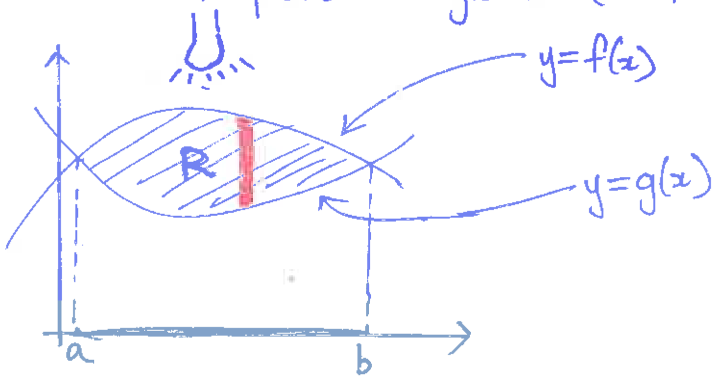
* Areas of planar regions (Week 4 p23–25, sheet #7 Q2)



$$\text{Area of } R = \int_a^b \underbrace{\overset{\text{top}}{f(x)} - \overset{\text{bottom}}{g(x)}}_{\text{area of red rectangle}}\, dx$$

a, b are where $y=f(x)$ and $y=g(x)$ intersect.
i.e. $f(x)=g(x)$

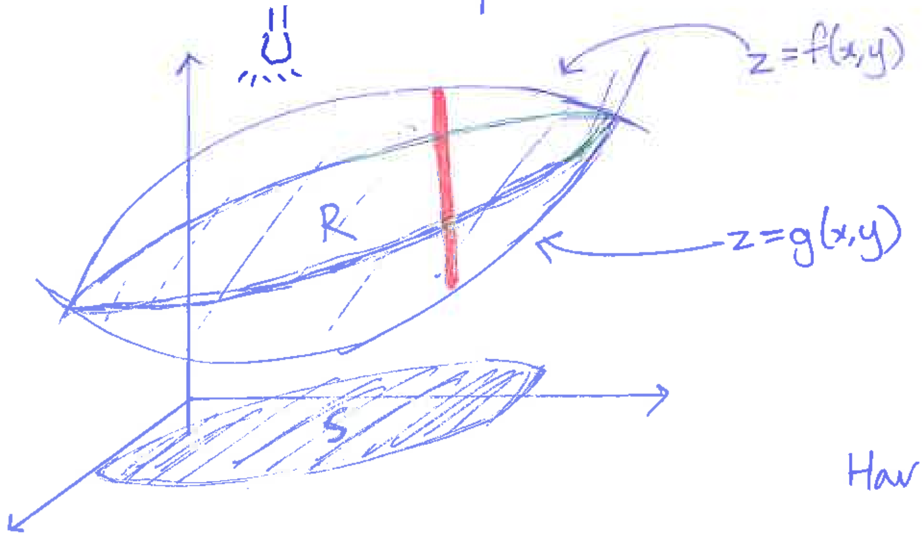
The domain of integration is $[a,b]$,
i.e. the "shadow" of R.

* Integrating over a 2D domain (Week 5 p17, sheet #8 Q2)

$$\text{e.g. mass of } R = \iint_R \delta(x,y)\, dA = \int_a^b \underbrace{\int_{g(x)}^{f(x)} \delta(x,y)\, dy}_{\text{mass of red rectangle}}\, dx$$

δ: density function

* Volumes of 3D regions (Week 5 p31,32, sheet #9Q2)
(in Cartesian or polar coordinates)



$$\text{Volume of } R = \iint_S \underbrace{\overset{\text{top}}{f(x,y)} - \overset{\text{bottom}}{g(x,y)}}_{\text{volume of red stick}} \, dA$$

↑ "shadow" of R on the xy-plane (projection)

How to find the shadow? We solve $f(x,y) = g(x,y)$ to find the boundary of S.

(Cases with ≥3 surfaces are more complicated (Week 5 p24.5, sheet #9Q1): usually, parts of the boundary of S: • cylinders (no z's)
• intersections of any surfaces with z's.

* Integrating over a 3D region

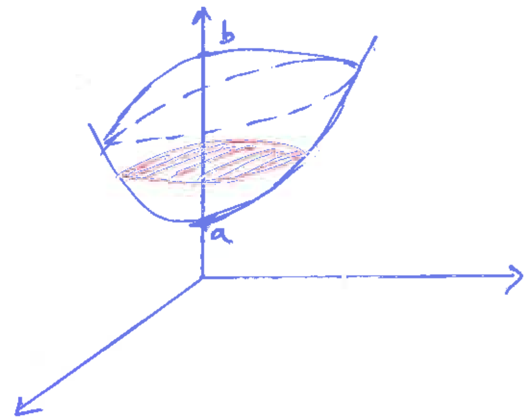
e.g. $\text{mass of } R = \iint_S \underbrace{\int_{g(x,y)}^{f(x,y)} \delta(x,y,z)\, dz}_{\text{mass of red stick}} \, dA$

Shadow method

Or: mass of $R = \int_a^b \underbrace{\iint_{CS(z)} \delta(x,y,z)\, dA}_{\text{mass of red plane}}\, dz$

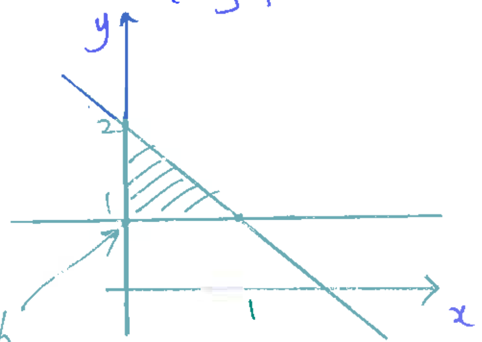
cross-section method

e.g. textbook §14.5 Ex.3



Tetrahedron: Sheet #10 Q1 $\underbrace{x=0,\ y=1}_{\text{no } z\text{'s: sides}},\ \underbrace{z=0,\ 2x+2y+z=4}_{\text{have } z\text{'s: tops/bottoms}}$.

Shadow: (xy-plane)



fourth corner above or below this point

boundary of shadow comes from

$x=0$

$y=1$

intersection of $z=0$ and $2x+2y+z=4$

i.e. $2x+2y=4$

To build the tetrahedron from the shadow, find the fourth corner: $(0,1,z)$ on $2x+2y+z=4$. $\rightarrow z=2$.

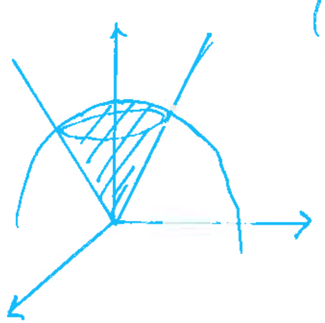
Example: Find the mass of the smaller region bounded by $z = \sqrt{3x^2 + 3y^2}$ and $x^2 + y^2 + z^2 = 1$, with density function $\delta(x, y, z) = x^2 z$.

$$\int_c^d \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta, z) r\, dr\, d\theta\, dz$$

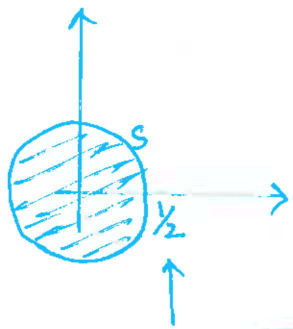
(from p32)

region is above the cone $z \geq \sqrt{3}r$

below the sphere $z \leq \sqrt{1-r^2}$



Projection to the xy plane:



from $\sqrt{3}r = \sqrt{1-r^2}$, this has solution $r = \frac{1}{2}$.

$$\text{mass} = \iint_S \int_{\sqrt{3}r}^{\sqrt{1-r^2}} (r\cos\theta)^2 z \, dz \, dA$$

$$= \int_0^{2\pi} \int_0^{1/2} (r\cos\theta)^2 \frac{z^2}{2} \Big|_{z=\sqrt{3}r}^{z=\sqrt{1-r^2}} r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{1/2} (r\cos\theta)^2 \left(\frac{1-r^2-3r^2}{2}\right) r\, dr\, d\theta$$

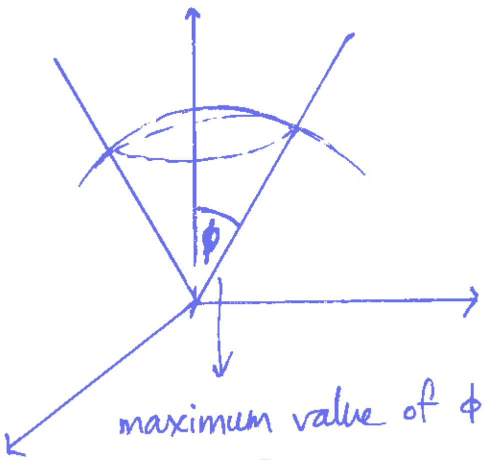
$$= \int_0^{2\pi} \int_0^{1/2} \cos^2\theta \left(\frac{r^3}{2} - 2r^5\right) dr\, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \left(\frac{r^4}{8} - \frac{2r^6}{6}\right)\Big|_{r=0}^{r=1/2} d\theta = \int_0^{2\pi} \cos^2\theta \frac{1}{384} d\theta$$

$$= \left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right)\frac{1}{384}\Big|_0^{2\pi} = \frac{\pi}{384}$$

$$\int_\gamma^\delta \int_\alpha^\beta \int_a^b f(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)R^2\sin\phi\, dR\, d\theta\, d\phi$$

Redo Example: (p46) Find the mass of the smaller region bounded by $z=\sqrt{3x^2+3y^2}$ and $x^2+y^2+z^2=1$, with density function $\delta(x,y,z)=x^2z$.



maximum value of ϕ:

$z=\sqrt{3}\,r$

$\frac{1}{\sqrt{3}}=\frac{r}{z}=\tan\phi$

$\phi=\frac{\pi}{6}$.

$$\int_0^{\frac{\pi}{6}}\int_0^{2\pi}\int_0^1 (R\sin\phi\cos\theta)^2(R\cos\phi)R^2\sin\phi\, dR\, d\theta\, d\phi$$

$$=\int_0^{\frac{\pi}{6}}\int_0^{2\pi} \frac{R^6}{6}\sin^3\phi\cos\phi\cos^2\theta\Big|_{R=0}^{R=1} d\theta\, d\phi$$

$$=\int_0^{\frac{\pi}{6}}\int_0^{2\pi}\frac{1}{6}\sin^3\phi\cos\phi\cos^2\theta\, d\theta\, d\phi$$

table

$$=\int_0^{\frac{\pi}{6}}\frac{1}{6}\sin^3\phi\cos\phi\,\frac{1}{2}(\theta+\sin\theta\cos\theta)\Big|_{\theta=0}^{\theta=2\pi} d\phi$$

$$=\frac{\pi}{6}\int_0^{\frac{\pi}{6}}\sin^3\phi\cos\phi\, d\phi$$

substitution $u=\sin\phi$
$du=\cos\phi$

$$=\frac{\pi}{6}\,\frac{\sin^4\phi}{4}\Big|_0^{\frac{\pi}{6}}=\frac{\pi}{6}\,\frac{1}{4}\left(\frac{1}{2}\right)^4=\frac{\pi}{384}$$