Tip: in $\mathbb{R}^n$, using standard basis $\mathcal{A}$:

If f is a symmetric bilinear form, $\{f\}_{\mathcal{A}} = A$

then $f\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\right) = (x_1 \cdots x_n)\{f\}_{\mathcal{A}} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$= \sum_{i,j=1}^{n} x_i y_j a_{ij}$$

so $q\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = \sum_{i,j=1}^{n} x_i x_j a_{ij}$

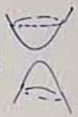
$$= \sum_{i=1}^{n} x_i^2 a_{ii} + \sum_{i<j} \underbrace{a_{ij} + a_{ji}}_{= 2a_{ij} \ \because A \text{ is symmetric}} x_i x_j$$

$\therefore$ To find A from q:

$$\left.\begin{aligned} a_{ii} &= \text{coefficient of } x_i^2 \\ a_{ij} = a_{ji} &= \tfrac{1}{2} \text{ coefficient of } x_i x_j. \end{aligned}\right\} *$$

Motivation for diagonalising quadratic forms:

In $\mathbb{R}^3$: $x_1^2 + x_2^2 + x_3^2 = 1$ is a sphere 

$x_1^2 - x_2^2 - x_3^2 = 1$ is a hyperboloid 

what shape is $2x_1x_2 + 2x_2x_3 + 4x_1x_3 = 1$?

Let $q\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = 2x_1x_2 + 2x_2x_3 + 4x_1x_3$.

corresponding matrix = $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$
(using $*$)

From diagonalisation from last time:

$q(y_1\beta_1 + y_2\beta_2 + y_3\beta_3) = 2y_1^2 - 2y_2^2 - 4y$

i.e. it's a hyperboloid

diagonal entries of D

BUT: there are many choices of P and D. How do we know there is no other choice of D where all diagonal entries are positive, i.e. there are no different coordinates where q is $z_1^2 + z_2^2 + z_3^2$?

Th. 9.5.1 Sylvester's law of inertia:

Let q be a quadratic form $\boxed{\text{over } \mathbb{R}}$

Let P, N denote the number of positive and negative entries respectively in a diagonal matrix representing q. Then P, N do not depend on the representing matrix.

So we can make the following definitions:

Def 9.5.3/9.5.6: Let q be a quadratic form over $\mathbb{R}$:

f be the related symmetric bilinear form

$A = \{f\}_{\mathcal{A}}$ a matrix representing f.

- The <u>rank</u> of q (or f, or A) is $P+N$, i.e. number of nonzero diagonal entries in a diagonal matrix representing q (or f).
- The <u>signature</u> of q (or f, or A) is $P-N$, or the signs of the diagonal entries in a diagonal matrix representing q (or f).

e.g. for previous example, signature is $1-2=-1$. or $+--$.
(possible to have signatures of $+0-$, $+00$ etc.)

q (or f, or A) is

<u>positive definite</u>	if $q(\alpha)>0$ $\forall \alpha \neq \vec{0}$	$\Leftrightarrow$	signature is $+\cdots+$.
<u>positive semidefinite</u>	$q(\alpha)\geqslant 0$	$\Leftrightarrow$	$+\cdots+0\cdots0$.
<u>negative definite</u>	$q(\alpha)<0$	$\Leftrightarrow$	$-\cdots-$
<u>negative semidefinite</u>	$q(\alpha)\leq 0$	$\Leftrightarrow$	$-\cdots-0\cdots0$
<u>indefinite</u>	$q(\alpha)>0$ for some α $q(\beta)<0$ for some β		$+,-,$ and others