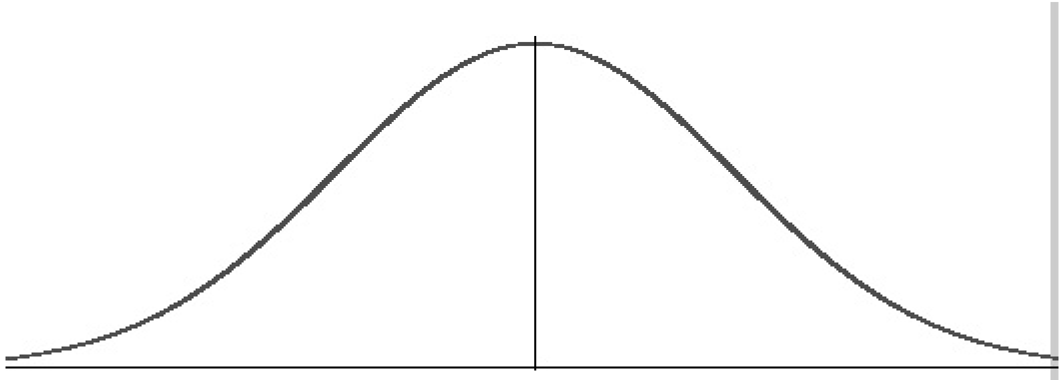
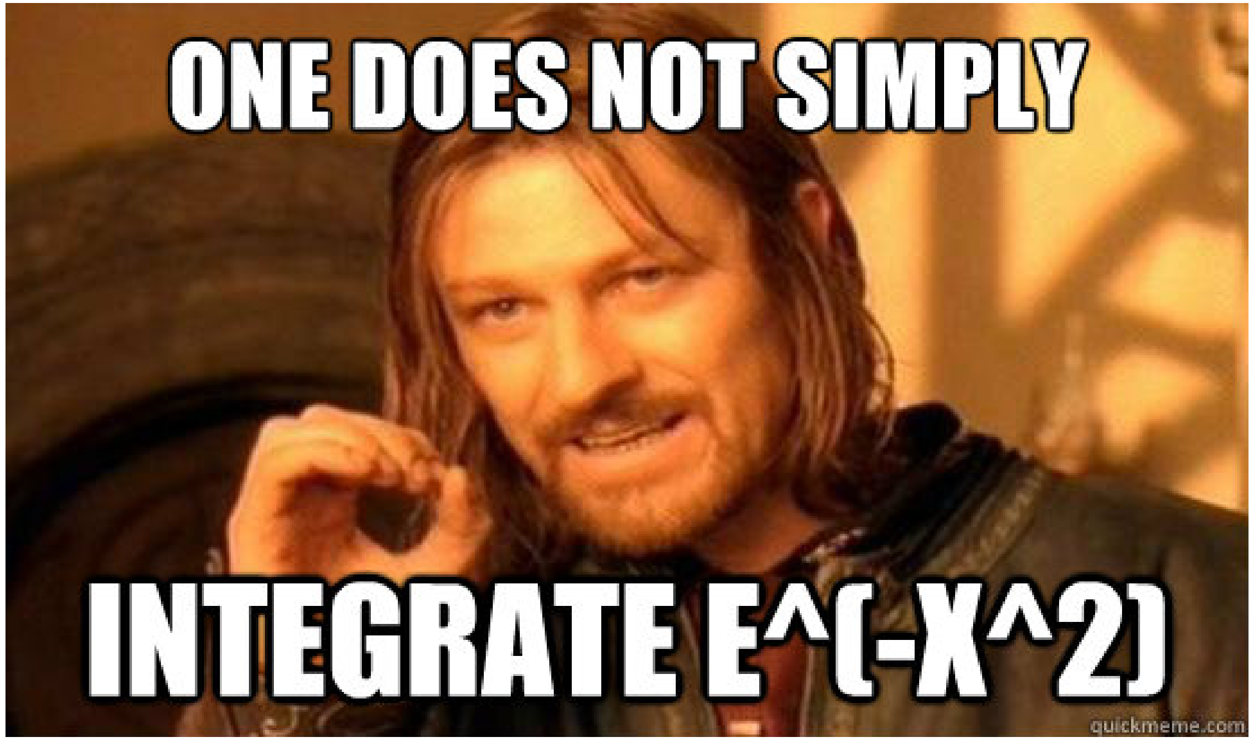
Application 1 of multivariate calculus to probability/statistics: the Gaussian integral

The normal distribution (also called the Gaussian distribution) has probability density function proportional to e^{-x^2}, i.e. $p(x) = \frac{1}{Z}e^{-x^2}$ for some Z. We need to choose the constant of proportionality Z so that the total probability is 1, i.e.

$$Z = \int_{-\infty}^{\infty} e^{-x^2}\, dx.$$





The problem is, we cannot write down the antiderivative of e^{-x^2} - it is not an elementary function. But multiple integration will help us in a surprising, clever way.

First, we need to show that the improper integral $Z = \int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges.

It will be enough to show that $Z' = \int_{1}^{\infty} e^{-x^2}\,dx$ converges, because then

$$Z = \int_{-\infty}^{-1} e^{-x^2}\,dx + \int_{-1}^{1} e^{-x^2}\,dx + \int_{1}^{\infty} e^{-x^2}\,dx = Z' + \int_{-1}^{1} e^{-x^2}\,dx + Z'.$$

We show that Z' converges using the (non-examinable) technique of integral estimation by inequalities:

e^{-x^2} is a non-negative function, so FTC1 says that $F(R) = \int_{1}^{R} e^{-x^2}\,dx$ is an increasing function (in R).

So, using some theorems from analysis, we know that, if there is a number M such that $M \geq F(R) = \int_{1}^{R} e^{-x^2}\,dx$ for every $R > 1$, then $\lim_{R\to\infty} F(R)$ exists, i.e. Z' converges.

For all $x \geq 1$, we have $e^{-x^2} \leq e^{-x}$, so $\int_{1}^{R} e^{-x^2}\,dx \leq \int_{1}^{R} e^{-x}\,dx = e^{-1} - e^{-R} \leq e^{-1}$,

so e^{-1} is the upper bound M that we want.

Reminder: we wish to evaluate $Z = \int_{-\infty}^{\infty} e^{-x^2}\,dx$.

Consider the double integral $\iint_{\mathbb{R}^2} e^{-x^2}e^{-y^2}\,dA$. The integrand is always positive, so we can calculate this improper integral using an iterated integral:

$$\iint_{\mathbb{R}^2} e^{-x^2}e^{-y^2}\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dy\,dx = \int_{-\infty}^{\infty} e^{-x^2} Z\,dx = Z^2$$

But we can also calculate this double integral using polar coordinates (yes, you can use change of variables on improper integrals):

$$\iint_{\mathbb{R}^2} e^{-x^2}e^{-y^2}\,dA = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

inner integral independent of θ, and substitution $u = r^2$

$$= 2\pi\int_0^{\infty}\frac{e^{-u}}{2}\,du = \pi\lim_{R\to\infty}\int_0^{R} e^{-u}\,du = \pi\left(\lim_{R\to\infty} 1 - e^{-R}\right) = \pi.$$

So $Z^2 = \pi$, i.e. $Z = \sqrt{\pi}$.