

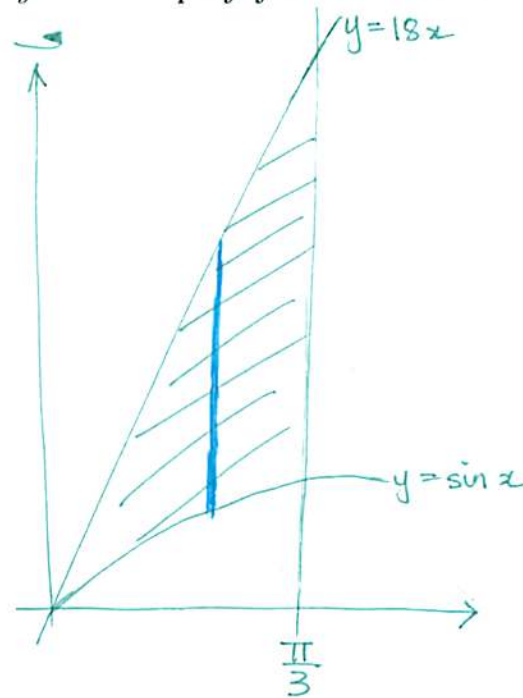
1. (7 points) Let  $R$  be the region bounded by the curves

$$y = \sin x, \quad y = 18x, \quad x = \frac{\pi}{3}.$$

Find the volume of the solid obtained by rotating  $R$  about the  $y$ -axis. Simplify your answer as much as possible.

Use cylindrical shells:

$$\begin{aligned} \text{Volume} &= \int_0^{\frac{\pi}{3}} 2\pi x (18x - \sin x) dx \\ &= \int_0^{\frac{\pi}{3}} 36\pi x^2 - 2\pi x \sin x dx \\ &= \left[ 36\pi \frac{x^3}{3} \right]_0^{\frac{\pi}{3}} - 2\pi \int_0^{\frac{\pi}{3}} x \sin x dx \\ &= 12\pi \left( \frac{\pi}{3} \right)^3 - 2\pi \int_0^{\frac{\pi}{3}} x \sin x dx \end{aligned}$$



Integration by parts:

$$\int_0^{\frac{\pi}{3}} x \sin x dx = \left[ x(-\cos x) \right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} -\cos x dx$$

$$= \frac{\pi}{3} \left( -\frac{1}{2} \right) - 0 - \left[ -\sin x \right]_0^{\frac{\pi}{3}}$$

$$= -\frac{\pi}{6} - \left( -\frac{\sqrt{3}}{2} + 0 \right) = -\frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

$$\begin{array}{ll} u = x & dv = \sin x dx \\ du = dx & v = -\cos x \end{array}$$

$$\text{So Volume} = \frac{12\pi^4}{3^3} - 2\pi \left( -\frac{\pi}{6} + \frac{\sqrt{3}}{2} \right) = \frac{4\pi^4}{9} + \frac{\pi^2}{3} - \sqrt{3}\pi$$

2. (7 points) Compute the following integral:

$$\int \frac{x+4}{x(x-1)^2} dx.$$

*Partial fractions:*

$$\frac{x+4}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$x+4 = A(x-1)^2 + Bx(x-1) + Cx$$

$$x=1: \quad 5 = C \quad \Rightarrow C=5$$

$$x=0: \quad 4 = A(-1)^2 \quad \Rightarrow A=4$$

$$\text{coeff of } x^2: \quad 0 = A+B \quad \Rightarrow B=-4$$

$$\begin{aligned} \therefore \int \frac{x+4}{x(x-1)^2} dx &= \int \frac{4}{x} - \frac{4}{x-1} + \frac{5}{(x-1)^2} dx \\ &= 4 \ln|x| - 4 \ln|x-1| + 5 \frac{(x-1)^{-1}}{-1} + C \\ &= 4 \ln|x| - 4 \ln|x-1| - \frac{5}{x-1} + C \end{aligned}$$

3. (7 points) Compute the following integral:

$$\int (x^2 + 9)^{-\frac{5}{2}} dx.$$

substitution:

$$x = 3 \tan \theta$$

$$dx = 3 \sec^2 \theta$$

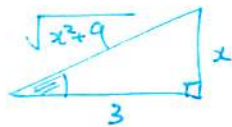
$$x^2 + 9 = 9 \tan^2 \theta + 9$$

$$= 9 \sec^2 \theta$$

substitution

$$u = \sin \theta$$

$$du = \cos \theta d\theta$$



$$\frac{x}{3} = \tan \theta = \frac{\text{opp}}{\text{adj}}$$

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{x^2 + 9}}$$

$$= \int (3 \sec \theta)^{-5} (3 \sec^2 \theta) d\theta$$

$$= \int 3^{-4} (\sec \theta)^{-3} d\theta$$

$$= \int \frac{1}{81} \cos^3 \theta d\theta$$

$$= \int \frac{1}{81} (1 - \sin^2 \theta) \cos \theta d\theta$$

$$= \frac{1}{81} \left( \sin \theta - \frac{\sin^3 \theta}{3} \right) + C$$

$$= \frac{1}{81} \frac{x}{\sqrt{x^2 + 9}} - \frac{1}{27} \left( \frac{x}{\sqrt{x^2 + 9}} \right)^3 + C$$