§1.8-1.9: Linear Transformations

This week's goal is to think of the equation $A\mathbf{x} = \mathbf{b}$ in terms of the "multiplication by A" function: its input is $\mathbf{x}$ and its output is $\mathbf{b}$.

Primary One:

$2^2 = 4$
$3^2 = 9$

Primary Four:

Think of this as: $2 \xrightarrow{\text{squaring}} 4$, $3 \xrightarrow{\text{squaring}} 9$

Last week:

$$\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Today:

Think of this as:

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \xrightarrow{\text{multiply by } A} \begin{bmatrix} 10 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{\text{multiply by } A} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Goal: think of the equation $A\mathbf{x} = \mathbf{b}$ in terms of the "multiplication by A" function: its input is $\mathbf{x}$ and its output is $\mathbf{b}$.

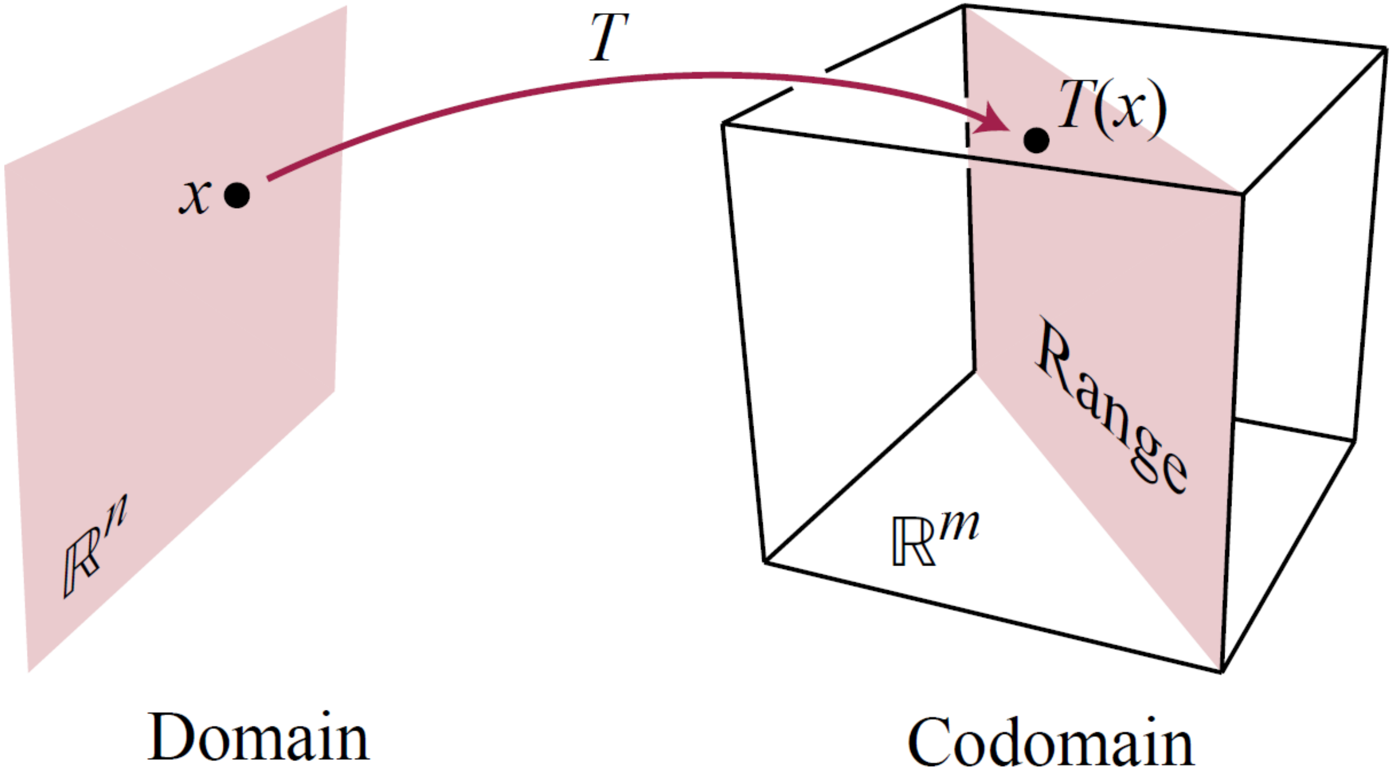
In this class, we are interested in functions that are linear (see p6 for the definition).
Key skills:

i Determine whether a function is linear (p7-9);
(This involves the important mathematical skill of "axiom checking", which also appears in other classes.)

ii Find the standard matrix of a linear function (p12-13);

iii Describe existence and uniqueness of solutions in terms of linear functions (p17-27).

Definition: A *function* f from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $f(\mathbf{x})$ in $\mathbb{R}^m$. We write $f : \mathbb{R}^n \to \mathbb{R}^m$.



$\mathbb{R}^n$ is the *domain* of f.

$\mathbb{R}^m$ is the *codomain* of f.

$f(\mathbf{x})$ is the *image of* $\mathbf{x}$ *under* f.

The *range* is the set of all images. It is a subset of the codomain.

Example: $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.
Its domain $=$ codomain $= \mathbb{R}$, its range $= \{y \in \mathbb{R} \mid y \geq 0\}$.

Examples:

$f : \mathbb{R}^2 \to \mathbb{R}^3$, defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2^3 \\ 2x_1 + x_2 \\ 0 \end{bmatrix}$.

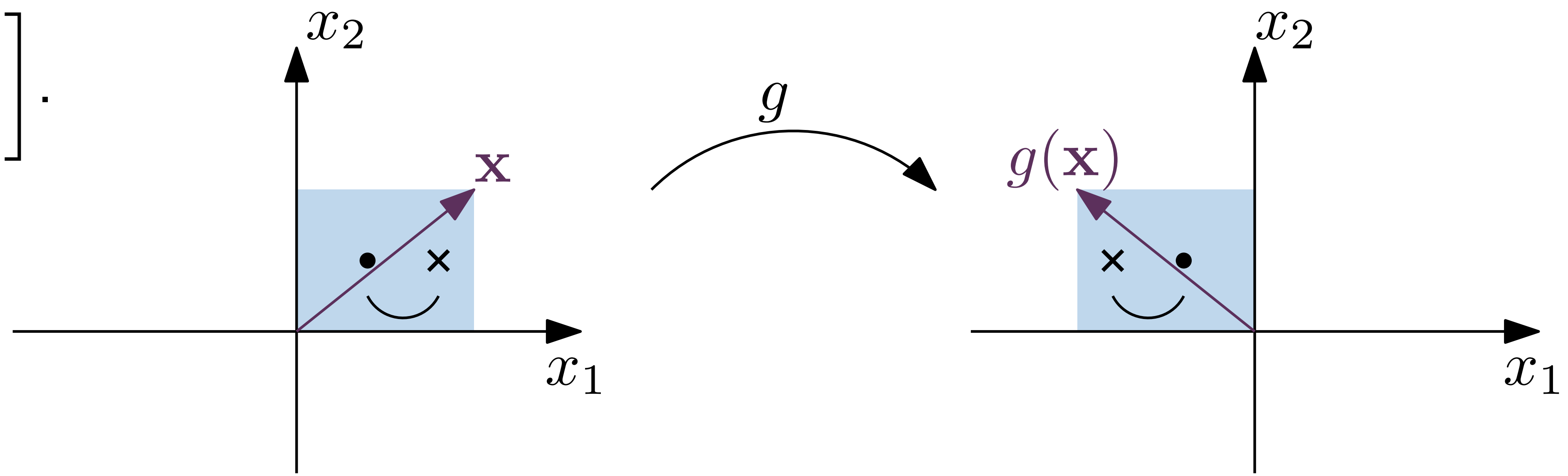
The range of f is the plane $z = 0$ (it is obvious that the range must be a subset of the plane $z = 0$, and with a bit of work (see p19), we can show that all points in $\mathbb{R}^3$ with $z = 0$ is the image of some point in $\mathbb{R}^2$ under f).

$h : \mathbb{R}^3 \to \mathbb{R}^2$, given by the matrix transformation $h(\mathbf{x}) = \begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \mathbf{x}$.

Geometric Examples:

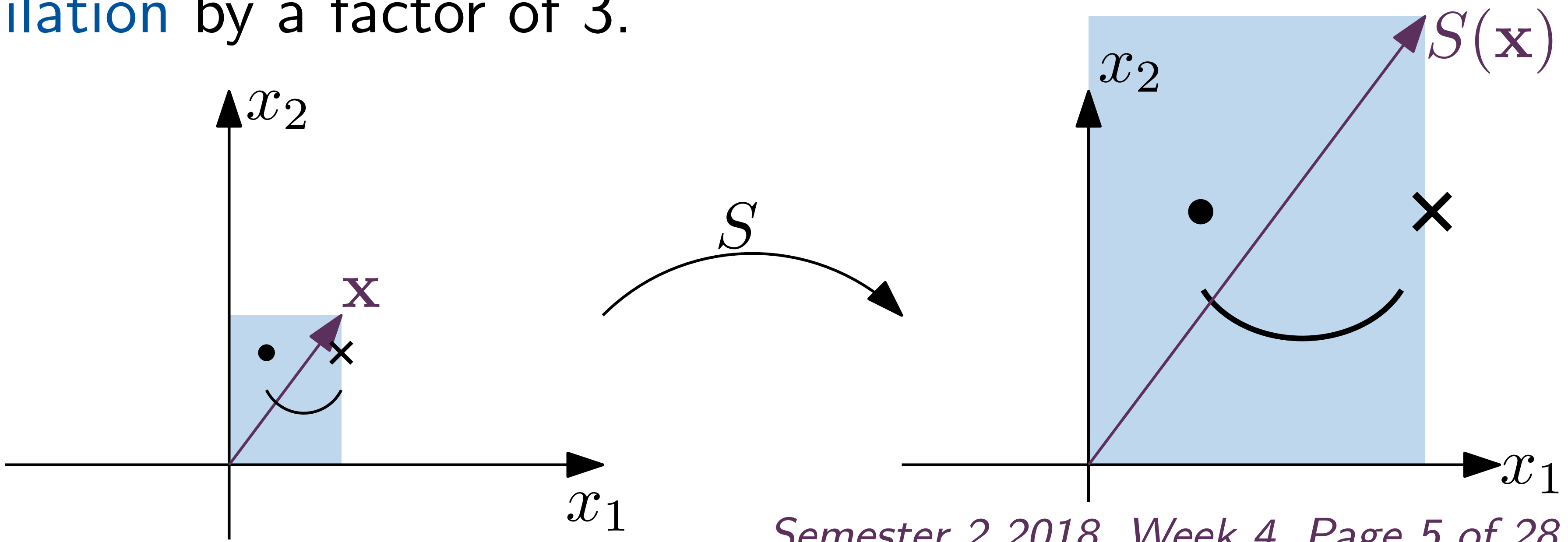
$g : \mathbb{R}^2 \to \mathbb{R}^2$, given by reflection through the x_2-axis.

$$g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}.$$



$S : \mathbb{R}^2 \to \mathbb{R}^2$, given by dilation by a factor of 3.

$$S(\mathbf{x}) = 3\mathbf{x}.$$



In this class, we will concentrate on functions that are linear. (For historical reasons, people like to say "linear transformation" instead of "linear function".)

Definition: A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* if:

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and for all $\mathbf{u}$ in the domain of T.

For your intuition: the name "linear" is because these functions preserve lines:
A line through the point $\mathbf{p}$ in the direction $\mathbf{v}$ is the set $\mathbf{p} + s\mathbf{v}$, where s is any number.
If T is linear, then the image of this set is

$$T(\mathbf{p} + s\mathbf{v}) \stackrel{1}{=} T(\mathbf{p}) + T(s\mathbf{v}) \stackrel{2}{=} T(\mathbf{p}) + sT(\mathbf{v}),$$

the line through the point $T(\mathbf{p})$ in the direction $T(\mathbf{v})$.
(If $T(\mathbf{v}) = \mathbf{0}$, then the image is just the point $T(\mathbf{p})$.)

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(If $T(\mathbf{v}) = \mathbf{0}$, then the image is just the point $T(\mathbf{p})$.)

Fact: A linear transformation T must satisfy $T(\mathbf{0}) = \mathbf{0}$.

Proof: Put $c = 0$ in condition 2.

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Example: Is $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2^3 \\ 2x_1 + x_2 \\ 0 \end{bmatrix}$ linear?

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Example: $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2^3 \\ 2x_1 + x_2 \\ 0 \end{bmatrix}$ is not linear:

Take $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $c = 2$:

$$f\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ 6 \\ 0 \end{bmatrix}.$$

$$2f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 8 \\ 6 \\ 0 \end{bmatrix}.$$

So condition 2 is false for f.

Exercise: find a $\mathbf{u}$ and a $\mathbf{v}$ to show that condition 1 is also false.

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Example: $g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$ (reflection through the x_2-axis) is linear:

1. $g\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right) = \begin{bmatrix} -u_1 - v_1 \\ u_2 + v_2 \end{bmatrix} = \begin{bmatrix} -u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix} = g\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + g\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right).$
2. $g\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) = \begin{bmatrix} -cu_1 \\ cu_2 \end{bmatrix} = c\begin{bmatrix} -u_1 \\ u_2 \end{bmatrix} = cg\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right).$

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Notice from the previous two examples:
To show that a function is linear, check both conditions for general $\mathbf{u}, \mathbf{v}, c$ (i.e. use variables).
To show that a function is not linear, show that one of the conditions is not satisfied for a particular numerical values of $\mathbf{u}$ and $\mathbf{v}$ (for 1) or of c and $\mathbf{u}$ (for 2).

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For simple functions, we can combine the two conditions at the same time, and check just one statement: $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$, for all scalars c, d and all vectors $\mathbf{u}, \mathbf{v}$. (Condition 1 is the case $c = d = 1$, condition 2 is the case $d = 0$. Exercise: show that if T satisfies conditions 1 and 2, then T satisfies the combined condition.)

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Example: $S(\mathbf{x}) = 3\mathbf{x}$ (dilation by a factor of 3) is linear:

$$S(c\mathbf{u} + d\mathbf{v}) = 3(c\mathbf{u} + d\mathbf{v}) = 3c\mathbf{u} + 3d\mathbf{v} = cS(\mathbf{u}) + dS(\mathbf{v}).$$

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Important Example: All matrix transformations $T(\mathbf{x}) = A\mathbf{x}$ are linear:

$$T(c\mathbf{u} + d\mathbf{v}) = A(c\mathbf{u} + d\mathbf{v}) = A(c\mathbf{u}) + A(d\mathbf{v}) = cA\mathbf{u} + dA\mathbf{v} = cT(\mathbf{u}) + dT(\mathbf{v}).$$

In general:

Write $\mathbf{e_i}$ for the vector with 1 in row i and 0 in all other rows.
(So $\mathbf{e}_i$ means a different thing depending on which $\mathbb{R}^n$ we are working in.)

For example, in $\mathbb{R}^3$, we have $\mathbf{e_1} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{e_2} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{e_3} = \begin{bmatrix}0\\0\\1\end{bmatrix}$.

$\{\mathbf{e_1}, \dots, \mathbf{e_n}\}$ span $\mathbb{R}^n$, and $\mathbf{x} = \begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = x_1\mathbf{e_1} + \cdots + x_n\mathbf{e_n}$.

So, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then

$$T(\mathbf{x}) = T(x_1\mathbf{e_1} + \cdots + x_n\mathbf{e_n}) = x_1T(\mathbf{e_1}) + \cdots + x_nT(\mathbf{e_n}) = \begin{bmatrix} | & & | \\ T(\mathbf{e}_1) & \dots & T(\mathbf{e}_n) \\ | & & | \end{bmatrix} \begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}.$$

Theorem 10: The matrix of a linear transformation: Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation: $T(\mathbf{x}) = A\mathbf{x}$ where A is the *standard matrix for T*, the $m \times n$ matrix given by

$$A = \begin{bmatrix} | & | & | \\ T(\mathbf{e}_1) & \dots & T(\mathbf{e}_n) \\ | & | & | \end{bmatrix}.$$

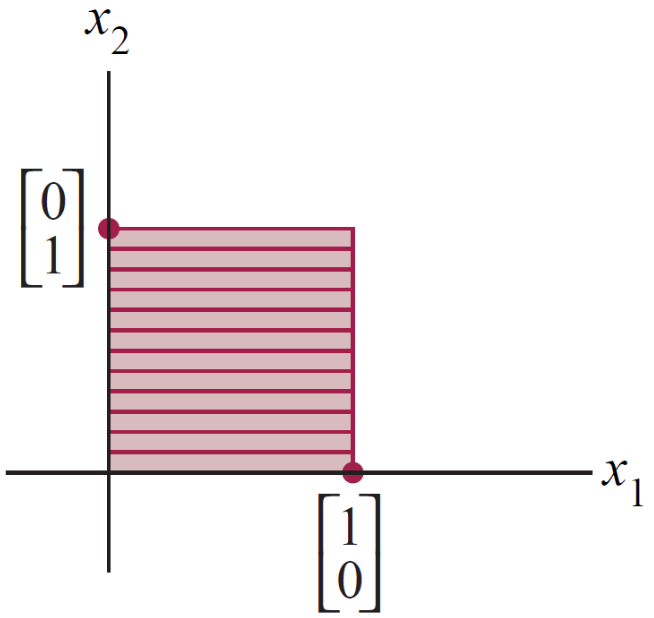
We can think of the standard matrix as a compact way of storing the information about T.

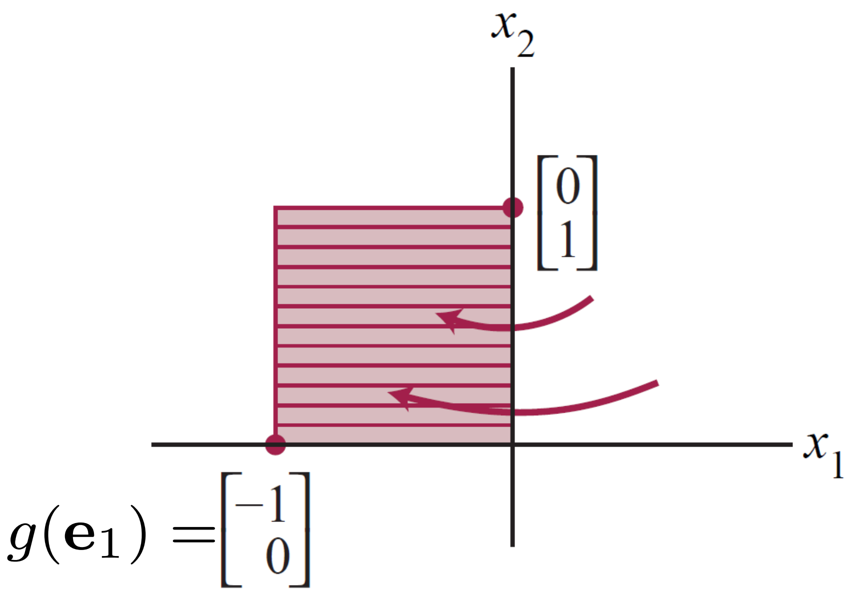
Example: $S : \mathbb{R}^2 \to \mathbb{R}^2$, given by dilation by a factor of 3, $S(\mathbf{x}) = 3\mathbf{x}$.

$$S(\mathbf{e_1}) = S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 3\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad S(\mathbf{e_2}) = S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 3\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

So the standard matrix of S is $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, i.e. $S(\mathbf{x}) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\mathbf{x}$.

Example: $g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$ (reflection through the x_2-axis):





The standard matrix of g is $\begin{bmatrix} | & | \\ g(\mathbf{e}_1) & g(\mathbf{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

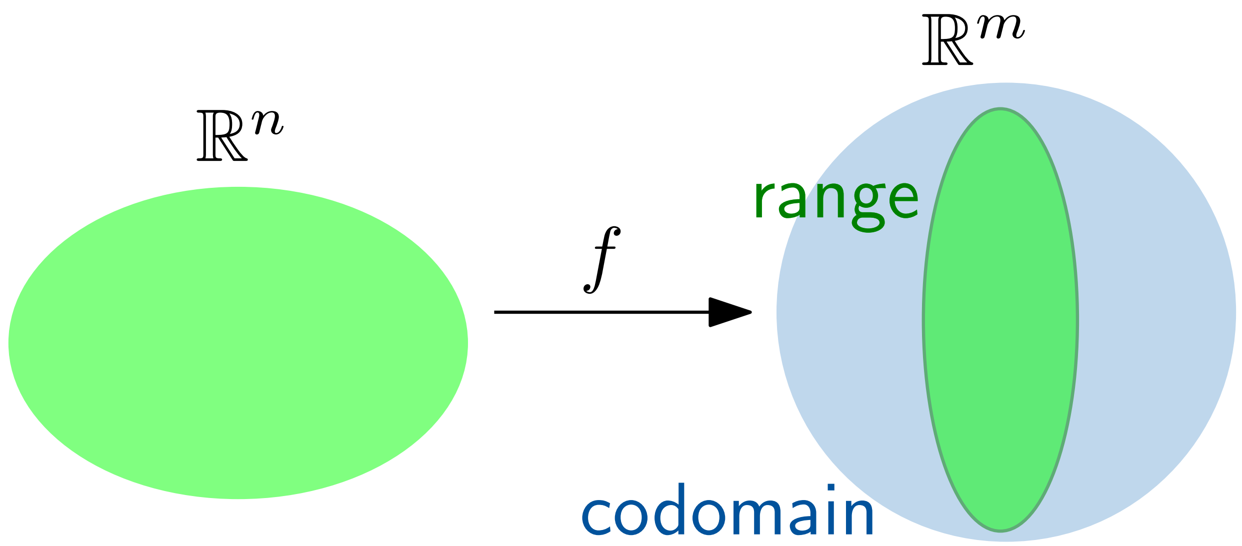
Indeed, $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$.

Now we rephrase our existence and uniqueness questions in terms of functions.

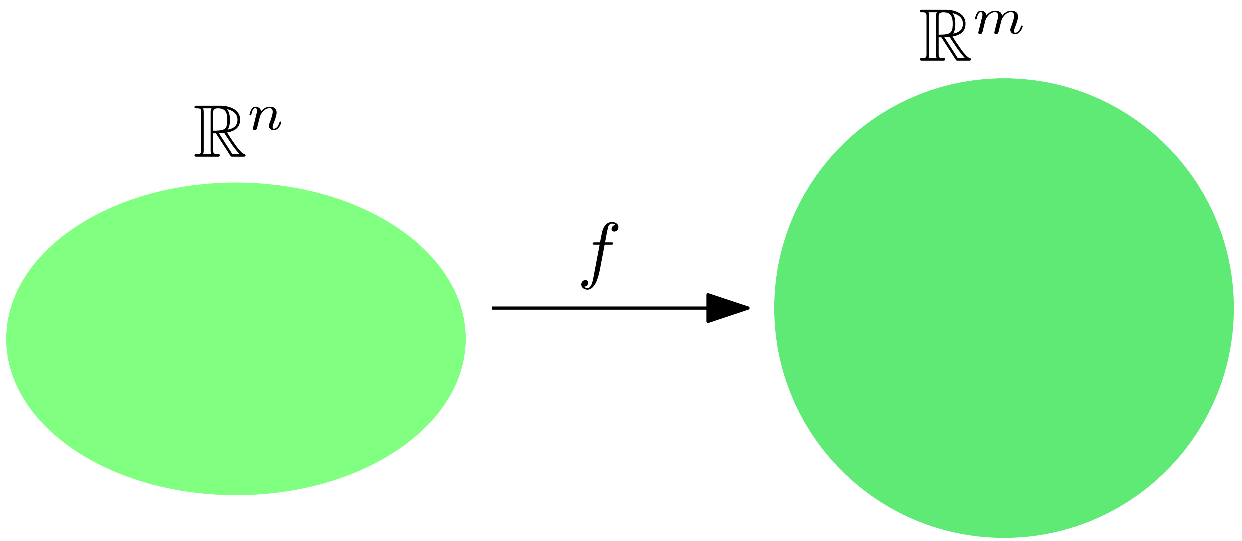
Definition: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *onto* (surjective) if each $\mathbf{y}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.

Other ways of saying this:

- The range is all of the codomain $\mathbb{R}^m$,
- The equation $f(\mathbf{x}) = \mathbf{y}$ has a solution for every $\mathbf{y}$ in $\mathbb{R}^m$.



f is not onto



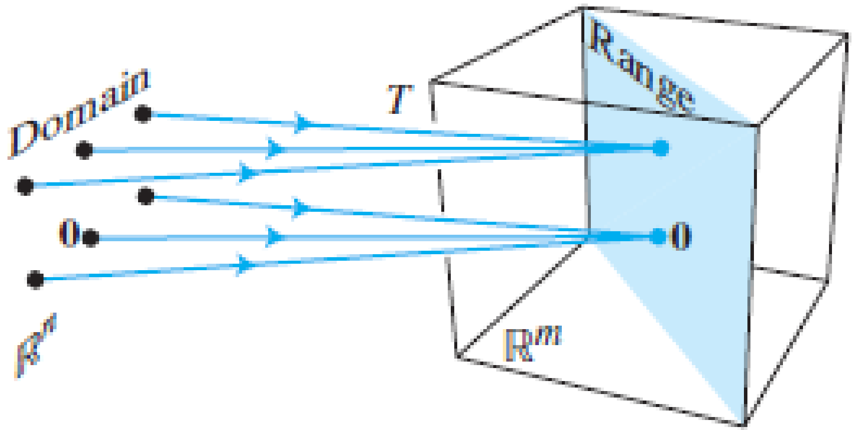
f is onto

Now we rephrase our existence and uniqueness questions in terms of functions.

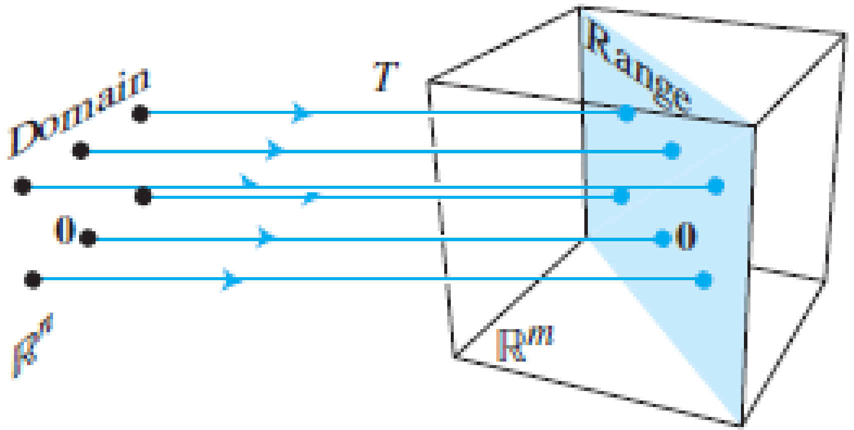
Definition: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *one-to-one* (injective) if each $\mathbf{y}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

Other ways of saying this:

- ??? (something that only works for linear transformations, see p22),
- The equation $f(\mathbf{x}) = \mathbf{y}$ has no solutions or a unique solution.



T is *not* one-to-one



T is one-to-one

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Example: $f : \mathbb{R}^2 \to \mathbb{R}^3$, defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2^3 \\ 2x_1 + x_2 \\ 0 \end{bmatrix}$.

Is f onto? Is f one-to-one?

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f is not onto, because $f(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ does not have a solution.

f is one-to-one:
if $y_3 \neq 0$, then $f(\mathbf{x}) = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ does not have a solution,

if $y_3 = 0$, then the unique solution to $f(\mathbf{x}) = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}$ is $x_2 = \sqrt[3]{y_1}$,
$x_1 = \frac{1}{2}(y_2 - x_2) = \frac{1}{2}(y_2 - \sqrt[3]{y_1})$.

There is an easier way to check if a linear transformation is one-to-one:

Definition: The *kernel* of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of solutions to $T(\mathbf{x}) = \mathbf{0}$.

Or, in set notation: $\ker T = \{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{0}\}$.

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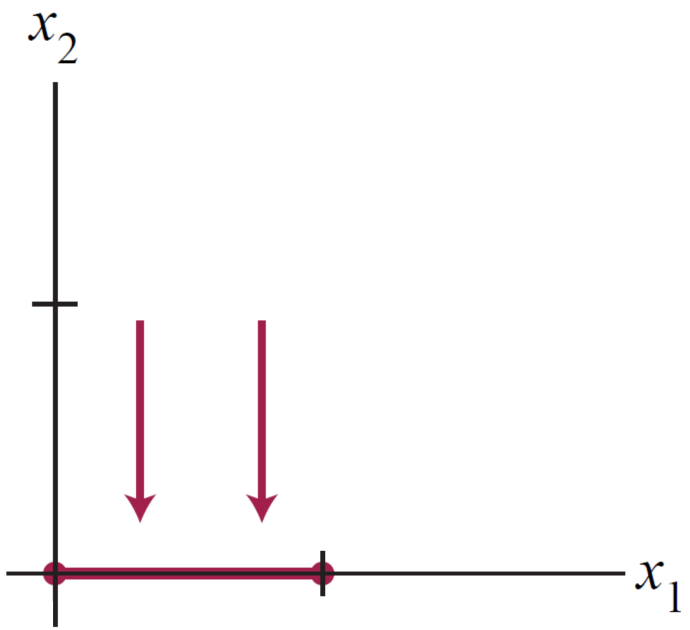
Example: Let T be projection onto the x_1-axis, whose standard matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (i.e. $T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x}$).



The kernel of T is the solution set of $T(\mathbf{x}) = \mathbf{0}$, i.e. the solution set of $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$. Using the usual algorithm, this solution set is $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} t \,\middle|\, t \in \mathbb{R} \right\}$, which is the x_2-axis.

It is also clear from the geometric description of projection that the x_2-axis is mapped to the origin.

There is an easier way to check if a linear transformation is one-to-one:

Recall: given $T : \mathbb{R}^n \to \mathbb{R}^m$ a linear transformation, $\ker T = \{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{0}\}$.

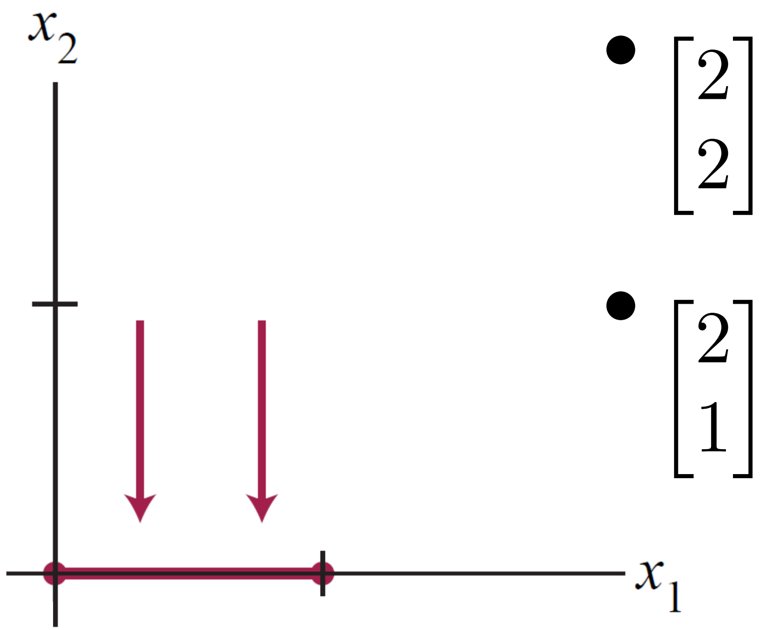
Fact: If $T(\mathbf{v_1}) = T(\mathbf{v_2})$, then $\mathbf{v_1} - \mathbf{v_2}$ is in the kernel of T.

Example: Let T be projection onto the x_1-axis.

The previous page showed that $\ker T$ is the x_2-axis.

Notice that $T\left(\begin{bmatrix}2\\2\end{bmatrix}\right) = T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}2\\0\end{bmatrix}$, and

$\begin{bmatrix}2\\2\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$, which is in the kernel.



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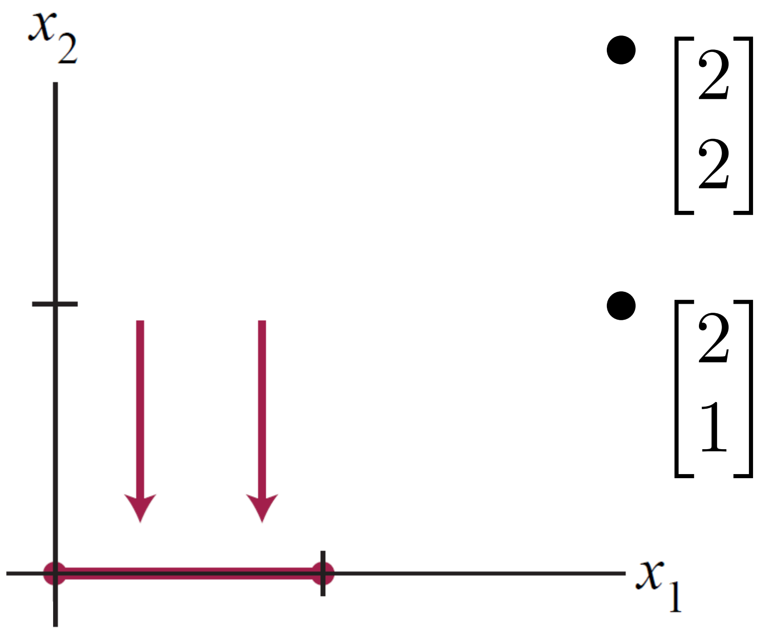
Fact: If $T(\mathbf{v_1}) = T(\mathbf{v_2})$, then $\mathbf{v_1} - \mathbf{v_2}$ is in the kernel of T.

Example: Let T be projection onto the x_1-axis.
The previous page showed that $\ker T$ is the x_2-axis.

Notice that $T\left(\begin{bmatrix}2\\2\end{bmatrix}\right) = T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}2\\0\end{bmatrix}$, and

$\begin{bmatrix}2\\2\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$, which is in the kernel.



uses linearity of T

Proof of Fact: If $T(\mathbf{v_1}) = T(\mathbf{v_2}) = \mathbf{y}$, then $T(\mathbf{v_1} - \mathbf{v_2}) = T(\mathbf{v_1}) - T(\mathbf{v_2}) = \mathbf{y} - \mathbf{y} = \mathbf{0}$, so $\mathbf{v_1} - \mathbf{v_2} \in \ker T$.

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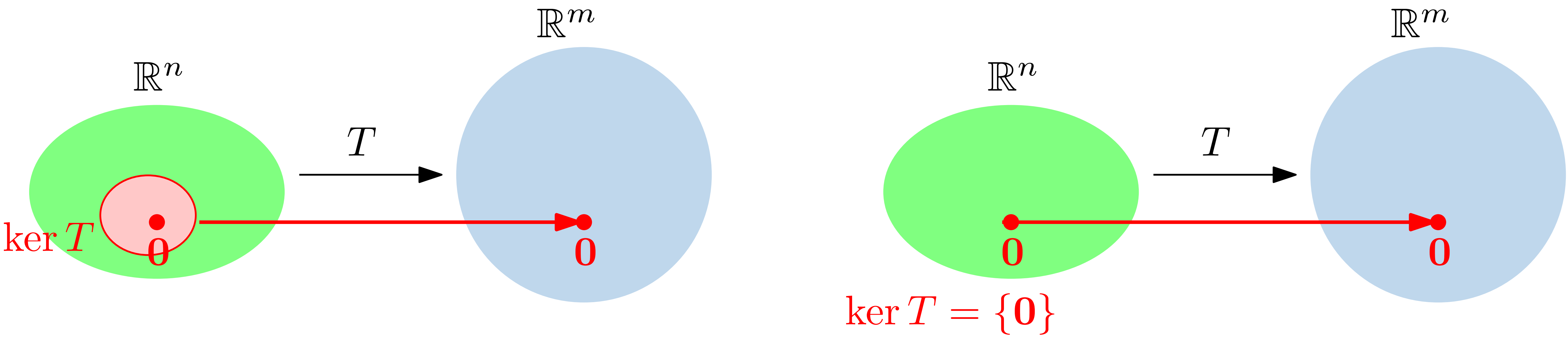
Theorem: A linear transformation is one-to-one if and only if its kernel is $\{\mathbf{0}\}$.



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Warning: the theorem is only for linear transformations. For other functions, the solution sets of $f(\mathbf{x}) = \mathbf{y}$ and $f(\mathbf{x}) = \mathbf{0}$ are not related.

Proof:

Suppose T is one-to-one. Taking $\mathbf{y} = \mathbf{0}$ in the definition of one-to-one shows $T(\mathbf{x}) = \mathbf{0}$ has at most one solution. Since $\mathbf{0}$ is a solution (because T is linear), it must be the only one. So its kernel is $\{\mathbf{0}\}$.

Suppose the kernel of T is $\{\mathbf{0}\}$.

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Suppose the kernel of T is $\{\mathbf{0}\}$. Then, from the Fact, if there are vectors $\mathbf{v_1}, \mathbf{v_2}$ with $T(\mathbf{v_1}) = T(\mathbf{v_2}) = \mathbf{y}$, then $\mathbf{v_1} - \mathbf{v_2} = \mathbf{0}$, i.e. $\mathbf{v_1} = \mathbf{v_2}$. So if $T(\mathbf{x}) = \mathbf{y}$ has a solution, then the solution is unique, i.e. T is one-to-one.

Theorem: A linear transformation is one-to-one if and only if its kernel is $\{\mathbf{0}\}$.

So a matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one if and only if the set of solutions to $A\mathbf{x} = \mathbf{0}$ is $\{\mathbf{0}\}$. This is equivalent to many other things:

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Theorem: Uniqueness of solutions to linear systems: For a matrix A, the following are equivalent:

a. $A\mathbf{x} = \mathbf{0}$ has no non-trivial solution (i.e. $\mathbf{x} = \mathbf{0}$ is the only solution).
b. If $A\mathbf{x} = \mathbf{b}$ is consistent, then it has a unique solution.
c. The columns of A are linearly independent.
d. rref(A) has a pivot in every column (i.e. all variables are basic).
e. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
f. The kernel of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $\{\mathbf{0}\}$.

Notice that e. is in terms of linear transformations, b. is in terms of matrices and linear equations, and they are the same thing.
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Now let's think about onto and existence of solutions.

Recall that the range of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of images, i.e. range$T = \{\mathbf{y} \in \mathbb{R}^m | \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^n\}$.

So, the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the set of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution.

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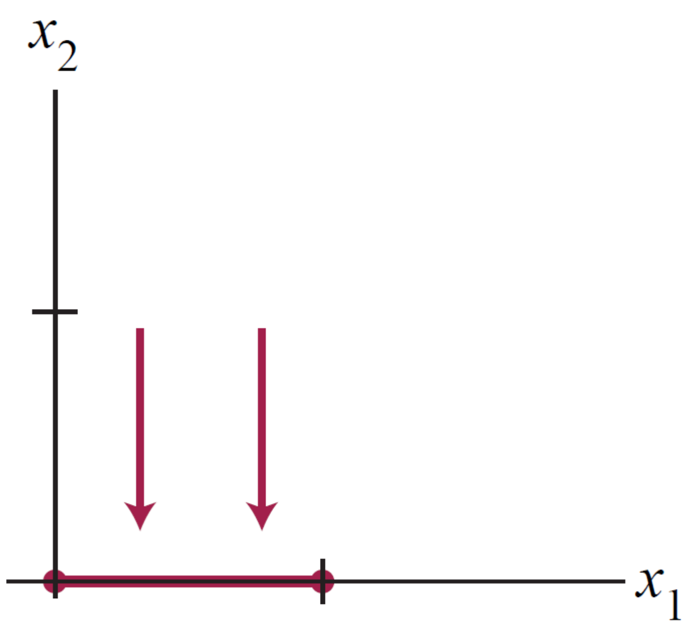
So, the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the set of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution.

So the range of T is the span of the columns of A (see week 2 p17).

Example: Let T be projection onto the x_1-axis, whose standard matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Its range is the span of the columns of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, i.e.

Span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, which is the x_1-axis.



It is also clear from the geometric description of projection that the set of images is the x_1-axis.

The range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the set of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution.
And a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ is onto if and only if its range is all of $\mathbb{R}^m$.
Putting these together: $\mathbf{x} \mapsto A\mathbf{x}$ is onto if and only if $A\mathbf{x} = \mathbf{b}$ is always consistent, and this is equivalent to many things:

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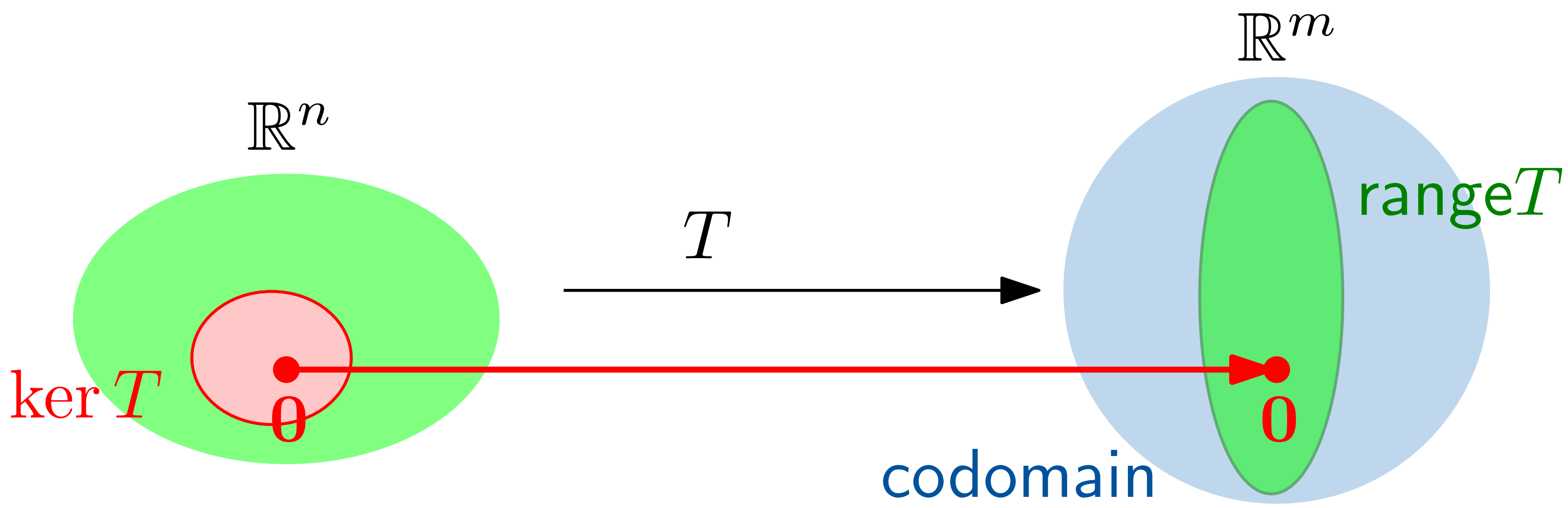
Theorem 4: Existence of solutions to linear systems: For an $m \times n$ matrix A, the following statements are logically equivalent (i.e. for any particular matrix A, they are all true or all false):

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. rref(A) has a pivot in every row.
e. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.
f. The range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $\mathbb{R}^m$.

The range and the kernel on one picture:



Remember from weeks 1-3 that existence and uniqueness are separate, unrelated concepts. Similarly, onto and one-to-one are unrelated:
Exercise 1: think of a linear transformation that is onto but not one-to-one, or both onto and one-to-one, or etc.

Exercise 2: consider the other linear transformations in this week's notes. Are they onto? Are they one-to-one?

Conceptual problems regarding linear independence and linear transformations:

In problems without specific numbers, it's often better not to use row-reduction.
The all-important equation: $T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p)$.

Example: Prove that, if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent and T is a linear transformation, then $\{T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})\}$ is linearly dependent.

Step 1 Rewrite the mathematical terms in the question as formulas.

What we know: there are scalars c_1, c_2, c_3 not all zero with $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$.
What we want to show: there are scalars d_1, d_2, d_3 not all zero such that $d_1T(\mathbf{u}) + d_2T(\mathbf{v}) + d_3T(\mathbf{w}) = \mathbf{0}$.

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Answer: We know $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$ for some scalars c_1, c_2, c_3 not all zero.
Apply T to both sides: $T(c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}) = T(\mathbf{0})$.

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Apply T to both sides: $T(c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}) = T(\mathbf{0})$.
Because T is a linear transformation: $c_1T(\mathbf{u}) + c_2T(\mathbf{v}) + c_3T(\mathbf{w}) = \mathbf{0}$.
Because c_1, c_2, c_3 are not all zero, this is a linear dependence relation among $T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})$.