Homework 2, due 12:00 Wednesday 27 February 2019 to Dr. Pang's mailbox

You must justify your answers to receive full credit.

1. a) (Proposition 6.5.14) Let $W_1, W_2, \ldots, W_n$ be subspaces of a vector space V. Prove that $W_1 + W_2 + \cdots + W_n$ is a direct sum if and only if, $\forall \alpha \in W_1 + W_2 + \cdots + W_n$, α can be expressed uniquely in the form $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ with $\alpha_i \in W_i$, $i = 1, 2, \ldots, n$. (Hint: modify appropriately the proof of Proposition 5.1.10, for $n = 2$. You can use induction for one of the directions, but I think it is easier without induction.)

   b) Find non-zero subspaces W_1, W_2, W_3 of $\mathbb{R}^3$ satisfying

$$W_1 \cap W_2 = \{\mathbf{0}\}, W_1 \cap W_3 = \{\mathbf{0}\}, W_2 \cap W_3 = \{\mathbf{0}\}, \tag{$*$}$$

   and some $\alpha \in \mathbb{R}^3$ which can be written as $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, with $\alpha_i \in W_i$, in two different ways. (This explains why $(*)$ is not the condition for a direct sum of three subspaces.)

2. Consider the following two subspaces of $M_{2,2}(\mathbb{R})$:

$$W_1 = \left\{ \begin{pmatrix} x & -x \\ y & z \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}, \quad W_2 = \left\{ \begin{pmatrix} a & b \\ -a & c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}.$$

   Find a basis of $W_1 \cap W_2$, and extend it to bases of W_1, W_2 and $W_1 + W_2$.

3. Let U, V, W be vector spaces, and take $\sigma, \sigma' \in L(U, V)$, $\tau \in L(V, W)$.

   a) (Proposition 7.1.8, axiom (V1) for $L(U, V)$) Prove that the sum $\sigma + \sigma'$ is a linear transformation.

   b) (Proposition 7.1.7) Prove that the composition $\tau \circ \sigma$ is in $L(?, ?)$ where the ?s are vector spaces that you should specify.

   c) (Proposition 7.1.14) Let V' be a subspace of V. Show that the preimage $\sigma^{-1}[V'] = \{\alpha \in U \mid \sigma(\alpha) \in V'\}$ is a subspace of U.

4. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$, and define $\sigma : M_{2,2}(\mathbb{R}) \to M_{2,2}(\mathbb{R})$ by $\sigma(X) = AX - XA$.

   a) Show that σ is a linear transformation.

   b) Find $[\sigma]_{\mathscr{A}}$, where $\mathscr{A} = \{E^{1,1}, E^{1,2}, E^{2,1}, E^{2,2}\}$ is the standard basis of $M_{2,2}(\mathbb{R})$.

   c) Find all $X \in M_{2,2}(\mathbb{R})$ such that $AX = XA$.

   d) Using part c) or otherwise, find the rank of σ.

   e) Let $U \subseteq M_{2,2}(\mathbb{R})$ be the subspace of symmetric matrices. Find a basis for the image $\sigma(U)$. (Click here for a hint)

5. Fill in the blanks and choose the correct words at the †s to complete the proof of the Rank Nullity Theorem (print this page and hand it in):

Theorem: if $\dim(U) < \infty$ **and** $\sigma \in L(U, V)$**, then** $\operatorname{rank} \sigma + \operatorname{nullity} \sigma = \dim U$**.**

Proof: Let $\dim U = n$. Because $\ker \sigma$ is a subspace of U, so $\ker \sigma$ is finite-dimensional. Let $\dim \ker \sigma = k$.

Take a basis $\{\alpha_1, \dots, \alpha_k\}$ of $\ker \sigma$, and a basis $\{\beta_1, \dots, \beta_{n-k}\}$ of a complement of $\ker \sigma$ in U, so $\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_{n-k}\}$ is a basis for U.

We show $\{\sigma(\beta_1), \dots, \sigma(\beta_{n-k})\}$ is a basis for $\operatorname{range} \sigma$ – that would mean $\operatorname{rank} \sigma = \dim \operatorname{range} \sigma = n - k = \dim U - \operatorname{nullity} \sigma$.

- We show $\{\sigma(\beta_1), \dots, \sigma(\beta_{n-k})\}$ is linearly independent:
  Suppose
  $$b_1\sigma(\beta_1) + \cdots + b_{n-k}\sigma(\beta_{n-k}) = \mathbf{0} \qquad (*)$$

  We need to show ______________________________.

  Because σ is linear, we can rewrite $(*)$ as
  $$\sigma\Big(\underline{\qquad\qquad\qquad\qquad\qquad\qquad}\Big) = \mathbf{0}.$$

  So
  $$\underline{\qquad\qquad\qquad\qquad\qquad\qquad} \in \ker \sigma.$$

  Because $\{\alpha_1, \dots, \alpha_k\}$ is a basis for $\ker \sigma$, we have, for some $a_i \in \mathbb{F}$,
  $$\underline{\qquad\qquad\qquad\qquad\qquad\qquad} = a_1\alpha_1 + \cdots + a_k\alpha_k.$$

  Move all terms to one side:
  $$-a_1\alpha_1\underline{\qquad\qquad\qquad\qquad\qquad\qquad} = \mathbf{0}.$$

  Because $\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_{n-k}\}$ is a basis for U, this set is linearly independent, so

  ______________________________.

- We show $\operatorname{Span}\{\sigma(\beta_1), \dots, \sigma(\beta_{n-k})\} = \operatorname{range} \sigma$:
  By the Lemma from class,
  $$\sigma(\operatorname{Span}\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_{n-k}\}) = \operatorname{Span}\{\sigma(\alpha_1), \dots, \sigma(\alpha_k), \sigma(\beta_1), \dots, \sigma(\beta_{n-k})\}.$$

  The left / right † hand side is $\operatorname{Span}\{\sigma(\beta_1), \dots, \sigma(\beta_{n-k})\}$ because ____________________
  ____.

  The left / right † hand side is $\operatorname{range} \sigma$ because ______________________________.

6. Let $P_{<n}(\mathbb{R})$ denotes the vector space of polynomials with degree less than n. Consider the bases

$$\begin{aligned} \mathscr{A} &= \left\{1+x-x^2, 1+x, 1+x^2\right\} &&\subseteq P_{<3}, \\ \mathscr{B} &= \{1, x\} &&\subseteq P_{<2}, \\ \mathscr{C} &= \{1+x, 1-x\} &&\subseteq P_{<2}. \end{aligned}$$

Let $\sigma : P_{<3}(\mathbb{R}) \to P_{<2}(\mathbb{R})$ be the linear transformation given by

$$\sigma(f) = f(2)x + \frac{df}{dx}.$$

a) Find $[2x+3x^2]_{\mathscr{A}}$, the $\mathscr{A}$-coordinates of $2x+3x^2$.

b) Find the polynomial f whose $\mathscr{C}$-coordinates are $[f]_{\mathscr{C}} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

c) Find the matrix P such that $[\alpha]_{\mathscr{B}} = P[\alpha]_{\mathscr{C}}$ for all $\alpha \in P_{<2}(\mathbb{R})$.

d) Find the matrix representation of σ relative to $\mathscr{A}$ and $\mathscr{C}$ (i.e. the matrix denoted $\underset{\mathscr{C} \leftarrow \mathscr{A}}{[\sigma]}$ or $[\sigma]_{\mathscr{C}}^{\mathscr{A}}$).

e) Is σ surjective? Explain your answer.

The following two questions are to prepare you for upcoming classes, and is unrelated to the material from recent classes.

7. Find all eigenvalues and eigenvectors of $A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ -2 & 2 & -1 \end{pmatrix}$, and hence diagonalise A, i.e. find a P and a diagonal D such that $A = PDP^{-1}$. You do **not** need to compute P^{-1}. (You may use an online RREF calculator, but remember you only have an ordinary calculator in the exams.)

8. a) Find the eigenvalues of $\begin{pmatrix} 5 & 2 & 0 & 0 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

Suppose A is a 3×3 matrix whose only eigenvalues are 1 and 2.

b) If A is diagonalisable, then what are the possibilities for the dimensions of its eigenspaces? (Answer in this way: "$\dim E_1 =?$ and $\dim E_2 =?$, or $\dim E_1 =?$ and $\dim E_2 =?$, or ...", and then give your reasons.)

c) If A is not diagonalisable, then what are the possibilities for the dimensions of its eigenspaces? (Answer in the same way as in part b.)

Optional questions If you attempted seriously all the above questions, then your scores for the following questions may replace any lower scores for two of the above questions.

9. Let V be a vector space, and take $\sigma \in L(V,V)$. Show that, if $\operatorname{rank}(\sigma) = \operatorname{rank}(\sigma \circ \sigma)$, then $\operatorname{range}(\sigma) \cap \ker(\sigma) = \{\mathbf{0}\}$.
(Hint: the equation $\operatorname{rank}(\sigma) = \operatorname{rank}(\sigma \circ \sigma)$ means these numbers are finite. You may assume V is finite-dimensional if you wish. It may be useful to show that $\operatorname{rank}(\sigma) \geq \operatorname{rank}(\sigma \circ \sigma)$.)

10. Let U, W_1, W_2 be subspaces of a vector space V, such that $W_1 \cap W_2 = \{\mathbf{0}\}$. Is $U \cap (W_1 \oplus W_2) = (U \cap W_1) \oplus (U \cap W_2)$? Give a proof or a counterexample.

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