Remember the addition and scalar multiplication of matrices:

$$(A+B)_{ij} = a_{ij} + b_{ij},$$

e.g $\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}.$

$$(cA)_{ij} = ca_{ij},$$

e.g. $(-3)\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -12 & 0 & -15 \\ 3 & -9 & -6 \end{bmatrix}.$

Is this really different from $\mathbb{R}^6$?

$$\begin{bmatrix} 4 \\ 0 \\ 5 \\ -1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 6 \\ 2 \\ 8 \\ 9 \end{bmatrix}. \qquad (-3)\begin{bmatrix} 4 \\ 0 \\ 5 \\ -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ 0 \\ -15 \\ 3 \\ -9 \\ -6 \end{bmatrix}.$$

Remember from calculus the addition and scalar multiplication of polynomials:

e.g $(2t^2 + 1) + (-t^2 + 3t + 2) = t^2 + 3t + 3.$

e.g $(-3)(-t^2 + 3t + 2) = 3t^2 - 9t - 6.$

Is this really different from $\mathbb{R}^3$?

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}. \qquad (-3)\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ -6 \end{bmatrix}.$$

§4.1: Abstract Vector Spaces

As the examples above showed, there are many objects in mathematics that "looks" and "feels" like $\mathbb{R}^n$. We will also call these vectors.

The real power of linear algebra is that everything we learned in Chapters 1-3 can be applied to all these abstract vectors, not just to column vectors.

You should think of abstract vectors as objects which can be added and multiplied by scalars. Their addition and scalar multiplication must obey some "sensible rules" called axioms (see next page).

The axioms guarantee that the proofs of every result and theorem from Chapters 1-3 will work for our new definition of vectors.

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms below. The axioms must hold for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and for all scalars c and d.

1. $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $(cd)\mathbf{u} = c(d\mathbf{u})$.
10. $1\mathbf{u} = \mathbf{u}$.

Examples of vector spaces:

$M_{2\times 3}$, the set of 2×3 matrices.

4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.

The zero vector for $M_{2\times 3}$ is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

You can check the other 9 axioms by using the properties of matrix addition and scalar multiplication (page 5 of week 5 slides, theorem 2.1 in textbook).

Similarly, $M_{m\times n}$, the set of all $m\times n$ matrices, is a vector space.

Is the set of all matrices (of all sizes) a vector space?

No, because we cannot add two matrices of different sizes, so axiom 1 does not hold.

Examples of vector spaces:

$\mathbb{P}_n$, the set of polynomials of degree at most n.

Each of these polynomials has the form

$$a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n,$$

for some numbers $a_0, a_1, \ldots a_n$.

4. There is a vector (called the zero vector) **0** in V such that **u** + **0** = **u**.

The zero vector for $\mathbb{P}_n$ is $0 + 0t + 0t^2 + \cdots + 0t^n$.

1. **u** + **v** is in V.

$(a_0 + a_1 t + a_2 t^2 + \cdots + a_n x^n) + (b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n)$
$= (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \cdots + (a_n + b_n)t^n$, which also has degree at most n.

Exercise: convince yourself that the other axioms are true.

Examples of vector spaces:

Warning: the set of polynomials of degree exactly n is not a vector space.

$$\underbrace{(t^3+t^2)}_{\text{degree 3}}+\underbrace{(-t^3+t^2)}_{\text{degree 3}}=\underbrace{2x^2}_{\text{degree2}}$$

$\mathbb{P}$, the set of all polynomials (no restriction on the degree) is a vector space.

$C(\mathbb{R})$, the set of all continuous functions is a vector space (because the sum of two continuous functions is continuous, the zero function is continuous, etc.)

These last two examples are a bit different from $M_{m\times n}$ and $\mathbb{P}_n$ because they are infinite-dimensional (more later).

(You do not have to remember the notation $M_{m\times n}, \mathbb{P}_n$, etc. for the vector spaces.)

Let W be the set of upper triangular 2×2 matrices. Is W a vector space?

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

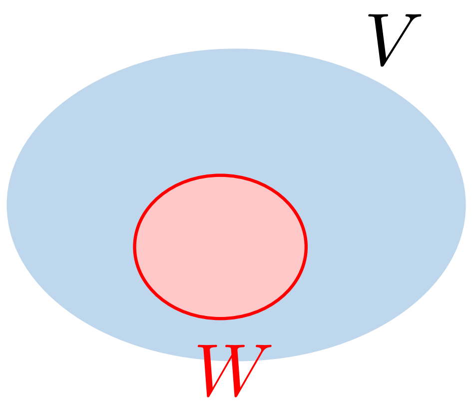
1. $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $(cd)\mathbf{u} = c(d\mathbf{u})$.
10. $1\mathbf{u} = \mathbf{u}$.

W is a subset of $M_{2\times 2}$.
Axioms 2, 3, 5, 7, 8, 9, 10 hold for W because they hold for $M_{2\times 2}$.
So we only need to check axioms 1, 4, 6.

Definition: A subset W of a vector space V is a *subspace* of V if the closure axioms 1,4,6 hold:

4. The zero vector is in W.
1. If $\mathbf{u}, \mathbf{v}$ are in W, then their sum $\mathbf{u} + \mathbf{v}$ is in W. (closed under addition)
6. If $\mathbf{u}$ is in W and c is any scalar, the scalar multiple $c\mathbf{u}$ is in W. (closed under scalar multiplication)

Fact: W is itself a vector space (with the same addition and scalar multiplication as V) if and only if W is a subspace of V.



Definition: A subset W of a vector space V is a *subspace* of V if the closure axioms 1,4,6 hold:

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Example: Let W be the set of vectors of the form $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$, where a, b can take any value.

(W is the x_1x_3-plane.) We show that W is a subspace of $\mathbb{R}^3$:

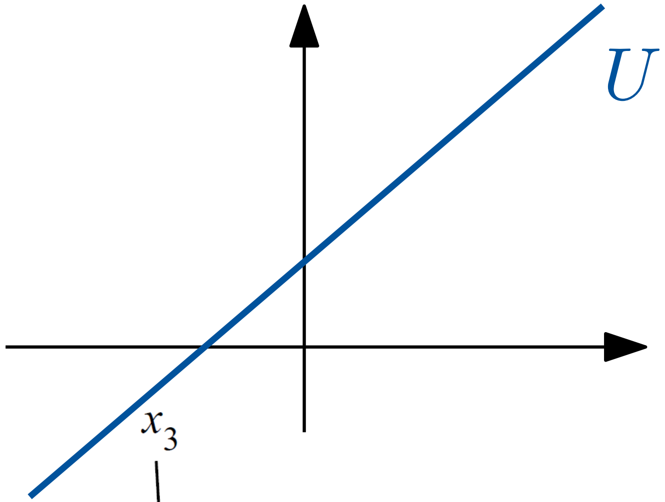
4. The zero vector is in W because it is the vector with $a = 0$, $b = 0$.
1. $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} x \\ 0 \\ y \end{bmatrix} = \begin{bmatrix} a+x \\ 0 \\ b+y \end{bmatrix}$ is in W.
6. $c \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} ca \\ 0 \\ cb \end{bmatrix}$ is in W.

Although W "feels like" $\mathbb{R}^2$, note that $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ - vectors in $\mathbb{R}^2$ have two entries, so they are not in $\mathbb{R}^3$.

Example: Let U be the set of vectors of the form $\begin{bmatrix} x \\ x+1 \end{bmatrix}$, where x can take any value.

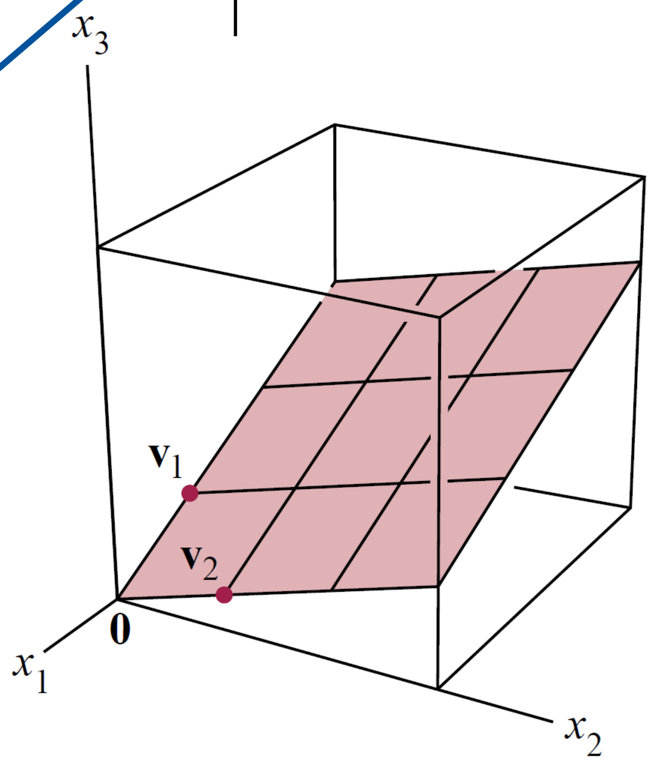
To show that U is not a subspace of $\mathbb{R}^2$, we need to find one counterexample to one of the axioms, e.g.

4. The zero vector is not in U, because there is no value of x with $\begin{bmatrix} x \\ x+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.



An alternative answer:

1. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are in U, but $\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is not of the form $\begin{bmatrix} x \\ x+1 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is not in U. So U is not closed under addition.



Best examples of a subspace: lines and planes containing the origin in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example: Let Q be the set of polynomials $\mathbf{p}(t)$ of degree at most 3 with $\mathbf{p}(2) = 0$. We show that Q is a subspace of $\mathbb{P}_3$:

4. The zero polynomial is in Q because $\mathbf{0}(2) = 0 + 0 \cdot 2 + 0 \cdot 2^2 + 0 \cdot 2^3 = 0$.
1. For $\mathbf{p}, \mathbf{q}$ in Q, we have $(\mathbf{p} + \mathbf{q})(2) = \mathbf{p}(2) + \mathbf{q}(2) = 0 + 0 = 0$, so $\mathbf{p} + \mathbf{q}$ is in Q.
6. For $\mathbf{p}$ in Q and any scalar c, we have $(c\mathbf{p})(2) = c(\mathbf{p}(2)) = c0 = 0$, so $c\mathbf{p}$ is in Q.

Example: In every vector space V, the set $\{\mathbf{0}\}$ containing only the zero vector is a subspace:

4. $\mathbf{0}$ is clearly in the subspace.
1. $\mathbf{0} + \mathbf{0} = \mathbf{0}$ (use axiom 4: $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u}$ in V).
6. $c\mathbf{0} = \mathbf{0}$ (use axiom 7: $c(\mathbf{0} + \mathbf{0}) = c\mathbf{0} + c\mathbf{0}$; and left hand side is $c\mathbf{0}$.)

$\{\mathbf{0}\}$ called the zero subspace.

Example: For every vector space V, the whole space V is a subspace.

A shortcut to show that a set is a subspace:

Theorem 1: Spans are subspaces: If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in a vector space V, then Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Redo Example: (p10) Let W be the set of vectors of the form $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$, where a, b can take any value. W is a subspace of $\mathbb{R}^3$ because $W =$ Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Redo Example: (p8) Let $UT_{2\times 2}$ be the set of upper triangular 2×2 matrices. $UT_{2\times 2}$ is a subspace of $M_{2\times 2}$ because $UT_{2\times 2} =$ Span $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

Warning: Theorem 1 does not help us show that a set is not a subspace.

THEOREM 1: Spans are subspaces

If $\mathbf{v}_1,\ldots,\mathbf{v}_p$ are vectors in a vector space V, then Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is a subspace of V.

Proof: We check axioms 4, 1 and 6 in the definition of a subspace.

4. $\mathbf{0}$ is in Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ since

$$\mathbf{0} = \_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\_\mathbf{v}_p$$

1. To show that Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is closed under addition, we choose two arbitrary vectors in Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$:

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p$$

and

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$\mathbf{u} + \mathbf{v} = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p) + (b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p)$$

$$= \_\_\_\_\ \_\_\_\_\mathbf{v}_1 + \_\_\_\_\ \_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\ \_\_\_\_\mathbf{v}_p$$

So $\mathbf{u} + \mathbf{v}$ is in Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$.

6. To show that Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is closed under scalar multiplication, choose an arbitrary number c and an arbitrary vector in Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$:

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$c\mathbf{v} = c(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p)$$

$$= \_\_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\_\_\mathbf{v}_p$$

So $c\mathbf{v}$ is in Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$.

Since 4,1,6 hold, Span$\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is a subspace of V.

One example of the power of abstract vector spaces - solving differential equations:

Question: What are all the polynomials $\mathbf{p}$ of degree at most 5 that satisfy

$$\frac{d^2}{dt^2}\mathbf{p}(t) - 4\frac{d}{dt}\mathbf{p}(t) + 3\mathbf{p}(t) = 3t - 1?$$

Answer: The differentiation function $D : \mathbb{P}_5 \to \mathbb{P}_5$ given by $D(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$ is a linear transformation (later, p27).

The function $T : \mathbb{P}_5 \to \mathbb{P}_5$ given by $T(\mathbf{p}) = \frac{d^2}{dt^2}\mathbf{p}(t) - 4\frac{d}{dt}\mathbf{p}(t) + 3\mathbf{p}(t)$ is a sum of compositions of linear transformations, so T is also linear.

We can check that the polynomial $t+1$ is a solution.

So, by the Solutions and Homogeneous Equations Theorem, the solution set to the above differential equation is all polynomials of the form $t+1+\mathbf{q}(t)$ where $T(\mathbf{q}) = 0$.

Extra: $\mathbb{P}_5$ is both the domain and codomain of T, so the Invertible Matrix Theorem applies. So, if the above equation has more than one solution, then there is a polynomial $\mathbf{g}$ such that $\frac{d^2}{dx^2}\mathbf{p}(t) - 4\frac{d}{dt}\mathbf{p}(t) + 3\mathbf{p}(t) = \mathbf{g}(t)$ has no solutions.

§4.2: Null Spaces and Column Spaces

Each linear transformation has two vector subspaces associated to it.

For each of these two subspaces, we are interested in two problems:

a. given a vector $\mathbf{v}$, is it in the subspace?

b. can we write this subspace as Span $\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ for some vectors $\mathbf{v}_1, \ldots \mathbf{v}_p$?

b*. can we write this subspace as Span $\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ for **linearly independent** vectors $\mathbf{v}_1, \ldots \mathbf{v}_p$? The set $\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ is then called a basis of the subspace.

Problem b is important because it means every vector in the subspace can be written as $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$. This allows us to prove statements about arbitrary vectors in the subspace.

We can get an answer to problem b* by applying the casting-out algorithm (see week 3 notes) to an answer to problem b, but sometimes there are better methods.

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

THEOREM 2: Null spaces are subspaces

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$.

Proof: Nul A is a subset of $\mathbf{R}^n$ since A has n columns. We check axioms 4,1,6 in the definition of a subspace.

4. $\mathbf{0}$ is in Nul A because

1. If $\mathbf{u}$ and $\mathbf{v}$ are in Nul A, we show that $\mathbf{u} + \mathbf{v}$ is in Nul A. Because $\mathbf{u}$ and $\mathbf{v}$ are in Nul A

Therefore

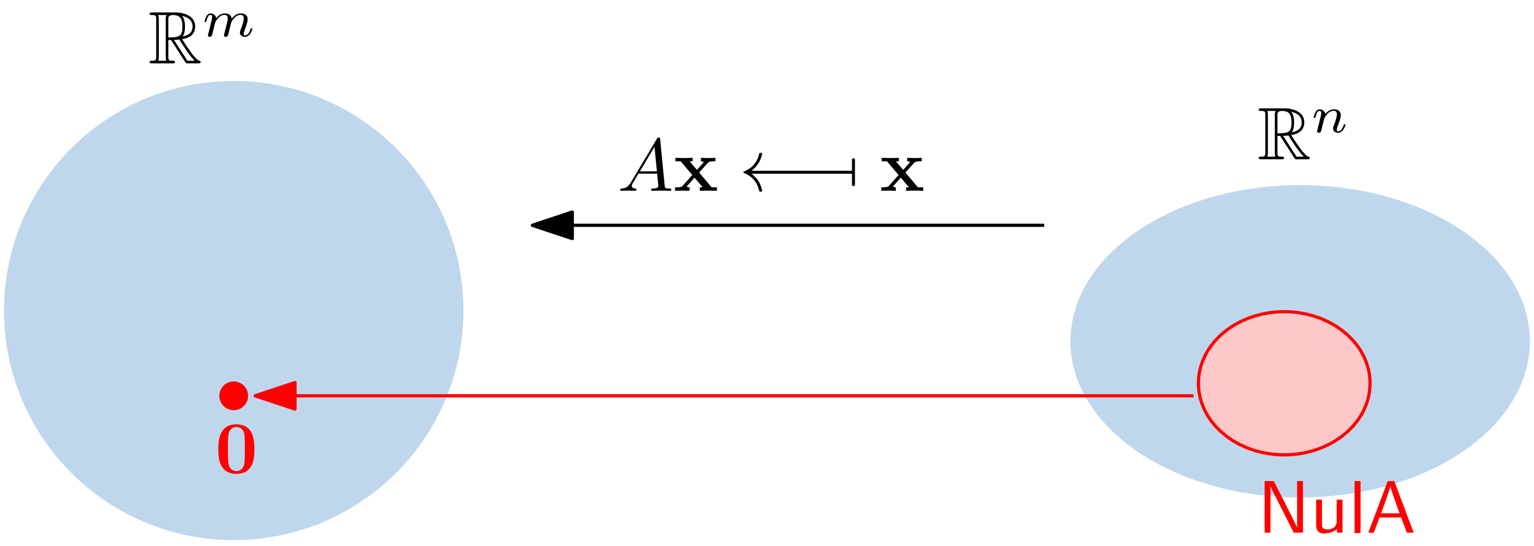
$$A(\mathbf{u} + \mathbf{v}) =$$

6. If $\mathbf{u}$ is in Nul A and c is a scalar, we show that $c\mathbf{u}$ iis in Nul A:

$$A(c\mathbf{u}) =$$

Since axioms 4,1,6 hold, Nul A is a subspace of $\mathbf{R}^n$.

Definition: The null space of a $m \times n$ matrix A, written Nul$A$, is the solution set to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.



Theorem 2 gives us a new way to show that a set is a subspace:

Example: Show that the line $y = x$ is a subspace of $\mathbb{R}^2$.
Answer: $y = x$ is the solution set to $x - y = 0$, which is the null space of $\begin{bmatrix} 1 & -1 \end{bmatrix}$.

Nul$A$ is implicitly defined - problem a is easy, problem b takes more work.

Example: Let $A = \begin{bmatrix} 1 & -3 & 4 & -3 \\ 3 & -7 & 8 & -5 \end{bmatrix}$. a. Is $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in Nul$A$?

b. Find vectors $\mathbf{v}_1, \ldots \mathbf{v}_p$ which span Nul(A).

Answer:

a. $A\mathbf{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq \mathbf{0}$, so $\mathbf{v}$ is not in Nul$A$.

b. $[A|\mathbf{0}] \xrightarrow{\text{row reduction}} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \end{array}\right] \longrightarrow \begin{aligned} x_1 &= 2x_3 - 3x_4 \\ x_2 &= 2x_3 - 2x_4 \\ x_3 &= \quad x_3 \\ x_4 &= \qquad\quad x_4 \end{aligned}$

So the solution set is $s\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ where s, t can take any value.

So Nul$A$ = Span $\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

linearly independent

In general: the solution to $A\mathbf{x} = \mathbf{0}$ in parametric form looks like $s_i\mathbf{w_i} + s_j\mathbf{w_j} + \ldots$, where $x_i, x_j, \ldots$ are the free variables (one vector for each free variable).
The vector $\mathbf{w_i}$ has a 1 in row i and a 0 in row j for every other free variable x_j, so $\{\mathbf{w_i}, \mathbf{w_j}, \ldots\}$ are automatically linearly independent (i.e. we don't need to use the casting-out algorithm).

b. $[A|\mathbf{0}] \xrightarrow{\text{row reduction}} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \end{array}\right] \longrightarrow \begin{aligned} x_1 &= 2x_3 - 3x_4 \\ x_2 &= 2x_3 - 2x_4 \\ x_3 &= \;\; x_3 \\ x_4 &= \qquad\;\; x_4 \end{aligned}$

So the solution set is $s\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ where s, t can take any value. So Nul$A$ = Span $\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

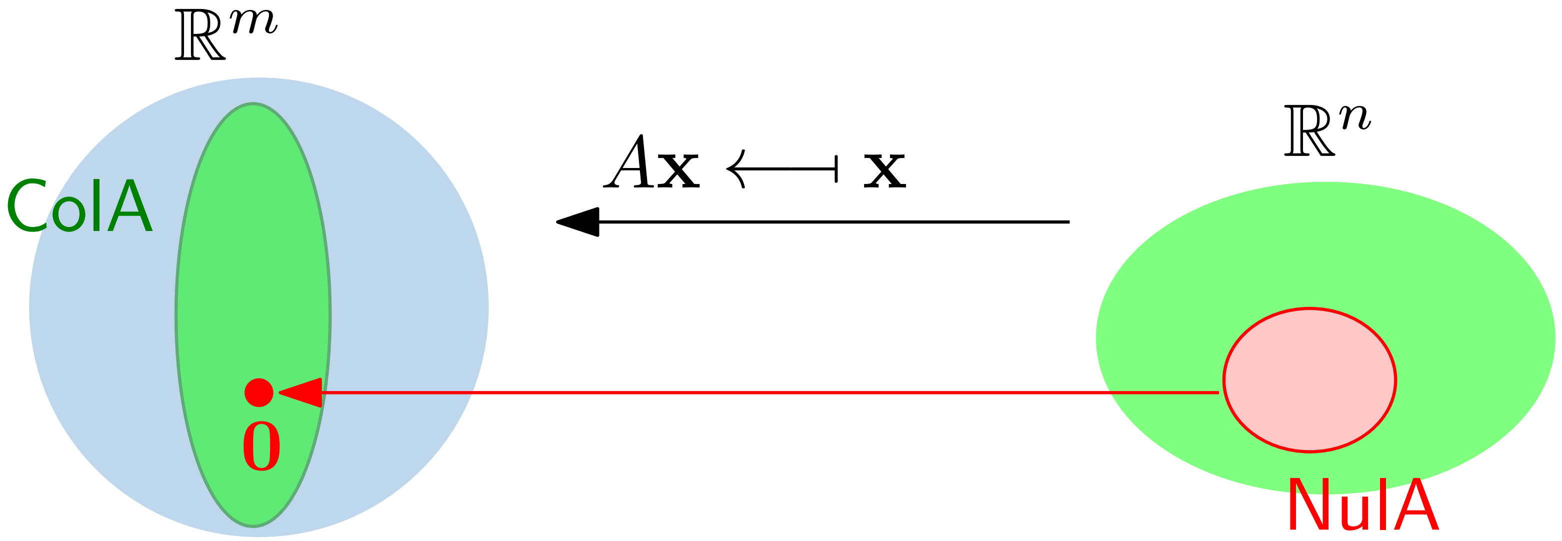
linearly independent

$\mathbf{w_3}$ $\mathbf{w_4}$

Definition: The column space of a $m \times n$ matrix A, written Col$A$, is the span of the columns of A.

Because spans are subspaces, it is obvious that Col(A) is a subspace of $\mathbb{R}^m$.

It follows from §1.3 that Col$A$ is the set of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has solutions.



Col$A$ is explicitly defined - problem a takes work, problem b is easy.

a. To test if $\mathbf{v}$ is in Col$A$, row-reduce $[A|\mathbf{v}]$.

b. An obvious set that spans Col$A$ are the columns of A.

b.* To obtain a linear independent set that spans Col$A$, find rref(A) and take the pivot columns of A (see week 3, casting-out algorithm).

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

p.222 of textbook

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	1. Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	2. Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \;\; \mid \mathbf{0}\,]$ are required.	3. It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them. ← problem b
4. There is no obvious relation between Nul A and the entries in A.	4. There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	5. A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	6. Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \mid \mathbf{v}\,]$ are required. ← problem a
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	7. Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	8. Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Definition: The row space of a $m \times n$ matrix A, written Row$A$, is the span of the rows of A. It is a subspace of $\mathbb{R}^n$.

Example: $A = \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 2 & 0 & 8 \\ 1 & 2 & -3 & 6 \end{bmatrix}$ $\qquad$ rref$(A) = \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\text{Row}A = \text{Span}\left\{(0,1,0,4),(0,2,0,8),(1,2,-3,6)\right\}.$$

Row$A$ is implicitly defined - indeed, it is equivalent to ColA^T.
So, to see if a vector $\mathbf{v}$ is in Row$A$, row-reduce $[A^T|\mathbf{v}^T]$.
To find a linear independent set that spans Row$A$, take the pivot columns of A^T, or...

Theorem 13: Row operations do not change the row space. In particular, the nonzero rows of rref(A) is a linearly independent set whose span is Row$A$.
E.g. for the above example, Row$A$ = Span$\{(1,0,-3,-2),(0,1,0,4)\}$.

Warning: the "pivot rows" of A do not usually span Row$A$:
e.g. here $(1,2,-3,6)$ is in Row$A$ but not in Span$\{(0,1,0,4),(0,2,0,8)\}$.

Theorem 13: Row operations do not change the row space. In particular, the nonzero rows of rref(A) is a linearly independent set whose span is Row$A$.

An example to explain why row operations do not change the row space:

$$A = \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 2 & 0 & 8 \\ 1 & 2 & -3 & 6 \end{bmatrix} \quad \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & -2 \end{bmatrix} \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(1, 4, -3, 14) = R_2 + R_3 = (R_2 - 2R_1) + (R_3 - R_1) + 3R_1$

Similarly, any linear combination of R_1, R_2, R_3 can be written as a linear combination of $R_1, R_2 - 2R_1, R_3 - R_1$.

Proof of the second sentence in Theorem 13:

From the first sentence, Row$(A) =$ Row$($rref$(A)) =$ Span of the nonzero rows of rref(A). Because each nonzero row has a 1 in one pivot column (different column for each row) and 0s in all other pivot columns, these rows are linearly independent.

Summary:

Axioms for a subspace:

4. The zero vector is in W.
1. If $\mathbf{u}, \mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W. (closed under addition)
6. If $\mathbf{u}$ is in W and c is a scalar, then $c\mathbf{u}$ is in W. (closed under scalar multiplication

Ways to show that a set W is a subspace:

- Show that $W = \text{Span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ for some $\mathbf{v}_1, \ldots, \mathbf{v}_p$.
- Show that W is Nul$A$ for some matrix A.
- Show that W is the kernel or range of a linear transformation (later, p27).
- Check the axioms directly.

To show that a set is not a subspace:

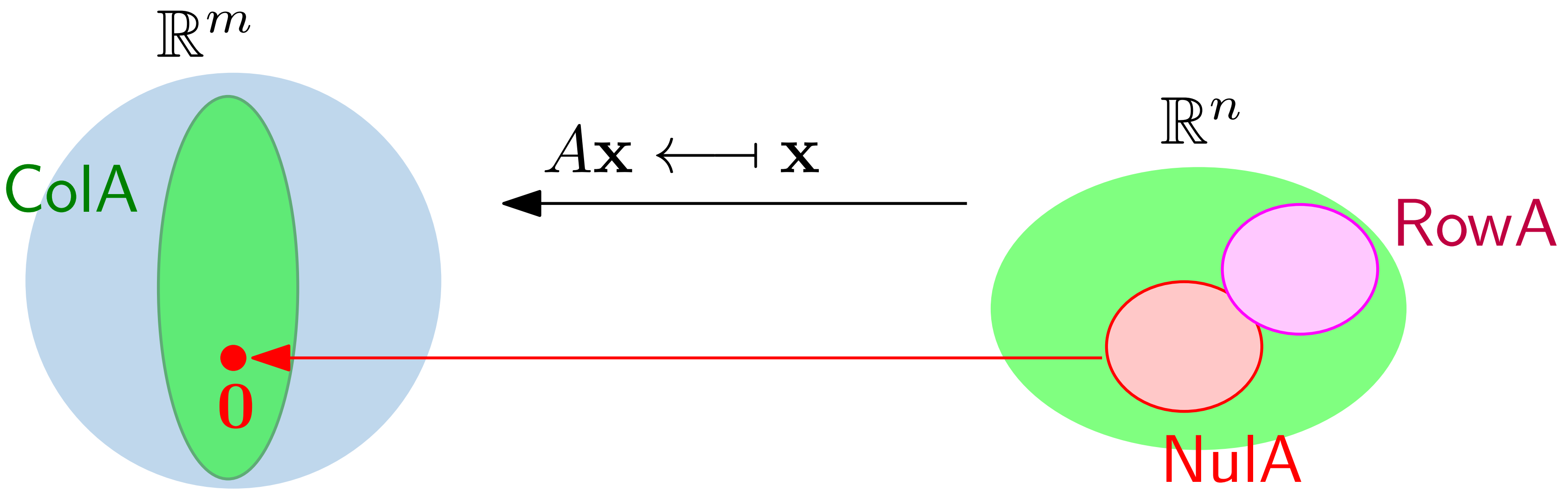
- Show that one of the axioms is false.

Best examples of a subspace: lines and planes containing the origin in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Summary (part 2):

A basis for W is a linearly independent set that spans W (more later).

- Nul$A$=solutions to $A\mathbf{x} = \mathbf{0}$, basis for Nul$A$: solve $A\mathbf{x} = \mathbf{0}$ via the rref.
- Col$A$=span of columns of A, basis for Col$A$: pivot columns of A.
- Row$A$=span of rows of A. basis for Row$A$: nonzero rows of rref(A).



Col$A$ is in $\mathbb{R}^m$. Nul$A$, Row$A$ are in $\mathbb{R}^n$.

In general, Col$A \neq$ Col(rref(A)). Nul$A =$ Nul(rref(A)), Row$A =$ Row(rref(A)).

§4.2 cont'd: Linear Transformations for Vector Spaces

Let V, W be vector spaces.

Definition: A function $T : V \to W$ $\cancel{T : \mathbb{R}^n \to \mathbb{R}^m}$ is a linear transformation if:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

Hard exercise: show that the set of all linear transformations $V \to W$ is a vector space.

Example: The differentiation function $D : \mathbb{P}_n \to \mathbb{P}_{n-1}$ given by $D(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$,

$$D(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) = a_1 + 2a_2 t + \cdots + n a_n t^{n-1}$$

is linear: $D(\mathbf{p} + \mathbf{q}) = D(\mathbf{p}) + D(\mathbf{q})$ and $D(c\mathbf{p}) = cD(\mathbf{p})$.

Our proof that null spaces are subspaces (p17) shows the kernel of a linear transformation is a subspace.
Exercise: show that the range of a linear transformation is a subspace.

Redo Example: (p12) Let Q be the set of polynomials $\mathbf{p}(t)$ of degree at most 3 with $\mathbf{p}(2) = 0$. We show that Q is a subspace of $\mathbb{P}_3$:
The evaluation-at-2 function $E_2 : \mathbb{P}_3 \to \mathbb{R}$ given by $E_2(\mathbf{p}) = \mathbf{p}(2)$,

$$E_2(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = a_0 + a_1 2 + a_2 2^2 + a_3 2^3$$

is a linear transformation because

1. For $\mathbf{p}, \mathbf{q}$ in $\mathbb{P}_3$, we have
   $E_2(\mathbf{p}+\mathbf{q}) = (\mathbf{p}+\mathbf{q})(2) = \mathbf{p}(2) + \mathbf{q}(2) = E_2(\mathbf{p}) + E_2(\mathbf{q})$.
2. For $\mathbf{p}$ in Q and any scalar c, we have $E_2(c\mathbf{p}) = (c\mathbf{p})(2) = c(\mathbf{p}(2)) = cE_2(\mathbf{p})$.

Q is the kernel of E_2, so Q is a subspace.

Can we write Q as Span $\{\mathbf{p}_1, \ldots, \mathbf{p}_p\}$ for some linearly independent polynomials $\mathbf{p}_1, \ldots, \mathbf{p}_p$?

One idea: associate a matrix A to E_2 and take a basis of Nul$A$ using the rref.
To do computations like this, we need coordinates.