

Remember the addition and scalar multiplication of matrices:

$$(A+B)_{ij} = a_{ij} + b_{ij},$$

$$\text{e.g. } \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}.$$

$$(cA)_{ij} = ca_{ij},$$

$$\text{e.g. } (-3) \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -12 & 0 & -15 \\ 3 & -9 & -6 \end{bmatrix}.$$

Is this really different from \mathbb{R}^6 ?

$$\begin{bmatrix} 4 \\ 0 \\ 5 \\ -1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 6 \\ 2 \\ 8 \\ 9 \end{bmatrix}.$$

$$(-3) \begin{bmatrix} 4 \\ 0 \\ 5 \\ -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ 0 \\ -15 \\ 3 \\ -9 \\ -6 \end{bmatrix}.$$

Remember from calculus the addition and scalar multiplication of polynomials:

$$\text{e.g. } (2t^2 + 1) + (-t^2 + 3t + 2) = t^2 + 3t + 3.$$

$$\text{e.g. } (-3)(-t^2 + 3t + 2) = 3t^2 - 9t - 6.$$

Is this really different from \mathbb{R}^3 ?

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}.$$

$$(-3) \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \\ 3 \end{bmatrix}.$$

\leftarrow coefficient of 1
 \leftarrow coefficient of t
 \leftarrow coefficient of t^2

§4.1, pp217-218: Abstract Vector Spaces

As the examples above showed, there are many objects in mathematics that “looks” and “feels” like \mathbb{R}^n . We will also call these **vectors**.

The real power of linear algebra is that everything we learned in Chapters 1-3 can be applied to all these abstract vectors, not just to column vectors.

You should think of abstract vectors as objects which can be added and multiplied by scalars - i.e. where the concept of “linear combination” makes sense. This addition and scalar multiplication must obey some “sensible rules” called **axioms** (see next page).

The axioms guarantee that the proof of every result and theorem from Chapters 1-3 will work for our new definition of vectors.

A **vector space** is a nonempty set V of objects, called **vectors**, on which are defined two operations, called **addition** and **multiplication by scalars** (real numbers), subject to the ten axioms below. The axioms must hold for all \mathbf{u} , \mathbf{v} and \mathbf{w} in V and for all scalars c and d .

1. $\mathbf{u} + \mathbf{v}$ is in V .
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each \mathbf{u} in V , there is vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V .
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $(cd)\mathbf{u} = c(d\mathbf{u})$.
10. $1\mathbf{u} = \mathbf{u}$.

Examples of vector spaces:

$M_{2 \times 3}$, the set of 2×3 matrices.

Let's check axiom 4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

The zero vector for $M_{2 \times 3}$ is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

You can check the other 9 axioms by using the properties of matrix addition and scalar multiplication (page 5 of week 4 slides, theorem 2.1 in textbook).

Similarly, $M_{m \times n}$, the set of all $m \times n$ matrices, is a vector space.

Is the set of all matrices (of all sizes) a vector space?

No, because we cannot add two matrices of different sizes, so axiom 1 does not hold.

Examples of vector spaces:

\mathbb{P}_n , the set of polynomials of degree at most n .

Each of these polynomials has the form

$$a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n,$$

for some numbers a_0, a_1, \dots, a_n .

Let's check axiom 4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

The zero vector for \mathbb{P}_n is $0 + 0t + 0t^2 + \dots + 0t^n$.

Let's check axiom 1. $\mathbf{u} + \mathbf{v}$ is in V .

$$\begin{aligned} & (a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n) + (b_0 + b_1 t + b_2 t^2 + \dots + b_n t^n) \\ &= (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \dots + (a_n + b_n)t^n, \end{aligned}$$

which also has degree at most n .

Exercise: convince yourself that the other axioms are true.

Examples of vector spaces:

Warning: the set of polynomials of degree exactly n is not a vector space, because axiom 1 does not hold:

$$\underbrace{(t^3 + t^2)}_{\text{degree 3}} + \underbrace{(-t^3 + t^2)}_{\text{degree 3}} = \underbrace{2t^2}_{\text{degree 2}}$$

\mathbb{P} , the set of all polynomials (no restriction on the degree) is a vector space.

$C(\mathbb{R})$, the set of all continuous functions is a vector space (because the sum of two continuous functions is continuous, the zero function is continuous, etc.)

These last two examples are a bit different from $M_{m \times n}$ and \mathbb{P}_n because they are infinite-dimensional (more later, see week 8 §4.5).

(You do not have to remember the notation $M_{m \times n}$, \mathbb{P}_n , etc. for the vector spaces.)

Let W be the set of symmetric 2×2 matrices. Is W a vector space?

1. $\mathbf{u} + \mathbf{v}$ is in V .

2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

5. For each \mathbf{u} in V , there is vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

6. $c\mathbf{u}$ is in V .

7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.

9. $(cd)\mathbf{u} = c(d\mathbf{u})$.

10. $1\mathbf{u} = \mathbf{u}$.

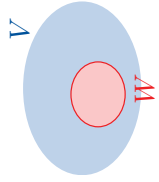
$$A = A^T, \text{ i.e. } A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \text{ for some } a, b, d$$

W is a subset of $M_{2 \times 2}$.
Axioms 2, 3, 5, 7, 8, 9, 10 hold for W
because they hold for $M_{2 \times 2}$.
So we only need to check axioms 1, 4, 6.

Definition: A subset W of a vector space V is a *subspace* of V if the closure axioms 1,4,6 hold:

4. The zero vector is in W .
1. If \mathbf{u}, \mathbf{v} are in W , then their sum $\mathbf{u} + \mathbf{v}$ is in W . (closed under addition)
6. If \mathbf{u} is in W and c is any scalar, the scalar multiple $c\mathbf{u}$ is in W . (closed under scalar multiplication)

Fact: W is itself a vector space (with the same addition and scalar multiplication as V) if and only if W is a subspace of V .



To show that W is a subspace, check **all** three axioms directly, for all $\mathbf{u}, \mathbf{v}, c$ (i.e. use variables).

To show that W is not a subspace, show that **one** of the axioms is false, for a particular value of $\mathbf{u}, \mathbf{v}, c$.

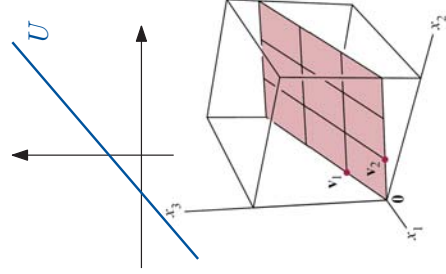
Example: Let U be the set of vectors of the form $\begin{bmatrix} x \\ x+1 \end{bmatrix}$, where x can take any value. To show that U is not a subspace of \mathbb{R}^2 , we need to find one counterexample to one of the axioms, e.g.

4. The zero vector is not in U , because there is no value of x with $\begin{bmatrix} x \\ x+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

An alternative answer:

1. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are in U , but $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is not of the form $\begin{bmatrix} x \\ x+1 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is not in U . So U is not closed under addition.

Best examples of a subspace: lines and planes containing the origin in \mathbb{R}^2 and \mathbb{R}^3 .



Definition: A subset W of a vector space V is a *subspace* of V if the closure axioms 1,4,6 hold:

4. The zero vector is in W .
1. If \mathbf{u}, \mathbf{v} are in W , then their sum $\mathbf{u} + \mathbf{v}$ is in W . (closed under addition)
6. If \mathbf{u} is in W and c is any scalar, the scalar multiple $c\mathbf{u}$ is in W . (closed under scalar multiplication)

Example: Let W be the set of vectors of the form $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$, where a, b can take any value. (W is the x_1x_3 -plane.) We show that W is a subspace of \mathbb{R}^3 :

4. The zero vector is in W because it is the vector with $a = 0, b = 0$.

$$1. \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} x \\ 0 \\ y \end{bmatrix} = \begin{bmatrix} a+x \\ 0 \\ b+y \end{bmatrix} \text{ is in } W.$$

$$6. c \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} ca \\ 0 \\ cb \end{bmatrix} \text{ is in } W.$$

Although W “feels like” \mathbb{R}^2 , note that \mathbb{R}^2 is **not** a subspace of \mathbb{R}^3 – vectors in \mathbb{R}^2 have two entries, so they are not in \mathbb{R}^3 .

Example: Let Q be the set of polynomials $\mathbf{p}(t)$ of degree at most 3 with $\mathbf{p}(2) = 0$. We show that Q is a subspace of \mathbb{P}_3 :

4. The zero polynomial is in Q because $\mathbf{0}(2) = 0 + 0 \cdot 2 + 0 \cdot 2^2 + 0 \cdot 2^3 = 0$.
1. For \mathbf{p}, \mathbf{q} in Q , we have $(\mathbf{p} + \mathbf{q})(2) = \mathbf{p}(2) + \mathbf{q}(2) = 0 + 0 = 0$, so $\mathbf{p} + \mathbf{q}$ is in Q .
6. For \mathbf{p} in Q and any scalar c , we have $(c\mathbf{p})(2) = c(\mathbf{p}(2)) = c(0) = 0$, so $c\mathbf{p}$ is in Q .

Example: In every vector space V , the set $\{\mathbf{0}\}$ containing only the zero vector is a subspace:

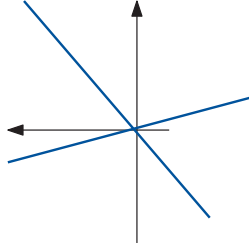
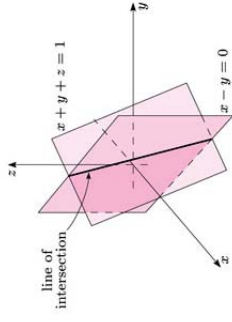
4. $\mathbf{0}$ is clearly in the subspace.
1. $\mathbf{0} + \mathbf{0} = \mathbf{0}$ (use axiom 4: $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for all \mathbf{u} in V).
6. $c\mathbf{0} = \mathbf{0}$ (use axiom 7: $c(\mathbf{0} + \mathbf{0}) = c\mathbf{0} + c\mathbf{0}$; and left hand side is $c\mathbf{0}$.) $\{\mathbf{0}\}$ called the zero subspace.

Example: For every vector space V , the whole space V is a subspace.

Non-examinable:

Fact: The intersection of two subspaces is a subspace (e.g. the intersection of two planes through the origin is a line through the origin).
Exercise: prove this from the axioms.

But the union of two subspaces is in general not a subspace (e.g. the union of two lines through the origin is not a line nor a plane through the origin, see also ex sheet Q1b).



The first of two shortcuts to show that a set is a subspace:

Theorem 1: Spans are subspaces: If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are vectors in a vector space V , then $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of V .

Redo Example: (p10) Let W be the set of vectors of the form $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$, where

a, b can take any value. We can rewrite such a vector as

$$\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Because a and b can take any value, this shows that

$$W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}. \text{ So } W \text{ is a subspace of } \mathbb{R}^3.$$

The first of two shortcuts to show that a set is a subspace:

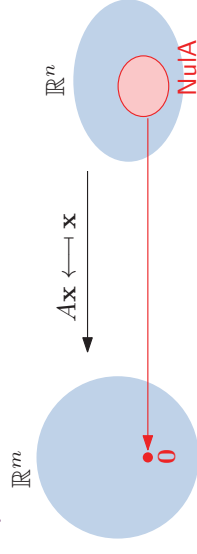
Theorem 1: Spans are subspaces: If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are vectors in a vector space V , then $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of V .

Redo Example: (p8) Let $\text{Sym}_{2 \times 2}$ be the set of symmetric 2×2 matrices, i.e. the set of matrices of the form $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$ where a, b, d can take any value. We can rewrite such a matrix as $\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. So $\text{Sym}_{2 \times 2} = \text{Span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and is therefore a subspace of $M_{2 \times 2}$.

Warning: Theorem 1 does not help us show that a set is **not** a subspace.

The second of two shortcuts to show that a set is a subspace:

Definition: The null space of a $m \times n$ matrix A , written $\text{Nul}A$, is the solution set to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.



Theorem 2: Null Spaces are Subspaces: The null space of an $m \times n$ matrix A is a subspace of \mathbb{R}^n .

This theorem is useful for showing that a set defined by conditions is a subspace.

Example: Show that the line $y = 2x$ is a subspace of \mathbb{R}^2 .

Answer: $y = 2x$ is the solution set to $2x - y = 0$, which in matrix form is $\begin{bmatrix} 2 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0}$. So this solution set is the null space of $\begin{bmatrix} 2 & -1 \end{bmatrix}$.

THEOREM 1: Spans are subspaces

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are vectors in a vector space V , then $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of V .

Proof: We check axioms 4, 1 and 6 in the definition of a subspace.

4. $\mathbf{0}$ is in $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ since

$$\mathbf{0} = \underline{\hspace{1cm}} \mathbf{v}_1 + \underline{\hspace{1cm}} \mathbf{v}_2 + \dots + \underline{\hspace{1cm}} \mathbf{v}_p$$

1. To show that $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is closed under addition, we choose two arbitrary vectors in $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$:

$$\mathbf{u} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \dots + a_p \mathbf{v}_p$$

and

$$\mathbf{v} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \dots + b_p \mathbf{v}_p.$$

Then

$$\mathbf{u} + \mathbf{v} = (a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \dots + a_p \mathbf{v}_p) + (b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \dots + b_p \mathbf{v}_p)$$

$$= \underline{\hspace{1cm}} \mathbf{v}_1 + \underline{\hspace{1cm}} \mathbf{v}_2 + \dots + \underline{\hspace{1cm}} \mathbf{v}_p$$

So $\mathbf{u} + \mathbf{v}$ is in $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

6. To show that $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is closed under scalar multiplication, choose an arbitrary number c and an arbitrary vector in $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$:

$$\mathbf{v} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \dots + b_p \mathbf{v}_p.$$

Then

$$\begin{aligned} c\mathbf{v} &= c(b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \dots + b_p \mathbf{v}_p) \\ &= \underline{\hspace{1cm}} \mathbf{v}_1 + \underline{\hspace{1cm}} \mathbf{v}_2 + \dots + \underline{\hspace{1cm}} \mathbf{v}_p \end{aligned}$$

So $c\mathbf{v}$ is in $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

Since 4, 1, 6 hold, $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of V .

The **null space** of an $m \times n$ matrix A , written as $\text{Nul } A$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

THEOREM 2: Null spaces are subspaces

The null space of an $m \times n$ matrix A is a subspace of \mathbf{R}^n .

Proof: $\text{Nul } A$ is a subset of \mathbf{R}^n since A has n columns. We check axioms 4, 1, 6 in the definition of a subspace.
4. $\mathbf{0}$ is in $\text{Nul } A$ because

1. If \mathbf{u} and \mathbf{v} are in $\text{Nul } A$, we show that $\mathbf{u} + \mathbf{v}$ is in $\text{Nul } A$. Because \mathbf{u} and \mathbf{v} are in $\text{Nul } A$

Therefore

$$A(\mathbf{u} + \mathbf{v}) =$$

6. If \mathbf{u} is in $\text{Nul } A$ and c is a scalar, we show that $c\mathbf{u}$ is in $\text{Nul } A$:

$$A(c\mathbf{u}) =$$

Since axioms 4, 1, 6 hold, $\text{Nul } A$ is a subspace of \mathbf{R}^n .

Summary:

Axioms for a subspace:

4. The zero vector is in W .

1. If \mathbf{u}, \mathbf{v} are in W , then $\mathbf{u} + \mathbf{v}$ is in W . (closed under addition)

6. If \mathbf{u} is in W and c is a scalar, then $c\mathbf{u}$ is in W . (closed under scalar multiplication)

Ways to show that a set W is a subspace:

- Show that $W = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ for some $\mathbf{v}_1, \dots, \mathbf{v}_p$ (if W is explicitly defined - i.e. its description has variables that can take any value).
- Show that W is $\text{Nul } A$ for some matrix A (if W is implicitly defined - i.e. by conditions that vectors must satisfy).
- Show that W is the kernel or range of a linear transformation (later, p42-43).
- Check all three axioms directly, for all $\mathbf{u}, \mathbf{v}, c$.

To show that a set is not a subspace:

- Show that one of the axioms is false, for a particular value of $\mathbf{u}, \mathbf{v}, c$.

Best examples of a subspace: lines and planes containing the origin in \mathbf{R}^2 and \mathbf{R}^3 .

One example of the power of abstract vector spaces - solving differential equations:

Question: What are all the polynomials \mathbf{p} of degree at most 5 that satisfy

$$\frac{d^2}{dt^2} \mathbf{p}(t) - 4 \frac{d}{dt} \mathbf{p}(t) + 3\mathbf{p}(t) = 3t - 1?$$

Answer: The differentiation function $D : \mathbb{P}_5 \rightarrow \mathbb{P}_5$ given by $D(\mathbf{p}) = \frac{d}{dt} \mathbf{p}$ is a linear transformation (later, p39).

The function $T : \mathbb{P}_5 \rightarrow \mathbb{P}_5$ given by $T(\mathbf{p}) = \frac{d^2}{dt^2} \mathbf{p}(t) - 4 \frac{d}{dt} \mathbf{p}(t) + 3\mathbf{p}(t)$ is a sum of compositions of linear transformations, so T is also linear.

We can check that the polynomial $t + 1$ is a solution.

So, by the Solutions and Homogeneous Equations Theorem, the solution set to the above differential equation is all polynomials of the form $t + 1 + \mathbf{q}(t)$ where $T(\mathbf{q}) = \mathbf{0}$.

Extra: \mathbb{P}_5 is both the domain and codomain of T , so the Invertible Matrix Theorem applies. So, if the above equation has more than one solution, then there is a polynomial \mathbf{g} such that $\frac{d^2}{dt^2} \mathbf{p}(t) - 4 \frac{d}{dt} \mathbf{p}(t) + 3\mathbf{p}(t) = \mathbf{g}(t)$ has no solutions.

§4.2, pp229-230, pp249-250: Subspaces and Matrices

Each linear transformation has two vector subspaces associated to it.

For each of these two subspaces, we are interested in two problems:

- given a vector \mathbf{v} , is it in the subspace?
 - can we write this subspace as $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ for some vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$?
- The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is then called a **spanning set** of the subspace.
- can we write this subspace as $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ for **linearly independent** vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$? The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is then called a **basis** of the subspace.

Problem b is important because it means every vector in the subspace can be written as $c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$. This allows us to calculate with and prove statements about arbitrary vectors in the subspace.

Problem b* is important because it means every vector in the subspace can be written **uniquely** as $c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$ (proof next week, §4.4).

We turn a spanning set into a basis by removing some vectors - this is the **Spanning Set Theorem** / **casting-out algorithm** (p27, also week 7 p10).

$\text{Nul}A$ is implicitly defined (i.e. defined by conditions) - problem a is easy, problem b takes more work.

Example: Let $A = \begin{bmatrix} 1 & -3 & 4 & -3 \\ 3 & -7 & 8 & -5 \end{bmatrix}$. a. Is $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in $\text{Nul}A$?

- Find vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ which span $\text{Nul}A$.

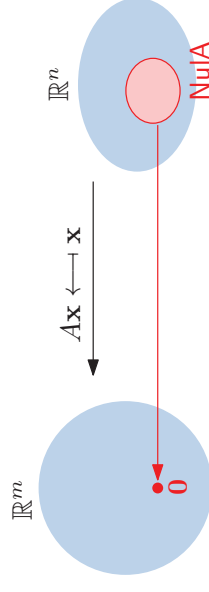
Answer:

- $A\mathbf{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq \mathbf{0}$, so \mathbf{v} is not in $\text{Nul}A$.

- $[A|\mathbf{0}]$ $\xrightarrow{\text{row reduction}}$ $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & -2 & 2 \end{array} \right] \begin{array}{l} \longrightarrow \\ \longrightarrow \end{array} \begin{array}{l} x_1 = 2x_3 - 3x_4 \\ x_2 = 2x_3 - 2x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{array} \begin{array}{l} \longrightarrow \\ \longrightarrow \end{array} \left\{ \begin{array}{l} 2 \\ 2 \\ 1 \\ 0 \end{array} \right\} \begin{array}{l} \longrightarrow \\ \longrightarrow \end{array} \left\{ \begin{array}{l} -3 \\ -2 \\ 0 \\ 1 \end{array} \right\}.$
So the solution set is $\begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ where s, t can take any value. So $\text{Nul}A = \text{Span} \left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.
linearly independent

Remember from p17:

Definition: The null space of a $m \times n$ matrix A , written $\text{Nul}A$, is the solution set to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.



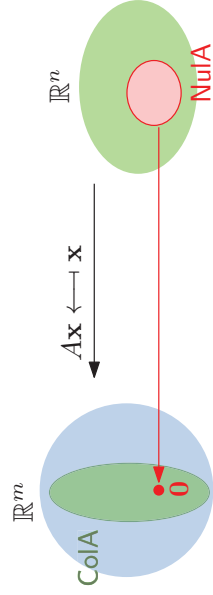
In general: the solution to $A\mathbf{x} = \mathbf{0}$ in parametric form looks like $s_i\mathbf{w}_i + s_j\mathbf{w}_j + \dots$, where x_i, x_j, \dots are the free variables (one vector for each free variable). The vector \mathbf{w}_i has a 1 in row i and a 0 in row j for every other free variable x_j , so $\{\mathbf{w}_i, \mathbf{w}_j, \dots\}$ are automatically linearly independent.

- $[A|\mathbf{0}]$ $\xrightarrow{\text{row reduction}}$ $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & -2 & 2 \end{array} \right] \begin{array}{l} \longrightarrow \\ \longrightarrow \end{array} \begin{array}{l} x_1 = 2x_3 - 3x_4 \\ x_2 = 2x_3 - 2x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{array} \begin{array}{l} \longrightarrow \\ \longrightarrow \end{array} \left\{ \begin{array}{l} 2 \\ 2 \\ 1 \\ 0 \end{array} \right\} \begin{array}{l} \longrightarrow \\ \longrightarrow \end{array} \left\{ \begin{array}{l} -3 \\ -2 \\ 0 \\ 1 \end{array} \right\}.$
So the solution set is $\begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ where s, t can take any value. So $\text{Nul}A = \text{Span} \left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.
linearly independent

Definition: The column space of a $m \times n$ matrix A , written $\text{Col}A$, is the span of the columns of A .

Because spans are subspaces, it is obvious that $\text{Col}A$ is a subspace of \mathbb{R}^m .

It follows from §1.3-1.4 that $\text{Col}A$ is the set of \mathbf{b} for which $A\mathbf{x} = \mathbf{b}$ has solutions.



Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of \mathbb{R}^n .	1. Col A is a subspace of \mathbb{R}^m .
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	2. Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[A \mid \mathbf{0}]$ are required.	3. It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	4. There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector \mathbf{v} in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	5. A typical vector \mathbf{v} in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector \mathbf{v} , it is easy to tell if \mathbf{v} is in Nul A. Just compute $A\mathbf{v}$.	6. Given a specific vector \mathbf{v} , it may take time to tell if \mathbf{v} is in Col A. Row operations on $[A \mid \mathbf{v}]$ are required.
7. Nul A = $\{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	7. Col A = \mathbb{R}^m if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every \mathbf{b} in \mathbb{R}^m .
8. Nul A = $\{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	8. Col A = \mathbb{R}^m if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps \mathbb{R}^n onto \mathbb{R}^m .

$\text{Col}A$ is explicitly defined - problem a takes work, problem b is easy.

Example: Let $A = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix}$. a. Is $\mathbf{v} = \begin{bmatrix} -5 \\ 9 \\ 5 \end{bmatrix}$ in $\text{Col}A$?

b. Find vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ which span $\text{Col}A$.

Answer:

$$\begin{array}{c} \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ \text{to echelon form} \end{array}$$

There is no row $[0 \dots 0 \mid *]$ with $* \neq 0$, so \mathbf{v} is in $\text{Col}A$.

b. By definition, $\text{Col}A$ is the span of the columns of A, so

$$\text{Col}A = \text{Span} \left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ 8 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} \right\}.$$

Note that this spanning set is not linearly independent (more than 3 vectors in \mathbb{R}^3).

p.222 of textbook

← problem b

← problem a

As we saw on p26, it is easy to obtain a spanning set for $\text{Col}A$ (just take all the columns of A), but usually this spanning set is not linearly independent.

To obtain a linearly independent set that spans $\text{Col}A$, take the pivot columns of A - this is called the casting-out algorithm.

Example: Let $A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix}$.

Find a linearly independent set that spans $\text{Col}A$.

Answer: $\begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ to echelon form

The pivot columns are 1, 2 and 5, so $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\} = \left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} \right\}$ is one answer.

(The answer from the casting-out algorithm is not the only answer - see p34.)

Casting-out algorithm: the pivot columns of A is a linearly independent set that spans $\text{Col}A$.

Why does the casting-out algorithm work part 1: why the pivot columns are linearly independent:

Example:

$$A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow{\text{row reduction to echelon form}} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

So $\begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_5 \\ | & | & | \end{bmatrix}$ is row-equivalent to $\begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, which has no free variables.

So $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$ is linearly independent.

$$A(2, 2, 1, 0, 0) = \mathbf{0} \longrightarrow 2\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0} \longrightarrow \mathbf{a}_3 = -2\mathbf{a}_1 - 2\mathbf{a}_2.$$

$$A(-3, -2, 0, 1, 0) = \mathbf{0} \longrightarrow -3\mathbf{a}_1 - 2\mathbf{a}_2 + \mathbf{a}_4 = \mathbf{0} \longrightarrow \mathbf{a}_4 = 3\mathbf{a}_1 + 2\mathbf{a}_2.$$

In other words: consider the solution to $A\mathbf{x} = \mathbf{0}$ where one free variable x_i is 1, and all other free variables are 0. This corresponds to a linear dependence relation among the columns of A , which can be rearranged to express the column \mathbf{a}_i as a linear combination of the pivot columns.

Why this is useful: any vector \mathbf{v} in $\text{Col}A$ has the form

$$\begin{aligned} \mathbf{v} &= c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 + c_4\mathbf{a}_4 + c_5\mathbf{a}_5, \\ \text{which we can rewrite as} \\ &= c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3(-2\mathbf{a}_1 - 2\mathbf{a}_2) + c_4(3\mathbf{a}_1 + 2\mathbf{a}_2) + c_5\mathbf{a}_5 \end{aligned}$$

a linear combination of the pivot columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5$. So \mathbf{v} is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$, and so $\text{Col}A = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$.

Casting-out algorithm: the pivot columns of A is a linearly independent set that spans $\text{Col}A$.

Why does the casting-out algorithm work part 2: why the pivot columns span $\text{Col}A$:

To explain this, we need to look at the solutions to $A\mathbf{x} = \mathbf{0}$:

Example:

$$A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow{\text{row reduction to rref}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So the solution set to $A\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 2 \\ 2 \\ 2 \\ 1+t \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ where s, t can take any value.

$$\begin{aligned} x_3 &= 1 & x_3 &= 0 \\ x_4 &= 0 & x_4 &= 1 \end{aligned}$$

These correspond respectively to the linear dependence relations

$$2\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3 = \mathbf{0} \text{ and } -3\mathbf{a}_1 - 2\mathbf{a}_2 + \mathbf{a}_4 = \mathbf{0}.$$

Rearranging: $\mathbf{a}_3 = -2\mathbf{a}_1 - 2\mathbf{a}_2$ and $\mathbf{a}_4 = 3\mathbf{a}_1 + 2\mathbf{a}_2$.

Another view: the casting-out algorithm as a greedy algorithm:

Example:

$$\begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow{\text{row reduction to rref}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\text{rref}\left(\begin{bmatrix} | & | \\ \mathbf{a}_1 \\ | \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a pivot in every column, so $\{\mathbf{a}_1\}$ is linearly independent, so we keep \mathbf{a}_1 .

$\text{rref}\left(\begin{bmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a pivot in every column, so $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent, so we keep \mathbf{a}_2 .

$\text{rref}\left(\begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ | & | & | \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ does not have a pivot in every column, so $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is linearly dependent, so we remove \mathbf{a}_3 .

Another view: the casting-out algorithm as a greedy algorithm (continued):

Example:

$$\begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to rref}]{\text{row reduction}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$\text{rref}\left(\begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_4 \\ | & | & | \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ does not have a pivot in every column, so $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is linearly dependent, so we remove \mathbf{a}_4 .

$\text{rref}\left(\begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_5 \\ | & | & | \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has a pivot in every column, so $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$ is linearly independent, so we keep \mathbf{a}_5 .

Definition: The row space of a $m \times n$ matrix A , written $\text{Row}A$, is the span of the rows of A . It is a subspace of \mathbb{R}^n .

Example: $A = \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 2 & 0 & 8 \\ 1 & 2 & -3 & 6 \end{bmatrix}$ $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\text{Row}A = \text{Span}\{(0, 1, 0, 4), (0, 2, 0, 8), (1, 2, -3, 6)\}.$$

$\text{Row}A$ is explicitly defined - indeed, it is equivalent to $\text{Col}A^T$.

So, to see if a vector \mathbf{v} is in $\text{Row}A$, row-reduce $[A^T \mid \mathbf{v}^T]$.

To find a linear independent set that spans $\text{Row}A$, take the pivot columns of A^T , or..

Theorem 13: Row operations do not change the row space. In particular, the nonzero rows of $\text{rref}(A)$ is a linearly independent set whose span is $\text{Row}A$.

E.g. for the above example, $\text{Row}A = \text{Span}\{(1, 0, -3, -2), (0, 1, 0, 4)\}$.

Warning: the “pivot rows” of A do not usually span $\text{Row}A$:

e.g. here $(1, 2, -3, 6)$ is in $\text{Row}A$ but not in $\text{Span}\{(0, 1, 0, 4), (0, 2, 0, 8)\}$.

So the casting-out algorithm is a greedy algorithm in that it prefers vectors that are earlier in the set.

Example: Let $A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix}$. Find a linearly independent set containing \mathbf{a}_3 that spans $\text{Col}A$.

Answer: To ensure that the set contains \mathbf{a}_3 , we should make it the leftmost

column - e.g. we row-reduce $\begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_3 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix}$ and take the pivot columns.

Warning: the example on the previous two pages is a little misleading: a subset of the columns of $\text{rref}(A)$ is **not** always the reduced echelon form of those columns of A , e.g. $\text{rref}\left(\begin{bmatrix} | & | \\ \mathbf{a}_2 & \mathbf{a}_3 \\ | & | \end{bmatrix}\right) \neq \begin{bmatrix} 0 & -2 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}$ (because this isn't in reduced echelon form).

The correct statement is that a subset of the columns of $\text{rref}(A)$ is **row equivalent** to those columns of A .

Theorem 13: Row operations do not change the row space. In particular, the nonzero rows of $\text{rref}(A)$ is a linearly independent set whose span is $\text{Row}A$.

An example to explain why row operations do not change the row space:

$$A = \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 2 & 0 & 8 \\ 1 & 2 & -3 & 6 \end{bmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & -2 \end{bmatrix} \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(1, 4, -3, 14) = R_2 + R_3 = (R_2 - 2R_1) + (R_3 - R_1) + 3R_1$$

Similarly, any linear combination of R_1, R_2, R_3 can be written as a linear combination of $R_1, R_2 - 2R_1, R_3 - R_1$.

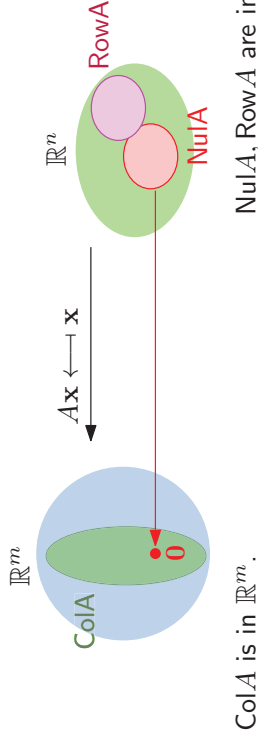
Proof of the second sentence in Theorem 13:

From the first sentence, $\text{Row}(A) = \text{Row}(\text{rref}(A)) = \text{Span of the nonzero rows of } \text{rref}(A)$. Because each nonzero row has a 1 in one pivot column (different column for each row) and 0s in all other pivot columns, these rows are linearly independent.

Summary:

A basis for W is a linearly independent set that spans W (more later).

- $\text{Nul}A$ =solutions to $Ax = 0$, basis for $\text{Nul}A$: solve $Ax = 0$ via the rref.
- $\text{Col}A$ =span of columns of A , basis for $\text{Col}A$: pivot columns of A .
- $\text{Row}A$ =span of rows of A , basis for $\text{Row}A$: nonzero rows of $\text{rref}(A)$.



$\text{Col}A$ is in \mathbb{R}^m .

$\text{Nul}A, \text{Row}A$ are in \mathbb{R}^n .

In general, $\text{Col}A \neq \text{Col}(\text{rref}(A))$.

$\text{Nul}A = \text{Nul}(\text{rref}(A))$, $\text{Row}A = \text{Row}(\text{rref}(A))$.

PP222-223: Linear Transformations for Vector Spaces

Recall (week 3 §1.8) the definition of a linear transformation:

Definition: A function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a *linear transformation* if:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all \mathbf{u}, \mathbf{v} in the domain of T ;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and for all \mathbf{u} in the domain of T .

Now consider a function $T: V \rightarrow W$, where V, W are abstract vector spaces. Because we can add and scalar-multiply in V , the left hand sides of the equations in 1,2 make sense.

Because we can add and scalar-multiply in W , the right hand sides of the equations in 1,2 make sense.

So we can ask if functions between abstract vector spaces are linear:

Definition: A function $T: V \rightarrow W$ is a *linear transformation* if:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all \mathbf{u}, \mathbf{v} in V ;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and for all \mathbf{u} in V .

Hard exercise: show that the set of all linear transformations $V \rightarrow W$ is a vector space.

Definition: A function $T: V \rightarrow W$ is a *linear transformation* if:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all \mathbf{u}, \mathbf{v} in V ;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and for all \mathbf{u} in V .

Example: The differentiation function $D: \mathbb{P}_n \rightarrow \mathbb{P}_{n-1}$ given by $D(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$,

$$D(a_0 + a_1t + a_2t^2 + \dots + a_nt^n) = a_1 + 2a_2t + \dots + na_nt^{n-1},$$

is linear.

If you've taken a calculus class, then you already know this:

$$\begin{aligned} \text{When you calculate} \quad & \frac{d}{dt}(3x + 2x^2) = 3 + 2 \cdot 2x \\ \text{you're really thinking} \quad & 3 \frac{d}{dt}x + 2 \frac{d}{dt}x^2 \end{aligned}$$

Method A to show that D is linear:

$$D(\mathbf{p} + \mathbf{q}) = \frac{d}{dt}(\mathbf{p} + \mathbf{q}) = \frac{d}{dt}\mathbf{p} + \frac{d}{dt}\mathbf{q} = D(\mathbf{p}) + D(\mathbf{q}); \text{ and}$$

$$D(c\mathbf{p}) = \frac{d}{dt}(c\mathbf{p}) = c \frac{d}{dt}\mathbf{p} = cD(\mathbf{p})$$

Definition: A function $T: V \rightarrow W$ is a *linear transformation* if:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all \mathbf{u}, \mathbf{v} in V ;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and for all \mathbf{u} in V .

Example: The differentiation function $D: \mathbb{P}_n \rightarrow \mathbb{P}_{n-1}$ given by $D(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$,

$$D(a_0 + a_1t + a_2t^2 + \dots + a_nt^n) = a_1 + 2a_2t + \dots + na_nt^{n-1},$$

is linear.

Method B to show that D is linear - use the formula:

$$D((a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \dots + (a_n + b_n)t^n)$$

$$= (a_1 + b_1) + 2(a_2 + b_2)t + \dots + n(a_n + b_n)t^{n-1}$$

$$= a_1 + 2a_2t + \dots + na_nt^{n-1} + b_1 + 2b_2t + \dots + nb_nt^{n-1}$$

$$= D(a_0 + a_1t + a_2t^2 + \dots + a_nt^n) + D(b_0 + b_1t + b_2t^2 + \dots + b_nt^n); \text{ and}$$

$$\begin{aligned} D((ca_0) + (ca_1)t + (ca_2)t^2 + \dots + (ca_n)t^n) &= (ca_1) + 2(ca_2)t + \dots + n(ca_n)t^{n-1} \\ &= c(a_1 + 2a_2t + \dots + na_nt^{n-1}) \\ &= cD(a_0 + a_1t + a_2t^2 + \dots + a_nt^n). \end{aligned}$$

Example: The “multiplication by t ” function $M : \mathbb{P}_n \rightarrow \mathbb{P}_{n+1}$ given by $M(\mathbf{p}(t)) = t\mathbf{p}(t)$,

$$M(a_0 + a_1t + \cdots + a_nt^n) = t(a_0 + a_1t + \cdots + a_nt^n),$$

is linear:

$$\begin{aligned} \text{Method A: } M(\mathbf{p} + \mathbf{q}) &= t[(\mathbf{p} + \mathbf{q})(t)] = t\mathbf{p}(t) + t\mathbf{q}(t) = M(\mathbf{p}) + M(\mathbf{q}); \text{ and} \\ M(c\mathbf{p}) &= t[(c\mathbf{p})(t)] = c[t(\mathbf{p}(t))] = cM(\mathbf{p}) \end{aligned}$$

$$\begin{aligned} \text{Method B: } M((a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n) &= t(a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n \\ &= t(a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n \\ &= t(a_0 + a_1t + \cdots + a_nt^n) + t(a_0 + a_1t + \cdots + a_nt^n) \\ &= M(a_0 + a_1t + \cdots + a_nt^n) + M(b_0 + b_1t + \cdots + b_nt^n); \text{ and} \\ M((ca_0) + (ca_1)t + \cdots + (ca_n)t^n) &= t((ca_0) + (ca_1)t + \cdots + (ca_n)t^n) \\ &= ct(a_0 + a_1t + \cdots + a_nt^n) \\ &= cM(a_0 + a_1t + \cdots + a_nt^n). \end{aligned}$$

Definition: A function $T : V \rightarrow W$ is a *linear transformation* if:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all \mathbf{u}, \mathbf{v} in V ;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and for all \mathbf{u} in V .

The concepts of kernel and range (week 3, §1.9) make sense for linear transformations between abstract vector spaces:

Definition: The *kernel* of T is the set of \mathbf{v} in V satisfying $T(\mathbf{v}) = \mathbf{0}$.

Definition: The *range* of T is the set of \mathbf{w} in W such that $\mathbf{w} = T(\mathbf{v})$ for some \mathbf{v} in V .

Example: The kernel of the differentiation function $D : \mathbb{P}_n \rightarrow \mathbb{P}_{n-1}$, given by $D(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$, is the set of constant polynomials $\mathbf{p}(t) = a_0$ for any number a_0 . The range of D is all of \mathbb{P}_{n-1} .

Our proof that null spaces are subspaces (p18) shows that the kernel of a linear transformation is a subspace.

Exercise: show that the range of a linear transformation is a subspace.

Redo Example: (p12) Let Q be the set of polynomials $\mathbf{p}(t)$ of degree at most 3 with $\mathbf{p}(2) = 0$. We show that Q is a subspace of \mathbb{P}_3 by showing that it is the kernel of a linear transformation. (This argument is hard; if you prefer the axiom-checking on p12 that is fine.)

The evaluation-at-2 function $E_2 : \mathbb{P}_3 \rightarrow \mathbb{R}$ given by $E_2(\mathbf{p}) = \mathbf{p}(2)$,

$$E_2(a_0 + a_1t + a_2t^2 + a_3t^3) = a_0 + a_12 + a_22^2 + a_32^3$$

is a linear transformation because

1. For \mathbf{p}, \mathbf{q} in \mathbb{P}_3 , we have

$$E_2(\mathbf{p} + \mathbf{q}) = (\mathbf{p} + \mathbf{q})(2) = \mathbf{p}(2) + \mathbf{q}(2) = E_2(\mathbf{p}) + E_2(\mathbf{q}).$$

2. For \mathbf{p} in \mathbb{P}_3 and any scalar c , we have $E_2(c\mathbf{p}) = (c\mathbf{p})(2) = c(\mathbf{p}(2)) = cE_2(\mathbf{p})$. So E_2 is a linear transformation. Q is the kernel of E_2 , so Q is a subspace.

Can we write Q as $\text{Span}\{\mathbf{p}_1, \dots, \mathbf{p}_p\}$ for some linearly independent polynomials $\mathbf{p}_1, \dots, \mathbf{p}_p$?

One idea: associate a matrix A to E_2 and take a basis of $\text{Nul} A$ using the rref.

To do computations like this, we need *coordinates*.