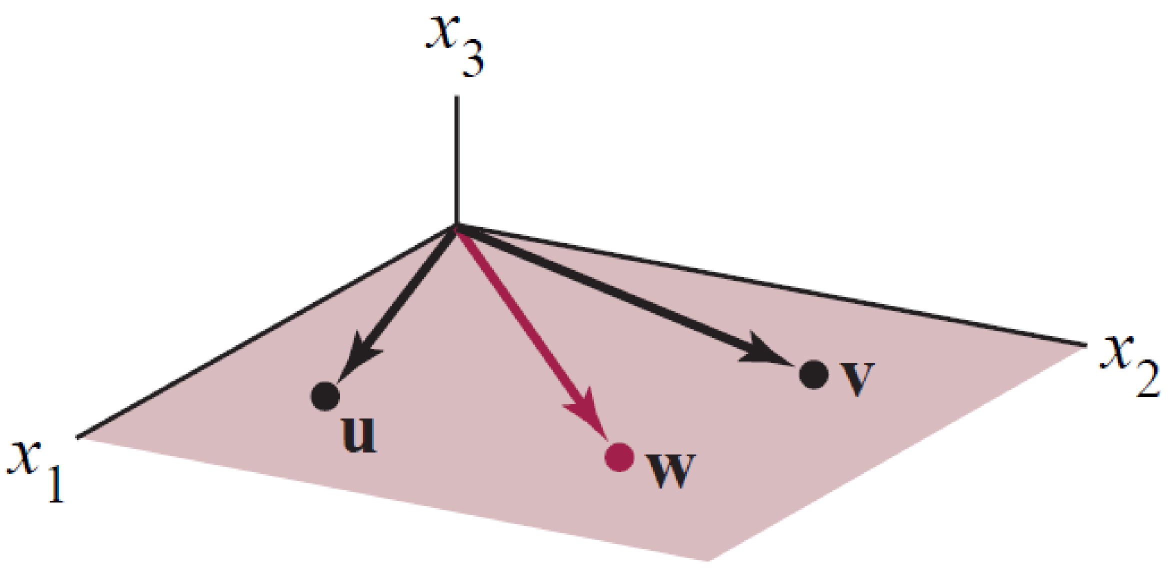
§1.7: Linear Independence



In this picture, the plane is Span $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ = Span $\{\mathbf{u}, \mathbf{v}\}$, so we do not need to include $\mathbf{w}$ to describe this plane.
We can think that $\mathbf{w}$ is "too similar" to $\mathbf{u}$ and $\mathbf{v}$ - and linear dependence is the way to make this idea precise.

Definition: A set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is *linearly independent* if the only solution to the vector equation

$$x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$$

is the trivial solution $(x_1 = \cdots = x_p = 0)$.

The opposite of linearly independent is linearly dependent:

Definition: A set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is *linearly dependent* if there are weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v_1} + \cdots + c_p\mathbf{v_p} = \mathbf{0}.$$

The equation $c_1\mathbf{v_1} + \cdots + c_p\mathbf{v_p} = \mathbf{0}$ is a linear dependence relation.

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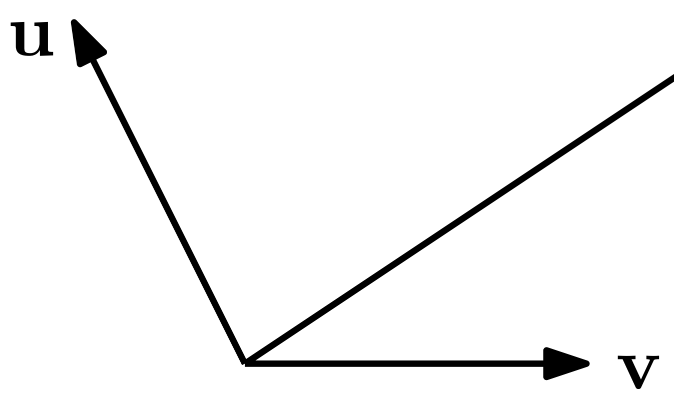
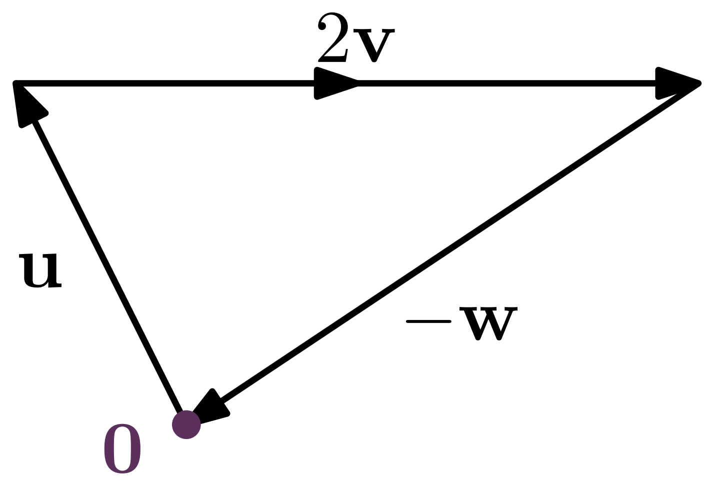
A picture of a linear dependence relation: " you can use the given directions to move in a circle".

$$\mathbf{u} + 2\mathbf{v} - \mathbf{w} = \mathbf{0}$$





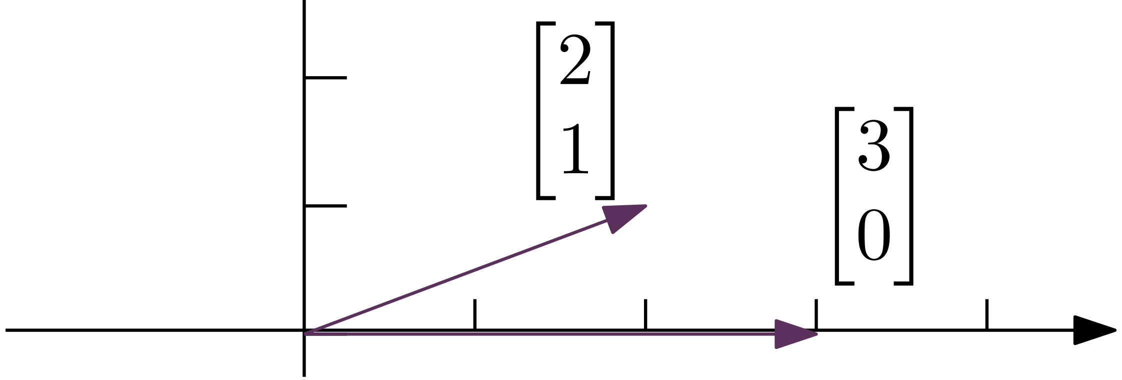
$$x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$$

The only solution is $x_1 = \cdots = x_p = 0$
$\rightarrow$ linearly independent

Example: $\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}\right\}$ is linearly independent because

$$x_1\begin{bmatrix}2\\1\end{bmatrix} + x_2\begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Longrightarrow \begin{aligned} 2x_1 + 3x_2 &= 0 \\ x_1 &= 0 \end{aligned}$$

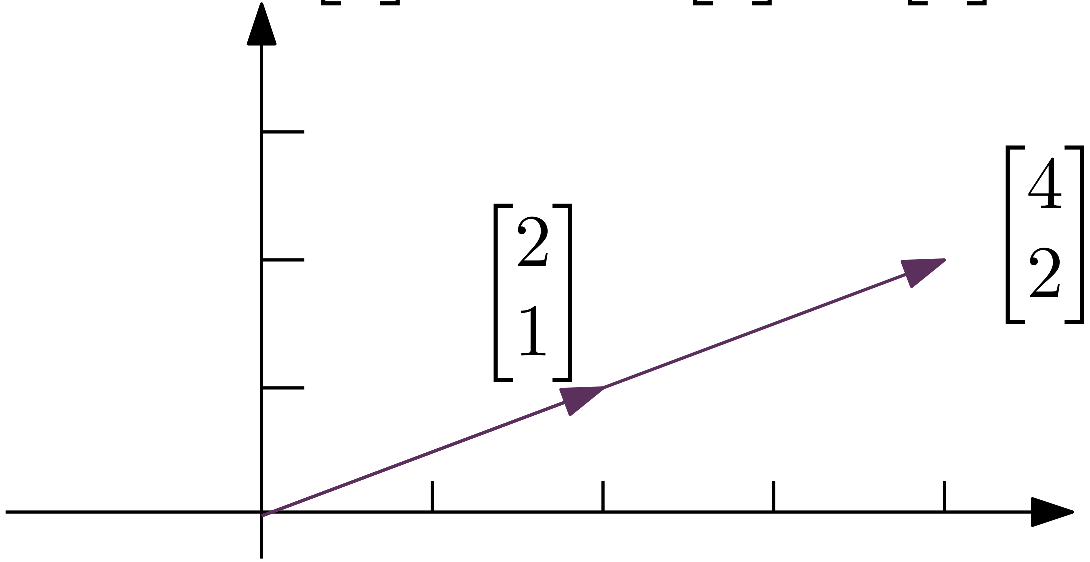
$$\Longrightarrow x_1 = 0, x_2 = 0.$$



There is a solution with some $x_i \neq 0$
$\rightarrow$ linearly dependent

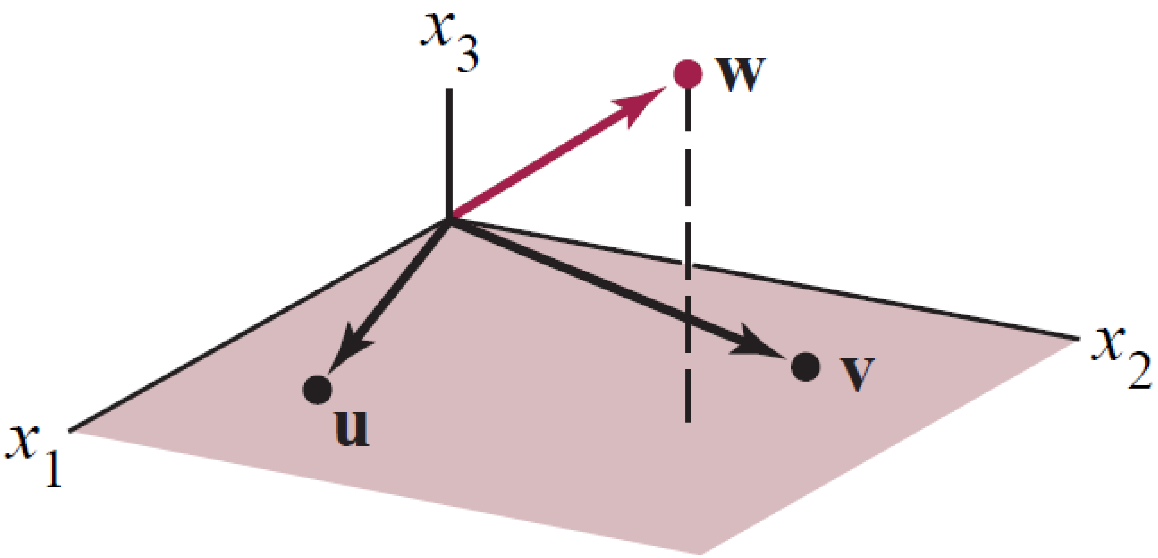
Example: $\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}4\\2\end{bmatrix}\right\}$ is linearly dependent because

$$2\begin{bmatrix}2\\1\end{bmatrix} + (-1)\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$



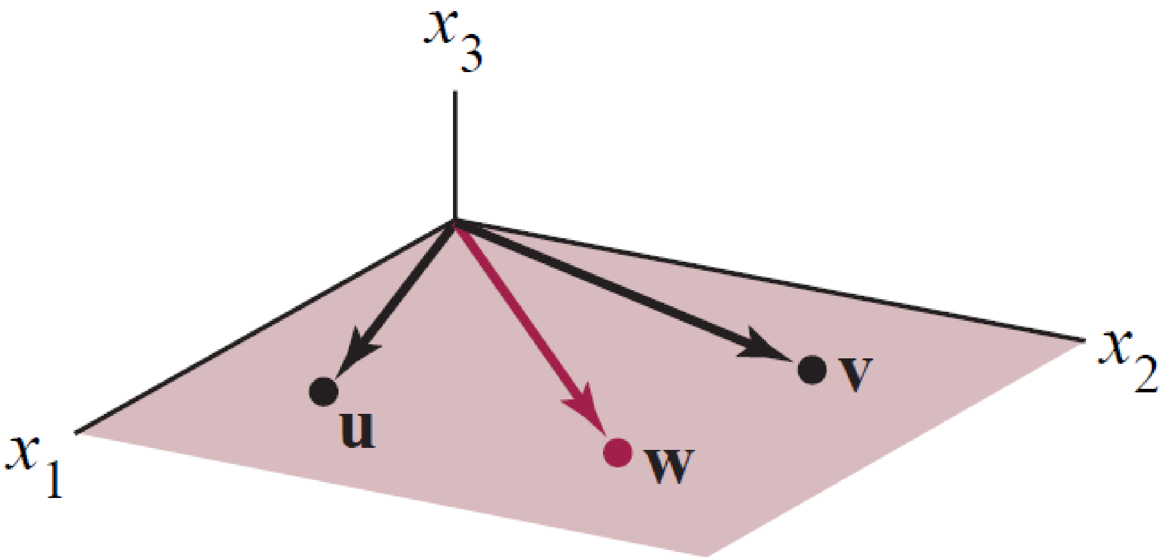
$$x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$$

The only solution is $x_1 = \cdots = x_p = 0$ (i.e. unique solution)
$\rightarrow$ linearly independent



Informally: $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in "totally different directions"; there is "no relationship" between $\mathbf{v}_1, \ldots \mathbf{v}_p$.

There is a solution with some $x_i \neq 0$ (i.e. infinitely many solutions)
$\rightarrow$ linearly dependent



Informally: $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in "similar directions"

Some easy cases:

- Sets containing the zero vector $\{\mathbf{0}, \mathbf{v_2}, \dots, \mathbf{v_p}\}$: then the linear dependence equation is
$$x_1\mathbf{0} + x_2\mathbf{v_2} + \cdots + x_p\mathbf{v_p} = \mathbf{0}.$$
A non-trivial solution is
$$(1)\mathbf{0} + (0)\mathbf{v_2} + \cdots + (0)\mathbf{v_p} = \mathbf{0},$$
so such a set is linearly dependent (it doesn't matter what $\mathbf{v}_2, \dots, \mathbf{v}_p$ are).

- Sets containing one vector $\{\mathbf{v}\}$: then the linear dependence equation is
$$x\mathbf{v} = \mathbf{0} \quad \text{i.e.} \quad \begin{bmatrix} xv_1 \\ \vdots \\ xv_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$
If some $v_i \neq 0$, then $x = 0$ is the only solution. So $\{\mathbf{v}\}$ is linearly independent if $\mathbf{v} \neq \mathbf{0}$.

Some easy cases:

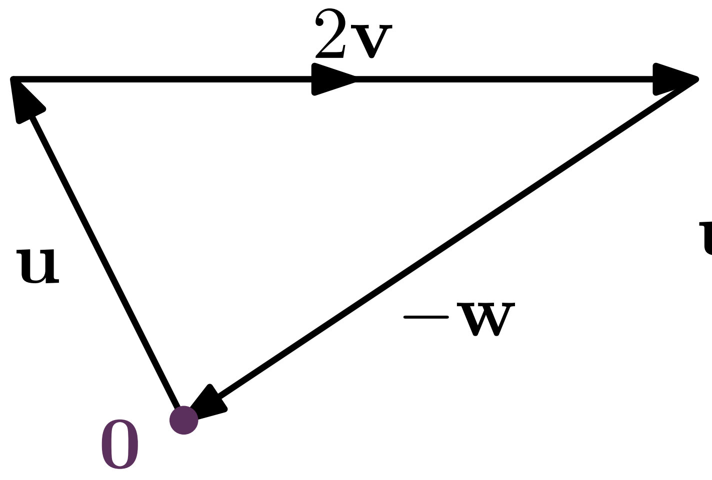
- Sets containing two vectors $\{\mathbf{u}, \mathbf{v}\}$: then the linear dependence equation is
$$x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}.$$
Using the same argument as in the example on p4, we can show that, if $\mathbf{v} = c\mathbf{u}$ for any c, then $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent:
$$\mathbf{v} = c\mathbf{u} \quad \text{means} \quad c\mathbf{u} + (-1)\mathbf{v} = \mathbf{0}.$$
The same argument applies if $\mathbf{u} = d\mathbf{v}$ for any d.
Is this the only way in which two vectors can be linearly dependent?

  Suppose we have $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}$ and x_1, x_2 are not both zero.
If $x_1 \neq 0$, then we can divide by it: $\mathbf{u} = \dfrac{-x_2}{x_1}\mathbf{v}$.
Similarly, if $x_2 \neq 0$, then $\mathbf{v} = \dfrac{-x_1}{x_2}\mathbf{u}$.

  So $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent if and only if one of the vectors is a multiple of the other , i.e. $\mathbf{u}, \mathbf{v}$ are in the same or opposite direction.

When there are more vectors, it is hard to tell quickly if a set is linearly independent or dependent.

As shown in this example from p3, three vectors can be linearly dependent without any of them being a multiple of any other vector.



$$\mathbf{u} + 2\mathbf{v} - \mathbf{w} = \mathbf{0}$$

The correct generalisation of the two-vector case is the following: a set of vectors is linearly dependent if and only if one of the vectors is a linear combination of the others. (More specifically: if the weight x_i in the linear dependency relation $x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$ is non-zero, then $\mathbf{v}_i$ is a linear combination of the other $\mathbf{v}$s, by the same argument as in the case of two vectors.)

How to determine if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent:

EXAMPLE Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$.

a. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
b. If possible, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Solution: (a) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent if ____

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{array}\right] \quad \text{row reduces to} \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is a free variable $\Rightarrow$ there are nontrivial solutions.

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is ____________________

(b) Reduced echelon form: $\left[\begin{array}{ccc|c} 1 & 0 & 33 & 0 \\ 0 & 1 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Let $x_3 =$ _____ (any nonzero number). Then $x_1 =$ _____ and $x_2 =$ _____.

$$\_\_\_\_ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\_\_\_\_\mathbf{v}_1 + \_\_\_\_\mathbf{v}_2 + \_\_\_\_\mathbf{v}_3 = \mathbf{0}$$

(one possible linear dependence relation)

A non-trivial solution to $A\mathbf{x} = \mathbf{0}$ is a linear dependence relation between the columns of A: $A\mathbf{x} = \mathbf{0}$ means $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$.

Theorem: Uniqueness of solutions for linear systems: For a matrix A, the following are equivalent:

a. $A\mathbf{x} = \mathbf{0}$ has no non-trivial solution.
b. If $A\mathbf{x} = \mathbf{b}$ is consistent, then it has a unique solution.
c. The columns of A are linearly independent.
d. rref(A) has a pivot in every column (i.e. all variables are basic).

In particular: the row reduction algorithm produces at most one pivot in each row of rref(A). So, if A has more columns than rows (a "fat" matrix), then rref(A) cannot have a pivot in every column.

So a set of more than n vectors in $\mathbb{R}^n$ is always linearly dependent.

Exercise: Combine this with the Theorem of Existence of Solutions (Week 2 p23) to show that a set of n linearly independent vectors span $\mathbb{R}^n$.

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Study tip: now that we're working with different types of mathematical objects (matrices, vectors, equations, numbers), you should be careful which properties apply to which objects: e.g. linear independence applies to a set of vectors, not to a matrix (at least not until Chapter 4). Do not say "$\underbrace{\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix}}_{\text{one object with 9 numbers}}$ is linearly dependent" when you mean "$\underbrace{\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} \right\}}_{\text{three objects each with 3 numbers}}$ are linearly dependent".

Abstract proofs of linear dependence and independence:

To prove that $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent, we need to find **one** choice of non-zero weights $c_1, \ldots, c_p$ such that $c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$. The technique that we saw in Week 2 applies here: express the information in the question as mathematical formulae, then reorganise the equations until we have something of the form $c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$.

A proof that $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent has a different structure. Now we need to show that the **only** choice of weights $c_1, \ldots, c_p$ such that $c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$ is $c_1 = \ldots = c_p = 0$. So we need to start with the equation $c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$, and solve it using the information given in the question.

(See p4 for this difference in a numerical example.)

EXAMPLE: Suppose $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent. Show that $\{\mathbf{u}, \mathbf{u}+\mathbf{v}\}$ is linearly independent.

What we know:

What we want to show:

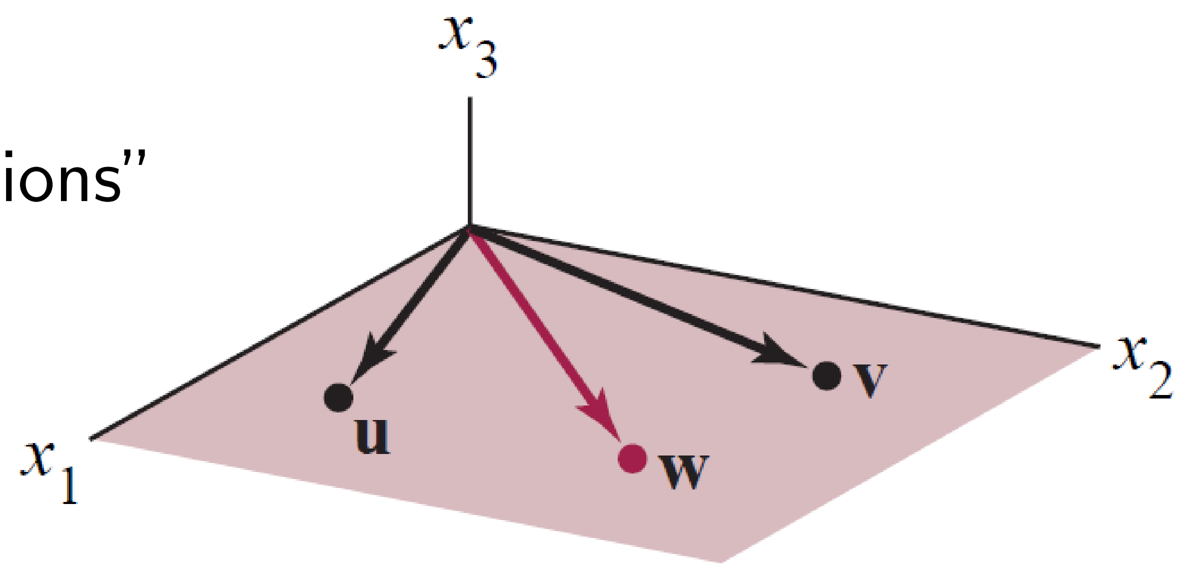
Partial summary of linear dependence:

The definition: $x_1\mathbf{v_1} + \cdots + x_p\mathbf{v_p} = \mathbf{0}$ has a non-trivial solution (not all x_i are zero); equivalently, it has infinitely many solutions.

Equivalently: one of the vectors is a linear combination of the others (see p8, also Theorem 7 in textbook). But it might not be the case that every vector in the set is a linear combination of the others (see ex. sheet #5 q2c).

Computation: rref $\left(\begin{bmatrix} | & & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_p \\ | & & | \end{bmatrix}\right)$ has at least one free variable.

Informal idea: the vectors are in "similar directions"



Partial summary of linear dependence (continued):

Easy examples:

- Sets containing the zero vector;
- Sets containing "too many" vectors (more than n vectors in $\mathbb{R}^n$);
- Multiples of vectors: e.g. $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$ (this is the only possibility if the set has two vectors);
- Other examples: e.g. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Make your own examples!

Adding vectors to a linearly dependent set still makes a linearly dependent set (see ex. sheet #5 Q2d).
Equivalent: removing vectors from a linearly independent set still makes a linearly independent set (because P implies Q is equivalent to (not Q) implies (not P) - this is the contrapositive).

Study tips:

- Linear independence will appear again in many topics throughout the class, so I suggest you add to this summary throughout the semester, so you can see the connections between linear independence and the other topics.

- Topic summaries like this one is useful for exam revision, but even more useful is making these summaries yourself. I encourage you to use my summary as a template for your own summaries of the other topics.

- Examples can be useful for solving true/false questions: if a true/false question is about a linear dependent set, try it on the examples on the previous page. Try to make a counterexample, and if you can't, it will give you some idea of why the statement is true.