§12.1: Visualising Functions of Several Variables

The simplest type of multivariate function takes a point $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ and returns a number $f(x_1, \ldots, x_n)$. More generally:

Definition: A *function of n variables* is a rule that takes each point $(x_1, \ldots, x_n)$ in the domain $\mathcal{D}(f) \subseteq \mathbb{R}^n$ and returns a point $f(x_1, \ldots, x_n)$ in $\mathbb{R}^m$.
In other words, this function returns m numbers, which we can call $f_1(x_1, \ldots, x_n), f_2(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)$ - these f_i are the *coordinate functions*.

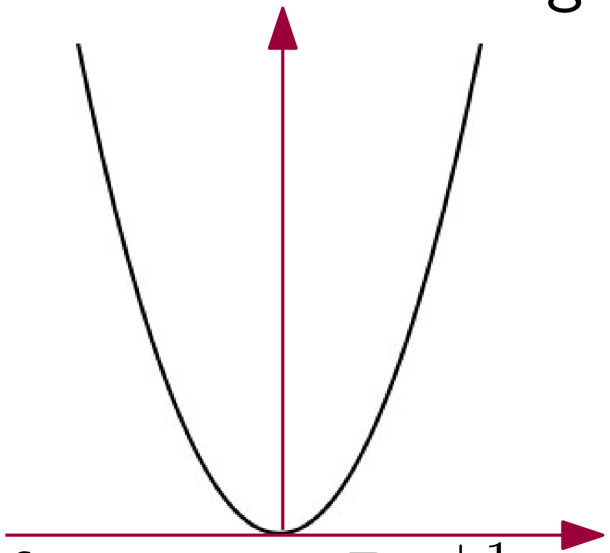
If $\mathcal{D}(f)$ is not explicitly given, then it is assumed to be the largest subset of $\mathbb{R}^n$ where the rule for f makes sense.

Whenever you analyse a function, you should think about its domain. For example, integrating over points not in the domain requires special techniques (improper integrals, see Week 7).

Example: Find the domain of the function

$$g(x, y) = \frac{\sqrt{x}}{1 + y}.$$

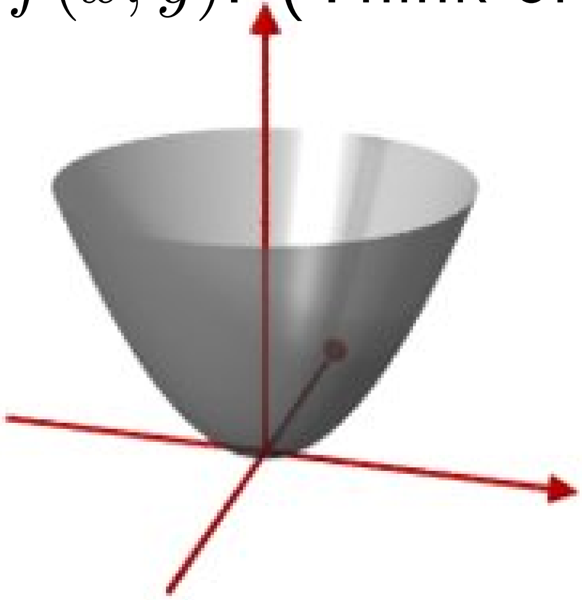
The standard way to visualise a single-variable function $f : \mathbb{R} \to \mathbb{R}$ is through its graph $y = f(x)$. This is a subset of $\mathbb{R}^2$.



Example: The graph of $f(x) = x^2$ is the parabola $y = x^2$.

Definition: The *graph* of a function $\mathbb{R}^n \to \mathbb{R}$ is the set of points in $\mathbb{R}^{n+1}$ satisfying $x_{n+1} = f(x_1, \ldots, x_n)$.

So the graph of $f(x, y)$ is the surface in $\mathbb{R}^3$ satisfying $z = f(x, y)$. (Think of z as the height at (x, y).)



Example: The graph of $f(x, y) = x^2 + y^2$ is the paraboloid $z = x^2 + y^2$.

(pictures from Wiktionary and Wikiwand)

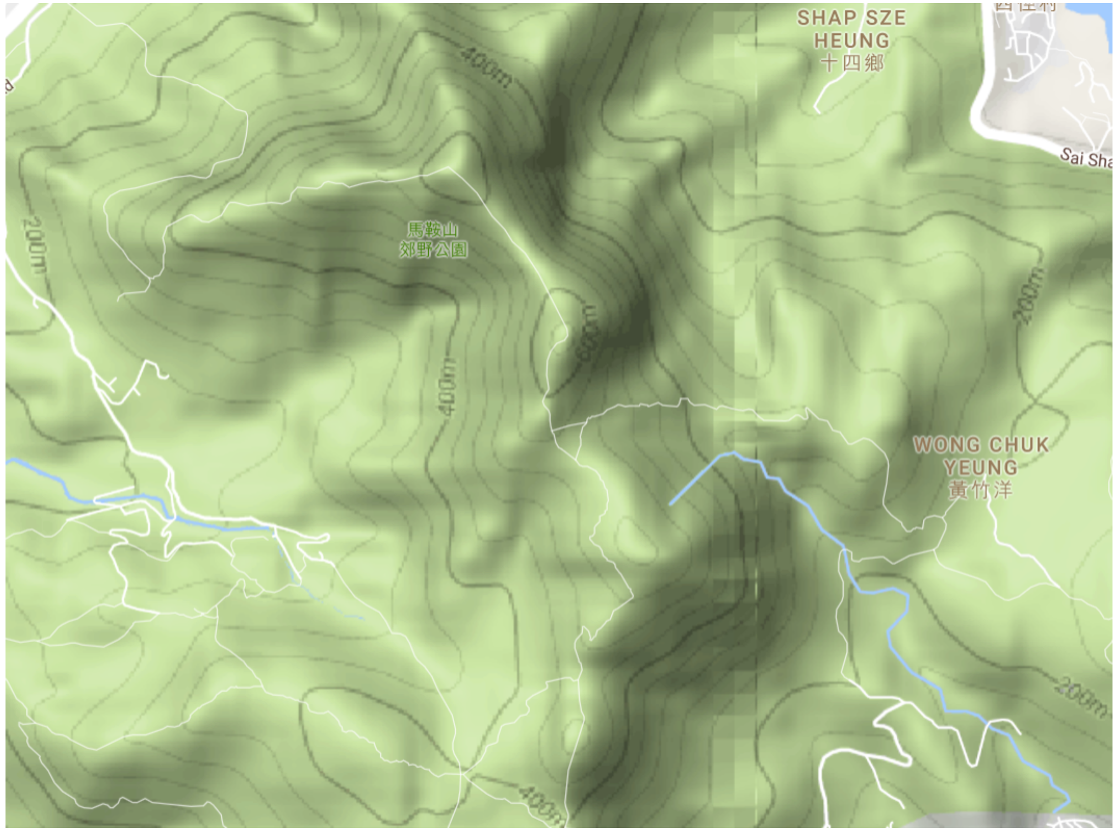
Example: Describe the graph of $k(x, y) = x - y$.

Definition: The *graph* of a function $\mathbb{R}^n \to \mathbb{R}$ is the set of points in $\mathbb{R}^{n+1}$ satisfying $x_{n+1} = f(x_1, \dots, x_n)$.

As shown in the previous example, the graph of a linear function is a plane (or hyperplane).
The graph of some 2-variable functions, such as $g(x, y) = \frac{\sqrt{x}}{1+y}$, can be very complicated.
Even worse: the graph of a 3-variable function $f(x, y, z)$ is in $\mathbb{R}^4$, which is not helpful at all.

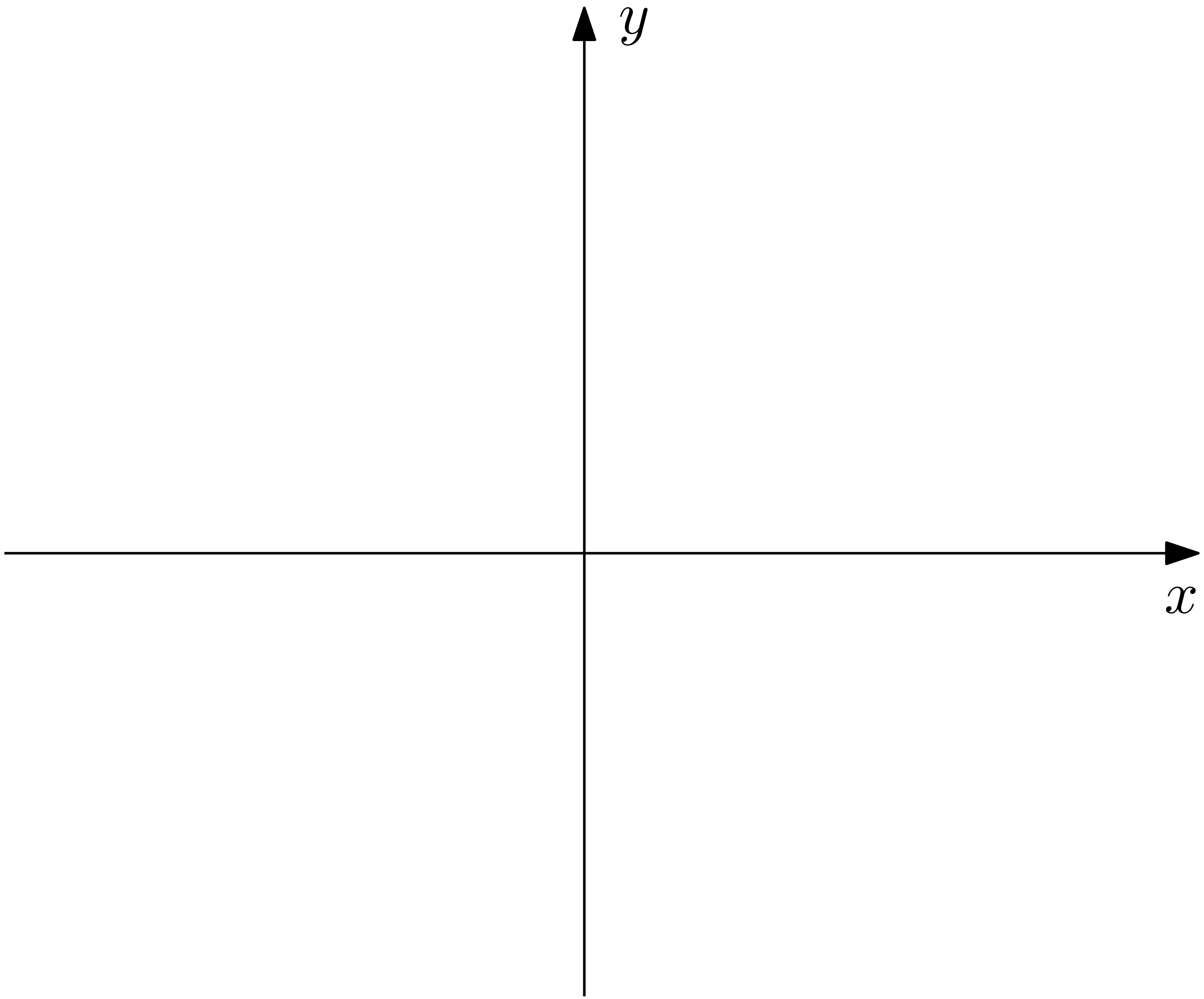
So here's another way to visualise a multivariate function:
Definition: The *level set* of a function $\mathbb{R}^n \to \mathbb{R}$ is the set of points in $\mathbb{R}^n$ satisfying $C = f(x_1, \dots, x_n)$, for some constant C.
Another view: the level sets are the intersections of the graph with the hyperplanes $x_{n+1} = C$. They are usually $n - 1$ dimensional objects in $\mathbb{R}^n$ (e.g. a 2-variable function has level curves in $\mathbb{R}^2$).

graph (from KK L on google)

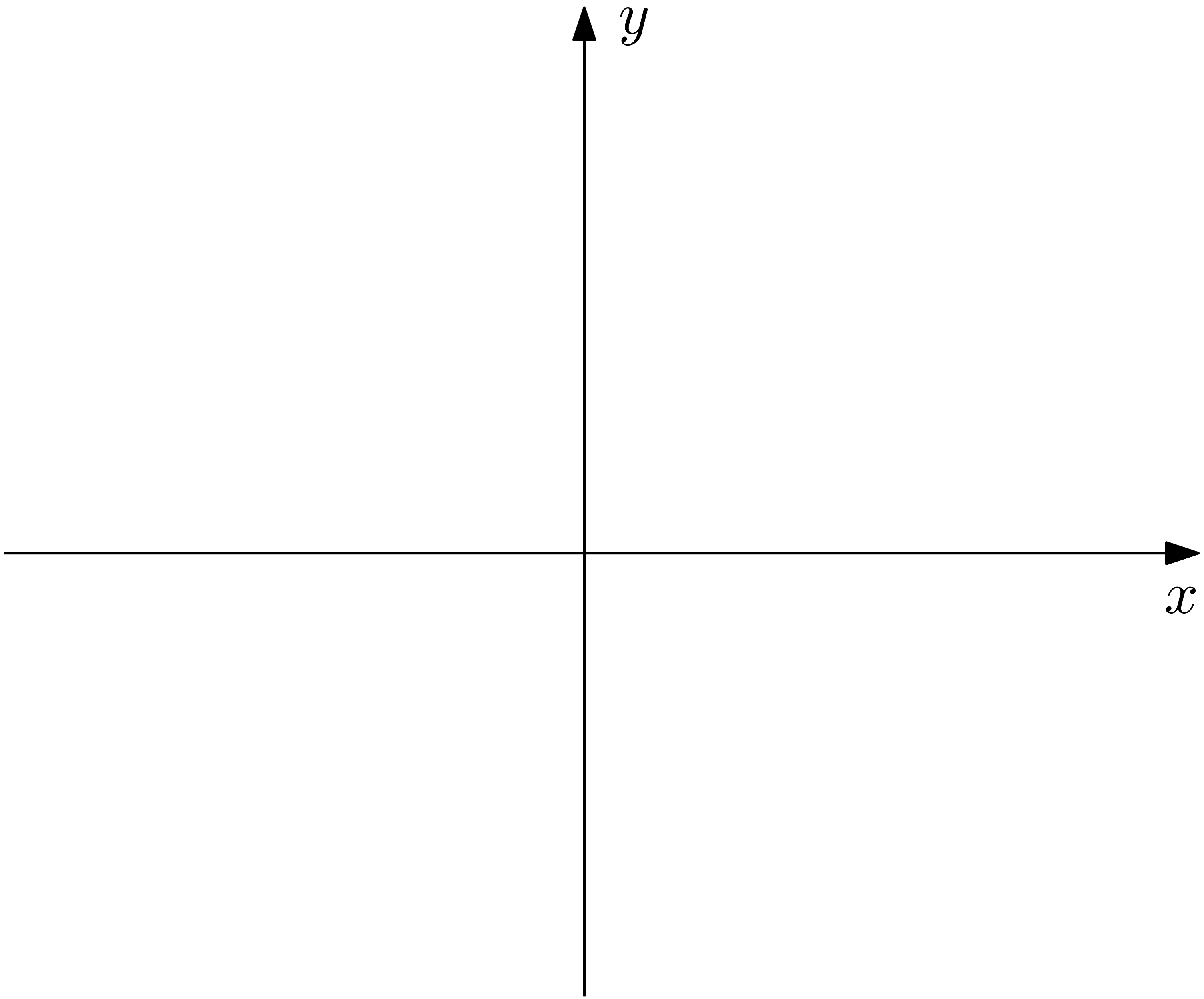


level set (from googlemaps)

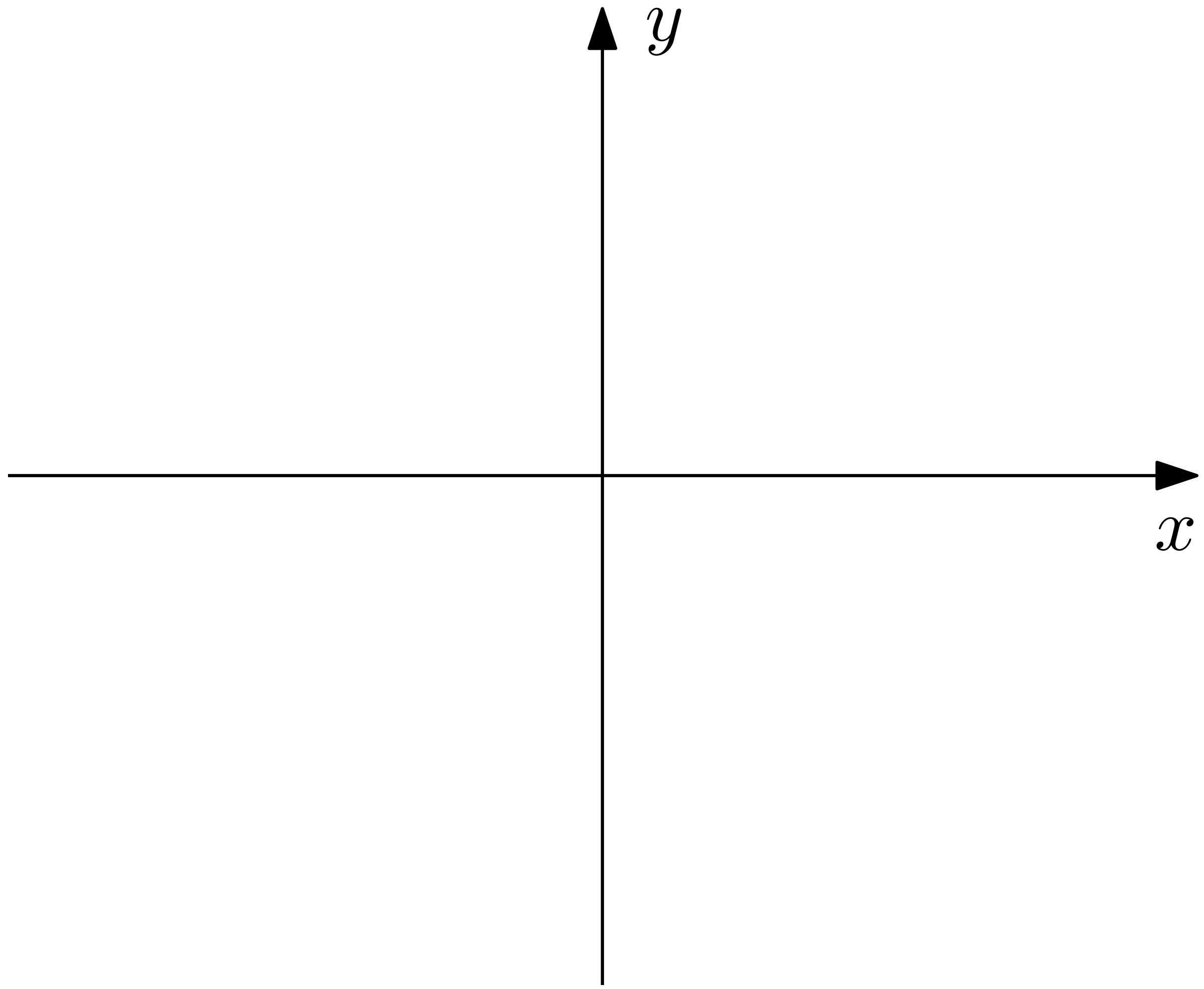
Example: Describe and sketch the level curves of $f(x, y) = x^2 + y^2$.

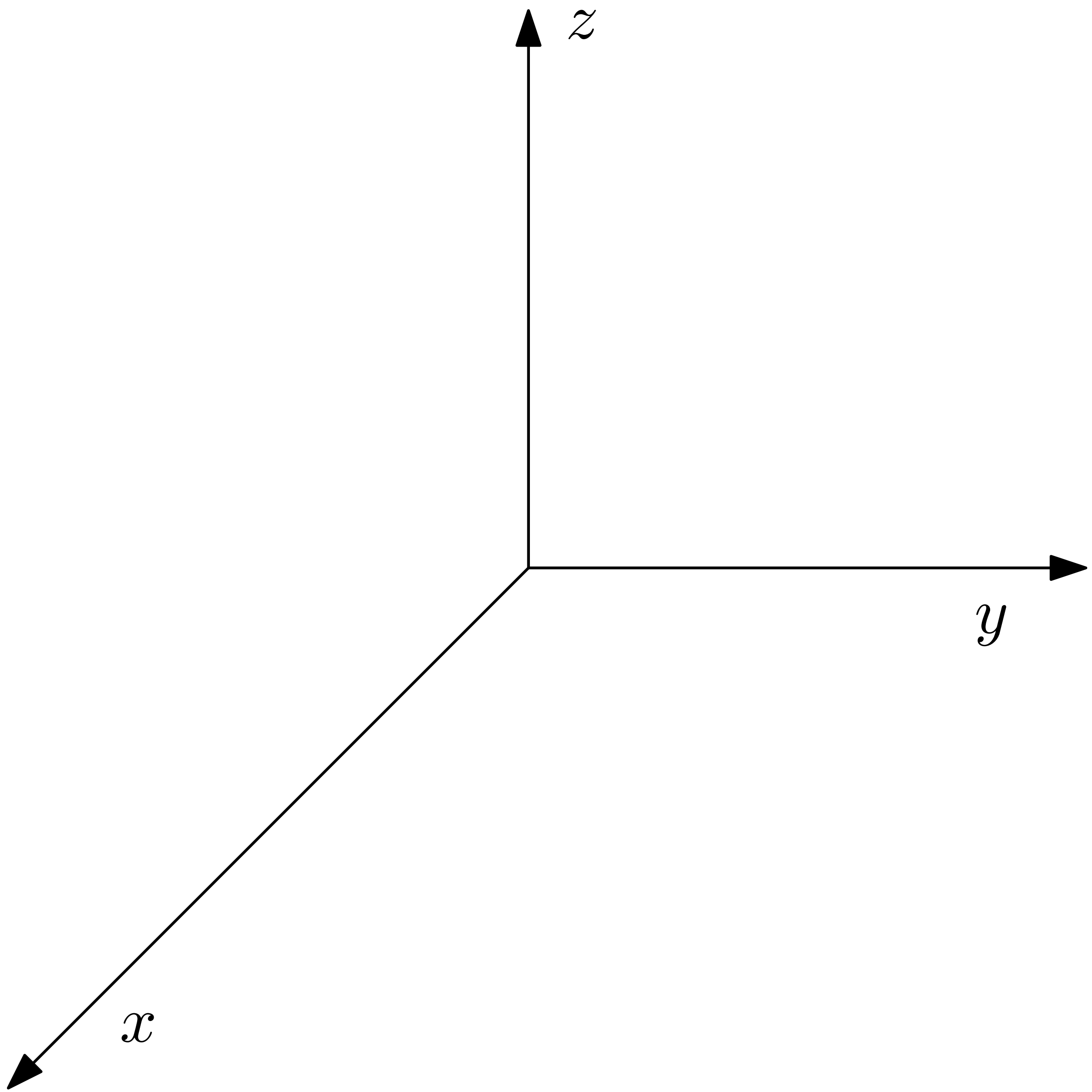


Example: Describe and sketch the level curves of $k(x,y) = x - y$.



Example: Describe and sketch the level curves of $h(x, y) = xy$, and use this to sketch the graph of h.



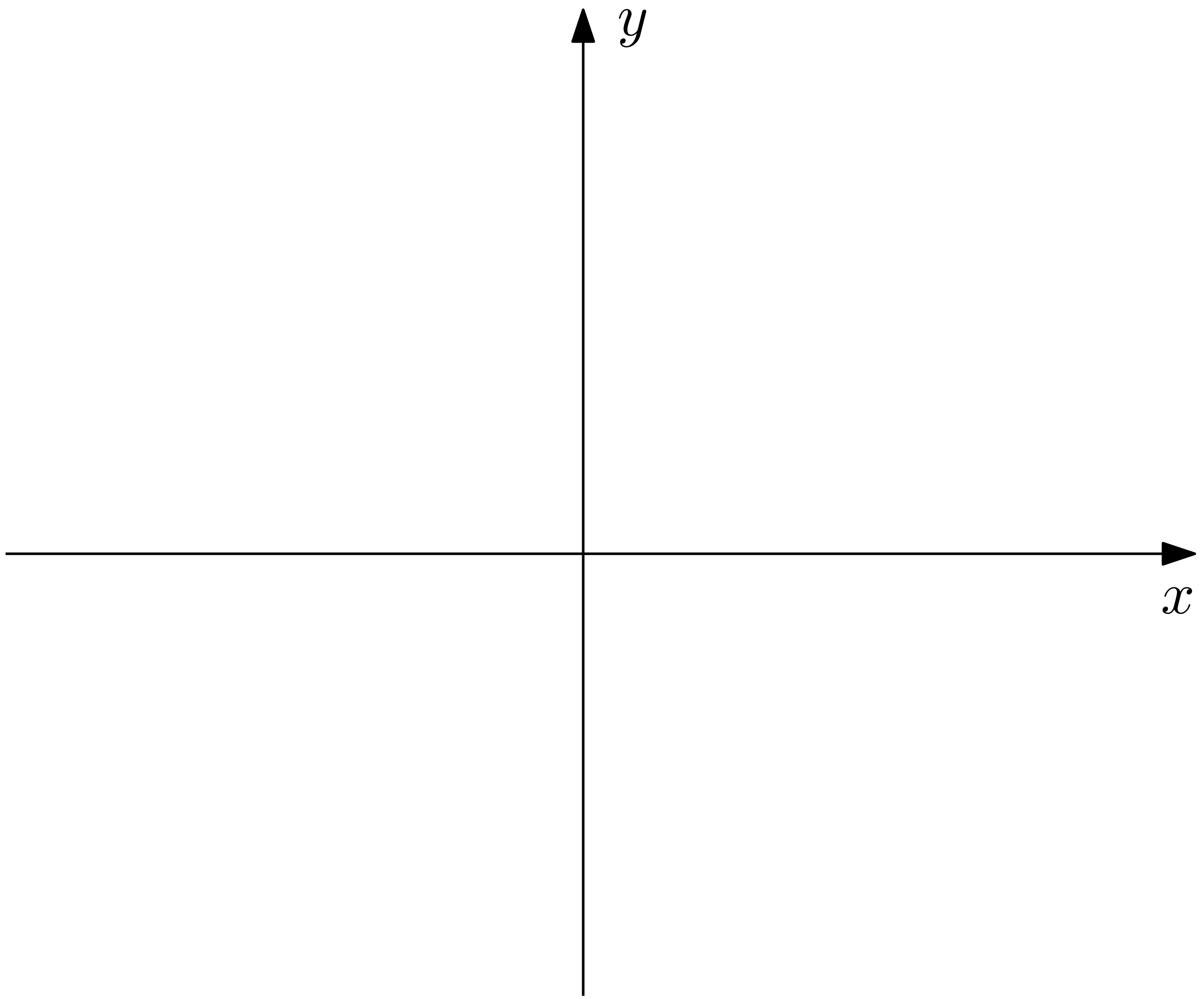


Another way to see that $z = xy$ is a hyperbolic paraboloid is by completing the square:

$$z = xy = \frac{1}{4}4xy = \frac{1}{4}(((x+y)^2 - (x-y)^2),$$

which is a difference of two squares.

Example: Describe and sketch the level curves of $g(x, y) = \dfrac{\sqrt{x}}{1+y}$.



Tips for drawing level sets:

- Make sure your level sets fill the domain. If you have an unfilled area, evaluate the function there to see what value of C will have a level set in that area. Or check negative or fractional values of C.
- Level sets corresponding to different function values C should not intersect (because an intersection would mean the function has two values at that point, which is not allowed).

For 3-variable functions, the level sets are generally level surfaces.

Example: Describe the level surfaces of

$$G(x, y, z) = e^{-x^2-y^2-z^2}.$$

Not every surface in $\mathbb{R}^3$ is the graph of a (2-variable) function, but most surfaces in $\mathbb{R}^3$ can be expressed as a level set of a (3-variable) function, and this is often useful (for finding tangent planes to the surfaces, see §12.7).

Example: Express the surface $2x + 2\ln y = 9 - z^2$ as the level set of a suitable function.