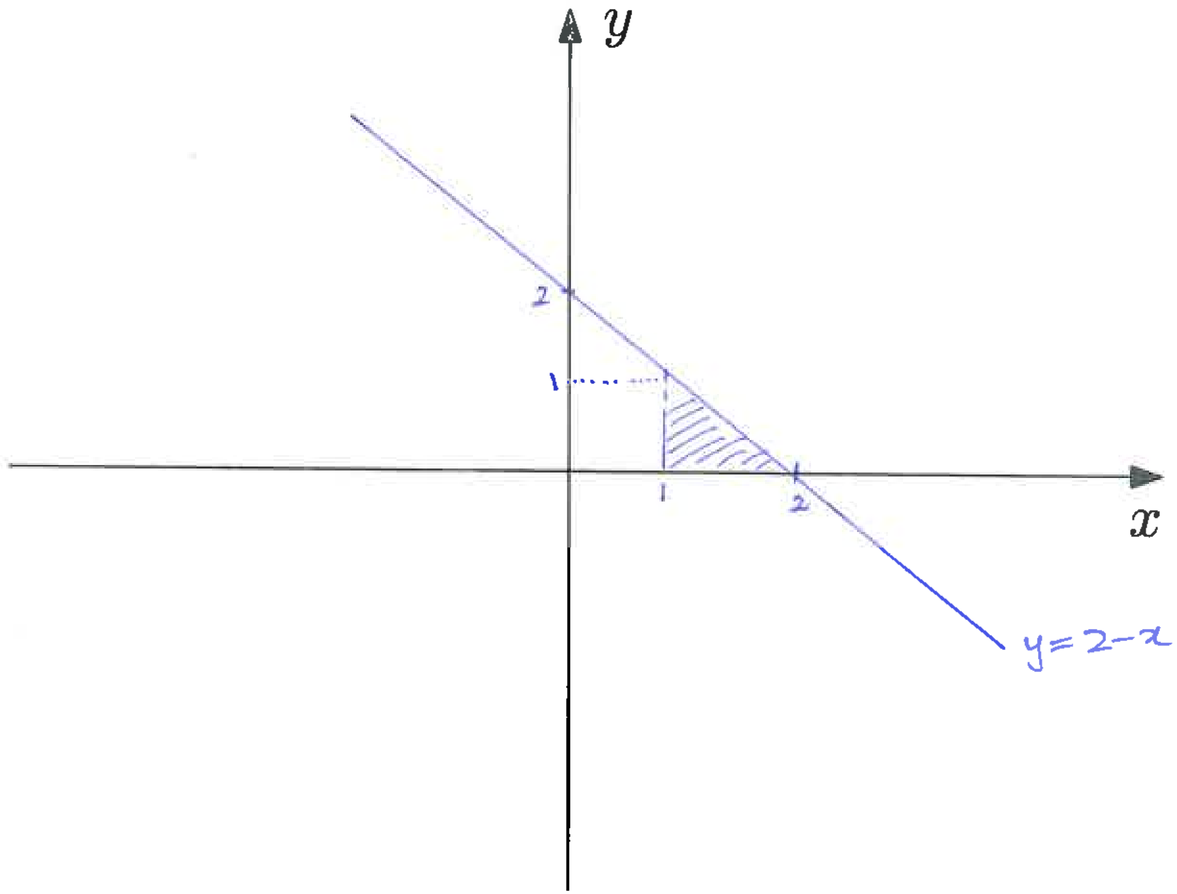
Example: By drawing a graph and using geometry, determine $\int_1^2 2 - x\,dx$.

$\int_1^2 2-x\,dx$

$= \text{area of the triangle}$

$= \frac{1}{2}\, 1 \cdot 1$

$= \frac{1}{2}$



$$\int_0^1 \sqrt{1-x^2}\,dx$$

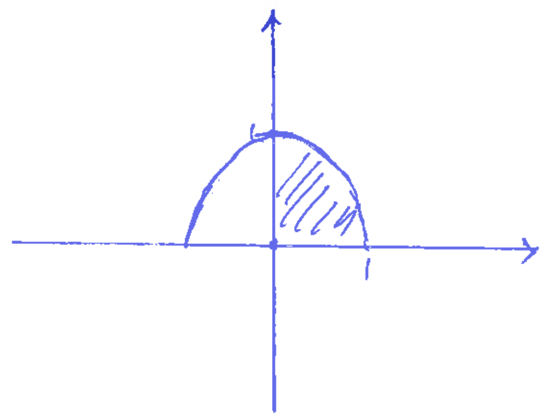
$= \text{area of quarter-circle}$

$= \dfrac{\pi}{4}.$

$y = \sqrt{1-x^2}$

$y^2 = 1-x^2$

$x^2 + y^2 = 1$



Even when we know how to calculate definite integrals using antiderivatives, it can be useful to use geometry.

To simplify the notation when using FTC2, we write $F(x)\big|_a^b$ to mean $F(b) - F(a)$. (The alternative notation $[F(x)]_a^b$ will also be accepted.)

Recall that the symbol $\int f(x)\,dx$ means the general antiderivative of f. So FTC2 says $\displaystyle\int_a^b f(x)\,dx = \left(\int f(x)\,dx\right)\Big|_a^b$.

Redo Example: (Q1 ex. sheet #5) Compute $\displaystyle\int_{-3}^{1} 2x\,dx$ using FTC2.

$\frac{d}{dx}(x^2) = 2x$

$$= x^2\Big|_{-3}^{1}$$

$$= 1^2 - (-3)^2 = -8$$

Redo Example: (p5-8) Compute $\int_0^1 x^2 + 1\,dx$ using FTC2.

$$= \int_0^1 x^2\,dx + \int_0^1 1\,dx$$

$$\frac{d}{dx}(x^3) = 3x^2$$

$$= \left(\frac{x^3}{3} + x\right)\Big|_0^1$$

$$= \left(\frac{1}{3} + 1\right) - \left(\frac{0}{3} + 0\right) = \frac{4}{3}$$

Redo Example: (p10) Compute $\int_0^2 2 + \cos x \, dx$ using FTC2.

$$= \left(2x + \sin x\right)\Big|_0^2$$

$$= (4 + \sin 2) - (0 + 0) = 4 + \sin 2$$

Our theorem (p15) says that piecewise continuous functions are integrable.
Here's an example of how to calculate such integrals:

Example: Compute $\int_1^5 f(x)\,dx$, where f is given by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } 1 \le x < 2 \\ \frac{x}{2} & \text{if } 2 \le x < 4 \\ 1 & \text{if } \quad x = 4 \\ -3x + 14 & \text{if } 4 < x \le 5. \end{cases}$$

$A_1 + A_2 + (A_3 - A_4)$

$$\int_1^5 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx + \int_4^5 f(x)\,dx$$

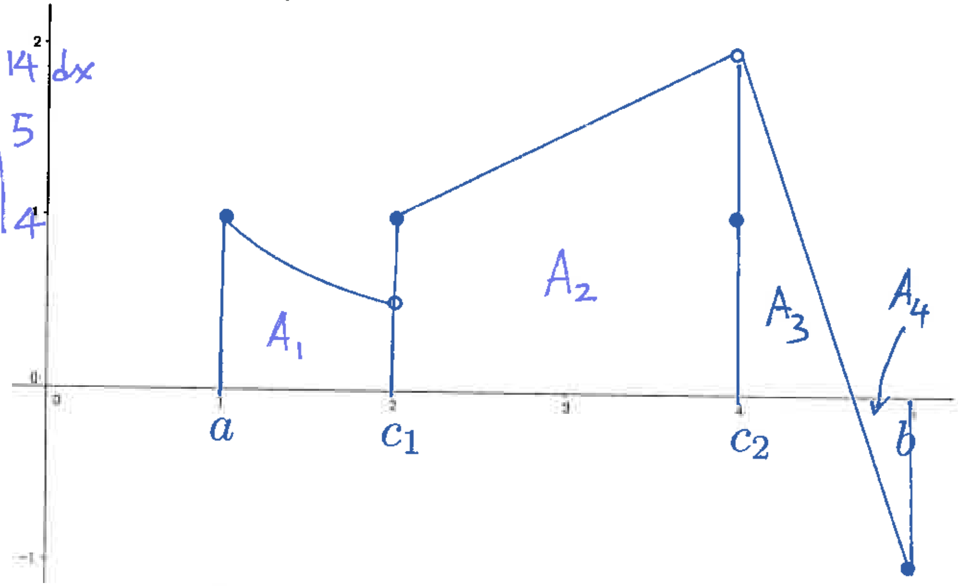
$$= \int_1^2 \frac{1}{x}\,dx + \int_2^4 \frac{x}{2}\,dx + \int_4^5 -3x+14\,dx$$

$$= \ln|x|\Big|_1^2 + \frac{x^2}{4}\Big|_2^4 + \left(\frac{-3x^2}{2} + 14x\right)\Big|_4^5$$

$$= \ln 2 - \cancel{\ln 1}^{\,0} + \left(\frac{16}{4} - \frac{4}{4}\right) - \frac{3}{2}(5^2 - 4^2) + 14(5-4)$$

$$= \ln 2 + \frac{7}{2}$$

$f(x) \neq \frac{1}{x}$ at $x = 2$ etc. This step will be explained on the next page



This techinque also works for continuous functions defined by different formulae on different subintervals, e.g. functions involving absolute values: recall

$$|g(x)| = \begin{cases} g(x) & \text{if } g(x) \geq 0 \\ -g(x) & \text{if } g(x) < 0. \end{cases}$$

how to find out when $g(x)$ is positive/negative: (if $g(x)$ is continuous): solve $g(x)=0$.

Example: Compute $\int_{-3}^{4} |x+1| + |x-1| \, dx$.

division points:
$x+1=0 \to x=-1$
$x-1=0 \to x=1$

integrand is $\begin{cases} -(x+1)-(x-1) = -2x & x \leq -1 \\ (x+1)-(x-1) = 2 & -1 < x \leq 1 \\ (x+1)+(x-1) = 2x & 1 < x \end{cases}$

integral is $\int_{-3}^{-1} -2x \, dx + \int_{-1}^{1} 2 \, dx + \int_{1}^{4} 2x \, dx$

$= -x^2 \Big|_{-3}^{-1} + 2x \Big|_{-1}^{1} + x^2 \Big|_{1}^{4} = -(1^2 - 3^2) + (2-(-2)) + (16-1) = 27$