Descent algebras

related topics: noncommutative symmetric functions / quasisymmetric functions
hyperplane arrangements
random walks on a group
reflection / Coxeter groups

Work in the symmetric group algebra $\mathbb{R}S_n$
(the permutations are a basis, add pointwise, multiply by linear extension of compositions of permutations.)
To interpret the multiplication in $\mathbb{R}S_n$: suppose $\alpha = \sum_{\sigma \in S_n} a_\sigma \sigma$; $\beta = \sum_{\sigma \in S_n} b_\sigma \sigma$.
then the coefficient of σ in $\alpha\beta$ is the probability of going from the identity to σ by multiplying by a random permutation with probability a_σ, then multiplying by another random permutation with probability b_σ.
e.g. if $\pi_\sigma^{(k)}$ is the chance of moving from the identity to σ in k repeats of the same process, and $\rho = \sum_{\sigma \in S_n} \pi_\sigma^{(1)} \sigma$, then $\rho^k = \sum_{\sigma \in S_n} \pi_\sigma^{(k)} \sigma$

Suppose ρ models a card-shuffle. Then a question of interest is, how many shuffles are necessary to mix the deck?
The deck is well-mixed if it is equally likely to be in any of the $n!$ orders
$\therefore$ define U to be $\sum_{\sigma \in S_n} \frac{1}{n!} \sigma$
Then the rigorous mathematical question is, given $\varepsilon > 0$, what value of k ensures $\|\rho^k - U\| < \varepsilon$, for some metric $\| \ \|$ on $\mathbb{R}S_n$?

The study of riffle-shuffles uses the following "magic" formula in $\mathbb{R}(S_n)[t]$:

$$\sum_{i=1}^{n} e_i t^i = \sum_{\sigma\in S_n} \frac{(t-d(\sigma))^{(n)}}{n!}\sigma$$

here, e_i are particular elements of $\mathbb{R}(S_n)$

$d(\sigma)$ is the number of descents in σ — i.e. $|\{i\in\{1,\dots,n\} \mid \sigma_{i+1}>\sigma_i\}|$

$x^{(n)}$ denotes the increasing factorial $x(x+1)\cdots(x+n-1)$.

e.g. in S_3, the right hand side is

$$\frac{t(t+1)(t+2)}{6}[\![123]\!] + \frac{(t-1)t(t+1)}{6}\left([\![132]\!]+[\![213]\!]+[\![231]\!]+[\![312]\!]\right) + \frac{(t-2)(t-1)t}{6}[\![321]\!]$$

so, collecting the powers of t, we get $e_1 = \frac{1}{3}[\![123]\!] - \frac{1}{6}$

$$e_2 = \frac{1}{2}[\![123]\!] - \frac{1}{2}[\![321]\!]$$

$$e_3 = \frac{1}{6}\left([\![123]\!]+[\![132]\!]+[\![213]\!]+[\![231]\!]+[\![312]\!]+[\![321]\!]\right)$$

It is always the case that $e_n = \frac{1}{n!}\sum_{\sigma\in S_n}\sigma$.

The e_i turn out to be orthogonal idempotents — i.e. $e_i^2 = e_i$, and $e_ie_j = 0$ if $i\neq j$.
Taking $t=1$ in the magic formula: $e_1+e_2+\dots+e_n = \mathrm{id}$

Taking $t=2$ in the magic formula:
because $x^{(n)}=0$ if $-n+1\le x\le 0$, the right hand side is $(n+1)\,\mathrm{id} + \sum_{\sigma\in S_n, d(\sigma)=1}\sigma$.
and, dividing this by 2^n gives the probabilities of the riffle-shuffle.
(see the paper by Bayer-Diaconis.)

So we are looking for the minimal k such that $\left(\frac{1}{2^n}\sum_{i=1}^{n} e_i 2^i\right)^k$ is close to uniform.

But, because the e_i are orthogonal idempotents, this is simply $\frac{1}{2^{nk}}\sum_{i=1}^{n} e_i 2^{ik}$.
Using the right-hand-side of the magic formula, we take the coefficients of each permutation, and discover that the k-step transition probabilities are

$$\Pi_\sigma^{(k)} = \frac{(2^k-d(\sigma))^{(n)}}{n!}$$

Note that this is non-zero only when $2^k > d(\sigma)$ $\therefore$ this is non-zero for all σ when $k\sim\log_2 n$. And indeed, Bayer-Diaconis shows that $\frac{3}{2}\log_2 n$ riffle shuffles are necessary to mix a deck.

(The idempotent e_1 is originally due to Michael Barr, from Hochschild cohomology)

The descent algebra is originally due to Solomon:

let $D(\sigma) := \{ i \mid 1 \le i \le n-1,\ \sigma_i > \sigma_{i+1} \} \subseteq \{1,2,\dots,n-1\}$

in $\mathbb{Q}S_n$, let $y_T := \sum_{D(\sigma)=T} \sigma$, where T is any of the 2^{n-1} subsets of $\{1,2,\cdots,n-1\}$

e.g. for $n=3$:

$$y_\emptyset = [\![123]\!] \qquad (y_\emptyset = \text{identity, always})$$
$$y_{\{1\}} = [\![213]\!] + [\![312]\!]$$
$$y_{\{2\}} = [\![132]\!] + [\![231]\!]$$
$$y_{\{1,2\}} = [\![321]\!]$$

Theorem of Solomon (which holds for all finite Coxeter groups)

the y_T span a subalgebra of $\mathbb{Q}S_n$: this is $\Sigma[S_n]$, the descent algebra

Equivalently, for two subsets T, R of $\{1,2,\cdots,n-1\}$, then $y_T y_R = \sum a^K_{TR} y_K$.

Indeed, the a^K_{TR} are integral, and Solomon gives an expression for them explicitly.

Since $y_\emptyset$ is the identity permutation, the descent algebra is an algebra with identity.

e.g. for $n=3$: $y_\emptyset$ acts as the identity.

$$y_{\{1,2\}}\, y_{\{1,2\}} = y_\emptyset$$
$$y_{\{1,2\}}\, y_{\{1\}} = y_{\{2\}}$$
$$y_{\{1,2\}}\, y_{\{2\}} = y_{\{1\}}$$
$$y_{\{1\}}\, y_{\{1\}} = y_\emptyset + y_{\{2\}} + y_{\{1,2\}}.$$

Coxeter groups: these are pairs (W,S) such that W is a finite group, and S generates W

$s,t \in S$ satisfy the relations $s^2 = \mathrm{id}$, $(st)^{m_{st}} = \mathrm{id}$ for some $m_{st} \in \mathbb{N}$, with $m_{st} \geq 2$.

if $m_{s,t} = 2$, then $(st)(st) = \mathrm{id} \Rightarrow \cancel{s}stst\cancel{t} = st \Rightarrow ts = st$. i.e. s,t commute.

If S is the disjoint union $S_1 \amalg S_2$, and $m_{st} = 2$ for all $s \in S_1$, $t \in S_2$

then the group W generated by S is a cartesian product $W_1 \times W_2$, where S_i generates W_i

$\therefore$ it is enough to understand irreducible Coxeter groups – these have been classified by Coxeter.

If we draw a graph whose vertices are S, and draw

$s \quad t$ if $m_{st} = 2$

s — t if $m_{st} = 3$

s = t if $m_{st} = 4$,

then the irreducible Coxeter groups correspond to connected graphs.

Example: the symmetric group S_n corresponds to the graph •—•—• ···· •—• with $n-1$ vertices

the generator s_i is the transposition $(i, i+1)$
the relations are $(s_i s_j)^2 = id$ if $|j-i| \geq 2$
$(s_i s_{i+1})^3 = id$.

For any $w \in W$, let $l(w)$ be the length of a minimal-length expression of w as a product of generators in S.
Then, for $w \in W$, its descent set is $D(w) = \{i \mid l(ws_i) < l(w)\} \subseteq S$
In the case of the symmetric group: right-multiplication by s_i exchanges the images of i and $i+1$ — i.e. it sends $\sigma_1 \sigma_2 \cdots \sigma_n$ to $\sigma_1 \sigma_2 \cdots \sigma_{i-1} \sigma_{i+1} \sigma_i \sigma_{i+2} \cdots \sigma_n$.
The length of a permutation is its number of inversions: $|\{(i,j) \mid i<j, \sigma_i > \sigma_j\}|$
And exchanging the images of i and $i+1$ reduces the number of inversions precisely when $\sigma_{i+1} > \sigma_i$ ∴ this more general definition of descents agree with our previous definition.

The general version of Solomon's theorem:
for any subset T of S, set $y_T := \sum_{D(w)=T} w$.
then the y_T span a subalgebra of $\mathbb{Q}W$ of dimension $2^{|S|}$, and there is an explicit description of the coefficients in the products $y_T y_R$.

In the case of the symmetric group, the basis elements y_T can also be indexed by compositions c of n (written $c \vDash n$), since there is a classical bijection between subsets of $n-1$ and compositions of n: $\{i_1, \dots, i_l\} \to (i_1, i_2-i_1, i_3-i_2, \dots)$

There is a second basis for $\Sigma[S_n]$: let $B_T := \sum_{D(\sigma) \subseteq T} \sigma$ — i.e. the permutations who might have a descent at T. The change from B_T to y_T is upper-unitriangular, so B_T is a basis. Its multiplication table is much easier, and proves that $\Sigma[S_n]$ is an algebra: $B_C B_D = \sum_M B_{C(M)}$ where the sum is over all matrices M with non-negative integral entries whose j^{th} column sums to c_j (j^{th} part of the composition C), and whose i^{th} row sums to d_i.
Then $C(M)$ is the composition formed by reading the matrix entries along the rows from left to right, from the top row to the bottom row, and deleting zeroes.

In $\mathbb{Q}[S_n]$, there are two anti-isomorphic algebras: the descent algebra $\Sigma[S_n]$, and l'algèbre de battage B_n.

the anti-isomorphism is given by $\sigma \to \sigma^{-1}$ (so $\alpha\beta \to \beta^{-1}\alpha^{-1}$, hence an anti-isomorphism).

The image of B_T under this anti-isomorphism (in B_n) is $(T=\{t_1, t_2, \cdots, t_k\})$

$$12\cdots t_1 \sqcup\!\sqcup\ t_1{+}1\ t_1{+}2 \cdots t_1{+}t_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup\ t_k{+}1,\ t_k{+}2 \cdots n.$$

(e.g. the image of $B_{\{3,5\}}$ is $123 \sqcup\!\sqcup 45 \sqcup\!\sqcup 678$)

(Here, $\sqcup\!\sqcup$ denotes the *shuffle product*: for α, β who between them have distinct letters, $\alpha \sqcup\!\sqcup \beta := \sum \gamma$ over all γ which contain α, β as complimentary substrings — i.e. $\alpha = \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k}$, $\beta = \gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_l}$, with $i_1 < i_2 < \cdots < i_k$; $j_1 < j_2 < \cdots < j_l$, $\{i_1, i_2, \cdots, i_k\} \sqcup \{j_1, j_2, \cdots, j_l\} = \{1, 2, \cdots\}$. This is an associative product.)

These images of B_T multiply by an analogous matrix rule, except the matrix composition is read down the columns, from the left column to the right column. (This is easy to prove, and the multiplication of B_T may be deduced from this.)

The same multiplication rule describes the Kronecker product (also called internal product) of complete symmetric functions of the same degree. (One definition of this product is that the power sums satisfy $p_\lambda \cdot p_\mu = 0$ if $\lambda \neq \mu$, and $p_\mu \cdot p_\mu = z_\mu p_\mu$, where z_μ is such that $\frac{n!}{z_\mu}$ is the number of permutations of cycle type μ. Equivalently, p_λ/z_λ are orthogonal idempotents.)

As a result, we can define a surjective algebra morphism $\Phi: \Sigma[S_n] \to \Lambda_n$, $\Phi(B_c) = h_c$. This surjection is in fact split — i.e. $\Sigma[S_n]$ is a direct sum of the kernel of S_n and a subspace which is isomorphic to Λ_n via Φ. So we can take the preimages of p_λ/z_λ in this subspace — these are the idempotents E_λ of Garsia and Reutenauer, and summing over all λ of the same number of parts give the e_i from before. More details to follow.

It's easy to verify that the sum of orthogonal idempotents is also an idempotent. We are interested in *primitive idempotents*, which cannot be written as the sum of orthogonal idempotents.

In a finite-dimensional algebra, orthogonal idempotents are necessarily linearly independent. So the number of mutually orthogonal idempotents is at most the dimension of the algebra.

Any set of mutually orthogonal idempotents $\{e_1, e_2, \cdots e_r\}$ can be extended so that $\sum e_i =$ unit of algebra: this is because, if e is an idempotent, then unit$-e$ is an idempotent orthogonal

to e. If $\sum e_i = 1$, then $\{e_1, \cdots, e_r\}$ is a complete family of idempotents.
Note that complete families of primitive orthogonal idempotents
for example, for the algebra of 2x2 upper-triangular matrices, $\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\}$
and $\{\begin{pmatrix}1&1\\0&0\end{pmatrix}, \begin{pmatrix}0&-1\\0&1\end{pmatrix}\}$ are both such families.
Idempotents are important for studying the representations of an algebra.
Expanding an algebra element in terms of idempotents is akin to a Fourier transform.

Let $p(x)$ be a polynomial $(x-a_1)(x-a_2)\cdots(x-a_n)$, where a_i are pairwise distinct.
Then, in the algebra, $\mathbb{C}[x]/\langle p(x)\rangle$, a complete family of primitive orthogonal idempotents are the Lagrange interpolation formulae : $e_k(x) = \frac{(x-a_1)\cdots(x-a_{k-1})(x-a_{k+1})\cdots(x-a_n)}{(a_k-a_1)\cdots(a_k-a_{k-1})(a_k-a_{k+1})\cdots(a_k-a_n)}$
This is because every element of $\mathbb{C}[x]/\langle p(x)\rangle$ is determined by its value on $a_1, a_2, \ldots, a_n$, and e_k is the indicator function on a_k.
So idempotents of algebras are often interesting.

Because $\{p_\lambda/z_\lambda\}$ gives a basis of Λ_n, they must be a complete family of primitive orthogonal idempotents. The unit of the Kronecker product of Λ_n is $\sum_{\lambda\vdash n} p_\lambda/z_\lambda = h_n$.
It happens that their preimages E_λ is a complete family of primitive orthogonal idempotents for $\Sigma[\mathfrak{S}_n]$.
Analogous families exist for other Coxeter groups — but what plays the role of partitions? what is the indexing set?