You must justify your answers to receive full credit.

1. For each set of vectors below, determine the value(s) of h for which the set is linearly dependent:

   a) $\left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ h \end{bmatrix} \right\}$

   b) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix} \right\}$

2. **Without doing any row-reduction**, determine whether the following sets are linearly independent, and explain why:

   a) $\left\{ \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \\ -1 \end{bmatrix} \right\}$

   b) $\left\{ \begin{bmatrix} -8 \\ 10 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}$

   c) $\left\{ \begin{bmatrix} 2 \\ -4 \\ 8 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -12 \\ 3 \end{bmatrix} \right\}$

   d) $\left\{ \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 4 \end{bmatrix} \right\}$

3. For each of the transformations below:
   (i) decide whether it is linear, and explain your answer,
   (ii) if it is linear, find its standard matrix,
   (iii) if it is linear, decide whether it is one-to-one and whether it is onto.

   a) $f : \mathbb{R}^4 \to \mathbb{R}^2$ given by $f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{2}x_1 + x_2 - x_3 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

   b) $g : \mathbb{R}^2 \to \mathbb{R}^4$ given by $g(\mathbf{x}) = \mathbf{0}$.

   c) $h : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection through the line $x_1 = 1$.

4. Let A be the matrix
$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix},$$
and T be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.

   a) What is the domain of T?
   b) What is the codomain of T?
   c) Is T onto? (Hint: a theorem may be useful.)
   d) Find the kernel of T.
   e) Is T one-to-one?
   f) Find the image of $\mathbf{e_1} + \mathbf{e_2}$ under T.

5. Let $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ be a linearly independent subset of $\mathbb{R}^n$, and $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

   a) Prove that, if T is one-to-one, then $\{T(\mathbf{v_1}), T(\mathbf{v_2}), T(\mathbf{v_3})\}$ is linearly independent.
   b) Show that the assumption that T is one-to-one is necessary - that is, find a numerical example of a linear transformation T that is not one-to-one, and a linearly independent set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$, such that $\{T(\mathbf{v_1}), T(\mathbf{v_2}), T(\mathbf{v_3})\}$ is linearly dependent.

6. State whether each of the following statements is (always) true or (sometimes) false. If it is true, give a brief justification; if it is false, give a counterexample with an explanation.

   a) If a set of non-zero vectors in $\mathbb{R}^n$ is linearly dependent, then the set contains at least n vectors.
   b) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$. If $\{\mathbf{u}, \mathbf{v}\}$, $\{\mathbf{u}, \mathbf{w}\}$ and $\{\mathbf{v}, \mathbf{w}\}$ are each linearly independent, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.
   c) If $A \begin{bmatrix} 4 \\ 0 \\ 2 \\ -3 \end{bmatrix} = \mathbf{0}$, then $A\mathbf{e}_4$ is a linear combination of the first three columns of A.
   d) $A\mathbf{x} = \mathbf{b}$ is a homogeneous equation if and only if $\mathbf{x} = \mathbf{0}$ is a solution.
   e) Let A be a 4 x 3 matrix with columns $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$. If $\mathbf{b}$ is a vector in $\mathbb{R}^4$ such that $A\mathbf{x} = \mathbf{b}$ has a solution, then $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{b}\}$ is linearly dependent.
   f) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $\{\mathbf{v_1}, \mathbf{v_2}\}$ spans $\mathbb{R}^2$, then $\{T(\mathbf{v_1}), T(\mathbf{v_2})\}$ also spans $\mathbb{R}^2$.

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