§1.8-1.9: Linear Transformations

Goal: think of the equation $A\mathbf{x} = \mathbf{b}$ in terms of the "multiplication by A" function: its input is $\mathbf{x}$ and its output is $\mathbf{b}$.

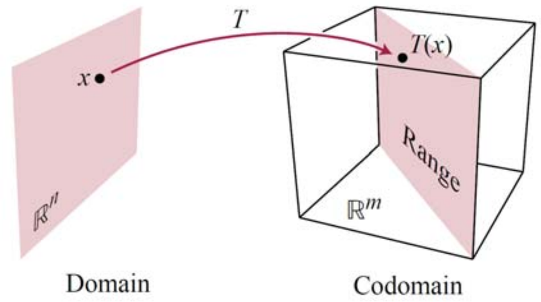
$2^2 = 4$
$3^2 = 9$

Think of this as: $2 \xrightarrow{\text{squaring}} 4$, $3 \longrightarrow 9$

$$\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \end{bmatrix} \qquad \text{Think of this as: } \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \xrightarrow{\text{multiply by } A} \begin{bmatrix} 10 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{\text{multiply by } A} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Definition: A *function* f from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $f(\mathbf{x})$ in $\mathbb{R}^m$. We write $f : \mathbb{R}^n \to \mathbb{R}^m$.



$\mathbb{R}^n$ is the *domain* of f.

$\mathbb{R}^m$ is the *codomain* of f.

$f(x)$ is the *image of x under f*.

The range is the set of all images. It is a subset of the codomain.

Example: $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.
Its domain = codomain = $\mathbb{R}$, its range = {zero and positive numbers}.

Examples:

$f : \mathbb{R}^2 \to \mathbb{R}^3$, defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^3 x_2 \\ 2x_2 \\ 0 \end{bmatrix}$.

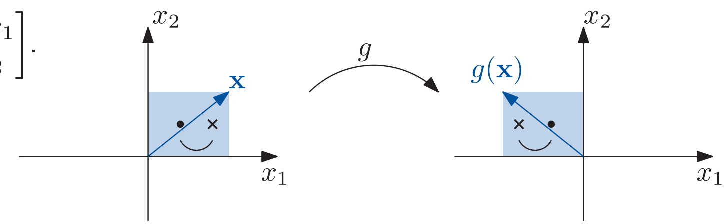
The range of f is the plane $z = 0$.

$h : \mathbb{R}^3 \to \mathbb{R}^2$, given by the matrix transformation $h(\mathbf{x}) = \begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix} \mathbf{x}$.

Examples:

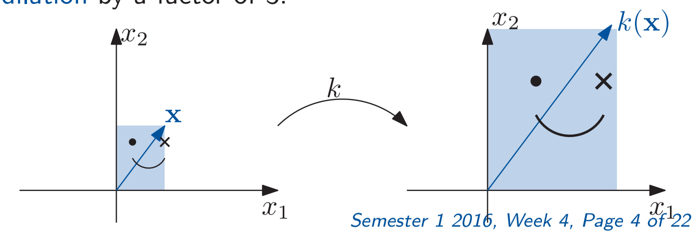
$g : \mathbb{R}^2 \to \mathbb{R}^2$, given by reflection through the x_2-axis.

$g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$.



$k : \mathbb{R}^2 \to \mathbb{R}^2$, given by dilation by a factor of 3.

$k(\mathbf{x}) = 3\mathbf{x}$.



In this class, we will concentrate on functions that are linear.

Definition: A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if:
1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and for all $\mathbf{u}$ in the domain of T.

For your intuition: the name "linear" is because these functions preserve lines:
A line through the point $\mathbf{p}$ in the direction $\mathbf{v}$ is the set $\mathbf{p} + s\mathbf{v}$, where s is any number.
If T is linear, then the image of this set is

$$T(\mathbf{p}+s\mathbf{v}) \overset{1}{=} T(\mathbf{p}) + T(s\mathbf{v}) \overset{2}{=} T(\mathbf{p}) + sT(\mathbf{v}),$$

the line through the point $T(\mathbf{p})$ in the direction $T(\mathbf{v})$.
(If $T(\mathbf{v}) = \mathbf{0}$, then the image is just the point $T(\mathbf{p})$.)

Fact: A linear transformation T must satisfy $T(\mathbf{0}) = \mathbf{0}$.

Proof: Put $c = 0$ in condition 2.

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Example: $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^3 x_2 \\ 2x_2 \\ 0 \end{bmatrix}$ is not linear:

Take $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $c = 2$:

$$f\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 16 \\ 4 \\ 0 \end{bmatrix}.$$

$$2f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 16 \\ 4 \\ 0 \end{bmatrix}.$$

So condition 2 is false for f.

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Example: $g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$ (reflection through the x_2-axis) is linear:

1. $g\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{bmatrix} -u_1-v_1 \\ u_2+v_2 \end{bmatrix} = \begin{bmatrix} -u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix} = g\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + g\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right).$

2. $g\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) = \begin{bmatrix} -cu_1 \\ cu_2 \end{bmatrix} = c\begin{bmatrix} -u_1 \\ u_2 \end{bmatrix} = cg\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right).$

Alternatively, we can combine the two conditions at the same time, and check just one statement: $T(c\mathbf{u}+d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$.

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Example: $k(\mathbf{x}) = 3\mathbf{x}$ (dilation by a factor of 3) is linear:

$$k(c\mathbf{u}+d\mathbf{v}) = 3(c\mathbf{u}+d\mathbf{v}) = 3cT(\mathbf{u}) + 3dT(\mathbf{v}) = k(c\mathbf{u}) + k(d\mathbf{v}).$$

Important Example: All matrix transformations $T(\mathbf{x}) = A\mathbf{x}$ are linear:

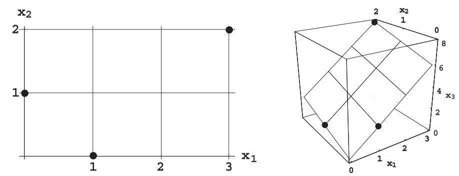
$$T(c\mathbf{u}+d\mathbf{v}) = A(c\mathbf{u}+d\mathbf{v}) = A(c\mathbf{u}) + A(d\mathbf{v}) = cA\mathbf{u} + dA\mathbf{v} = cT(\mathbf{u}) + dT(\mathbf{v}).$$

EXAMPLE: Let $\mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$. Suppose $T : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear transformation with

$$T(\mathbf{e}_1) = \begin{bmatrix}1\\0\\2\end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

Find the image of $\begin{bmatrix}3\\2\end{bmatrix}$.

Solution:



$T(3\mathbf{e}_1 + 2\mathbf{e}_2) = 3T(\mathbf{e}_1) + 2T(\mathbf{e}_2)$

In general:

Write $\mathbf{e_i}$ for the vector with 1 in row i and 0 in all other rows.

For example, in $\mathbb{R}^3$, we have $\mathbf{e_1} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{e_2} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{e_3} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$\{\mathbf{e_1}, \dots, \mathbf{e_n}\}$ span $\mathbb{R}^n$, and $\mathbf{x} = \begin{bmatrix}x_1\\ \dots \\ x_n\end{bmatrix} = x_1\mathbf{e_1} + \dots + x_n\mathbf{e_n}$.

So, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then

$$T(\mathbf{x}) = T(x_1\mathbf{e_1} + \dots x_n\mathbf{e_n}) = x_1T(\mathbf{e_1}) + \dots x_nT(\mathbf{e_n}) = \begin{bmatrix} | & & | \\ T(\mathbf{e}_1) & \dots & T(\mathbf{e}_n) \\ | & & | \end{bmatrix} \begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix}.$$

Theorem 10: The matrix of a linear transformation: Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation: $T(\mathbf{x}) = A\mathbf{x}$ where A is the *standard matrix for T*, the $m \times n$ matrix given by

$$A = \begin{bmatrix} | & & | \\ T(\mathbf{e}_1) & \dots & T(\mathbf{e}_n) \\ | & & | \end{bmatrix}.$$

Example: $k : \mathbb{R}^2 \to \mathbb{R}^2$, given by dilation by a factor of 3, $k(\mathbf{x}) = 3\mathbf{x}$.

$$k(\mathbf{e_1}) = k\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = 3\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\0\end{bmatrix}, \quad k(\mathbf{e_2}) = k\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = 3\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\3\end{bmatrix}.$$

So the standard matrix of k is $\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$, i.e. $k(\mathbf{x}) = \begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}\mathbf{x}$.

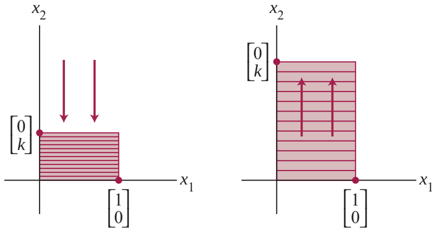
Example: $g\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}-x_1\\x_2\end{bmatrix}$ (reflection in the x_2-axis):

The standard matrix of g is $\begin{bmatrix} | & | \\ g(\mathbf{e}_1) & g(\mathbf{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$.

Indeed, $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-x_1\\x_2\end{bmatrix}$.

Vertical Contraction and Expansion

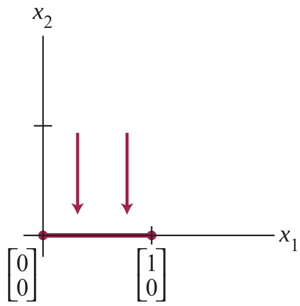
Image of the Unit Square	Standard Matrix



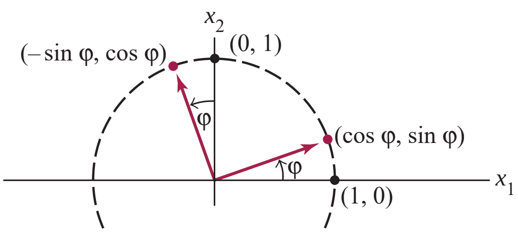
$0 < k < 1$ $\quad$ $k > 1$

Projection onto the x_1-axis

Image of the Unit Square	Standard Matrix



EXAMPLE: $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation clockwise through an angle φ:

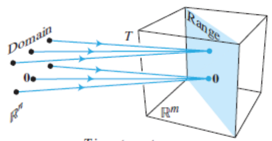


Definition: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *onto* (surjective) if each $\mathbf{y}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.

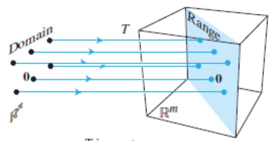
Other ways of saying this:
- The range is all of the codomain $\mathbb{R}^m$,
- The equation $f(\mathbf{x}) = \mathbf{y}$ always has a solution.

Definition: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *one-to-one* (injective) if each $\mathbf{y}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

Other ways of saying this:
- ??? (something that only works for linear transformations),
- The equation $f(\mathbf{x}) = \mathbf{y}$ has no solutions or a unique solution.



T is *not* one-to-one



T is one-to-one

Definition: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *onto* (surjective) if each $\mathbf{y}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.

Definition: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *one-to-one* (injective) if each $\mathbf{y}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

Example: $f : \mathbb{R}^2 \to \mathbb{R}^3$, defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^3 x_2 \\ 2x_2 \\ 0 \end{bmatrix}$.

f is not onto, because $f(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ does not have a solution. Indeed, the range of f is the plane $z = 0$.

f is one-to-one: the solution to $f(\mathbf{x}) = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}$ is $x_2 = \frac{1}{2} y_2$, $x_1 = \sqrt[3]{\frac{2y_1}{y_2}}$.

There is an easier way to check if a linear transformation is one-to-one:

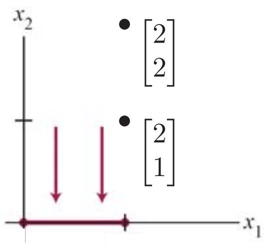
Definition: The *kernel* of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of solutions to $T(\mathbf{x}) = \mathbf{0}$.

Fact: If $T(\mathbf{v_1}) = T(\mathbf{v_2})$, then $\mathbf{v_1} - \mathbf{v_2}$ is in the kernel of T.

Example: Let T be projection onto the x_1-axis.
The kernel of T is the x_2-axis.

$$T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

$\begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which is in the kernel.



Proof of Fact: If $T(\mathbf{v_1}) = T(\mathbf{v_2}) = \mathbf{y}$, then $T(\mathbf{v_1} - \mathbf{v_2}) = T(\mathbf{v_1}) - T(\mathbf{v_2}) = \mathbf{y} - \mathbf{y} = \mathbf{0}$.

There is an easier way to check if a linear transformation is one-to-one:

Definition: The *kernel* of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of solutions to $T(\mathbf{x}) = \mathbf{0}$.

Fact: If $T(\mathbf{v_1}) = T(\mathbf{v_2})$, then $\mathbf{v_1} - \mathbf{v_2}$ is in the kernel of T.

Theorem: A linear transformation is one-to-one if and only if its kernel is $\{\mathbf{0}\}$.

Warning: this only works for linear transformations. For other functions, the solution sets of $f(\mathbf{x}) = \mathbf{y}$ and $f(\mathbf{x}) = \mathbf{0}$ are not related.

Proof:

Suppose T is one-to-one. So $T(\mathbf{x}) = \mathbf{0}$ has at most one solution. Since $\mathbf{0}$ is a solution, it must be the only one. So its kernel is $\{\mathbf{0}\}$.

Suppose the kernel of T is $\{\mathbf{0}\}$. Then, from the Fact, if there are vectors $\mathbf{v_1}, \mathbf{v_2}$ with $T(\mathbf{v_1}) = T(\mathbf{v_2}) = \mathbf{y}$, then $\mathbf{v_1} - \mathbf{v_2} = \mathbf{0}$, i.e. $\mathbf{v_1} = \mathbf{v_2}$.

For a linear transformation T whose standard matrix is A

Theorem: Uniqueness of solutions to linear systems: ~~For a matrix A,~~ the following are equivalent:

a. $A\mathbf{x} = \mathbf{0}$ has no non-trivial solution.
b. If $A\mathbf{x} = \mathbf{b}$ is consistent, then it has a unique solution.
c. The columns of A are linearly independent.
d. rref(A) has a pivot in every column (i.e. all variables are basic).
e. T is a one-to-one function.

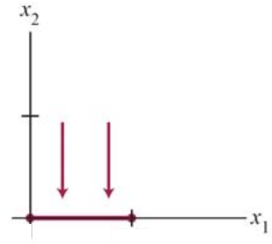
The range of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the set of images, i.e. the set of $\mathbf{y}$ in $\mathbb{R}^m$ with $T(\mathbf{x}) = \mathbf{y}$ for some $\mathbf{x}$.

So, if A is the standard matrix of T, then the range of T is the set of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution.

So the range of T is the span of the columns of A.

Example: The standard matrix of projection onto the x_1-axis is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Its range is the x_1-axis, which is also Span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.



So the range of T is the span of the columns of A.

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ whose standard matrix is A

Theorem 4: Existence of solutions to linear systems: ~~For an $m \times n$ matrix A,~~ the following statements are logically equivalent (i.e. for any particular matrix A, they are all true or all false):

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. rref(A) has a pivot in every row.
e. T is onto