

§4.5: Dimension

From last week:

- Given a vector space V , a basis for V is a linearly independent set that spans V .
- If $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V , then the \mathcal{B} -coordinates of \mathbf{x} are the weights c_i in the linear combination $\mathbf{x} = c_1\mathbf{b}_1 + \dots + c_p\mathbf{b}_p$.
- Coordinate vectors allow us to test for spanning / linear independence, to solve linear systems, and to test for one-to-one / onto by working in \mathbb{R}^n .

Another example of this idea:

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V .

- Any set in V containing more than n vectors must be linearly dependent (theorem 9 in textbook).
- Any set in V containing fewer than n vectors cannot span V .

We prove this (next page) using coordinate vectors, and the fact that we already know it is true for $V = \mathbb{R}^n$.

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V .

- i Any set in V containing more than n vectors must be linearly dependent.
- ii Any set in V containing fewer than n vectors cannot span V .

Proof: Let our set of vectors in V be $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$, and consider the matrix

$$A = \begin{bmatrix} | & & | \\ [\mathbf{u}_1]_{\mathcal{B}} & \cdots & [\mathbf{u}_p]_{\mathcal{B}} \\ | & & | \end{bmatrix},$$

which has p columns and n rows.

- i If $p > n$, then $\text{rref}(A)$ cannot have a pivot in every column, so $\{[\mathbf{u}_1]_{\mathcal{B}}, \dots, [\mathbf{u}_p]_{\mathcal{B}}\}$ is linearly dependent in \mathbb{R}^n , so $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is linearly dependent in V .
- ii If $p < n$, then $\text{rref}(A)$ cannot have a pivot in every row, so the set of coordinate vectors $\{[\mathbf{u}_1]_{\mathcal{B}}, \dots, [\mathbf{u}_p]_{\mathcal{B}}\}$ cannot span \mathbb{R}^n , so $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ cannot span V .

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V .

- i Any set in V containing more than n vectors must be linearly dependent.
- ii Any set in V containing fewer than n vectors cannot span V .

As a consequence:

Theorem 10: Every basis has the same size: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

So the following definition makes sense:

Definition: Let V be a vector space.

- If V is spanned by a finite set, then V is *finite-dimensional*.
The *dimension* of V , written $\dim V$, is the number of vectors in a basis for V .
(This number is finite because of the spanning set theorem.)
- If V is not spanned by a finite set, then V is *infinite-dimensional*.

Note that the definition does not involve “infinite sets”.

Definition: (or convention) The dimension of the zero vector space $\{\mathbf{0}\}$ is 0.

Definition: The *dimension* of V is the number of vectors in a basis for V .

Examples:

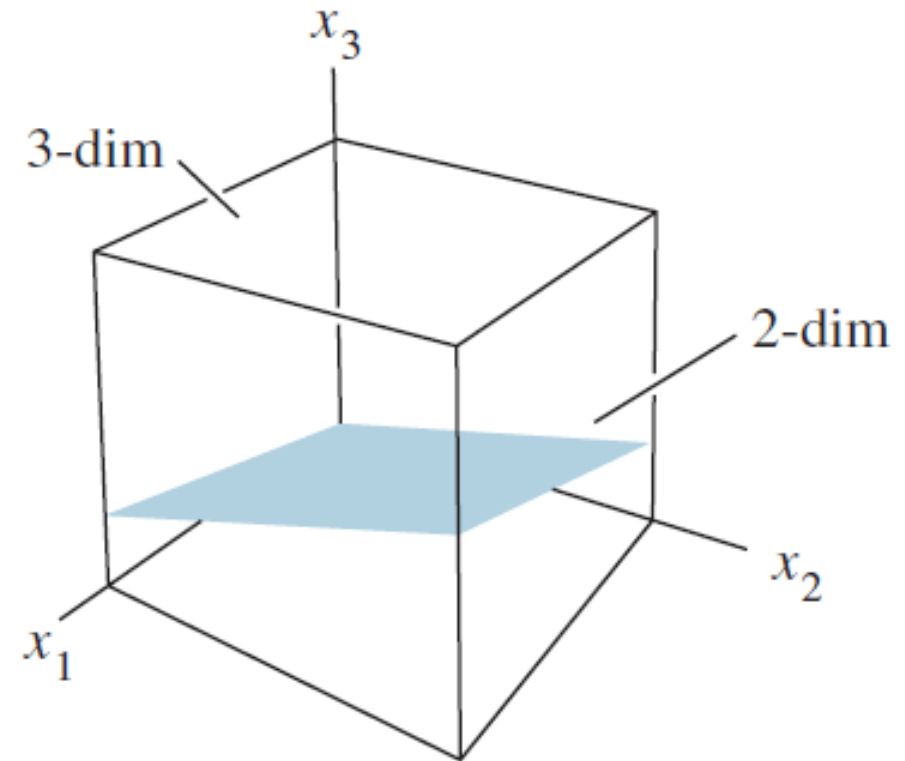
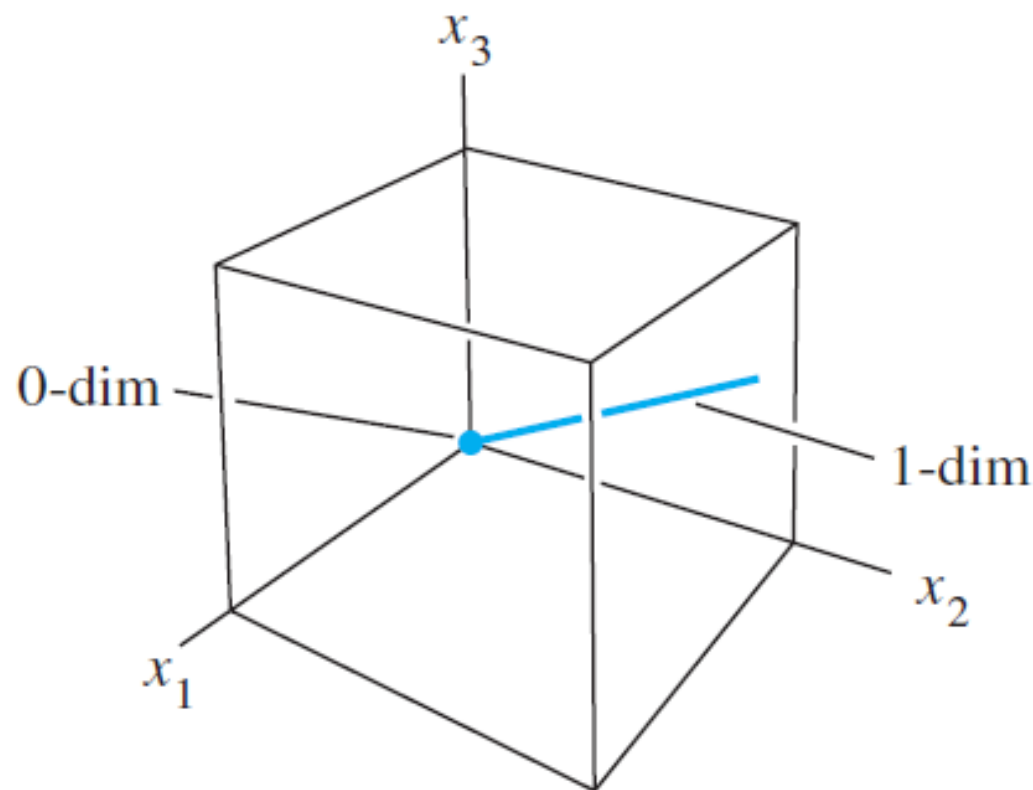
- The standard basis for \mathbb{R}^n is $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, so $\dim \mathbb{R}^n = n$.
- The standard basis for \mathbb{P}_n is $\{1, t, \dots, t^n\}$, so $\dim \mathbb{P}_n = n + 1$.
- Exercise: Show that $\dim M_{m \times n} = mn$.

Example: Let W be the set of vectors of the form $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$, where a, b can take any value. We showed (week 8 p20) that a basis for W is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. So $\dim W = 2$.

From the theorem on p2, we know that any set of 3 vectors in W must be linearly dependent, because $3 > \dim W$.

Example: We classify the subspaces of \mathbb{R}^3 by dimension:

- 0-dimensional: only the zero subspace $\{\mathbf{0}\}$.
- 1-dimensional, i.e. $\text{Span}\{\mathbf{v}\}$: lines through the origin.
- 2-dimensional, i.e. $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ where $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent: planes through the origin.
- 3-dimensional: by Invertible Matrix Theorem, 3 linearly independent vectors in \mathbb{R}^3 spans \mathbb{R}^3 , so the only 3-dimensional subspace of \mathbb{R}^3 is \mathbb{R}^3 itself.



Here is a counterpart to the spanning set theorem (week 8 p10):

Theorem 11: Linearly Independent Set Theorem: Let W be a subspace of a finite-dimensional vector space V . If $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a linearly independent set in W , we can find $\mathbf{v}_{p+1}, \dots, \mathbf{v}_n$ so that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for W .

Proof:

- If $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = W$, then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis for W .
- Otherwise $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ does not span W , so there is a vector \mathbf{v}_{p+1} in W that is not in $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$. Adding \mathbf{v}_{p+1} to the set still gives a linearly independent set. Continue adding vectors in this way until the set spans W . This process must stop after at most $\dim V - p$ additions, because a set of more than $\dim V$ elements must be linearly dependent.

The above logic proves something stronger:

Theorem 11 part 2: Subspaces of Finite-Dimensional Spaces: If W is a subspace of a finite-dimensional vector space V , then W is also finite-dimensional and $\dim W \leq \dim V$.

Because of the spanning set theorem and linearly independent set theorem:

Theorem 12: Basis Theorem: If V is a p -dimensional vector space, then

- i Any linearly independent set of exactly p elements in V is a basis for V .
- ii Any set of exactly p elements that span V is a basis for V .

In other words, to prove that \mathcal{B} is a basis of a p -dimensional vector space V , we only need to show **two of the following three** things (the third will be automatic):

- \mathcal{B} contains exactly p vectors;
 - \mathcal{B} is linearly independent;
 - $\text{Span}\mathcal{B} = V$.
- } If V is a subspace of U , these two statements are usually easier to check because we can work in the big space U (see p9 and p14).

Proof:

- i By the linearly independent set theorem, we can add elements to any linearly independent set to obtain a basis for V . But that larger set must contain exactly $\dim V = p$ elements. So our starting set must already be a basis.
- ii By the spanning set theorem, we can remove elements from any set that spans V to obtain a basis for V . But that smaller set must contain exactly $\dim V = p$ elements. So our starting set must already be a basis.

Summary:

- If V is spanned by a finite set, then V is finite-dimensional and $\dim V$ is the number of vectors in any basis for V .
- If V is not spanned by a finite set, then V is infinite-dimensional.
- If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ spans V , then some subset is a basis for V (week 8 p10).
- If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent and V is finite-dimensional, then it can be expanded to a basis for V (p4).

If $\dim V = p$ (so V and \mathbb{R}^p are isomorphic):

- Any set of more than p vectors in V is linearly dependent (p2).
- Any set of fewer than p vectors in V cannot span V (p2).
- Any linearly independent set of exactly p elements in V is a basis for V (p7).
- Any set of exactly p elements that span V is a basis for V (p7).

To prove that \mathcal{B} is a basis of V , show two of the following three things:

- \mathcal{B} contains exactly p vectors;
- \mathcal{B} is linearly independent;
- $\text{Span}\mathcal{B} = V$.

The basis theorem is useful for finding bases of subspaces:

Example:

Let $W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Is $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ a basis for W ?

Answer: We are given that $W = \text{Span} \{ \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4 \}$ and $\{ \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4 \}$ is a linearly independent set, so $\{ \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4 \}$ is a basis for W , and so $\dim W = 3$.

The vectors in \mathcal{B} are all in W , and \mathcal{B} consists of exactly 3 vectors, so it's enough to check whether \mathcal{B} is linearly independent.

Row reduction: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 3 & 5 & 2 \end{bmatrix} \xrightarrow{R_4 - 3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -7 \end{bmatrix} \xrightarrow{R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has a pivot

in each column, so \mathcal{B} is linearly independent, and is therefore a basis.

Note that we never had to work in W , only in \mathbb{R}^4 .

§4.6: Rank

Next we look at how the idea of dimension can help us answer questions about existence and uniqueness of solutions to linear systems.

Definition: The *rank* of a matrix A is the dimension of its column space.
The *nullity* of a matrix A is the dimension of its null space.

Example: Let $A = \begin{bmatrix} 5 & -3 & 10 \\ 7 & 2 & 14 \end{bmatrix}$, $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$.

A basis for $\text{Col}A$ is $\left\{ \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$ \longleftarrow one vector per pivot

A basis for $\text{Nul}A$ is $\left\{ \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$ \longleftarrow one vector per free variable

A basis for $\text{Row}A$ is $\{(1, 0, 1/2), (0, 1, 0)\}$. \longleftarrow one vector per pivot

So $\text{rank}A = 2$, $\text{nullity}A = 1$.

So $\text{rank}A + \text{nullity}A = ?$

Theorem 14:

Rank Theorem: $\text{rank}A = \dim \text{Col}A = \dim \text{Row}A = \text{number of pivots in } \text{rref}(A).$

Rank-Nullity Theorem: For an $m \times n$ matrix A ,

$$\text{rank}A + \text{nullity}A = n.$$

Proof: From our algorithms for bases of $\text{Col}A$ and $\text{Nul}A$ (see week 7 slides):
 $\text{rank}A = \text{number of pivots in } \text{rref}(A) = \text{number of basic variables},$
 $\text{nullity}A = \text{number of free variables}.$

Each variable is either basic or free, and the total number of variables is n , the number of columns.

An application of the Rank-Nullity theorem:

Example: Suppose a homogeneous system of 10 equations in 12 variables has a solution set that is spanned by two linearly independent vectors. Then the nullity of this system is 2, so the rank is $12 - 2 = 10$. So this system has 10 pivots. Since there are ten equations, there must be a pivot in every row, so any nonhomogeneous system with the same coefficients always has a solution.

Using our new ideas of dimension, we can add more statements to the Existence theorem, the Uniqueness theorem, and the Invertible Matrix Theorem:

Theorem 8: Invertible Matrix Theorem (IMT): For a square $n \times n$ matrix A , the following are equivalent:

$\text{rref}(A)$ has a pivot in every row.

$A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n .

The columns of A span \mathbb{R}^n .

The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

There is a matrix D such that $AD = I_n$.

$\text{Col}A = \mathbb{R}^n$.

$\text{rank}A = n$.

$\text{rref}(A)$ has a pivot in every column.

$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

The columns of A are linearly independent.

The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

There is a matrix C such that $CA = I_n$.

$\text{Nul}A = \{\mathbf{0}\}$.

$\text{nullity}A = 0$.

$\det A \neq 0$.

$\text{rref}(A) = I_n$.

$A\mathbf{x} = \mathbf{b}$ has a unique solution for each \mathbf{b} in \mathbb{R}^n .

The columns of A form a basis for \mathbb{R}^n .

The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is an invertible function.

A is an invertible matrix.

Advanced application of the Rank-Nullity Theorem and the Basis Theorem:

Redo Example: (p10) Let $A = \begin{bmatrix} 5 & -3 & 10 \\ 7 & 2 & 14 \end{bmatrix}$. Find a basis for $\text{Nul}A$ and $\text{Col}A$.

Answer: (a clever trick without any row-reduction)

- Observe that $2 \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$, so $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{0}$. So $\text{nullity}A \geq 1$.
- The first two columns of A are linearly independent (not multiples of each other), so $\left\{ \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$ is a linearly independent set in $\text{Col}A$, so $\text{rank}A \geq 2$.
- But $\text{rank}A + \text{nullity}A = 3$, so in fact $\text{rank}A = 2$ and $\text{nullity}A = 1$, and, by the Basis Theorem, the linearly independent sets we found above are bases:
so $\left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \right\}$ is a basis for $\text{Nul}A$, $\left\{ \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$ is a basis for $\text{Col}A$.

So for a general $m \times n$ matrix, it's enough to find k linearly independent vectors in $\text{Nul}A$ and $n - k$ linearly independent vectors in $\text{Col}A$.

The Rank-Nullity theorem also holds for linear transformations $T : V \rightarrow W$ whenever V is finite-dimensional (to prove it yourself, work through q8 of homework 5 from 2015):

$$\dim \text{range of } T + \dim \text{kernel of } T = \dim V.$$

Advanced application:

Example: Find a basis for $Q = \{\mathbf{p} \in \mathbb{P}_3 \mid \mathbf{p}(2) = 0\}$, i.e. polynomials $\mathbf{p}(t)$ of degree at most 3 with $\mathbf{p}(2) = 0$.

Answer: Remember (week 7 p43) that Q is the kernel of the evaluation-at-2 function $E_2 : \mathbb{P}_3 \rightarrow \mathbb{R}$ given by $E_2(\mathbf{p}) = \mathbf{p}(2)$,

$$E_2(a_0 + a_1t + a_2t^2 + a_3t^3) = a_0 + a_12 + a_22^2 + a_32^3.$$

E_2 is onto, so its range has dimension 1. So $\dim Q = \dim \mathbb{P}_3 - 1 = 4 - 1 = 3$.

Now $\mathcal{B} = \{(2 - t), (2 - t)^2, (2 - t)^3\}$ is a subset of Q , and is linearly independent (check with coordinate vectors relative to the standard basis of \mathbb{P}_3 , or because these three polynomials have different degrees - see week 8 p14-15). Since \mathcal{B} contains exactly 3 vectors, it is a basis for Q .