MATH 2207: Linear Algebra
Homework 1, due 15:45 Monday 29 January 2018

You must justify your answers to receive full credit.

1. Find a polynomial of degree 3 (a polynomial of the form $f(t) = a + bt + ct^2 + dt^3$) whose graph passes through the points $(0, 1)$, $(1, 0)$, $(-1, 0)$ and $(2, -15)$.

2. Consider a linear system whose augmented matrix is

$$\left[\begin{array}{ccccc|c} 3 & 0 & 6 & 0 & 0 & 9 \\ 2 & 2 & 4 & -1 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2 & 4 & 3 \end{array}\right].$$

   a) Find the reduced echelon form of this matrix.

   Now consider a linear system whose augmented matrix is

$$\left[\begin{array}{ccccc|c} 3 & 0 & 6 & 0 & 0 & 9 \\ 2 & 2 & 4 & -1 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & 3 & a & b \end{array}\right].$$

   b) For what values of a and b will the system have infinitely many solutions? (You may use your computations from part a.)

   c) For what values of a and b will the system be inconsistent?

3. Let

$$A = \begin{bmatrix} | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 4 & -2 \\ -2 & 8 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}.$$

   a) Is $\mathbf{b}$ in $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$? How many vectors are in $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$?

   b) Is $\mathbf{b}$ in Span $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$? How many vectors are in Span $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$?

   c) Is $\mathbf{a}_1$ in Span $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$? Explain your answer. (Hint: you do not need to do any calculation.)

4. Give, in parametric form, the solution to the linear system associated to the following augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 & 6 \end{array}\right].$$

5. Prove the following: if $\mathbf{w}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$ and $\mathbf{x}$ is in Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, then $\mathbf{x}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

6. State whether each of the following statements is always true or sometimes false. If it is true, give a brief justification (e.g. by referring to results from the textbook or from class); if it is false, give a numerical counterexample with an explanation.

   a) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be vectors in $\mathbb{R}^n$. If $\mathbf{v}_3$ is in Span$\{\mathbf{v_1}\}$, then $\mathbf{v}_3$ is in Span$\{\mathbf{v_1}, \mathbf{v_2}\}$.

   b) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be vectors in $\mathbb{R}^n$. If $\mathbf{v}_3$ is in Span$\{\mathbf{v_1}, \mathbf{v_2}\}$, then $\mathbf{v}_3$ is in Span$\{\mathbf{v_1}\}$.

   c) If rref(A) contains a row of zeros, then $A\mathbf{x} = \mathbf{b}$ is an inconsistent system.

   d) If $A\mathbf{x} = \mathbf{b}$ is an inconsistent system, then rref(A) contains a row of zeros.

   e) If A is an $m \times n$ matrix whose columns do not span $\mathbb{R}^m$, then $A\mathbf{x} = \mathbf{b}$ is inconsistent for all $\mathbf{b}$ in $\mathbb{R}^m$.

   f) Suppose A is an $m \times n$ matrix and there is a vector $\mathbf{b}$ in $\mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Then, if $\mathbf{c}$ is any vector such that $A\mathbf{x} = \mathbf{c}$ is consistent, then $A\mathbf{x} = \mathbf{c}$ has a unique solution.

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