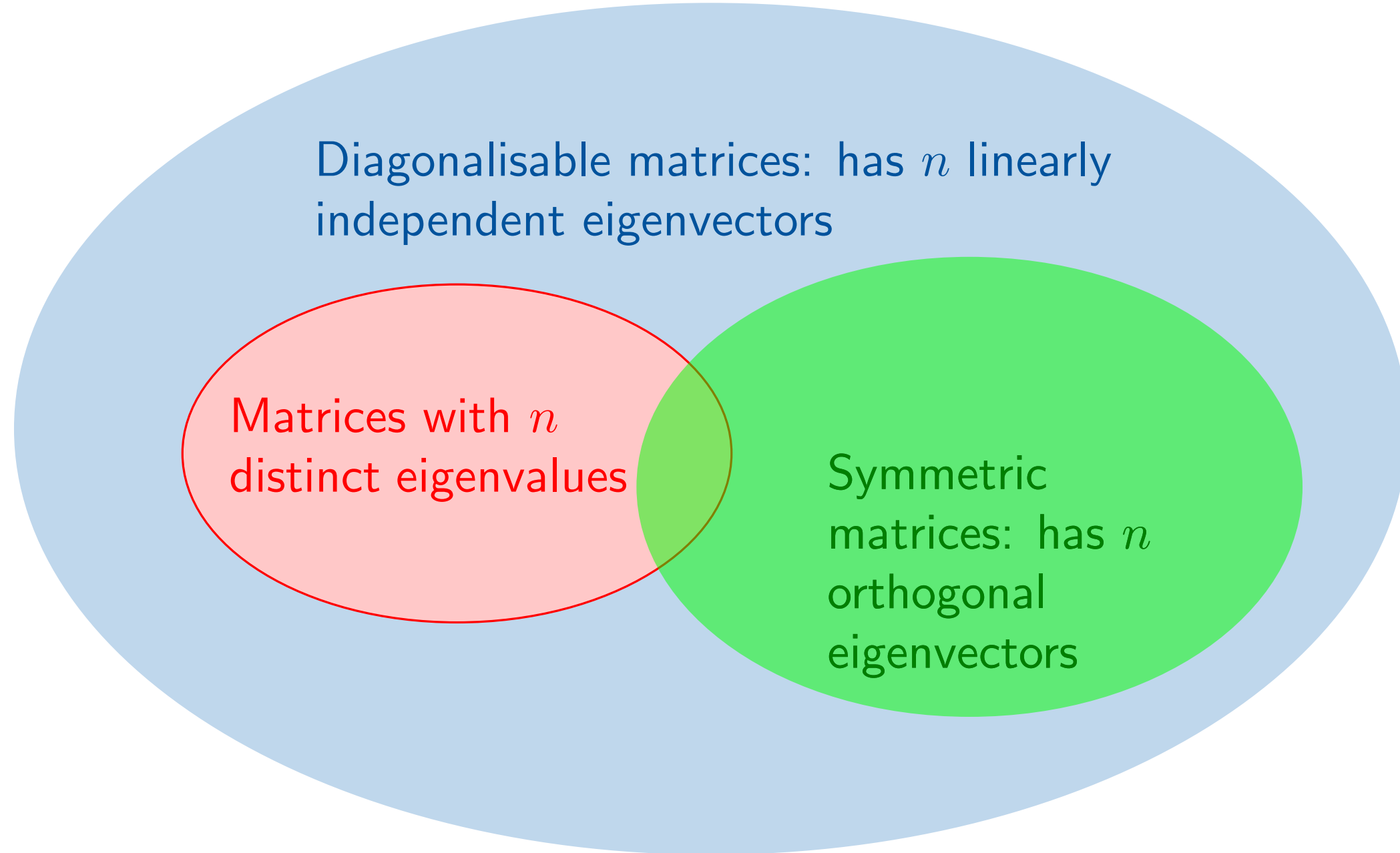
**Theorem 2: Orthogonally diagonalisable matrices are symmetric:** If  $A = PDP^{-1}$  and  $P$  is orthogonal and  $D$  is diagonal, then  $A$  is symmetric.

**Proof:**

$$A^T = (PDP^{-1})^T = (PD^T P^T)^T = (P^T)^T D^T P^T = PD^T P^T = A.$$

$P$  is orthogonal                       $D$  is diagonal

A diagram to summarise what we know about diagonalisation:





Non-examinable: ideas behind the proof of the spectral theorem

Because we need to work on subspaces of  $\mathbb{R}^n$  in the proof, we consider self-adjoint linear transformations  $((T\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (T\mathbf{v}))$  instead of symmetric matrices. So we want to show: a self-adjoint linear transformation has an orthogonal basis of eigenvectors.

The key ideas are:

1. Every linear transformation (on any vector space) has a complex eigenvector.

Proof: Every polynomial has a solution if we allow complex numbers. Apply this to the characteristic polynomial.

2. Any complex eigenvector of a (real) self-adjoint linear transformation is a real eigenvector corresponding to a real eigenvalue. (We won't comment on the proof.)

3. Let  $\mathbf{v}$  be an eigenvector of a self-adjoint linear transformation  $T$ , and  $\mathbf{w}$  be any vector orthogonal to  $\mathbf{v}$ . Then  $T(\mathbf{w})$  is still orthogonal to  $\mathbf{v}$ .

Proof:  $\mathbf{v} \cdot (T(\mathbf{w})) = (T(\mathbf{v})) \cdot \mathbf{w} = \lambda \mathbf{v} \cdot \mathbf{w} = \lambda 0 = 0$ .

Putting these together: if  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is self-adjoint, then by 1 and 2 it has a real eigenvector  $\mathbf{v}$ . Let  $W = (\text{Span}\{\mathbf{v}\})^\perp$ , the subspace of vectors orthogonal to  $\mathbf{v}$ . By 3, any vector in  $W$  stays in  $W$  after applying  $T$  (i.e.  $W$  is an **invariant subspace** under  $T$ ), so we can consider the restriction  $T : W \rightarrow W$ , which is self-adjoint. So repeat this argument on  $W$  (i.e. use induction on the dimension of the domain of  $T$ ).