Th 9.4.13 Polarisation identity — to find the symmetric bilinear form f given q with $q(\alpha)=f(\alpha,\alpha)$.

$$2f(\alpha,\beta) = q(\alpha+\beta)-q(\alpha)-q(\beta)$$

$\therefore$ If $1+1\neq 0$ in $\mathbb{F}$ (i.e. can divide by 2), then quadratic forms are in bijection with symmetric bilinear forms.

Proof:
$$\begin{aligned} q(\alpha+\beta) &= f(\alpha+\beta,\alpha+\beta) \\ &= f(\alpha,\alpha+\beta)+f(\beta,\alpha+\beta) \\ &= f(\alpha,\alpha)+f(\alpha,\beta)+f(\beta,\alpha)+f(\beta,\beta) \\ &= q(\alpha)+2f(\alpha,\beta)+q(\beta) \end{aligned}$$

Change of coordinates for bilinear forms

$\boxed{?}$ How is $\{f\}_{\mathcal{A}}$ and $\{f\}_{\mathcal{B}}$ related?

$$\begin{aligned} q(\alpha) &= [\alpha]_{\mathcal{A}}^T\{f\}_{\mathcal{A}}[\alpha]_{\mathcal{A}} \\ &= \left([\iota]_{\mathcal{A}\leftarrow\mathcal{B}}[\alpha]_{\mathcal{B}}\right)^T\{f\}_{\mathcal{A}}\left([\iota]_{\mathcal{A}\leftarrow\mathcal{B}}[\alpha]_{\mathcal{B}}\right) \\ &= [\alpha]_{\mathcal{B}}^T\left([\iota]_{\mathcal{A}\leftarrow\mathcal{B}}^T\{f\}_{\mathcal{A}}[\iota]_{\mathcal{A}\leftarrow\mathcal{B}}\right)[\alpha]_{\mathcal{B}} \end{aligned}$$

Compare $q(\alpha)=[\alpha]_{\mathcal{B}}^T\{f\}_{\mathcal{B}}[\alpha]_{\mathcal{B}}$

Th: $\{f\}_{\mathcal{B}} = [\iota]_{\mathcal{A}\leftarrow\mathcal{B}}^T\{f\}_{\mathcal{A}}[\iota]_{\mathcal{A}\leftarrow\mathcal{B}}$

Def 9.4.4: A and B are <u>congruent</u> if $B = P^T A P$ for some invertible P.

Goal: Given a symmetric matrix A, find a diagonal D and invertible P such that $D = P^T A P$.

Diagonalising a quadratic form by row and column operatio

Observe:

$$\begin{array}{r} \\ R_1+kR_3 \\ \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & \cdot & k \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a+kg & b+kh & c+ki \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ k & \cdot & 1 \end{pmatrix} = \begin{pmatrix} a+kc & b & c \\ d+kf & e & f \\ g+ki & h & i \end{pmatrix}$$

$$C_1+kC_3$$

In general: $i=1,\ j=3$ in example above

$$(I+kE_{ij})X = \text{do } (R_i \longrightarrow R_i + kR_j) \text{ to } X$$

$$X\underbrace{(I+kE_{ji})}_{=(I+kE_{ij})^T} = \text{do } (C_i \longrightarrow C_i + kC_j) \text{ to } X$$

$$i \neq j$$

$\therefore$ if S is of the form $I + kE_{ij}$ (i.e. an elementary matrix)

then SXS^T is the result of doing the corresponding row and column operations to X.

Goal: do row and column operations to make A diagonal, i.e. find $S_1, S_2, \cdots, S_r$ such that $S_r \cdots S_2 S_1 A S_1^T S_2^T \cdots S_r^T = D$

$$\underbrace{(S_r \cdots S_1)}_{P^T} A \underbrace{(S_r \cdots S_1)^T}_{P} = D$$

$\therefore$ if we do the same row operations to I, then the result $S_r \cdots S_1 I$, so transposing then gives P.

- Start with $(A \mid I)$
  - this is NOT a linear system.
- Do row operations to both "sides"
- Do column operations to left side only

repeat

- Result is $(D \mid P^T)$

Tip: use replacement operation only!

interchange and scaling have strange effects, $\because$ we are doing both row and column operations

Ex 9.4.20: $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} \boxed{0} & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

put pivot here

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) R_1 + R_2$$

$C_1 + C_2$

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

check "left side" is symmetric

make 0s under pivot

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{9}{2} & -\frac{3}{2} & -\frac{3}{2} & 1 \end{array}\right) \begin{array}{l} R_2 - \frac{1}{2}R_1 \\ R_3 - \frac{3}{2}R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} \boxed{2} & 0 & 0 & 1 & 1 & 0 \\ 0 & \boxed{-\frac{1}{2}} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{9}{2} & -\frac{3}{2} & -\frac{3}{2} & 1 \end{array}\right) \begin{array}{l} C_2 - \frac{1}{2}C_1 \\ C_3 - \frac{3}{2}C_1 \end{array}$$

next pivot

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -4 & -1 & -2 & 1 \end{array}\right)_{R_3 - R_2} \quad \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -4 & -1 & -2 & 1 \end{array}\right) C_3 - C_2$$

$$\underbrace{}_{D} \quad \underbrace{}_{P^T}$$

$$\therefore D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -4 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 1 & \frac{1}{2} & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

(P, D not unique)

If the diagonal entries are never 0 in this process, then P is upper-triangular — this is related to Cholesky factorisation $A = LDL^T$ — see numerical linear algebra — twice as fast as row-reduction.