You must justify your answers to receive full credit.

1. Let V be a vector space over a field $\mathbb{F}$. (In all parts below, please state clearly which axiom or property is being used in each step.)

   a) Axiom (V4) states that $\exists\, \mathbf{0} \in V$ such that
   $$\forall \alpha \in V, \quad \alpha + \mathbf{0} = \alpha. \tag{$*$}$$
   Show that such $\mathbf{0}$ is unique.
   (Hint: suppose $\mathbf{0}$ and $\mathbf{0}'$ both satisfy $(*)$, then show $\mathbf{0} = \mathbf{0}'$.)

   b) Now let $\mathbb{F} = \mathbb{R}$, i.e. V is a real vector space. Suppose $a \in \mathbb{R}$ and $\alpha \in V$ satisfy $a\alpha = \mathbf{0}$. Show that, either $a = 0$ or $\alpha = \mathbf{0}$. (Hint: Suppose $a \neq 0$, and show $\alpha = \mathbf{0}$.)

2. Let $V = \mathbb{R}^2$ with the following strange operations:
   $$\begin{pmatrix} x \\ y \end{pmatrix} \boxplus \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \end{pmatrix}, \quad c \boxdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cy \\ cx \end{pmatrix}.$$

   a) Does V satisfy Axiom (V9)? Explain your answer.

   b) Show that V does not satisfy one of the other axioms.

3. a) Show that $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ is in the span (over $\mathbb{R}$) of the matrices
   $$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

   b) Determine if
   $$\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
   are linearly independent over $\mathbb{R}$.

4. a) Show that if $\{\alpha_1, \ldots, \alpha_n\}$ is linearly independent over $\mathbb{F}$ and if $\{\alpha_1, \ldots, \alpha_n, \beta\}$ is linearly dependent over $\mathbb{F}$, then β is a linear combination of $\alpha_1, \ldots, \alpha_n$.

   b) Show that, if $\{\alpha, \beta, \gamma\}$ is linearly independent over $\mathbb{R}$, then $\{\alpha + \beta, \beta + \gamma, \gamma\}$ is linearly independent over $\mathbb{R}$.

   c) Use Definition 6.3.6 or its equivalent in Remark 6.3.8 to show that, if $A \subseteq B$, then $\mathrm{Span}(A) \subseteq \mathrm{Span}(B)$. Do **not** use Theorem 6.3.9 about linear combinations.

5. Let $P_{<n}(\mathbb{R})$ be the set of polynomials over $\mathbb{R}$ of degree less than n. Let
   $W = \{ax^2 + bx + 2a + 3b \mid a, b \in \mathbb{F}\}$.

   a) Show that W is a subspace of $P_{<3}(\mathbb{F})$.

   b) Is W a subspace of $P_{<4}(\mathbb{F})$?

   c) Find a basis for W.

6. Consider, in $\mathbb{R}^4$, the vectors

$$\beta_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \beta_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

and let $W = \operatorname{Span}\{\alpha_1, \alpha_2, \alpha_3\}$.

a) Use the casting-out algorithm to obtain a basis for W that contains β_1 and β_2.

b) Carry out the algorithm in the proof of the Steinitz Replacement Theorem, i.e. write out explicitly the linear combinations involved (the equations marked 1, 2, 3, 4 in the class notes) and each relabelling of the α_i. (You will **not** be asked explicitly to apply this algorithm in an exam. This question is to help you understand the proof. Your final answer may be different from that in part a, but that's fine because there are many bases for each subspace.)

The following two questions are to prepare you for upcoming classes, and is unrelated to the material from recent classes.

7. Let

$$A = \begin{pmatrix} 2 & -3 & -7 & 5 \\ 1 & -2 & 4 & 3 \\ 2 & 0 & -4 & 2 \\ 1 & -5 & -7 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} -4 \\ -3 \\ 2 \\ -9 \end{pmatrix}.$$

a) Find all solutions $X \in \mathbb{R}^4$ to $AX = B$. Please show all steps in your computation.

b) Find, with justification, a basis for the column space of A.

c) Find, with justification, a basis for the null space of A.

d) Let $\sigma : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by $\sigma(\alpha) = A\alpha$ (i.e. the standard matrix of σ is A). Let $\mathscr{B}$ be the basis of $\mathbb{R}^4$ given by

$$\mathscr{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \\ 0 \end{pmatrix} \right\}.$$

Write down $[\sigma]_{\mathscr{B}}$, the matrix for σ relative to $\mathscr{B}$, as the product of three matrices and/or their inverses. (You do **not** need to invert or multiply the three matrices.)

8. Let $P_{<3}(\mathbb{R})$ be the set of polynomials over $\mathbb{R}$ of degree less than 3. Consider the function $\sigma : P_{<3}(\mathbb{R}) \to P_{<3}(\mathbb{R})$ given by $\sigma(a + bx + cx^2) = (a - b) + (b + c)x^2$.

   a) Show that σ is a linear transformation.

   b) Find the matrix representing σ relative to the standard basis $\{1, x, x^2\}$ of $P_{<3}(\mathbb{R})$.

   c) Find a linearly independent set of polynomials that span the kernel of σ.

Optional questions. If you attempted seriously all the above questions, then your scores for the following questions may replace any lower scores for two of the above questions.

9. Let $V = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$ be vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Assume $f, g \in V$, prove that the set $\{f, g\}$ is linearly dependent if and only if $\forall\, a, b \in \mathbb{R}$, $f(a)g(b) = g(a)f(b)$.

10. Let S_1 and S_2 be subsets of a vector space V. Assume that $S_1 \cap S_2 \neq \emptyset$. Is $\operatorname{span}(S_1 \cap S_2) = \operatorname{span}(S_1) \cap \operatorname{span}(S_2)$? Give a proof or a counterexample.

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