

Remember that there are two rules for differentiating complicated functions: the chain rule and the product rule. (The quotient rule is a combination of these two rules, since $\frac{u}{v} = uv^{-1}$.)

Since FTC says that integration is antidifferentiation, we can derive from these differentiation rules two techniques of integration:

chain rule \longrightarrow method of substitution (this week, §5.6)

product rule \longrightarrow integration by parts (Week 7, §6.1)

These techniques are **not** rules. They do not give us the answer; they only **change our integral to a new integral**, which we hope will be easier to evaluate. There are no rules in integration: there is no guaranteed algorithm to integrate a function. Using the techniques require some creativity, and there are often multiple efficient ways to calculate the same integral.

§5.6: The Method of Substitution

(The letters used here are different from in the textbook.)

Recall the chain rule for differentiation:

$$\frac{d}{dx} F(g(x)) = F'(g(x))g'(x)$$

Take the antiderivative of both sides:

$$F(g(x)) + C = \int F'(g(x))g'(x) dx$$

Write u for $g(x)$:

$$F(u) + C = \int F'(u) \frac{du}{dx} dx$$

Write f for F' :

$$\int f(u) du = \int f(u) \frac{du}{dx} dx.$$

Hence, if we can identify a function $u(x)$ such that our integrand is a product, of the composition $f(u(x))$ and the derivative $\frac{du}{dx}$ then we can rewrite our integral as $\int f(u) du$.

$$\int f(u) \frac{du}{dx} dx = \int f(u) du.$$

(i.e. we can treat $\frac{du}{dx}$ formally like a fraction)

Example: Evaluate $\int \cos(x^3) 3x^2 dx$.

$$\int f(u) \frac{du}{dx} dx = \int f(u) du.$$

Example: Evaluate $\int e^{3x} dx$ using the substitution $u = 3x$.

$$\int f(u) \frac{du}{dx} dx = \int f(u) du.$$

Example: Evaluate $\int x\sqrt{1+x^2} dx$ using the substitution $u = 1 + x^2$.

$$\int f(u) \frac{du}{dx} dx = \int f(u) du.$$

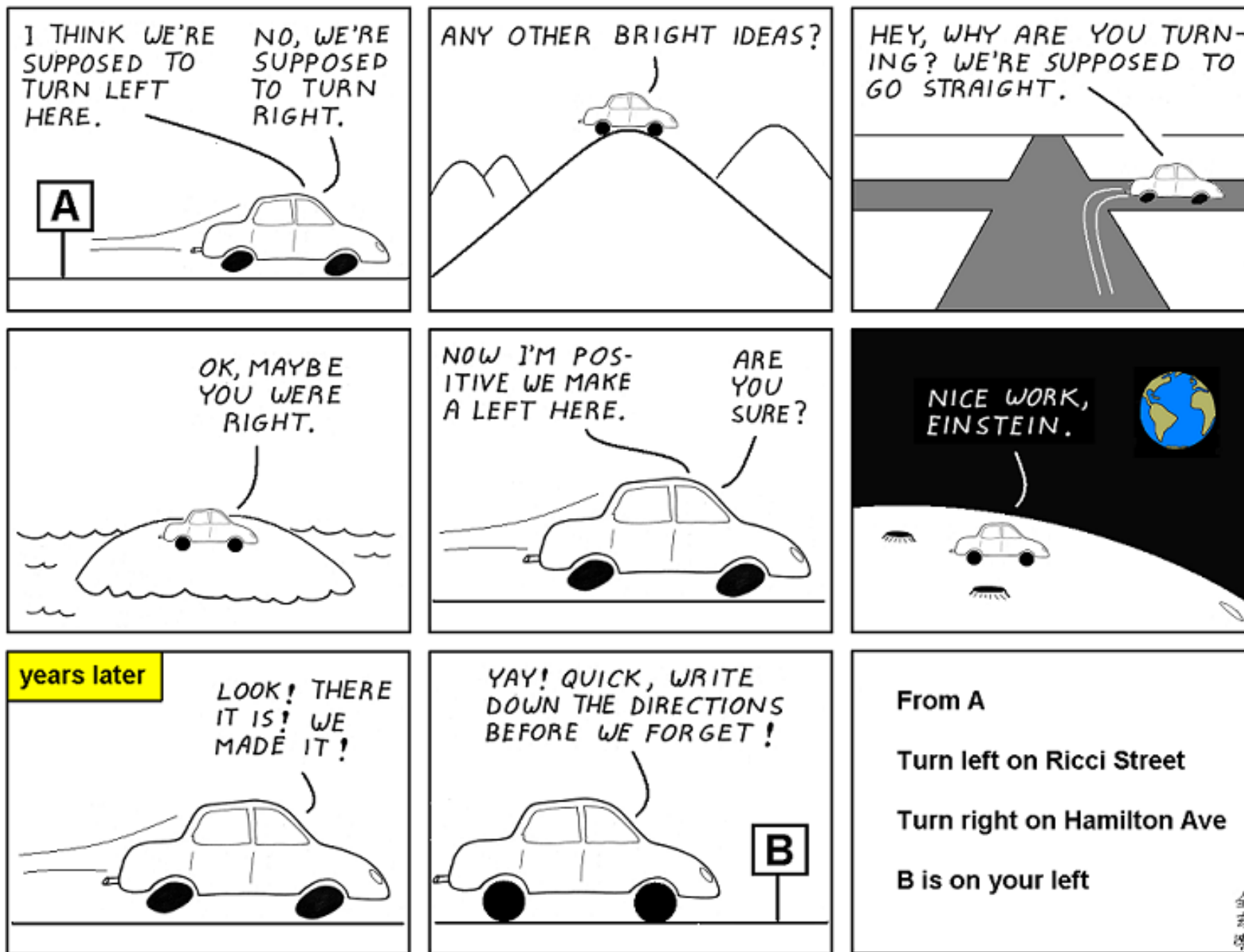
There are two skills involved in the method of substitution:

1. **Using** a substitution: first make sure you can get the right answer when you are given u (for indefinite and definite integrals, see p8-10). Cover the examples in the textbook except the first line to see their u , then try to finish the integral by yourself.

Very important: make sure your integrand is **entirely in terms of u** (no x s) before you start integrating.

2. **Choosing** a substitution: the only way to get better at choosing u is to do lots of problems, and **think about** why your chosen u was effective. Some tips:

- If the integrand contains a composite function e.g. $e^{g(x)}$, $\cos(g(x))$, $\sin(g(x))$, $\sqrt{g(x)}$, $\frac{1}{g(x)}$, try $u = g(x)$.
- If the integrand does not contain a composite function, or the tip above didn't work, then choose u to be “some part” of the integrand, preferably one where $\frac{du}{dx}$ also appears in the integrand.
- When you do more examples, you will discover more tips.



This is how most mathematical proofs are written.

Integration is problem-solving; there are no rules to follow. You have to try many ideas, and usually some of them will end up not being useful. If your friend or a textbook has a short, simple answer, that's only because they don't show all the ideas that didn't work.

(picture from Abstruse Goose)

Harder example: Evaluate $\int \frac{x^2}{1+x^6} dx$.

There are two ways to calculate a definite integral by substitution:

1. Find the indefinite integral and then substitute in the limits for x ;
2. (Usually faster) Change the limits into limits for u .

Example: (see p5) Evaluate $\int_0^1 x \sqrt{1+x^2} dx$.

Two other correct ways to use method 1:

$$\begin{aligned} & \int x \sqrt{1+x^2} \, dx \\ &= \int \frac{1}{2} \sqrt{u} \, du \\ &= \frac{u^{3/2}}{2(3/2)} + C \\ &= \frac{1}{3} \sqrt{1+x^2}^3 + C, \end{aligned}$$

so

$$\begin{aligned} & \int_0^1 x \sqrt{1+x^2} \, dx \\ &= \left. \frac{1}{3} \sqrt{1+x^2}^3 \right|_0^1 = \frac{1}{3} (\sqrt{2}^3 - 1). \end{aligned}$$

$$\begin{aligned} & \int_0^1 x \sqrt{1+x^2} \, dx \\ &= \int_{x=0}^{x=1} \frac{1}{2} \sqrt{u} \, du \\ &= \left. \frac{u^{3/2}}{2(3/2)} \right|_{x=0}^{x=1} \\ &= \left. \frac{1}{3} \sqrt{1+x^2}^3 \right|_0^1 = \frac{1}{3} (\sqrt{2}^3 - 1). \end{aligned}$$

Do **not** write $\int_0^1 \frac{1}{2} \sqrt{u} \, du$ - that would mean you want to evaluate at $u = 0, 1$.

Note that the final two steps in method 1 are to change the indefinite integral from us to x , then substitute the limits of x . In method 2 below, we combine these two steps – simply substitute the corresponding limits for u .

Redo Example: (p9) Evaluate $\int_0^1 x \sqrt{1 + x^2} dx$.

Harder example: Evaluate $\int_0^1 x^3 \sqrt{1-x^2} \, dx$.

Using various trigonometric identities and the method of substitution, we can obtain the integrals of many trigonometric functions - these are on the course formula sheet, and will be given to you on the exams.

Examples:

$$\begin{aligned}\int \cos^2 x \, dx &= \int \frac{1}{2}(1 + \cos(2x)) \, dx \\ &= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C \\ &= \frac{1}{2}x + \frac{1}{2}\sin x \cos x + C.\end{aligned}$$

by the identity $\cos(2x) = 2\cos^2 x - 1$

substitution $u = 2x$ in the second term

by the identity $\sin(2x) = 2\sin x \cos x$

$$\begin{aligned}\int \cos^3 x \, dx &= \int \cos x(1 - \sin^2 x) \, dx \\ &= \int \cos x - \cos x \sin^2 x \, dx \\ &= \sin x - \frac{1}{3}\sin^3 x + C.\end{aligned}$$

by the identity $\cos^2 x + \sin^2 x = 1$

substitution $u = \sin x$ in the second term

The full list of trigonometric-power integrals on the formula sheet:

$$\int \sin^2 x \, dx = \frac{1}{2}(x - \sin x \cos x) + C,$$

$$\int \sin^3 x \, dx = -\cos x + \frac{1}{3} \cos^3 x + C.$$

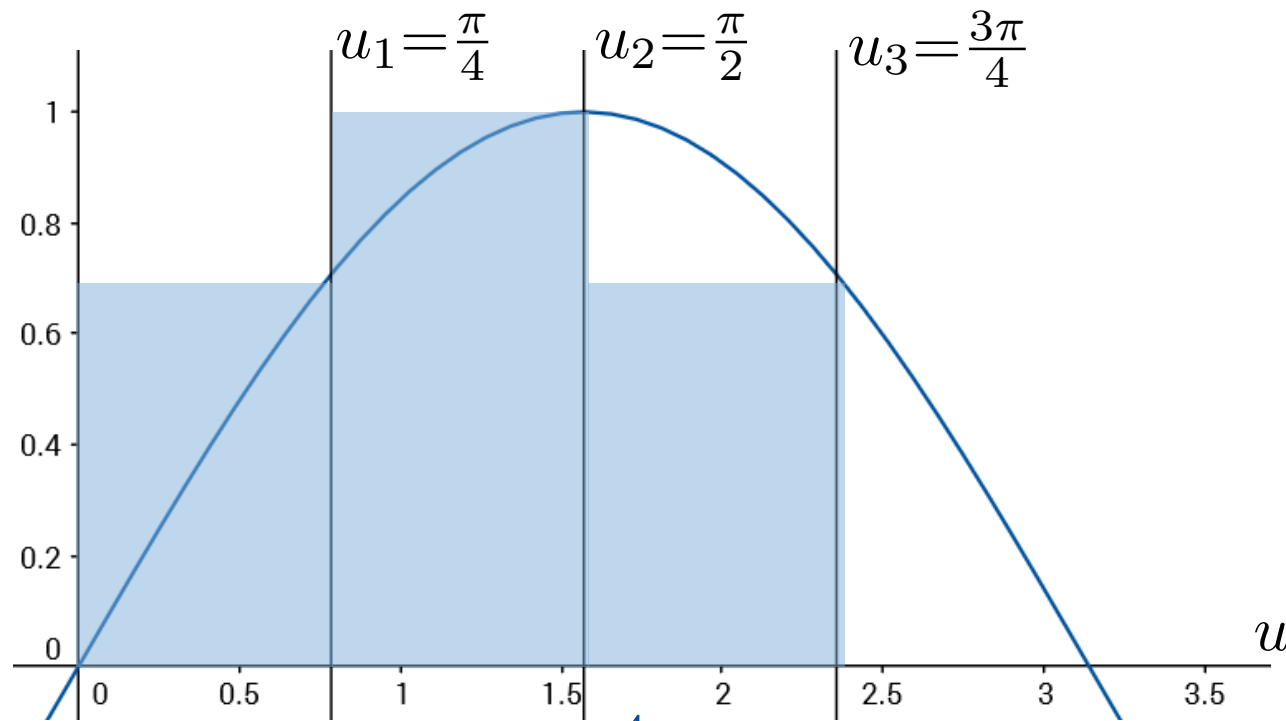
$$\int \sin^4 x \, dx = \frac{1}{8}(3x - 3 \sin x \cos x - 2 \sin^3 x \cos x) + C,$$

$$\int \cos^2 x \, dx = \frac{1}{2}(x + \sin x \cos x) + C,$$

$$\int \cos^3 x \, dx = \sin x - \frac{1}{3} \sin^3 x + C.$$

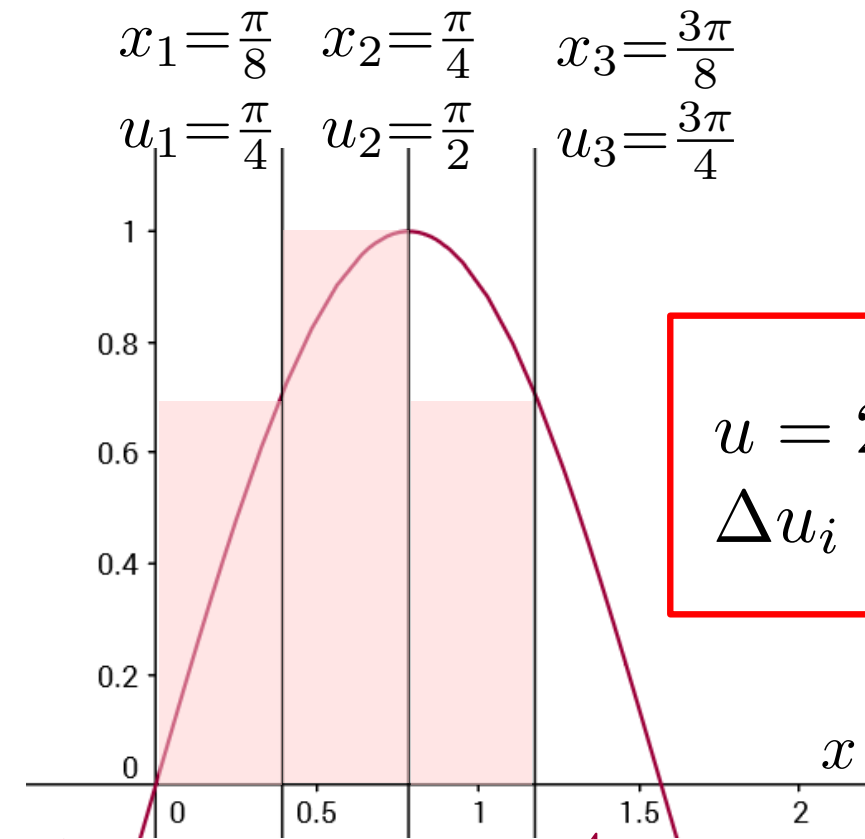
$$\int \cos^4 x \, dx = \frac{1}{8}(3x + 3 \sin x \cos x + 2 \cos^3 x \sin x) + C.$$

To prepare for a multivariate version of substitution (see the final week), we need to understand single-variable substitution geometrically, i.e. in terms of approximating the area under a curve with Riemann sums:



$$\int_0^{\pi} \sin u \, du \approx \sum_{i=1}^4 \sin(u_i) \Delta u_i.$$

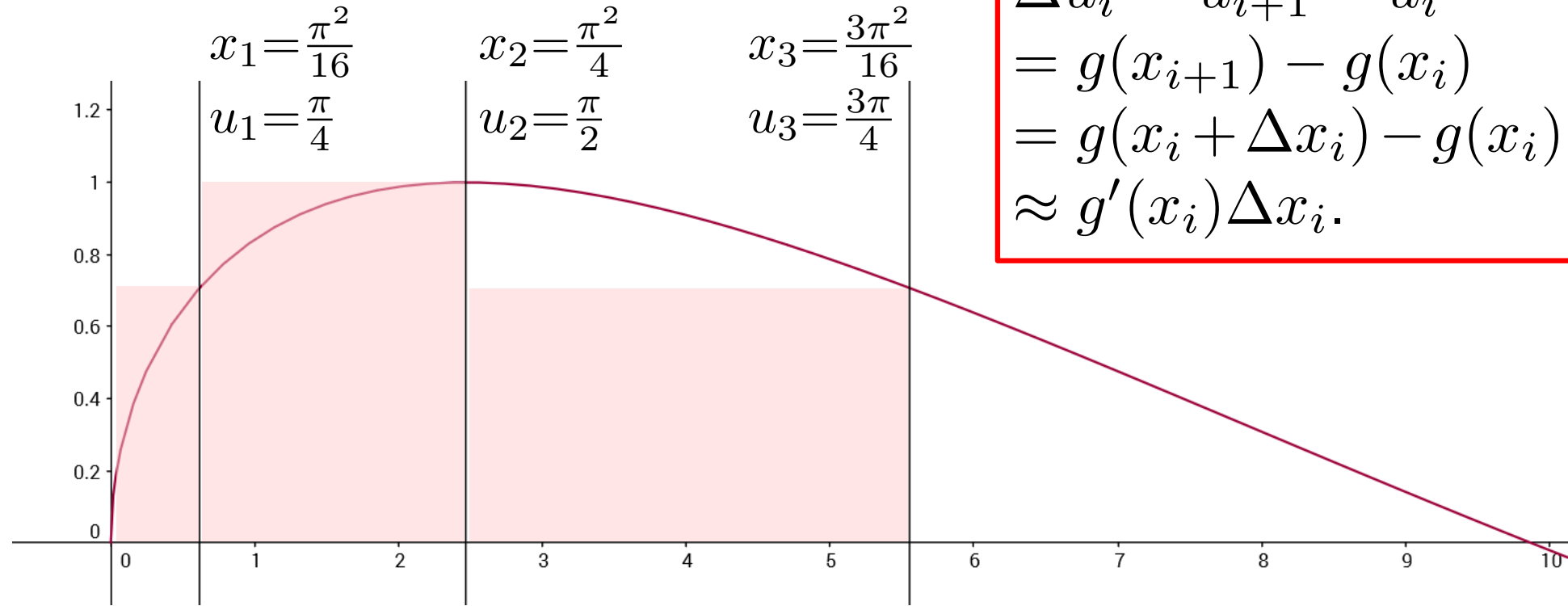
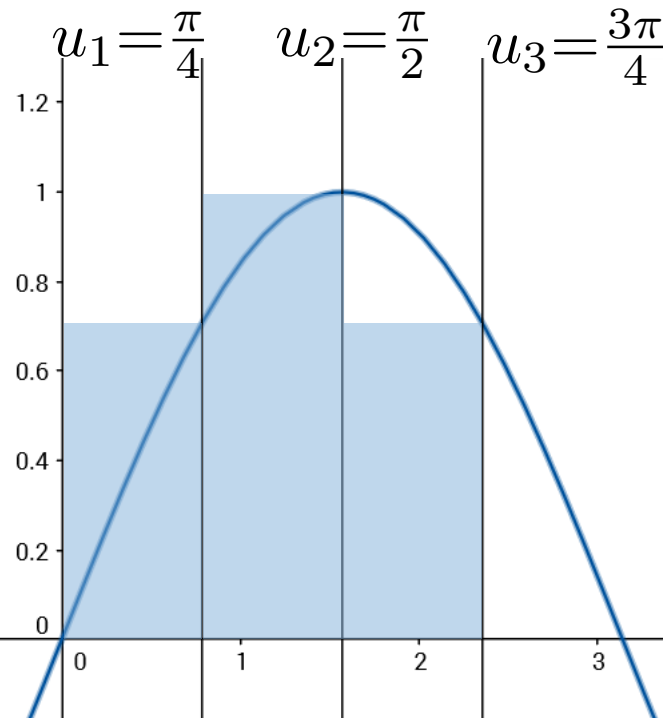
The heights of the two sets of approximating rectangles are the same, but on the right the rectangles are half as wide.



$$u = 2x, \text{ and } \Delta u_i = 2\Delta x_i.$$

$$\begin{aligned} \int_0^{\pi/2} \sin(2x) \, dx &\approx \sum_{i=1}^4 \sin(2x_i) \Delta x_i \\ &= \sum_{i=1}^4 \sin(u_i) \frac{1}{2} \Delta u_i \approx \int_0^{\pi} \sin u \, \frac{1}{2} du. \end{aligned}$$

When u is not a linear function of x , the widths of the rectangles stretch by different amounts.



When $u = g(x)$, then
 $\Delta u_i = u_{i+1} - u_i$
 $= g(x_{i+1}) - g(x_i)$
 $= g(x_i + \Delta x_i) - g(x_i)$
 $\approx g'(x_i) \Delta x_i.$

$$\int_0^\pi \sin u \, du \approx \sum_{i=1}^4 \sin(u_i) \Delta u_i.$$

In this example, $u = \sqrt{x}$, so

$$\Delta u_i \approx \frac{1}{2\sqrt{x_i}} \Delta x_i = \frac{1}{2u} \Delta x_i.$$

$$\int_0^{\pi^2} \sin \sqrt{x} \, dx \approx \sum_{i=1}^4 \sin \sqrt{x_i} \Delta x_i$$

$$\approx \sum_{i=1}^4 \sin(u_i) 2u \Delta u_i \approx \int_0^\pi \sin u \, 2u \, du.$$