You must justify your answers to receive full credit.

1. Let $A$ be the matrix
$$A = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 1 & 0 \\ 1 & 3 & 4 \\ 2 & -2 & -3 \end{bmatrix},$$
and $T$ be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.
   a) What is the domain of $T$?
   b) What is the codomain of $T$?
   c) Find the kernel of $T$.
   d) Is $T$ one-to-one?
   e) Find the image of $\mathbf{e_1} + \mathbf{e_2}$ under $T$.
   f) Let $T^*$ be the linear transformation $T^*(\mathbf{x}) = A^T\mathbf{x}$. What is the domain of $T^*$?

2. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation satisfying
$$F\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 2 \\ 8 \end{bmatrix}, \quad F\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \\ 12 \end{bmatrix}.$$
   a) Find the standard matrix of $F$.
   b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection through the $x_2$-axis. Find the standard matrix of the composition $F \circ T$.

3. Let
$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$
Calculate the following quantities, or explain why they are not defined.

| | | | |
|---|---|---|---|
| a) $AB$ | b) $BAB^T$ | c) $B + I_3$ | d) $x^T B$ |
| e) $B^3$ | f) $A^3$ | g) $A^{301}$ | h) $\det B$ |

4. Let
$$A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & -2 & 3 & 5 \\ 1 & p & 1 & 3 \end{bmatrix}.$$
   a) Find all values of $p$ such that $A$ is invertible.
   b) Find the inverse of $A$ when $p = 0$.

5. Suppose
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$
Find the following determinants, and explain your answers:

a) $\begin{vmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{vmatrix}$ b) $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$

c) $\begin{vmatrix} a & b & c \\ d & e & f \\ g+3a & h+3b & i+3c \end{vmatrix}$ d) $\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}$ e) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$

6. State whether each of the following statements is (always) true or (sometimes) false. If it is true, give a brief justification; if it is false, give a counterexample with an explanation.

   a) If $A$ is a $n \times n$ matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$ has more than one solution, then the columns of $A$ span $\mathbb{R}^n$.
   b) If $A$ is a square matrix with linearly independent columns, then $A^2\mathbf{x} = \mathbf{b}$ has a solution for all $b$.
   c) If $A$ is a square matrix, then $(A^2)^T = (A^T)^2$.
   d) Each column of the matrix product $AB$ is a linear combination of the columns of $B$.
   e) The determinant of a square matrix is the product of its diagonal entries.
   f) If $A$ is a square matrix, then $\det(-A) = -\det(A)$.

7. **Optional Problem**: Consider a group of 5 students. Student 1 is friends on Facebook with each of the other four students. Also, Student 4 is Facebook friends with Student 3 and Student 5. There are no other Facebook friendships among the 5 students.

   Let $A$ be a $5 \times 5$ matrix, where the entry in row $i$ and column $j$ is 1 if Student $i$ and Student $j$ are Facebook friends, and 0 if Student $i$ and Student $j$ are not Facebook friends. So $A$ is a symmetric matrix. (We assume Facebook does not allow a student to be friends with himself or herself, so all diagonal entries of $A$ are zero.)

   a) Write down the matrix $A$.
   b) Let $u$ be the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Calculate $A\mathbf{u}$. What is the meaning of the $i$th entry of $A\mathbf{u}$?
   c) Calculate $A^2$. What is the meaning of the $(i, j)$-entry of $A^2$, when $i \neq j$?
   *This is the beginnings of the subject of "algebraic graph theory".*

— END —