Important theorems from Chapters 1-3:

Theorem: Existence and Uniqueness:

A linear system is consistent if and only if an echelon form of its augmented matrix has **no** row of the form $[0 \ldots 0|*]$ with $* \neq 0$.
If a linear system is consistent, then:
- it has a unique solution if there are no free variables;
- it has infinitely many solutions if there are free variables.

Theorem: Solution sets and homogeneous systems:

Suppose $\mathbf{p}$ is a solution to $A\mathbf{x} = \mathbf{b}$. Then the solution set to $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v_h}$, where $\mathbf{v_h}$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Theorem: Existence of solutions:

For an $m \times n$ matrix A, the following statements are logically equivalent:

- For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
- Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
- The columns of A span $\mathbb{R}^m$.
- $\mathrm{rref}(A)$ has a pivot in every row.
- The function $\mathbf{x} \mapsto a\mathbf{x}$ is onto.

Theorem: Uniqueness of solutions:

For a matrix A, the following are equivalent:

- $A\mathbf{x} = \mathbf{0}$ has no non-trivial solution.
- If $A\mathbf{x} = \mathbf{b}$ is consistent, then it has a unique solution.
- The columns of A are linearly independent.
- $\mathrm{rref}(A)$ has a pivot in every column (i.e. all variables are basic).
- The function $\mathbf{x} \mapsto a\mathbf{x}$ is one-to-one.

Invertible Matrix Theorem:

For a square matrix, all statements in the above two theorems are logically equivalent, and they are also logically equivalent to:

- A is an invertible matrix.
- $\mathrm{rref}(A) = I_n$.
- $\det A \neq 0$.

- A^T is an invertible matrix (so these statements are also equivalent to the above statements if we replace "column" by "row").

Properties of Determinants:

- Replacement $R_i \to R_i + cR$: determinant does not change.
- Interchange $R_i \to R_j$, $R_j \to R_i$: determinant multiplies by -1.
- Scaling $R_i \to cR_i$, $c \neq 0$: determinant multiplies by c.
- $\det A = \det A^T$.
- $\det(AB) = \det A \det B$.
- $\det(A^{-1}) = \frac{1}{\det A}$.
- $\det(cA) = c^n \det A$.

Important calculations from Chapters 1-3:

- Row-reduction to echelon form to determine existence / uniquness of solutions.
- Row-reduction to reduced echelon form to determine solutions, and express them in parametric form.
- The standard matrix of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the $m \times n$ matrix
$$\begin{bmatrix} | & | & | \\ T(\mathbf{e}_1) & \dots & T(\mathbf{e}_n) \\ | & | & | \end{bmatrix}.$$
- Products of matrices.
- Inverse of a matrix: $[A|I_n] \xrightarrow{\text{row reduction}} [I_n|A^{-1}]$.
- Determinants.

Other tips:

- $\mathbf{b}$ is in the span of $\mathbf{v}_1, \dots, \mathbf{v}_p$ if and only if
$$\begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_p \\ | & | & | \end{bmatrix} \mathbf{x} = \mathbf{b}$$
has a solution.
- Many things about linear independence - see the handwritten sheet (from the class after the quiz).
- To prove that T is a linear transformation:
  - Show $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;
  - Show $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.
- To decide if T is a linear transformation:
  - If $T(\mathbf{0}) \neq \mathbf{0}$, then T is not linear.
  - If $T(\mathbf{0}) = \mathbf{0}$, see if the two properties above hold. Prove **both** properties, or find a $\mathbf{u}, \mathbf{v}, c$ such that **one** of the two properties fails.
- Conceptual problems about linear transformations and span / linear independence / one-to-one / onto, without numbers: use
$$T(c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \dots + c_pT(\mathbf{v}_p).$$