If f is an elementary function, then we can use our single-variable differentiation rules to calculate $\frac{\partial f}{\partial x}$, by treating y as a constant (and similarly for $\frac{\partial f}{\partial y}$).

Example: Find the first-order partial derivatives of $f(x,y) = \frac{xy}{x+1}$ at $(1,2)$.

quotient rule

$$\frac{\partial f}{\partial x} = \frac{(x+1)y - xy1}{(x+1)^2} = \frac{y}{(x+1)^2}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = \frac{2}{(1+1)^2} = \frac{2}{4} = \frac{1}{2}$$

no quotient rule: no y's in denominator

$$\frac{\partial f}{\partial y} = \frac{x}{x+1}$$

x is a constant

$$\left.\frac{\partial f}{\partial y}\right|_{(1,2)} = \frac{1}{1+1} = \frac{1}{2}$$

When f is defined by different formulae around (a, b), we need to use the limit definition to calculate the partial derivatives at (a, b).

Example: Let $f(x, y) = \begin{cases} \dfrac{y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$. Find $f_y(0, 0)$.

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0, 0+k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{1}{k}\left(\frac{k^3}{0^2 + k^2} - 0\right) = \lim_{k \to 0} \frac{1}{k}(k - 0) = \lim_{k \to 0} 1 = 1$$

Example: Find the second-order partial derivatives of $f(x,y) = \dfrac{xy}{x+1}$ at $(1,2)$.

from p17: $\dfrac{\partial f}{\partial x} = \dfrac{y}{(x+1)^2}$, $\dfrac{\partial f}{\partial y} = \dfrac{x}{x+1}$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{y}{(x+1)^2}\right) = \frac{\partial}{\partial x}\left(y(x+1)^{-2}\right) = -2y(x+1)^{-3}$$

chain rule

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(1,2)} = -4(2)^{-3} = \frac{-1}{2}$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{y}{(x+1)^2}\right) = \frac{1}{(x+1)^2} \qquad \left.\frac{\partial^2 f}{\partial y\partial x}\right|_{(1,2)} = \frac{1}{4}.$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{x}{x+1}\right) = \frac{(x+1)-x}{(x+1)^2} = \frac{1}{(x+1)^2} \qquad \left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(1,2)} = \frac{1}{4}.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x}{x+1}\right) = 0$$

no y's.

Before continuing with the theory of differentiability, let us make sure we understand the linearisation and its applications:

Example: Calculate the linearisation of $f(x,y) = x^2 y$ at $(1,2)$, and use it to estimate $f(1.1, 1.8)$.

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = \left.2xy\right|_{(1,2)} = 4 \qquad \left.\frac{\partial f}{\partial y}\right|_{(1,2)} = \left.x^2\right|_{(1,2)} = 1$$

$$L(x,y) = f(1,2) + \left.\frac{\partial f}{\partial x}\right|_{(1,2)}(x-1) + \left.\frac{\partial f}{\partial y}\right|_{(1,2)}(y-2)$$

$$= 1^2 2 + 4(x-1) + 1(y-2)$$

$$= 2 + 4(x-1) + 1(y-2)$$

check that this is a linear function

$$L(1.1, 1.8) = 2 + 4(0.1) + 1(-0.2) = 2.2$$

For your interest: $1.1^2 \times 1.8 = 2.178$

$\vec{f}: \mathbb{R}^2 \to \mathbb{R}^3$

Example: Calculate the Jacobian matrix of $\mathbf{f}(x, y) = \left(\frac{xy}{x+1}, x^2y, x \right)$ at $(1, 2)$, and use it to estimate $\mathbf{f}(1.1, 2.3)$.

from p17, p26

$$D\vec{f}(1,2) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{pmatrix} \Bigg|_{(1,2)} = \begin{pmatrix} \frac{y}{(x+1)^2} & \frac{x}{x+1} \\ 2xy & x^2 \\ 1 & 0 \end{pmatrix} \Bigg|_{(1,2)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 4 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\vec{f}(1.1, 2.3) \approx \vec{L}(1.1, 2.3) = \vec{f}(1,2) + [D\vec{f}(1,2)] \begin{pmatrix} 1.1 - 1 \\ 2.3 - 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1.2}{1+1} \\ 1^2 2 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 4 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0.1 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \cdot 0.1 + \frac{1}{2} 0.3 \\ 4(0.1) + 1(0.3) \\ 1(0.1) + 0(0.3) \end{pmatrix}$$

$$= \begin{pmatrix} 1.2 \\ 2.7 \\ 1.1 \end{pmatrix}$$