Remember that there are two rules for differentiating complicated functions: the chain rule and the product rule. (The quotient rule is a combination of these two rules, since $\frac{u}{v} = uv^{-1}$.)

Since FTC says that integration is antidifferentiation, we can derive from these differentiation rules two techniques of integration:

chain rule $\longrightarrow$ method of substitution (this week, §5.6)
product rule $\longrightarrow$ integration by parts (Week 7, §6.1)

These techniques are not rules. They do not give us the answer; they only change our integral to a new integral, which we hope will be easier to evaluate. There are no rules in integration: there is no guaranteed algorithm to integrate a function. Using the techniques require some creativity, and there are often multiple efficient ways to calculate the same integral.

§5.6: The Method of Substitution

(The letters used here are different from in the textbook.)

Recall the chain rule for differentiation: $$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x)$$

Take the antiderivative of both sides: $$F(g(x)) + C = \int F'(g(x))g'(x)\,dx$$

Write u for $g(x)$: $$F(u) + C = \int F'(u)\frac{du}{dx}\,dx$$

Write f for F': $$\int f(u)\,du = \int f(u)\frac{du}{dx}\,dx.$$

Hence, if we can identify a function $u(x)$ such that our integrand is a product, of the composition $f(u(x))$ and the derivative $\frac{du}{dx}$ then we can rewrite our integral as $\int f(u)\,du$.

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du.$$

(i.e. we can treat $\frac{du}{dx}$ formally like a fraction)

Example: Evaluate $\int \cos(x^3)\, 3x^2\, dx$.

$$\int f(u) \frac{du}{dx}\, dx = \int f(u)\, du.$$

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du.$$

Example: Evaluate $\displaystyle\int e^{3x}dx$ using the substitution $u = 3x$.

Example: Evaluate $\int x\sqrt{1+x^2}\,dx$ using the substitution $u = 1 + x^2$.

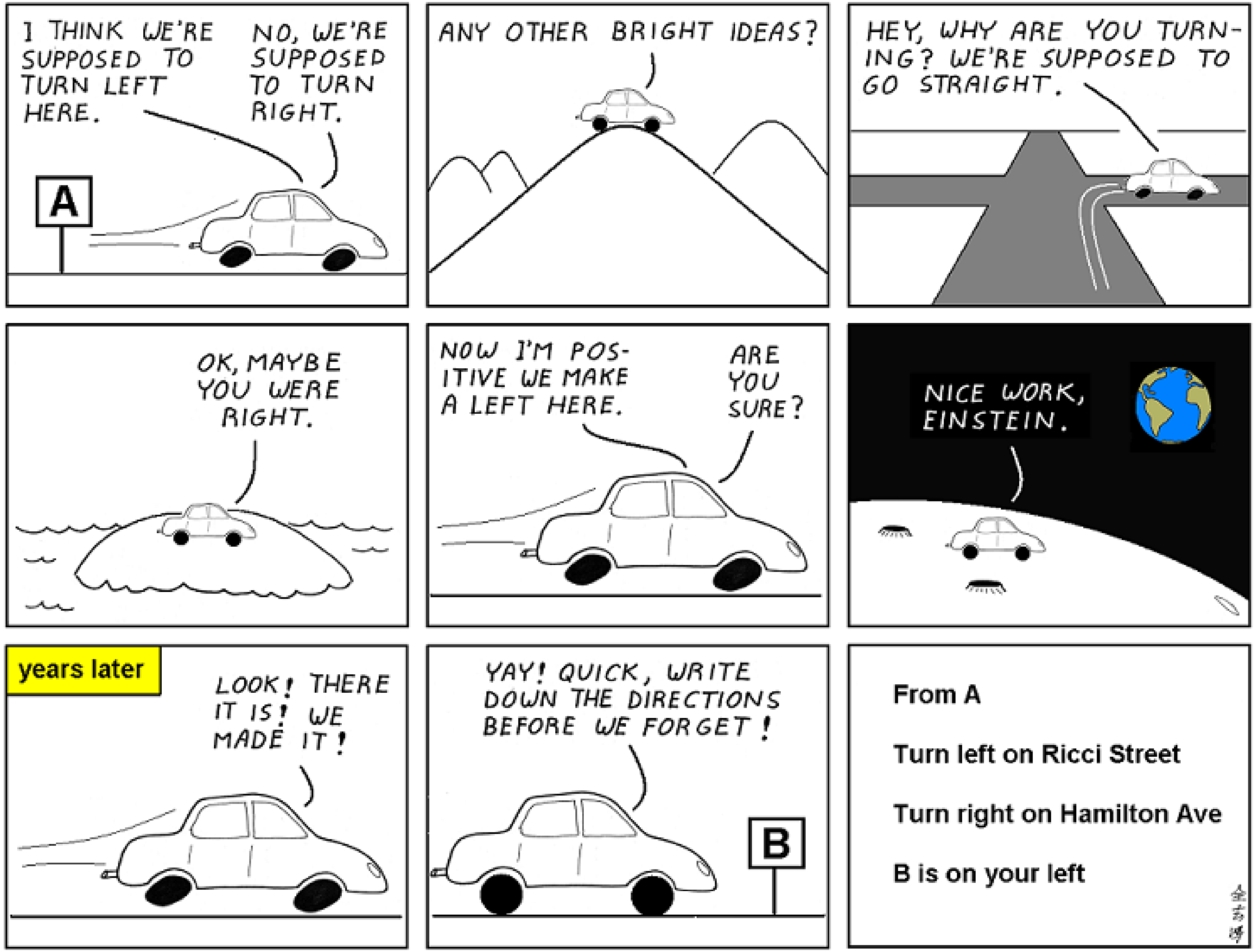
$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du.$$

There are two skills involved in the method of substitution:
1. Using a substitution: first make sure you can get the right answer when you are given u (for indefinite and definite integrals, see p9-11). Cover the examples in the textbook except the first line to see their u, then try to finish the integral by yourself.

Very important: make sure your integrand is entirely in terms of u (no xs) before you start integrating.

2. Choosing a substitution: the only way to get better at choosing u is to do lots of problems, and think about why your chosen u was effective. Some tips:

- If the integrand contains a composite function e.g. $e^{g(x)}$, $\cos(g(x))$, $\sin(g(x))$, $\sqrt{g(x)}$, $\frac{1}{g(x)}$, try $u = g(x)$.
- If the integrand does not contain a composite function, or the tip above didn't work, then choose u to be "some part" of the integrand, preferably one where $\frac{du}{dx}$ also appears in the integrand.
- When you do more examples, you will discover more tips.



This is how most mathematical proofs are written.

Integration is problem-solving; there are no rules to follow. You have to try many ideas, and usually some of them will end up not being useful. If your friend or a textbook has a short, simple answer, that's only because they don't show all the ideas that didn't work.

(picture from Abstruse Goose)

Harder example: Evaluate $\displaystyle\int \frac{x^2}{1+x^6}\,dx$.

There are two ways to calculate a definite integral by substitution:

1. Find the indefinite integral and then substitute in the limits for x;
2. (Usually faster) Change the limits into limits for u.

Example: (see p5) Evaluate $\displaystyle\int_0^1 x\sqrt{1+x^2}\,dx$.

Two other correct ways to use method 1:

$$\int x\sqrt{1+x^2}\,dx$$
$$=\int \frac{1}{2}\sqrt{u}\,du$$
$$=\frac{u^{3/2}}{2(3/2)}+C$$
$$=\frac{1}{3}\sqrt{1+x^2}^3+C,$$

so
$$\int_0^1 x\sqrt{1+x^2}\,dx$$
$$=\frac{1}{3}\sqrt{1+x^2}^3\Big|_0^1=\frac{1}{3}(\sqrt{2}^3-1).$$

$$\int_0^1 x\sqrt{1+x^2}\,dx$$
$$=\int_{x=0}^{x=1}\frac{1}{2}\sqrt{u}\,du$$
$$=\frac{u^{3/2}}{2(3/2)}\Big|_{x=0}^{x=1}$$
$$=\frac{1}{3}\sqrt{1+x^2}^3\Big|_0^1=\frac{1}{3}(\sqrt{2}^3-1).$$

Do not write $\int_0^1 \frac{1}{2}\sqrt{u}\,du$ - that would mean you want to evaluate at $u=0,1$.

Note that the final two steps in method 1 are to change the indefinite integral from us to x, then substitute the limits of x. In method 2 below, we combine these two steps – simply substitute the corresponding limits for u.

Redo Example: (p9) Evaluate $\displaystyle\int_0^1 x\sqrt{1+x^2}\,dx$.

Harder example: Evaluate $\displaystyle\int_0^1 x^3\sqrt{1-x^2}\,dx$.

Using various trigonometric identities and the method of substitution, we can obtain the integrals of many trigonometric functions - these are on the course formula sheet, and will be given to you on the exams.

Examples:

$$\int \cos^2 x\,dx = \int \frac{1}{2}(1+\cos(2x))\,dx \qquad \text{by the identity } \cos(2x) = 2\cos^2 x - 1$$

$$= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C \qquad \text{substitution } u = 2x \text{ in the second term}$$

$$= \frac{1}{2}x + \frac{1}{2}\sin x \cos x + C. \qquad \text{by the identity } \sin(2x) = 2\sin x \cos x$$

$$\int \cos^3 x\,dx = \int \cos x(1-\sin^2 x)\,dx \qquad \text{by the identity } \cos^2 x + \sin^2 x = 1$$

$$= \int \cos x - \cos x \sin^2 x\,dx$$

$$= \sin x - \frac{1}{3}\sin^3 x + C. \qquad \text{substitution } u = \sin x \text{ in the second term}$$

The full list of trigonometric-power integrals on the formula sheet:

$$\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x \cos x) + C,$$

$$\int \sin^3 x\,dx = -\cos x + \frac{1}{3}\cos^3 x + C.$$

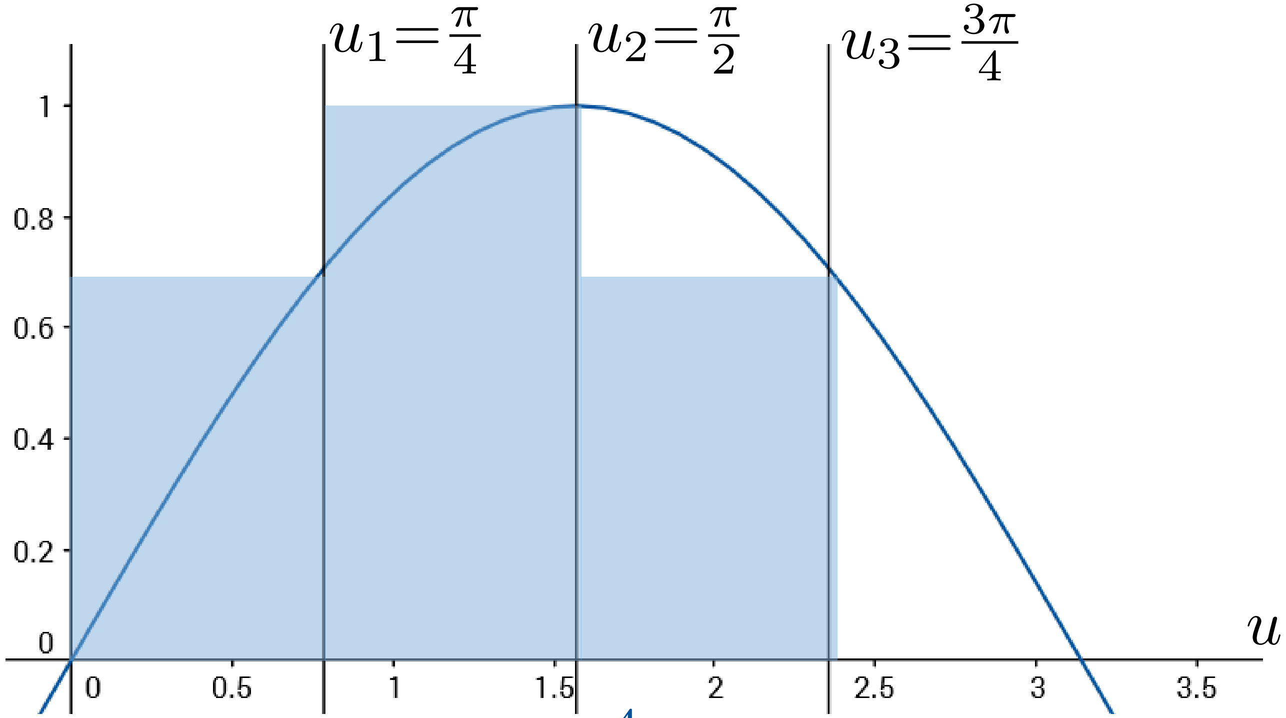
$$\int \sin^4 x\,dx = \frac{1}{8}(3x - 3\sin x \cos x - 2\sin^3 x \cos x) + C,$$

$$\int \cos^2 x\,dx = \frac{1}{2}(x + \sin x \cos x) + C,$$

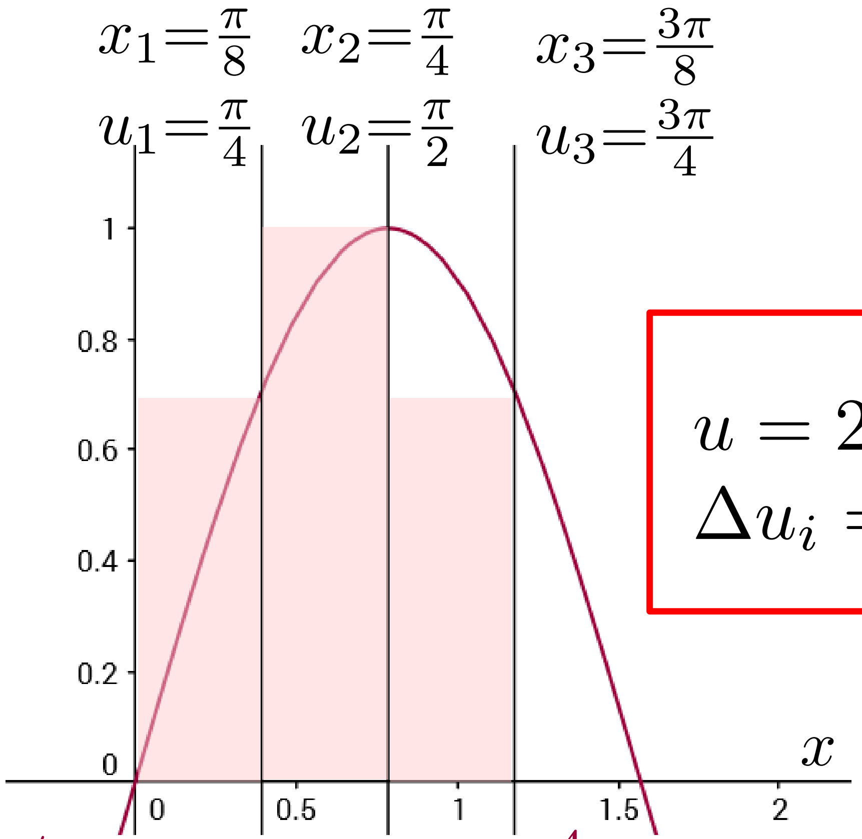
$$\int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x + C.$$

$$\int \cos^4 x\,dx = \frac{1}{8}(3x + 3\sin x \cos x + 2\cos^3 x \sin x) + C.$$

To prepare for a multivariate version of substitution (see the final week), we need to understand single-variable substitution geometrically, i.e. in terms of approximating the area under a curve with Riemann sums:



$$\int_0^{\pi} \sin u \, du \approx \sum_{i=1}^{4} \sin(u_i) \Delta u_i.$$



$u = 2x$, and
$\Delta u_i = 2\Delta x_i$.

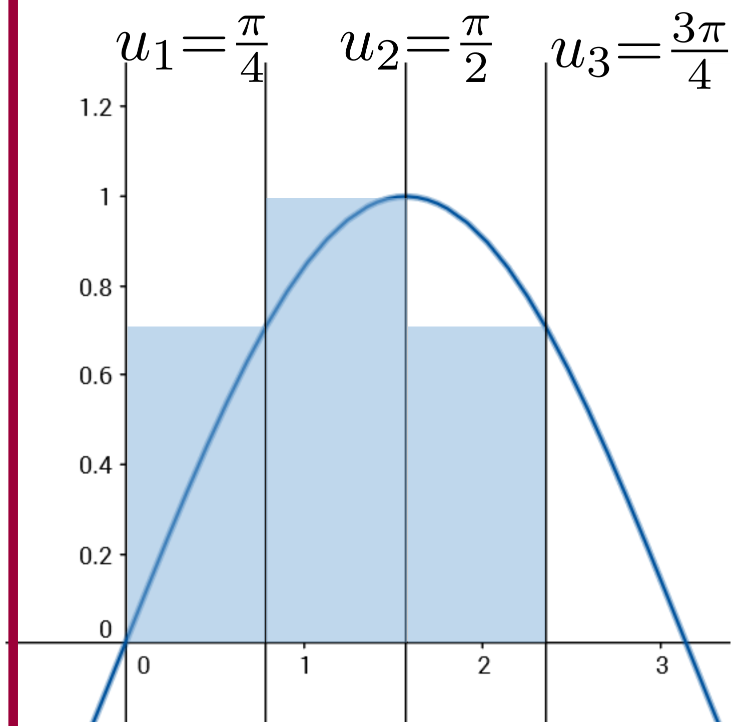
$$\int_0^{\pi/2} \sin(2x) \, dx \approx \sum_{i=1}^{4} \sin(2x_i) \Delta x_i$$

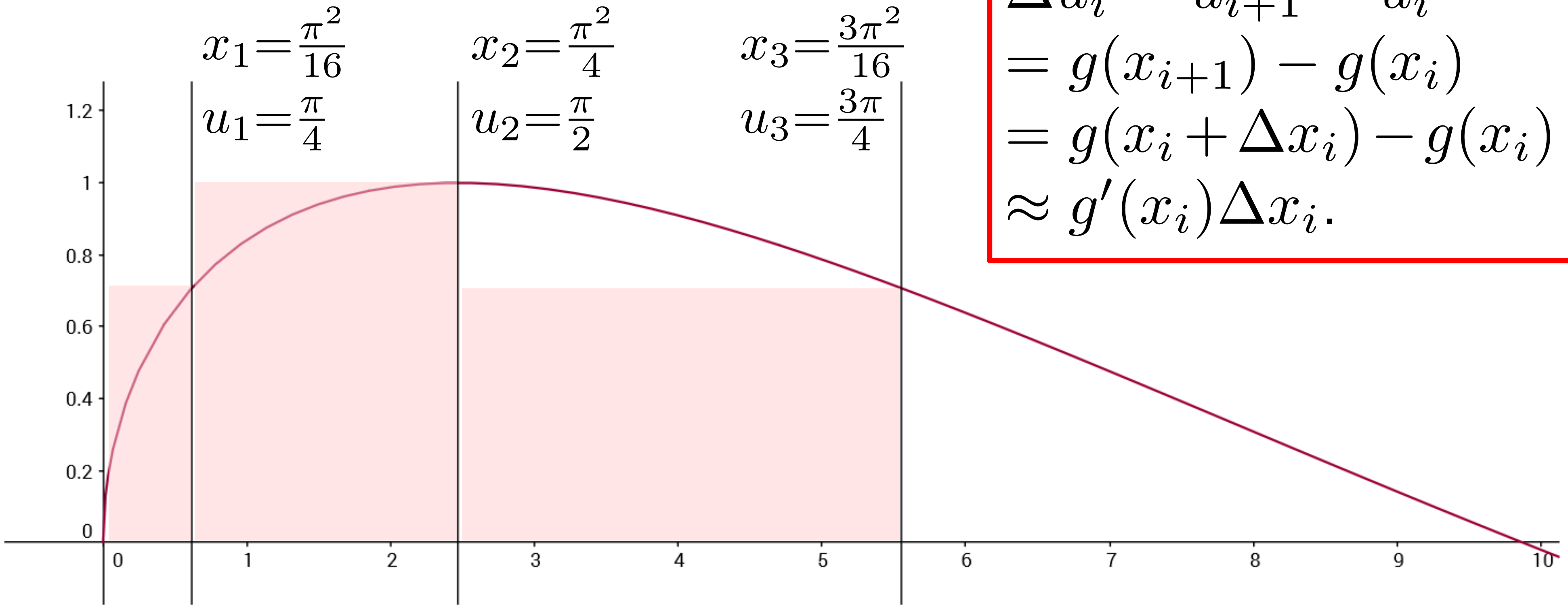
$$= \sum_{i=1}^{4} \sin(u_i) \frac{1}{2} \Delta u_i \approx \int_0^{\pi} \sin u \, \frac{1}{2} du.$$

The heights of the two sets of approximating rectangles are the same, but on the right the rectangles are half as wide.

When u is not a linear function of x, the widths of the rectangles stretch by different amounts.

When $u = g(x)$, then
$$\begin{aligned}\Delta u_i &= u_{i+1} - u_i \\ &= g(x_{i+1}) - g(x_i) \\ &= g(x_i + \Delta x_i) - g(x_i) \\ &\approx g'(x_i)\Delta x_i.\end{aligned}$$





$$\int_0^{\pi} \sin u \, du \approx \sum_{i=1}^{4} \sin(u_i)\Delta u_i.$$

$$\int_0^{\pi^2} \sin\sqrt{x}\, dx \approx \sum_{i=1}^{4} \sin\sqrt{x_i}\,\Delta x_i$$

In this example, $u = \sqrt{x}$, so

$$\Delta u_i \approx \frac{1}{2\sqrt{x_i}}\Delta x_i = \frac{1}{2u}\Delta x_i.$$

$$\approx \sum_{i=1}^{4} \sin(u_i) 2u\,\Delta u_i \approx \int_0^{\pi} \sin u\, 2u\, du.$$