§8.1: Diagonal and Triangular form:

Review / Update: Let $\sigma \in L(V,V)$, $A = [\sigma]_{\mathcal{A}} \left(= [\sigma]_{\mathcal{A} \leftarrow \mathcal{A}} \right)$

Def: A is similar to B if $\exists$ invertible P such that $A = PBP^{-1}$

i.e. $\exists$ basis $\mathcal{B}$ such that $[\sigma]_{\mathcal{B}} = B$

$\left(P = [\iota]_{\mathcal{A} \leftarrow \mathcal{B}} \right)$

Def: A is diagonalisable if $\exists$ invertible P, diagonal D with $A = PDP^{-1}$.

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Def 8.1.2: (if $\dim V < \infty$) σ is diagonalisable if $\exists$ basis $\mathcal{B}$ such that $[\sigma]_{\mathcal{B}} = D$, a diagonal matrix i.e. Let $\mathcal{B}=\{\beta_1,\cdots,\beta_n\}$, then $[\sigma]_{\mathcal{B}} = \begin{pmatrix} | & & | \\ [\sigma(\beta_1)]_{\mathcal{B}} & \cdots & [\sigma(\beta_n)]_{\mathcal{B}} \\ | & & | \end{pmatrix} = \begin{pmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_n \end{pmatrix}$

first column: $[\sigma(\beta_1)]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longrightarrow \sigma(\beta_1) = \lambda_1\beta_1 + 0\beta_2 + \cdots + 0\beta_n$

similar in other columns: $\sigma(\beta_i) = \lambda_i\beta_i$

$\therefore$ σ is diagonalisable if and only if $\exists$ basis $\mathcal{B}=\{\beta_1,\cdots,\beta_n\}$ such that $\sigma(\beta_i)=\lambda_i\beta_i$ for some $\lambda_i \in F$.

α is a <u>λ-eigenvector</u> of A if $A\alpha=\lambda\alpha$ and $\alpha \neq \vec{0}$.

$\mathcal{E}_\lambda = \mathrm{Nul}(A-\lambda I)$ is the <u>λ-eigenspace</u> of A.

Def 8.1.2 β is λ-eigenvector of σ if $\sigma(\beta)=\lambda\beta$ and $\beta\neq\vec{0}$

$\mathcal{E}_\lambda = \ker(\sigma-\lambda\iota)$ is the λ-eigenspace of σ.

Note: β is a λ-eigenvector of $\sigma \Leftrightarrow [\beta]_{\mathcal{A}}$ is a λ-eigenvector of $A = {}_{\mathcal{A}}[\sigma]_{\mathcal{A}}$

if β is an eigenvector, then $\sigma(\mathrm{Span}\{\beta\}) \subseteq \mathrm{Span}\{\beta\}$.

To find eigenvectors, first find eigenvalues: solve the characteristic polynomial

C_A in textbook $\to$ $\chi_A(x) = \det(A - xI)$

Def 8.1.1: The characteristic polynomial of σ is $\chi_\sigma(x) = \det([\sigma]_{\mathcal{A}} - xI)$ for any basis $\mathcal{A}$

(it does not depend on $\mathcal{A}$: see 2207 week 10 p31)

Def: A is <u>triangularisable</u> if $\exists$ invertible P such that $P^{-1}AP$ is upper-triangular

σ is <u>triangularisable</u> if $\exists$ basis $\mathcal{B}$ such that $[\sigma]_{\mathcal{B}\leftarrow\mathcal{B}}$ is upper-triangular.

i.e. $$\begin{pmatrix} | & & | \\ [\sigma(\beta_1)]_{\mathcal{B}} & \cdots & [\sigma(\beta_n)]_{\mathcal{B}} \\ | & & | \end{pmatrix} = \begin{pmatrix} \lambda_1 & * & * & * \\ & \lambda_2 & & \\ & & \ddots & * \\ 0 & & & \lambda_n \end{pmatrix}$$

first column: $\sigma(\beta_1) = \lambda_1\beta_1 + 0\beta_2 + \cdots + 0\beta_n \in \operatorname{Span}\{\beta_1\}$.

second column: $\sigma(\beta_2) = *\beta_1 + \lambda_2\beta_2 + 0\beta_3 + \cdots 0\beta_n \in \operatorname{Span}\{\beta_1, \beta_2\}$

similarly: $\sigma(\beta_k) = *\beta_1 + \cdots + *\beta_{k-1} + \lambda_k\beta_k + 0\beta_{k+1} + \cdots + 0\beta_n \in \operatorname{Span}\{\beta_1, \beta_2, \cdots, \beta_k\}$

$\therefore \sigma(\operatorname{Span}\{\beta_1, \cdots, \beta_k\}) \overset{\text{Lemma}}{=} \operatorname{Span}\{\sigma(\beta_1), \cdots, \sigma(\beta_k)\} \subseteq \operatorname{Span}\{\beta_1, \cdots, \beta_k\}$ ($\because$ each $\sigma(\beta_i) \in \operatorname{Span}\{\beta_1, \cdots, \beta_k\}$ for $i \leq k$)

i.e. $\operatorname{Span}\{\beta_1, \cdots, \beta_k\}$ is an invariant subspace.

Def 8.1.4: A subspace $W \subseteq V$ is <u>invariant under σ</u> if $\sigma(W) \subseteq W$.

Advantages over diagonalisation:

- The eigenvalues are on the diagonal.
- Schur Theorem: every linear transformation over $\mathbb{C}$ is triangularisable (orthogonally, see §10.4).
(for other fields, e.g. $\{0,1\} = \mathbb{Z}_2$ – use a bigger field where all polynomials have solutions. – find an eigenvector, use induction on complement)
- Triangularisation is more stable on computers