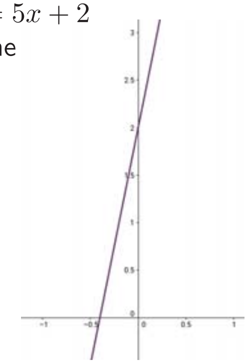
What is Linear Algebra?

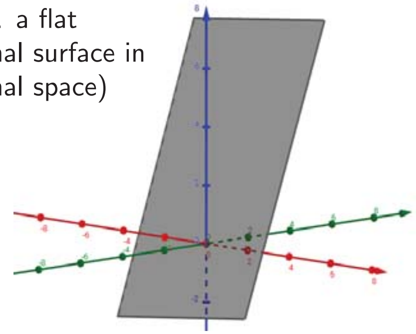
Linear algebra is the study of linear equations.

We will think about linear equations in many different ways in this class, e.g. geometrically. (You will NOT be tested on drawing, but it is useful to imagine the pictures.)

e.g. $y = 5x + 2$
a line



$z = 2x + 3y$
a plane (i.e. a flat 2-dimensional surface in 3-dimensional space)



To do well in this class, you must understand the connections between the different points of view.

This class is about more than computations. From the official syllabus:

Course Intended Learning Outcomes (CILOs):

Upon successful completion of this course, students should be able to:

No.	Course Intended Learning Outcomes (CILOs)
1	Explain the $\boxed{\text{concept/theory}}$ in linear algebra, to develop dynamic and graphical views to the related issues of the chosen topics as outlined in "course content," and to $\boxed{\text{formally prove theorems}}$

This class will introduce you to some basic proof techniques and some ways to think about abstract concepts (this is good preparation for Math 2215 Mathematical Analysis, and good training for your brain).

(Week 1 is straightforward computation; we will start the abstract theory in Week 2.)

§1.1: Systems of Linear Equations

Linear Algebra is the study of linear equations.

Example: $y = 5x + 2$ is a linear equation. We can take all the variables to the left hand side and rewrite this as $(-5)x + (1)y = 2$.

Example: $3(x_1 + 2x_2) + 1 = x_1 + 1 \longrightarrow (2)x_1 + (6)x_2 = 0$

Example: $x_2 = \sqrt{2}(\sqrt{6} - x_1) + x_3 \longrightarrow \sqrt{2}x_1 + (1)x_2 + (-1)x_3 = 2\sqrt{3}$

The following two equations are not linear, why?

$x_2 = 2\sqrt{x_1}$ $\qquad$ $xy + x = e^5$ $\qquad$ The problem is that the variables are not only multiplied by numbers.

In general, a linear equation is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

$x_1, x_2, \dots x_n$ are the variables. $\qquad$ $a_1, a_2, \dots a_n$ are the coefficients.

A linear equation has the form $a_1x_1 + a_2x_2 + \dots a_nx_n = b$.

Definition: A *system of linear equations* (or a *linear system*) is a collection of linear equations involving the same set of variables.

Example:

$$\begin{aligned} x \quad + y \qquad\quad &= 3 \\ 3x \qquad\quad + 2z &= -2 \end{aligned}$$

is a system of 2 equations in 3 variables, x, y, z. Notice that not every variable appears in every equation.

Definition: A *solution* of a linear system is a list $(s_1, s_2, \dots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, \dots, s_n$ are substituted for $x_1, x_2, \dots, x_n$ respectively.

Definition: The *solution set* of a linear system is the set of all possible solutions.

Example: One solution to the above system is $(x, y, z) = (2, 1, -4)$, because $2 + 1 = 3$ and $3(2) + 2(-4) = -2$.

Question: Is there another solution? How many solutions are there?

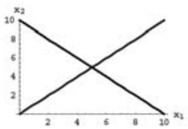
Definition: A linear system is *consistent* if it has a solution,
and *inconsistent* if it does not have a solution.

Fact: (which we will prove in the next class) A linear system has either

- exactly one solution — consistent
- infinitely many solutions — consistent
- no solutions — inconsistent

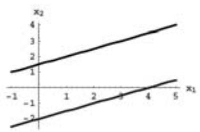
EXAMPLE Two equations in two variables:

$$\begin{aligned} x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0 \end{aligned}$$



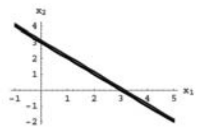
one unique solution
consistent

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8 \end{aligned}$$



no solution
inconsistent

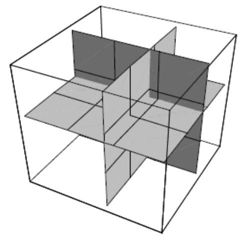
$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$



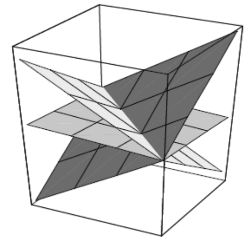
infinitely many solutions
consistent

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

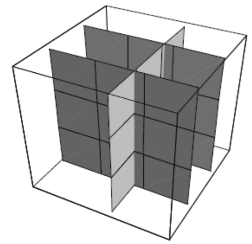
i) The planes intersect in one point. *(one solution)*



ii) The planes intersect in one line. *(infinitely many solutions)*



iii) There is no point in common to all three planes. *(no solution)*



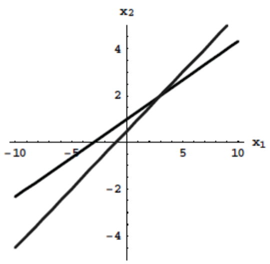
Which of these cases are consistent?

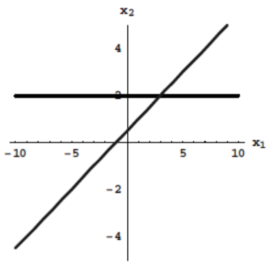
consistent — consistent — inconsistent

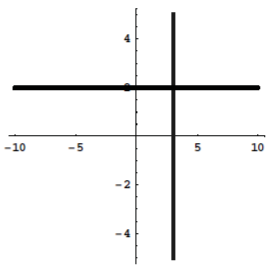
Our goal for this week is to develop an efficient algorithm to solve a linear system.

Example:

$$\begin{array}{lrcl} R_1 & x_1 - 2x_2 &=& -1 \\ R_2 & -x_1 + 3x_2 &=& 3 \end{array} \xrightarrow{} \begin{array}{lrcl} & x_1 - 2x_2 &=& -1 \\ R_2 + R_1 \to & x_2 &=& 2 \end{array} \quad \begin{array}{lrcl} R_1 + 2R_2 \to & x_1 &=& 3 \\ & x_2 &=& 2 \end{array}$$







Definition: Two linear systems are *equivalent* if they have the same solution set.

So the three linear systems above are different but equivalent.

A general strategy for solving a linear system: replace one system with an equivalent system that is easier to solve.

We simplify the writing by using matrix notation, recording only the coefficients and not the variables

$$\begin{array}{lrcl} R_1 & x_1 - 2x_2 &=& -1 \\ R_2 & -x_1 + 3x_2 &=& 3 \end{array} \xrightarrow{} \begin{array}{lrcl} & x_1 - 2x_2 &=& -1 \\ R_2 + R_1 \to & x_2 &=& 2 \end{array} \quad \begin{array}{lrcl} R_1 + 2R_2 \to & x_1 &=& 3 \\ & x_2 &=& 2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

coefficient of x_1 — coefficient of x_2 — right hand side

The augmented matrix of a linear system contains the right hand side: $\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$

The coefficient matrix of a linear system is the left hand side only: $\left[\begin{array}{cc} 1 & -2 \\ -1 & 3 \end{array}\right]$

(The textbook does not put a vertical line between the coefficient matrix and the right hand side, but I recommend that you do to avoid confusion.)

$$\begin{array}{lrcrcr} R_1 & x_1 & - & 2x_2 & = & -1 \\ R_2 & -x_1 & + & 3x_2 & = & 3 \end{array} \quad \xrightarrow{R_2+R_1\to} \quad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array} \quad \xrightarrow{R_1+2R_2\to} \quad \begin{array}{rcrcr} x_1 & & & = & 3 \\ & & x_2 & = & 2 \end{array}$$

$$\left[\begin{array}{rr|r} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{rr|r} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

In this example, we solved the linear system by applying elementary row operations to the augmented matrix (we only used 1. above, the others will be useful later):

1. Replacement: add a multiple of one row to another row. $R_i \to R_i + cR_j$
2. Interchange: interchange two rows. $R_i \to R_j,\ R_j \to R_i$
3. Scaling: multiply all entries in a row by a nonzero constant. $R_i \to cR_i,\ c \neq 0$

Definition: Two matrices are *row equivalent* if one can be transformed into the other by a sequence of elementary row operations.

Fact: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set, i.e. they are equivalent linear systems.

General strategy for solving a linear system: do row operations to its augmented matrix to get an equivalent system that is easier to solve.

EXAMPLE:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ & & & \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ & & & \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & x_2 & & & = & \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ & & & \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & & & & & = & \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} & & & \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Solution: $(x_1, x_2, x_3) = (29, 16, 3)$

Check: Is $(29, 16, 3)$ a solution of the ***original*** system?

Warning: Do not do multiple elementary row operations at the same time, except adding multiples of the same row to several rows.

$$\begin{aligned} x_1 - 2x_2 &= 1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \qquad \begin{aligned} x_2 &= 2 \\ x_2 &= 2 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \begin{array}{l} \leftarrow R_1 + R_2 \\ \leftarrow R_2 + R_1 \end{array}$$

These are NOT equivalent systems: in the system on the right, x_1 can take any value, which is not true for the system on the left.

$$\begin{aligned} x_1 - 2x_2 \quad &= -3 \\ x_2 \quad &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} \leftarrow R_1 - R_3 \\ \leftarrow R_2 + 4R_3 \\ \end{array}$$

$$\begin{aligned} x_1 \quad &= 29 \\ x_2 \quad &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Sometimes we are not interested in the exact value of the solutions, just the number of solutions. In other words:

1. Existence of solutions: is the system consistent?
2. Uniqueness of solutions: if a solution exists, is it the only one?

Answering this requires less work than finding the solution.

Example:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

We can stop here: back-substitution shows that we can find a unique solution.

EXAMPLE: Is this system consistent?

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \\ 3x_2 - 6x_3 &= 8 \end{aligned}$$

EXAMPLE: For what values of h will the following system be consistent?

$$x_1 - 3x_2 = 4$$

$$-2x_1 + 6x_2 = h$$

Section 1.2: Row Reduction and Echelon Forms

Motivation: it is easy to solve a linear system whose augmented matrix is in reduced echelon form

Echelon form (or **row echelon form):**

1. All nonzero rows are above any rows of all zeros.
2. Each *leading entry* (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

EXAMPLE: Echelon forms

(a) $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$

Reduced echelon form: Add the following conditions to conditions 1, 2, and 3 above:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

EXAMPLE (continued):

Reduced echelon form :

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

EXAMPLE: Are these matrices in echelon form, reduced echelon form, or neither?

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \boxed{\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]}$$

echelon form

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \boxed{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]}$$

reduced echelon form

Here is the example from p10. Notice that we use row operations to first put the matrix into echelon form, and then into reduced echelon form.

Can we always do this for any linear system?

Theorem: Any matrix A is row-equivalent to exactly one reduced echelon matrix, which is called its reduced echelon form and written $\text{rref}(A)$.

So our general strategy for solving a linear system is: apply row operations to its augmented matrix to obtain its rref.

And our general strategy for determining existence/uniqueness of solutions is: apply row operations to its augmented matrix to obtain an echelon form, i.e. a row-equivalent echelon matrix.

Warning: an echelon form is not unique. Its entries depend on the row operations we used. But its pattern of ■ and $*$ is unique.

These processes of row operations (to get to echelon or reduced echelon form) are called row reduction.

Row reduction:

augmented matrix of linear system → echelon form → reduced echelon form

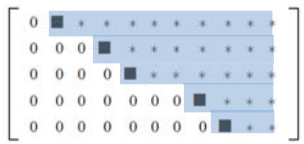
echelon form ↓ existence and uniqueness of solutions

reduced echelon form ↓ solution set

The rest of this section:
- The row reduction algorithm
- Getting the solution, existence/uniqueness from the (reduced) echelon form

Important terms in the row reduction algorithm:
- pivot position: the position of a leading entry in a row-equivalent echelon matrix.
- pivot: a nonzero entry of the matrix that is used in a pivot position to create zeroes below it.
- pivot column: a column containing a pivot position.

The black squares are the pivot positions.



Row reduction algorithm:

EXAMPLE:
$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{array}\right]$$

1. The top of the leftmost nonzero column is a pivot position.
2. Put a pivot in this position, by scaling or interchanging rows.

$$\begin{array}{c} \left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right] \end{array} \begin{array}{c} R_3 \\ \\ R_1 \end{array}$$

3. Create zeroes in all positions below the pivot, by adding multiples of the top row to each row.

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ & & & & & \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

4. Ignore this row and all rows above, and repeat steps 1-3.

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

1. The top of the leftmost nonzero column is a pivot position.
2. Put a pivot in this position, by scaling or interchanging rows.

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ & & & & & \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

3. Create zeroes in all positions below the pivot, by adding multiples of the top row to each row.

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ & & & & & \end{array}\right]$$

4. Ignore this row and all rows above, and repeat steps 1-3.

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

1. The top of the leftmost nonzero column is a pivot position.
2. Put a pivot in this position, by scaling or interchanging rows.
3. Create zeroes in all positions below the pivot, by adding multiples of the top row to each row.

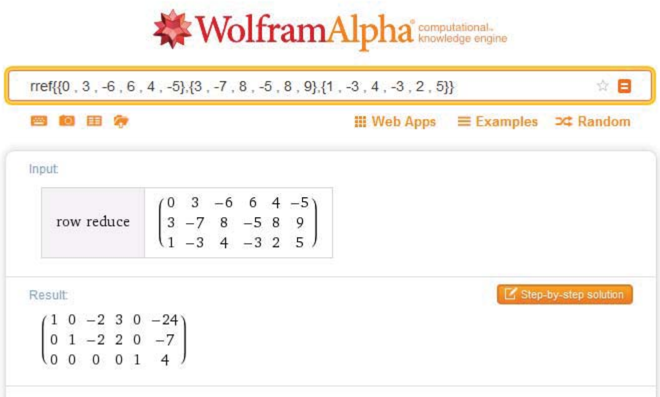
We are at the bottom row, so we don't need to repeat anymore. We have arrived at an echelon form.

5. To get from echelon to reduced echelon form (back substitution):
   Starting from the bottom row: for each pivot, add multiples of the row with the pivot to the other rows to create zeroes above the pivot.

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \begin{array}{l} R_1 - 2R_3 \\ R_2 - R_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} & & & & & \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Check your answer: www.wolframalpha.com



Getting the solution set from the reduced echelon form:

A basic variable is a variable corresponding to a pivot column.
All other variables are free variables.

6. Write each row of the augmented matrix as a linear equation.

Example:

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \qquad \begin{aligned} x_1 \quad -2x_3 + 3x_4 &= -24 \\ x_2 - 2x_3 + 2x_4 &= -7 \\ x_5 &= 4 \end{aligned}$$

basic variables: x_1, x_2, x_5, free variables: x_3, x_4.

The free variables can take any value. These values then uniquely determine the basic variables.

Example:

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \qquad \begin{aligned} x_1 \quad -2x_3 + 3x_4 &= -24 \\ x_2 - 2x_3 + 2x_4 &= -7 \\ x_5 &= 4 \end{aligned}$$

basic variables: x_1, x_2, x_5, free variables: x_3, x_4.

The free variables can take any value. These values then uniquely determine the basic variables.

7. Take the free variables in the equations to the right hand side, and add equations of the form "free variable = itself", so we have equations for each variable in terms of the free variables.

Example:

$$\begin{aligned} x_1 &= -24 + 2x_3 - 3x_4 \\ x_2 &= -7 + 2x_3 - 2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \\ x_5 &= 4 \end{aligned}$$

So the solution set is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -24 + 2s - 3t \\ -7 + 2s - 2t \\ s \\ t \\ 4 \end{pmatrix}$$

where s and t can take any value.

Example: Suppose we found that the reduced echelon form of the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array}\right]$$

The last equation says $0x_1 + 0x_2 + 0x_3 = 15$, so this system is inconsistent.

Theorem 2: Existence and Uniqueness:
A linear system is consistent if and only if an echelon form of its augmented matrix has no row of the form $[0 \dots 0|*]$ with $* \neq 0$.

If a linear system is consistent, then:
- it has a unique solution if there are no free variables;
- it has infinitely many solutions if there are free variables.

In particular, this proves the fact we saw earlier, that a linear system has either a unique solution, infinitely many solutions, or no solutions.
Warning: In general, the existence of solutions is unrelated to the uniqueness of solutions. (We will meet an important exception in §2.3.)