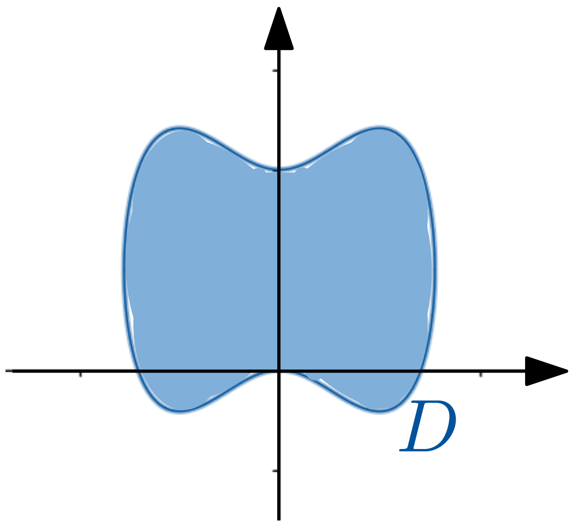
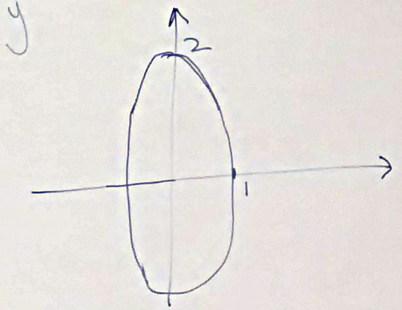
Some properties of multiple integrals, analogous to properties for 1D definite integrals (same labelling as in Week 3 p19-20):

g. If $f(x, y)$ is an odd function in x (i.e. $f(x, y) = -f(-x, y)$) and D is symmetric about the y-axis (i.e. replacing x by $-x$ in the definition of D doesn't change D), then $\int_D f(x, y)\, dA = 0$ (and similarly for an odd function in y and a domain symmetric about the x-axis).



Example: Find $\iint_D y + \sin x \cos y \, dA$, where $D = \{(x, y) \in \mathbb{R}^2 | 4x^2 + y^2 \leq 4\}$.

$\iint_D y \, dA = 0$ $\because$ y is an odd function in y and D is symmetric about the x-axis

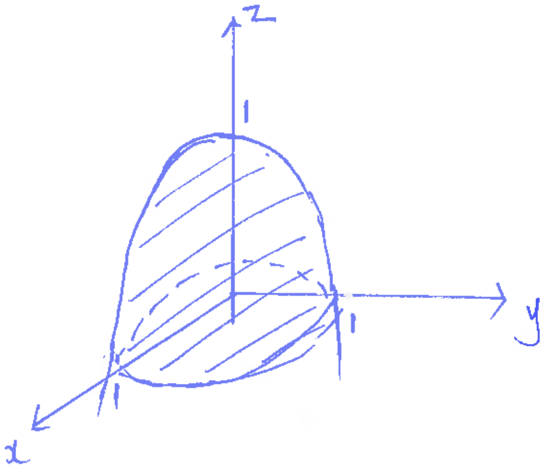


$\iint_D \sin x \cos y \, dA = 0$ $\because$ $\sin x \cos y$ is odd function in x, and D is symmetric about the y-axis

$\therefore$ So the integral is 0.

The following example leads to a very complicated integral that we will redo (p31) in a much easier way.

Example: Find the volume of the region bounded by $z = 1 - x^2 - y^2$ and $z = 0$.

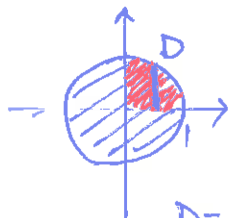


$\because$ In general, given two surfaces, set their z-values equal to find the boundary of the domain of integration (generalising Week 4 p24 for areas between "curves")

Intersection: $0 = 1 - x^2 - y^2 \implies x^2 + y^2 = 1$

Integrand = top surface − bottom surface

2D domain of integration (view from top)



$D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

This 3D region is symmetric in x and y.

So its volume is $4 \int_0^1 \int_0^{\sqrt{1-x^2}} 1 - x^2 - y^2 \, dy \, dx$

$$= 4 \int_0^1 y - x^2 y - \frac{y^3}{3} \Bigg|_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= 4 \int_0^1 (1-x^2)\sqrt{1-x^2} - \frac{\sqrt{1-x^2}^{\,3}}{3} \, dx$$

substitution $u=1-x^2$ doesn't work

substitution

$x=\sin t$

$dx=\cos t\,dt$

$x=0 \to t=0$

$x=1 \to t=\frac{\pi}{2}$

$$= 4\int_0^1 \frac{2}{3}\sqrt{1-x^2}^{\,3}\,dx$$

$$= 4\int_0^{\frac{\pi}{2}} \frac{2}{3}\cos^3 t\,\cos t\,dt$$

$$= \frac{8}{3}\int_0^{\frac{\pi}{2}} \cos^4 t\,dt$$

use table

$$= \frac{8}{3}\left(\frac{1}{8}3t + \underbrace{3\sin t\cos t + 2\cos^3 t\sin t}_{=0}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

$\because \sin 0 = 0$

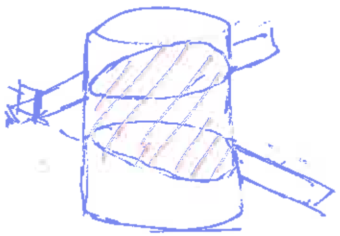
$\cos\frac{\pi}{2} = 0$

An example of a simple region bounded by 3 surfaces

Example: Express as an iterated integral the volume of the region bounded by

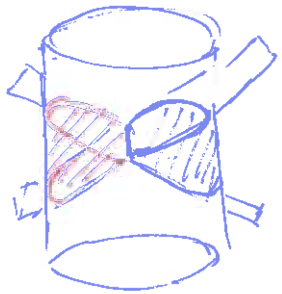
$$\underbrace{9x^2+y^2=9}_{\text{elliptic cylinder}}, \quad \underbrace{y+z=1}_{\text{plane}} \quad \text{and} \quad \underbrace{z=0}_{\text{plane}} \quad \text{with } z\geq 0$$

2 possibilities:



planes intersect outside the cylinder

$$\iint_{\text{ellipse}} \text{top plane} - \text{bottom plane}$$



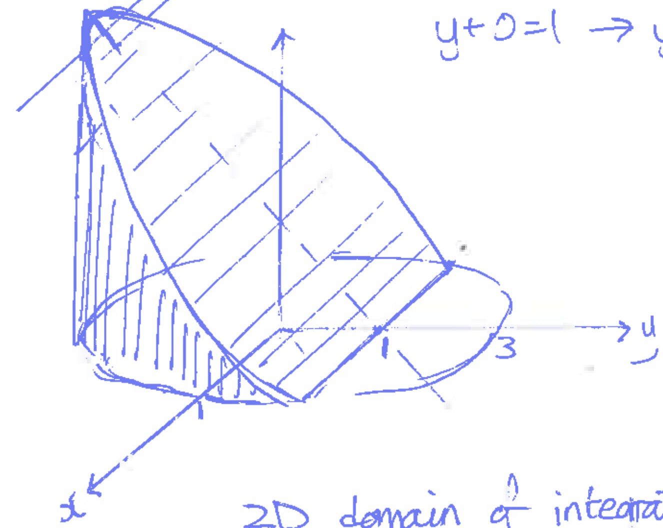
planes intersect inside the cylinder.

$$\iint \text{top plane} - \text{bottom plane}$$

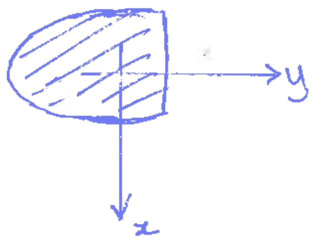
↑ part of the ellipse

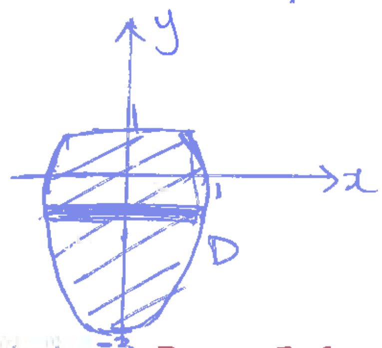
To work out which case: find the x,y values of the intersection of the planes.

The planes $y+z=1$ and $z=0$ intersect at $y+0=1 \to y=1$



2D domain of integration: (view from top)





Top plane $y+z=1$
$z=1-y$

Bottom plane $z=0$

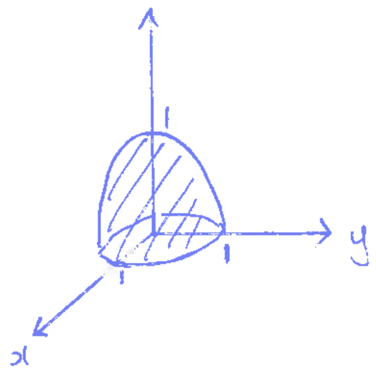
$$\iint_D (1-y)-0 \, dA = \int_{-3}^{1}\int_{-\sqrt{1-y^2/9}}^{\sqrt{1-y^2/9}} 1-y \, dx \, dy$$

$$9x^2+y^2=9$$
$$9x^2=9-y^2$$
$$x=\pm\sqrt{1-y^2/9}$$

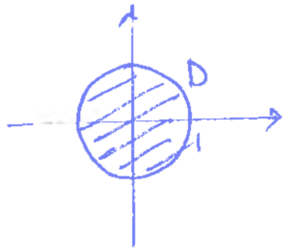
$$\int_0^{2\pi}\int_0^b f(r\cos\theta, r\sin\theta)r\,drd\theta$$

Redo Example: (p23) Find the volume of the region bounded by $z = 1 - x^2 - y^2$ and $z = 0$.

(from p23)



2D domain



$$\iint_D 1-x^2-y^2\,dA$$

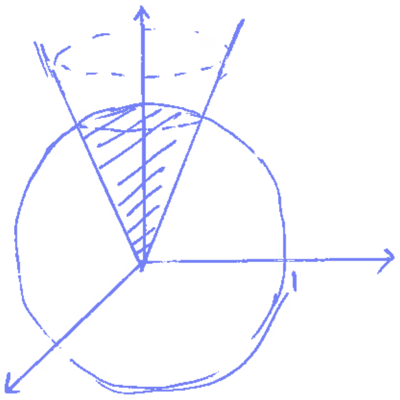
$$= \int_0^{2\pi} \underbrace{\int_0^1 (1-r^2)r\,dr}_{\text{independent of } \theta}\,d\theta$$

$$= 2\pi \int_0^1 r - r^3\,dr$$

$$= 2\pi\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1 = 2\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\pi}{2}$$

Example: Find the volume of the smaller region bounded by $z = \sqrt{3x^2 + 3y^2}$ and $x^2 + y^2 + z^2 = 1$.

$$\int_0^{2\pi} \int_0^b f(r\cos\theta, r\sin\theta) r \, dr d\theta$$



region is inside/above the cone and inside/below the sphere.

$z \geq \sqrt{3x^2+3y^2} = \sqrt{3}\, r$

$x^2+y^2+z^2 \leq 1$

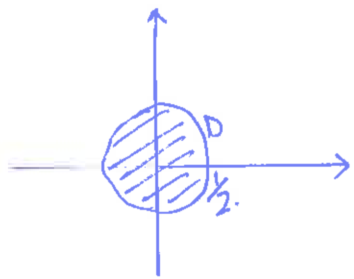
$z^2 \leq 1-x^2-y^2$

$z \leq \sqrt{1-x^2-y^2} = \sqrt{1-r^2}$

Intersection: $x^2+y^2+\left(\sqrt{3x^2+3y^2}\right)^2 = 1$

$4x^2+4y^2 = 1$ circle of radius ½

domain of integration



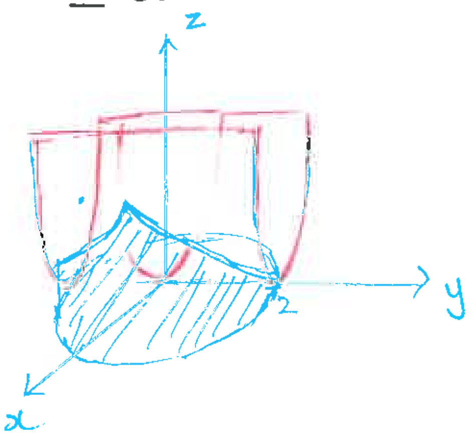
$$\text{Volume} = \iint_D \sqrt{1-x^2-y^2} - \sqrt{3x^2+3y^2} \, dA$$

$$= \int_0^{2\pi} \int_0^{1/2} \left(\sqrt{1-r^2} - \sqrt{3}\, r\right) r \, dr \, d\theta$$

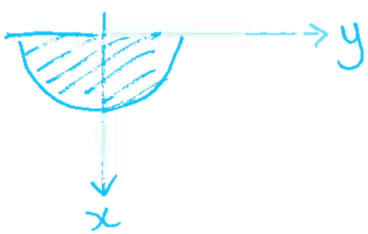
$$\int_{\alpha}^{\beta} \int_{0}^{b} f(r\cos\theta, r\sin\theta) r \, dr d\theta$$

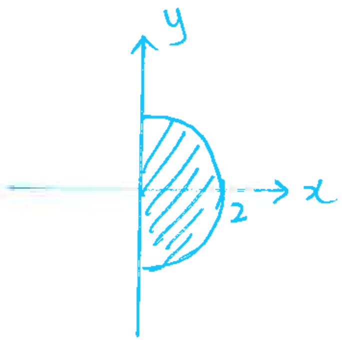
Polar coordinates are also useful for integrating over sectors:

Example: Find the volume of the region bounded by $z = x^2$, $x^2 + y^2 = 4$ and $z = 0$ with $x \geq 0$.



domain





top surface − bottom surface ...

$$\text{Volume} = \iint_D x^2 - 0 \; dA$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 (r\cos\theta)^2 \, r \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^4}{4} \cos^2\theta \Big|_{r=0}^{r=2} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos^2\theta \; d\theta$$

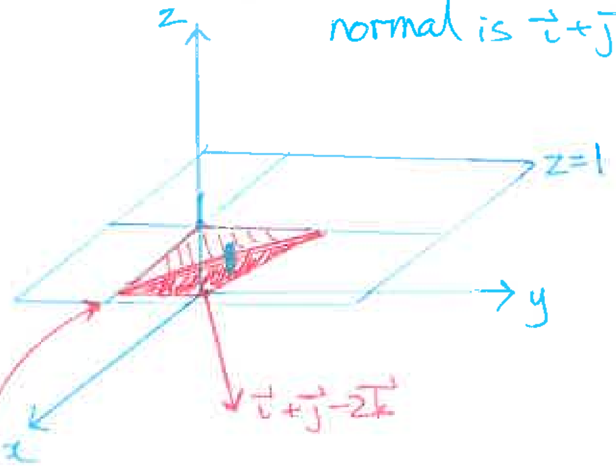
table

$$= 4 \cdot \frac{1}{2}(\theta + \sin\theta\cos\theta) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 2\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = 2\pi$$

Example: Find the mass of the tetrahedron bounded by the planes $x=0$, $y=0$, $z=1$ and $x+y-2z=0$, whose density function is $\rho(x,y,z)=x$.

normal is $\vec{i}+\vec{j}-2\vec{k}$

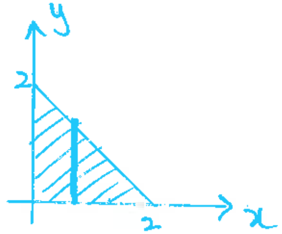


this line is the intersection of $z=1$ and $x+y-2z=0$

(other surfaces don't involve z)

$x+y-2=0$ i.e. $x+y=2$.

Projection to the xy-plane (view from top)



$$\text{Mass} = \int_0^2 \int_0^{2-x} \int_{\frac{x+y}{2}}^{1} x \, dz \, dy \, dx$$

limits for dy, dx integrals are from the projection to the xy plane - like a 2D integral

limits for dz integral are top surface and bottom surface

$x+y-2z=0$

$\frac{x+y}{2}=z$

$$= \int_0^2 \int_0^{2-x} xz \Big|_{z=\frac{x+y}{2}}^{z=1} dy \, dx$$

$$= \int_0^2 \int_0^{2-x} x - \frac{x^2+xy}{2} \, dy \, dx$$

$$= \int_0^2 xy - \frac{x^2y}{2} - \frac{xy^2}{4} \Big|_{y=0}^{y=2-x} dx$$

$$= \int_0^2 x(2-x) - \frac{x^2(2-x)}{2} - \frac{x(2-x)^2}{4} \, dx$$

$$= \int_0^2 \frac{(2-x)^2 x}{2} - \frac{x(2-x)^2}{4} \, dx$$

$$= \frac{1}{4} \int_0^2 4x - 4x^2 + x^3 \, dx = \frac{1}{4}\left(2x^2 - \frac{4}{3}x^3 + \frac{x^4}{4}\right)\Big|_0^2 = \frac{1}{3}$$

∴ Check that answer is positive ∵ it is a mass

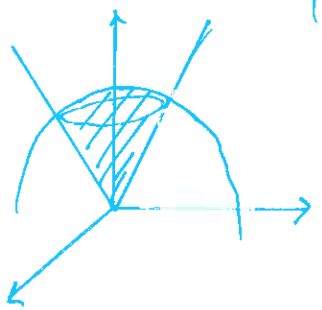
Example: Find the mass of the smaller region bounded by $z = \sqrt{3x^2 + 3y^2}$ and $x^2 + y^2 + z^2 = 1$, with density function $\delta(x, y, z) = x^2 z$.

$$\int_c^d \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta, z) r\,dr\,d\theta\,dz$$

(from p32)

region is above the cone $z \geq \sqrt{3}\,r$

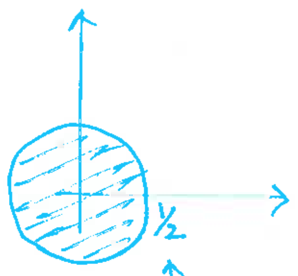
below the sphere $z \leq \sqrt{1-r^2}$



$$\text{mass} = \int_0^{2\pi} \int_0^{\frac{1}{2}} \int_{\sqrt{3}\,r}^{\sqrt{1-r^2}} (r\cos\theta)^2 z \; r \; dz\, dr\, d\theta$$

$=$

Projection to the xy plane:



from $\sqrt{3}\,r = \sqrt{1-r^2}$, this has solution $r = \frac{1}{2}$.