1. (7 points) Compute the following two improper integrals, or explain why they do not converge. **Simplify your answer as much as possible.**

(a)

$$\int_e^\infty \frac{1}{x(1+\ln x)^2}dx.$$

$$= \lim_{t\to\infty} \int_e^t \frac{1}{x(1+\ln x)^2}\,dx$$

substitution $u = 1+\ln x$, $du = \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \left[\frac{-1}{1+\ln x}\right]_e^t$$

$$= \lim_{t\to\infty} \frac{-1}{1+\ln t} + \frac{1}{1+1}$$

$$= \frac{1}{2}$$

because, as $t \to \infty$, $\ln t \to \infty$, so $\frac{1}{1+\ln x} \to 0$

(b)

$$\int_{\frac{1}{e}}^1 \frac{1}{x(1+\ln x)^2}dx.$$

$$= \lim_{t\to\frac{1}{e}^+} \int_t^1 \frac{1}{x(1+\ln x)^2}\,dx$$

$$= \lim_{t\to\frac{1}{e}^+} \left[\frac{-1}{1+\ln x}\right]_t^1$$

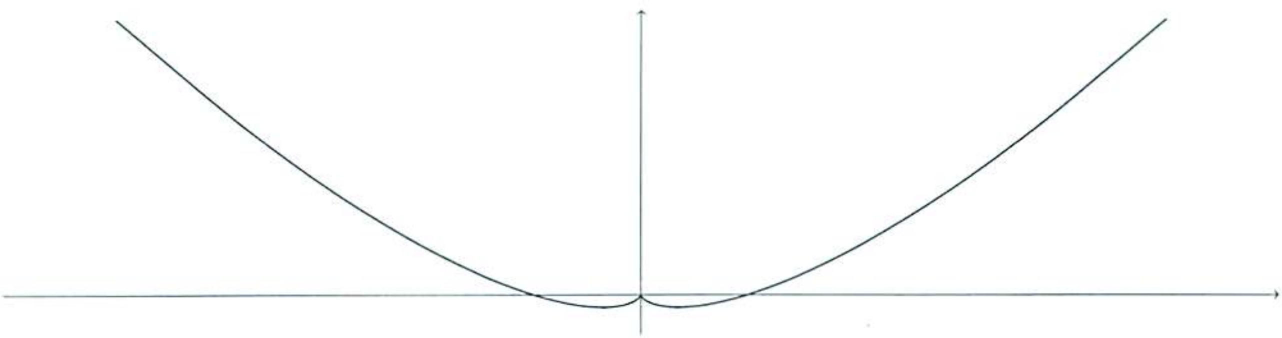
$$= \lim_{t\to\frac{1}{e}^+} -\frac{1}{1} + \frac{1}{1+\ln t}.$$

as $t \to \frac{1}{e}^+$, $\ln t \to -1^+$, so $\frac{1}{1+\ln t} \to \infty$ so this integral diverges.

2. (14 points) Let C be the parametrised curve with equation

$$x = \frac{4}{3}t^3, \quad y = t^4 - \frac{t^2}{2},$$

as shown in the diagram below.



(a) Find the point(s) where C has a horizontal tangent. **Simplify your answer as much as possible.**

C has a horizontal tangent when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

$\frac{dy}{dt} = 0$ when $4t^3 - t = 0$

$4t(t^2 - 1/4) = 0$

$t = 0$ or $t = \frac{1}{2}$ or $t = -\frac{1}{2}$

when $t = 0$: $\frac{dx}{dt} = 4t^2 = 0$ — $t=0$ corresponds to the point $(0,0)$, and from the diagram we see there is no horizontal tangent there.

when $t = \frac{1}{2}$: $\frac{dx}{dt} = 4\left(\frac{1}{2}\right)^2 \neq 0$

When $t = -\frac{1}{2}$: $\frac{dx}{dt} = 4\left(-\frac{1}{2}\right)^2 \neq 0$

$\Big\}$ ∴ these do give horizontal tangents.

Corresponding (x,y) coordinates are:

$t = \frac{1}{2}$: $x = \frac{4}{3}\,\frac{1}{8} = \frac{1}{6}$ $\quad y = \frac{1}{16} - \frac{1}{2}\,\frac{1}{4} = -\frac{1}{16}$

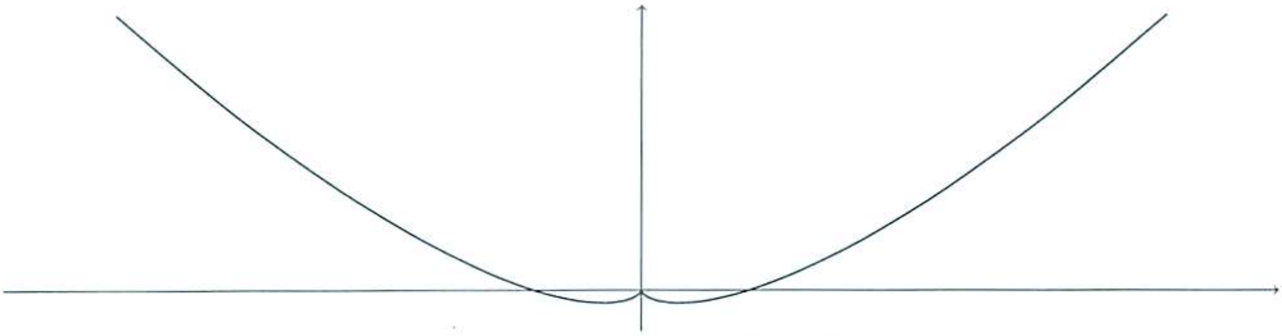
$t = -\frac{1}{2}$: $x = \frac{4}{3}\left(-\frac{1}{8}\right) = -\frac{1}{6}$ $\quad y = \frac{1}{16} - \frac{1}{2}\,\frac{1}{4} = -\frac{1}{16}$

∴ C has horizontal tangents at $\left(\frac{1}{6}, -\frac{1}{16}\right)$ and $\left(-\frac{1}{6}, -\frac{1}{16}\right)$.

(Question 2 continues on the next page.)

(b) For your convenience, here again is the information about the parametrised curve C:

$$x = \frac{4}{3}t^3, \quad y = t^4 - \frac{t^2}{2}.$$



Find the length of the part of C with $-3 \leq t \leq -1$. **Simplify your answer as much as possible.**

$$\text{length} = \int_{-3}^{-1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_{-3}^{-1} \sqrt{(4t^2)^2 + (4t^3 - t)^2}\, dt$$

$$= \int_{-3}^{-1} \sqrt{16t^4 + 16t^6 - 8t^4 + t^2}\, dt$$

$$= \int_{-3}^{-1} \sqrt{16t^6 + 8t^4 + t^2}\, dt$$

$$= \int_{-3}^{-1} \sqrt{t^2(4t^2+1)^2}\, dt$$

$$= \int_{-3}^{-1} |t|\,|4t^2+1|\, dt$$

$4t^2+1 > 0$ always,
$t < 0$ for $-3 \leq t \leq -1$.

$$= \int_{-3}^{-1} -t(4t^2+1)\, dt$$

$$= \left[-4 \cdot \frac{t^4}{4} - \frac{t^2}{2}\right]_{-3}^{-1} = \left(-1 - \frac{1}{2}\right) - \left(-81 - \frac{9}{2}\right) = 84$$