Section 1.1: Systems of Linear Equations

A **linear equation:**

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

EXAMPLE:

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2(\sqrt{6} - x_1) + x_3$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

rearranged   rearranged

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$3x_1 - 5x_2 = -2 \qquad\qquad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

Not linear:

$$4x_1 - 6x_2 = x_1x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 7$$

A **system of linear equations (**or a **linear system):**

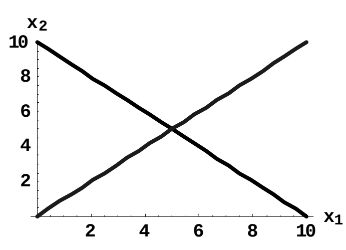
A collection of one or more linear equations involving the same set of variables, say, $x_1, x_2, \ldots, x_n$.

A **solution** of a linear system:

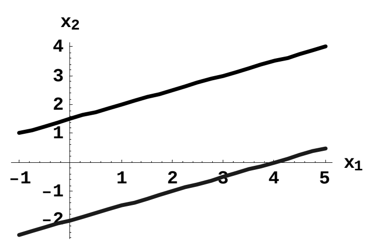
A list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

EXAMPLE Two equations in two variables:

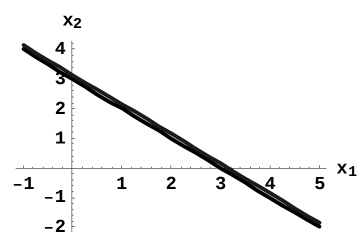
$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 10 \\ -x_1 & + & x_2 & = & 0 \end{array} \qquad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -3 \\ 2x_1 & - & 4x_2 & = & 8 \end{array} \qquad \begin{array}{rcrcr} x_1 & + & x_2 & = & 3 \\ -2x_1 & - & 2x_2 & = & -6 \end{array}$$



one unique solution



no solution



infinitely many solutions

BASIC FACT: A system of linear equations has either

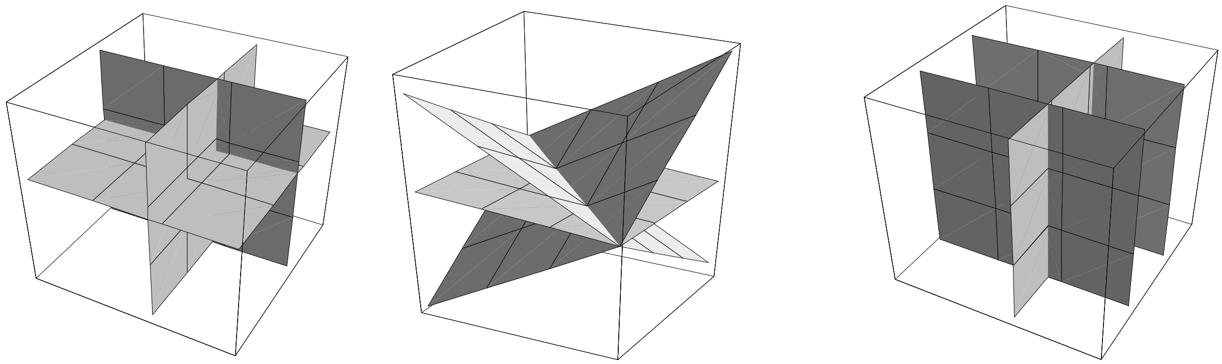
(i) exactly one solution *(consistent)* or

(ii) infinitely many solutions *(consistent)* or

(iii) no solution *(inconsistent)*.

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

i) The planes intersect in one point. *(one solution)* ii) The planes intersect in one line. *(infinitely many solutions)* iii) There is not point in common to all three planes. *(no solution)*



The **solution set:**

- The set of all possible solutions of a linear system.

Equivalent systems:

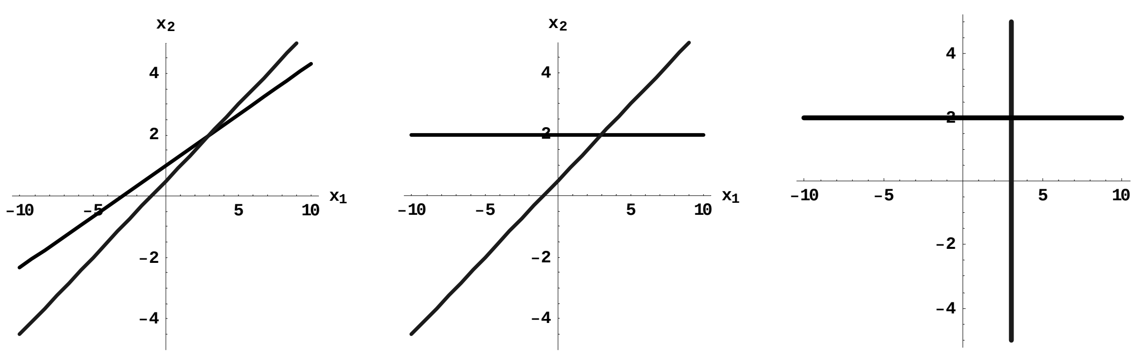
- Two linear systems with the same solution set.

STRATEGY FOR SOLVING A SYSTEM:

- *Replace one system with an equivalent system that is easier to solve.*

EXAMPLE:

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \quad \rightarrow \quad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array} \quad \rightarrow \quad \begin{array}{rcrcr} x_1 & & & = & 3 \\ & & x_2 & = & 2 \end{array}$$



$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array} \qquad \begin{array}{rcrcr} x_1 & & & = & 3 \\ & & x_2 & = & 2 \end{array}$$

Matrix Notation

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

(coefficient matrix)

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \quad \left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right] \rightarrow \begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned} \quad \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

(augmented matrix)

$$\downarrow$$

$$\begin{aligned} x_1 \quad &= 3 \\ x_2 &= 2 \end{aligned} \quad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

Elementary Row Operations:
1. *(Replacement)* Add one row to a multiple of another row.
2. *(Interchange)* Interchange two rows.
3. *(Scaling)* Multiply all entries in a row by a nonzero constant.

Row equivalent matrices: Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Fact about Row Equivalence: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

EXAMPLE:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \quad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \quad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \quad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \quad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \quad \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & & & & & = & 29 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \quad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Solution: $(29, 16, 3)$

Check: Is $(29, 16, 3)$ a solution of the ***original*** system?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

$$\begin{array}{rcl} (29) - 2(16) + 3 & = 29 - 32 + 3 & = 0 \\ 2(16) - 8(3) & = 32 - 24 & = 8 \\ -4(29) + 5(16) + 9(3) & = -116 + 80 + 27 & = -9 \end{array}$$

Two Fundamental Questions (Existence and Uniqueness)

1) Is the system consistent; (i.e. does a solution **exist**?)

2) If a solution exists, is it **unique**? (i.e. is there one & only one solution?)

EXAMPLE: Is this system consistent?

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

In the last example, this system was reduced to the triangular form:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This is sufficient to see that the system is consistent and unique. Why?

EXAMPLE: Is this system consistent?

$$\begin{aligned} 3x_2 - 6x_3 &= 8 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \end{aligned} \qquad \left[\begin{array}{ccc|c} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{array}\right]$$

Solution: Row operations produce:

$$\left[\begin{array}{ccc|c} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 3 & -6 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

Equation notation of triangular form:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -1 \\ 3x_2 - 6x_3 &= 8 \\ 0x_3 &= -3 \quad \leftarrow \textit{Never true} \end{aligned}$$

The original system is inconsistent!

EXAMPLE: For what values of h will the following system be consistent?

$$\begin{aligned} 3x_1 - 9x_2 &= 4 \\ -2x_1 + 6x_2 &= h \end{aligned}$$

Solution: Reduce to triangular form.

$$\left[\begin{array}{cc|c} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h + \frac{8}{3} \end{array}\right]$$

The second equation is $0x_1 + 0x_2 = h + \frac{8}{3}$. System is consistent only if $h + \frac{8}{3} = 0$ or $h = \frac{-8}{3}$.

Section 1.2: Row Reduction and Echelon Forms

Echelon form (or **row echelon form):**

1. All nonzero rows are above any rows of all zeros.
2. Each *leading entry* (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

EXAMPLE: Echelon forms

$$\text{(a)} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{(c)} \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$$

Reduced echelon form: Add the following conditions to conditions 1, 2, and 3 above:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

EXAMPLE (continued):

Reduced echelon form :

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

Theorem 1 (Uniqueness of The Reduced Echelon Form):

Each matrix is row-equivalent to one and only one reduced echelon matrix.

Important Terms:

- **pivot position:** a position of a leading entry in an echelon form of the matrix.

- **pivot:** a nonzero number that either is used in a pivot position to create 0's or is changed into a leading 1, which in turn is used to create 0's.

- **pivot column:** a column that contains a pivot position.

(See the Glossary at the back of the textbook.)

EXAMPLE: Row reduce to echelon form and locate the pivot columns.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution

pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$\uparrow$

pivot column

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$ Possible Pivots:________

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Original Matrix: } \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

$$\begin{array}{lccccc} & \uparrow & \uparrow & & \uparrow & \\ \text{pivot columns:} & 1 & 2 & & 4 & \end{array}$$

Note: There is no more than one pivot in any row. There is no more than one pivot in any column.

EXAMPLE: Row reduce to echelon form and then to reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Cover the top row and look at the remaining two rows for the left-most nonzero column.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

(echelon form)

Final step to create the reduced echelon form:

Beginning with the rightmost leading entry, and working upwards to the left, create zeros above each leading entry and scale rows to transform each leading entry into 1.

$$\left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

SOLUTIONS OF LINEAR SYSTEMS

- **basic variable:** any variable that corresponds to a pivot column in the augmented matrix of a system.
- **free variable:** all nonbasic variables.

EXAMPLE:

$$\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \qquad \begin{array}{rrrrl} x_1 & +6x_2 & & +3x_4 & & = 0 \\ & & x_3 & -8x_4 & & = 5 \\ & & & & x_5 & = 7 \end{array}$$

pivot columns:

basic variables:

free variables:

Final Step in Solving a Consistent Linear System: After the augmented matrix is in **reduced** echelon form and the system is written down as a set of equations:

Solve each equation for the basic variable in terms of the free variables (if any) in the equation.

EXAMPLE:

$$\begin{array}{rrrrl} x_1 & +6x_2 & & +3x_4 & & = 0 \\ & & x_3 & -8x_4 & & = 5 \\ & & & & x_5 & = 7 \end{array} \qquad \left\{ \begin{array}{l} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{array} \right. \quad \text{(general solution)}$$

The **general solution** of the system provides a parametric description of the solution set. (The free variables act as parameters.) The above system has **infinitely many solutions.** Why?

Warning: Use only the reduced echelon form to solve a system.

Existence and Uniqueness Questions

EXAMPLE:

$$\begin{bmatrix} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = 15 \end{bmatrix}$$

In an earlier example, we obtained the echelon form:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad (x_5 = 4)$$

No equation of the form $0 = c$, where $c \neq 0$, so the system is consistent.

Free variables: x_3 and x_4

Consistent system with free variables $\Rightarrow$ **infinitely many solutions.**

EXAMPLE:

$$\begin{array}{rrcr} 3x_1 & +4x_2 & = & -3 \\ 2x_1 & +5x_2 & = & 5 \\ -2x_1 & -3x_2 & = & 1 \end{array} \rightarrow \left[\begin{array}{rr|r} 3 & 4 & -3 \\ 2 & 5 & 5 \\ -2 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{rr|r} 3 & 4 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{l} 3x_1 + 4x_2 = -3 \\ x_2 = 3 \end{array}$$

Consistent system, no free variables $\Rightarrow$ **unique solution.**

Theorem 2 (Existence and Uniqueness Theorem)

1. A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, i.e., if and only if an echelon form of the augmented matrix has no row of the form

$$\left[\begin{array}{ccc|c} 0 & \dots & 0 & b \end{array}\right] \text{ (where } b \text{ is nonzero).}$$

2. If a linear system is consistent, then the solution contains either
(i) a unique solution (when there are no free variables) or
(ii) infinitely many solutions (when there is at least one free variable).

Using Row Reduction to Solve Linear Systems

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If not, stop; otherwise go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. State the solution by expressing each basic variable in terms of the free variables and declare the free variables.

EXAMPLE:

a) What is the largest possible number of pivots a 4×6 matrix can have? Why?

b) What is the largest possible number of pivots a 6×4 matrix can have? Why?

c) How many solutions does a consistent linear system of 3 equations and 4 unknowns have? Why?

d) Suppose the coefficient matrix corresponding to a linear system is 4×6 and has 3 pivot columns. How many pivot columns does the augmented matrix have if the linear system is inconsistent?

1.3 VECTOR EQUATIONS

Key concepts to master: linear combinations of vectors and a spanning set.

Vector: A matrix with only one column.

Vectors in R^n (vectors with n entries):

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Geometric Description of R^2

Vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the point (x_1, x_2) in the plane.

R^2 is the set of all points in the plane.

Parallelogram rule for addition of two vectors:

If **u** and **v** in **R**^{2} are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are **0**, **u** and **v**. (Note that $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.)

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Graphs of $\mathbf{u}, \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ are given below:

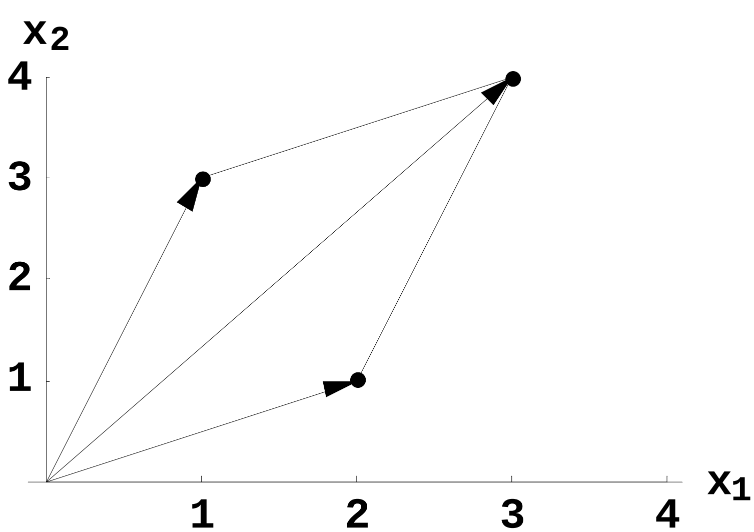
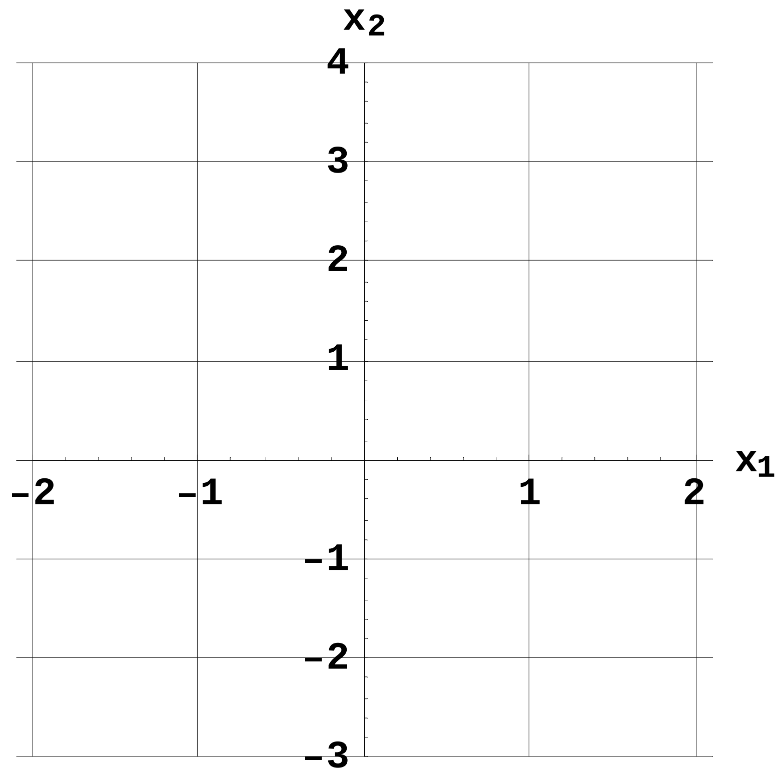


Illustration of the Parallelogram Rule

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Express $\mathbf{u}$, $2\mathbf{u}$, and $\frac{-3}{2}\mathbf{u}$ on a graph.



Linear Combinations

DEFINITION

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbf{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

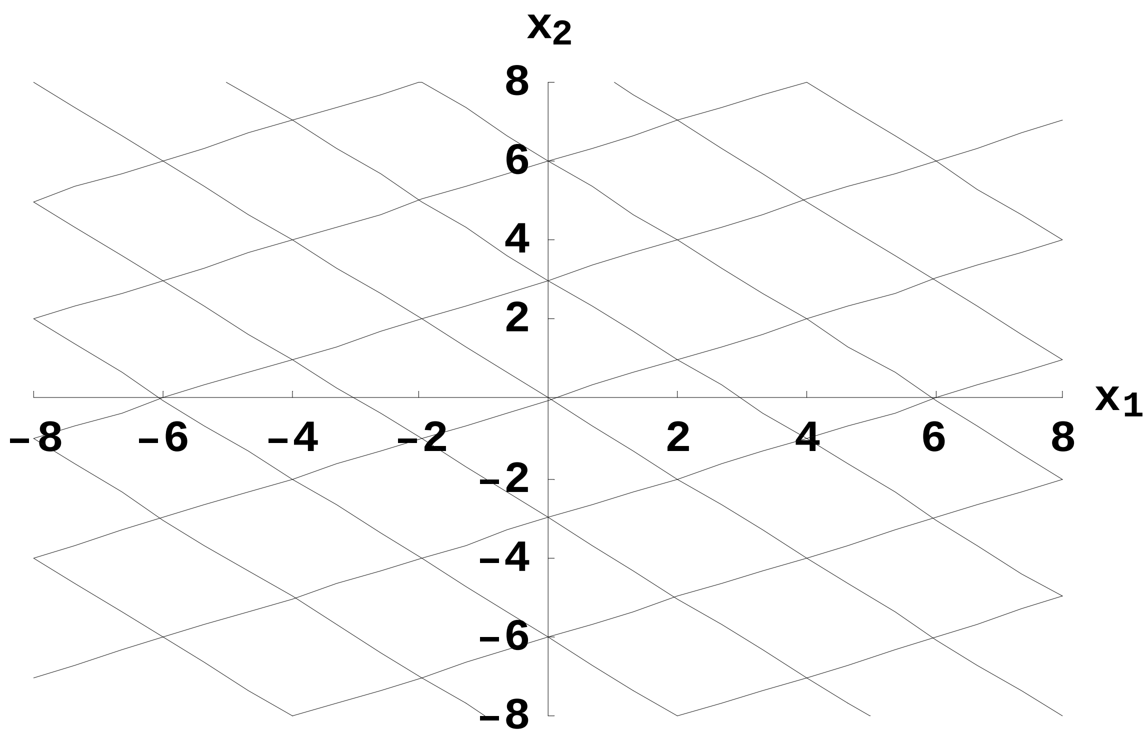
is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ using weights $c_1, c_2, \ldots, c_p$.

Examples of linear combinations of $\mathbf{v}_1$and $\mathbf{v}_2$:

$$3\mathbf{v}_1 + 2\mathbf{v}_2, \qquad \tfrac{1}{3}\mathbf{v}_1, \qquad \mathbf{v}_1 - 2\mathbf{v}_2, \qquad \mathbf{0}$$

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Express each of the following as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \ \mathbf{d} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$



EXAMPLE: Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$.

Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

Solution: Vector $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ if can we find weights x_1, x_2, x_3 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$$

Vector Equation (fill-in):

Corresponding System:

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= -1 \\ 2x_2 + 6x_3 &= 8 \\ 3x_1 + 14x_2 + 10x_3 &= -5 \end{aligned}$$

Corresponding Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = \underline{\quad} \\ x_2 = \underline{\quad} \\ x_3 = \underline{\quad} \end{array}$$

Review of the last example: $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{b}$ are columns of the augmented matrix

$$\begin{array}{cccc} \left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{array}\right] \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b} \end{array}$$

Solution to

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$$

is found by solving the linear system whose augmented matrix is

$$\left[\begin{array}{ccc|c} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{array}\right].$$

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

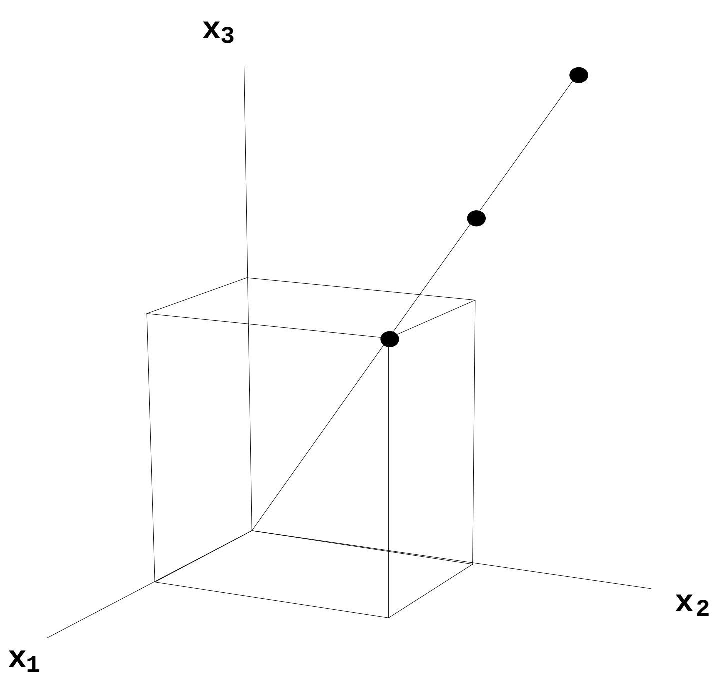
has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \,|\, \mathbf{b} \end{bmatrix}.$$

In particular, **b** can be generated by a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ if and only if there is a solution to the linear system corresponding to the augmented matrix.

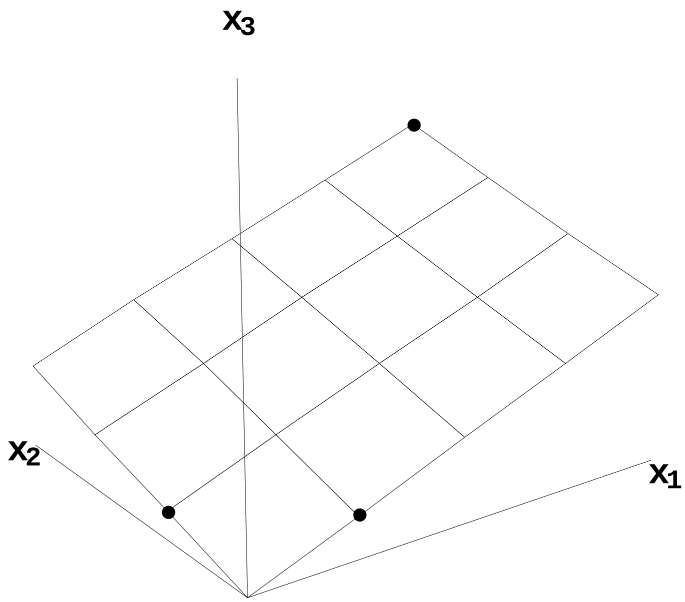
The Span of a Set of Vectors

EXAMPLE: Let $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. Label the origin $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ together with **v**, 2**v** and 1.5**v** on the graph below.



v, 2**v** and 1.5**v** all lie on the same line.
Span$\{\mathbf{v}\}$ is the set of all vectors of the form $c\mathbf{v}$.
Here, **Span**$\{\mathbf{v}\} =$ a line through the origin.

EXAMPLE: Label **u**, **v**, **u** + **v** and 3**u** +4**v** on the graph below.



u, **v**, **u** + **v** and 3**u** +4**v** all lie in the same plane.
Span$\{\mathbf{u},\mathbf{v}\}$ is the set of all vectors of the form $x_1\mathbf{u} + x_2\mathbf{v}$.
Here, **Span**$\{\mathbf{u},\mathbf{v}\}$ = a plane through the origin.

Definition

Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are in $\mathbf{R}^n$; then

Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ = set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.

Stated another way: **Span**$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written as

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p$$

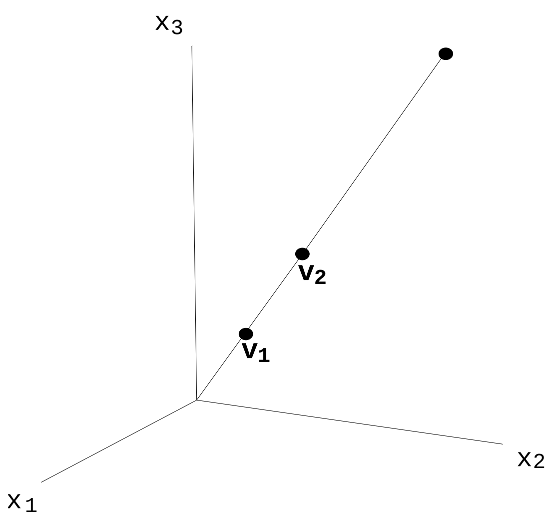
where $x_1, x_2, \ldots, x_p$ are scalars.

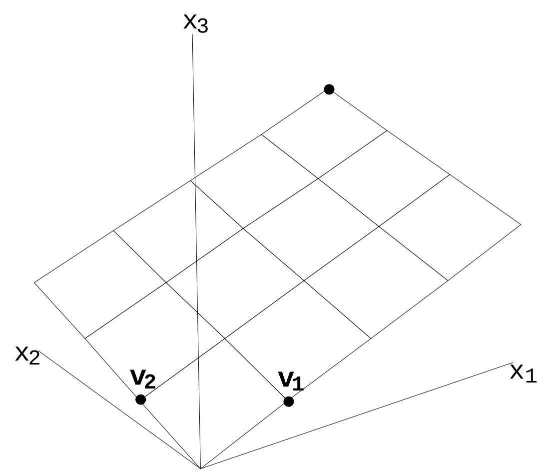
EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

(a) Find a vector in **Span**$\{\mathbf{v}_1, \mathbf{v}_2\}$.

(b) Describe **Span**$\{\mathbf{v}_1, \mathbf{v}_2\}$ geometrically.

Spanning Sets in $\mathbf{R}^3$





$\mathbf{v}_2$ is a multiple of $\mathbf{v}_1$
Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ $=$**Span**$\{\mathbf{v}_1\}$ $=$**Span**$\{\mathbf{v}_2\}$
(line through the origin)

$\mathbf{v}_2$ is **not** a multiple of $\mathbf{v}_1$
Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ $=$plane through the origin

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$. Is **Span**$\{\mathbf{v}_1, \mathbf{v}_2\}$ a line or a plane?

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$. Is **b** in the plane spanned by the columns of A?

Solution:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$$

Do x_1 and x_2 exist so that

Corresponding augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & 1 & 3 \\ 0 & 5 & 17 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 5 & 17 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 0 & -4 \end{array}\right]$$

So **b** is not in the plane spanned by the columns of A

1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

Linear combinations can be viewed as a matrix-vector multiplication.

Definition

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbf{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is the **linear combination of the columns of A using the corresponding entries in x as weights**. I.e.,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_n \mathbf{a}_n$$

EXAMPLE:

$$\begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -6 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + -6 \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 21 \\ 0 \end{bmatrix} + \begin{bmatrix} 24 \\ -12 \\ -30 \end{bmatrix} = \begin{bmatrix} 31 \\ 9 \\ -30 \end{bmatrix}$$

EXAMPLE: Write down the system of equations corresponding to the augmented matrix below and then express the system of equations in vector form and finally in the form $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b}$ is a 3×1 vector.

$$\begin{bmatrix} 2 & 3 & 4 & 9 \\ -3 & 1 & 0 & -2 \end{bmatrix}$$

Solution: Corresponding system of equations (fill-in)

Vector Equation (fill-in):

$$\_\_ \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \_\_ \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \_\_ \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}.$$

Matrix equation (fill-in):

Three equivalent ways of viewing a linear system:

1. as a system of linear equations;
2. as a vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$**; or**
3. as a matrix equation $A\mathbf{x} = \mathbf{b}$.

THEOREM 3

If A is a $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbf{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & | \mathbf{b} \end{bmatrix}.$$

Useful Fact:

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a

______________ _______________________ of the columns of A.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b}$?

Solution: Augmented matrix corresponding to $A\mathbf{x} = \mathbf{b}$:

$$\left[\begin{array}{ccc|c} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 5 & b_1 \\ 0 & 1 & 1 & 3b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + b_3 \end{array}\right]$$

$A\mathbf{x} = \mathbf{b}$ is ________ consistent for all $\mathbf{b}$ since some choices of $\mathbf{b}$ make $-2b_1 + b_3$ nonzero.

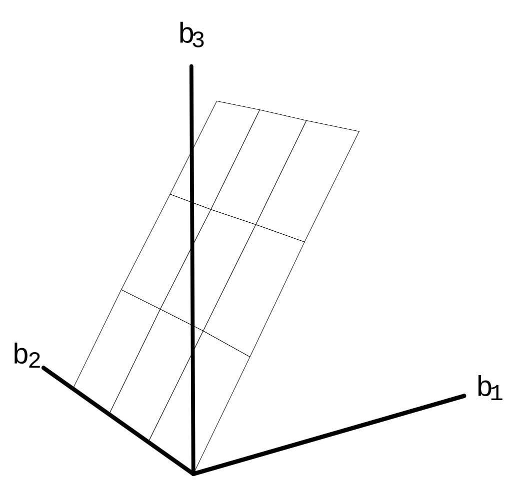
$$A = \begin{array}{c} \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{array} \end{array}$$

The equation $A\mathbf{x} = \mathbf{b}$ is consistent if

$$-2b_1 + b_3 = 0.$$

(equation of a plane in $\mathbf{R}^3$)

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_3 + x_3\mathbf{a}_3 = \mathbf{b} \text{ if and only if } b_3 - 2b_1 = 0.$$



Columns of A span a plane
in $\mathbf{R}^3$ through $\mathbf{0}$

Instead, if *any* $\mathbf{b}$ in $\mathbf{R}^3$ (not just those lying on a particular line or in a plane) can be expressed as a linear combination of the columns of A, then we say that the columns of A span $\mathbf{R}^3$.

Definition

We say that **the columns of** $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_p \end{bmatrix}$ **span** $\mathbf{R}^m$ if ***every*** vector $\mathbf{b}$ in $\mathbf{R}^m$ is a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_p$

(i.e. Span$\{\mathbf{a}_1, \ldots, \mathbf{a}_p\} = \mathbf{R}^m$).

THEOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

a. For each $\mathbf{b}$ in $\mathbf{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbf{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbf{R}^m$.

d. A has a pivot position in every row.

Proof (outline): Statements (a), (b) and (c) are logically equivalent.

To complete the proof, we need to show that (a) is true when (d) is true and (a) is false when (d) is false.

Suppose (d) is ______________. Then row-reduce the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$:

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} \sim \cdots \sim \begin{bmatrix} U & \mathbf{d} \end{bmatrix}$$

and each row of U has a pivot position and so there is no pivot in the last column of $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$.

So (a) is ______________.

Now suppose (d) is ______________. Then the last row of $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ contains all zeros.

Suppose $\mathbf{d}$ is a vector with a 1 as the last entry. Then $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ represents an inconsistent system.

Row operations are reversible: $\begin{bmatrix} U & \mathbf{d} \end{bmatrix} \sim \cdots \sim \begin{bmatrix} A & \mathbf{b} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ is inconsistent also. So (a) is _____________. ■

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible **b**?

Solution: A has only ______ columns and therefore has at most ______ pivots.

Since A does not have a pivot in every _________, $A\mathbf{x} = \mathbf{b}$

is __________ ________________________ for all possible **b**,

according to Theorem 4.

EXAMPLE: Do the columns of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{bmatrix}$ span $\mathbf{R}^3$?

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{bmatrix} \sim$$

(no pivot in row 2)

By Theorem 4, the columns of A ____.

Another method for computing $A\mathbf{x}$:

Read Example 4 on page 54 through Example 5 on page 55 to learn this rule for computing the product $A\mathbf{x}$.

Theorem 5

If A is an $m \times n$ matrix, **u** and **v** are vectors in $\mathbf{R}^n$, and c is a scalar, then:

a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = cA\mathbf{u}$.

1.5 Solutions Sets of Linear Systems

Homogeneous System:

$$A\mathbf{x} = \mathbf{0}$$

(A is $m \times n$ and $\mathbf{0}$ is the zero vector in $\mathbf{R}^m$)

EXAMPLE:

$$\begin{array}{rcrcl} x_1 & + & 10x_2 & = & 0 \\ 2x_1 & + & 20x_2 & = & 0 \end{array}$$

Corresponding matrix equation $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 10 \\ 2 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Trivial solution:

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{or} \qquad \mathbf{x} = \mathbf{0}$$

The homogeneous system $A\mathbf{x} = \mathbf{0}$ ***always*** has the **trivial solution**, $\mathbf{x} = \mathbf{0}$.

Nonzero vector solutions are called **nontrivial solutions.**

Do **nontrivial** solutions exist?

$$\left[\begin{array}{cc|c} 1 & 10 & 0 \\ 2 & 20 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 10 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Consistent system with a free variable has infinitely many solutions.

A homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions if and only if the

system of equations has____________________________________.

EXAMPLE: Determine if the following homogeneous system has nontrivial solutions and then describe the solution set.

$$\begin{aligned} 2x_1 + 4x_2 - 6x_3 &= 0 \\ 4x_1 + 8x_2 - 10x_3 &= 0 \end{aligned}$$

Solution:

There is at least one free variable (why?)

$\Rightarrow$ nontrivial solutions exist

$$\begin{bmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 4 & 8 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

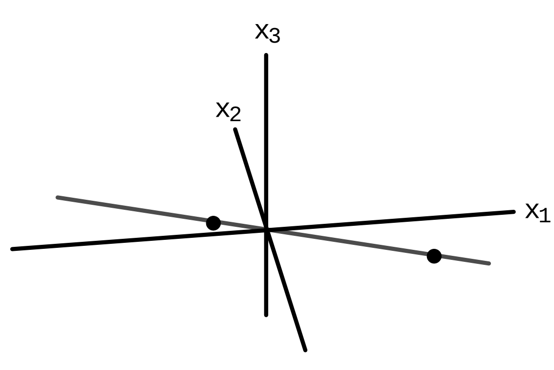
$x_1 =$ ___

x_2 is free

$x_3 =$ ___

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = \_\_\_\_ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_2\mathbf{v}$$

Graphical representation:



solution set $=$ span$\{\mathbf{v}\}$ $=$ line through $\mathbf{0}$ in $\mathbf{R}^3$

EXAMPLE: Describe the solution set of

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & - & 6x_3 & = & 0 \\ 4x_1 & + & 8x_2 & - & 10x_3 & = & 4 \end{array}$$

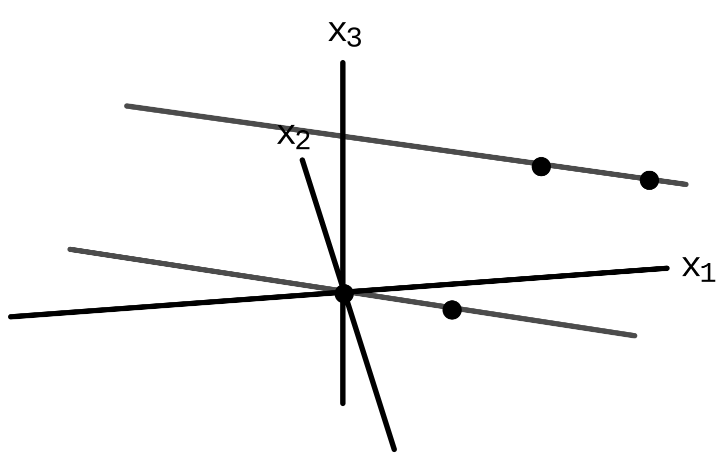
(same left side as in the previous example)

Solution:

$$\begin{bmatrix} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 4 \end{bmatrix} \quad \text{row reduces to} \quad \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

$$\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{p} + x_2 \mathbf{v}$$



Parallel solution sets of $A\mathbf{x} = \mathbf{0}$ & $A\mathbf{x} = \mathbf{b}$

Recap of Previous Two Examples

Solution of $A\mathbf{x} = \mathbf{0}$

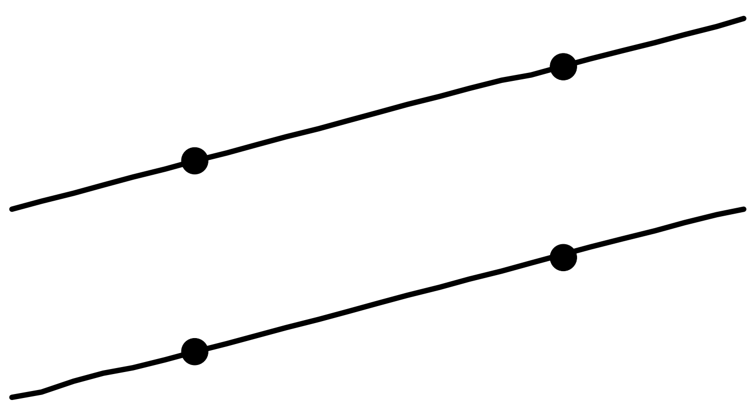
$$\mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_2\mathbf{v}$$

$\mathbf{x} = x_2\mathbf{v}$ = parametric equation of line passing through $\mathbf{0}$ and $\mathbf{v}$

Solution of $A\mathbf{x} = \mathbf{b}$

$$\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{p} + x_2\mathbf{v}$$

$\mathbf{x} = \mathbf{p} + x_2\mathbf{v}$ = parametric equation of line passing through $\mathbf{p}$ parallel to $\mathbf{v}$



Parallel solution sets of
$A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$

THEOREM 6

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

EXAMPLE: Describe the solution set of $2x_1 - 4x_2 - 4x_3 = 0$; compare it to the solution set $2x_1 - 4x_2 - 4x_3 = 6$.

Solution: Corresponding augmented matrix to $2x_1 - 4x_2 - 4x_3 = 0$:

$$\left[\begin{array}{ccc|c} 2 & -4 & -4 & 0 \end{array}\right] \sim$$ (fill-in)

Vector form of the solution:

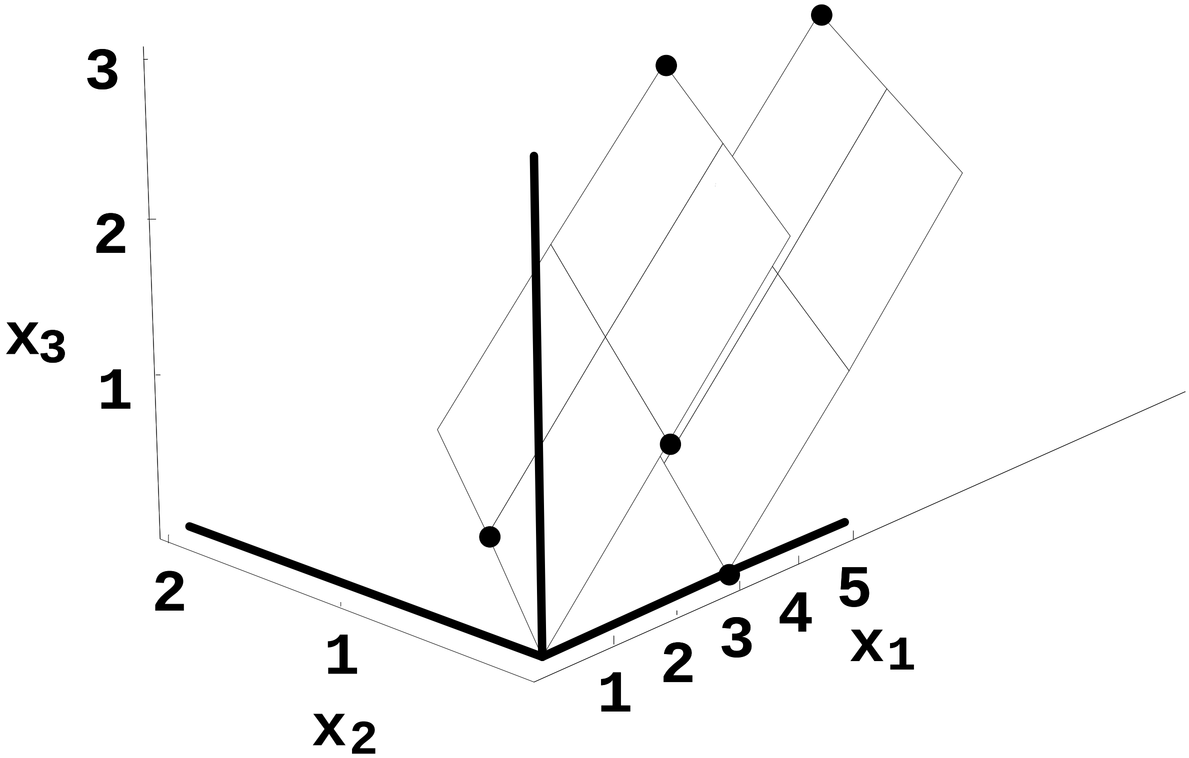
$$\mathbf{v} = \begin{bmatrix} 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \_\_\_\_ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Corresponding augmented matrix to $2x_1 - 4x_2 - 4x_3 = 6$:

$$\left[\begin{array}{ccc|c} 2 & -4 & -4 & 6 \end{array}\right] \sim$$ (fill -in)

Vector form of the solution:

$$\mathbf{v} = \begin{bmatrix} 3 + 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$



Parallel Solution Sets of $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$

1.7 Linear Independence

A homogeneous system such as

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

can be viewed as a vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The vector equation has the trivial solution ($x_1 = 0$, $x_2 = 0$, $x_3 = 0$), but is this the *only solution*?

Definition

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists weights $c_1, \ldots, c_p$, not all 0, such that

$$\underbrace{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}}_{\uparrow}.$$

linear dependence relation
(when weights are not all zero)

EXAMPLE Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$.

a. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
b. If possible, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Solution: (a)

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & -1 & 18 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is a free variable $\Rightarrow$ there are nontrivial solutions.

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set

(b) Reduced echelon form: $\left[\begin{array}{ccc|c} 1 & 0 & 33 & 0 \\ 0 & 1 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = \_\_\_ \\ x_2 = \_\_\_ \\ x_3 \ \_\_\_\_\_\_ \end{array}$

Let $x_3 =$ _____ (any nonzero number). Then $x_1 =$ _____ and $x_2 =$ _____.

$$\_\_\_\_ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\_\_\_\_\mathbf{v}_1 + \_\_\_\_\mathbf{v}_2 + \_\_\_\_\mathbf{v}_3 = \mathbf{0}$$

(one possible linear dependence relation)

Linear Independence of Matrix Columns

A linear dependence relation such as

$$-33\begin{bmatrix}1\\3\\5\end{bmatrix}+18\begin{bmatrix}2\\5\\9\end{bmatrix}+1\begin{bmatrix}-3\\9\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

can be written as the matrix equation:

$$\begin{bmatrix}1 & 2 & -3\\3 & 5 & 9\\5 & 9 & 3\end{bmatrix}\begin{bmatrix}-33\\18\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution to $A\mathbf{x}=\mathbf{0}$.

> The columns of matrix A are linearly independent if and only if the equation $A\mathbf{x}=\mathbf{0}$ has *only* the trivial solution.

Special Cases

Sometimes we can determine linear independence of a set with minimal effort.

1. *A Set of One Vector*

Consider the set containing one nonzero vector: $\{\mathbf{v}_1\}$

The only solution to $x_1\mathbf{v}_1=0$ is $x_1=$ _____.

$\boxed{\text{So } \{\mathbf{v}_1\} \text{ is linearly independent when } \mathbf{v}_1 \neq \mathbf{0}.}$

2. ***A Set of Two Vectors***

EXAMPLE Let

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

a. Determine if $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly dependent set or a linearly independent set.

b. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly dependent set or a linearly independent set.

Solution: (a) Notice that $\mathbf{u}_2 =$ _____$\mathbf{u}_1$. Therefore

$$\_\_\_\_\_\mathbf{u}_1 + \_\_\_\_\_\mathbf{u}_2 = 0$$

This means that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly _________________ set.

(b) Suppose

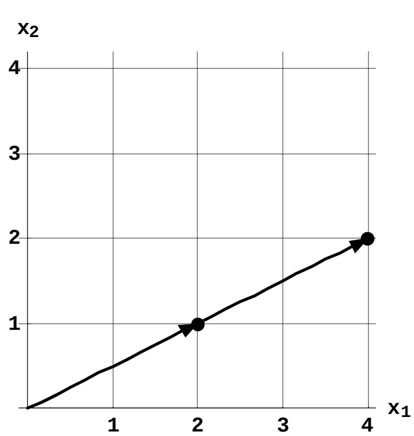
$$c\mathbf{v}_1 + d\mathbf{v}_2 = \mathbf{0}.$$

Then $\mathbf{v}_1 = \text{———}\mathbf{v}_2$ if $c \neq 0$. But this is impossible since $\mathbf{v}_1$ is ______ a multiple of $\mathbf{v}_2$ which means $c =$ _____.

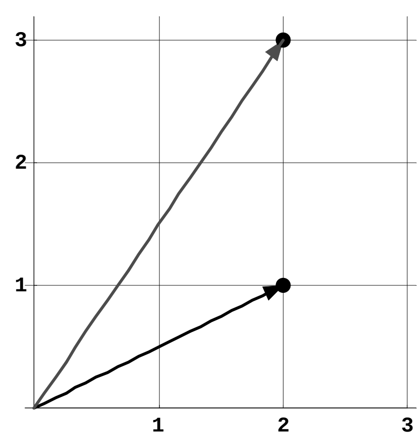
Similarly, $\mathbf{v}_2 = \text{———}\mathbf{v}_1$ if $d \neq 0$. But this is impossible since $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$ and so $d = 0$. This means that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly __________________ set.

A set of two vectors is linearly dependent if at least one vector is a multiple of the other.

A set of two vectors is linearly independent if and only if neither of the vectors is a multiple of the other.



linearly ____________________



linearly ____________________

3. *A Set Containing the* **0** *Vector*

Theorem 9

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ containing the zero vector is linearly dependent.

Proof: Renumber the vectors so that $\mathbf{v}_1 =$ ____. Then

$$\_\_\_\_\mathbf{v}_1 + \_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\mathbf{v}_p = \mathbf{0}$$

which shows that S is linearly ________________.

4. *A Set Containing Too Many Vectors*

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. I.e. any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is linearly dependent if $p > n$.

Outline of Proof:

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p \end{bmatrix} \text{ is } n \times p$$

Suppose $p > n$.

$\Rightarrow A\mathbf{x} = \mathbf{0}$ has more variables than equations

$\Rightarrow A\mathbf{x} = \mathbf{0}$ has nontrivial solutions

$\Rightarrow$columns of A are linearly dependent

EXAMPLE With the least amount of work possible, decide which of the following sets of vectors are linearly independent and give a reason for each answer.

a. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}$

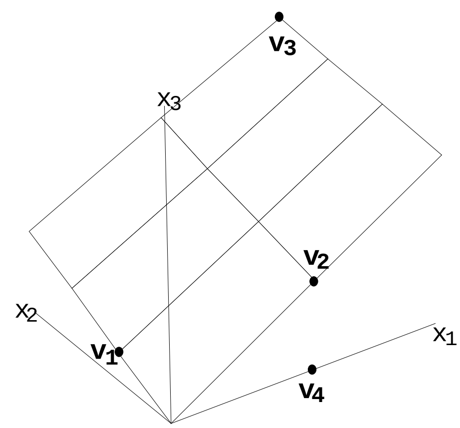
b. Columns of $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 \\ 9 & 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 & 8 \end{bmatrix}$

c. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d. $\left\{ \begin{bmatrix} 8 \\ 2 \\ 1 \\ 4 \end{bmatrix} \right\}$

Characterization of Linearly Dependent Sets

EXAMPLE Consider the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ in $\mathbf{R}^3$ in the following diagram. Is the set linearly dependent? Explain



Theorem 7

An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent, and $\mathbf{v}_1 \neq \mathbf{0}$, then some vector $\mathbf{v}_j$ ($j \geq 2$) is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

1.8 Introduction to Linear Transformations

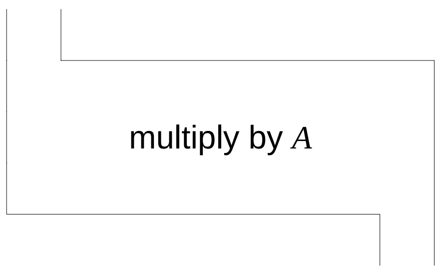
Another way to view $A\mathbf{x} = \mathbf{b}$:

Matrix A is an object acting on $\mathbf{x}$ by multiplication to produce a new vector $A\mathbf{x}$ or $\mathbf{b}$.

EXAMPLE:

$$\begin{bmatrix} 2 & -4 \\ 3 & -6 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ -12 \\ -4 \end{bmatrix} \qquad \begin{bmatrix} 2 & -4 \\ 3 & -6 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

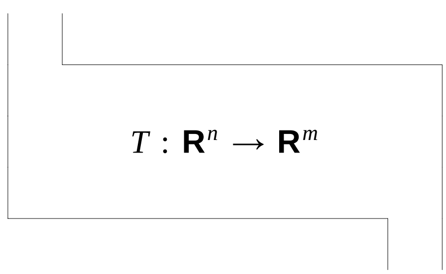
Suppose A is $m \times n$. Solving $A\mathbf{x} = \mathbf{b}$ amounts to finding all ____ in $\mathbf{R}^n$ which are transformed into vector $\mathbf{b}$ in $\mathbf{R}^m$ through multiplication by A.



transformation
"machine"

Matrix Transformations

A **transformation** T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbf{R}^n$ a vector $T(\mathbf{x})$ in $\mathbf{R}^m$.



Terminology:

$\mathbf{R}^n$: **domain** of T $\qquad$ $\mathbf{R}^m$: **codomain** of T

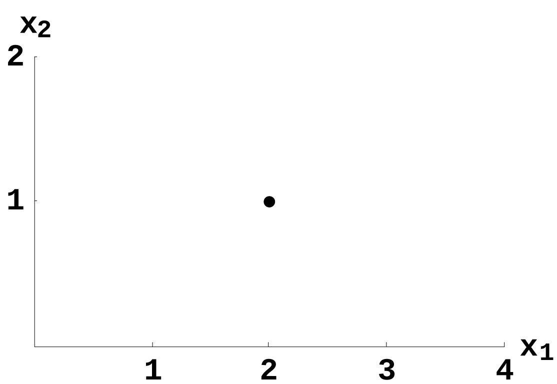
$T(\mathbf{x})$ in $\mathbf{R}^m$ is the **image** of $\mathbf{x}$ under the transformation T

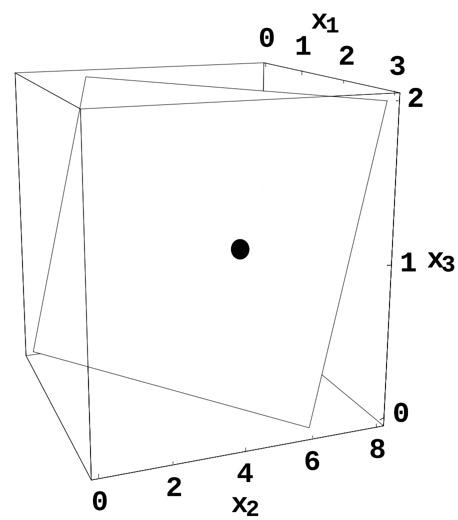
Set of all images $T(\mathbf{x})$ is the **range** of T

EXAMPLE: Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$. Define a transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

Then if $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$,

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$





EXAMPLE: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Then define a transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$.

a. Find an $\mathbf{x}$ in $\mathbf{R}^3$ whose image under T is $\mathbf{b}$.
b. Is there more than one $\mathbf{x}$ under T whose image is $\mathbf{b}$. *(uniqueness problem)*
c. Determine if $\mathbf{c}$ is in the range of the transformation T. *(existence problem)*

Solution: (a) Solve ________=______ for $\mathbf{x}$. I.e., solve ________=______ or

$$\begin{bmatrix} 1 & -2 & 3 \\ -5 & 10 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$$

Augmented matrix:

$$\begin{bmatrix} 1 & -2 & 3 & 2 \\ -5 & 10 & -15 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 2x_2 - 3x_3 + 2$
x_2 is free
x_3 is free

Let $x_2 =$ _____ and $x_3 =$ _____. Then $x_1 =$ _____.

$$\text{So } \mathbf{x} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

(b) Is there an $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{b}$?

Free variables exist $\Rightarrow$ There is more than one $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{b}$

(c) Is there an $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{c}$? This is another way of asking if $A\mathbf{x} = \mathbf{c}$ is ________________.

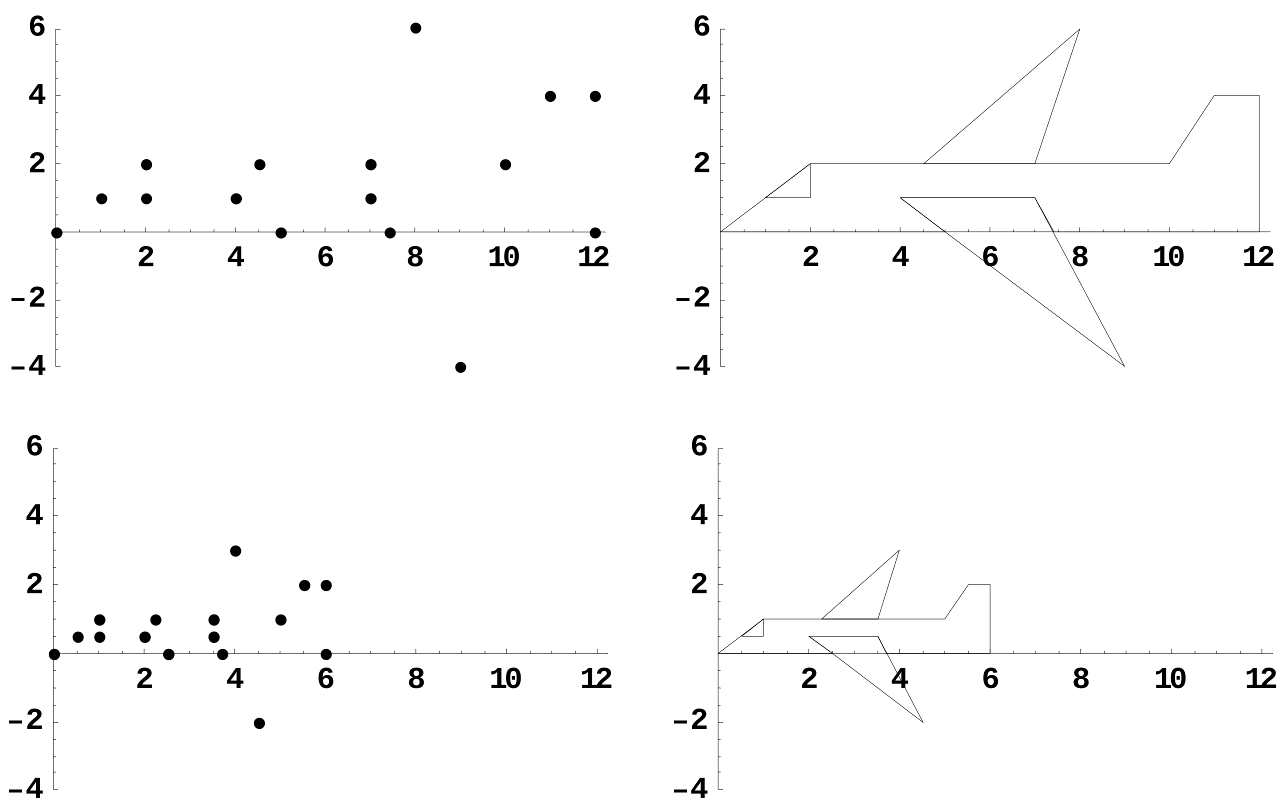
Augmented matrix: $\begin{bmatrix} 1 & -2 & 3 & 3 \\ -5 & 10 & -15 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\mathbf{c}$ is not in the ________________ of T.

Matrix transformations have many applications - including *computer graphics*.

EXAMPLE: Let $A = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}$. The transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is an example of a **contraction** transformation. The transformation $T(\mathbf{x}) = A\mathbf{x}$ can be used to move a point $\mathbf{x}$.

$$\mathbf{u} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} \qquad T(\mathbf{u}) = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$



Linear Transformations

If A is $m \times n$, then the transformation $T(\mathbf{x}) = A\mathbf{x}$ has the following properties:

$$T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = \underline{\hspace{2cm}} + \underline{\hspace{2cm}}$$

$$= \underline{\hspace{2cm}} + \underline{\hspace{2cm}}$$

and

$$T(c\mathbf{u}) = A(c\mathbf{u}) = \underline{\hspace{1.5cm}}A\mathbf{u} = \underline{\hspace{1.5cm}}T(\mathbf{u})$$

for all $\mathbf{u}, \mathbf{v}$ in $\mathbf{R}^n$ and all scalars c.

DEFINITION

A transformation T is **linear** if:

i. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T.

ii. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in the domain of T and all scalars c.

Every matrix transformation is a **linear** transformation.

RESULT If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0} \quad \text{and} \quad T(c\mathbf{u}+d\mathbf{v}) = c\mathbf{T}(\mathbf{u}) + d\mathbf{T}(\mathbf{v}).$$

Proof:

$$T(\mathbf{0}) = T(0\mathbf{u}) = \underline{\hspace{1.2cm}}T(\mathbf{u}) = \underline{\hspace{1.5cm}}.$$

$$T(c\mathbf{u}+d\mathbf{v}) = T\left(\hspace{1.5cm}\right) + T\left(\hspace{1.5cm}\right) = \underline{\hspace{1.5cm}}T\left(\hspace{1cm}\right) + \underline{\hspace{1.5cm}}T\left(\hspace{1cm}\right)$$

EXAMPLE: Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Suppose $T : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear transformation which maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution: First, note that

$$T(\mathbf{e}_1) = \_\_\_\_\_\_ \quad \text{and} \quad T(\mathbf{e}_2) = \_\_\_\_\_\_.$$

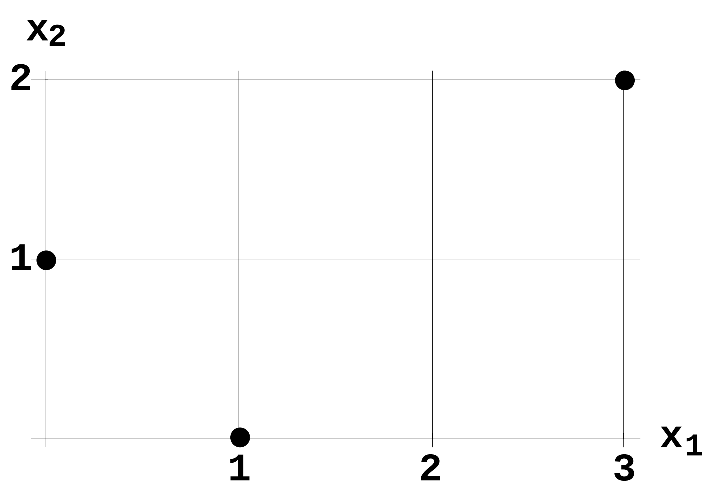
Also

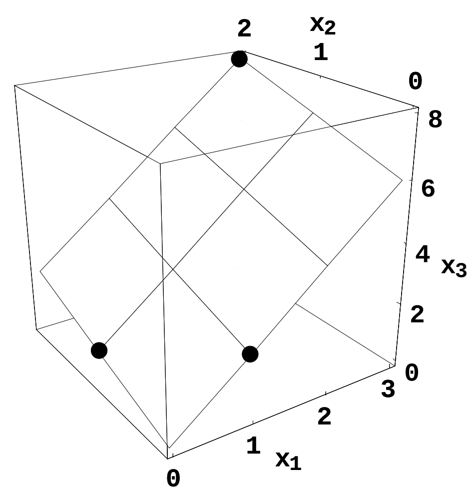
$$\_\_\_\mathbf{e}_1 + \_\_\_\mathbf{e}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Then

$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = T(\_\_\_\mathbf{e}_1 + \_\_\_\mathbf{e}_2) = \_\_\_T(\mathbf{e}_1) + \_\_\_T(\mathbf{e}_2)$$

$$=$$





$$T(3\mathbf{e}_1 + 2\mathbf{e}_2) = 3T(\mathbf{e}_1) + 2T(\mathbf{e}_2)$$

Also

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T(\_\_\_\_\_\mathbf{e}_1 + \_\_\_\_\_\mathbf{e}_2) = \_\_\_\_T(\mathbf{e}_1) + \_\_\_\_T(\mathbf{e}_2) =$$

EXAMPLE: Define $T : \mathbf{R}^3 \to \mathbf{R}^2$ such that $T(x_1, x_2, x_3) = (|x_1 + x_3|, 2 + 5x_2)$. Show that T is a not a linear transformation.

Solution: Another way to write the transformation:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} |x_1 + x_3| \\ 2 + 5x_2 \end{bmatrix}$$

Provide a **counterexample** - example where $T(\mathbf{0}) = \mathbf{0}$, $T(c\mathbf{u}) = c\mathbf{T}(\mathbf{u})$ or $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ is violated.

A counterexample:

$$T(\mathbf{0}) = T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \quad \\ \quad \end{bmatrix} \neq \underline{\qquad}$$

which means that T is not linear.

Another counterexample: Let $c = -1$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then

$$T(c\mathbf{u}) = T\left(\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} |-1 + -1| \\ 2 + 5(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

and

$$cT(\mathbf{u}) = -1T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = -1\begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}.$$

Therefore $T(c\mathbf{u}) \neq \underline{\qquad} T(\mathbf{u})$ and therefore T is not __________________.

Section 1.9 (Through Theorem 10) The Matrix of a Linear Transformation

Identity Matrix I_n is an $n \times n$ matrix with 1's on the main left to right diagonal and 0's elsewhere. The ith column of I_n is labeled $\mathbf{e}_i$.

EXAMPLE:

$$I_3 = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that

$$I_3\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \underline{\qquad} \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} + \underline{\qquad} \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} + \underline{\qquad} \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} = \underline{\qquad}.$$

In general, for $\mathbf{x}$ in $\mathbf{R}^n$,

$$I_n\mathbf{x} = \underline{\quad}$$

From Section 1.8, if $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, then $T(c\mathbf{u} + d\mathbf{v}) = c\mathbf{T}(\mathbf{u}) + d\mathbf{T}(\mathbf{v})$.

Generalized Result:

$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p).$$

EXAMPLE: The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose T is a linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^3$ where

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \text{ and } T(\mathbf{e}_2) = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$$

Compute $T(\mathbf{x})$ for any $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution: A vector $\mathbf{x}$ in $\mathbf{R}^2$ can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{\qquad}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \underline{\qquad}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underline{\qquad}\mathbf{e}_1 + \underline{\qquad}\mathbf{e}_2$$

Then

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = \underline{\qquad}T(\mathbf{e}_1) + \underline{\qquad}T(\mathbf{e}_2)$$

$$= \underline{\qquad}\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} + \underline{\qquad}\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \qquad\qquad\qquad \\ \\ \\ \end{bmatrix}.$$

Note that

$$T(\mathbf{x}) = \begin{bmatrix} \qquad\qquad \\ \\ \\ \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

So

$$T(\mathbf{x}) = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix}\mathbf{x} = A\mathbf{x}$$

To get A, replace the identity matrix $\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix}$ with $\begin{bmatrix} T(\mathbf{e}_2) & T(\mathbf{e}_2) \end{bmatrix}$.

Theorem 10

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n.$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbf{R}^n$.

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)]$$

$$\uparrow$$

standard matrix for the linear transformation T

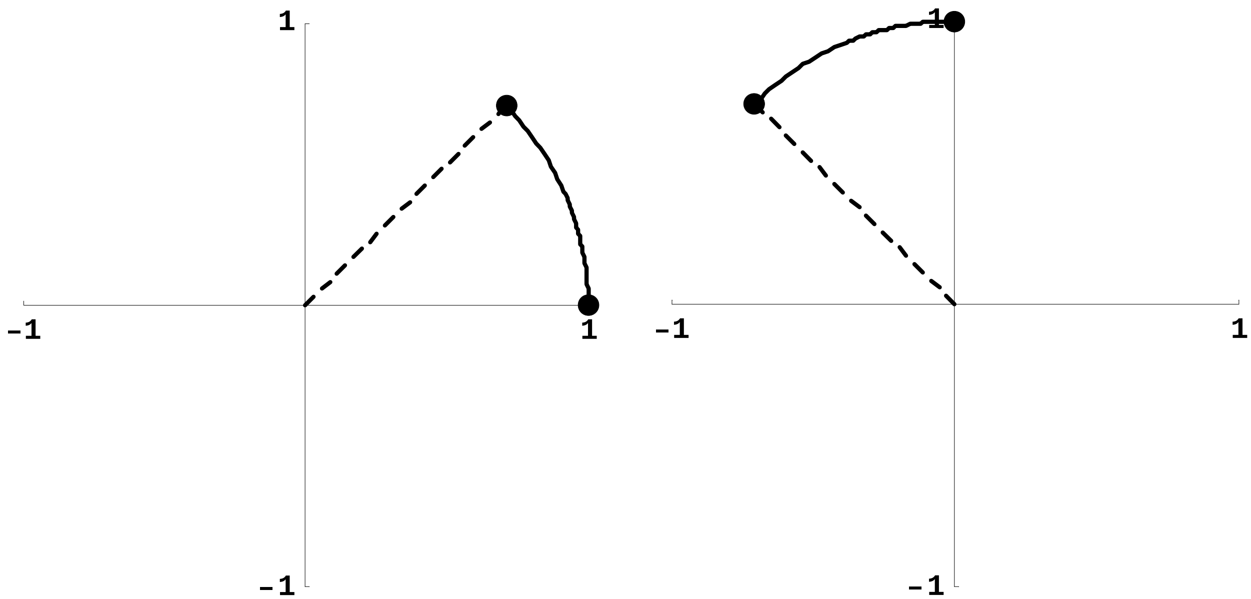
EXAMPLE: $$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ 4x_1 \\ 3x_1 + 2x_2 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} = \text{standard matrix of the linear transformation } T$$

$$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} =$$ (fill-in)

EXAMPLE: Find the standard matrix of the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ which rotates a point about the origin through an angle of $\frac{\pi}{4}$ radians (counterclockwise).



$$T(\mathbf{e}_1) = \begin{bmatrix} \\ \\ \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$\Rightarrow \qquad A = \begin{bmatrix} & \\ & \end{bmatrix}$$

One-to-one, onto: see pages 92-94 of textbook.

2.1 Matrix Operations

Matrix Notation:

Two ways to denote $m \times n$ matrix A:

In terms of the *columns* of A:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

In terms of the *entries* of A:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Main diagonal entries:____________________

Zero matrix:

$$0 = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

THEOREM 1

Let A, B, and C be matrices of the same size, and let r and s be scalars. Then

a. $A + B = B + A$
b. $(A + B) + C = A + (B + C)$
c. $A + 0 = A$
d. $r(A + B) = rA + rB$
e. $(r + s)A = rA + sA$
f. $r(sA) = (rs)A$

Matrix Multiplication

Multiplying B and $\mathbf{x}$ transforms $\mathbf{x}$ into the vector $B\mathbf{x}$. In turn, if we multiply A and $B\mathbf{x}$, we transform $B\mathbf{x}$ into $A(B\mathbf{x})$. So $A(B\mathbf{x})$ is the composition of two mappings.

Define the product AB so that $A(B\mathbf{x}) = (AB)\mathbf{x}$.

Suppose A is $m \times n$ and B is $n \times p$ where

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}.$$

Then

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p$$

and

$$A(B\mathbf{x}) = A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p)$$

$$= A(x_1\mathbf{b}_1) + A(x_2\mathbf{b}_2) + \cdots + A(x_p\mathbf{b}_p)$$

$$= x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \cdots + x_pA\mathbf{b}_p = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}.$$

Therefore,

$$A(B\mathbf{x}) = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}\mathbf{x}.$$

and by defining

$$\boxed{AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}}$$

we have $A(B\mathbf{x}) = (AB)\mathbf{x}$.

EXAMPLE: Compute AB where $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$.

Solution:

$$A\mathbf{b}_1 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ -24 \\ 6 \end{bmatrix}, \qquad A\mathbf{b}_2 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ 26 \\ -7 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} -4 & 2 \\ -24 & 26 \\ 6 & -7 \end{bmatrix}$$

Note that $A\mathbf{b}_1$ is a linear combination of the columns of A and $A\mathbf{b}_2$ is a linear combination of the columns of A.

Each column of AB is a linear combination of the columns of A using weights from the corresponding columns of B.

EXAMPLE: If A is 4×3 and B is 3×2, then what are the sizes of AB and BA?

Solution:

$$AB = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ & \\ & \\ & \end{bmatrix}$$

BA would be $\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$ which is ____________________.

If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$.

Row-Column Rule for Computing AB (alternate method)

The definition

$$\boxed{AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}}$$

is good for theoretical work.

When A and B have small sizes, the following method is more efficient when working by hand.

If AB is defined, let $(AB)_{ij}$ denote the entry in the ith row and jth column of AB. Then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

$$\begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix} \begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix} = \begin{bmatrix} & & \\ & (AB)_{ij} & \\ & & \end{bmatrix}$$

EXAMPLE $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute AB, if it is defined.

Solution: Since A is 2×3 and B is 3×2, then AB is defined and AB is _____×_____.

$$AB = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} \mathbf{28} & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix}, \quad \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & \mathbf{-3} \\ 0 & \mathbf{1} \\ 4 & \mathbf{-7} \end{bmatrix} = \begin{bmatrix} 28 & \mathbf{-45} \\ \blacksquare & \blacksquare \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 6 \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ \mathbf{2} & \blacksquare \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & 6 \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} 2 & \mathbf{-3} \\ 0 & \mathbf{1} \\ 4 & \mathbf{-7} \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ 2 & \mathbf{-4} \end{bmatrix}$$

So $AB = \begin{bmatrix} 28 & -45 \\ 2 & -4 \end{bmatrix}$.

THEOREM 2

Let A be $m \times n$ and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B+C) = AB + AC$ (left - distributive law)

c. $(B+C)A = BA + CA$ (right-distributive law)

d. $r(AB) = (rA)B = A(rB)$ for any scalar r

e. $I_m A = A = A I_n$ (identity for matrix multiplication)

WARNINGS

Properties above are analogous to properties of real numbers. But **NOT ALL** real number properties correspond to matrix properties.

1. It is not the case that AB always equal BA. (see Example 7, page 116)
2. Even if $AB = AC$, then B may not equal C. (see Exercise 10, page 118)
3. It is possible for $AB = 0$ even if $A \neq 0$ and $B \neq 0$. (see Exercise 12, page 119)

Powers of A

$$A^k = \underbrace{A \cdots A}_{k}$$

EXAMPLE:

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 21 & 8 \end{bmatrix}$$

If A is $m \times n$, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 8 \\ 7 & 6 & 5 & 4 & 3 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 7 & 6 \\ 3 & 8 & 5 \\ 4 & 9 & 4 \\ 5 & 8 & 3 \end{bmatrix}$$

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$. Compute AB, $(AB)^T$, A^TB^T and B^TA^T.

Solution:

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$$

$$A^TB^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 10 \\ 2 & 0 & -4 \\ 2 & 1 & 4 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$$

THEOREM 3

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$ (I.e., the transpose of A^T is A)

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^TA^T$ (I.e. the transpose of a product of matrices equals the product of their transposes in reverse order.)

EXAMPLE: Prove that $(ABC)^T =$ _________.

Solution: By Theorem 3d,

$$(ABC)^T = ((AB)C)^T = C^T(\qquad)^T = C^T(\qquad) = \_\_\_\_\_\_\_\_\_.$$

2.2 The Inverse of a Matrix

The inverse of a real number a is denoted by a^{-1}. For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the **inverse** of A .

FACT If A is invertible, then the inverse is unique.

Proof: Assume B and C are both inverses of A. Then

$$B = BI = B(\_\_\_\_\_) = (\_\_\_\_\_\_)\_\_\_\_\_\_ = I\_\_\_\_\_\_ = C.$$

So the inverse is unique since any two inverses coincide.$\blacksquare$

The inverse of A is usually denoted by A^{-1}.

We have

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

Not all $n \times n$ ***matrices are invertible.*** A matrix which is *not* invertible is sometimes called a **singular** matrix. An invertible matrix is called **nonsingular** matrix.

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

Assume A is any invertible matrix and we wish to solve $A\mathbf{x} = \mathbf{b}$. Then

$$\_\_\_\_\_A\mathbf{x} = \_\_\_\_\_\mathbf{b} \qquad \text{and so}$$

$$I\mathbf{x} = \_\_\_\_\_\_\_\_ \text{ or } \mathbf{x} = \_\_\_\_\_\_\_\_.$$

Suppose $\mathbf{w}$ is also a solution to $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{w} = \mathbf{b}$ and

$$\_\_\_\_\_A\mathbf{w} = \_\_\_\_\_\mathbf{b} \qquad \text{which means} \qquad \mathbf{w} = A^{-1}\mathbf{b}.$$

So, $\mathbf{w} = A^{-1}\mathbf{b}$, which is in fact the same solution.

We have proved the following result:

Theorem 5

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbf{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

EXAMPLE: Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ to solve $\begin{array}{rcrcr} -7x_1 & + & 3x_2 & = & 2 \\ 5x_1 & - & 2x_2 & = & 1 \end{array}$.

Solution: Matrix form of the linear system: $\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{14-15}\begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}.$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Theorem 6 Suppose A and B are invertible. Then the following results hold:

a. A^{-1} is invertible and $(A^{-1})^{-1} = A$ (i.e. A is the inverse of A^{-1}).

b. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

c. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Partial proof of part b:

$$(AB)(B^{-1}A^{-1}) = A(\underline{\qquad\qquad})A^{-1}$$

$$= A(\underline{\qquad\qquad})A^{-1} = \underline{\qquad\qquad} = \underline{\qquad\quad}.$$

Similarly, one can show that $(B^{-1}A^{-1})(AB) = I$.

Theorem 6, part b can be generalized to three or more invertible matrices:

$$(ABC)^{-1} = \underline{\qquad\qquad}$$

Algorithm for finding $\mathbf{A}^{-1}$

Place A and I side-by-side to form an augmented matrix $\left[A\,|\,I\right]$. Then perform row operations on this matrix (which will produce identical operations on A and I).

$$\left[A\,|\,I\right] \text{ will row reduce to } \left[I\,|A^{-1}\right]$$

or A is not invertible.

EXAMPLE: Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution:

$$\left[A \quad I\right] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$

Order of multiplication is important!

EXAMPLE Suppose A, B, C, and D are invertible $n \times n$ matrices and $A = B(D - I_n)C$.

Solve for D in terms of A, B, C and D.

Solution:

$$\_\_\_\_\_A\_\_\_\_\_ = \_\_\_\_\_B(D - I_n)C\_\_\_\_\_$$

$$D - I_n = B^{-1}AC^{-1}$$

$$D - I_n + \_\_\_\_ = B^{-1}AC^{-1} + \_\_\_\_$$

$$D = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

2.3 Characterizations of Invertible Matrices

Theorem 8 (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The the following statements are equivalent (i.e., for a given A, they are either all true or all false).

a. A is an invertible matrix.

b. A is row equivalent to I_n.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \to A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbf{R}^n$.

h. The columns of A span $\mathbf{R}^n$.

i. The linear transformation $\mathbf{x} \to A\mathbf{x}$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I_n$.

k. There is an $n \times n$ matrix D such that $AD = I_n$.

l. A^T is an invertible matrix.

EXAMPLE: Use the Invertible Matrix Theorem to determine if A is invertible, where

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -4 & 11 & 1 \\ 2 & 7 & 3 \end{bmatrix}.$$

Solution

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -4 & 11 & 1 \\ 2 & 7 & 3 \end{bmatrix} \sim \cdots \sim \underbrace{\begin{bmatrix} 1 & -3 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 16 \end{bmatrix}}_{\text{3 pivots positions}}$$

Conclusion: Matrix A ________ invertible.

EXAMPLE: Suppose H is a 5×5 matrix and suppose there is a vector $\mathbf{v}$ in $\mathbf{R}^5$ which is not a linear combination of the columns of H. What can you say about the number of solutions to $H\mathbf{x} = \mathbf{0}$?

Solution Since $\mathbf{v}$ in $\mathbf{R}^5$ is not a linear combination of the

columns of H, the columns of H do not ____________ $\mathbf{R}^5$.

So by the Invertible Matrix Theorem, $H\mathbf{x} = \mathbf{0}$ has

____.

Invertible Linear Transformations

For an invertible matrix A,

$$A^{-1}A\mathbf{x} = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n$$
and
$$AA^{-1}\mathbf{x} = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n.$$

Pictures:

A linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ is said to be **invertible** if there exists a function $S : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n$$
and
$$T(S(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n.$$

Theorem 9

Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying

$$S(T(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n$$
and
$$T(S(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n.$$

3.1 Introduction to Determinants

Notation: A_{ij} is the submatrix obtained from matrix A by deleting the ith row and jth column of A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

Recall that $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and we let $\det[a] = a$.

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{1+n}a_{1n}\det A_{1n}$$

$$= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det A_{1j}$$

EXAMPLE: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

Solution

$$\det A = 1\det\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} + 0\det\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

$$= \underline{\hspace{12em}} = \underline{\hspace{3em}}$$

Common notation: $\det\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}$.

So

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix}$$

THEOREM 1 The determinant of an $n \times n$ matrix A can be computed by expanding across any row or down any column:

$$\det A = (-1)^{i+1}a_{i1}\det A_{i1} + (-1)^{i+2}a_{i2}\det A_{i2} + \cdots + (-1)^{i+n}a_{in}\det A_{in}$$

$$= \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\det A_{ij} \qquad \text{(expansion across row } i\text{)}$$

$$\det A = (-1)^{1+j}a_{1j}\det A_{1j} + (-1)^{2+j}a_{2j}\det A_{2j} + \cdots + (-1)^{n+j}a_{nj}\det A_{nj}$$

$$= \sum_{i=1}^{n}(-1)^{i+j}a_{ij}\det A_{ij} \qquad \text{(expansion down column } j\text{)}$$

Use a matrix of signs to determine $(-1)^{i+j}$ $\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$

EXAMPLE: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ by expansion down column 3.

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1.$$

EXAMPLE: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix}$

Solution

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 3 & 5 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 1 \cdot 2\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 14$$

Method of expansion is not practical for large matrices - see Numerical Note on page 185

Triangular Matrices:

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$

(upper triangular) (lower triangular)

THEOREM 2: If A is a triangular matrix, then $\det A$ is the product of the main diagonal entries of A.

EXAMPLE:

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{vmatrix} = \underline{\hspace{6cm}} = -24$$

3.2 Properties of Determinants

THEOREM 3 Let A be a square matrix.

a. If a multiple of one row of A is added to another row of A to produce a matrix B, then $\det A = \det B$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

EXAMPLE: Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix} = 5\begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 10 \\ 2 & 7 & 11 \end{vmatrix} = 5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 2 & 7 & 11 \end{vmatrix}$$

$$= 5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{vmatrix} = \underline{\hspace{2cm}} = \underline{\hspace{1cm}}.$$

Theorem 3(c) indicates that $\begin{vmatrix} * & * & * \\ -2k & 5k & 4k \\ * & * & * \end{vmatrix} = k\begin{vmatrix} * & * & * \\ -2 & 5 & 4 \\ * & * & * \end{vmatrix}$.

EXAMPLE: Compute $\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix}$$

$$= 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} = 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 2(-4)(1)(1)(5) = -40$$

EXAMPLE: Compute $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix}$ using a combination of row reduction and expanding across rows or down columns

Solution $$\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 & 1 \\ 4 & 7 & 3 \\ 1 & 2 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 0 & -6 \end{vmatrix} = -2(1)(-1)(-6) = -12.$$

Suppose A has been reduced to $U = \begin{bmatrix} \blacksquare & * & * & \cdots & * \\ 0 & \blacksquare & * & \cdots & * \\ 0 & 0 & \blacksquare & \cdots & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$ by row replacements and row interchanges (no row scaling), then

$$\det A = \begin{cases} (-1)^r \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

THEOREM 4 A square matrix is invertible if and only if $\det A \neq 0$.

THEOREM 5 If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Partial proof (2×2 case)

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \qquad \text{and}$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$$

$$\Rightarrow \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

(3×3 case)

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\det\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = a\begin{vmatrix} e & h \\ f & i \end{vmatrix} - b\begin{vmatrix} d & g \\ f & i \end{vmatrix} + c\begin{vmatrix} d & g \\ e & h \end{vmatrix}$$

$$\Rightarrow \det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \det\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}.$$

Implications of Theorem 5*?*

Theorem 3 still holds if the word *row* is replaced

with ____________________.

THEOREM 6 (Multiplicative Property)

For $n \times n$ matrices A and B, $\det(AB) = (\det A)(\det B)$.

EXAMPLE: Compute $\det A^3$ if $\det A = 5$.

Solution: $\det A^3 = \det(AAA) = (\det A)(\det A)(\det A)$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_.$$

EXAMPLE: For $n \times n$ matrices A and B, show that A is singular if $\det B \neq 0$ and $\det AB = 0$.

Solution: *Since*

$$(\det A)(\det B) = \det AB = 0$$

and

$$\det B \neq 0,$$

then $\det A = 0$. Therefore A is singular.

4.1 Vector Spaces & Subspaces

Many concepts concerning vectors in $\mathbf{R}^n$ can be extended to other mathematical systems.

We can think of a *vector space* in general, as a collection of objects that behave as vectors do in $\mathbf{R}^n$. The objects of such a set are called *vectors*.

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms below. The axioms must hold for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and for all scalars c and d.

1. $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $(cd)\mathbf{u} = c(d\mathbf{u})$.
10. $1\mathbf{u} = \mathbf{u}$.

Vector Space Examples

EXAMPLE: Let $M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \text{ are real} \right\}$

In this context, note that the $\mathbf{0}$ vector is $\begin{bmatrix} & \\ & \end{bmatrix}$.

EXAMPLE: Let $n \geq 0$ be an integer and let

$$\mathbf{P}_n = \text{the set of all polynomials of degree at most } n \geq 0.$$

Members of $\mathbf{P}_n$ have the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

where $a_0, a_1, \ldots, a_n$ are real numbers and t is a real variable. The set $\mathbf{P}_n$ is a vector space.

We will just verify 3 *out of the* 10 *axioms here.*

Let $\mathbf{p}(t) = a_0 + a_1 t + \cdots + a_n t^n$ and $\mathbf{q}(t) = b_0 + b_1 t + \cdots + b_n t^n$. Let c be a scalar.

Axiom* 1*:

The polynomial $\mathbf{p} + \mathbf{q}$ is defined as follows: $(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t)$. Therefore,

$$(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t)$$

$$= (\_\_\_\_\_\_\_\_\_\_) + (\_\_\_\_\_\_\_\_\_\_)t + \cdots + (\_\_\_\_\_\_\_\_\_\_)t^n$$

which is also a ________________________ of degree at most _________. So $\mathbf{p} + \mathbf{q}$ is in $\mathbf{P}_n$.

Axiom* 4*:

$$\mathbf{0} = 0 + 0t + \cdots + 0t^n$$

(zero vector in $\mathbf{P}_n$)

$$(\mathbf{p} + \mathbf{0})(t) = \mathbf{p}(t) + \mathbf{0} = (a_0 + 0) + (a_1 + 0)t + \cdots + (a_n + 0)t^n$$

$$= a_0 + a_1 t + \cdots + a_n t^n = \mathbf{p}(t)$$

and so $\mathbf{p} + \mathbf{0} = \mathbf{p}$

Axiom* 6*:

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = (\_\_\_\_\_\_\_\_) + (\_\_\_\_\_\_\_\_)t + \cdots + (\_\_\_\_\_\_\_\_)t^n$$

which is in $\mathbf{P}_n$.

The other 7 axioms also hold, so $\mathbf{P}_n$ is a vector space.

Subspaces

Vector spaces may be formed from subsets of other vectors spaces. These are called *subspaces*.

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

b. For each **u** and **v** are in H, $\mathbf{u}+\mathbf{v}$ is in H. (In this case we say H is closed under vector addition.)

c. For each **u** in H and each scalar c, $c\mathbf{u}$ is in H. (In this case we say H is closed under scalar multiplication.)

If the subset H satisfies these three properties, then H itself is a vector space.

EXAMPLE: Let $H = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a \text{ and } b \text{ are real} \right\}$. Show that H is a subspace of $\mathbf{R}^3$.

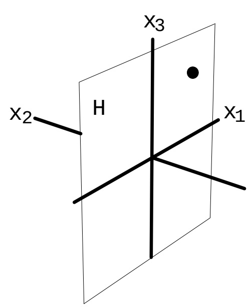
Solution: Verify properties a, b and c of the definition of a subspace.

a. The zero vector of $\mathbf{R}^3$ is in H (let $a =$ _______ and $b =$ _______).

b. Adding two vectors in H always produces another vector whose second entry is ______ and therefore the sum of two vectors in H is also in H. (H is closed under addition)

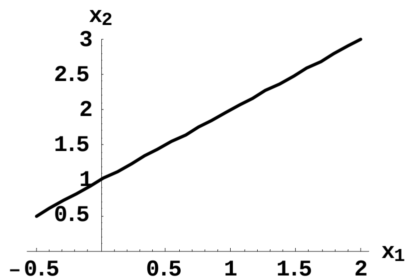
c. Multiplying a vector in H by a scalar produces another vector in H (H is closed under scalar multiplication).

Since properties a, b, and c hold, V is a subspace of $\mathbf{R}^3$. **Note:** Vectors $(a, 0, b)$ in H look and act like the points (a,b) in $\mathbf{R}^2$.



EXAMPLE: Is $H = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} : \; x \text{ is real} \right\}$ a subspace of ________?

I.e., does H satisfy properties a, b and c?



Graphical Depiction of H

Solution:

All three properties must hold in order for H to be a subspace of $\mathbf{R}^2$.

Property (a) is not true because

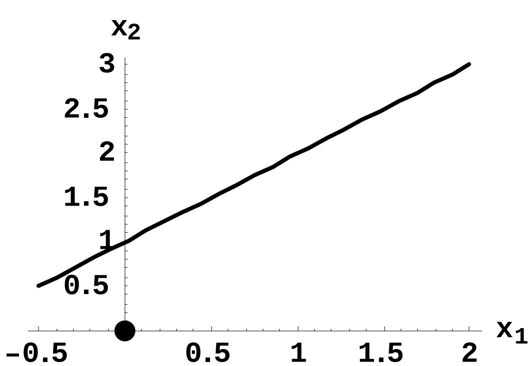
____.

Therefore H is not a subspace of $\mathbf{R}^2$.

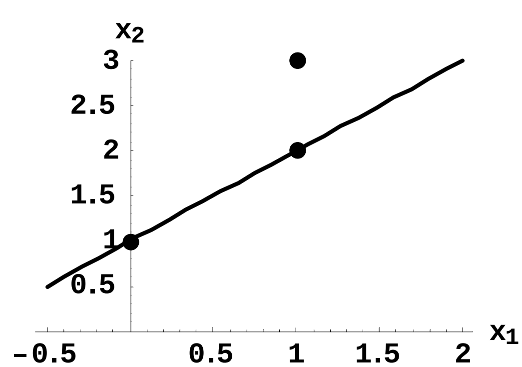
Another way to show that H is not a subspace of $\mathbf{R}^2$: Let

$$\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ then } \mathbf{u} + \mathbf{v} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

and so $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, which is _____ in H. So property (b) fails and so H is not a subspace of $\mathbf{R}^2$.



Property (a) fails



Property (b) fails

A Shortcut for Determining Subspaces

THEOREM 1

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Proof: In order to verify this, check properties a, b and c of definition of a subspace.

a. $\mathbf{0}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ since

$$\mathbf{0} = \underline{\qquad}\mathbf{v}_1 + \underline{\qquad}\mathbf{v}_2 + \cdots + \underline{\qquad}\mathbf{v}_p$$

b. To show that Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ closed under vector addition, we choose two arbitrary vectors in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$:

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p$$

and

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$\mathbf{u} + \mathbf{v} = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p) + (b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p)$$

$$= (\underline{\quad}\mathbf{v}_1 + \underline{\qquad}\mathbf{v}_1) + (\underline{\qquad}\mathbf{v}_2 + \underline{\qquad}\mathbf{v}_2) + \cdots + (\underline{\qquad}\mathbf{v}_p + \underline{\qquad}\mathbf{v}_p)$$

$$= (a_1 + b_1)\mathbf{v}_1 + (a_2 + b_2)\mathbf{v}_2 + \cdots + (a_p + b_p)\mathbf{v}_p.$$

So $\mathbf{u} + \mathbf{v}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

c. To show that Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ closed under scalar multiplication, choose an arbitrary number c and an arbitrary vector in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$:

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.$$

Then

$$c\mathbf{v} = c(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p)$$

$$= \underline{\qquad\quad}\mathbf{v}_1 + \underline{\qquad\quad}\mathbf{v}_2 + \cdots + \underline{\qquad\quad}\mathbf{v}_p$$

So $c\mathbf{v}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Since properties a, b and c hold, Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Recap

1. To show that H is a subspace of a vector space, use Theorem 1.
2. To show that a set is not a subspace of a vector space, provide a specific example showing that at least one of the axioms a, b or c (from the definition of a subspace) is violated.

EXAMPLE: Is $V = \{(a+2b, 2a-3b) : a \text{ and } b \text{ are real}\}$ a subspace of $\mathbf{R}^2$? Why or why not?
Solution: Write vectors in V in column form:

$$\begin{bmatrix} a+2b \\ 2a-3b \end{bmatrix} = \begin{bmatrix} a \\ 2a \end{bmatrix} + \begin{bmatrix} 2b \\ -3b \end{bmatrix} = \_\_\_\_\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \_\_\_\_\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

So $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and therefore V is a subspace of _____ by Theorem 1.

EXAMPLE: Is $H = \left\{ \begin{bmatrix} a+2b \\ a+1 \\ a \end{bmatrix} : a \text{ and } b \text{ are real} \right\}$ a subspace of $\mathbf{R}^3$? Why or why not?

Solution: $\mathbf{0}$ is not in H since $a = b = 0$ or any other combination of values for a and b does not produce the zero vector. So property _____ fails to hold and therefore H is not a subspace of $\mathbf{R}^3$.

EXAMPLE: Is the set H of all matrices of the form $\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}$ a subspace of $M_{2\times 2}$? Explain.

Solution: Since

$$\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 3a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 3b \end{bmatrix}$$

$$= a\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} + b\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

Therefore $H = \text{Span}\left\{ \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \right\}$ and so H is a subspace of $M_{2\times 2}$.

4.2 Null Spaces, Column Spaces, & Linear Transformations

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbf{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\} \qquad \text{(set notation)}$$

THEOREM 2

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbf{R}^n$.

Proof: Nul A is a subset of $\mathbf{R}^n$ since A has n columns. Must verify properties a, b and c of the definition of a subspace.

Property (a) Show that $\mathbf{0}$ is in Nul A. Since __________, $\mathbf{0}$ is in ___________.

Property (b) If $\mathbf{u}$ and $\mathbf{v}$ are in Nul A, show that $\mathbf{u} + \mathbf{v}$ is in Nul A. Since $\mathbf{u}$ and $\mathbf{v}$ are in Nul A,

___________ and ___________.

Therefore

$$A(\mathbf{u} + \mathbf{v}) = \_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_ + \_\_\_\_\_\_\_ = \_\_\_\_.$$

Property (c) If $\mathbf{u}$ is in Nul A and c is a scalar, show that $c\mathbf{u}$ in Nul A:

$$A(c\mathbf{u}) = \_\_\_A(\mathbf{u}) = c\mathbf{0} = \mathbf{0}.$$

Since properties a, b and c hold, A is a subspace of $\mathbf{R}^n$.

Solving $A\mathbf{x} = \mathbf{0}$ yields an ***explicit description*** *of Nul A.*

EXAMPLE: Find an explicit description of Nul A where $A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$.

Solution: Row reduce augmented matrix corresponding to $A\mathbf{x} = \mathbf{0}$:

$$\left[\begin{array}{ccccc|c} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}$$

Then

$$\text{Nul } A = \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$$

Observations:

1. Spanning set of Nul A, found using the method in the last example, is automatically linearly independent:

$$c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = \_\_\_\_\_ \quad c_2 = \_\_\_\_\_ \quad c_3 = \_\_\_\_\_$$

2. If Nul A$\neq \{\mathbf{0}\}$, the the number of vectors in the spanning set for Nul A equals the number of free variables in $A\mathbf{x} = \mathbf{0}$.

The **column space** of an $m \times n$ matrix A (Col A) is the set of all linear combinations of the columns of A.

If $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$, then

$$\boxed{\text{Col } A = \text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}}$$

THEOREM 3

The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.

Why? (Theorem 1, page 212)

Recall that if $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b}$ is a linear combination of the columns of A. Therefore

$$\boxed{\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbf{R}^n\}}$$

EXAMPLE: Find a matrix A such that $W = Col\ A$ where $W = \left\{ \begin{bmatrix} x-2y \\ 3y \\ x+y \end{bmatrix} : x, y \text{ in } \mathbf{R} \right\}$.

Solution:

$$\begin{bmatrix} x-2y \\ 3y \\ x+y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{Therefore } A = \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

By Theorem 4 (Chapter 1),

> The column space of an $m \times n$ matrix A is all of $\mathbf{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^m$.

The Contrast Between Nul A and Col A

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) The column space of A is a subspace of $\mathbf{R}^k$ where $k =$_____.

(b) The null space of A is a subspace of $\mathbf{R}^k$ where $k =$_____.

(c) Find a nonzero vector in Col A. (There are infinitely many possibilities.)

$$\underline{\quad}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \underline{\quad}\begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix} + \underline{\quad}\begin{bmatrix} 3 \\ 7 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$$

(d) Find a nonzero vector in Nul A. Solve $A\mathbf{x} = \mathbf{0}$ and pick one solution.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 4 & 7 & 0 \\ 3 & 6 & 10 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \text{ row reduces to } \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -2x_2$

x_2 is free

$x_3 = 0$

Let $x_2 =$ ___ and then

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Contrast Between Nul A and Col A where A is $m \times n$ (see page 232)

Review

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

b. For each **u** and **v** in H, $\mathbf{u}+\mathbf{v}$ is in H. (In this case we say H is closed under vector addition.)

c. For each **u** in H and each scalar c, $c\mathbf{u}$ is in H. (In this case we say H is closed under scalar multiplication.)

If the subset H satisfies these three properties, then H itself is a vector space.

THEOREM 1, 2 and 3 (Sections 4.1 & 4.2)

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$.

The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.

EXAMPLE: Determine whether each of the following sets is a vector space or provide a counterexample.

(a) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x - y = 4 \right\}$. *Solution*: *Since* _____$= \begin{bmatrix} \quad \end{bmatrix}$ is not in H, H is not a vector space.

(b) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{l} x - y = 0 \\ y + z = 0 \end{array} \right\}$. *Solution:* Rewrite $\begin{array}{l} x - y = 0 \\ y + z = 0 \end{array}$ as

$$\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $V =$Nul A where $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Since Nul A is a subspace of $\mathbf{R}^2$, V is a vector space.

(c) $S = \left\{ \begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} : x, y, z \text{ are real} \right\}$

One Solution: Since

$$\begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

$S = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \right\}$; therefore S is a vector space by Theorem 1.

Another Solution: Since

$$\begin{bmatrix} x+y \\ 2x-3y \\ 3y \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

$S = \text{Col } A$ where $A = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 3 \end{bmatrix}$; therefore S is a vector space, since a column space is a vector space.

Kernal and Range of a Linear Transformation

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

i. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V;

ii. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in tin V and all scalars c.

The ***kernel*** (or **null space**) of T is the set of all vectors $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$. The ***range*** of T is the set of all vectors in W of the form $T(\mathbf{u})$ where $\mathbf{u}$ is in V.

So if $T(\mathbf{x}) = A\mathbf{x}$, col A = range of T.

4.3 Linearly Independent Sets; Bases

Definition

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in a vector space V is said to be **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution $c_1 = 0, \ldots, c_p = 0$.

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists weights $c_1, \ldots, c_p$, not all 0, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$

The following results from Section 1.7 are still true for more general vectors spaces.

A set containing the zero vector is linearly dependent.

A set of two vectors is linearly dependent if and only if one is a multiple of the other.

EXAMPLE: $\left\{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 3 & 0 \end{bmatrix}\right\}$ is a linearly ________________ set.

EXAMPLE: $\left\{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ 9 & 11 \end{bmatrix}\right\}$ is a linearly ________________ set since $\begin{bmatrix} 3 & 6 \\ 9 & 11 \end{bmatrix}$ is not a multiple of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Theorem 4

An indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some vector $\mathbf{v}_j$ $(j > 1)$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

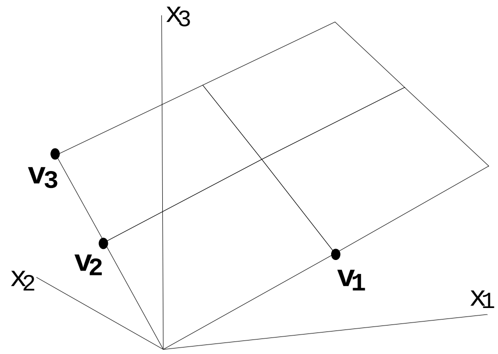
EXAMPLE: Let $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ be a set of vectors in $\mathbf{P}_2$ where $\mathbf{p}_1(t) = t$, $\mathbf{p}_2(t) = t^2$, and $\mathbf{p}_3(t) = 4t + 2t^2$. Is this a linearly dependent set?

Solution: Since $\mathbf{p}_3 =$ _____$\mathbf{p}_1 +$ _____$\mathbf{p}_2$, $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a linearly ____________________ set.

A Basis Set

Let H be the plane illustrated below. Which of the following are valid descriptions of H?

(a) $H =$Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ (b) $H =$Span$\{\mathbf{v}_1, \mathbf{v}_3\}$
(c) $H =$Span$\{\mathbf{v}_2, \mathbf{v}_3\}$ (d) $H =$Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$



A *basis set* is an "efficient" spanning set containing no unnecessary vectors. In this case, we would consider the linearly independent sets $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_3\}$ to both be examples of basis sets or bases (plural for basis) for H.

DEFINITION

Let H be a subspace of a vector space V. An indexed set of vectors $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a basis for H if

(i) β is a linearly independent set, and

(ii) $H =$ Span$\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$.

EXAMPLE: Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Show that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathbf{R}^3$. The set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is called a **standard basis** for $\mathbf{R}^3$.

Solutions: (Review the IMT, page 139) Let

$A = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Since A has 3 pivots, the columns of A are linearly

________________________ by the IMT and the columns of A ______________ __________

by IMT. Therefore, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathbf{R}^3$.

EXAMPLE: Let $S = \{1,\ t,\ t^2,\ \ldots, t^n\}$. Show that S is a basis for $\mathbf{P}_n$.

Solution: Any polynomial in $\mathbf{P}_n$ is in span of S. To show that S is linearly independent, assume

$$c_0 \cdot 1 + c_1 \cdot t + \cdots + c_n \cdot t^n = \mathbf{0}$$

Then $c_0 = c_1 = \cdots = c_n = 0$. Hence S is a basis for $\mathbf{P}_n$.

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a **basis** for $\mathbf{R}^3$?

Solution: Again, let $A = [\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3] = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. Using row reduction,

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

and since there are 3 pivots, the columns of A are linearly independent and they span $\mathbf{R}^3$ by the IMT. Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a **basis** for $\mathbf{R}^3$. (Faster: use determinant.)

EXAMPLE: Explain why each of the following sets is **not** a basis for $\mathbf{R}^3$.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$

Bases for Nul A

EXAMPLE: Find a basis for Nul A where $A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$.

Solution: Row reduce $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{array}\right] \qquad \begin{aligned} x_1 &= -2x_2 - 13x_4 - 33x_5 \\ x_3 &= 6x_4 + 15x_5 \end{aligned}$$

x_2, x_4 and x_5 are free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\uparrow\ \mathbf{u}} + x_4 \underbrace{\begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}}_{\uparrow\ \mathbf{v}} + x_5 \underbrace{\begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}}_{\uparrow\ \mathbf{w}}$$

Therefore $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a spanning set for Nul A. In the last section we observed that this set is linearly independent. Therefore $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for Nul A. The technique used here always provides a linearly independent set.

The Spanning Set Theorem

A basis can be constructed from a spanning set of vectors by discarding vectors which are linear combinations of preceding vectors in the indexed set.

EXAMPLE: Suppose $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$.

Solution: If $\mathbf{x}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(\_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\mathbf{v}_2)$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\_\_\_\_\_\_\_\_\mathbf{v}_2$$

Therefore,

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}.$$

THEOREM 5 The Spanning Set Theorem

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V and let $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

a. If one of the vectors in S - say $\mathbf{v}_k$ - is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

Bases for Col A

EXAMPLE: Find a basis for Col A, where

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

Solution: Row reduce:

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{bmatrix}$$

Note that

$$\mathbf{b}_2 = \underline{\qquad}\mathbf{b}_1 \quad \text{and} \quad \mathbf{a}_2 = \underline{\qquad}\mathbf{a}_1$$

$$\mathbf{b}_4 = 4\mathbf{b}_1 + 5\mathbf{b}_3 \quad \text{and} \quad \mathbf{a}_4 = 4\mathbf{a}_1 + 5\mathbf{a}_3$$

$\mathbf{b}_1$ and $\mathbf{b}_3$ are not multiples of each other

$\mathbf{a}_1$ and $\mathbf{a}_3$ are not multiples of each other

Elementary row operations on a matrix do not affect the linear dependence relations among the columns of the matrix.

Therefore $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} = \text{Span}\{\mathbf{a}_1, \mathbf{a}_3\}$ and $\{\mathbf{a}_1, \mathbf{a}_3\}$ is a basis for Col A.

THEOREM 6

The pivot columns of a matrix A form a basis for Col A.

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ -4 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$. Find a basis for Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ -3 & 6 & 9 \end{bmatrix}$ and note that Col $A =$ Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

By row reduction, $A \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore a basis

for Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is $\left\{ \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}, \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \right\}$.

Review:

1. To find a basis for Nul A, use elementary row operations to transform $\begin{bmatrix} A \mid \mathbf{0} \end{bmatrix}$ to an equivalent reduced row echelon form $\begin{bmatrix} B \mid \mathbf{0} \end{bmatrix}$. Use the reduced row echelon form to find parametric form of the general solution to $A\mathbf{x} = \mathbf{0}$. The vectors found in this parametric form of the general solution form a basis for Nul A.

2. A basis for Col A is formed from the pivot columns of A. **Warning: Use the pivot columns of A, not the pivot columns of B, where B is in reduced echelon form and is row equivalent to** A.