Example of a strange vector space over $\mathbb{R}$:

$\mathbb{R}^2$ with strange operations:

$$\begin{pmatrix} x \\ y \end{pmatrix} \boxplus \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + x' + 1 \\ y + y' \end{pmatrix}$$

$$c \boxdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx + c - 1 \\ cy \end{pmatrix}$$

Check some axioms

$(\alpha + \beta) + \gamma$

V2 LHS: $\left[\begin{pmatrix} x \\ y \end{pmatrix} \boxplus \begin{pmatrix} x' \\ y' \end{pmatrix} \right] \boxplus \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

$$= \begin{pmatrix} x + x' + 1 \\ y + y' \end{pmatrix} \boxplus \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$= \begin{pmatrix} (x + x' + 1) + x_1 + 1 \\ (y + y') + y_1 \end{pmatrix}$$

$$= \begin{pmatrix} x + x' + x_1 + 2 \\ y + y' + y_1 \end{pmatrix}$$

RHS: $\alpha + (\beta + \gamma)$

$$\begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \left[\begin{pmatrix} x' \\ y' \end{pmatrix} \boxed{+} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \begin{pmatrix} x' + x_1 + 1 \\ y' + y_1 \end{pmatrix}$$

$$= \begin{pmatrix} x + (x' + x_1 + 1) + 1 \\ y + (y' + y_1) \end{pmatrix} = \text{LHS} \checkmark$$

V4 $\vec{0}$ for this vector space is NOT $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$:

$$\begin{pmatrix} x \\ y \end{pmatrix} \boxplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x+0+1 \\ y+0 \end{pmatrix} \neq \begin{pmatrix} x \\ y \end{pmatrix}.$$

To find $\vec{0}$: solve $\begin{pmatrix} x \\ y \end{pmatrix} \boxplus \begin{pmatrix} ?_1 \\ ?_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \forall x, y.$

$$\begin{pmatrix} x + ?_1 + 1 \\ y + ?_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \forall x, y.$$

$$\Rightarrow \vec{0} = \begin{pmatrix} ?_1 \\ ?_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \checkmark$$

V5: $\forall \alpha$, $\exists (-\alpha)$ such that $\alpha + (-\alpha) = \vec{0}$.

Taking $-\alpha$ to be $(-1)\alpha$ always works.

$$\begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \left(-1 \boxed{\cdot} \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \begin{pmatrix} (-1)x + (-1) - 1 \\ -1y \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \begin{pmatrix} -x-2 \\ -y \end{pmatrix}$$

$$= \begin{pmatrix} x + (-x-2) + 1 \\ y + (-y) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \vec{0} \quad \checkmark$$

Exercise: check V7–V10.

Non-example: strange operations that don't make a vector space:

$$\begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \max\{x, x'\} \\ \max\{y, y'\} \end{pmatrix}$$

e.g. $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \boxed{+} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

$$c \boxed{\cdot} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ cy \end{pmatrix}$$

V2, V3 are true (check yourself)

V4 does not hold:

$\vec{0} = \begin{pmatrix} a \\ b \end{pmatrix}$ must satisfy

$$\begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \forall x, y \in \mathbb{R}.$$

i.e. $\begin{pmatrix} \max\{x, a\} \\ \max\{y, b\} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

i.e. $a \leq x \quad \forall x \in \mathbb{R}$ — no a, b
$b \leq y \quad \forall y \in \mathbb{R}$ satisfies this

$\therefore$ this set does not contain a $\vec{0}$.

Lemma 6.1.2 : Cancellation law

small theorem if $\alpha + \beta = \alpha + \gamma$, then $\beta = \gamma$.

Proof:

$$\alpha + \beta = \alpha + \gamma$$

Add $-\alpha$: $-\alpha + (\alpha + \beta) = -\alpha + (\alpha + \gamma)$

V2: $(-\alpha + \alpha) + \beta = (-\alpha + \alpha) + \gamma$

V3: $(\alpha+(-\alpha))+\beta=(\alpha+(-\alpha))+\gamma$

V5: $\vec{0}+\beta=\vec{0}+\gamma$

V4: $\beta=\gamma$

Corollary 6.1.3: if $\alpha+\beta=\alpha$, then $\beta=\vec{0}$.

follows from other results

(A way to prove something is the zero vector)

Proof: we know $\alpha+\beta=\alpha \underset{V4}{=} \alpha+\vec{0}$

Now use cancellation law with $\gamma=\vec{0}$.

Lemma 6.1.4 a,c multiply-by-zero law:

a. $\forall \alpha\in V$, $0\alpha=\vec{0}$

c. $\forall a\in \mathbb{F}$, $a\vec{0}=\vec{0}$

Proof: by 6.1.3, it's enough to show

$\gamma + 0\alpha = \gamma$ and $\gamma + a\vec{0} = \gamma$ $\forall \alpha \in V, a \in \mathbb{F},$

and some $\gamma \in V$.

a. $(\gamma = 0\alpha)$ $\quad 0\alpha + 0\alpha = (0+0)\alpha$ V9

$= 0\alpha$

c. $(\gamma = a\vec{0})$ $\quad a\vec{0} + a\vec{0} = a(\vec{0} + \vec{0})$ V8

$= a\vec{0}$ V4