A Hopf-algebraic lift of the down-up-chain on partitions to permutations

Amy Pang, LaCIM

based on the preprint arXiv:1508.01570

slides available at tinyurl.com/HopfLift

Fulman's down-up chain on partitions

A chain on irreducible representations of $\mathfrak{S}_n$

1.
2.
3.

Fulman's down-up chain on partitions

A chain on irreducible representations of $\mathfrak{S}_n$

1. restrict to $\mathfrak{S}_{n-1}$
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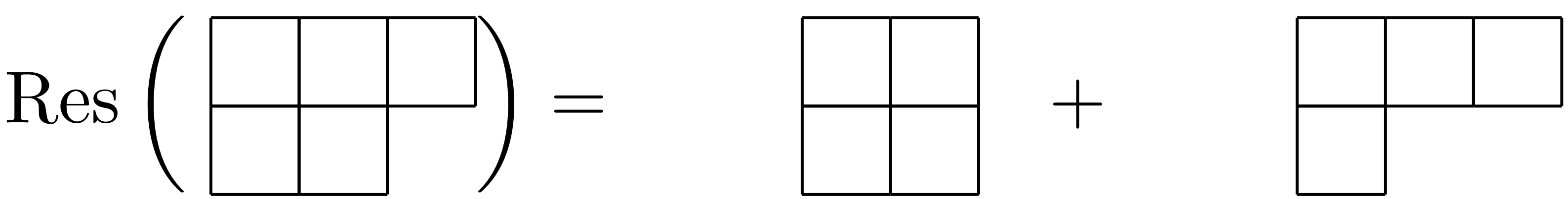

Fulman's down-up chain on partitions

A chain on irreducible representations of $\mathfrak{S}_n$

1. restrict to $\mathfrak{S}_{n-1}$
2. induce to $\mathfrak{S}_n$
3.

$$\mathrm{Ind} \circ \mathrm{Res} \left(\ydiagram{3,2} \right) = \mathrm{Ind} \left(\ydiagram{2,2} \right) + \mathrm{Ind} \left(\ydiagram{3,1} \right)$$

$$= \ydiagram{2,2,1} + \ydiagram{3,2} + \ydiagram{3,2} + \ydiagram{4,1} + \ydiagram{3,1,1}$$

Fulman's down-up chain on partitions

A chain on irreducible representations of $\mathfrak{S}_n$

1. restrict to $\mathfrak{S}_{n-1}$
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3. choose an irreducible with probability $\dfrac{\dim(\text{irreducible})}{n\dim(\text{original})}$

$$\mathrm{Ind}\circ\mathrm{Res}\left(\ydiagram{3,2}\right) = \mathrm{Ind}\left(\ydiagram{2,2}\right) + \mathrm{Ind}\left(\ydiagram{3,1}\right)$$

$$= \underset{\frac{5}{25}}{\ydiagram{2,2,1}} + \underset{\frac{5}{25}}{\ydiagram{3,2}} + \underset{\frac{5}{25}}{\ydiagram{3,2}} + \underset{\frac{4}{25}}{\ydiagram{4,1}} + \underset{\frac{6}{25}}{\ydiagram{3,1,1}}$$

Fulman's down-up chain on partitions

A chain on irreducible representations of $\mathfrak{S}_n$

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Fulman, 2004: Stationary distribution is Plancherel: $\frac{(\dim\lambda)^2}{n!}$

Fulman's down-up chain on partitions

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Fulman, 2004: Stationary distribution is Plancherel: $\frac{(\dim \lambda)^2}{n!}$

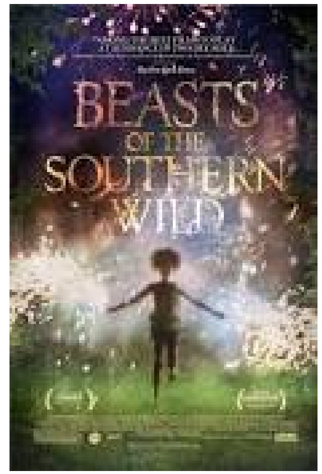
Question: Is there a chain X_t on permutations, with uniform stationary distribution, so that

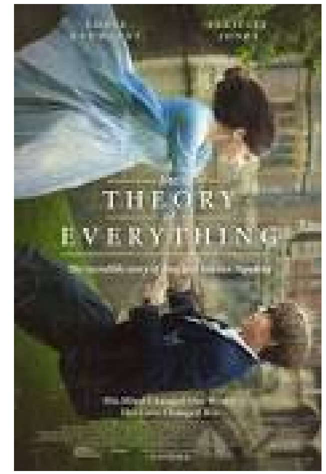
$$\mathrm{shape}(\mathrm{RSK}(X_t)) = \text{down-up on partitions?}$$

lift lumping

My weekly film list

Sunday afternoon:





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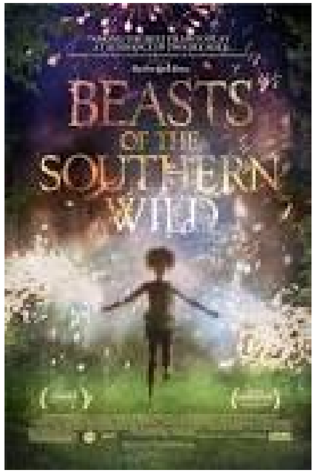

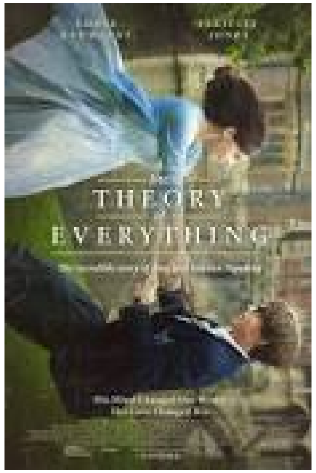

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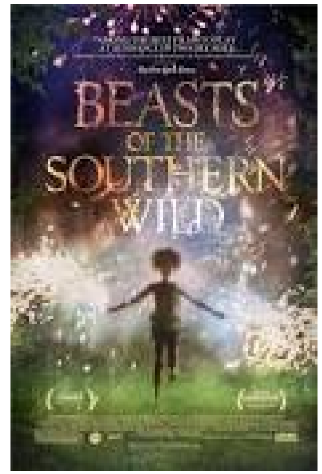

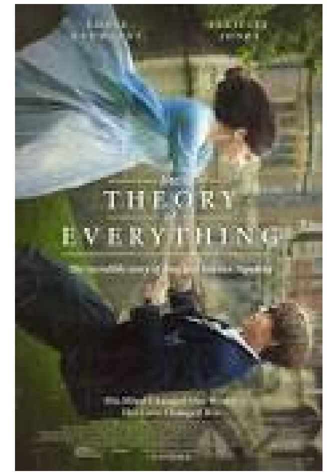

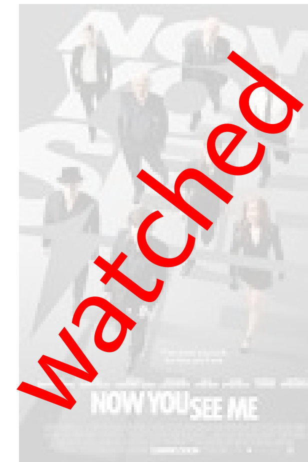

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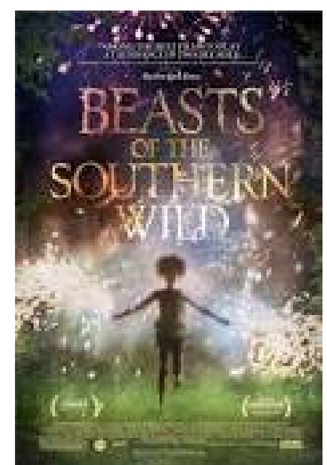

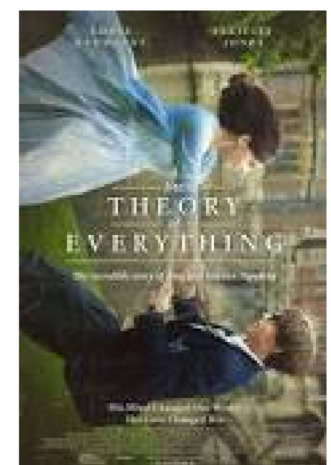

My weekly film list

Sunday afternoon:

1

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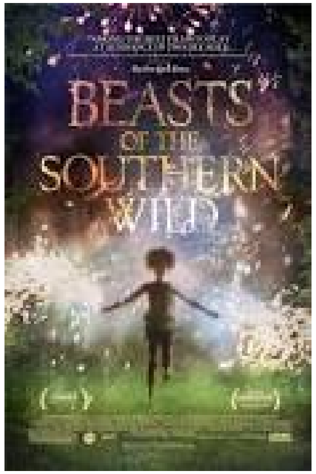

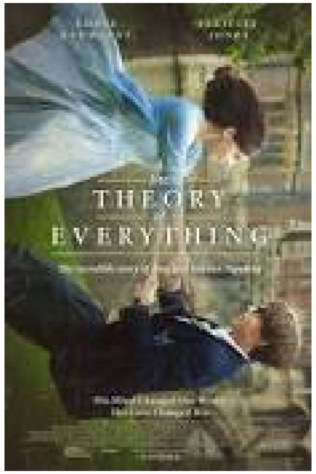

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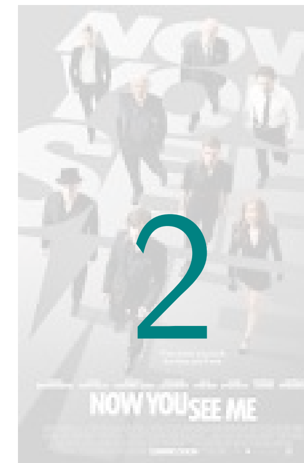

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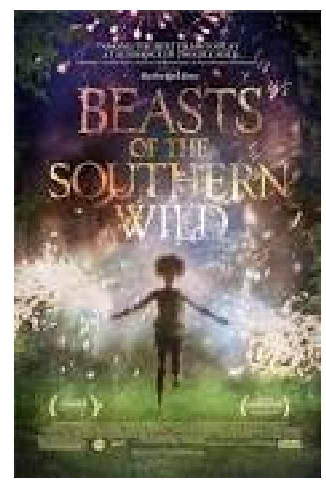

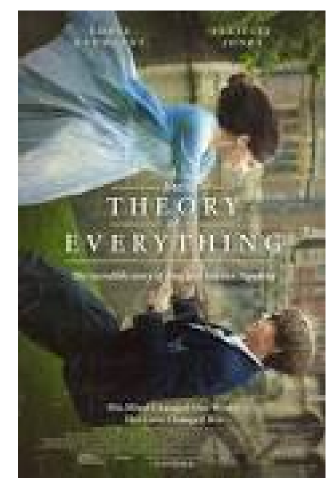

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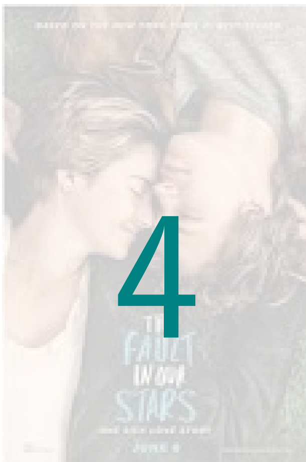

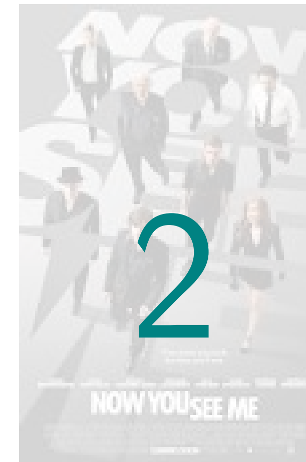

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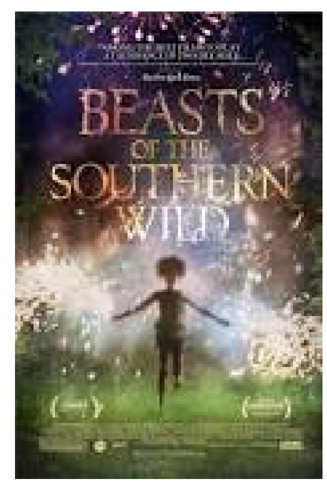

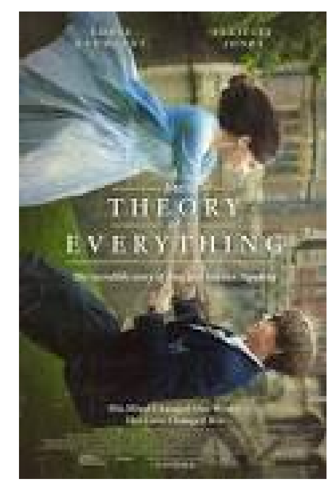

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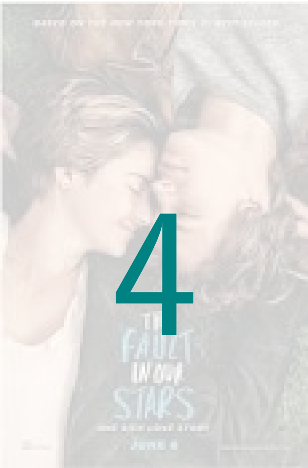

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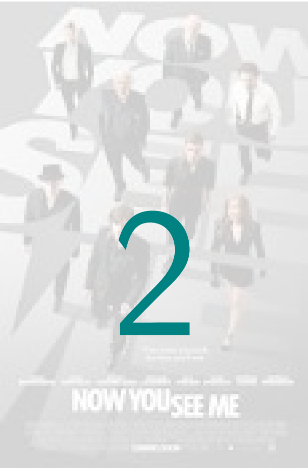

2

Monday morning:

1

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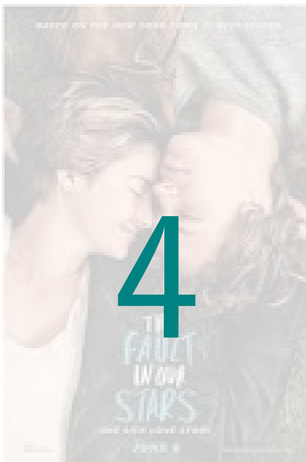

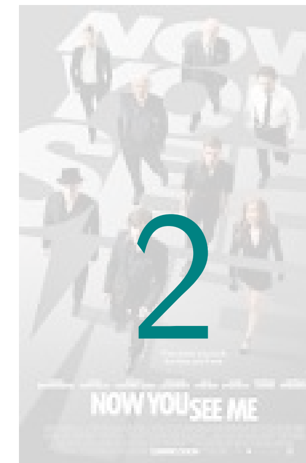

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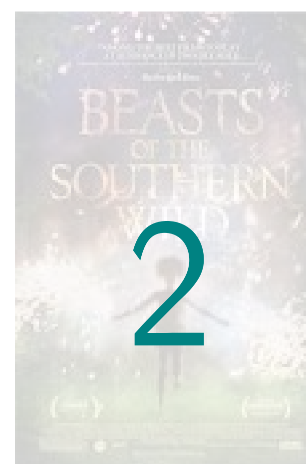

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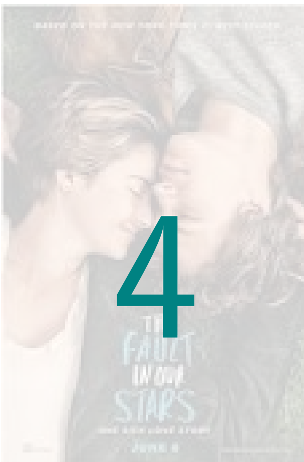

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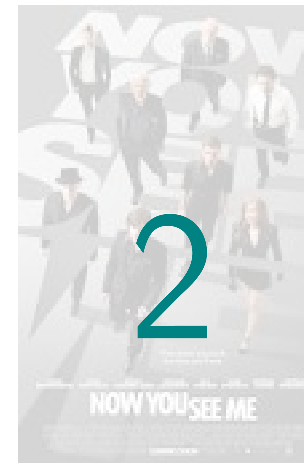

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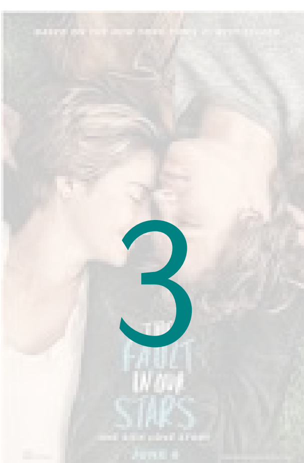

3

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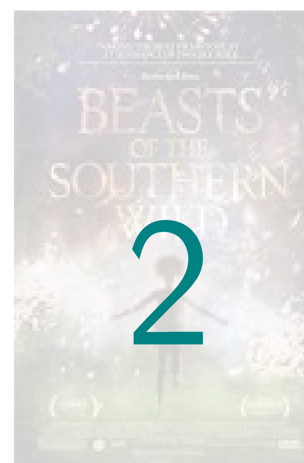
5



2

4

Lift: bottom-to-random-with-standardisation

Theorem (2015): Taking RSK-shape of the following chain gives the down-up chain on partitions, under an initial condition.

1.
2.
3.

$(4, 1, 3, 5, 2)$

Lift: bottom-to-random-with-standardisation

Theorem (2015): Taking RSK-shape of the following chain gives the down-up chain on partitions, under an initial condition.

1. remove last letter
2.
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$$(4,1,3,5,2) \xrightarrow{\text{remove}} (4,1,3,5)$$

Lift: bottom-to-random-with-standardisation

Theorem (2015): Taking RSK-shape of the following chain gives the down-up chain on partitions, under an initial condition.

1. remove last letter
2. standardise
3.

$$\begin{tikzcd}[column sep=small] (4,1,3,5,2) \arrow[dr, "\text{remove}"'] & & \\ & (4,1,3,5) \arrow[r, "\text{std}"] & {\color{red}(3,1,2,4)} \end{tikzcd}$$

Lift: bottom-to-random-with-standardisation

Theorem (2015): Taking RSK-shape of the following chain gives the down-up chain on partitions, under an initial condition.

1. remove last letter
2. standardise
3. insert n at random position

$(4, 1, 3, 5, 2)$ $\qquad\qquad$ $(3, 1, 5, 2, 4)$

remove $\qquad\qquad$ insert

$$(4, 1, 3, 5) \xrightarrow{\text{std}} (3, 1, 2, 4)$$

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& (4,1,3,5) \arrow[r, "\text{std}"] & (3,1,2,4) \arrow[ur, dashed, "\text{insert}"] & & {\color{red}(3,1,5,2)}
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Lift: bottom-to-random-with-standardisation

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$$\begin{tikzcd}
(4,1,3,5,2) \arrow[dr, "\text{remove}"] & & & (3,1,5,2,4) \arrow[dr, "\text{remove}"] & & & \color{red}{(5,3,1,4,2)} \\
& (4,1,3,5) \arrow[r, "\text{std}"] & (3,1,2,4) \arrow[ur, dashed, "\text{insert}"] & & (3,1,5,2) \arrow[r, "\text{std}"] & (3,1,4,2) \arrow[ur, dashed, red, "\text{insert}"] &
\end{tikzcd}$$

Proof

$$\Lambda \xtwoheadleftarrow{\text{shape}} \mathbf{FSym} \xhookrightarrow{\mathrm{RSK}^*} \mathbf{FQSym}$$

Proof

symmetric functions	Poirier-Reutenauer	Malvenuto-Reutanauer
partitions	standard tableaux	permutations

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partitions	standard tableaux	permutations

$$
\begin{tikzcd}
\Lambda \arrow[d, rightsquigarrow] & \mathbf{FSym} \arrow[l, two heads, "\text{shape}"'] \arrow[r, hook, "\text{RSK}^*"] \arrow[d, rightsquigarrow] & \mathbf{FQSym} \arrow[d, rightsquigarrow] \\
\text{down-up on partitions} & \text{down-up on standard tableaux} & \text{bottom-to-random-with-standardisation}
\end{tikzcd}
$$

Proof

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$$\begin{tikzcd}[column sep=huge]
\Lambda \arrow[d, rightsquigarrow] & \mathbf{FSym} \arrow[l, two heads, "\text{shape}"'] \arrow[r, hook, "\text{RSK}^*"] \arrow[d, rightsquigarrow] & \mathbf{FQSym} \arrow[d, rightsquigarrow] \\
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\end{tikzcd}$$

... and a new lumping theorem for Markov chains from combinatorial Hopf algebras

Markov chains from combinatorial Hopf algebras

- graded Hopf algebra: $\mathcal{H} = \bigoplus \mathcal{H}_n$
- basis of $\mathcal{H}_n$ is $\mathcal{B}_n$
- product $\mathrm{mult} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$
- "partial" coproduct $\Delta_{n-1,1} : \mathcal{H} \to \mathcal{H}_{n-1} \otimes \mathcal{H}_1$

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For $x, y \in \mathcal{B}_n$:

Prob$(x \to y) =$ coefficient of y in $\frac{1}{n} \mathrm{mult} \circ \Delta_{n-1,1}(x)$

Markov chains from combinatorial Hopf algebras

For $x, y \in \mathcal{B}_n$:

$$\text{Prob}(x \to y) = \text{coefficient of } y \text{ in } \frac{1}{n} \mathrm{mult} \circ \Delta_{n-1,1}(x)$$

$\mathbf{FQSym}$, $\mathbf{F}$ basis $\rightsquigarrow$ bottom-to-random-with-standardisation:

$$\Delta_{4,1}((4,1,3,5,2)) = \qquad (3,1,2,4) \otimes (1)$$

Markov chains from combinatorial Hopf algebras

For $x, y \in \mathcal{B}_n$:

$\text{Prob}(x \to y) =$ coefficient of y in $\dfrac{1}{n} \operatorname{mult} \circ \Delta_{n-1,1}(x)$

FQSym, **F** basis $\rightsquigarrow$ bottom-to-random-with-standardisation:

$$\begin{aligned}
\operatorname{mult} \circ \Delta_{4,1}((4,1,3,5,2)) &= \operatorname{mult}((3,1,2,4) \otimes (1)) \\
&= (5,3,1,2,4) + (3,5,1,2,4) \\
&\quad + (3,1,5,2,4) + (3,1,2,5,4) \\
&\quad + (3,1,2,4,5)
\end{aligned}$$

Markov chains from combinatorial Hopf algebras

For $x, y \in \mathcal{B}_n$:

$$\text{Prob}(x \to y) = \text{coefficient of } y \text{ in } \frac{1}{n} \operatorname{mult} \circ \Delta_{n-1,1}(x)$$

FQSym, F basis $\rightsquigarrow$ bottom-to-random-with-standardisation:

$$\begin{aligned}
\tfrac{1}{5} \operatorname{mult} \circ \Delta_{4,1}((4,1,3,5,2)) &= \tfrac{1}{5} \operatorname{mult}((3,1,2,4) \otimes (1)) \\
&= \tfrac{1}{5}(5,3,1,2,4) + \tfrac{1}{5}(3,5,1,2,4) \\
&\quad + \tfrac{1}{5}(3,1,5,2,4) + \tfrac{1}{5}(3,1,2,5,4) \\
&\quad + \tfrac{1}{5}(3,1,2,4,5)
\end{aligned}$$

Markov chains from combinatorial Hopf algebras

For $x, y \in \mathcal{B}_n$: (Doob h-transform)

$$\text{Prob}(x \to y) = \text{coefficient of } y \text{ in } \frac{1}{n} \operatorname{mult} \circ \Delta_{n-1,1}(x) \frac{\dim y}{\dim x}$$

Λ, Schur basis $\rightsquigarrow$ down-up on partitions:

1. restrict to $\mathfrak{S}_{n-1}$
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Markov chains from combinatorial Hopf algebras

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$$\text{Prob}(x \to y) = \text{coefficient of } y \text{ in } \frac{1}{n} \operatorname{mult} \circ \Delta_{n-1,1}(x) \frac{\dim y}{\dim x}$$

$$\dim(x) = \text{sum of coefficients in } \Delta_{1,\dots,1}(x)$$

Λ, Schur basis $\rightsquigarrow$ down-up on partitions:

1. restrict to $\mathfrak{S}_{n-1}$
2. induce to $\mathfrak{S}_n$
3. choose an irreducible with probability $\dfrac{\dim(\text{irreducible})}{n \dim(\text{original})}$

Strong lumping theorem

$$\Lambda \twoheadleftarrow\!\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!- \mathbf{FSym}$$

$$s_{\mathrm{shape}(T)} \longleftarrow\!\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!- \mathbf{S}_T$$

Strong lumping theorem

$$\Lambda \twoheadleftarrow\!\!\longrightarrow \mathbf{FSym}$$

$$s_{\mathrm{shape}(T)} \longleftarrow\!\!\!\longrightarrow \mathbf{S}_T$$

In general:

$$\bar{\mathcal{B}} \longleftarrow\!\!\!\longrightarrow \mathcal{B} : \mathrm{lump}$$

Theorem (2014): If a map on the bases $\mathrm{lump} : \mathcal{B} \to \bar{\mathcal{B}}$ extends linearly to a surjective Hopf morphism, then

$$\mathrm{lump}(\text{chain on } \mathcal{B}_n) = \text{chain on } \bar{\mathcal{B}}_n.$$

Weak lumping theorem

$$\mathbf{FSym} \not\twoheadleftarrow \mathbf{FQSym}$$

Weak lumping theorem

$$\mathbf{FSym} \hookrightarrow \mathbf{FQSym}$$

$$\mathbf{S}_T \longrightarrow \sum_{\mathrm{RSK}(\sigma)=T} \mathbf{F}_\sigma$$

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In general:

$$\mathcal{B} \longleftarrow \tilde{\mathcal{B}} : \mathrm{lump}$$

$$\mathrm{lump}^* : x \longrightarrow \sum_{\mathrm{lump}(\tilde{x})=x} \tilde{x}$$

Weak lumping theorem

$$\mathbf{FSym} \hookrightarrow \mathbf{FQSym}$$

$$\mathbf{S}_T \longmapsto \sum_{\mathrm{RSK}(\sigma)=T} \mathbf{F}_\sigma$$

In general:

$$\mathcal{B} \longleftarrow \tilde{\mathcal{B}} : \mathrm{lump}$$

$$\mathrm{lump}^* : x \longmapsto \sum_{\mathrm{lump}(\tilde{x})=x} \tilde{x}$$

Theorem (2015): If this "preimage sum map" extends linearly to a Hopf morphism, then

$$\mathrm{lump}(\text{chain on } \tilde{\mathcal{B}}_n) = \text{chain on } \mathcal{B}_n,$$

Weak lumping theorem

$$\mathbf{FSym} \hookrightarrow \mathbf{FQSym}$$

$$\mathbf{S}_T \longmapsto \sum_{\mathrm{RSK}(\sigma)=T} \mathbf{F}_\sigma$$

In general:

$$\mathcal{B} \longleftarrow \tilde{\mathcal{B}} : \mathrm{lump}$$

$$\mathrm{lump}^* : x \longmapsto \sum_{\mathrm{lump}(\tilde{x})=x} \tilde{x}$$

Theorem (2015): If this "preimage sum map" extends linearly to a Hopf morphism, then

$$\mathrm{lump}(\text{chain on } \tilde{\mathcal{B}}_n) = \text{chain on } \mathcal{B}_n,$$

provided the starting distribution X_0 on $\tilde{\mathcal{B}}_n$ satisfies $\frac{X_0(\tilde{x})}{\dim(\tilde{x})} = \frac{X_0(\tilde{y})}{\dim(\tilde{y})}$ whenever $\mathrm{lump}(\tilde{x}) = \mathrm{lump}(\tilde{y})$.

In general

$$\Lambda \xtwoheadleftarrow{\text{shape}} \mathbf{FSym}$$

$$\mathbf{FSym} \xhookrightarrow{\text{RSK}^*} \mathbf{FQSym}$$

In general

$$\Lambda \xleftarrow{\text{shape}} \mathbf{FSym} \qquad\qquad \mathbf{FSym} \xhookrightarrow{\text{RSK}^*} \mathbf{FQSym}$$

$$\begin{tikzcd}
\Lambda \arrow[d, "\frac{1}{n}\,\mathrm{mult}\circ\Delta_{n-1,1}"'] & \mathbf{FSym} \arrow[l, two heads, "\text{shape}"'] \arrow[d, "\frac{1}{n}\,\mathrm{mult}\circ\Delta_{n-1,1}"] \\
\Lambda & \mathbf{FSym} \arrow[l, two heads, "\text{shape}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{FSym} \arrow[r, hook, "\text{RSK}^*"] \arrow[d, "\frac{1}{n}\,\mathrm{mult}\circ\Delta_{n-1,1}"'] & \mathbf{FQSym} \arrow[d, "\frac{1}{n}\,\mathrm{mult}\circ\Delta_{n-1,1}"] \\
\mathbf{FSym} \arrow[r, hook, "\text{RSK}^*"] & \mathbf{FQSym}
\end{tikzcd}$$

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$$\mathbf{FSym} \xhookrightarrow{\mathrm{RSK}^*} \mathbf{FQSym}$$

$$\begin{tikzcd}
\bar{V} \arrow[d, "\bar{\mathbf{T}}"'] & V \arrow[l, two heads, "\text{lump}"'] \arrow[d, "\mathbf{T}"] \\
\bar{V} & V \arrow[l, two heads, "\text{lump}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, hook, "\text{lump}^*"] \arrow[d, "\mathbf{T}"'] & \tilde{V} \arrow[d, "\tilde{\mathbf{T}}"] \\
V \arrow[r, hook, "\text{lump}^*"] & \tilde{V}
\end{tikzcd}$$

$\bar{\mathbf{T}}$ chain $=$ lump ($\mathbf{T}$ chain)

$\mathbf{T}$ chain $=$ lump ($\tilde{\mathbf{T}}$ chain),

with condition on starting distribution

In general

$$\Lambda \xtwoheadleftarrow{\text{shape}} \mathbf{FSym}$$

$$\mathbf{FSym} \xhookrightarrow{\text{RSK}^*} \mathbf{FQSym}$$

$$\begin{tikzcd}
\bar{V} \arrow[d, "\bar{\mathbf{T}}"'] & V \arrow[l, two heads, "\text{lump}"'] \arrow[d, "\mathbf{T}"] \\
\bar{V} & V \arrow[l, two heads, "\text{lump}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, hook, "\text{lump}^*"] \arrow[d, "\mathbf{T}"'] & \tilde{V} \arrow[d, "\tilde{\mathbf{T}}"] \\
V \arrow[r, hook, "\text{lump}^*"] & \tilde{V}
\end{tikzcd}$$

$\bar{\mathbf{T}}$ chain $= \text{lump}$ ($\mathbf{T}$ chain)

$\mathbf{T}$ chain $= \text{lump}$ ($\tilde{\mathbf{T}}$ chain),
with condition on starting distribution

Question: a simpler application without Hopf algebras?

A final question

Fulman (2004): The bottom-to-random shuffle is a lift of the down-up chain on partitions, if starting at the identity.

$(5, 2, 4, 1, 3)$

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Fulman (2004): The bottom-to-random shuffle is a lift of the down-up chain on partitions, if starting at the identity.

$$(5, 2, 4, 1, 3) \longrightarrow (5, 2, 4, 1)$$

A final question

Fulman (2004): The bottom-to-random shuffle is a lift of the down-up chain on partitions, if starting at the identity.

$$\begin{tikzcd}
(5,2,4,1,3) \arrow[dr] & & (5,2,3,4,1) \\
& (5,2,4,1) \arrow[ur, dashed] &
\end{tikzcd}$$

A final question

Fulman (2004): The bottom-to-random shuffle is a lift of the down-up chain on partitions, if starting at the identity.

$$
\begin{tikzcd}
(5,2,4,1,3) \arrow[dr] & & (5,2,3,4,1) \arrow[dr, red] & \\
& (5,2,4,1) \arrow[ur, dashed] & & \color{red}{(5,2,3,4)}
\end{tikzcd}
$$

A final question

Fulman (2004): The bottom-to-random shuffle is a lift of the down-up chain on partitions, if starting at the identity.

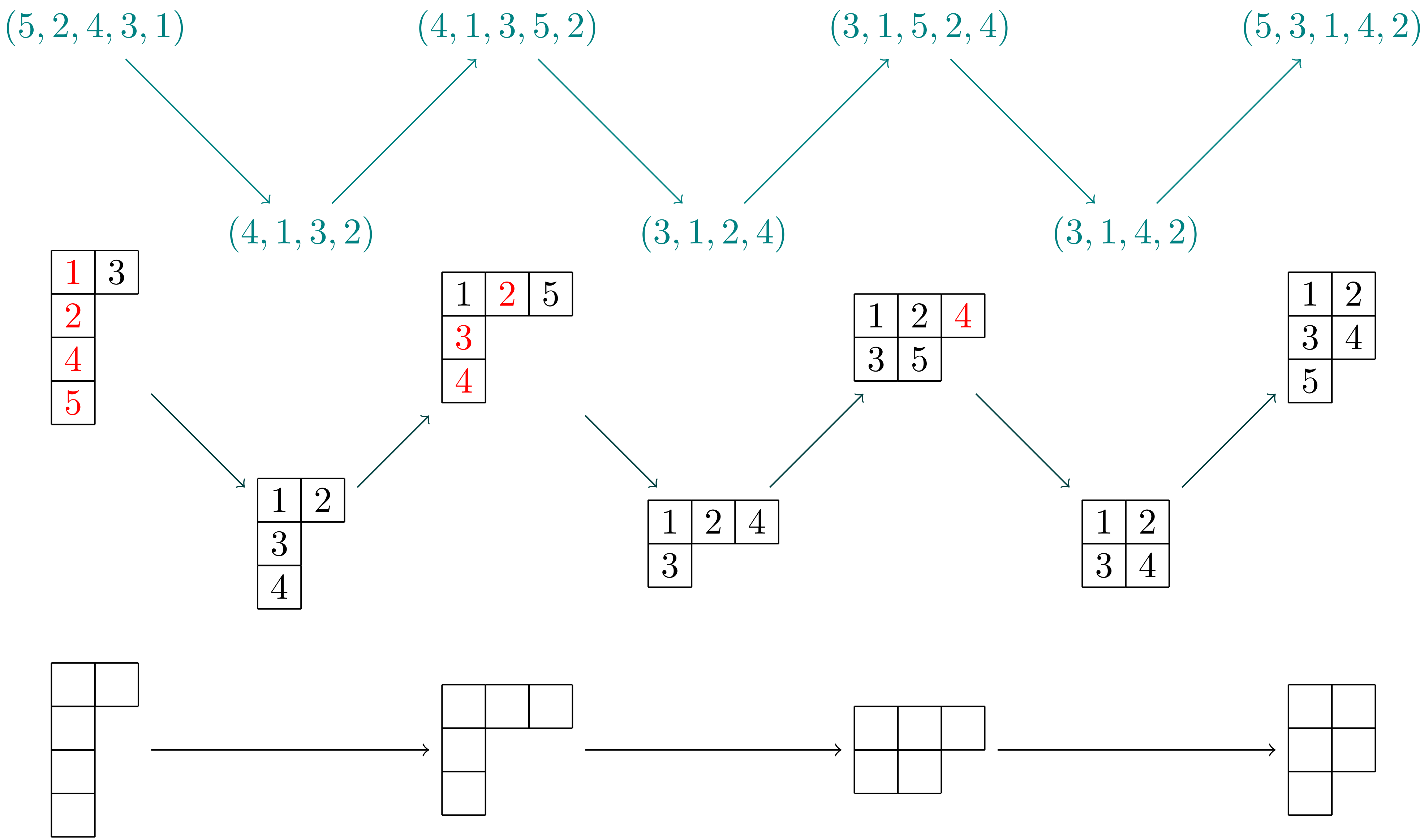
$$\begin{tikzcd}
(5,2,4,1,3) \arrow[dr] & & (5,2,3,4,1) \arrow[dr] & & (1,5,2,3,4) \\
& (5,2,4,1) \arrow[ur, dashed] & & (5,2,3,4) \arrow[ur, dashed] &
\end{tikzcd}$$

A final question

Fulman (2004): The bottom-to-random shuffle is a lift of the down-up chain on partitions, if starting at the identity.

$$
\begin{tikzcd}
(5,2,4,1,3) \arrow[dr] & & (5,2,3,4,1) \arrow[dr] & & \color{red}{(1,5,2,3,4)} \\
& (5,2,4,1) \arrow[ur, dashed] & & (5,2,3,4) \arrow[ur, dashed, red] &
\end{tikzcd}
$$

Question: is there a Hopf proof? $\Lambda \twoheadleftarrow \dots \hookrightarrow$ shuffle algebra?



Thank you!

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