Remember from last week:

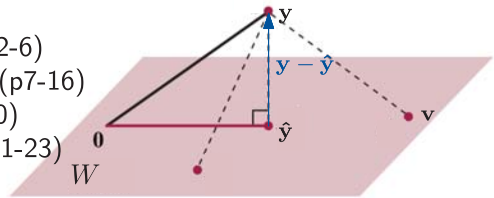
**Theorem 9: Best Approximation Theorem**: Let $W$ be a subspace of $\mathbb{R}^n$, and $\mathbf{y}$ a vector in $\mathbb{R}^n$. Then the closest point in $W$ to $\mathbf{y}$ is the unique point $\hat{\mathbf{y}}$ in $W$ such that $\mathbf{y}-\hat{\mathbf{y}}$ is in $W^\perp$. In other words, $\|\mathbf{y}-\hat{\mathbf{y}}\| < \|\mathbf{y}-\mathbf{v}\|$ for all $\mathbf{v}$ in $W$ with $\mathbf{v} \neq \hat{\mathbf{y}}$.

We proved last week that, if $\hat{\mathbf{y}}$ is in $W$, and $\mathbf{y}-\hat{\mathbf{y}}$ is in $W^\perp$, then $\hat{\mathbf{y}}$ is the unique closest point in $W$ to $\mathbf{y}$. But we did not prove that a $\hat{\mathbf{y}}$ satisfying these conditions always exist.

We will show that the function $\mathbf{y} \mapsto \hat{\mathbf{y}}$ is a linear transformation, called the orthogonal projection onto $W$, and calculate it using an orthogonal basis for $W$.

Our remaining goals:

§6.2 The properties of orthogonal bases (p2-6)
§6.3 Calculating the orthogonal projection (p7-16)
§6.4 Constructing orthogonal bases (p17-20)
§6.2 Matrices with orthogonal columns (p21-23)






# §6.2-6.3: Orthogonal Bases, Orthogonal Projections

**Definition**:
- A set of vectors $\{\mathbf{v}_1,\dots,\mathbf{v}_p\}$ is an *orthogonal set* if each pair of distinct vectors from the set is orthogonal, i.e. if $\mathbf{v}_i\cdot\mathbf{v}_j=0$ whenever $i\neq j$.
- A set of vectors $\{\mathbf{u}_1,\dots,\mathbf{u}_p\}$ is an *orthonormal set* if it is an orthogonal set and each $\mathbf{u}_i$ is a unit vector.

**Example**: $\left\{\begin{bmatrix}3\\0\\-1\end{bmatrix},\begin{bmatrix}-1\\5\\-3\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$ is an orthogonal set, because

$$\begin{bmatrix}3\\0\\-1\end{bmatrix}\cdot\begin{bmatrix}-1\\5\\-3\end{bmatrix}=-3+0+3=0,\quad \begin{bmatrix}3\\0\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}=3+0-3=0,\quad \begin{bmatrix}-1\\5\\-3\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}=-1+10-9=0.$$

To obtain an orthonormal set, we normalise each vector in the set:

$$\left\{\begin{bmatrix}3/\sqrt{10}\\0\\-1/\sqrt{10}\end{bmatrix},\begin{bmatrix}-1/\sqrt{35}\\5/\sqrt{35}\\-3/\sqrt{35}\end{bmatrix},\begin{bmatrix}1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14}\end{bmatrix}\right\} \text{ is an orthonormal set.}$$



**Example**: In $\mathbb{R}^6$, the set $\{\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_5,\mathbf{e}_6,\mathbf{0}\}$ is an orthogonal set, because $\mathbf{e}_i\cdot\mathbf{e}_j=0$ for all $i\neq j$, and $\mathbf{e}_i\cdot\mathbf{0}=0$.

So an orthogonal set may contain the zero vector. But when it doesn't:

**Theorem 4: Nonzero Orthogonal sets are Linearly Independent**: If $\{\mathbf{v}_1,\dots,\mathbf{v}_p\}$ is an orthogonal set of nonzero vectors, then it is linearly independent.

**Proof**: We need to show that $c_1=\dots=c_p=0$ is the only solution to

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+\dots+c_p\mathbf{v}_p=\mathbf{0}. \qquad (*)$$

Take the dot product of both sides with $\mathbf{v}_1$:

$$(c_1\mathbf{v}_1+c_2\mathbf{v}_2+\dots+c_p\mathbf{v}_p)\cdot\mathbf{v}_1=\mathbf{0}\cdot\mathbf{v}_1$$
$$c_1\mathbf{v}_1\cdot\mathbf{v}_1+c_2\mathbf{v}_2\cdot\mathbf{v}_1+\dots+c_p\mathbf{v}_p\cdot\mathbf{v}_1=0.$$

Using that $\mathbf{v}_j\cdot\mathbf{v}_1=0$ whenever $j\neq 1$:

$$c_1\mathbf{v}_1\cdot\mathbf{v}_1+c_2 0 \quad +\dots+c_p 0 \quad =0.$$

Since $\mathbf{v}_1$ is nonzero, $\mathbf{v}_1\cdot\mathbf{v}_1$ is nonzero, so it must be that $c_1=0$.
By taking the dot product of $(*)$ with each of the other $\mathbf{v}_i$s and using this argument, each $c_i$ must be 0.



Let $\{\mathbf{v}_1,\dots,\mathbf{v}_p\}$ is an orthogonal set of nonzero vectors, as before, and use the same idea with

$$\mathbf{y}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\dots+c_p\mathbf{v}_p. \qquad (*)$$

Take the dot product of both sides with $\mathbf{v}_1$:

$$\mathbf{v}_1\cdot\mathbf{y}=\mathbf{v}_1\cdot(c_1\mathbf{v}_1+c_2\mathbf{v}_2+\dots+c_p\mathbf{v}_p)$$
$$\mathbf{v}_1\cdot\mathbf{y}=c_1\mathbf{v}_1\cdot\mathbf{v}_1+c_2\mathbf{v}_2\cdot\mathbf{v}_1+\dots+c_p\mathbf{v}_p\cdot\mathbf{v}_1.$$

Using that $\mathbf{v}_j\cdot\mathbf{v}_1=0$ whenever $j\neq 1$:

$$\mathbf{v}_1\cdot\mathbf{y}=c_1\mathbf{v}_1\cdot\mathbf{v}_1+c_2 0 \quad +\dots+c_p 0$$

Since $\mathbf{v}_1$ is nonzero, $\mathbf{v}_1\cdot\mathbf{v}_1$ is nonzero, we can divide both sides by $\mathbf{v}_1\cdot\mathbf{v}_1$:

$$\frac{\mathbf{v}_1\cdot\mathbf{y}}{\mathbf{v}_1\cdot\mathbf{v}_1}=c_1$$

By taking the dot product of $(*)$ with each of the other $\mathbf{v}_j$s and using this argument, we obtain $c_j=\dfrac{\mathbf{y}\cdot\mathbf{v}_j}{\mathbf{v}_j\cdot\mathbf{v}_j}$.

So finding the weights in a linear combination of orthogonal vectors is much easier than for arbitrary vectors (where we would have to row-reduce $\left[\begin{array}{ccc|c}\mathbf{v}_1&\dots&\mathbf{v}_p&\mathbf{y}\end{array}\right]$.)

**Definition**:
- A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an *orthogonal basis* for a subspace $W$ if it is both an orthogonal set and a basis for $W$.
- A set of vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an *orthonormal basis* for a subspace $W$ if it is both an orthonormal set and a basis for $W$.

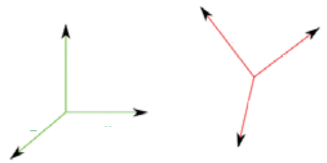
**Example**: The standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is an orthonormal basis for $\mathbb{R}^n$.
By the previous theorem, any orthogonal set of nonzero vectors is an orthogonal basis for its span.

As proved on the previous page, a big advantage of orthogonal bases is:
**Theorem 5: Weights for Orthogonal Bases**: If $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthogonal basis for $W$, then, for each $\mathbf{y}$ in $W$, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}.$$



In particular, if $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthonormal basis, then the weights are $c_j = \mathbf{y} \cdot \mathbf{u}_j$.



**Example**: We showed on p2 that $\left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is an orthogonal set. Since these vectors are nonzero, the set is linearly independent, and is therefore a basis for $\mathbb{R}^3$.

To express $\begin{bmatrix} 10 \\ 9 \\ 0 \end{bmatrix}$ as a linear combination of these three basis vectors: $\boxed{c_j = \frac{\mathbf{y} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}}$

$$c_1 = \frac{\begin{bmatrix} 10 \\ 9 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}} = \frac{30+0+0}{9+0+1} = 3, \quad c_2 = \frac{\begin{bmatrix} 10 \\ 9 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}}{\begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}} = \frac{-10+45+0}{1+25+9} = 1, \quad c_3 = \frac{\begin{bmatrix} 10 \\ 9 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} = \frac{10+18+0}{1+4+9} = 2$$

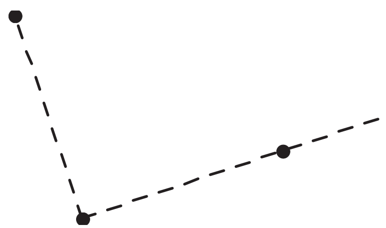
So $\begin{bmatrix} 10 \\ 9 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.



From the Weights for Orthogonal Bases Theorem: if $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthonormal basis for a subspace $W$ in $\mathbb{R}^n$, then each $\mathbf{y}$ in $W$ is

$$\mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p.$$

A geometric interpretation of this decomposition in $\mathbb{R}^2$:



From the picture, the closest point in the line $\mathrm{Span}\{\mathbf{u}_1\}$ to $\mathbf{y}$ is ...

Algebraic proof: this satisfies the hypothesis of the Best Approximation Theorem because:

1.

2.

Similarly in $\mathbb{R}^3$:
**Example**: Let $\mathbf{u}_1 = \begin{bmatrix} 3/\sqrt{10} \\ 0 \\ -1/\sqrt{10} \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{35} \\ 5/\sqrt{35} \\ -3/\sqrt{35} \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$. We showed on p2 that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$. Let $W$ be the plane spanned by $\{\mathbf{u}_1, \mathbf{u}_2\}$.

Let $\mathbf{y} = \begin{bmatrix} 10 \\ 9 \\ 0 \end{bmatrix}$. We can write (see p6)

$$\mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2 + (\mathbf{y} \cdot \mathbf{u}_3)\mathbf{u}_3$$
$$= \underbrace{3\sqrt{10}\mathbf{u}_1 + \sqrt{35}\mathbf{u}_2}_{\text{Call this vector } \hat{\mathbf{y}}.\ \text{It is in } W.} + \underbrace{2\sqrt{14}\mathbf{u}_3}_{\text{Then this is } \mathbf{y}-\hat{\mathbf{y}}.\ \text{It is in } W^\perp,\text{ because it is orthogonal to a spanning set for } W.}.$$

So, by the Best Approximation Theorem, $\hat{\mathbf{y}} = 3\sqrt{10}\mathbf{u}_1 + \sqrt{35}\mathbf{u}_2 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}$ is the closest point in $W$ to $\mathbf{y}$. Notice that $\mathbf{u}_3$ was not necessary to calculate $\hat{\mathbf{y}}$.
The distance from $\mathbf{y}$ to $W$ is $\|\mathbf{y} - \hat{\mathbf{y}}\| = \|2\sqrt{14}\mathbf{u}_3\| = 2\sqrt{14}$.

In general:

**Theorem 8: Orthogonal Decomposition Theorem**: Let $W$ be a subspace of $\mathbb{R}^n$. Then every $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely as $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ with $\hat{\mathbf{y}}$ in $W$ and $\mathbf{z}$ in $W^\perp$. In fact, if $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is any orthogonal basis for $W$, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{v}_p}{\mathbf{v}_p\cdot\mathbf{v}_p}\mathbf{v}_p \quad \text{and} \quad \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}.$$

**Definition**: The *orthogonal projection onto $W$* is the function $\text{proj}_W : \mathbb{R}^n \to \mathbb{R}^n$ such that $\text{proj}_W(\mathbf{y})$ is the unique $\hat{\mathbf{y}}$ in the above theorem. The image vector $\text{proj}_W(\mathbf{y})$ is the *orthogonal projection of $\mathbf{y}$ onto $W$*.

The uniqueness part of the theorem means that the $\text{proj}_W(\mathbf{y})$ does not depend on the orthogonal basis used to calculate it.

Note that $\text{proj}_W(\mathbf{y})$ satisfies the hypotheses of the Best Approximation Theorem, so $\text{proj}_W(\mathbf{y})$ is the closest point in $W$ to $\mathbf{y}$. This implies the uniqueness of the orthogonal decomposition, but we will give another proof not using the Best Approximation Theorem.



**Theorem 8: Orthogonal Decomposition Theorem**: Let $W$ be a subspace of $\mathbb{R}^n$. Then every $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely as $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ with $\hat{\mathbf{y}}$ in $W$ and $\mathbf{z}$ in $W^\perp$. In fact, if $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is any orthogonal basis for $W$, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{v}_p}{\mathbf{v}_p\cdot\mathbf{v}_p}\mathbf{v}_p \quad \text{and} \quad \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}.$$

**Proof**: We first show that the formulas for $\hat{\mathbf{y}}$ and $\mathbf{z}$ above indeed give an orthogonal decomposition.

$\hat{\mathbf{y}}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$, so it is in $W$.

To show $\mathbf{z}$ is in $W^\perp$:

$$\begin{aligned} \mathbf{z}\cdot\mathbf{v}_1 &= (\mathbf{y}-\hat{\mathbf{y}})\cdot\mathbf{v}_1 \\ &= \mathbf{y}\cdot\mathbf{v}_1 - \frac{\mathbf{y}\cdot\mathbf{v}_1}{\cancel{\mathbf{v}_1\cdot\mathbf{v}_1}}\cancel{\mathbf{v}_1\cdot\mathbf{v}_1} - \frac{\mathbf{y}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\overbrace{\mathbf{v}_2\cdot\mathbf{v}_1}^{=0} - \cdots - \frac{\mathbf{y}\cdot\mathbf{v}_p}{\mathbf{v}_p\cdot\mathbf{v}_p}\overbrace{\mathbf{v}_p\cdot\mathbf{v}_1}^{=0} \\ &= \mathbf{y}\cdot\mathbf{v}_1 - \mathbf{y}\cdot\mathbf{v}_1 = 0, \end{aligned}$$

and the same argument shows that $\mathbf{z}\cdot\mathbf{v_i} = 0$ for all $i$, so $\mathbf{z}$ is orthogonal to a spanning set for $W$, and therefore in $W^\perp$.



**Theorem 8: Orthogonal Decomposition Theorem**: Let $W$ be a subspace of $\mathbb{R}^n$. Then every $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely as $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ with $\hat{\mathbf{y}}$ in $W$ and $\mathbf{z}$ in $W^\perp$. In fact, if $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is any orthogonal basis for $W$, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{v}_p}{\mathbf{v}_p\cdot\mathbf{v}_p}\mathbf{v}_p \quad \text{and} \quad \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}.$$

**Proof**: (continued) We show the uniqueness of $\hat{\mathbf{y}}$ and $\mathbf{z}$.

Suppose $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ and $\mathbf{y} = \hat{\mathbf{y}}_1 + \mathbf{z}_1$ are two such decompositions, so $\hat{\mathbf{y}}, \hat{\mathbf{y}}_1$ are in $W$, and $\mathbf{z}, \mathbf{z}_1$ are in $W^\perp$, and

$$\hat{\mathbf{y}} + \mathbf{z} = \hat{\mathbf{y}}_1 + \mathbf{z}_1$$
$$\hat{\mathbf{y}} - \hat{\mathbf{y}}_1 = \mathbf{z}_1 - \mathbf{z}.$$

LHS: Because $\hat{\mathbf{y}}, \hat{\mathbf{y}}_1$ are in $W$ and $W$ is a subspace, the difference $\hat{\mathbf{y}} - \hat{\mathbf{y}}_1$ is in $W$.

RHS: Because $\mathbf{z}, \mathbf{z}_1$ are in $W^\perp$ and $W^\perp$ is a subspace, the difference $\mathbf{z}_1 - \mathbf{z}$ is in $W^\perp$.

So the vector $\hat{\mathbf{y}} - \hat{\mathbf{y}}_1 = \mathbf{z}_1 - \mathbf{z}$ is in both $W$ and $W^\perp$, this vector is the zero vector (property 1 on week 12, p10). So $\hat{\mathbf{y}} = \hat{\mathbf{y}}_1$ and $\mathbf{z}_1 = \mathbf{z}$.

(Technically, to complete the proof, we need to show that every subspace has an orthogonal basis - see p17-20 for an explicit construction.)



**Theorem 8: Orthogonal Decomposition Theorem**: Let $W$ be a subspace of $\mathbb{R}^n$. Then every $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely as $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ with $\hat{\mathbf{y}}$ in $W$ and $\mathbf{z}$ in $W^\perp$. In fact, if $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is any orthogonal basis for $W$, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{v}_p}{\mathbf{v}_p\cdot\mathbf{v}_p}\mathbf{v}_p \quad \text{and} \quad \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}.$$

How can we discover the formula for $\hat{\mathbf{y}}$ if we did not consider orthogonal bases for $\mathbb{R}^n$?

We want a $\hat{\mathbf{y}}$ in $W$, and $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis for $W$, so $\hat{\mathbf{y}} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$ for some weights $c_1, \dots, c_p$.

We want $\mathbf{y} - \hat{\mathbf{y}}$ to be in $W^\perp$. By the properties of $W^\perp$, it's enough to show that $(\mathbf{y} - \hat{\mathbf{y}})\cdot\mathbf{v}_i = 0$ for each $i$. We can use this condition to solve for $c_i$:

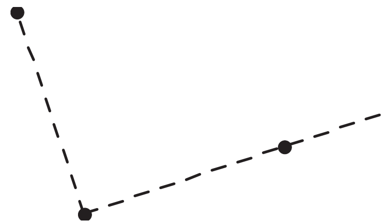
$$\begin{aligned} (\mathbf{y} - \hat{\mathbf{y}})\cdot\mathbf{v}_1 &= 0 \\ (\mathbf{y} - c_1\mathbf{v}_1 - c_2\mathbf{v}_2 - \cdots - c_p\mathbf{v}_p)\cdot\mathbf{v}_1 &= 0 \\ \mathbf{y}\cdot\mathbf{v}_1 - c_1\mathbf{v}_1\cdot\mathbf{v}_1 - c_2\mathbf{v}_2\cdot\mathbf{v}_1 - \cdots - c_p\mathbf{v}_p\cdot\mathbf{v}_1 &= 0 \\ \mathbf{y}\cdot\mathbf{v}_1 - c_1\mathbf{v}_1\cdot\mathbf{v}_1 - \cdots - +c_2 0 \qquad + \cdots + c_p 0 \qquad &= 0 \end{aligned}$$

so $c_1 = \dfrac{\mathbf{y}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}$. Similarly, $c_i = \dfrac{\mathbf{y}\cdot\mathbf{v}_i}{\mathbf{v}_i\cdot\mathbf{v}_i}$

Let $W$ be a subspace of $\mathbb{R}^n$. If $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthonormal basis for $W$, then, for every $\mathbf{y}$ in $\mathbb{R}^n$,

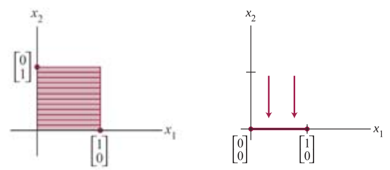
$$\text{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + \dots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p.$$

Thinking about $\text{proj}_W : \mathbb{R}^n \to \mathbb{R}^n$ as a function:



We saw a special case in Week 4 §1.8-1.9:

Projection onto the $x_1$-axis



Properties of the function $\text{proj}_W : \mathbb{R}^n \to \mathbb{R}^n$:

a. $\text{proj}_W$ is a linear transformation.
b. $\text{proj}_W(\mathbf{y}) = \mathbf{y}$ if and only if $\mathbf{y}$ is in $W$.
c. The range of $\text{proj}_W$ is $W$.
d. The kernel of $\text{proj}_W$ is $W^\perp$.
e. $\text{proj}^2_W =$ .......... .
f. $\text{proj}_W + \text{proj}_{W^\perp}$ is the identity transformation.

It is easy to prove a,b,c,d,e using the formula, but we can also prove them from the existence and uniqueness of the orthogonal decomposition, e.g. to see a: if we have orthogonal decompositions $\mathbf{y}_1 = \text{proj}_W(\mathbf{y}_1) + \mathbf{z}_1$ and $\mathbf{y}_2 = \text{proj}_W(\mathbf{y}_2) + \mathbf{z}_2$, then

$$\begin{aligned} c\mathbf{y}_1 + d\mathbf{y}_2 &= c(\text{proj}_W(\mathbf{y}_1) + \mathbf{z}_1) + d(\text{proj}_W(\mathbf{y}_2) + \mathbf{z}_2) \\ &= \underbrace{c\text{proj}_W(\mathbf{y}_1) + d\text{proj}_W(\mathbf{y}_2)}_{\text{in } W} + \underbrace{c\mathbf{z}_1 + d\mathbf{z}_2}_{\text{in } W^\perp} \end{aligned}$$

Since the orthogonal decomposition is unique, this shows
$\text{proj}_W(c\mathbf{y}_1 + d\mathbf{y}_2) = c\text{proj}_W(\mathbf{y}_1) + d\text{proj}_W(\mathbf{y}_2)$.



Properties of the function $\text{proj}_W : \mathbb{R}^n \to \mathbb{R}^n$:

a. $\text{proj}_W$ is a linear transformation.
b. $\text{proj}_W(\mathbf{y}) = \mathbf{y}$ if and only if $\mathbf{y}$ is in $W$.
c. The range of $\text{proj}_W$ is $W$.
d. The kernel of $\text{proj}_W$ is $W^\perp$.
e. $\text{proj}^2_W =$ .......... .
f. $\text{proj}_W + \text{proj}_{W^\perp}$ is the identity transformation.

To see f: Write $U$ for $W^\perp$. Then,

$$\mathbf{y} = \underbrace{\hat{\mathbf{y}}}_{\text{in } W = U^\perp} + \underbrace{\mathbf{z}}_{\text{in } W^\perp = U}.$$

By uniqueness of the orthogonal decomposition, $\mathbf{z} = \text{proj}_U(\mathbf{y})$. So
$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} = \text{proj}_W(\mathbf{y}) + \text{proj}_{W^\perp}(\mathbf{y})$ for each $\mathbf{y}$ in $\mathbb{R}^n$, so $\text{proj}_W + \text{proj}_{W^\perp}$ is the identity transformation.



The orthogonal projection is a linear transformation, so we can ask for its standard matrix. (It is faster to compute orthogonal projections by taking dot products (formula on p9) than using the standard matrix, but this result is useful theoretically.)

**Theorem 10: Matrix for Orthogonal Projection**: Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be an orthonormal basis for a subspace $W$, and $U$ be the matrix $U = \begin{bmatrix} | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_p \\ | & & | \end{bmatrix}$.

Then the standard matrix for $\text{proj}_W$ is $[\text{proj}_W]_{\mathcal{E}} = UU^T$.

**Proof**:

$$UU^T\mathbf{y} = \begin{bmatrix} | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_p \\ | & & | \end{bmatrix} \begin{bmatrix} -- & \mathbf{u}_1 & -- \\ -- & \vdots & -- \\ -- & \mathbf{u}_p & -- \end{bmatrix} \mathbf{y} = \begin{bmatrix} | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_p \\ | & & | \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{y} \\ \vdots \\ \mathbf{u}_p \cdot \mathbf{y} \end{bmatrix}$$
$$= (\mathbf{u}_1 \cdot \mathbf{y})\mathbf{u}_1 + \dots + (\mathbf{u}_p \cdot \mathbf{y})\mathbf{u}_p.$$

Tip: to remember that $[\text{proj}_W]_{\mathcal{E}} = UU^T$ and not $U^TU$ (which is important too, see p21), make sure this matrix is $n \times n$.

# §6.4: The Gram-Schmidt Process

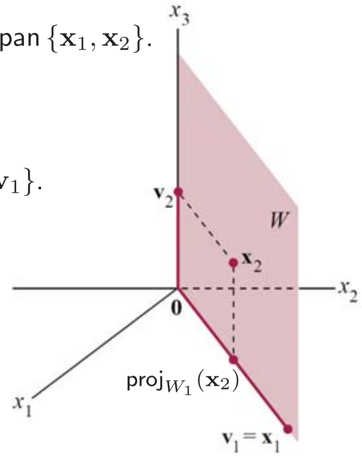
This is an algorithm to make an orthogonal basis out of an arbitrary basis.

**Example**: Let $\mathbf{x}_1 = \begin{bmatrix}4\\2\\0\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}2\\1\\3\end{bmatrix}$ and let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$.
Construct an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for $W$.

**Answer**: Let $\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix}4\\2\\0\end{bmatrix}$, and let $W_1 = \text{Span}\{\mathbf{v}_1\}$.

By the Orthogonal Decomposition Theorem,
$\mathbf{x}_2 - \text{proj}_{W_1}(\mathbf{x}_2)$ is orthogonal to $W_1$.
So let $\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1}(\mathbf{x}_2) = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$

$$= \begin{bmatrix}2\\1\\3\end{bmatrix} - \frac{8+2+0}{4^2+2^2+0}\begin{bmatrix}4\\2\\0\end{bmatrix} = \begin{bmatrix}0\\0\\3\end{bmatrix}.$$






For subspaces of dimension $p > 2$, we repeat this idea $p$ times, like this:

**Example**: Let $\mathbf{x}_1 = \begin{bmatrix}3\\0\\-1\\0\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}8\\5\\-6\\0\end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix}-6\\7\\2\\1\end{bmatrix}$ and let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.
Construct an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $W$.

**Answer**: Let $\mathbf{v}_1 = \mathbf{x}_1$, $W_1 = \text{Span}\{\mathbf{v}_1\}$.
By the Orthogonal Decomposition Theorem, $\mathbf{x}_2 - \text{proj}_{W_1}(\mathbf{x}_2)$ is orthogonal to $W_1$.

So let $\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1}(\mathbf{x}_2) = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix}3\\0\\-1\\0\end{bmatrix} - \frac{24+0-6+0}{3^2+0+1^2+0}\begin{bmatrix}3\\0\\-1\\0\end{bmatrix} = \begin{bmatrix}1\\-5\\3\\0\end{bmatrix}$.

Let $W_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. By the Orthogonal Decomposition Theorem, $\mathbf{x}_3 - \text{proj}_{W_2}(\mathbf{x}_3)$ is orthogonal to $W_2$, and in particular to $\mathbf{v}_1$ and $\mathbf{v}_2$. So let $\mathbf{v}_3 = \mathbf{x}_3 - \text{proj}_{W_2}(\mathbf{x}_3) =$...



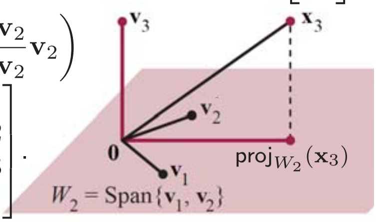
**Example**: Let $\mathbf{x}_1 = \begin{bmatrix}3\\0\\-1\\0\end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix}8\\5\\-6\\0\end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix}-6\\7\\2\\1\end{bmatrix}$ and let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.
Construct an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $W$.

**Answer**: (continued) So far we have $\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix}3\\0\\-1\\0\end{bmatrix}$, $\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{W_1}(\mathbf{x}_2) = \begin{bmatrix}1\\-5\\3\\0\end{bmatrix}$

and $W_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

Let $\mathbf{v}_3 = \mathbf{x}_3 - \text{proj}_{W_2}(\mathbf{x}_3) = \mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1\mathbf{x}_3 + \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2\right)$

$$= \begin{bmatrix}-6\\7\\2\\1\end{bmatrix} - \tfrac{-18+0-2+0}{3^2+0+1^2+0}\begin{bmatrix}3\\0\\-1\\0\end{bmatrix} - \tfrac{-6-35+6+0}{1^2+(-5)^2+3^2+0}\begin{bmatrix}1\\-5\\3\\0\end{bmatrix} = \begin{bmatrix}1\\2\\3\\1\end{bmatrix}.$$




Check our answer: $\mathbf{v}_1 \cdot \mathbf{v}_2 = 3 + 0 - 3 + 0 = 0$, $\mathbf{v}_1 \cdot \mathbf{v}_3 = \cdots = 0$, $\mathbf{v}_2 \cdot \mathbf{v}_3 = \cdots = 0$



In general:
**Theorem 11: Gram-Schmidt**: Given a basis $\{\mathbf{x}_1, \ldots \mathbf{x}_p\}$ for a subspace $W$ of $\mathbb{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1\\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1\\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2\\
&\vdots\\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for $W$, and
$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for each $k$ between 1 and $p$.

In fact, you can apply this algorithm to any spanning set (not necessarily linearly independent). Then some $\mathbf{v}_k$s might be zero, and you simply remove them.

# pp361-362: Matrices with orthonormal columns

This is an important class of matrices.

**Theorem 6: Matrices with Orthonormal Columns**: A matrix $U$ has orthonormal columns if and only if $U^TU = I$.

**Proof**: Let $\mathbf{u}_i$ denote the $i$th column of $U$. From the row-column rule of matrix multiplication (week 12 p14):

$$\begin{bmatrix} -- & \mathbf{u}_1 & -- \\ -- & \vdots & -- \\ -- & \mathbf{u}_p & -- \end{bmatrix} \begin{bmatrix} | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_p \\ | & & | \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{u}_1 & \dots & \mathbf{u}_1 \cdot \mathbf{u}_p \\ \vdots & & \vdots \\ \mathbf{u}_p \cdot \mathbf{u}_1 & \dots & \mathbf{u}_p \cdot \mathbf{u}_p \end{bmatrix}.$$

so $U^TU = I$ if and only if $\mathbf{u}_i \cdot \mathbf{u}_i = 1$ for each $i$ (diagonal entries), and $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ for each pair $i \neq j$ (non-diagonal entries).



**Theorem 7: Matrices with Orthonormal Columns represent Length-Preserving Linear Transformations**: Let $U$ be an $m \times n$ matrix with orthonormal columns. Then, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}.$$

In particular, $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$, and $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

**Proof**:

$$(U\mathbf{x}) \cdot (U\mathbf{y}) = (U\mathbf{x})^T(U\mathbf{y}) = \mathbf{x}^T U^T U \mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

because $U^TU = I_n$, by the previous theorem

Length-preserving linear transformations are sometimes called isometries.
Exercise: prove that an isometry also preserves angles; that is, if $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$, then $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y}$. (Hint: think about $\mathbf{x} + \mathbf{y}$.)



An important special case:
**Definition**: A matrix $U$ is *orthogonal* if it is a square matrix with orthonormal columns. Equivalently, $U^{-1} = U^T$.

**Warning**: An orthogonal matrix has orthonormal columns, not simply orthogonal columns.

**Example**: The standard matrix of a rotation in $\mathbb{R}^2$ is $U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, and this is an orthogonal matrix because

$$U^TU = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It can be shown that every orthogonal $2 \times 2$ matrix $U$ with determinant 1 represents a rotation. So an orthogonal $n \times n$ matrix with determinant 1 is a high-dimensional generalisation of a rotation.



Non-examinable: distances for abstract vector spaces
On an abstract vector space, a function that takes two vectors to a scalar satisfying the symmetry, linearity and positivity properties (week 12 p5) is called an inner product. The inner product of $\mathbf{u}$ and $\mathbf{v}$ is often written $\langle \mathbf{u}, \mathbf{v} \rangle$ or $\langle \mathbf{u} | \mathbf{v} \rangle$. (So the dot product is one example of an inner product on $\mathbb{R}^n$, but other useful inner products exist; these can be used to compute weighted regression lines, see §6.8 of the textbook)

Many common inner products on $C([0,1])$, the vector space of continuous functions, have the form

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(t)g(t)w(t)\,dt$$

for some weight function $w(t)$. This inner product can be used to find polynomial approximations and Fourier approximations to functions, see §6.7-6.8 of the textbook.

Applying Gram-Schmidt to $\{1, t, t^2, \dots\}$ produces various families of orthogonal polynomials, which is a big field of study.

With the concept of inner product, we can understand what the transpose means for linear transformations:

First notice: if $A$ is an $m \times n$ matrix, then, for all $\mathbf{v}$ in $\mathbb{R}^n$ and all $\mathbf{u}$ in $\mathbb{R}^m$:

$$\underbrace{(A^T\mathbf{u}) \cdot \mathbf{v}}_{\text{dot product in } \mathbb{R}^n} = (A^T\mathbf{u})^T\mathbf{v} = \mathbf{u}^T A\mathbf{v} = \underbrace{\mathbf{u} \cdot (A\mathbf{v})}_{\text{dot product in } \mathbb{R}^m}.$$

So, if $A$ is the standard matrix of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, then $A^T$ is the standard matrix of its adjoint $T^* : \mathbb{R}^m \to \mathbb{R}^n$ , which satisfies

$$(T^*\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (T\mathbf{v}).$$

or, for abstract vector space with an inner product:

$$\langle T^*\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, T\mathbf{v}\rangle.$$

So symmetric matrices ($A^T = A$) represent self-adjoint linear transformations ($T^* = T$). For example, on $C([0, 1])$ with any integral inner product, the multiplication-by-$x$ function $\mathbf{f} \mapsto x\mathbf{f}$ is self-adjoint.