

[?] Given  $\psi \in \widehat{P_{\leq 2}(\mathbb{R})}$ , how to write  $\psi$  as a linear combination of  $\phi_1, \phi_2, \phi_3$ ?

Answer:  $\psi = a'\phi_1 + b'\phi_2 + c'\phi_3$ .

i.e.  $\psi(p) = a'\phi_1(p) + b'\phi_2(p) + c'\phi_3(p) \quad \forall p \in P_{\leq 2}(\mathbb{R})$ .

in particular:  $\psi(1) = a'\phi_1(1) + b'\phi_2(1) + c'\phi_3(1)$   
 $= a' \cdot 1 + b' \cdot 0 + c' \cdot 0 = a'$

$x = \alpha_2$   
 $\psi(x) = a'\phi_1(x) + b'\phi_2(x) + c'\phi_3(x)$   
 $= a' \cdot 0 + b' \cdot 1 + c' \cdot 0 = b'$

For some reason:  $\psi(x^2) = c'$

For general  $V$ :

Prop: if  $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$  is a basis of  $V$ ,  
 and  $\hat{\mathcal{A}} = \{\phi_1, \dots, \phi_n\}$  is the dual basis of  $\mathcal{A}$ ,

then  $\beta = \phi_1(\beta)\alpha_1 + \dots + \phi_n(\beta)\alpha_n$   
 $\psi = \psi(\alpha_1)\phi_1 + \dots + \psi(\alpha_n)\phi_n. \quad (*)$

Ex: (continue from before)

Let  $\psi(f) = \int_0^1 f(t) dt$

Write  $\psi$  as a linear combination of  $\hat{\mathcal{A}}$ :

$\psi(1) = \int_0^1 1 dt = 1 \quad \psi(x^2) = \int_0^1 t^2 dt = \frac{1}{3}$ .

$\psi(x) = \int_0^1 t dt = \frac{1}{2} \quad \therefore \psi = 1\phi_1 + \frac{1}{2}\phi_2 + \frac{1}{3}\phi_3$ .  
 i.e.  $\psi(ax+bx^2) = a + \frac{1}{2}b + \frac{1}{3}c$ .

Note:  $\psi$  as a vector:  $[\psi]_{\hat{\mathcal{A}}} = \begin{pmatrix} \psi(\alpha_1) \\ \vdots \\ \psi(\alpha_n) \end{pmatrix}$  from  $(*)$

$\psi$  as a linear transformation:  $V \rightarrow \mathbb{F}$ :

$[\psi]_{\hat{\mathcal{A}}} = (\psi(\alpha_1) \dots \psi(\alpha_n))$

so  $[\psi]_{\hat{\mathcal{A}}} = \left( [\psi]_{\mathcal{A}} \right)^T$

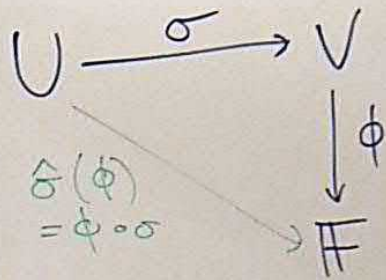
$\{1\}$  is the standard basis of  $\mathbb{F} = \mathbb{R}$   
 $\alpha_1 = 1$

(more §9.2 later)

### §9.3 The dual of a linear transformation:

Given  $\sigma \in L(U, V)$  and  $\phi \in \hat{V} \in L(V, \mathbb{F})$

we can define  $\phi \circ \sigma \in L(U, \mathbb{F}) = \hat{U}$



This works for every  $\phi \in \hat{V}$  — so we have a function  $\hat{\sigma} : \hat{V} \rightarrow \hat{U}$  given by  $\hat{\sigma}(\phi) = \phi \circ \sigma$  i.e.  $\hat{\sigma}$  is "precomposition by  $\sigma$ ".

Def 9.3.2.  $\hat{\sigma}$  is the dual of  $\sigma$ .

Ex ①:  $\sigma: P_{<3}(\mathbb{R}) \rightarrow P_{<2}(\mathbb{R})$  given by differentiation  $\sigma(f) = f'$

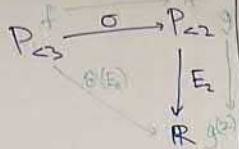
$E_2: P_{<2}(\mathbb{R}) \rightarrow \mathbb{R}$  given by

$$E_2(g) = g(2).$$

Then  $\hat{\sigma}(E_2): P_{<3}(\mathbb{R}) \rightarrow \mathbb{R}$

$$\hat{\sigma}(E_2) = E_2 \circ \sigma$$

$$\begin{aligned} \text{i.e. } [\hat{\sigma}(E_2)](f) &= (E_2 \circ \sigma)(f) \\ &= E_2(\sigma(f)) \\ &= E_2(f') = f'(2). \end{aligned}$$

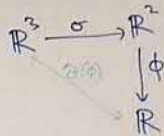


Ex ②:  $\sigma: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ ,  $\sigma(x) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \end{pmatrix} x$

$$\text{i.e. } \sigma \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+3z \\ 4x+5z \end{pmatrix}$$

$\phi \in \widehat{\mathbb{R}^2}$  i.e.  $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\phi \begin{pmatrix} u \\ v \end{pmatrix} = au + bv$$



$\hat{\sigma}(\phi) \in \widehat{\mathbb{R}^3}$ , i.e.  $\hat{\sigma}(\phi): \mathbb{R}^3 \rightarrow \mathbb{R}$

$$\begin{aligned} [\hat{\sigma}(\phi)] \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \phi \left( \sigma \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \phi \begin{pmatrix} x+2y+3z \\ 4x+5z \end{pmatrix} \\ &= a(x+2y+3z) + b(4x+5z) \end{aligned}$$

Th 9.3.1 for  $\sigma \in L(U, V)$ , the dual  $\hat{\sigma}: \hat{V} \rightarrow \hat{U}$  is linear (i.e.  $\hat{\sigma} \in L(\hat{V}, \hat{U})$ )

Proof: We need to show, for all  $\phi, \psi \in \hat{V}$ , and  $a \in \mathbb{F}$

$$\hat{\sigma}(a\phi + \psi) = a\hat{\sigma}(\phi) + \hat{\sigma}(\psi).$$

$$\hat{\sigma}(a\phi + \psi) = (a\phi + \psi) \circ \sigma \quad [\text{definition of } \hat{\sigma}]$$

$$\begin{aligned} \text{So } [\hat{\sigma}(a\phi + \psi)](\alpha) &= (a\phi + \psi)(\sigma(\alpha)) \\ &= a\phi(\sigma(\alpha)) + \psi(\sigma(\alpha)) \quad [\text{definition of } a\phi + \psi] \\ &= a(\phi \circ \sigma)(\alpha) + (\psi \circ \sigma)(\alpha) \\ &= (a\phi \circ \sigma + \psi \circ \sigma)(\alpha) \quad [\text{definition of } a\phi \circ \sigma + \psi \circ \sigma] \\ &= (a\hat{\sigma}(\phi) + \hat{\sigma}(\psi))(\alpha) \end{aligned}$$

This holds  $\forall \alpha \in U$ , so  $\hat{\sigma}(a\phi + \psi) = a\hat{\sigma}(\phi) + \hat{\sigma}(\psi)$ .