You must justify your answers to receive full credit.

1. For each of the sets W_i below:
(i) determine, with explanation, whether it is a subspace of $\mathbb{R}^3$;
(ii) if it is a subspace, give a basis for it. (Hint: if you use the theorem "null spaces are subspaces" to show that W_i is a subspace, then you can use the algorithm for computing a basis of the null space to find a basis of W_i.)

   a) $W_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x, y, z \geq 0 \right\}$.

   b) $W_2 = \left\{ \begin{bmatrix} 5b + 3c \\ b - c \\ c \end{bmatrix} \middle| b, c \in \mathbb{R} \right\}$.

   c) $W_3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| a + 3b = c \text{ and } b + c + a = 0 \right\}$.

2. Let $\mathbb{P}_3$ denote the set of polynomials of degree at most 3. For each of the sets W_i below:
(i) determine, with explanation, whether it is a subspace of $\mathbb{P}_3$;
(ii) if it is a subspace, give a basis for it.

   a) $W_4 = \{\mathbf{p} \in \mathbb{P}_3 | \mathbf{p}(0) = 1\}$.

   b) $W_5 = \{a + bt + at^2 | a, b \in \mathbb{R}\}$.

3. Suppose

$$A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 11 & -3 \\ 2 & 4 & -1 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \end{bmatrix}, \quad \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 1 & -7 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   a) Find a basis for the null space of A.

   b) Find a basis for the column space of A.

   c) Find a basis for the row space of A.

   d) Is $\{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ a basis for the column space of A? Explain your answer.

4. Let $\mathbb{P}_3$ denote the set of polynomials of degree at most 3, with the standard basis $\mathcal{B} = \{1, t, t^2, t^3\}$.

a) Use coordinate vectors to determine if the set of polynomials

$$\left\{1+2t^3, 2+t-3t^2, -t+3t^2+4t^3\right\}$$

is linearly independent.

Let $T : \mathbb{P}_3 \to \mathbb{P}_3$ be the function given by

$$T(a_0+a_1t+a_2t^2+a_3t^3) = t(a_0+a_1t+a_2t^2)+a_3(t^2-1),$$

(i.e. multiply by t and then replace t^4 with t^2-1 - this type of function is common in abstract algebra).

b) Show that T is a linear transformation.

c) Find the matrix of T relative to the standard basis $\mathcal{B}$.

5. Let $\mathcal{B} = \{1+t^2, t+t^2, 1+2t+t^2\}$.

   a) Show that $\mathcal{B}$ is a basis for $\mathbb{P}_2$.

   b) Find $[1+4t+7t^2]_{\mathcal{B}}$.

   c) If $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix}1\\0\\1\end{bmatrix}$, find $\mathbf{p}$.

6. State whether each of the following statements is (always) true or (sometimes) false. If it is true, give a brief justification; if it is false, give a counterexample with an explanation.

   a) $\left\{\begin{bmatrix}a\\0\\b+1\end{bmatrix}\middle|\, a,b\in\mathbb{R}\right\}$ is a subspace of $\mathbb{R}^3$.

   b) Let $M_{2\times 2}$ denote the vector space of 2×2 matrices. The determinant function $\det : M_{2\times 2} \to \mathbb{R}$ is a linear transformation.

   c) The differentiation function $D : \mathbb{P}_3 \to \mathbb{P}_2$ is onto.

   d) The differentiation function $D : \mathbb{P}_3 \to \mathbb{P}_3$ is onto.

   e) For every 3×5 matrix A, there is a vector in Col$A$ that is not a column of A.

   f) The determinant of a square matrix is the product of its diagonal entries.

7. **Optional problem** (challenging, within syllabus): Let $M_{m\times n}$ be the vector space of $m\times n$ matrices, and B be a fixed $3\times n$ matrix.

   a) Show that the "right-multiplication by B" function $T : M_{3\times 3} \to M_{3\times n}$ given by $T(A) = AB$ is a linear transformation.

   b) Show that $\{A \in M_{3\times 3} | AB = 0\}$ is a subspace of $M_{3\times 3}$. (Hint: how is this set related to T?)

c) By choosing a suitable matrix B, show that $\left\{ A \in M_{3\times 3} \middle| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathrm{Nul}A \right\}$ is a subspace of $M_{3\times 3}$.

d) Show that $\{A \in M_{3\times 3} | (1,1,1) \in \mathrm{Row}A\}$ is not a subspace of $M_{3\times 3}$.

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