Important consequences of Steinitz:

Cor 6.4.7: All bases of V have the same number of vectors.

Proof: If $\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\}$, $\mathcal{B} = \{\beta_1, \cdots, \beta_m\}$ are both bases of V.

$\mathcal{A}$ spans V, $\mathcal{B}$ is linearly independent: $m \leq n$

$\mathcal{B}$ spans V, $\mathcal{A}$ is linearly independent: $n \leq m$ $\quad \therefore m = n$.

Def 6.4.8: If a basis of V has n vectors, then V is <u>finite-dimensional</u> and <u>$\dim V = n$</u> ($\dim V$ may depend on $\mathbb{F}$.)

If V has no finite basis, then V is <u>infinite-dimensional</u>.

Ex: Using standard bases

$\dim \mathbb{F}^n = n$, $\dim P_{<n}(\mathbb{F}) = n$, $\dim M_{m,n}(\mathbb{F}) = mn$

$\mathbb{F}[x]$ is infinite-dimensional.

How to find a basis 3: Use dimension.

Th 6.4.11 Basis theorem: If $\mathcal{A} \subseteq V$ and $|\mathcal{A}| = \dim V$ $(\neq \infty)$, then $\mathcal{A}$ is linearly independent if and only if $\mathcal{A}$ spans V.

$\therefore$ only need to check one of .

(2207 Week 8.5 p7)

Other useful things:

Cor 6.4.10: If $\mathcal{A} \subseteq V$ and $|\mathcal{A}| > \dim V$, then $\mathcal{A}$ is linearly dependent.

Th. 6.4.16: If W is a subspace of V, then $\dim W \leq \dim V$.

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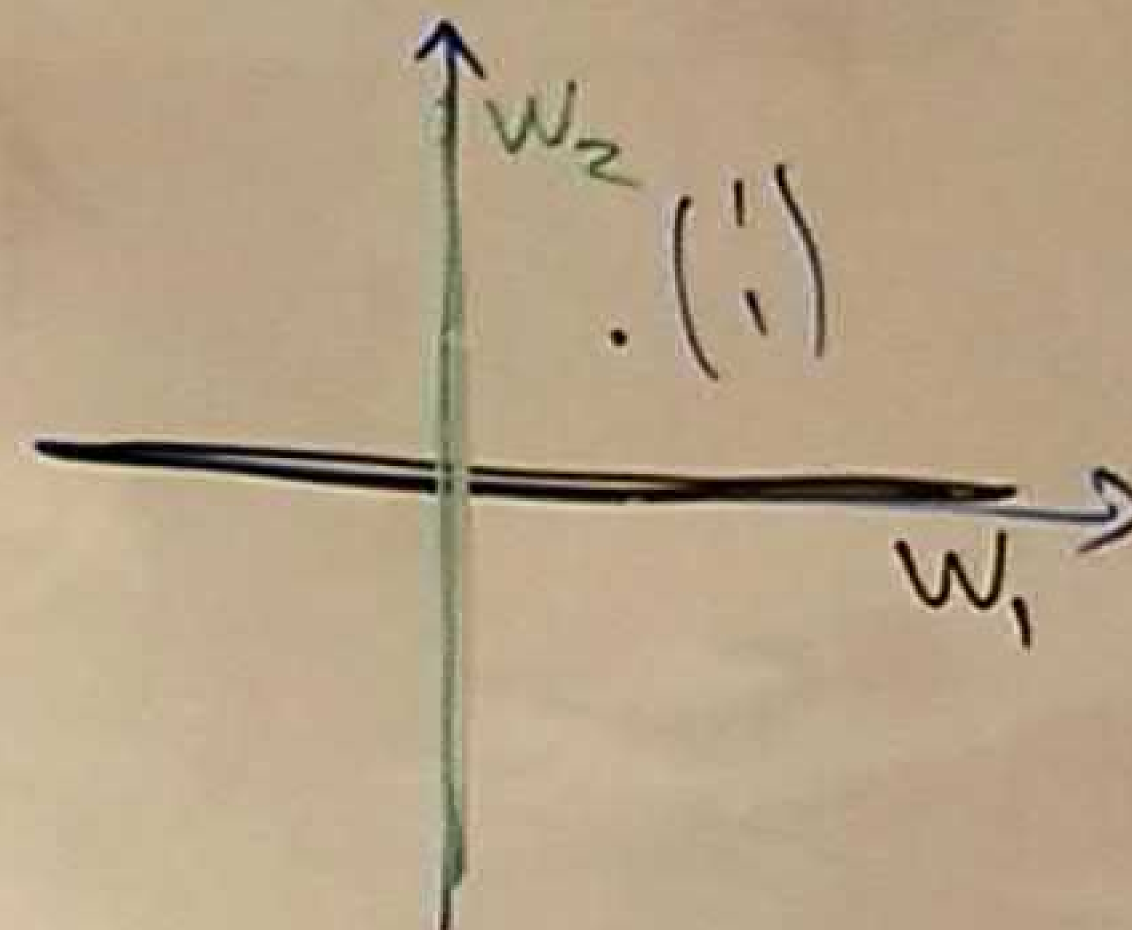
6.5: Sums and direct sums of subspaces

? How to make a big subspace out of small ones?

<u>Ex</u> in $\mathbb{R}^2$: $W_1 = \mathrm{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$

$W_2 = \mathrm{Span}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$



$W_1 \cup W_2$ is not a subspace:

$\because$ $\underset{W_1}{\underset{\cap}{\begin{pmatrix}1\\0\end{pmatrix}}} + \underset{W_2}{\underset{\cap}{\begin{pmatrix}0\\1\end{pmatrix}}} = \begin{pmatrix}1\\1\end{pmatrix} \notin W_1 \cup W_2$

$\therefore$ To make a big subspace W containing W_1 and W_2, we must include sums like $\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}$.

Def 6.5.1/6.5.4: Let $W_1, \cdots, W_k$ be subspaces of V.

The <u>sum</u> of $W_1, \cdots, W_k$ is:

$$W_1 + \cdots + W_k = \sum_{i=1}^{k} W_i = \left\{ \sum_{i=1}^{k} \alpha_i \,\middle|\, \alpha_i \in W_i \quad 1 \le i \le k \right\}$$

Ex: from above: $W_1 + W_2 = \mathbb{R}^2 \because \begin{pmatrix} x \\ y \end{pmatrix} = \underset{\in W_1}{\begin{pmatrix} x \\ 0 \end{pmatrix}} + \underset{\in W_2}{\begin{pmatrix} 0 \\ y \end{pmatrix}}$

in $\mathbb{R}^3$: $U_1 = \text{span}\{e_1\}$ line

$U_2 = \text{span}\{e_2\}$ line

$V_1 = \text{span}\{e_1, e_2\}$ plane

$V_2 = \text{span}\{e_2, e_3\}$ plane

$U_1 + U_2 = V_1 \because \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \underset{\in U_1}{\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}} + \underset{\in U_2}{\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}}$

$V_1 + V_2 = \mathbb{R}^3 \because \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underset{\in V_1}{\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}} + \underset{\in V_2}{\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}}$

Note: $\cdot$ summing is commutative: $W_1 + W_2 = W_2 + W_1$

$\cdot$ $W_1 + \cdots + W_k \supseteq W_i$ for $1 \le i \le k$,

$\because$ given $w_i \in W_i$,

$$w_i = \underset{\substack{\cap\\ W_1}}{\vec{0}} + \cdots + \underset{\substack{\cap\\ W_{i-1}}}{\vec{0}} + \underset{\substack{\cap\\ W_i}}{w_i} + \underset{\substack{\cap\\ W_{i+1}}}{\vec{0}} + \cdots + \underset{\substack{\cap\\ W_k}}{\vec{0}}.$$

$\in W_1 + \cdots + W_k$.

<u>Prop. 6.5.3</u>: $W_1 + \cdots + W_k$ is a subspace.

Proof: $\vec{0} \in W_1 + \cdots + W_k$ $\because$ $\vec{0} = \vec{0} + \cdots + \vec{0}$

and $\vec{0} \in$ each W_i $\because$ W_i is a subspace.

if $\alpha, \beta \in W_1 + \cdots + W_k$, then $\alpha = \alpha_1 + \cdots + \alpha_k$,

$\beta = \beta_1 + \cdots + \beta_k$, for $\alpha_i, \beta_i \in W_i$ $(1 \le i \le k)$.

$$a\alpha + \beta = a(\alpha_1 + \cdots + \alpha_k) + \beta_1 + \cdots + \beta_k$$
$$= (a\alpha_1 + \beta_1) + \cdots + (a\alpha_k + \beta_k)$$

and $a\alpha_i + \beta_i \in W_i$

$\because$ W_i is a subspace.

Another view: $W_1 \cup W_2$ is a set

$\therefore$ to make a subspace, take

$\mathrm{Span}(W_1 \cup W_2)$.

Th. 6.5.5: a) $W_1 + \cdots + W_k = \text{Span}(W_1 \cup \cdots \cup W_k)$

b) if $W_i = \text{Span}(A_i)$, then

$W_1 + \cdots + W_k = \text{Span}(A_1 \cup \cdots \cup A_k)$

Proof: for simplicity, we show only $k=2$.

To show $W_1 + W_2 \supseteq \text{Span}(W_1 \cup W_2)$:

$W_1 + W_2$ is a subspace, and contains W_1 and W_2

$\therefore$ contains $W_1 \cup W_2$

(6.3.8) $\therefore$ contains $\text{Span}(W_1 \cup W_2)$

To show $\text{Span}(W_1 \cup W_2) \supseteq W_1 + W_2$