For the next few weeks, we focus on differentiation of multivariate functions (in a different order from the textbook):

$\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ (i.e. vector-valued functions)

- This week: differentiating a multivariate function (§12.2, 12.3 first two pages, 12.4 first two pages, 12.6 first two pages, fifth and sixth pages)
- Week 8: the chain rule, for differentiating compositions (§12.5, and the matrix version in §12.6)

$f : \mathbb{R}^n \to \mathbb{R}$ (i.e. scalar-valued functions)

- Week 9: direction of greatest increase, tangent planes, Taylor polynomials (§12.7, 12.9 first four pages)
- Week 10: classifying critical points (§13.1, the subsection "Classiyfing Critical Points" until Example 7; the rest is in Week 11)
- Week 11: finding maxima and minima (§13.1-13.5 8E, §13.1-13.4 7E)

Notation: we call the m "outputs" of $\mathbf{f}$ by $f_1, f_2, \ldots, f_m$, these are the coordinate functions. e.g. $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$ is denoted $\mathbf{f}(x, y) = (f_1(x, y), f_2(x, y), f_3(x, y))$. We will often analyze $\mathbf{f}$ by analysing its coordinate functions separately.

For one-variable functions, the derivative is the limit of a difference quotient:

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

To discuss the differentiability of multivariate functions, we must first define the limit of a multivariate function $\displaystyle\lim_{(x,y)\to(a,b)} g(x,y)$.

Unlike the limits of 2-variable Riemann sums that we saw in multiple integration, the limit of a 2-variable function cannot be calculated by taking 1D limits separately in the x and y directions. It requires a more careful analysis.

On the next page we give an informal definition of a limit for functions $f : \mathbb{R}^n \to \mathbb{R}^m$, but we will concentrate on when the domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}$.

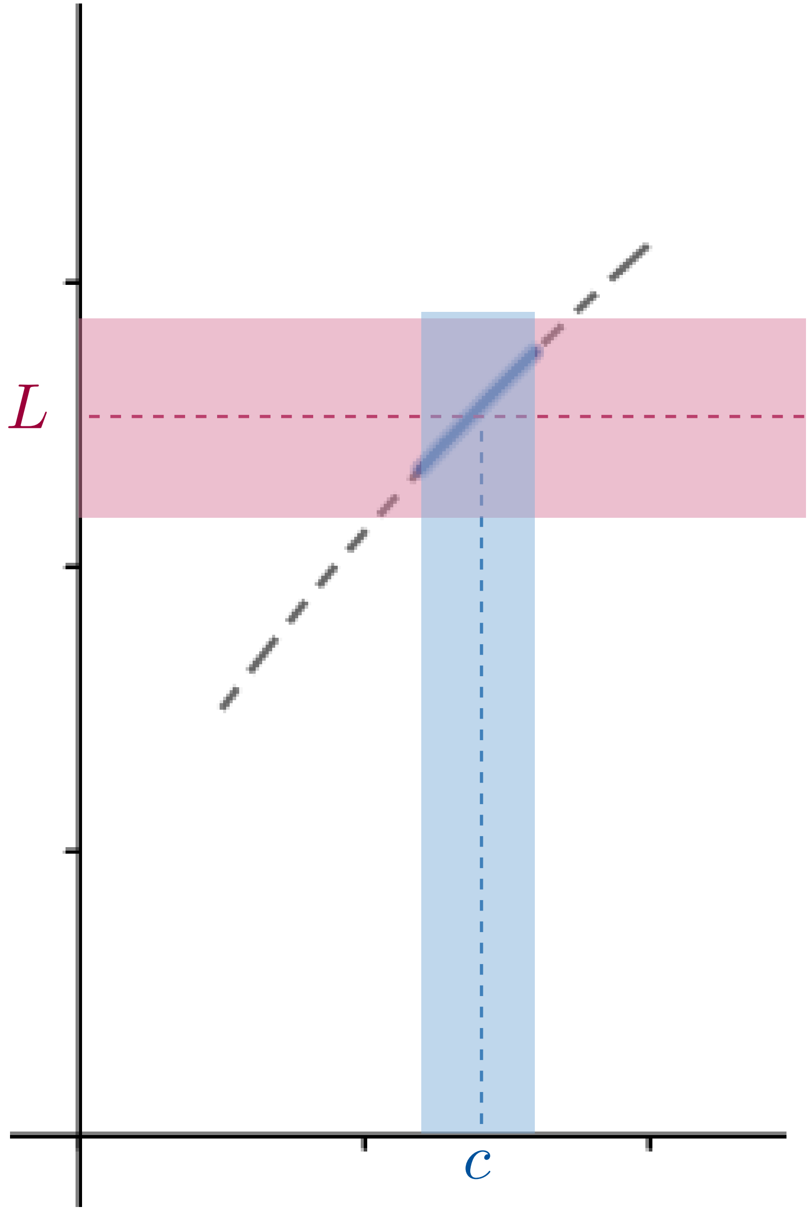
We will first discuss ways to show that a limit does not exist (p5-9), and then ways to evaluate a limit that does exist (p10-13).

§12.2: Limits and Continuity

Remember the informal definition of a single-variable limit:

Definition: Given a function $f(x)$ defined near a point c, the statement $\lim_{x \to c} f(x) = L$ means: we can ensure that $f(x)$ is as close as we want to L by taking x close enough (but not equal) to c.

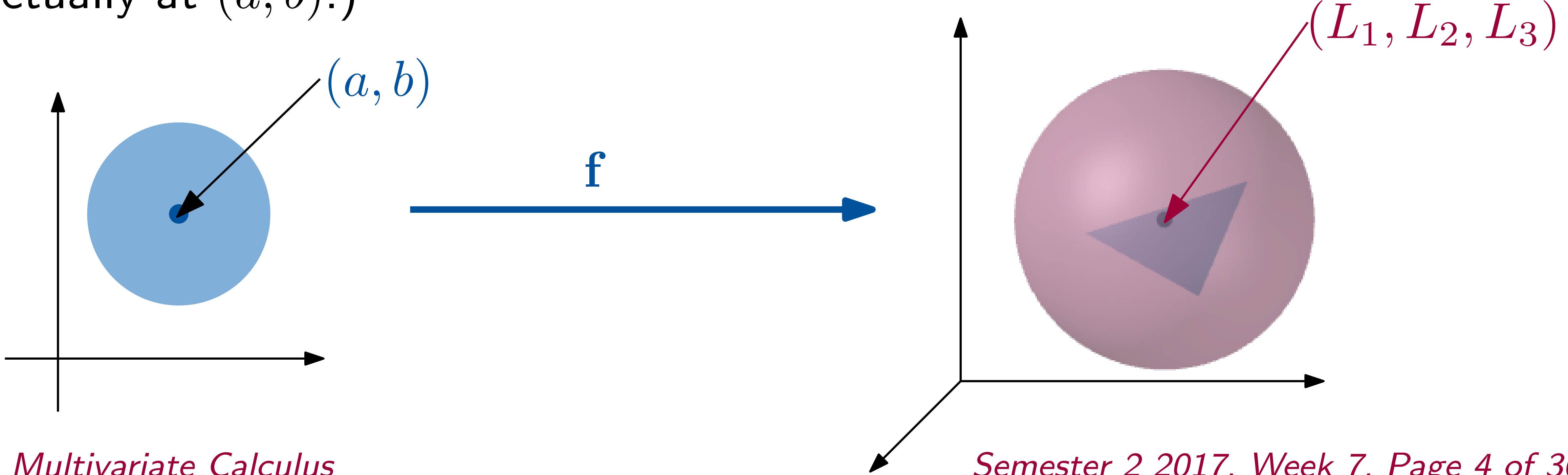
In other words: given any small interval around L (the height of the red rectangle), we can find a small interval around c (the width of the blue rectangle) so the values of $f(x)$ when x is in this small "blue" interval all lie in the "red" interval around L (i.e. the part of the graph of f in the blue rectangle is also in the red rectangle).



A limit of a multivariate function is the same idea, using balls and spheres instead of intervals:

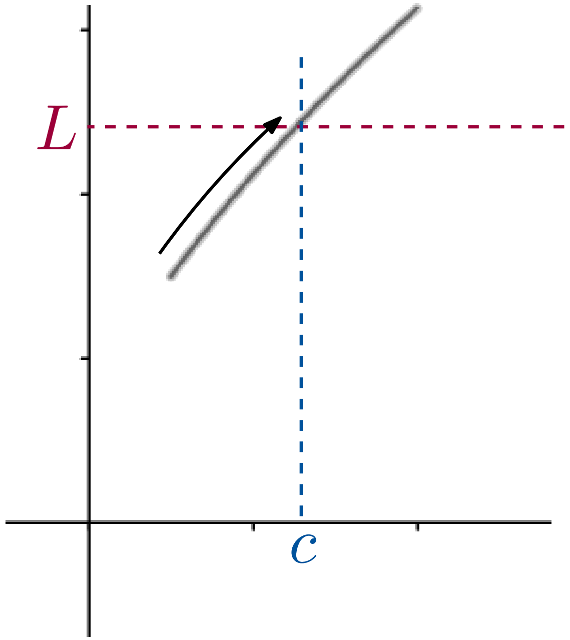
Example: Given a 2-variable function $\mathbf{f} : \mathcal{D} \to \mathbb{R}^3$ whose domain $\mathcal{D}$ contains points close to (a, b), the statement $\lim\limits_{(x,y)\to(a,b)} \mathbf{f}(x, y) = (L_1, L_2, L_3)$ means:

given a small sphere around (L_1, L_2, L_3), we can find a small disk around (a, b) such that the image of this disk under $\mathbf{f}$ is entirely contained in the sphere. (Strictly speaking, $\mathbf{f}(a, b)$ does not need to be in the sphere for the limit statement to hold, because a limit is about how a function behaves around (a, b) but not actually at (a, b).)

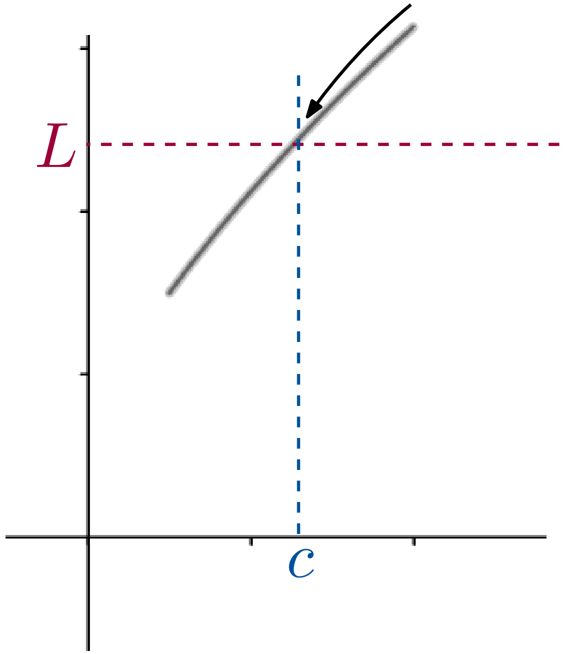


Another informal way to think about single-variable limits is: how does $f(x)$ behave as x moves towards c? This is the one-sided limit:

$$\lim_{x \to c^-} f(x) = L$$

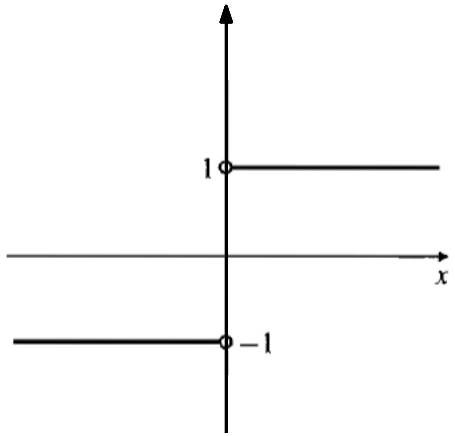


$$\lim_{x \to c^+} f(x) = L$$

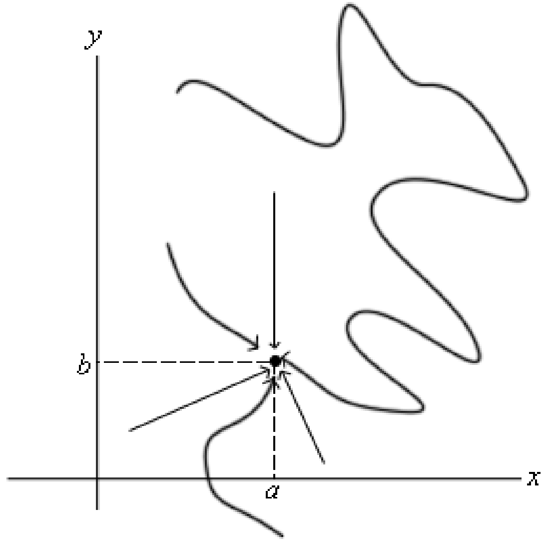


It is a theorem that $\lim_{x \to c} f(x)$ exists if and only if the two one-sided limits exist and are equal - i.e. $f(x)$ "goes towards" the same number no matter how we move towards c.

Example: $\lim_{x \to 0} \frac{x}{|x|}$ does not exist because $\lim_{x \to 0^-} \frac{x}{|x|} = -1$, $\lim_{x \to 0^+} \frac{x}{|x|} = 1$, and these limits are not equal.



The same is true for multivariate limits, but there are now many more ways for (x, y) to approach (a, b).



Each way of approach can be formalised as a path, i.e. a function $t \mapsto (x(t), y(t))$ such that $x(c) = a, y(c) = b$. (Imagine drawing one of the paths in the diagram, and recording the position of your pen at time t. Write c for the time that your pen reaches the point of interest (a, b).) We then study f by considering the values that f takes along the path, i.e. by considering the composition $f(x(t), y(t))$.

Theorem: Multivariate Limits and Paths: Let $f : \mathcal{D} \to \mathbb{R}$ be a two-variable function, and suppose $\mathcal{D}$ contains points arbitrarily close to (a, b). We have $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$ if and only if, for all paths $t \mapsto (x(t), y(t))$ such that $x(c) = a, y(c) = b$, the limits $\lim\limits_{t\to c} f(x(t), y(t))$ all exist and are equal to L.

Because the existence of the 2D limit is equivalent to the existence of 1D limits along infinitely many paths, it is not practical to use this theorem to prove the existence of a 2D limit. However, the theorem is useful for showing a 2D limit doesn't exist: simply find two paths along which the limits are different.

Example: Show that the limit $\displaystyle\lim_{(x,y)\to(-1,2)} \frac{x^2-1}{4x^2-y^2}$ does not exist.

Example: Show that the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^3}$ does not exist.

Example: Show that the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4}{x^2y+y^2}$ does not exist.

Now we give some strategies for showing that a 2-variable limit does exist. This will use the concept of continuity, which has the same definition as the 1D case.

Definition: An n-variable function $f : \mathcal{D} \to \mathbb{R}^m$ is *continuous* at a point $(a_1, \dots, a_n)$ in the domain $\mathcal{D}$ if

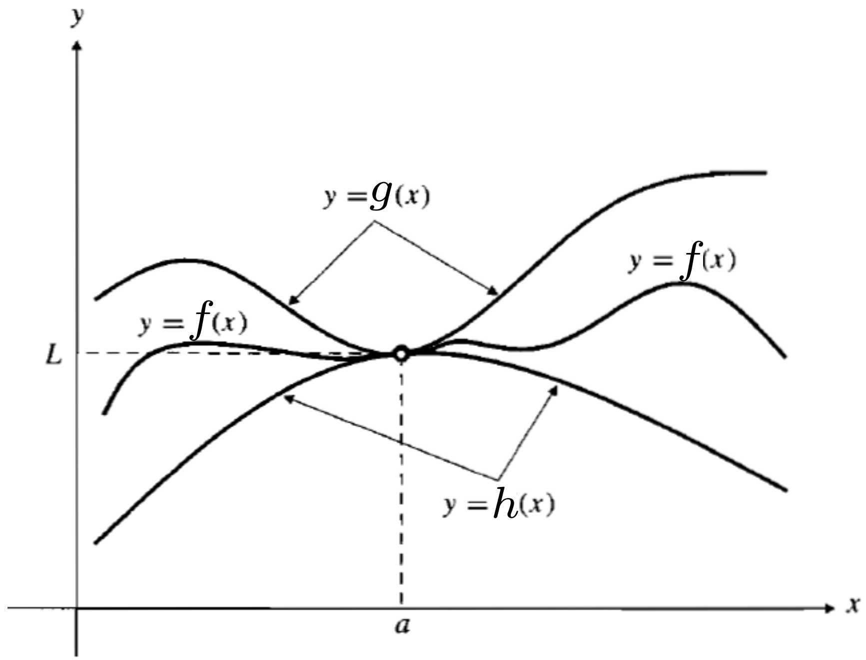
$$\lim_{(x_1,\dots x_n)\to(a_1,\dots,a_n)} f(x_1, \dots, x_n) = f(a_1, \dots, a_n).$$

As in the 1D case, elementary functions (i.e. sums, products and compositions of $x^n, e^x, \ln x, \sin x, \cos x$) are continuous. So the following example is easy:

Example: Evaluate the limit $\displaystyle\lim_{(x,y)\to(-1,2)} \frac{x^2-2}{y^2-1}$, or prove that it does not exist.

In more complicated examples, our main tool for evaluating limits is the squeeze theorem. The multivariate squeeze theorem is a very simple extension of the 1D statement. (The diagram is in 1D, but we can easily imagine a 2D version.)

Squeeze Theorem: Suppose there are functions $g(x, y)$ and $h(x, y)$ such that, for all points (x, y) in the domain of f that are near (a, b), we have the inequality $h(x, y) \leq f(x, y) \leq g(x, y)$. Suppose also that $\lim_{(x,y)\to(a,b)} g(x, y) = \lim_{(x,y)\to(a,b)} h(x, y) = L$. Then $\lim_{(x,y)\to(a,b)} f(x, y)$ exists and is also equal to L.



We will often choose our squeezing functions $g(x, y)$ and $h(x, y)$ to be elementary functions, so their limits are easy to calculate.

In the special case where we want to show $\lim_{(x,y)\to(a,b)} f(x, y) = 0$, it is enough to find one squeezing function $g(x, y)$ such that $\lim_{(x,y)\to(a,b)} g(x, y) = 0$ and $-g(x, y) \leq f(x, y) \leq g(x, y)$ - this inequality is equivalent to $|f(x, y)| \leq g(x, y)$.

Here's a simple 2-dimensional version of the standard 1D squeeze theorem example (see Homework 3 final question).

Example: Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} xy^2 \sin\left(\frac{1}{y}\right)$, or prove that the limit does not exist.

Most applications of the Squeeze Theorem in 2D don't involve trigonometric functions, and look more like this next example:

Example: Evaluate the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{yx^2}{x^2+y^2}$, or prove that it does not exist.

Summary on analysing the limit $\lim_{(x,y)\to(a,b)} f(x,y)$:

Step 1 If f is a continuous function that is defined at (a,b), then the limit is $f(a,b)$ (see p10).

Step 2 Evaluate the limit of f along straight line paths: $\lim_{x\to a} f(x,b)$, $\lim_{y\to b} f(a,y)$, $\lim_{x\to 0} f(x,mx)$ (if $(a,b)=(0,0)$). If you find two different limits, then f does not have a limit at (a,b) (see p8).

Step 3 Evaluate the limit of f along paths of the form $y=x^n$ or $(x,y)=(t^i,t^j)$ (if $(a,b)=(0,0)$). If these give different limits from the paths in Step 2, then f does not have a limit at (a,b) (see p9).

Step 4 Try to prove that $\lim_{(x,y)\to(a,b)} f(x,y)$ is the value of the limits found in Steps 2 and 3, by using the Squeeze Theorem (see p13).

(Note that the squeeze theorem only applies to scalar-valued functions, because there is no concept of $<$ or $>$ in $\mathbb{R}^m$ for $m>1$. But we can apply the squeeze theorem to the coordinate functions of a vector-valued function - $\mathbf{f}$ has a limit if and only if each coordinate function f_i has a limit.)

§12.3-4: Partial Derivatives

Remember that the derivative of a single-variable function is the limit of a difference quotient, measuring the rate of change of f as we change the input variable x:

$$f'(x) = \frac{df}{dx} = \frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

The derivative of a 2-variable function will be a more complicated concept because there are many different ways to change the input variables (x, y). We start with the simplest way, where we fix one variable and change the other:

Definition: The *first-order partial derivatives* of the function $f(x, y)$, with respect to the variables x and y respectively, are given by:

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}f(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}f(x,y) = \lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{k}$$

These are 1D limits, not the 2D limits of the previous section.

Definition: The *first-order partial derivatives* of the function $f(x, y)$, with respect to the variables x and y respectively, are given by:

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

It is clear how to define first-order partial derivatives for an n-variable function $f : \mathbb{R}^n \to \mathbb{R}$ - there will be n of them, one for each variable (which will change that variable and keep the other $n-1$ variables fixed).

And for a vector-valued function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$, we can take partial derivatives of each coordinate function, so there will be nm partial derivatives altogether (see p32).

(The textbook writes $f_1(x, y)$ to mean "differentiate with respect to the first variable", i.e. what we are calling $f_x(x, y)$. But this causes problems when $\mathbf{f}$ is vector-valued, because f_i is also the ith coordinate function of $\mathbf{f}$.)

If f is an elementary function, then we can use our single-variable differentiation rules to calculate $\frac{\partial f}{\partial x}$, by treating y as a constant (and similarly for $\frac{\partial f}{\partial y}$).

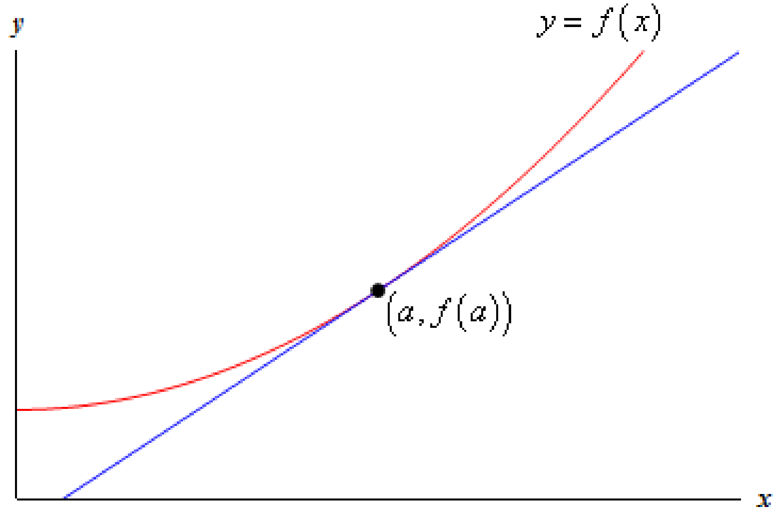
Example: Find the first-order partial derivatives of $f(x, y) = \frac{xy}{x+1}$ at $(1, 2)$.

When f is defined by different formulae around (a, b), we need to use the limit definition to calculate the partial derivatives at (a, b) .

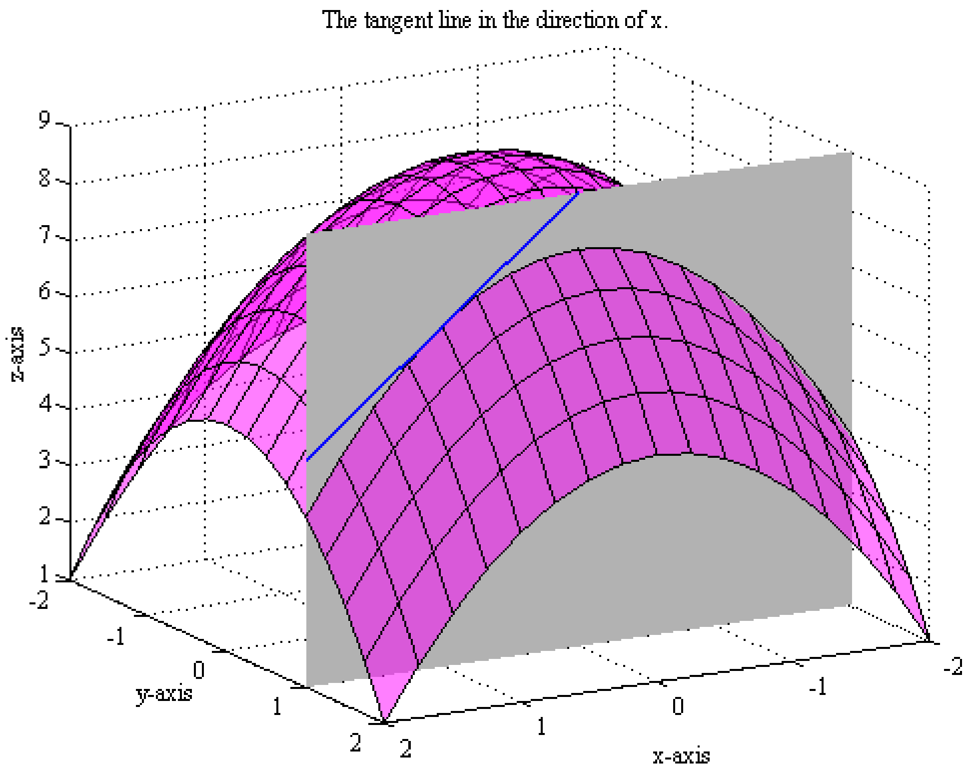
Example: Let $f(x,y) = \begin{cases} \dfrac{y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. Find $f_y(0,0)$.

The geometric meaning of partial derivatives

Recall that the derivative $f'(a)$ of a single-variable function f is the slope of the tangent line to the graph of f at the point $(a, f(a))$.



The partial derivatives $f_x(a,b)$ and $f_y(a,b)$ are also the slopes of tangent lines to certain curves: these curves are the graphs of $f(x,b)$ and $f(a,y)$, which are the intersections of the graph of f with the planes $y = b$ and $x = a$ respectively.



(pictures from Paul's Online Math Notes and WikiHow)

Higher order partial derivatives

Remember that the second derivative of a single-variable function comes from taking the derivative twice: $f''(x) = \frac{d^2 f}{dx^2} = \frac{d^2}{dx^2} f(x) = \frac{d}{dx}\left(\frac{df}{dx}\right)$.

Similarly, we can take the derivative twice of a 2-variable function $f(x, y)$ - now there are many combinations depending on which variable we differentiate with respect to, as they can be different variables for the first time and the second time:

Definition: The *second-order partial derivatives* of the function $f(x, y)$ are:

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2}{\partial x^2} f(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

mixed partial derivatives

$$\left\{\begin{aligned} f_{xy}(x,y) &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2}{\partial y \partial x} f(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\ f_{yx}(x,y) &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \end{aligned}\right.$$

$$f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2}{\partial y^2} f(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

It is clear how to define third-order partial derivatives, by repeated partial differentiation.

Example: Find the second-order partial derivatives of $f(x, y) = \dfrac{xy}{x+1}$ at $(1, 2)$.

In the previous example, we found that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. This is a general fact for well-behaved functions:

Theorem 1: Equality of Mixed Partial Derivatives: (Clairaut's Theorem) Suppose p, q are two kth-order partial derivatives of f obtained by differentiating with respect to the same set of k variables (possibly with repetition) but in different orders. Suppose also that p, q are continuous at (a, b), and all $k-1$th-order partial derivatives of f are continuous around (a, b). Then $p(a, b) = q(a, b)$.

Example: Clairaut's Theorem says that, if $f : \mathbb{R}^2 \to \mathbb{R}$ has third-order partial derivatives that are continuous everywhere, then there are only four different third-order partial derivatives:

$$f_{xxx};$$
$$f_{xxy} = f_{xyx} = f_{yxx};$$
$$f_{xyy} = f_{yxy} = f_{yyx};$$
$$f_{yyy}.$$

The proof of Clairaut's Theorem uses the 1D mean value theorem separately in the x and y directions - see the textbook.

§12.6: Linear Approximation and Differentiability

Remember that a single-variable function f is said to be differentiable at a point a if the derivative $f'(a)$ exists.

It is not a good idea to say that a multivariate function is differentiable if all its partial derivatives exist, since there are discontinuous functions that have partial derivatives. One example is $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$ On ex sheet #13 Q1, you showed that f does not have a limit at $(0,0)$, so f is not continuous, but because f is always 0 on the x and y axes, its partial derivatives do exist and are 0. (You will prove this carefully on Homework 4.)

The existence of partial derivatives means that f is well-behaved in the x and y directions, but f can still be horrible in other directions. A good definition of differentiability at (a,b) must be a statement about all points around (a,b).

We will say that f is differentiable if it is locally well-approximated by a linear function.

Let's first understand what this means for a single-variable function $f : \mathbb{R} \to \mathbb{R}$. Remember that the *linearisation* of f at a is

$$L(x) = f(a) + f'(a)(x-a).$$

This linear function is important because we can use it to approximate $f(x)$ when x is near a, i.e. when $x = a + h$ and h is small. This approximation is good because the error satisfies

$$\begin{aligned}
f(x) - L(x) &= f(x) - f(a) - f'(a)(x-a) \\
f(a+h) - L(a+h) &= f(a+h) - f(a) - f'(a)h \\
\frac{f(a+h) - L(a+h)}{h} &= \frac{f(a+h) - f(a)}{h} - f'(a) \\
\lim_{h\to 0} \frac{f(a+h) - L(a+h)}{h} &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} - f'(a) = 0
\end{aligned}$$

In other words, the error is small compared to h, the distance from a to x.

Remember again that the linearisation of $f:\mathbb{R}\to\mathbb{R}$ at a is $L(x) = f(a) + f'(a)(x-a)$.
A straightforward multivariate generalisation is:

Definition: The *linearisation* of a function $f(x,y)$ at (a,b) is

$$L(x,y) = f(a,b) + \underset{\substack{\uparrow \\ \text{rate of change of } f \\ \text{with respect to } x}}{f_x(a,b)}\ \underset{\substack{\uparrow \\ \text{change} \\ \text{in } x}}{(x-a)} + \underset{\substack{\uparrow \\ \text{rate of change of } f \\ \text{with respect to } y}}{f_y(a,b)}\ \underset{\substack{\uparrow \\ \text{change} \\ \text{in } y}}{(y-b)}.$$

And then f is differentiable if the error from using $L(x,y)$ to approximate $f(x,y)$ is small compared to the distance from (x,y) to (a,b):

Definition: A function $f(x,y)$ is *differentiable* at (a,b) if

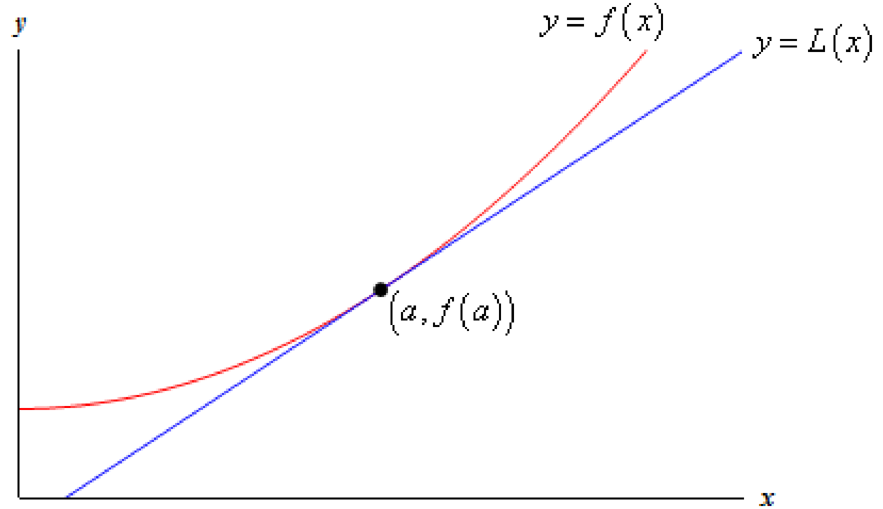
$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b-k) - f(a,b) - f_x(a,b)h - f_y(a,b)k}{\sqrt{h^2+k^2}} = 0.$$

It is clear how to generalise these definitions for n-variable scalar-valued functions. They also make sense for vector-valued functions, using addition and scalar-multiplication in $\mathbb{R}^m$ (see p35).

Before continuing with the theory of differentiability, let us make sure we understand the linearisation and its applications:

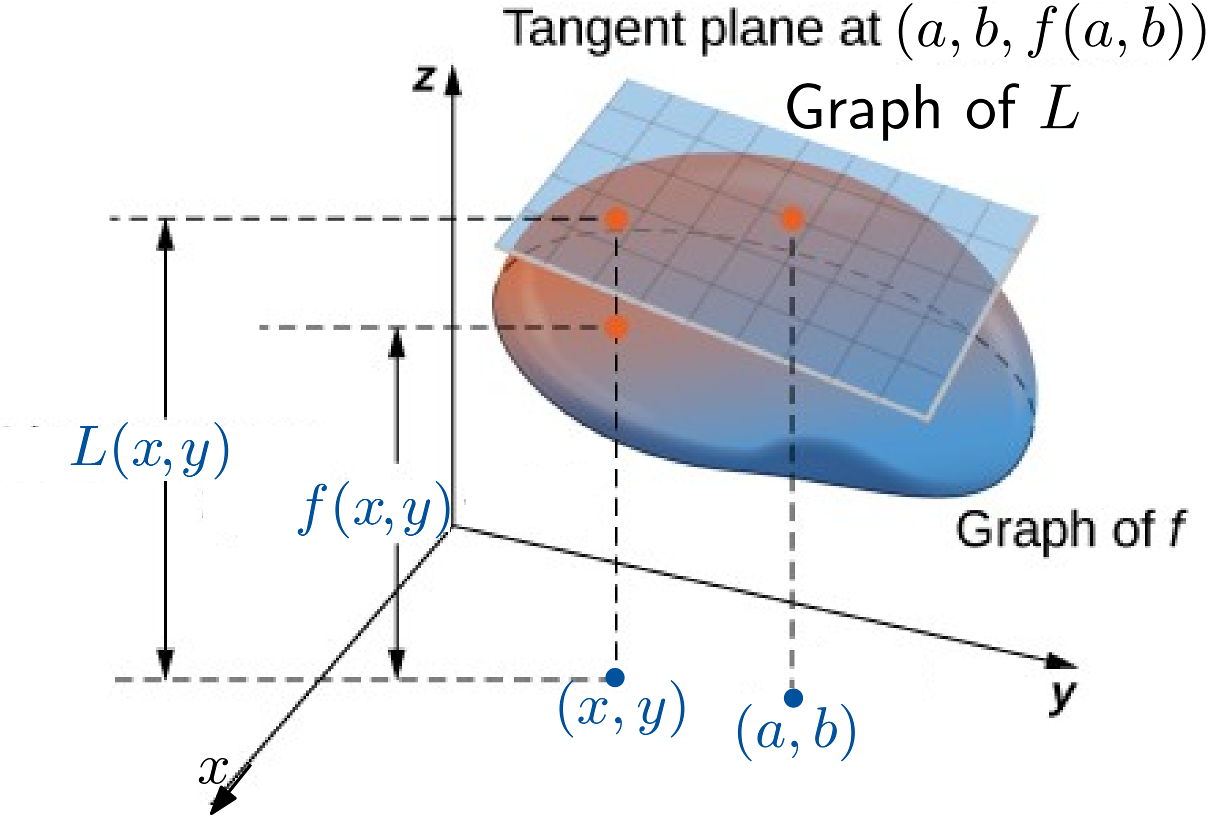
Example: Calculate the linearisation of $f(x, y) = x^2 y$ at $(1, 2)$, and use it to estimate $f(1.1, 1.8)$.

To motivate the second main application of linearisations, recall that the graph of a single-variable linearisation $y = f(a) + f'(a)(x - a)$ is the tangent line to the graph of f at $(a, f(a))$.



Consider the linearisation of a 2-variable function $f(x, y)$ at (a, b). Its graph is $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$, and this is the tangent plane to the graph of f at $(a, b, f(a, b))$.

In Week 9 (§12.7) we will see a more general method that computes tangent planes to any surface, not just to a graph.



(pictures from Paul's Online Math Notes and archive.cnx.org)

Back to differentiability: remember that

Definition: A function $f(x, y)$ is *differentiable* at (a, b) if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a,b) - f_x(a,b)h - f_y(a,b)k}{\sqrt{h^2+k^2}} = 0.$$

We will rarely need to use this definition to check if functions are differentiable, thanks to the following theorem:

Theorem 4: Continuous Partial Derivatives guarantee Differentiability:
Given a function $f : \mathbb{R}^n \to \mathbb{R}$, if all its partial derivatives $\dfrac{\partial f}{\partial x_i}$ are continuous around (a, b), then f is differentiable at (a, b).

The main idea of the proof (p30) is to write $f(a+h, b+k) - f(a, b)$ in the definition of differentiable in terms of partial derivatives, using a multivariate mean value theorem. On the next page we state precisely the 2D mean value theorem, you can imagine the analogous statement when the domain is $\mathbb{R}^n$.

Theorem 3: Mean Value Theorem (MVT): Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives f_x and f_y on a small disk around (a, b). If h, k are small enough that $(a + h, b + k)$ are in this disk, then there exist numbers θ_1, θ_2 between 0 and 1 such that

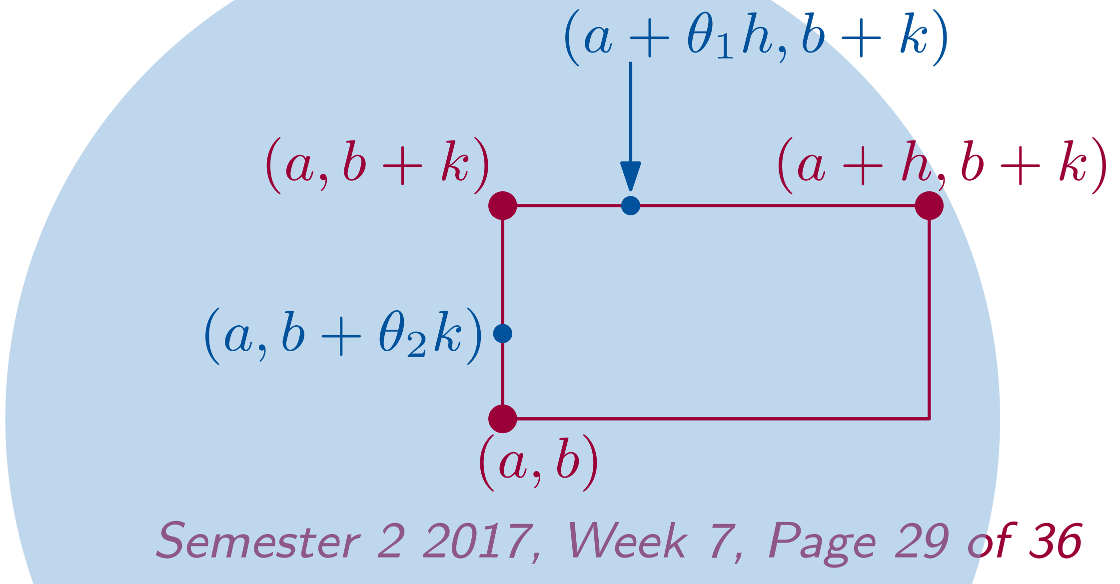
$$f(a+h, b+k) - f(a, b) = hf_x(a + \theta_1 h, b + k) + kf_y(a, b + \theta_2 k).$$

Proof:

$$f(a+h,b+k)-f(a,b) = \underbrace{f(a+h,b+k)-f(a,b+k)}_{\text{1D MVT in } x\text{-direction}} + \underbrace{f(a,b+k)-f(a,b)}_{\text{1D MVT in } y\text{-direction}}$$

$$= hf_x(a+\theta_1 h, b+k) \qquad + kf_y(a, b+\theta_2 k).$$

We are using the 1D MVT phrased as follows: if g is differentiable on $[a, a+h]$, then there is a point between a and $a+h$, i.e. a point $a + \theta h$ for θ between 0 and 1, such that $f(a+h) - f(a) = hf'(a + \theta h)$.



Proof: (of Theorem 4, sketch):
Recall that MVT says there are numbers θ_1, θ_2 between 0 and 1 with
$f(a+h, b+k) - f(a,b) = hf_x(a+\theta_1 h, b+k) + kf_y(a, b+\theta_2 k)$.

We now use this to show that, if $f(x,y)$ has continuous partial derivatives, then f is differentiable. So we need to show that

$$\frac{f(a+h, b-k) - f(a,b) - f_x(a,b)h - f_y(a,b)k}{\sqrt{h^2+k^2}} \to 0 \text{ as } (h,k) \to (0,0).$$

Using MVT to replace the first two terms in the numerator:

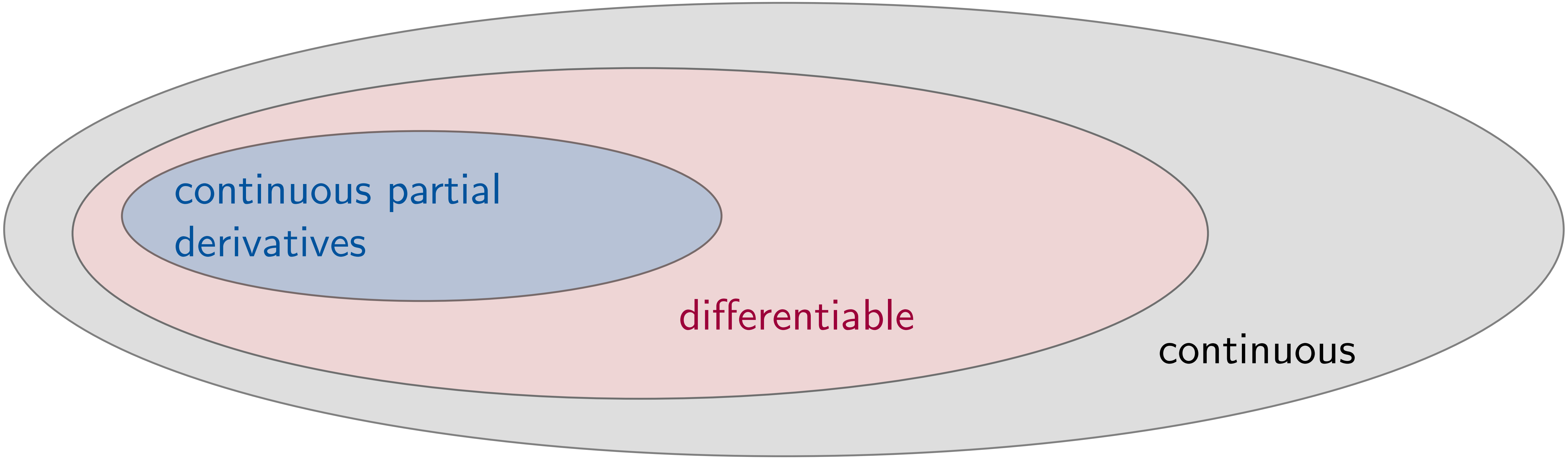
$$\frac{hf_x(a+\theta_1 h, b+k) + kf_y(a, b+\theta_2 k) - f_x(a,b)h - f_y(a,b)k}{\sqrt{h^2+k^2}}$$

$$= \underbrace{\frac{h}{\sqrt{h^2+k^2}}}_{\text{is finite.}} \underbrace{(f_x(a+\theta_1 h, b+k) - f_x(a,b))}_{\substack{\text{goes to 0 because} \\ f_x \text{ is continuous}}} + \underbrace{\frac{k}{\sqrt{h^2+k^2}}}_{\text{is finite.}} \underbrace{(f_y(a, b+\theta_2 k) - f_y(a,b))}_{\substack{\text{goes to 0 because} \\ f_y \text{ is continuous}}}$$

There is one more important result about differentiability - for simplicity we state it below for 2-variable functions, but it holds for any $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$:

Theorem: Differentiable Functions are Continuous: If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b), then it is continuous at (a, b).

So the hierachy of functions is as follows:



Theorem: Differentiable Functions are Continuous: If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b), then it is continuous at (a, b).

Proof: (sketch, same as the 1D proof):
We show that $\lim\limits_{(h,k)\to(0,0)} f(a+h, b+k) - f(a, b) = 0$.

$$
\begin{aligned}
& f(a+h,b+k) - f(a,b) \\
=& f(a+h,b+k) - f(a,b) - f_x(a,b)h - f_y(a,b)k + f_x(a,b)h + f_y(a,b)k \\
=& \underbrace{\sqrt{h^2+k^2}}_{\substack{\text{goes to 0}\\ \text{because it is}\\ \text{a continuous}\\ \text{function of}\\ (h,k)}} \underbrace{\frac{f(a+h,b+k) - f(a,b) - f_x(a,b)h - f_y(a,b)k}{\sqrt{h^2+k^2}}}_{\text{goes to 0 because } f \text{ is differentiable}} + \underbrace{f_x(a,b)h + f_y(a,b)k}_{\substack{\text{goes to 0 because}\\ \text{it is a continuous}\\ \text{function of } (h,k)}}
\end{aligned}
$$

Derivatives of vector-valued functions

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}^3$. We can compute the linearisation of its coordinate functions separately:

$$
\begin{aligned}
L_1(x,y) &= f_1(a,b) + \boxed{\left.\frac{\partial f_1}{\partial x}\right|_{(a,b)} (x-a) + \left.\frac{\partial f_1}{\partial y}\right|_{(a,b)} (y-b)} \\
L_2(x,y) &= f_2(a,b) + \boxed{\left.\frac{\partial f_2}{\partial x}\right|_{(a,b)} (x-a) + \left.\frac{\partial f_2}{\partial y}\right|_{(a,b)} (y-b)} \\
L_3(x,y) &= f_3(a,b) + \boxed{\left.\frac{\partial f_3}{\partial x}\right|_{(a,b)} (x-a) + \left.\frac{\partial f_3}{\partial y}\right|_{(a,b)} (y-b)}
\end{aligned}
$$

This is matrix multiplication $\begin{pmatrix} \left.\frac{\partial f_1}{\partial x}\right|_{(a,b)} & \left.\frac{\partial f_1}{\partial y}\right|_{(a,b)} \\ \left.\frac{\partial f_2}{\partial x}\right|_{(a,b)} & \left.\frac{\partial f_2}{\partial y}\right|_{(a,b)} \\ \left.\frac{\partial f_3}{\partial x}\right|_{(a,b)} & \left.\frac{\partial f_3}{\partial y}\right|_{(a,b)} \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix}$

So one way to organise the mn partial derivatives of a function $\mathbf{f}: \mathbb{R}^n \to \mathbb{R}^m$, that makes sense with linear algebra, is with an $m \times n$ matrix:

Definition: The *Jacobian matrix* $D\mathbf{f}(\mathbf{x})$ of a function $\mathbf{f}: \mathbb{R}^n \to \mathbb{R}^m$ is the $m \times n$ matrix with $\dfrac{\partial f_i}{\partial x_j}$ in row i and column j:

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

As observed on the previous page, we can write the linearisation of a vector-valued function using the Jacobian matrix:

$$\mathbf{L}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + \underbrace{D\mathbf{f}(\mathbf{a})(\mathbf{x}-\mathbf{a})}_{\text{matrix-vector multiplication}}.$$

Example: Calculate the Jacobian matrix of $\mathbf{f}(x, y) = \left(\frac{xy}{x+1}, x^2y, x\right)$ at $(1, 2)$, and use it to estimate $\mathbf{f}(1.1, 2.3)$.

Non-examinable: the derivative as a linear transformation

Recall that the Jacobian matrix is

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix},$$

and the linearisation is:

$$\mathbf{L}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x}-\mathbf{a}).$$

The linear transformation represented by the Jacobian matrix is called *the derivative* of $\mathbf{f}$. It allows a definition of differentiability without reference to coordinates: $\mathbf{f}$ is differentiable at $\mathbf{a}$ if there is a linear transformation $D\mathbf{f}(\mathbf{a})$ such that

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - D\mathbf{f}(\mathbf{a})(\mathbf{x}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|} = \mathbf{0}.$$