7. If $\mathcal{A}, \mathcal{B}$ are bases of U, V $(\dim U, V < \infty)$
so $\hat{\mathcal{A}}, \hat{\mathcal{B}}$ bases of $\hat{U}, \hat{V}$

then what is ${}_{\hat{\mathcal{A}} \leftarrow}[\hat{\sigma}]_{\hat{\mathcal{B}}}$?

In Ex② above: $U = \mathbb{R}^3$ $\mathcal{A} = \{e_1, e_2, e_3\}$, $\hat{\mathcal{A}} = \{\phi_1, \phi_2, \phi_3\}$
$V = \mathbb{R}^2$ $\mathcal{B} = \{e_1, e_2\}$ $\hat{\mathcal{B}} = \{\psi_1, \psi_2\}$.

so, if $\phi\begin{pmatrix} u \\ v \end{pmatrix} = au + bv$, then $\phi = a\psi_1 + b\psi_2$

From before: $[\hat{\sigma}(\phi)]\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (a+4b)x + 2ay + (3a+5b)z$.

$$\hat{\sigma}(\phi) = (a+4b)\phi_1 + 2a\phi_2 + (3a+5b)\phi_3.$$

$$[\hat{\sigma}(\phi)]_{\hat{\mathcal{A}}} = \begin{pmatrix} a+4b \\ 2a \\ 3a+5b \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 0 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

${}_{\hat{\mathcal{A}} \leftarrow}[\hat{\sigma}]_{\hat{\mathcal{B}}} = {}_{\mathcal{B} \leftarrow}[\sigma]^T_{\mathcal{A}}$

$\begin{pmatrix} a \\ b \end{pmatrix} = [\phi]_{\hat{\mathcal{B}}}$

Th. 9.3.3: For $\sigma \in L(U, V)$, $\dim U, \dim V < \infty$.
$\mathcal{A}, \mathcal{B}$ bases of U, V.
Then ${}_{\hat{\mathcal{A}} \leftarrow}[\hat{\sigma}]_{\hat{\mathcal{B}}} = {}_{\mathcal{B} \leftarrow}[\sigma]^T_{\mathcal{A}}$

Proof: $\hat{\mathcal{A}} = \{\phi_1, \cdots, \phi_n\}$, $\hat{\mathcal{B}} = \{\psi_1, \cdots, \psi_m\}$.
$\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\}$ $\mathcal{B} = \{\beta_1, \cdots, \beta_m\}$

column j of ${}_{\hat{\mathcal{A}} \leftarrow}[\hat{\sigma}]_{\hat{\mathcal{B}}}$ is $[\hat{\sigma}(\psi_j)]_{\hat{\mathcal{A}}}$

so row i, column j of ${}_{\hat{\mathcal{A}} \leftarrow}[\hat{\sigma}]_{\hat{\mathcal{B}}}$ is
row i of $[\hat{\sigma}(\psi_j)]_{\hat{\mathcal{A}}}$, is coefficient of ϕ_i in $\hat{\sigma}(\psi_j)$.

From Prop: coefficient of ϕ_i is given by evaluation at α_i.
i.e. we want $[\hat{\sigma}(\psi_j)](\alpha_i) = \psi_j(\sigma(\alpha_i))$

ψ_j is a coordinate function $\therefore$ $\psi_j(\sigma(\alpha_i))$ is the coefficient of β_j in $\sigma(\alpha_i)$ i.e. the jth entry of $[\sigma(\alpha_i)]_{\mathcal{B}}$

i.e. the jth row of the ith column of ${}_{\mathcal{B}\leftarrow}[\sigma]_{\mathcal{A}}$.

<u>Another proof:</u> ${}_{\hat{\mathcal{A}}\leftarrow}[\hat{\sigma}]_{\hat{\mathcal{B}}}$ is the matrix X satisfying $[\hat{\sigma}(\phi)]_{\hat{\mathcal{A}}} = X[\phi]_{\hat{\mathcal{B}}}$ $\forall \phi \in \hat{V}$.

so we show $[\hat{\sigma}(\phi)]_{\hat{\mathcal{A}}} = \left({}_{\mathcal{B}\leftarrow}[\sigma]_{\mathcal{A}}\right)^T [\phi]_{\hat{\mathcal{B}}}$.

$$
\begin{aligned}
(\mathrm{RHS})^T &= [\phi]_{\hat{\mathcal{B}}}^T \; {}_{\mathcal{B}\leftarrow}[\sigma]_{\mathcal{A}} \\
&= {}_{\{1\}\leftarrow}[\phi]_{\mathcal{B}} \; {}_{\mathcal{B}\leftarrow}[\sigma]_{\mathcal{A}} = {}_{\{1\}\leftarrow}[\phi\circ\sigma]_{\mathcal{A}} = {}_{\{1\}\leftarrow}[\hat{\sigma}(\phi)]_{\mathcal{A}} = [\hat{\sigma}(\phi)]_{\hat{\mathcal{A}}}^T \\
&= (\mathrm{LHS})^T.
\end{aligned}
$$

§9.2 The double dual

Recall $\hat{V} = L(V, \mathbb{F})$

So $\hat{\hat{V}} = L(\hat{V}, \mathbb{F})$

i.e. each $f \in \hat{\hat{V}}$ is a function $\phi \mapsto$ number,

where the input ϕ is a function on V.

One example: $f =$ evaluation at some fixed $\alpha \in V$.

i.e. f is $\phi \mapsto \phi(\alpha)$ i.e. $f(\phi) = \phi(\alpha)$.

Check this f is in $\hat{\hat{V}}$, i.e. f is linear (in its input in $\hat{V}$)