10.5 Orthogonal diagonalisation of normal operators

Def 10.5.1 $\sigma : V \to V$ is <u>normal</u> if $\sigma \circ \sigma^* = \sigma^* \circ \sigma$

Def 10.5.13: $A \in M_{n \times n}(\mathbb{F})$ is <u>normal</u> if $A\overline{A^T} = \overline{A^T}A$

Normal operators include:

- self-adjoint operators (symmetric / Hermitian matrices)
- <u>unitary</u> operators: $\sigma^* = \sigma^{-1}$ (orthogonal / <u>unitary</u> matrices: $\overline{A^T} = A^{-1}$)

Spectral theorem for normal operators :

Th 10.5.10: if $\sigma : V \to V$ is normal, then σ has an orthonormal basis of eigenvectors.

Th 10.5.11: furthermore, if σ is self-adjoint, then all eigenvalues of σ are real.

(Actually, both are iff.)

Proof: Th. 10.5.4, a stronger version of "furthermore":

if $\sigma : V \to V$ is normal, and ξ is a λ-eigenvector of σ, then ξ is a $\bar{\lambda}$-eigenvector of σ^*

($\therefore$ if $\sigma = \sigma^*$ then $\lambda = \bar{\lambda}$)

Proof: we show $\sigma^*(\xi)=\bar{\lambda}\xi$ i.e. $\sigma^*(\xi)-\bar{\lambda}\xi=\vec{0}$

$$\|\sigma^*(\xi)-\bar{\lambda}\xi\|^2 = \langle \sigma^*(\xi)-\bar{\lambda}\xi,\ \sigma^*(\xi)-\bar{\lambda}\xi\rangle$$
$$=\langle\sigma^*(\xi),\sigma^*(\xi)\rangle-\langle\bar{\lambda}\xi,\sigma^*(\xi)\rangle-\langle\sigma^*(\xi),\bar{\lambda}\xi\rangle+\langle\bar{\lambda}\xi,\bar{\lambda}\xi\rangle$$
$$=\langle\sigma^*(\xi),\sigma^*(\xi)\rangle-\lambda\langle\xi,\sigma^*(\xi)\rangle-\bar{\lambda}\langle\sigma^*(\xi),\xi\rangle+\lambda\bar{\lambda}\langle\xi,\xi\rangle$$
$$=\langle\sigma^*(\xi),\sigma^*(\xi)\rangle-\lambda\langle\xi,\sigma^*(\xi)\rangle-\bar{\lambda}\langle\sigma^*(\xi),\xi\rangle+\lambda\bar{\lambda}\langle\xi,\xi\rangle$$
$$=\langle\xi,\sigma\circ\sigma^*(\xi)\rangle-\lambda\langle\sigma(\xi),\xi\rangle-\bar{\lambda}\langle\xi,\sigma(\xi)\rangle+\lambda\bar{\lambda}\langle\xi,\xi\rangle$$

$$\langle\xi,\sigma\circ\sigma^*(\xi)\rangle = \langle\xi,\sigma^*\circ\sigma(\xi)\rangle\ [\because \sigma \text{ is normal}] = \langle\sigma(\xi),\sigma(\xi)\rangle$$

$$=\langle\sigma(\xi),\sigma(\xi)\rangle-\langle\sigma(\xi),\lambda\xi\rangle-\langle\lambda\xi,\sigma(\xi)\rangle+\langle\lambda\xi,\lambda\xi\rangle$$
$$=\|\sigma(\xi)-\lambda\xi\|^2=0 \quad \therefore \xi \text{ is a } \lambda\text{-eigenvector.}$$

(Equivalently: if τ is normal, then $\ker\tau=\ker\tau^*$ — Apply to $\tau=\sigma-\lambda\iota$, $\tau^*=\sigma^*-\bar{\lambda}\iota$.

Now back to Th10.5.10:

(see also point 3 from last page of 2207)

idea: find an eigenvector ξ_1

consider $W_1=(\mathrm{Span}\{\xi_1\})^\perp$

In W_1, find an eigenvector ξ_2

$\therefore \xi_2\in W_1$, we have $\langle\xi_1,\xi_2\rangle=0$

Consider $W_2=(\mathrm{Span}\{\xi_1,\xi_2\})^\perp$

In W_2, find an eigenvector...

use induction on $\dim V$

i.e. apply inductive hypothesis to

$\sigma|_{W_1}: W_1\longrightarrow W_1$

To apply inductive hypothesis to $\sigma|_{W_1}$, we need:

a) W_1 is invariant under σ

(i.e. we can let the codomain of $\sigma|_{W_1}$ be W_1 instead of V)

b) $\sigma|_{W_1}$ is normal — which follows from W_1 being invariant under σ^* also (then $(\sigma|_{W_1})^*=\sigma^*|_{W_1}$)

To show W_1 is invariant under σ, i.e.

if $\alpha\in W_1=(\mathrm{Span}\{\xi_1\})^\perp$, then $\sigma(\alpha)\in W_1$

$$\langle\sigma(\alpha),\xi_1\rangle=\langle\alpha,\sigma^*(\xi_1)\rangle \quad [\text{property of adjoint}]$$
$$=\langle\alpha,\bar{\lambda}\xi_1\rangle \quad [\text{Th.10.5.4, writing } \lambda \text{ for the eigenvalue of } \xi_1 \text{ for } \sigma]$$
$$=\bar{\lambda}\langle\alpha,\xi_1\rangle$$
$$=0 \quad [\because \alpha\in W_1=(\mathrm{Span}\{\xi_1\})^\perp]$$

Same argument shows W_1 invariant under σ^*.