§2.1: Matrix Operations

We have several ways to combine functions to make new functions:

- Addition: if f, g have the same domains and codomains, then we can set $(f+g)\mathbf{x} = f(\mathbf{x}) + g(\mathbf{x})$,
- Composition: if the codomain of f is the domain of g, then we can set $(g \circ f)\mathbf{x} = g(f(\mathbf{x}))$,
- Inverse (§2.2): if f is one-to-one and onto, then we can set $f^{-1}(\mathbf{y})$ to be the unique solution to $f(\mathbf{x}) = \mathbf{y}$.

It turns out that the sum, composition and inverse of linear transformations are also linear (exercise: prove it!), and we can ask how the standard matrix of the new function is related to the standard matrices of the old functions.

Notation:

The (i, j)-entry of a matrix A is the entry in row i, column j, and is written a_{ij} or $(A)_{ij}$.

e.g. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$

The diagonal entries of A are the entries $a_{11}, a_{22}, \ldots$.

A square matrix has the same number of rows as columns. The associated linear transformation has the same domain and codomain.

A diagonal matrix is a square matrix whose nondiagonal entries are 0.

e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The identity matrix I_n is the $n \times n$ matrix whose diagonal entries are 1 and whose nondiagonal entries are 0. It is the standard matrix for the identity transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{x}) = \mathbf{x}$.

e.g. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Addition:

If A, B are the standard matrices for some linear transformations $S, T : \mathbb{R}^n \to \mathbb{R}^m$, then what is $A + B$, the standard matrix of $S + T$?

Proceed column by column:
First column of the standard matrix of $S + T$
$= (S + T)(\mathbf{e_1})$
$= S(\mathbf{e_1}) + T(\mathbf{e_1})$
$=$ first column of A $+$ first column of B.
i.e. $(i, 1)$-entry of $A + B$=$a_{i1} + b_{i1}$.

The same is true of all the other columns, so $(A + B)_{ij}$=$a_{ij} + b_{ij}$.

Example: $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$, $\quad B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$, $\quad A + B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$.

Scalar multiplication:

If A is the standard matrix for a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^m$, and c is a scalar, then $(cS)\mathbf{x} = c(S\mathbf{x})$ is a linear transformation. What is its standard matrix cA?

Proceed column by column:
First column of the standard matrix of cS
$= (cS)(\mathbf{e_1})$
$= c(S\mathbf{e_1})$
$=$ first column of A multiplied by c.
i.e. $(i, 1)$-entry of cA=ca_{i1}.

The same is true of all the other columns, so $(cA)_{ij} = ca_{ij}$.

Example: $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, \quad c = -3, \quad cA = \begin{bmatrix} -12 & 0 & -15 \\ 3 & -9 & -6 \end{bmatrix}.$

Addition and scalar multiplication satisfy some familiar rules of arithmetic:

Let A, B, and C be matrices of the same size, and let r and s be scalars. Then

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

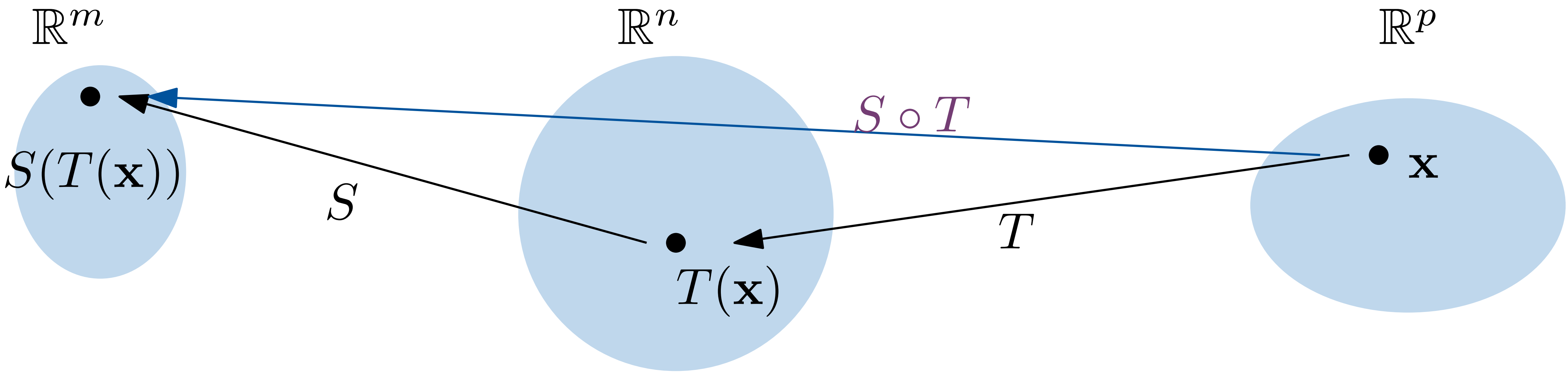
f. $r(sA) = (rs)A$

0 denotes the zero matrix:

$$0 = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

Composition:

If A is the standard matrix for a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^m$
and B is the standard matrix for a linear transformation $T : \mathbb{R}^p \to \mathbb{R}^n$
then the composition $S \circ T$ (T first, then S) is linear.
What is its standard matrix AB?



A is a $m \times n$ matrix,
B is a $n \times p$ matrix,
AB is a $m \times p$ matrix - so the (i, j)-entry of AB cannot simply be $a_{ij}b_{ij}$.

Composition:

Proceed column by column:

First column of the standard matrix of $S \circ T$

$= (S \circ T)(\mathbf{e_1})$

$= S(T(\mathbf{e_1}))$

$= S(\mathbf{b_1})$ (writing $\mathbf{b}_j$ for column j of B)

$= A\mathbf{b_1}$, and similarly for the other columns.

So $$AB = A \begin{bmatrix} | & & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_p \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ A\mathbf{b}_1 & \dots & A\mathbf{b}_p \\ | & & | \end{bmatrix}.$$

The jth column of AB is a linear combination of the columns of A using weights from the jth column of B.

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The jth column of AB is a linear combination of the columns of A using weights from the jth column of B.

Another view is the row-column method: the (i, j)-entry of AB is $a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.

Some familiar rules of arithmetic hold for matrix multiplication...

Let A be $m \times n$ and let B and C have sizes for which the indicated sums and products are defined (different sizes for each statement).

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B+C) = AB + AC$ (left - distributive law)

c. $(B+C)A = BA + CA$ (right-distributive law)

d. $r(AB) = (rA)B = A(rB)$

for any scalar r

e. $I_m A = A = AI_n$ (identity for matrix multiplication)

... but not all of them:

- Usually, $AB \neq BA$ (because order matters for function composition: $S \circ T \neq T \circ S$);
- It is possible for $AB = 0$ even if $A \neq 0$ and $B \neq 0$.

A fun application of matrix multiplication:

Consider rotations counterclockwise about the origin.
Rotation through $(\theta + \varphi)$ = (rotation through θ) $\circ$ (rotation through φ).

$$\begin{bmatrix}\cos(\theta+\varphi) & -\sin(\theta+\varphi)\\ \sin(\theta+\varphi) & \cos(\theta+\varphi)\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}$$
$$= \begin{bmatrix}\cos\theta\cos\varphi - \sin\theta\sin\varphi & -\cos\theta\sin\varphi - \sin\theta\cos\varphi\\ \sin\theta\cos\varphi + \cos\theta\sin\varphi & -\sin\theta\sin\varphi + \cos\theta\cos\varphi\end{bmatrix}.$$

So, equating the entries in the first column:
$\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$
$\sin(\theta+\varphi) = \cos\theta\sin\varphi + \sin\theta\cos\varphi$

Powers:

For a square matrix A, the kth power of A is $A^k = \underbrace{A \dots A}_{k \text{ times}}$.

If A is the standard matrix for a linear transformation T, then A^k is the standard matrix for T^k, the function that "applies T k times".

Examples:

$$\begin{bmatrix}1 & 2\\3 & 0\end{bmatrix}^3 = \begin{bmatrix}1 & 2\\3 & 0\end{bmatrix}\left(\begin{bmatrix}1 & 2\\3 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 0\end{bmatrix}\right) = \begin{bmatrix}1 & 2\\3 & 0\end{bmatrix}\begin{bmatrix}7 & 2\\3 & 6\end{bmatrix} = \begin{bmatrix}13 & 14\\21 & 6\end{bmatrix}.$$

$$\begin{bmatrix}3 & 0\\0 & -2\end{bmatrix}^3 = \begin{bmatrix}3 & 0\\0 & -2\end{bmatrix}\left(\begin{bmatrix}3 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}3 & 0\\0 & -2\end{bmatrix}\right) = \begin{bmatrix}3 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}9 & 0\\0 & 4\end{bmatrix} = \begin{bmatrix}27 & 0\\0 & -8\end{bmatrix} = \begin{bmatrix}3^3 & 0\\0 & (-2)^3\end{bmatrix}.$$

Exercise: show that $\begin{bmatrix}a_{11} & 0\\0 & a_{22}\end{bmatrix}^k = \begin{bmatrix}a_{11}^k & 0\\0 & a_{22}^k\end{bmatrix}$, and similarly for larger diagonal matrices.

We can consider polynomials involving square matrices:

Example: Let $p(x) = x^3 - 2x^2 + x - 2$, and $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ as on the previous page. Then

use the identity matrix instead of constants

$$p(A) = A^3 - 2A^2 + A - 2I_2 = \begin{bmatrix} 13 & 14 \\ 21 & 6 \end{bmatrix} - \begin{bmatrix} 14 & 4 \\ 6 & 12 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 12 \\ 18 & -8 \end{bmatrix}.$$

$$p(D) = D^3 - 2D^2 + D - 2I_2 = \begin{bmatrix} 27 & 0 \\ 0 & -8 \end{bmatrix} - \begin{bmatrix} 18 & 0 \\ 0 & 8 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & -20 \end{bmatrix} = \begin{bmatrix} p(3) & 0 \\ 0 & p(2) \end{bmatrix}.$$

For a polynomial involving a single matrix, we can factorise and expand as usual:

Example: $x^3 - 2x^2 + x - 2 = (x^2 + 1)(x - 2)$, and

$$(A^2 + I_2)(A - 2I_2) = \begin{bmatrix} 8 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 12 \\ 18 & -8 \end{bmatrix}.$$

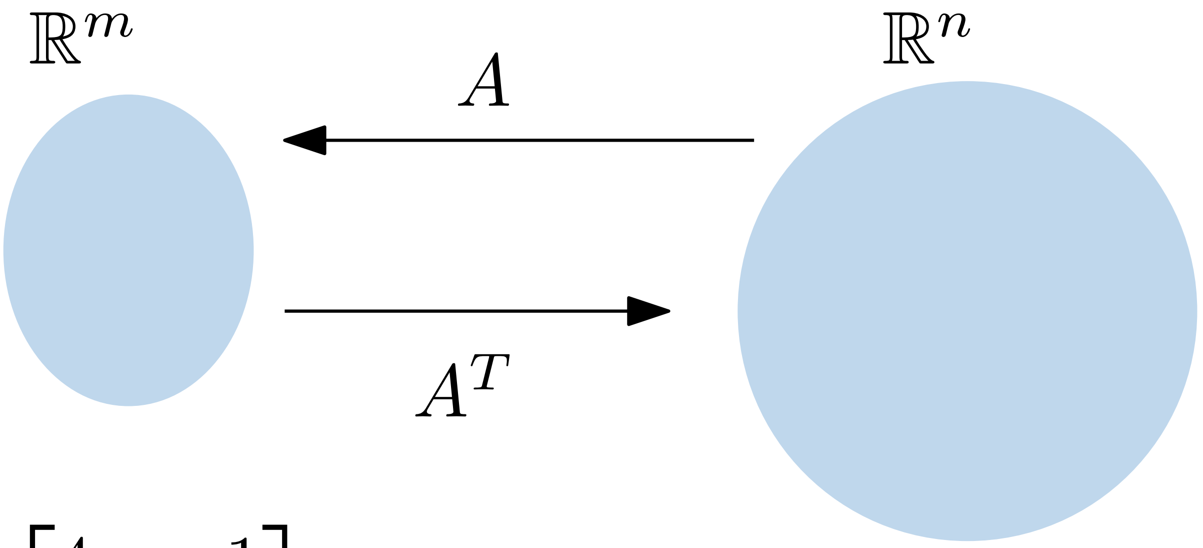
But be careful with the order when there are two or more matrices:

Example: $x^2 - y^2 = (x + y)(x - y)$, but

$$(A + D)(A - D) = A^2 - AD + DA - D^2 \neq A^2 + D^2.$$

Transpose:

The transpose of A is the matrix A^T whose (i,j)-entry is a_{ji}.
i.e. we obtain A^T by "flipping A through the main diagonal".



Example: $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, \quad A^T = \begin{bmatrix} 4 & -1 \\ 0 & 3 \\ 5 & 2 \end{bmatrix}.$

We will be interested in square matrices A such that
$A = A^T$ (symmetric matrix, self-adjoint linear transformation, §7.1), or
$A = -A^T$ (skew-symmetric matrix), or
$A^{-1} = A^T$ (orthogonal matrix, or isometric linear transformation, §6.2).

Properties of the transpose:

Let A and B denote matrices whose sizes are appropriate for the following sums and products (different sizes for each statement).

a. $(A^T)^T =$

b. $(A+B)^T =$

c. For any scalar r, $(rA)^T =$

d. $(AB)^T =$

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Let A and B denote matrices whose sizes are appropriate for the following sums and products (different sizes for each statement).

a. $(A^T)^T = A$ (I.e., the transpose of A^T is A)

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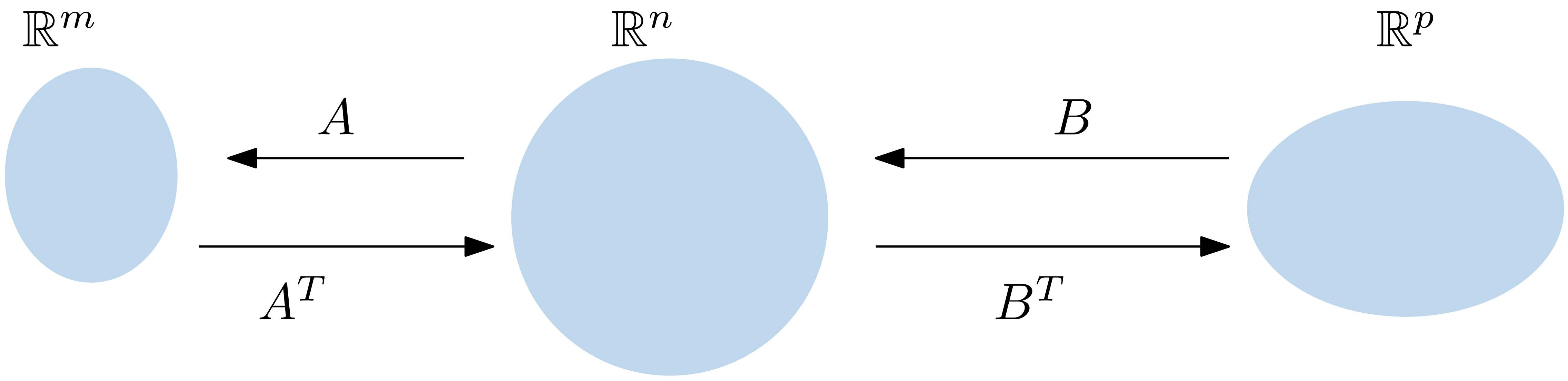

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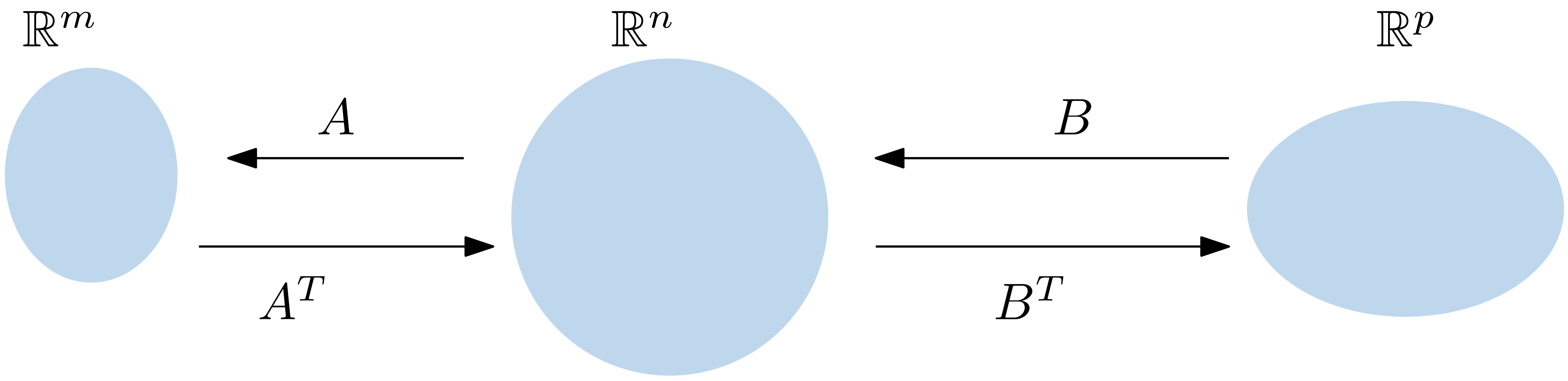

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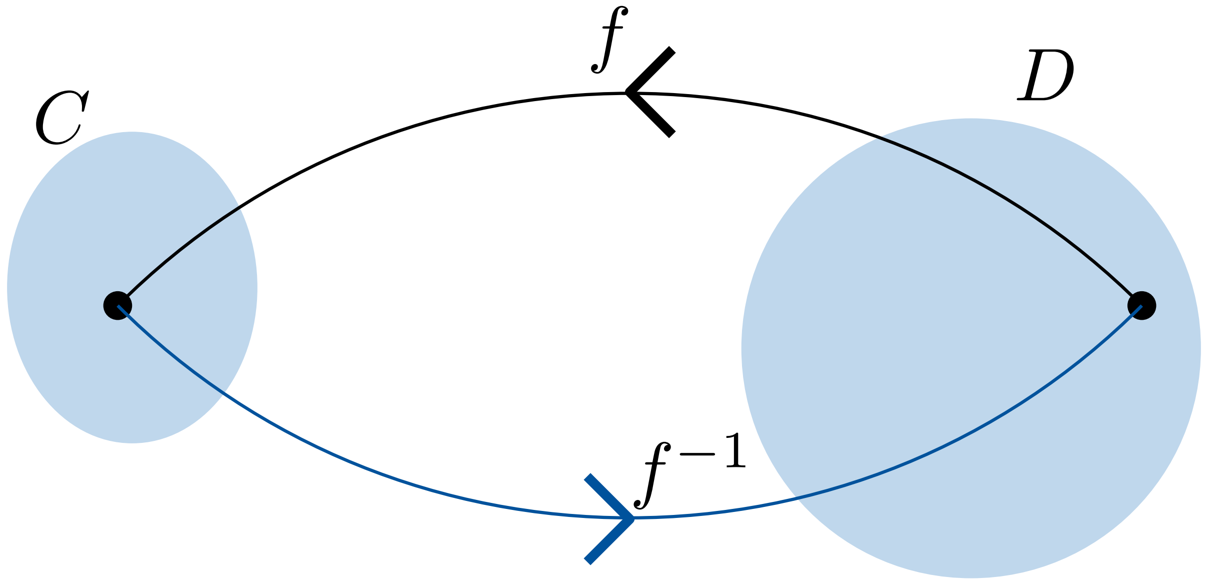
Proof: (i,j)-entry of $(AB)^T$

$$\begin{aligned} &= (j,i)\text{-entry of } AB \\ &= a_{j1}b_{1i} + a_{j2}b_{2i} \cdots + a_{jn}b_{ni} \\ &= (A^T)_{1j}(B^T)_{i1} + (A^T)_{2j}(B^T)_{i2} \cdots + (A^T)_{nj}(B^T)_{in} \\ &= (i,j)\text{-entry of } B^T A^T. \end{aligned}$$

§2.2: The Inverse of a Matrix

Remember from calculus that the inverse of a function $f : D \to C$ is the function $f^{-1} : C \to D$ such that $f^{-1} \circ f =$ identity map on D and $f \circ f^{-1} =$ identity map on C.

Equivalently, $f^{-1}(y)$ is the unique solution to $f(x) = y$.
So f^{-1} exists if and only if f is one-to-one and onto. Then we say f is invertible.



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Let T be a linear transformation whose standard matrix is A. From last week:

- T is one-to-one if and only if rref(A) has a pivot in every
- T is onto if and only if rref(A) has a pivot in every

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Let T be a linear transformation whose standard matrix is A. From last week:

- T is one-to-one if and only if rref(A) has a pivot in every column.
- T is onto if and only if rref(A) has a pivot in every row.

So if T is invertible, then A must be a square matrix.

Warning: not all square matrices come from invertible linear transformations, e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

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Definition: A $n \times n$ matrix A is *invertible* if there is a $n \times n$ matrix C satisfying $CA = AC = I_n$.

Fact: A matrix C with this property is unique:
if $BA = AC = I_n$, then $BAC = BI_n = B$ and $BAC = I_nC = C$ so $B = C$.

The matrix C is called the inverse of A, and is written A^{-1}. So

$$A^{-1}A = AA^{-1} = I_n.$$

A matrix that is not invertible is sometimes called singular.

Remember from calculus that the inverse of a function $f : D \to C$ is the function $f^{-1} : C \to D$ such that $f^{-1} \circ f =$ identity map on D and $f \circ f^{-1} =$ identity map on C.

Equivalently, $f^{-1}(y)$ is the unique solution to $f(x) = y$.

Theorem 5: Solving linear systems with the inverse: If A is an invertible $n \times n$ matrix, then, for each $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof: For any $\mathbf{b}$ in $\mathbb{R}^n$, we have $A(A^{-1}\mathbf{b}) = \mathbf{b}$, so $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution.

And, if $\mathbf{u}$ is any solution, then $\mathbf{u} = A^{-1}(A\mathbf{u}) = A^{-1}\mathbf{b}$, so $A^{-1}\mathbf{b}$ is the unique solution.

In particular, if A is an invertible $n \times n$ matrix, then rref$(A) =$?

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In particular, if A is an invertible $n \times n$ matrix, then $\text{rref}(A) = I_n$.

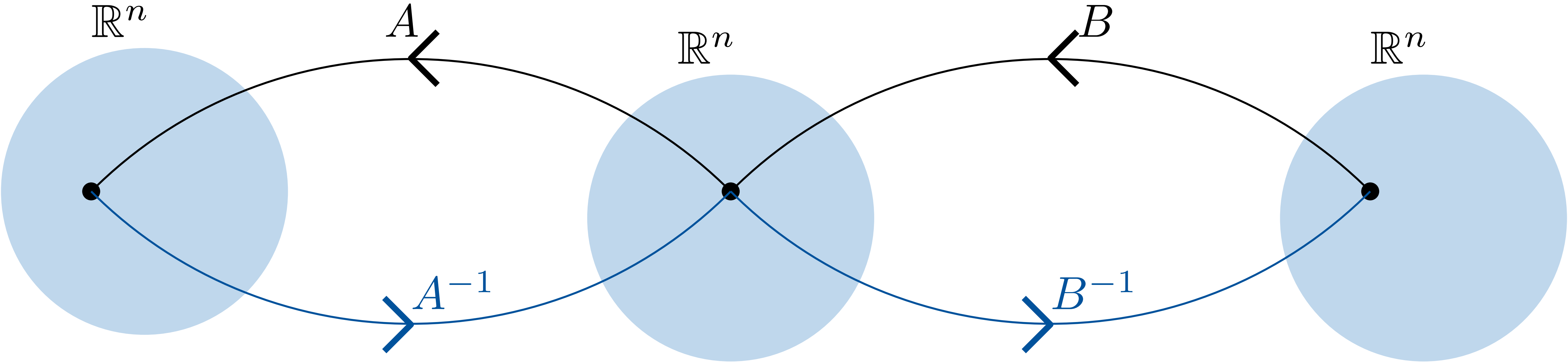
Properties of the inverse:

Suppose A and B are invertible. Then the following results hold:

a. A^{-1} is invertible and $(A^{-1})^{-1} = A$ (i.e. A is the inverse of A^{-1}).

b. AB is invertible and $(AB)^{-1} = ?$

c. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$



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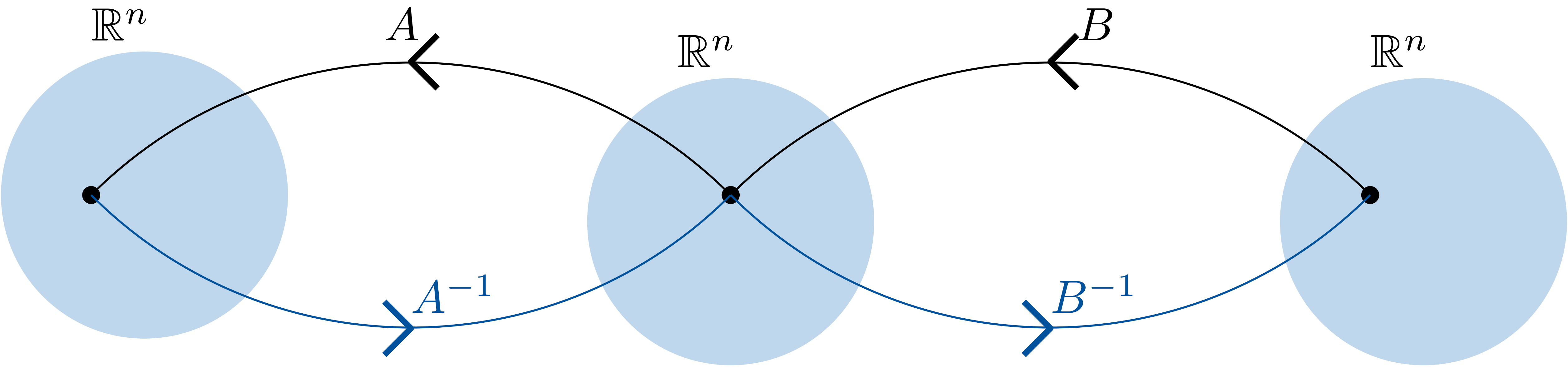

Inverse of a 2×2 matrix:

Fact: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

i) if $ad - bc \neq 0$, then A is invertible and $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$,

ii) if $ad - bc = 0$, then A is not invertible.

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Proof of i):

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & -ab+ba \\ cd-dc & -cb+da \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} da-bc & db-bd \\ -ca+ac & -cb+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Proof of ii): next week.

Inverse of a 2×2 matrix:

Example: Let $A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$, the standard matrix of rotation about the origin through an angle φ counterclockwise.

Example: Let $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, the standard matrix of projection to the x_1-axis.

Inverse of a 2×2 matrix:

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$\cos\varphi\cos\varphi - (-\sin\varphi)\sin\varphi = \cos^2\varphi + \sin^2\varphi = 1$ so A is invertible, and

$A^{-1} = \frac{1}{1}\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} = \begin{bmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{bmatrix}$, the standard matrix of rotation about the origin through an angle φ clockwise.

Example: Let $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, the standard matrix of projection to the x_1-axis.

$1 \cdot 0 - 0 \cdot 0 = 0$ so B is not invertible.

Exercise: choose a matrix C that is the standard matrix of a reflection, and check that C is invertible and $C^{-1} = C$.

Inverse of a $n \times n$ matrix:

If A is the standard matrix of an invertible linear transformation T, then A^{-1} is the standard matrix of T^{-1}. So

$$A^{-1} = \begin{bmatrix} | & & | \\ T^{-1}(\mathbf{e}_1) & \dots & T^{-1}(\mathbf{e}_n) \\ | & & | \end{bmatrix}.$$

$T^{-1}(\mathbf{e}_i)$ is the unique solution to the equation $T(\mathbf{x}) = \mathbf{e_i}$, or equivalently $A\mathbf{x} = \mathbf{e_i}$. So if we row-reduce the augmented matrix $[A|\mathbf{e_i}]$, we should get $[I_n|T^{-1}(\mathbf{e}_i)]$. (Remember rref$(A) = I_n$.)

We carry out this row-reduction for all $\mathbf{e_i}$ at the same time:

$$[A|I_n] = \left[\begin{array}{c|ccc} & | & & | \\ A & \mathbf{e}_1 & \dots & \mathbf{e}_n \\ & | & & | \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{c|ccc} & | & & | \\ I_n & T^{-1}(\mathbf{e}_1) & \dots & T^{-1}(\mathbf{e}_n) \\ & | & & | \end{array}\right] = [I_n|A^{-1}].$$

We showed that, if A is invertible, then $[A|I_n]$ row-reduces to $[I_n|A^{-1}]$.
In other words, if we already knew that A was invertible, then we can find its inverse by row-reducing $[A|I_n]$.
It would be useful if we could apply this without first knowing that A is invertible.

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Indeed, we can:
Fact: If $[A|I_n]$ row-reduces to $[I_n|C]$, then A is invertible and $C = A^{-1}$.

Proof: (different from textbook, not too important)
If $[A|I_n]$ row-reduces to $[I_n|C]$, then $\mathbf{c}_i$ is the unique solution to $A\mathbf{x} = \mathbf{e}_i$, so $AC\mathbf{e}_i = A\mathbf{c}_i = \mathbf{e}_i$ for all i, so $AC = I_n$.
Also, by switching the left and right sides, and reading the process backwards, $[C|I_n]$ row-reduces to $[I_n|A]$. So $\mathbf{a}_i$ is the unique solution to $C\mathbf{x} = \mathbf{e}_i$, so $CA\mathbf{e}_i = C\mathbf{a}_i = \mathbf{e}_i$ for all i, so $CA = I_n$.

We showed that, if A is invertible, then $[A|I_n]$ row-reduces to $[I_n|A^{-1}]$.
In other words, if we already knew that A was invertible, then we can find its inverse by row-reducing $[A|I_n]$.
It would be useful if we could apply this without first knowing that A is invertible.

Indeed, we can:
Fact: If $[A|I_n]$ row-reduces to $[I_n|C]$, then A is invertible and $C = A^{-1}$.

Proof: (different from textbook, not too important)
If $[A|I_n]$ row-reduces to $[I_n|C]$, then $\mathbf{c}_i$ is the unique solution to $A\mathbf{x} = \mathbf{e}_i$, so $AC\mathbf{e}_i = A\mathbf{c}_i = \mathbf{e}_i$ for all i, so $AC = I_n$.
Also, by switching the left and right sides, and reading the process backwards, $[C|I_n]$ row-reduces to $[I_n|A]$. So $\mathbf{a}_i$ is the unique solution to $C\mathbf{x} = \mathbf{e}_i$, so $CA\mathbf{e}_i = C\mathbf{a}_i = \mathbf{e}_i$ for all i, so $CA = I_n$.

In particular: an $n \times n$ matrix A is invertible if and only if $\text{rref}(A) = I_n$.
Also equivalent: $\text{rref}(A)$ has a pivot in every row and column.
For a square matrix, having a pivot in each row is the same as having a pivot in each column.

§2.3: Characterisations of Invertible Matrices

For a square $n \times n$ matrix A, the following are equivalent:

- A is invertible.
- rref(A) $= I_n$.
- rref(A) has a pivot in every row.
- rref(A) has a pivot in every column.

Theorem 8 (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The the following statements are equivalent (i.e., for a given A, they are either all true or all false).

a. A is an invertible matrix.

b. A is row equivalent to I_n.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \to A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbf{R}^n$.

h. The columns of A span $\mathbf{R}^n$.

i. The linear transformation $\mathbf{x} \to A\mathbf{x}$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I_n$.

k. There is an $n \times n$ matrix D such that $AD = I_n$.

l. A^T is an invertible matrix.

follows from ex. 1a from Monday

ex. 1b from Monday

Important consequences:

- A set of n vectors in $\mathbb{R}^n$ span $\mathbb{R}^n$ if and only if the set is linearly independent (h $\Leftrightarrow$ e).
- If A is a $n \times n$ matrix and $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$, then $A\mathbf{x} = \mathbf{c}$ has a unique solution for all $\mathbf{c}$ in $\mathbb{R}^n$ ($\sim$ d $\Longrightarrow \sim$ g).
- If A is a $n \times n$ matrix and $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution, then there is a $\mathbf{b}$ in $\mathbb{R}^n$ for which $A\mathbf{x} = \mathbf{b}$ has no solution (not d $\implies$ not g).
- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one if and only if it is onto (f $\Leftrightarrow$ i).

Important consequences:

- A set of n vectors in $\mathbb{R}^n$ span $\mathbb{R}^n$ if and only if the set is linearly independent (h $\Leftrightarrow$ e).
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- If A is a $n \times n$ matrix and $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution, then there is a $\mathbf{b}$ in $\mathbb{R}^n$ for which $A\mathbf{x} = \mathbf{b}$ has no solution (not d $\Longrightarrow$ not g).
- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one if and only if it is onto (f $\Leftrightarrow$ i).

Other applications:

Example: Is the matrix $\begin{bmatrix} 1 & -1 & 4 \\ 2 & 0 & 8 \\ 3 & 8 & 12 \end{bmatrix}$ invertible?

Important consequences:

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Other applications:

Example: Is the matrix $\begin{bmatrix} 1 & -1 & 4 \\ 2 & 0 & 8 \\ 3 & 8 & 12 \end{bmatrix}$ invertible?

Answer: No: the third column is 4 times the first column, so the columns are not linearly independent. So the matrix is not invertible (not e $\Longrightarrow$ not a).

a. A is invertible $\Leftrightarrow$ l. A^T is invertible. (Proof: Check that $(A^T)^{-1} = (A^{-1})^T$.)

This means that the statements in the Invertible Matrix Theorem are equivalent to the corresponding statements with "row" instead of "column", for example:

- The columns of an $n \times n$ matrix are linearly independent if and only if its rows span $\mathbb{R}^n$ (e $\Leftrightarrow$ h^T). (This is in fact also true for rectangular matrices.)
- If A is a square matrix and $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$, then the rows of A are linearly independent ($\sim$d $\implies$ e^T).

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Advanced application (important for probability):

Let A be a square matrix. If the entries in each column of A sum to 1, then there is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{v}$.

Hint: $(A - I)^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \mathbf{0}$.