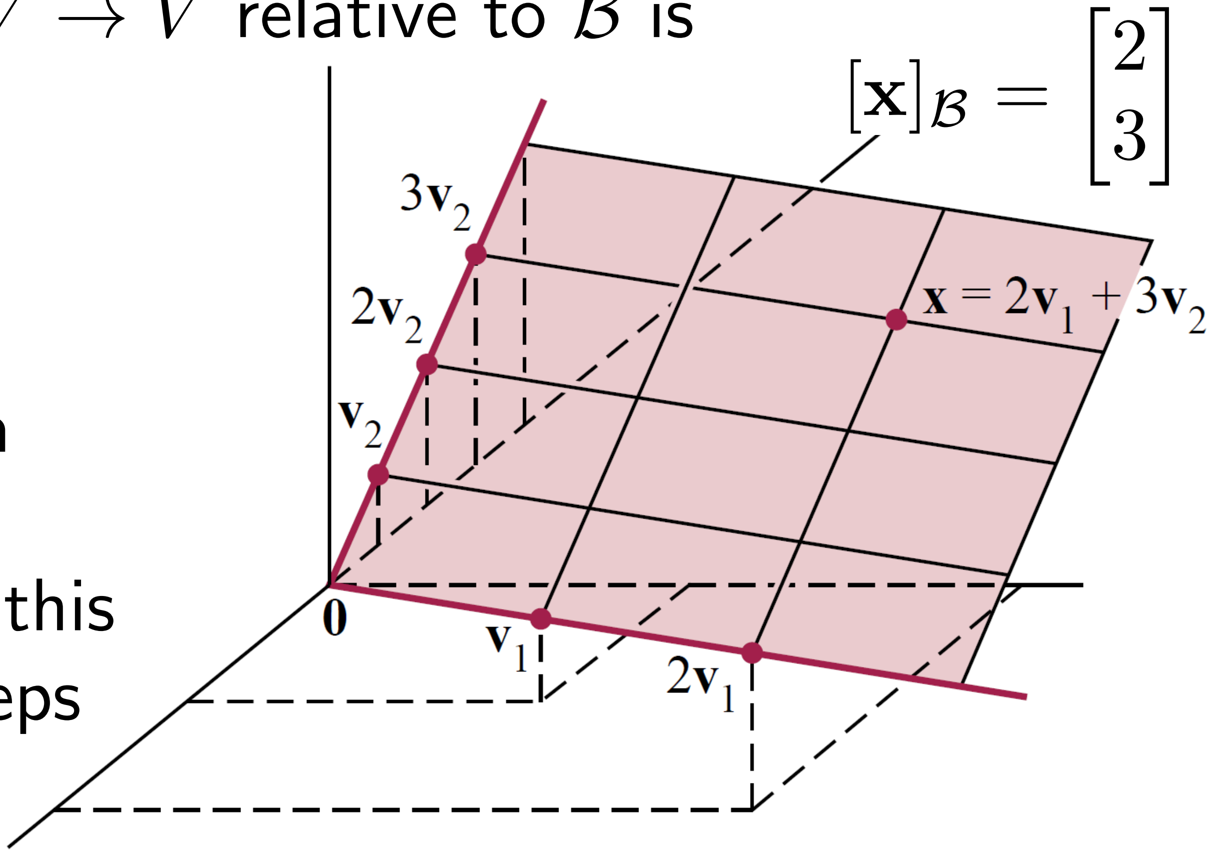
§4.4, 4.7, 5.4: Change of Basis

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for V. Remember:

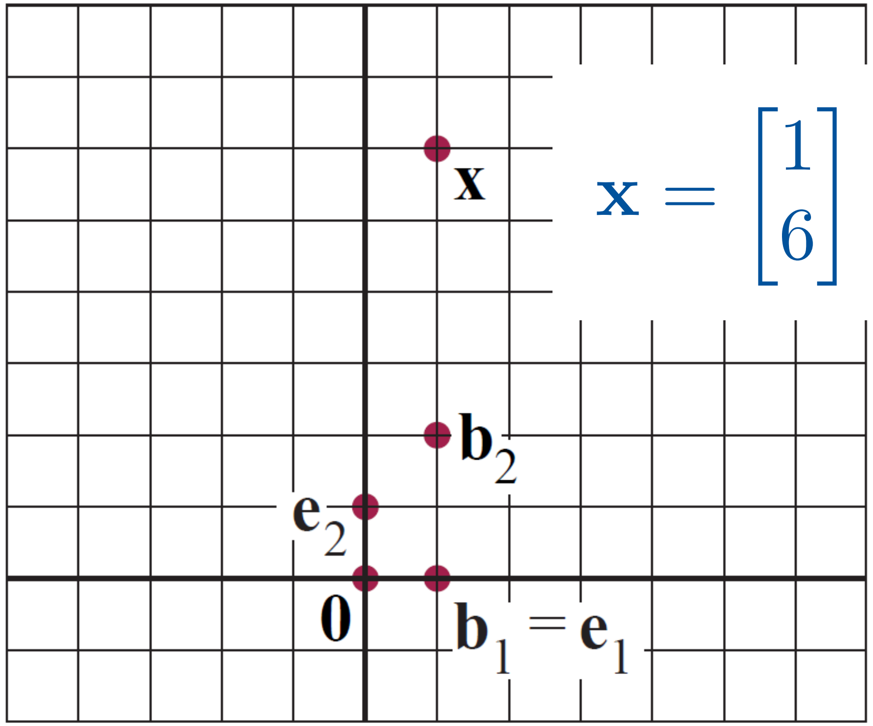
- The $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ where $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.
- The matrix for a linear transformation $T: V \to V$ relative to $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & \dots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix}.$$

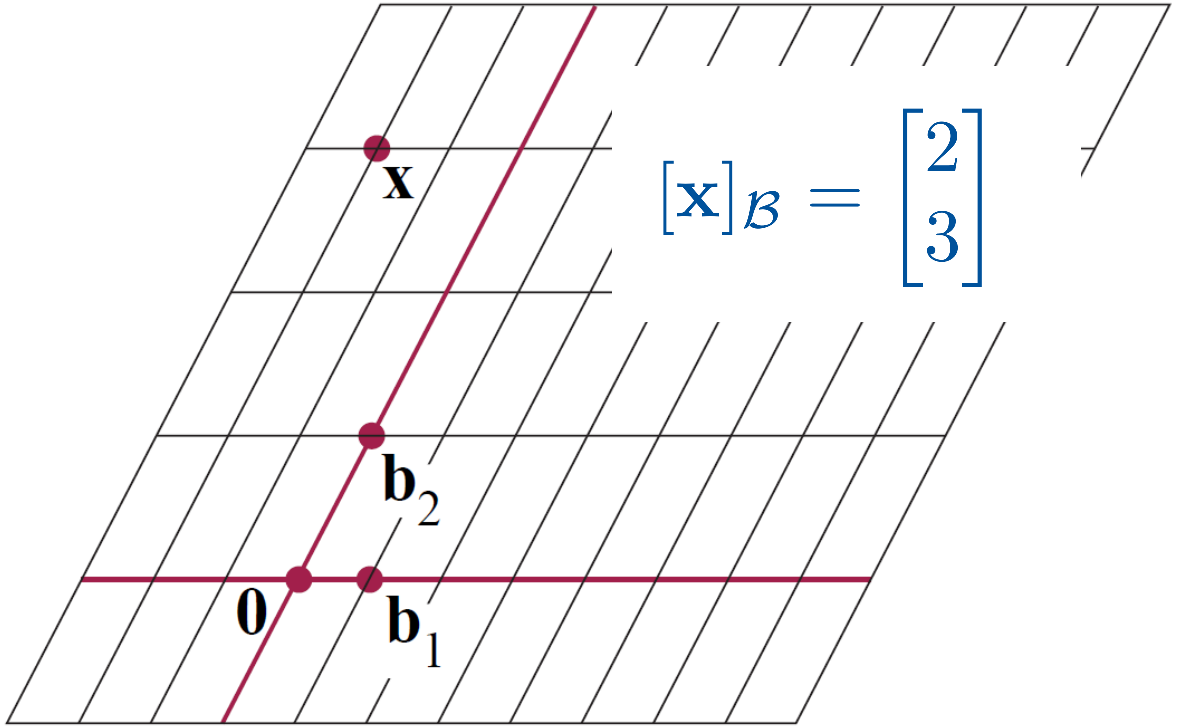


A basis for this plane in $\mathbb{R}^3$ allows us to draw a coordinate grid on the plane. The coordinate vectors then describe the location of points on this plane relative to this coordinate grid (e.g. 2 steps in $\mathbf{v}_1$ direction, 3 steps in $\mathbf{v}_2$ direction.)

Although we already have the standard coordinate grid on $\mathbb{R}^n$, some computations are much faster and more accurate in a different basis i.e. using a different coordinate grid (later, p17-19).



standard coordinate grid



$\mathcal{B}$-coordinate grid

Important questions:

i how are $\mathbf{x}$ and $[\mathbf{x}]_{\mathcal{B}}$ related (p3-6, §4.4 in textbook);

ii how are $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{F}}$ related for two bases $\mathcal{B}$ and $\mathcal{F}$ (p7-10, §4.7);

iii how are the standard matrix of T and the matrix $[T]_{\mathcal{B}}$ related (p11-14, §5.4).

Changing from any basis to the standard basis of $\mathbb{R}^n$

EXAMPLE: (see the picture on p3) Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis of $\mathbb{R}^2$.

a. If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, then what is $\mathbf{x}$?

b. If $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then what is $\mathbf{v}$?

Solution: (a) Use the definition of coordinates:

$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ means that $\mathbf{x} =$ ______ $\mathbf{b}_1 +$ ______ $\mathbf{b}_2 =$

(b) Use the definition of coordinates:

$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ means that $\mathbf{v} =$ ______ $\mathbf{b}_1 +$ ______ $\mathbf{b}_2 =$

In general, if $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$, and $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, then

$\mathbf{x} =$ ______ $\mathbf{b}_1 +$ ______ $\mathbf{b}_2 + \cdots +$ ______ $\mathbf{b}_n = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}$.

This is the **change-of-coordinates matrix from $\mathcal{B}$ to the standard basis** ($\mathcal{P}_{\mathcal{B}}$ in textbook).

In the opposite direction

Changing from the standard basis to any other basis of $\mathbb{R}^n$

EXAMPLE: (see the picture on p3) Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis of $\mathbb{R}^2$.

a. If $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$, then what are its $\mathcal{B}$-coordinates $[\mathbf{x}]_{\mathcal{B}}$?

b. If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then what are its $\mathcal{B}$-coordinates $[\mathbf{v}]_{\mathcal{B}}$?

Solution: (a) Suppose $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. This means that

$$\begin{bmatrix} 1 \\ 6 \end{bmatrix} = \mathbf{x} =$$

So (c_1, c_2) is the solution to the linear system $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 6 \end{array}\right]$.

Row reduction: $\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \end{array}\right]$

So $[\mathbf{x}]_{\mathcal{B}} =$

(b) The $\mathcal{B}$-coordinate vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ of $\mathbf{v}$ satisfies $\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

So $[\mathbf{v}]_{\mathcal{B}}$ is the solution to

In general, if $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$, and $\mathbf{v}$ is any vector in $\mathbb{R}^n$, then

$[\mathbf{v}]_\mathcal{B}$ is a solution to $\underbrace{\begin{bmatrix} | & | & | \\ \mathbf{b}_1 & \ldots & \mathbf{b}_n \\ | & | & | \end{bmatrix}}_{\mathcal{P}_\mathcal{B}} \mathbf{x} = \mathbf{v}.$

Because $\mathcal{B}$ is a basis, the columns of $\mathcal{P}_\mathcal{B}$ are linearly independent, so by the Invertible Matrix Theorem, $\mathcal{P}_\mathcal{B}$ is invertible, and the unique solution to $\mathcal{P}_\mathcal{B}\mathbf{x} = \mathbf{v}$ is

$$[\mathbf{v}]_\mathcal{B} = \begin{bmatrix} | & | & | \\ \mathbf{b}_1 & \ldots & \mathbf{b}_n \\ | & | & | \end{bmatrix}^{-1} \mathbf{v}.$$

In other words, the change-of-coordinates matrix from the standard basis to $\mathcal{B}$ is $\mathcal{P}_\mathcal{B}^{-1}$.

Indeed, in the previous example, $\mathcal{P}_\mathcal{B}^{-1}\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ -0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$

A very common mistake is to get the direction wrong:

Does multiplication by $\mathcal{P}_{\mathcal{B}}$ change from standard coordinates to $\mathcal{B}$-coordinates, or from $\mathcal{B}$-coordinates to standard coordinates?

Don't memorise the formulas. Instead, remember the definition of coordinates:

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \text{ means } \quad \mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n = \begin{bmatrix} | & & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_n \\ | & & | \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}$$

and you won't go wrong.

ii: Changing between two non-standard bases:

Example: As before, $\mathbf{b}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix}1\\2\end{bmatrix}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$.

Another basis: $\mathbf{f}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{f}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2\}$.

If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}-2\\3\end{bmatrix}$, then what are its $\mathcal{F}$-coordinates $[\mathbf{x}]_{\mathcal{F}}$?

Answer 1: $\mathcal{B}$ to standard to $\mathcal{F}$ - works only in $\mathbb{R}^n$, in general easiest to calculate.

$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}-2\\3\end{bmatrix}$ means $\mathbf{x} = -2\mathbf{b}_1 + 3\mathbf{b}_2 = -2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix}$.

So if $[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix}d_1\\d_2\end{bmatrix}$, then $d_1\begin{bmatrix}1\\1\end{bmatrix} + d_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix}$.

Row-reducing $\left[\begin{array}{cc|c}1 & 0 & 1\\1 & 1 & 6\end{array}\right]$ shows $d_1 = 1, d_2 = 5$ so $[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix}1\\5\end{bmatrix}$.

In other words, $\mathbf{x} = \mathcal{P}_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{F}} = \mathcal{P}_{\mathcal{F}}^{-1}\mathbf{x}$, so $[\mathbf{x}]_{\mathcal{F}} = \mathcal{P}_{\mathcal{F}}^{-1}\mathcal{P}_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$.

Answer 2: A different view that works for abstract vector spaces (without reference to a standard basis) - important theoretically, but may be hard to calculate for general examples in $\mathbb{R}^n$.

$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ means $\mathbf{x} = -2\mathbf{b}_1 + 3\mathbf{b}_2$.

So $[\mathbf{x}]_{\mathcal{F}} = [-2\mathbf{b}_1 + 3\mathbf{b}_2]_{\mathcal{F}} = -2[\mathbf{b}_1]_{\mathcal{F}} + 3[\mathbf{b}_2]_{\mathcal{F}} = \begin{bmatrix} | & | \\ [\mathbf{b}_1]_{\mathcal{F}} & [\mathbf{b}_2]_{\mathcal{F}} \\ | & | \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

because $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{F}}$ is an isomorphism, so every vector space calculation is accurately reproduced using coordinates.

$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{f}_1 - \mathbf{f}_2$ so $[\mathbf{b}_1]_{\mathcal{F}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{f}_1 + \mathbf{f}_2$ so $[\mathbf{b}_2]_{\mathcal{F}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

This step can be hard to calculate if the $\mathbf{b}_i$ are not "easy" linear combinations of the $\mathbf{f}_i$. But if you need to change bases in a practical application, the bases are probably "nicely" related.

So $[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Theorem 15: Change of Basis: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{F} = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ be two bases of a vector space V. Then, for all $\mathbf{x}$ in V,

$$[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix} | & & | \\ [\mathbf{b}_1]_{\mathcal{F}} & \ldots & [\mathbf{b}_n]_{\mathcal{F}} \\ | & & | \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}.$$

Notation: write $\underset{\mathcal{F} \leftarrow \mathcal{B}}{\mathcal{P}}$ for the matrix $\begin{bmatrix} | & & | \\ [\mathbf{b}_1]_{\mathcal{F}} & \ldots & [\mathbf{b}_n]_{\mathcal{F}} \\ | & & | \end{bmatrix}$, the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{F}$.

A tip to get the direction correct:

$$[\mathbf{x}]_{\mathcal{F}} = \underbrace{\underset{\mathcal{F} \leftarrow \mathcal{B}}{\mathcal{P}} [\mathbf{x}]_{\mathcal{B}}}$$

A $\mathcal{F}$-coordinate vector

a linear combination of columns of $\underset{\mathcal{F} \leftarrow \mathcal{B}}{\mathcal{P}}$, so these columns should be $\mathcal{F}$-coordinate vectors

Theorem 15: Change of Basis: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{F} = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ be two bases of a vector space V. Then, for all $\mathbf{x}$ in V,

$$[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix} | & & | \\ [\mathbf{b}_1]_{\mathcal{F}} & \ldots & [\mathbf{b}_n]_{\mathcal{F}} \\ | & & | \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}.$$

Properties of the change-of-coordinates matrix $\underset{\mathcal{F}\leftarrow\mathcal{B}}{\mathcal{P}} = \begin{bmatrix} | & & | \\ [\mathbf{b}_1]_{\mathcal{F}} & \ldots & [\mathbf{b}_n]_{\mathcal{F}} \\ | & & | \end{bmatrix}$:

- $\underset{\mathcal{F}\leftarrow\mathcal{B}}{\mathcal{P}} = \underset{\mathcal{B}\leftarrow\mathcal{F}}{\mathcal{P}}^{-1}$.
- If V is $\mathbb{R}^n$ and $\mathcal{E}$ is the standard basis $\{\mathbf{e}_1, \ldots \mathbf{e}_n\}$, then $$\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}} = \mathcal{P}_{\mathcal{B}} = \begin{bmatrix} | & & | \\ \mathbf{b}_1 & \ldots & \mathbf{b}_n \\ | & & | \end{bmatrix}$$, because $[\mathbf{b}_i]_{\mathcal{E}} = \mathbf{b}_i$. Also $\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}} = \mathcal{P}_{\mathcal{B}}^{-1}$.
- If V is $\mathbb{R}^n$, then $\underset{\mathcal{F}\leftarrow\mathcal{B}}{\mathcal{P}} = \mathcal{P}_{\mathcal{F}}^{-1}\mathcal{P}_{\mathcal{B}}$ (see p8).