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Rearrangement: $(2)x_1 + (6)x_2 = 0$

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In general, a linear equation is an equation of the form

$$a_1x_1 + a_2x_2 + \ldots a_nx_n = b.$$

$x_1, x_2, \ldots x_n$ are the variables. $a_1, a_2, \ldots a_n$ are the coefficients.

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Definition: A *system of linear equations* (or a *linear system*) is a collection of linear equations involving the same set of variables.

Example:

$$\begin{array}{rrrcr} x & +y & & = & 3 \\ 3x & & +2z & = & -2 \end{array}$$

This is a system of 2 equations in 3 variables, x, y, z.

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A solution is: $(x, y, z) = (2, 1, -4)$.

This is a system of 2 equations in 3 variables, x, y, z.

Definition: A *solution* of a linear system is a list $(s_1, s_2, \dots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, \dots, s_n$ are substituted for $x_1, x_2, \dots, x_n$ respectively.

Definition: The *solution set* of a linear system is the set of all possible solutions.

Definition: A linear system is *consistent* if it has a solution,

and *inconsistent* if it does not have a solution.

Fact: A linear system has either

- exactly one solution — consistent
- infinitely many solutions — consistent
- no solutions — inconsistent

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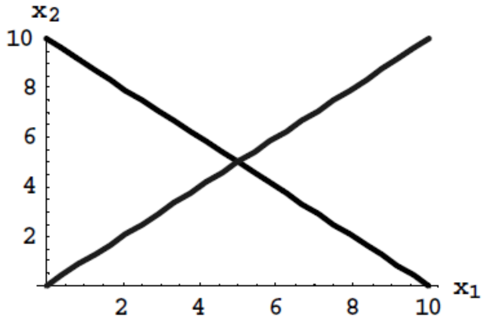
EXAMPLE Two equations in two variables:

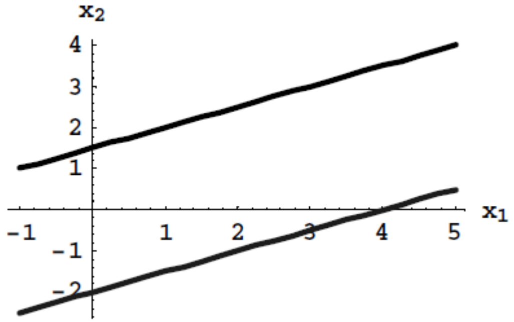
$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 10 \\ -x_1 & + & x_2 & = & 0 \end{array}$$



one unique solution

consistent

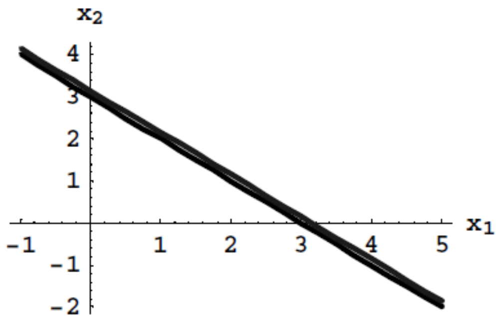
$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -3 \\ 2x_1 & - & 4x_2 & = & 8 \end{array}$$



no solution

inconsistent

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 3 \\ -2x_1 & - & 2x_2 & = & -6 \end{array}$$



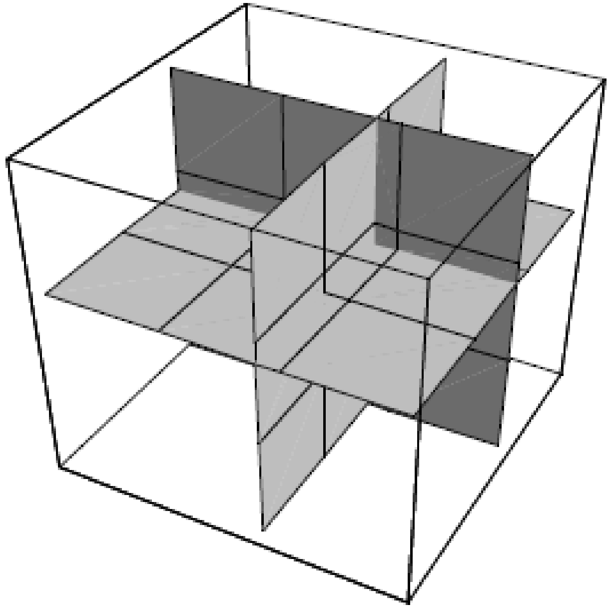
infinitely many solutions

consistent

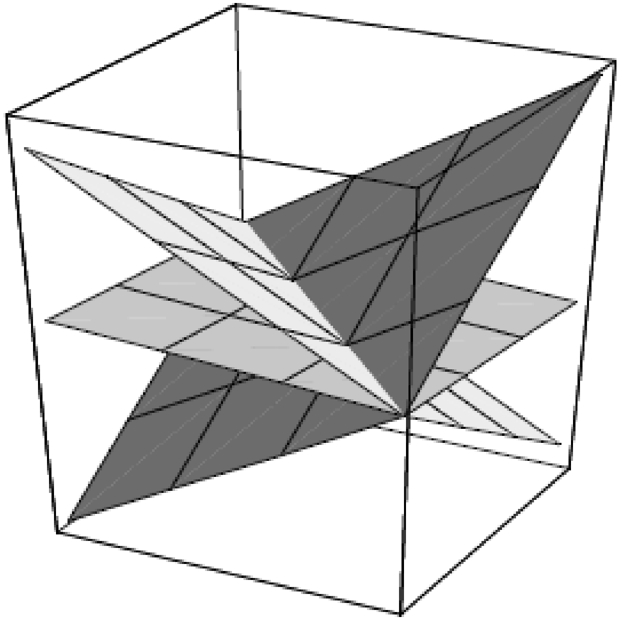
$$ax + by + cz = d, \text{ or } z = a'x + b'y + d'$$

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

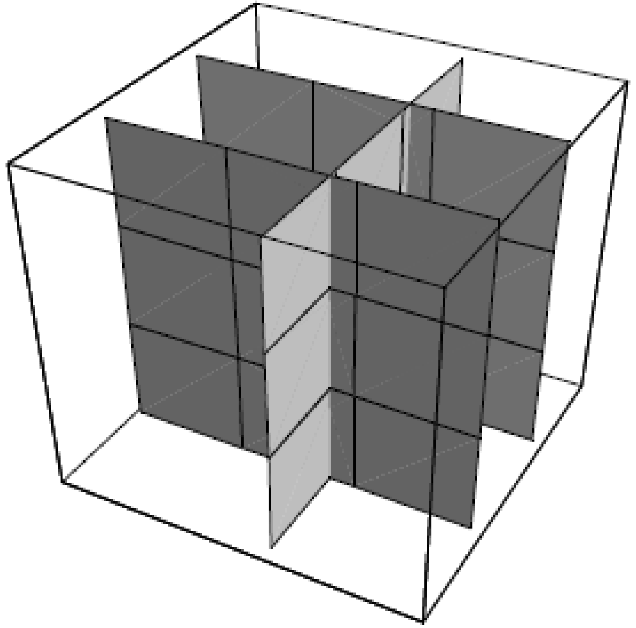
i) The planes intersect in one point. *(one solution)*



ii) The planes intersect in one line. *(infinitely many solutions)*



iii) There is no point in common to all three planes. *(no solution)*



Which of these cases are consistent?

$ax + by + cz = d$, or $z = a'x + b'y + d'$

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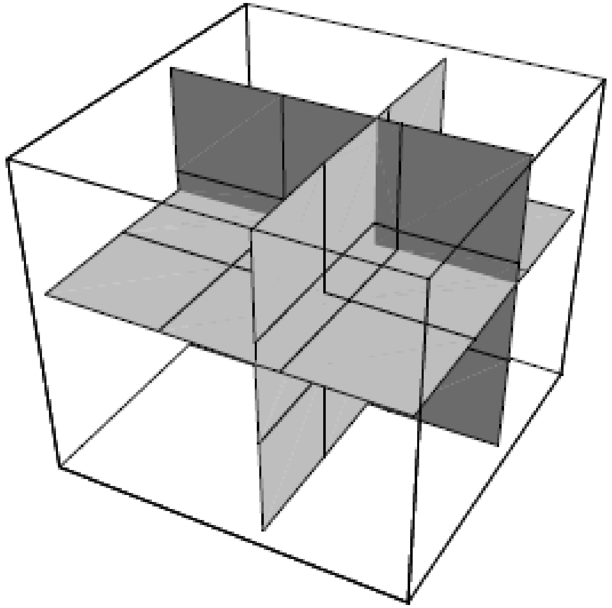

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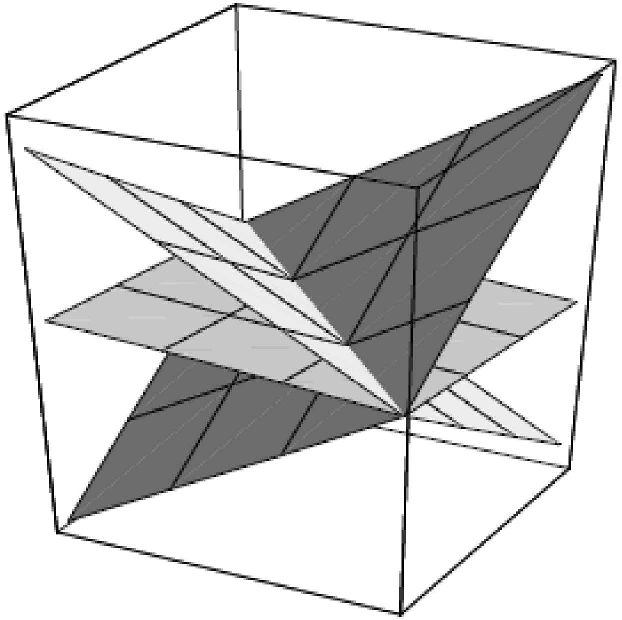

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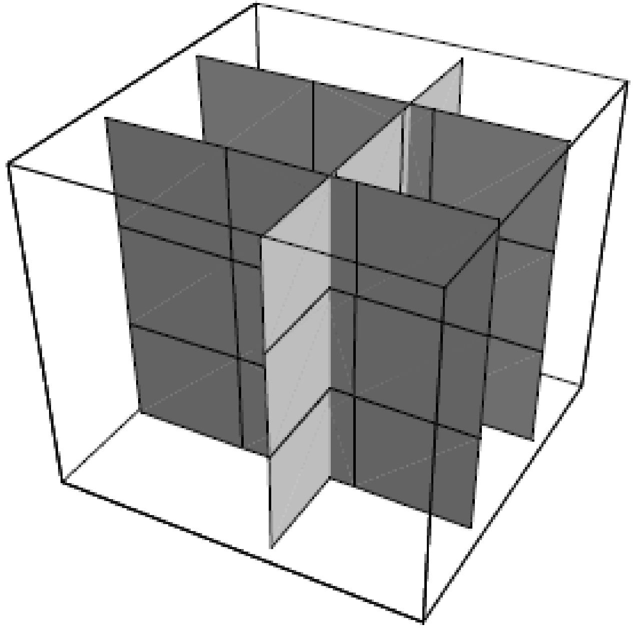

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consistent | consistent | inconsistent

How to solve a linear system? **Example**:

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

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$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \quad \longrightarrow \quad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array} \quad \longrightarrow \quad \begin{array}{rcrcr} x_1 & & & = & 3 \\ & & x_2 & = & 2 \end{array}$$

How to solve a linear system? **Example**:

$$\begin{array}{lrcrcr} R_1 & x_1 & - & 2x_2 & = & -1 \\ R_2 & -x_1 & + & 3x_2 & = & 3 \end{array} \longrightarrow \begin{array}{rrcrcr} & x_1 & - & 2x_2 & = & -1 \\ R_1+R_2 \rightarrow & & & x_2 & = & 2 \end{array} \longrightarrow \begin{array}{rrcr} R_1+2R_2 \rightarrow & x_1 & = & 3 \\ & x_2 & = & 2 \end{array}$$

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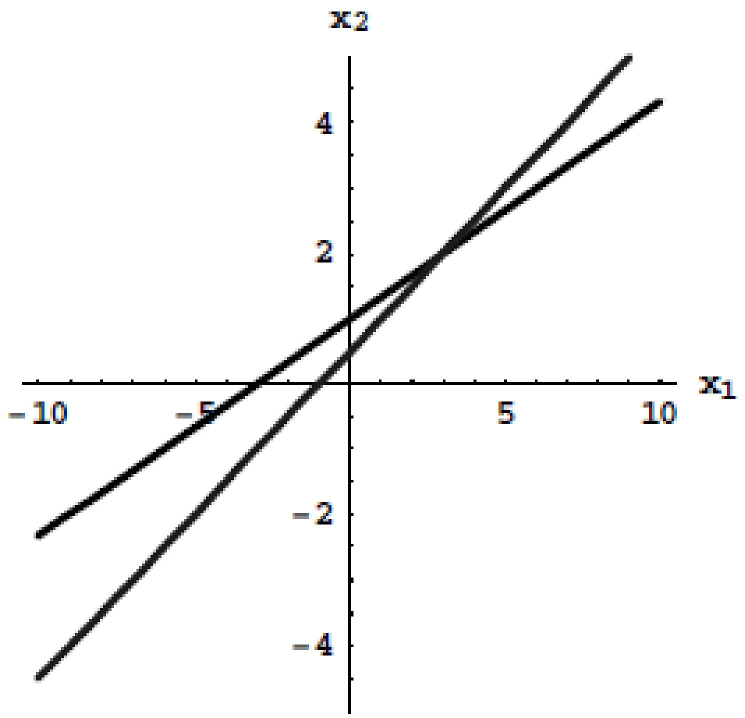
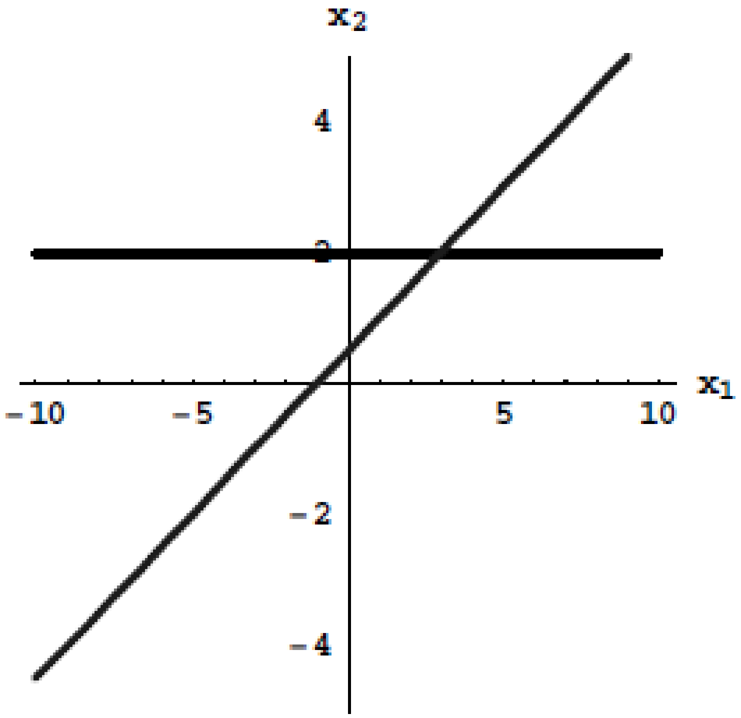
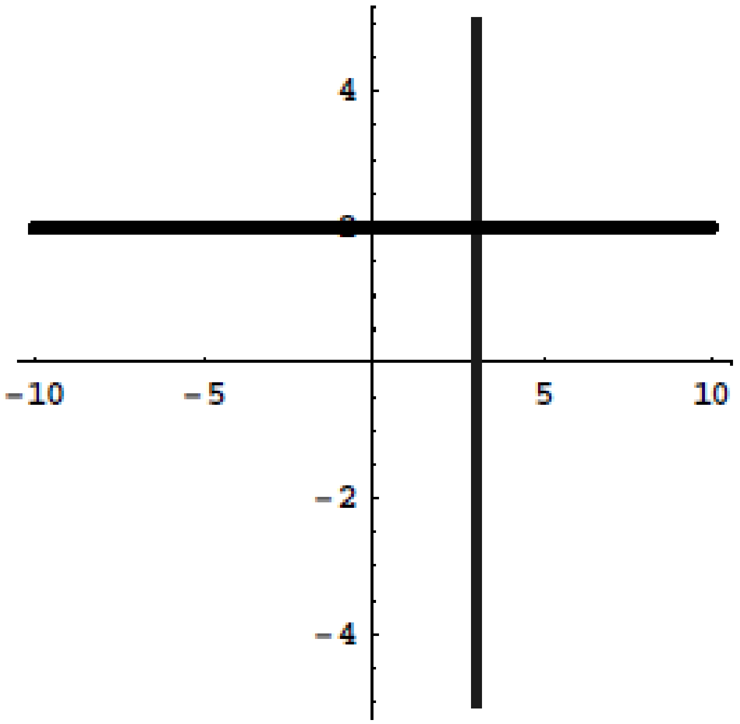
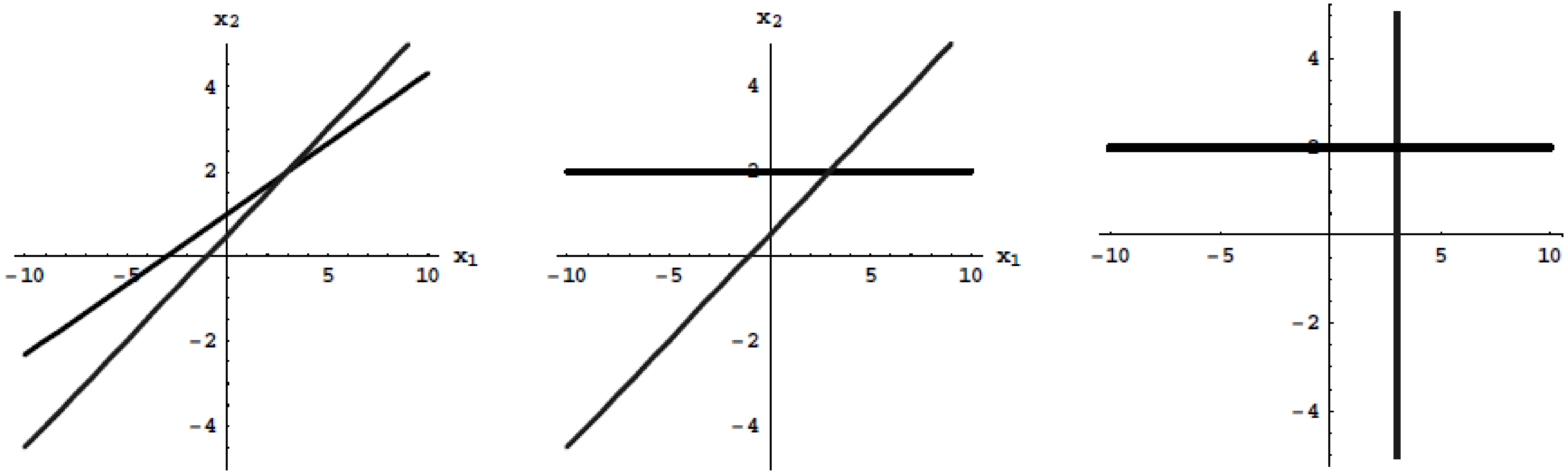






How to solve a linear system? **Example**:

$$
\begin{array}{lrcrcr}
R_1 & x_1 & - & 2x_2 & = & -1 \\
R_2 & -x_1 & + & 3x_2 & = & 3
\end{array}
\quad \rightarrow \quad
\begin{array}{rrcrcr}
 & x_1 & - & 2x_2 & = & -1 \\
R_1 + R_2 \rightarrow & & & x_2 & = & 2
\end{array}
\quad
\begin{array}{rrcr}
R_1 + 2R_2 \rightarrow & x_1 & = & 3 \\
 & x_2 & = & 2
\end{array}
$$



Definition: Two linear systems are *equivalent* if they have the same solution set.

General strategy for solving a linear system: replace one system with an equivalent system that is easier to solve.

Simplify the writing by using matrix notation:

$$\begin{array}{rrcrcr} R_1 & x_1 & - & 2x_2 & = & -1 \\ R_2 & -x_1 & + & 3x_2 & = & 3 \end{array} \longrightarrow \begin{array}{rrcrcr} & x_1 & - & 2x_2 & = & -1 \\ R_1 + R_2 \to & & & x_2 & = & 2 \end{array} \longrightarrow \begin{array}{rrcr} R_1 + 2R_2 \to & x_1 & = & 3 \\ & x_2 & = & 2 \end{array}$$

$$\left[\begin{array}{rr|r} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{rr|r} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

coefficient of x_1

coefficient of x_2

right hand side

Simplify the writing by using matrix notation:

$$
\begin{array}{l}
R_1 \\
R_2
\end{array}
\begin{array}{rcrcr}
x_1 & - & 2x_2 & = & -1 \\
-x_1 & + & 3x_2 & = & 3
\end{array}
\;\xrightarrow{R_1 + R_2 \to R_2}\;
\begin{array}{rcrcr}
x_1 & - & 2x_2 & = & -1 \\
 & & x_2 & = & 2
\end{array}
\;\xrightarrow{R_1 + 2R_2 \to R_1}\;
\begin{array}{rcr}
x_1 & = & 3 \\
x_2 & = & 2
\end{array}
$$

$$
\left[\begin{array}{rr|r} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]
\longrightarrow
\left[\begin{array}{rr|r} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right]
\longrightarrow
\left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]
$$

coefficient of x_1

coefficient of x_2

right hand side

The augmented matrix of a linear system contains the right hand side: $\left[\begin{array}{rr|r} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$

The coefficient matrix of a linear system is the left hand side only: $\left[\begin{array}{rr} 1 & -2 \\ -1 & 3 \end{array}\right]$

$$\begin{array}{lrcrcr} R_1 & x_1 & - & 2x_2 & = & -1 \\ R_2 & -x_1 & + & 3x_2 & = & 3 \end{array} \quad \xrightarrow{} \quad \begin{array}{lrcrcr} & x_1 & - & 2x_2 & = & -1 \\ R_1 + R_2 \rightarrow & & & x_2 & = & 2 \end{array} \quad \begin{array}{lrcr} R_1 + 2R_2 \rightarrow & x_1 & = & 3 \\ & x_2 & = & 2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

Elementary row operations:

1. Replacement: add a multiple of one row to another row. $R_i \rightarrow R_i + cR_j$
2. Interchange: interchange two rows. $R_i \rightarrow R_j,\ R_j \rightarrow R_i$
3. Scaling: multiply all entries in a row by a nonzero constant. $R_i \rightarrow cR_i,\ c \neq 0$

$$\begin{array}{llcrcr} R_1 & x_1 & - & 2x_2 & = & -1 \\ R_2 & -x_1 & + & 3x_2 & = & 3 \end{array} \longrightarrow \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ R_1 + R_2 \rightarrow & & x_2 & = & 2 \end{array} \quad R_1 + 2R_2 \rightarrow \begin{array}{rcr} x_1 & = & 3 \\ x_2 & = & 2 \end{array}$$

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Definition: Two matrices are *row equivalent* if one can be trans- formed into the other by a sequence of elementary row operations.

$$\begin{array}{l} R_1 \\ R_2 \end{array} \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \quad \longrightarrow \quad \begin{array}{r} \\ R_1 + R_2 \rightarrow \end{array} \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array} \quad \begin{array}{r} R_1 + 2R_2 \rightarrow \\ \\ \end{array} \begin{array}{rcr} x_1 & = & 3 \\ x_2 & = & 2 \end{array}$$

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Fact: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set, i.e. they are equivalent linear systems.

Warning: Do not do multiple elementary row operations at the same time, except adding multiples of the same row to several rows.

$$\begin{aligned} x_1 - 2x_2 &= 1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \qquad \begin{aligned} x_2 &= 2 \\ x_2 &= 2 \end{aligned}$$

These are NOT equivalent systems.

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \begin{array}{l} \leftarrow R_1 + R_2 \\ \leftarrow R_2 + R_1 \end{array}$$

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} \leftarrow R_1 - R_3 \\ \leftarrow R_2 + 4R_3 \\ \end{array}$$

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Two fundamental questions:

1. Existence of solutions: is the system consistent?
2. Uniqueness of solutions: if a solution exists, is it the only one?

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Answering this requires less work than finding the solution.

Example:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

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We can stop here: back-substitution shows that we can find a unique solution.

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echelon form

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reduced echelon form

Theorem: Any matrix A is row-equivalent to exactly one reduced echelon matrix, which is called its reduced echelon form and written $\text{rref}(A)$.

General strategy for solving a linear system: apply row operations to its augmented matrix to obtain its rref.

General strategy for determining existence/uniqueness of solutions: apply row operations to its augmented matrix to obtain an echelon form, i.e. a row-equivalent echelon matrix.

These processes of row operations (to get to echelon or reduced echelon form) are called row reduction.

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General strategy for determining existence/uniqueness of solutions: apply row operations to its augmented matrix to obtain an echelon form, i.e. a row-equivalent echelon matrix.

Warning: an echelon form is not unique. Its entries depend on the row operations we used. But its pattern of $\blacksquare$ and $*$ is unique.

These processes of row operations (to get to echelon or reduced echelon form) are called row reduction.

Row reduction:

$$\begin{tikzcd}
\text{augmented matrix of linear system} \arrow[r] & \text{echelon form} \arrow[r] \arrow[d] & \text{reduced echelon form} \arrow[d] \\
& \text{existence and uniqueness of solutions} & \text{solution set}
\end{tikzcd}$$

Row reduction:

augmented matrix of linear system → echelon form → reduced echelon form

echelon form ↓ existence and uniqueness of solutions

reduced echelon form ↓ solution set

The rest of this section:

- The row reduction algorithm
- Getting the solution, existence/uniqueness from the (reduced) echelon form

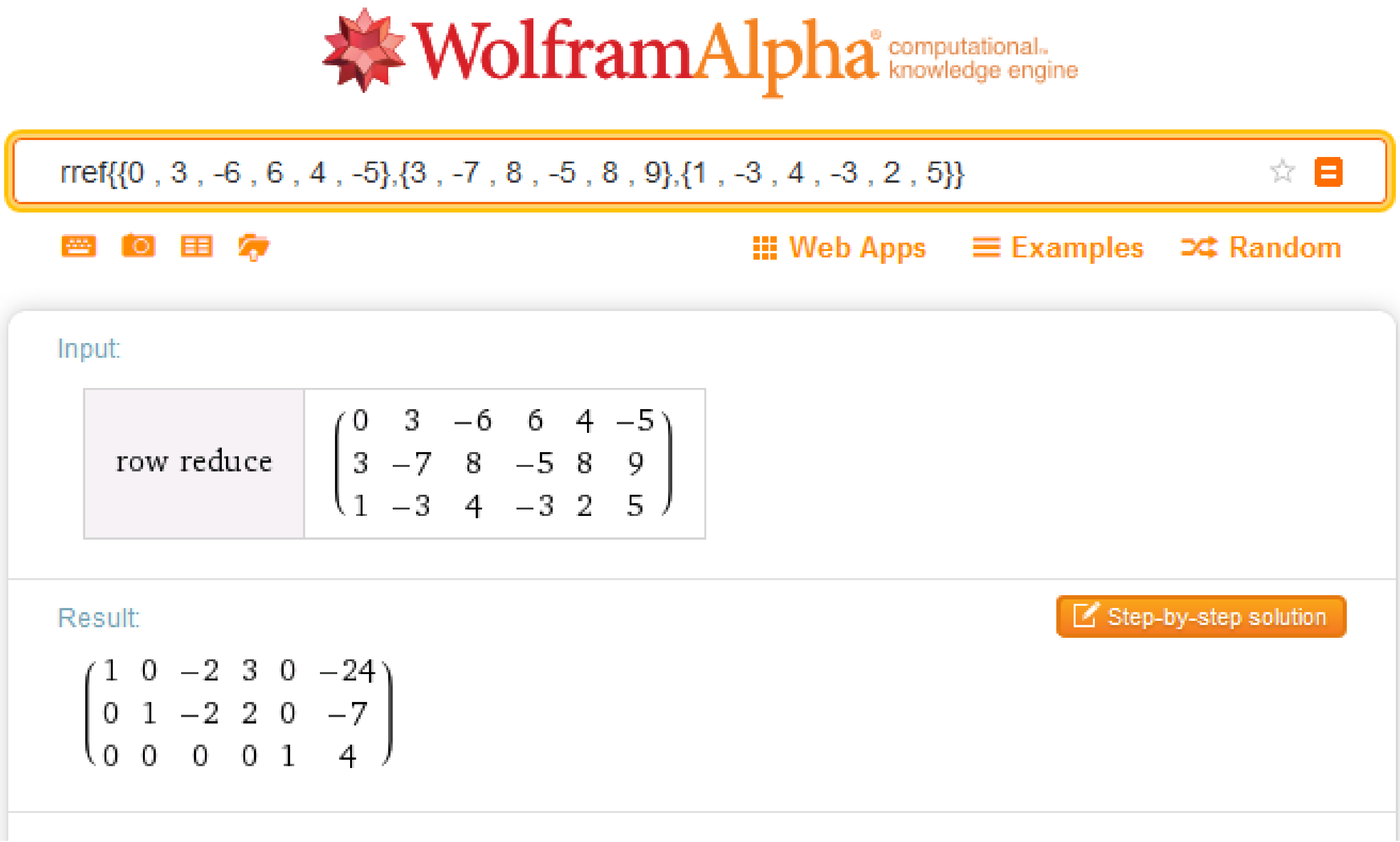
Important terms in the row reduction algorithm:

- pivot position: the position of a leading entry in a row-equivalent echelon matrix.
- pivot: a nonzero entry of the matrix that is used in a pivot position to create zeroes below it.
- pivot column: a column containing a pivot position.

The black squares are the pivot positions.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$$

Check your answer: www.wolframalpha.com



Getting the solution set from the reduced echelon form:

A basic variable is a variable corresponding to a pivot column.
All other variables are free variables.

Example:

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \qquad \begin{aligned} x_1 \quad -2x_3 + 3x_4 \quad &= -24 \\ x_2 - 2x_3 + 2x_4 \quad &= -7 \\ x_5 &= 4 \end{aligned}$$

basic variables: x_1, x_2, x_5, free variables: x_3, x_4.

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The free variables can take any value. These values then uniquely determine the basic variables.

Example:

$$\begin{aligned} x_1 &= -24 + 2x_3 - 3x_4 \\ x_2 &= -7 + 2x_3 - 2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \\ x_5 &= 4 \end{aligned}$$

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$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -24 + 2s - 3t \\ -7 + 2s - 2t \\ s \\ t \\ 4 \end{pmatrix}$$

Getting the solution set from the reduced echelon form:

Another example: reduced echelon form is $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array}\right]$

Getting the solution set from the reduced echelon form:

Another example: reduced echelon form is

$$0x_1 + 0x_2 + 0x_3 = 15 \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array}\right]$$

the system is inconsistent

Getting the solution set from the reduced echelon form:

Another example: reduced echelon form is

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the system is inconsistent

Theorem 2: Existence and Uniqueness:
A linear system is consistent if and only if an echelon form of its augmented matrix has no row of the form $[0 \ldots 0|*]$ with $* \neq 0$.

If a linear system is consistent, then:
- it has a unique solution if there are no free variables;
- it has infinitely many solutions if there are free variables.

Next week: we talk about this: