Remember that there are two rules for differentiating complicated functions: the chain rule and the product rule. (The quotient rule is a combination of these two rules, since $\frac{u}{v} = uv^{-1}$.)

Since FTC says that integration is antidifferentiation, we can derive from these differentiation rules two techniques of integration:

- chain rule $\longrightarrow$ method of substitution (p2-15, §5.6)
- product rule $\longrightarrow$ integration by parts (p16-22, §6.1)

These techniques are not rules. They do not give us the answer; they only change our integral to a new integral, which we hope will be easier to evaluate. There are no rules in integration: there is no guaranteed algorithm to integrate a function. Using the techniques require some creativity, and there are often multiple efficient ways to calculate the same integral.

§5.6: The Method of Substitution

(The letters used here are different from in the textbook.)

Recall the chain rule for differentiation: $\frac{d}{dx}F(g(x)) = F'(g(x))g'(x)$

Take the antiderivative of both sides: $F(g(x)) + C = \int F'(g(x))g'(x)\,dx$

Write u for $g(x)$: $F(u) + C = \int F'(u)\frac{du}{dx}\,dx$

Write f for F': $\int f(u)\,du = \int f(u)\frac{du}{dx}\,dx.$

Hence, if we can identify a function $u(x)$ such that our integrand is a product, of the composition $f(u(x))$ and the derivative $\frac{du}{dx}$ then we can rewrite our integral as $\int f(u)\,du$.

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du.$$

(i.e. we can treat $\frac{du}{dx}$ formally like a fraction)

Example: Evaluate $\int \cos(x^3)\,3x^2\,dx$.

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du.$$

Example: Evaluate $\int e^{3x}dx$.

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du.$$

Example: Evaluate $\int x\sqrt{1+x^2}\,dx$.

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du.$$

There are two ways to calculate a definite integral by substitution:

1. Find the indefinite integral and then substitute in the limits for x;
2. (Usually faster) Change the limits into limits for u.

Example: Evaluate $\int_0^1 x\sqrt{1+x^2}\,dx$.

Two other correct ways to use method 1:

$$\begin{aligned}&\int x\sqrt{1+x^2}\,dx\\ =&\int \frac{1}{2}\sqrt{u}\,du\\ =&\frac{u^{3/2}}{2(3/2)}+C\\ =&\frac{1}{3}\sqrt{1+x^2}^3+C,\end{aligned}$$

so

$$\begin{aligned}&\int_0^1 x\sqrt{1+x^2}\,dx\\ =&\frac{1}{3}\sqrt{1+x^2}^3\Big|_0^1=\frac{1}{3}(\sqrt{2}^3-1).\end{aligned}$$

$$\begin{aligned}&\int_0^1 x\sqrt{1+x^2}\,dx\\ =&\int_{x=0}^{x=1}\frac{1}{2}\sqrt{u}\,du\\ =&\frac{u^{3/2}}{2(3/2)}\Big|_{x=0}^{x=1}\\ =&\frac{1}{3}\sqrt{1+x^2}^3\Big|_0^1=\frac{1}{3}(\sqrt{2}^3-1).\end{aligned}$$

Do not write $\int_0^1 \frac{1}{2}\sqrt{u}\,du$ - that would mean you want to evaluate at $u=0,1$.

Note that the final two steps in method 1 are to change the indefinite integral from us to x, then substitute the limits of x. In method 2 below, we combine these two steps – simply substitute the corresponding limits for u.

Example: Evaluate $\int_0^1 x\sqrt{1+x^2}\,dx$.

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du.$$

Tips for choosing a good u:

- If the integrand contains a composite function e.g. $e^{g(x)}$, $\cos(g(x))$, $\sin(g(x))$, $\sqrt{g(x)}$, $\frac{1}{g(x)}$, try $u = g(x)$.
- Choose a u for which $\frac{du}{dx}$ appears in the integrand.

The best way to get better at choosing u is to do lots of problems, and think about why your chosen u was effective.

Very important: make sure your integrand is entirely in terms of u (no xs) before you start integrating.

Harder example: Evaluate $\int_0^1 x^3\sqrt{1-x^2}\,dx$.

Harder example: Evaluate $\int_0^1 \frac{x^2}{1+x^6}\,dx$.

Using various trigonometric identities and the method of substitution, we can obtain the integrals of many trigonometric functions - these will be given to you on the exams.

Examples:

$$\begin{aligned}
\int \cos^2 x\,dx &= \int \frac{1}{2}(1+\cos(2x))\,dx && \text{by the identity } \cos(2x) = 2\cos^2 x - 1\\
&= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C && \text{substitution } u = 2x \text{ in the second term}\\
&= \frac{1}{2}x + \frac{1}{2}\sin x\cos x + C. && \text{by the identity } \sin(2x) = 2\sin x\cos x
\end{aligned}$$

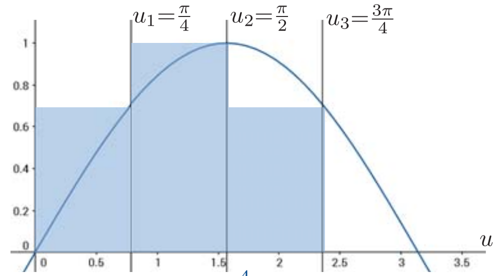
$$\begin{aligned}
\int \cos^3 x\,dx &= \int \cos x(1-\sin^2 x)\,dx && \text{by the identity } \cos^2 x + \sin^2 x = 1\\
&= \int \cos x - \cos x\sin^2 x\,dx\\
&= \sin x - \frac{1}{3}\sin^3 x + C. && \text{substitution } u = \sin x \text{ in the second term}
\end{aligned}$$

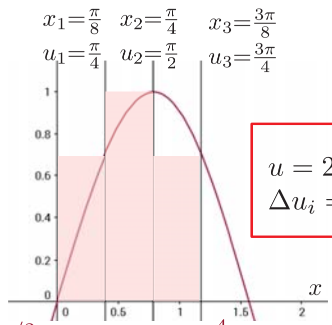
The full list of trigonometric-power integrals you will be given in exams:

$$\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x \cos x) + C,$$
$$\int \sin^3 x\,dx = -\cos x + \frac{1}{3}\cos^3 x + C.$$
$$\int \sin^4 x\,dx = \frac{1}{8}(3x - 3\sin x\cos x - 2\sin^3 x \cos x) + C,$$

$$\int \cos^2 x\,dx = \frac{1}{2}(x + \sin x \cos x) + C,$$
$$\int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x + C.$$
$$\int \cos^4 x\,dx = \frac{1}{8}(3x + 3\sin x\cos x + 2\cos^3 x \sin x) + C.$$

To prepare for a multivariate version of substitution (see the final week), we need to understand single-variable substitution geometrically, i.e. in terms of approximating the area under a curve with Riemann sums:





$u = 2x$, and $\Delta u_i = 2\Delta x_i$.

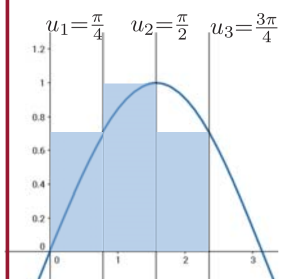
$$\int_0^{\pi} \sin u\,du \approx \sum_{i=1}^{4} \sin(u_i)\Delta u_i.$$

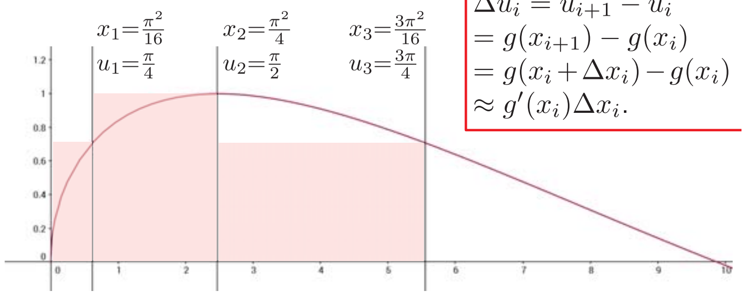
$$\int_0^{\pi/2} \sin(2x)\,dx \approx \sum_{i=1}^{4} \sin(2x_i)\Delta x_i$$
$$= \sum_{i=1}^{4} \sin(u_i)\frac{1}{2}\Delta u_i \approx \int_0^{\pi} \sin u\,\frac{1}{2}du.$$

The heights of the two sets of approximating rectangles are the same, but on the right the rectangles are half as wide.

When u is not a linear function of x, the widths of the rectangles stretch by different amounts.

When $u = g(x)$, then
$$\Delta u_i = u_{i+1} - u_i$$
$$= g(x_{i+1}) - g(x_i)$$
$$= g(x_i + \Delta x_i) - g(x_i)$$
$$\approx g'(x_i)\Delta x_i.$$





$$\int_0^{\pi} \sin u\,du \approx \sum_{i=1}^{4} \sin(u_i)\Delta u_i.$$

$$\int_0^{\pi^2} \sin\sqrt{x}\,dx \approx \sum_{i=1}^{4} \sin\sqrt{x_i}\Delta x_i$$

In this example, $u = \sqrt{x}$, so
$$\Delta u_i \approx \frac{1}{2\sqrt{x_i}}\Delta x_i = \frac{1}{2u}\Delta x_i. \longrightarrow \approx \sum_{i=1}^{4} \sin(u_i)2u\,\Delta u_i \approx \int_0^{\pi} \sin u\,2u\,du.$$

§6.1: Integration by Parts

Recall the product rule for differentiation:
$$\frac{d}{dx}(U(x)V(x)) = U(x)\frac{dV}{dx} + V\frac{dU}{dx}$$
Take the antiderivative of both sides:
$$U(x)V(x) = \int U(x)\frac{dV}{dx}\,dx + \int V\frac{dU}{dx}\,dx$$
Rearranging:
$$\int U(x)\frac{dV}{dx}\,dx = U(x)V(x) - \int V\frac{dU}{dx}\,dx$$
A shorthand that is easy to remember:
$$\int U\,dV = UV - \int V\,dU$$

Standard example: Evaluate $\int xe^x\,dx$.

$$\int U\,dV = UV - \int V\,dU$$

Standard example: Evaluate $\int x\sin x\,dx$.

$$\int U\,dV = UV - \int V\,dU$$

Standard example: Evaluate $\int x\ln x\,dx$.

$$\int U\,dV = UV - \int V\,dU$$

$$\int U\,dV = UV - \int V\,dU$$

The technique of integration by parts relies on separating your integrand into two parts, a U and a $\frac{dV}{dx}$. Because we need to calculate $\int V\,dU$, we want U to be easy to differentiate and V to be easy to integrate. One good strategy to choose these parts is the DETAIL rule:

dV should be the part of the integrand that appears highest in this list:

$$\left.\begin{array}{l}\text{Exponential: } e^x\\ \text{Trigonometric: } \sin x,\ \cos x\end{array}\right\}\text{nice to integrate}$$

Algebraic: x^n

$$\left.\begin{array}{l}\text{Inverse trigonometric: } \sin^{-1} x,\ \tan^{-1} x\\ \text{Logarithmic: } \ln x\end{array}\right\}\text{hard to integrate}$$

In our previous examples:
xe^x (p17) is a product of an algebraic and an exponential function, and exponential is higher on the list, so $dV = e^x dx$ and $U = x$.
$x\ln x$ (p19) is a product of an algebraic and a logarithmic function, and algebraic is higher on the list, so $dV = xdx$ and $U = \ln x$.

Sometimes, after integration by parts, our new integral again requires integration by parts:

Example: Evaluate $\int_0^2 (xe^x)^2\,dx$.

$$\int U\,dV = UV - \int V\,dU$$

Some integrals are best calculated using a substitution and then integration by parts. (It can also happen that, after integration by parts, the new integral requires a substitution.)

Example: Evaluate $\int x^3 e^{x^2}\,dx$.

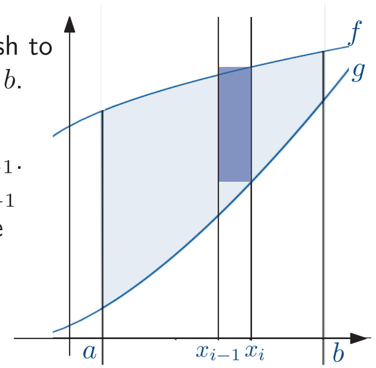
$$\int U\,dV = UV - \int V\,dU$$

§5.7: Areas of Plane Regions

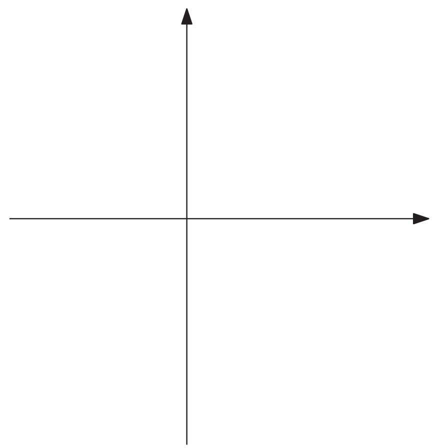
This is a simple, first example where we calculate a geometric quantity by writing it as a limit of a Riemann sum, identifying it as an integral, and then using FTC2. (We will do more of this in higher dimensions).

Given functions $f, g : [a, b] \to \mathbb{R}$ with $f(x) \geq g(x)$ we wish to find the area bounded by $y = f(x), y = g(x), x = a, x = b$.

1. Divide [a,b] into n subintervals by choosing x_i with $a = x_0 < x_1 < \cdots < x_n = b$, and let $\Delta x_i = x_i - x_{i-1}$.
2. Approximate the part of the desired area between x_{i-1} and x_i by a rectangle, whose width is Δx_i and whose height is $f(x_i^*) - g(x_i^*)$, for some $x_i^* \in [x_{i-1}, x_i]$.
3. So the area is
$$\lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\Delta x_i = \int_a^b f(x) - g(x)\,dx.$$



Example: Find the area of the region bounded by $y = x^2 - 4$ and $y = -x^2 + 2x$.



Example: Find the area of the region bounded by $y = 2\sqrt{x}$, $y = 3 - x$ and $y = 0$.