Homework 2, due 13:45 Friday 17 February 2017

You must justify your answers to receive full credit.

1. For each set of vectors below, determine the value(s) of h for which the set is linearly dependent:

   a) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ h \end{bmatrix} \right\}$

   b) $\left\{ \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix} \right\}$

2. Without doing any row-reduction, determine whether the following sets are linearly independent, and explain why:

   a) $\left\{ \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \end{bmatrix} \right\}$

   b) $\left\{ \begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}$

   c) $\left\{ \begin{bmatrix} 2 \\ -4 \\ 8 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -12 \\ -15 \end{bmatrix} \right\}$

   d) $\left\{ \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

3. For each matrix A in parts a), b), c) below, determine:

   (i) does the equation $A\mathbf{x} = 0$ have a nontrivial solution; and,

   (ii) does the equation $A\mathbf{x} = \mathbf{b}$ have at least one solution for every possible $\mathbf{b}$?

   a) A is a 3×3 matrix with three pivot positions.

   b) A is a 3×3 matrix with two pivot positions.

   c) A is a 3×2 matrix with two pivot positions.

4. For each of the transformations below:
(i) decide whether it is linear, and explain your answer,
(ii) if it is linear, find its standard matrix,
(iii) if it is linear, decide whether it is one-to-one and whether it is onto.

a) $f : \mathbb{R}^4 \to \mathbb{R}^2$ given by $f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{2}x_1 + x_2 - x_3 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

b) $g : \mathbb{R}^2 \to \mathbb{R}^3$ given by $g(\mathbf{x}) = \mathbf{0}$.

c) $h : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation through an angle of $\frac{\pi}{2}$ counterclockwise about the point (-1,1).

5. Let $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be a linearly independent subset of $\mathbb{R}^n$, and $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

a) Prove that, if T is one-to-one, then $\{T(\mathbf{v_1}), T(\mathbf{v_2}), T(\mathbf{v_3})\}$ is linearly independent.

b) Show that the assumption that T is one-to-one is necessary - that is, find a numerical example of a linear transformation T that is not one-to-one, and a linearly independent set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$, such that $\{T(\mathbf{v_1}), T(\mathbf{v_2}), T(\mathbf{v_3})\}$ is linearly dependent.

6. State whether each of the following statements is (always) true or (sometimes) false. If it is true, give a brief justification; if it is false, give a counterexample with an explanation.

a) If $\mathbf{w}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$ and $\mathbf{x}$ is in Span $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, then $\mathbf{x}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$.

b) If $A \begin{bmatrix} 2 \\ 0 \\ 4 \\ -3 \end{bmatrix} = \mathbf{0}$, then $A\mathbf{e}_4$ is a linear combination of the first three columns of A.

c) $A\mathbf{x} = \mathbf{b}$ is a homogeneous equation if and only if $\mathbf{x} = \mathbf{0}$ is a solution.

d) Let A be a 4 x 3 matrix with columns $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$, and suppose $\mathbf{b}$ is a vector in $\mathbb{R}^4$ such that $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{b}\}$ is linearly dependent. Then $A\mathbf{x} = \mathbf{b}$ has a solution.

e) Let A be a 4 x 3 matrix with columns $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$. If $\mathbf{b}$ is a vector in $\mathbb{R}^4$ such that $A\mathbf{x} = \mathbf{b}$ has a solution, then $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{b}\}$ is linearly dependent.

f) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\{\mathbf{v_1}, \mathbf{v_2}\}$ spans $\mathbb{R}^2$, then $\{T(\mathbf{v_1}), T(\mathbf{v_2})\}$ also spans $\mathbb{R}^2$.

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