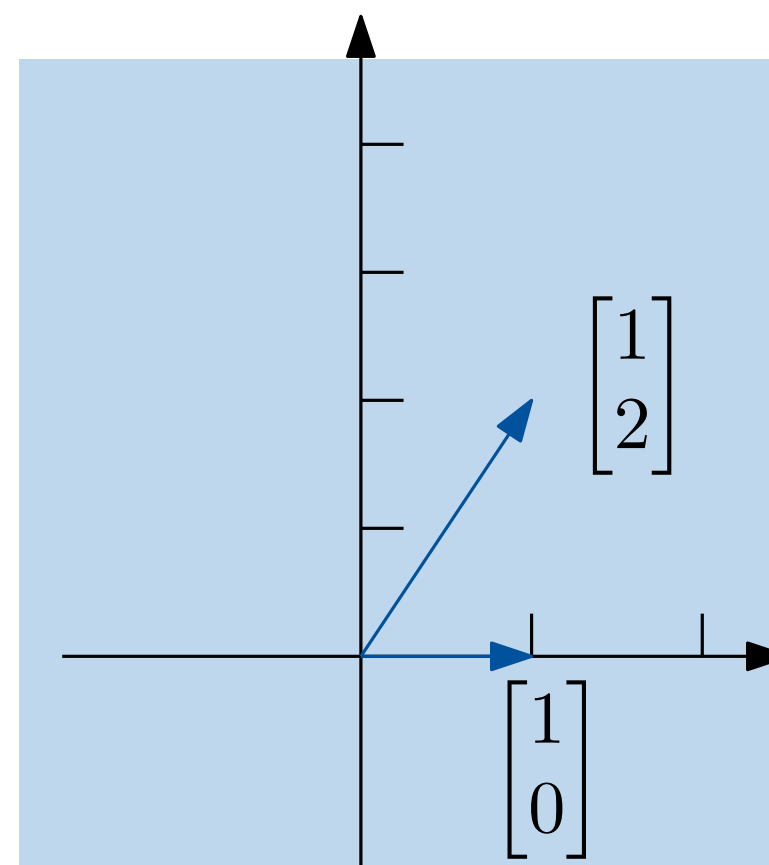
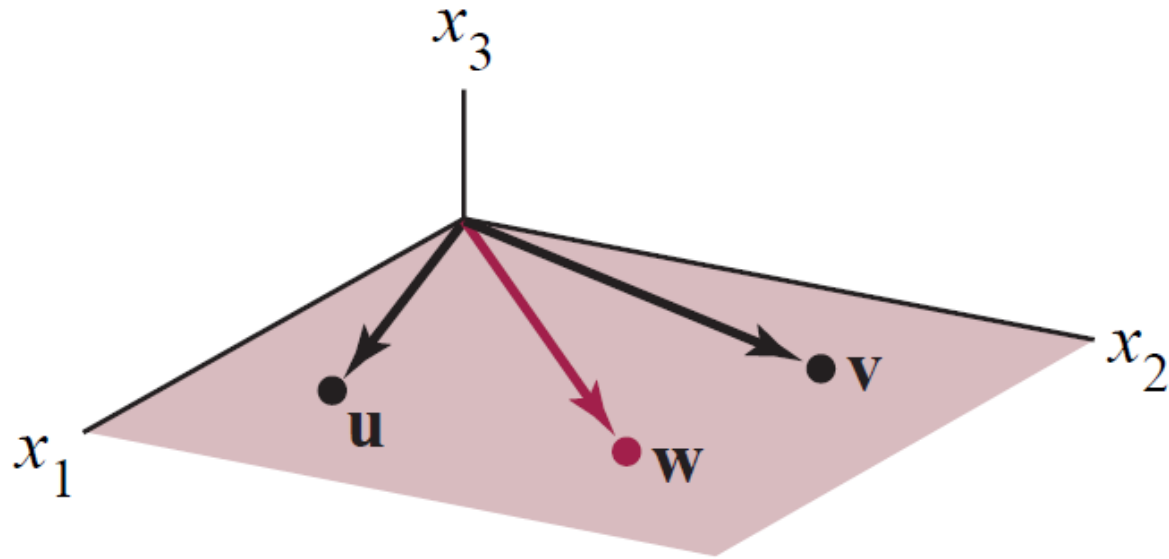


$$\text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\} = \text{a line}$$



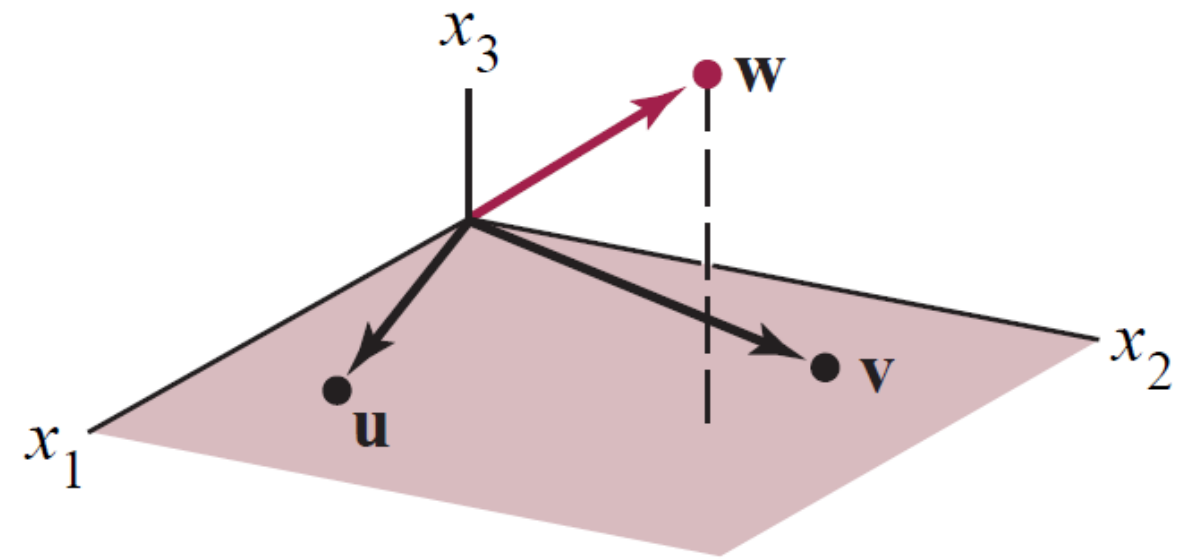
$$\text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = \mathbb{R}^2$$



$\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \text{Span}\{\mathbf{u}, \mathbf{v}\} = \text{a plane}$

When do n vectors span \mathbb{R}^n ?

How to find an efficient spanning set?



$\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathbb{R}^3$

When they are a **linearly independent** set.

The **casting out algorithm**.

§1.7: Linear Independence

Definition: A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is *linearly independent* if the only solution to the vector equation

$$x_1\mathbf{v}_1 + \dots + x_p\mathbf{v}_p = \mathbf{0}$$

is the *trivial solution* ($x_1 = \dots = x_p = 0$).

The opposite of linearly independent is linearly dependent:

Definition: A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is *linearly dependent* if there are weights c_1, \dots, c_p , *not all zero*, such that

$$c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p = \mathbf{0}.$$

The equation $c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p = \mathbf{0}$ is a *linear dependence relation*.

$$x_1 \mathbf{v}_1 + \cdots + x_p \mathbf{v}_p = \mathbf{0}$$

Only solution is $x_1 = \cdots = x_p = 0$
 \rightarrow linearly independent

There is a solution with some $x_i \neq 0$
 \rightarrow linearly dependent

Example: The set $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$ is linearly dependent because

$$2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Example: The set $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is linearly independent because

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{array}{lcl} x_1 + x_2 & = & 0 \\ 2x_1 & = & 0 \end{array} \implies x_1 = 0, x_2 = 0.$$

Some easy cases:

- Sets containing the zero vector $\{\mathbf{0}, \mathbf{v}_2, \dots, \mathbf{v}_p\}$:

$$(1)\mathbf{0} + (0)\mathbf{v}_2 + \dots + (0)\mathbf{v}_p = \mathbf{0} \quad \text{linearly dependent}$$

- Sets containing one vector $\{\mathbf{v}\}$:

$$x\mathbf{v} = \mathbf{0} \quad \text{linearly independent if } \mathbf{v} \neq \mathbf{0}$$

$$\begin{bmatrix} xv_1 \\ \vdots \\ xv_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \text{ If some } v_i \neq 0, \text{ then } x = 0 \text{ is the only solution.}$$

Some easy cases:

- Sets containing two vectors $\{\mathbf{u}, \mathbf{v}\}$:

$$x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}$$

if $x_1 \neq 0$, then $\mathbf{u} = (-x_2/x_1)\mathbf{v}$.

if $x_2 \neq 0$, then $\mathbf{v} = (-x_1/x_2)\mathbf{u}$.

So $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent if and only if one of the vectors is a multiple of the other.

- Sets containing more vectors:

A set of vectors is linearly dependent if and only if one of the vectors is a linear combination of the others. (Any vector with nonzero weight in the linear dependency relation will work.)

EXAMPLE Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$.

- a. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
 b. If possible, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Solution: (a)

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{array} \right] \quad \text{row reduces to} \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

x_3 is a free variable \Rightarrow there are nontrivial solutions.

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set

(b) Reduced echelon form: $\left[\begin{array}{ccc|c} 1 & 0 & 33 & 0 \\ 0 & 1 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$

Let $x_3 = \underline{\hspace{1cm}}$ (any nonzero number). Then $x_1 = \underline{\hspace{1cm}}$ and $x_2 = \underline{\hspace{1cm}}$.

$$\underline{\hspace{1cm}} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \underline{\hspace{1cm}} \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + \underline{\hspace{1cm}} \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\underline{\hspace{1cm}} \mathbf{v}_1 + \underline{\hspace{1cm}} \mathbf{v}_2 + \underline{\hspace{1cm}} \mathbf{v}_3 = \mathbf{0}$$

(one possible linear dependence relation)

A non-trivial solution to $A\mathbf{x} = \mathbf{0}$ is a linear dependence relation between the columns of A .

Theorem: Uniqueness of solutions for linear systems: For a matrix A , the following are equivalent:

- a. $A\mathbf{x} = \mathbf{0}$ has no non-trivial solution.
- b. If $A\mathbf{x} = \mathbf{b}$ is consistent, then it has a unique solution.
- c. The columns of A are linearly independent.
- d. $\text{rref}(A)$ has a pivot in every column (i.e. all variables are basic).

In particular: the row reduction algorithm produces at most one pivot in each row of $\text{rref}(A)$. So, if A has more columns than rows (a “fat” matrix), then $\text{rref}(A)$ cannot have a pivot in every column.

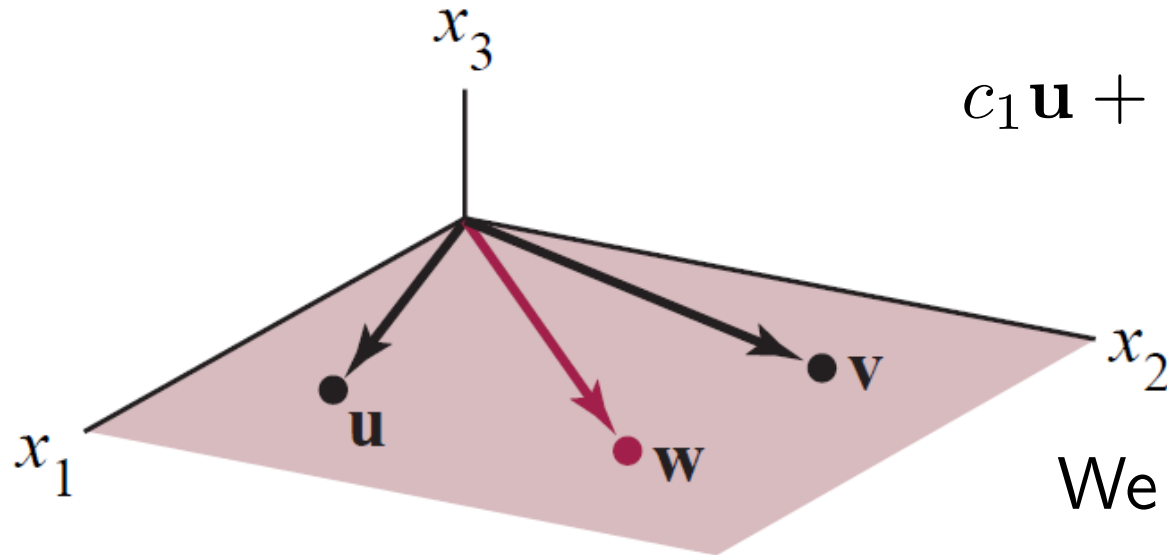
So a set of **more than n vectors in \mathbb{R}^n** is always **linearly dependent**.

Exercise: Combine this with Theorem 4 to show that a set of n linearly independent vectors span \mathbb{R}^n .

Problem: if $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly dependent, then $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the span of fewer vectors.

E.g. if $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$, then $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \text{Span}\{\mathbf{u}, \mathbf{v}\}$:

$$\begin{aligned} c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} &= c_1\mathbf{u} + c_2\mathbf{v} + c_3(a\mathbf{u} + b\mathbf{v}) \\ &= (c_1 + c_3a)\mathbf{u} + (c_2 + c_3b)\mathbf{v}. \end{aligned}$$



We want to remove from $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ some vectors that are linear combinations of other \mathbf{v}_i s.

One answer ([casting-out algorithm](#)):

Row reduce $\begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_p \\ | & | & & | \end{bmatrix}$ and keep the vectors in the pivot columns.

The casting-out algorithm:

Example: Let

$$S = \left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ 8 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} \right\}.$$

Find a linearly independent subset R of S such that $\text{Span}R = \text{Span}S$.

Answer:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to echelon form}]{\text{row reduction}} \begin{bmatrix} 1 & -3 & 4 & 3 & 2 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The pivot columns are 1, 2 and 5, so $R = \left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} \right\}$ is one answer.

(The answer from the casting out algorithm is not the only answer.)

Why the casting-out algorithm works:

Example:

$$\begin{bmatrix} | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to rref}]{\text{row reduction}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\text{rref} \left(\begin{bmatrix} | \\ \mathbf{v}_1 \\ | \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a pivot in every column, so $\{\mathbf{v}_1\}$ is linearly independent.

$\text{rref} \left(\begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a pivot in every column, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

Why the casting-out algorithm works:

Example:

$$\left[\begin{array}{c|c|c|c|c} | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \\ | & | & | & | & | \end{array} \right] = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to rref}]{\text{row reduction}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rref} \left(\left[\begin{array}{c|c|c} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{array} \right] \right) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \text{ does not have a pivot in every column.}$$

$$\text{The solution set to } \left[\begin{array}{c|c|c} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{array} \right] \mathbf{x} = \mathbf{0} \text{ is } \mathbf{x} = s \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \text{ where } s \text{ can take any value.}$$

$$\text{Take } s = 1: \left[\begin{array}{c|c|c} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{array} \right] \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \mathbf{0}.$$

So $2\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$, so $\mathbf{v}_3 = -2\mathbf{v}_1 - 2\mathbf{v}_2$, a linear combination of \mathbf{v}_1 and \mathbf{v}_2 . So we don't need \mathbf{v}_3 to get the same span.

Why the casting-out algorithm works:

Example:

$$\left[\begin{array}{c|c|c|c|c} | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \\ | & | & | & | & | \end{array} \right] = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to rref}]{\text{row reduction}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The solution set to $\left[\begin{array}{c|c|c|c} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ | & | & | & | \end{array} \right] \mathbf{x} = \mathbf{0}$ is $\mathbf{x} = s \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ where s, t can take any value.

Take $s = 0, t = 1$: $\left[\begin{array}{c|c|c|c} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ | & | & | & | \end{array} \right] \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$. So $-3\mathbf{v}_1 - 2\mathbf{v}_2 + 0\mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$, so $\mathbf{v}_4 = 3\mathbf{v}_1 + 2\mathbf{v}_2$, a linear combination of the **pivot columns**.

Why the casting-out algorithm works:

The row reduction algorithm writes the solution set of

$$\begin{bmatrix} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_p \\ | & | & | & | \end{bmatrix} \mathbf{x} = \mathbf{0}$$

in the form $s_i \mathbf{w}_i + s_j \mathbf{w}_j + \dots$, where x_i, x_j, \dots are the free variables.

For each column \mathbf{v}_i corresponding to a free variable, the solution $A\mathbf{w}_i = \mathbf{0}$ allows you to write \mathbf{v}_i as a linear combination of the earlier pivot columns.

So $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is the same as the span of the pivot columns.

The casting-out algorithm is a “greedy algorithm”: it prefers vectors that are earlier in the set.

E.g. if you want a linearly independent subset of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ with the same span, and you want \mathbf{w} to be in this set, you should row-reduce $[\mathbf{w} \quad \mathbf{u} \quad \mathbf{v}]$.