§4.4, 4.7, 5.4: Change of Basis

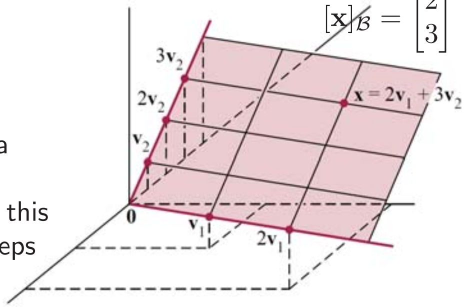
Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for V. Remember:

- The $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ where $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.
- The matrix for a linear transformation $T : V \to V$ relative to $\mathcal{B}$ is

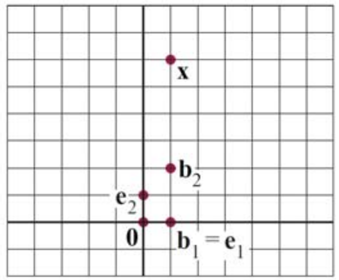
$$[T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & \dots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix}.$$

A basis for this plane in $\mathbb{R}^3$ allows us to draw a coordinate grid on the plane. The coordinate vectors then describe the location of points on this plane relative to this coordinate grid (e.g. 2 steps in $\mathbf{v}_1$ direction, 3 steps in $\mathbf{v}_2$ direction.)

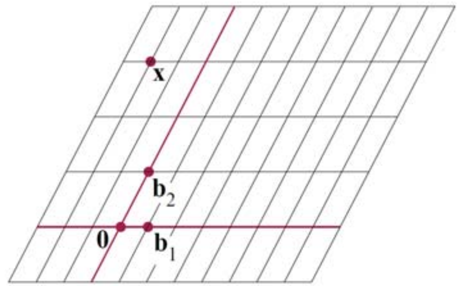
$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$



Although we already have the standard coordinate grid on $\mathbb{R}^n$, some computations are much faster and more accurate in a different basis i.e. using a different coordinate grid (later, p17-19).



standard coordinate grid



$\mathcal{B}$-coordinate grid

Important questions:

i how are $\mathbf{x}$ and $[\mathbf{x}]_{\mathcal{B}}$ related (p3-7, §4.4 in textbook);
ii how are $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{F}}$ related for two bases $\mathcal{B}$ and $\mathcal{F}$ (p8-11, §4.7);
iii how are the standard matrix of T and the matrix $[T]_{\mathcal{B}}$ related (p12-15, §5.4).

i: Changing between the standard basis and another basis in $\mathbb{R}^n$:

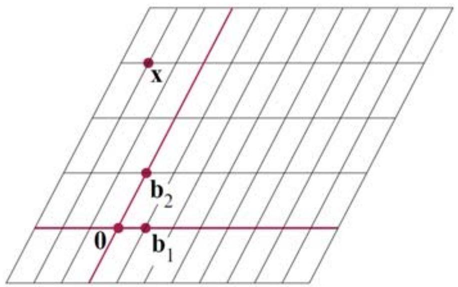
Example: Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis of $\mathbb{R}^2$.

If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, then what is $\mathbf{x}$?

Answer:
$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ means that

$\mathbf{x} = -2\mathbf{b}_1 + 3\mathbf{b}_2 = -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

(You can check this from the picture on the previous page.)



So, if $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = \begin{bmatrix} | & | \\ \mathbf{b}_1 & \mathbf{b}_2 \\ | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

In general, if $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$, and $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, then

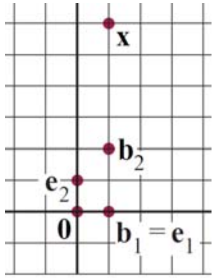
$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n = \begin{bmatrix} | & & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_n \\ | & & | \end{bmatrix} [\mathbf{x}]_{\mathcal{B}}.$$

Notation: write $\mathcal{P}_{\mathcal{B}}$ for the matrix $\begin{bmatrix} | & & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_n \\ | & & | \end{bmatrix}$. This is the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis.

Going the other direction:

Example: Let $\mathbf{b}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix}1\\2\end{bmatrix}$ and let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis of $\mathbb{R}^2$.

If $\mathbf{x} = \begin{bmatrix}1\\6\end{bmatrix}$, then what are its $\mathcal{B}$-coordinates $[\mathbf{x}]_\mathcal{B}$?



Answer:
$[\mathbf{x}]_\mathcal{B} = \begin{bmatrix}c_1\\c_2\end{bmatrix}$ where $\begin{bmatrix}1\\6\end{bmatrix} = c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\2\end{bmatrix}$ i.e. (c_1, c_2) is the solution to the linear system $\left[\begin{array}{cc|c}1 & 1 & 1\\0 & 2 & 6\end{array}\right]$. (Exercise: do the row-reduction and show that $c_1 = -2, c_2 = 3$ as on the last pages.)

So the $\mathcal{B}$-coordinate vector $\begin{bmatrix}c_1\\c_2\end{bmatrix}$ of $\mathbf{v}$ satisfies $\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = \begin{bmatrix}| & |\\\mathbf{b}_1 & \mathbf{b}_2\\| & |\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$.

So $[\mathbf{v}]_\mathcal{B}$ is the solution to $\begin{bmatrix}| & |\\\mathbf{b}_1 & \mathbf{b}_2\\| & |\end{bmatrix}\mathbf{x} = \mathbf{v}$.

In general, if $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$, and $\mathbf{v}$ is any vector in $\mathbb{R}^n$, then $[\mathbf{v}]_\mathcal{B}$ is a solution to $\underbrace{\begin{bmatrix}| & & |\\\mathbf{b}_1 & \dots & \mathbf{b}_n\\| & & |\end{bmatrix}}_{\mathcal{P}_\mathcal{B}}\mathbf{x} = \mathbf{v}$.

Because $\mathcal{B}$ is a basis, the columns of $\mathcal{P}_\mathcal{B}$ are linearly independent, so by the Invertible Matrix Theorem, $\mathcal{P}_\mathcal{B}$ is invertible, and the unique solution to $\mathcal{P}_\mathcal{B}\mathbf{x} = \mathbf{v}$ is

$$[\mathbf{v}]_\mathcal{B} = \begin{bmatrix}| & & |\\\mathbf{b}_1 & \dots & \mathbf{b}_n\\| & & |\end{bmatrix}^{-1}\mathbf{v}.$$

In other words, the change-of-coordinates matrix from the standard basis to $\mathcal{B}$ is $\mathcal{P}_\mathcal{B}^{-1}$.

Indeed, in the previous example, $\mathcal{P}_\mathcal{B}^{-1}\mathbf{x} = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}^{-1}\begin{bmatrix}1\\6\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & -1\\-0 & 1\end{bmatrix}\begin{bmatrix}1\\6\end{bmatrix} = \begin{bmatrix}-2\\3\end{bmatrix}$.

A very common mistake is to get the direction wrong:

Does multiplication by $\mathcal{P}_\mathcal{B}$ change from standard coordinates to $\mathcal{B}$-coordinates, or from $\mathcal{B}$-coordinates to standard coordinates?

Don't memorise the formulas. Instead, remember the definition of coordinates:

$$[\mathbf{x}]_\mathcal{B} = \begin{bmatrix}c_1\\\vdots\\c_n\end{bmatrix} \text{ means } \quad \mathbf{x} = c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n = \begin{bmatrix}| & & |\\\mathbf{b}_1 & \dots & \mathbf{b}_n\\| & & |\end{bmatrix}[\mathbf{x}]_\mathcal{B}$$

and you won't go wrong.

ii: Changing between two non-standard bases:

Example: As before, $\mathbf{b}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix}1\\2\end{bmatrix}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$.

Another basis: $\mathbf{f}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{f}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2\}$.

If $[\mathbf{x}]_\mathcal{B} = \begin{bmatrix}-2\\3\end{bmatrix}$, then what are its $\mathcal{F}$-coordinates $[\mathbf{x}]_\mathcal{F}$?

Answer 1: $\mathcal{B}$ to standard to $\mathcal{F}$ - works only in $\mathbb{R}^n$, in general easiest to calculate.

$[\mathbf{x}]_\mathcal{B} = \begin{bmatrix}-2\\3\end{bmatrix}$ means $\mathbf{x} = -2\mathbf{b}_1 + 3\mathbf{b}_2 = -2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix}$.

So if $[\mathbf{x}]_\mathcal{F} = \begin{bmatrix}d_1\\d_2\end{bmatrix}$, then $d_1\begin{bmatrix}1\\1\end{bmatrix} + d_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix}$.

Row-reducing $\left[\begin{array}{cc|c}1 & 0 & 1\\1 & 1 & 6\end{array}\right]$ shows $d_1 = 1, d_2 = 5$ so $[\mathbf{x}]_\mathcal{F} = \begin{bmatrix}1\\5\end{bmatrix}$.

In other words, $\mathbf{x} = \mathcal{P}_\mathcal{B}[\mathbf{x}]_\mathcal{B}$ and $[\mathbf{x}]_\mathcal{F} = \mathcal{P}_\mathcal{F}^{-1}\mathbf{x}$, so $[\mathbf{x}]_\mathcal{F} = \mathcal{P}_\mathcal{F}^{-1}\mathcal{P}_\mathcal{B}[\mathbf{x}]_\mathcal{B}$.

Answer 2: A different view that works for abstract vector spaces (without reference to a standard basis) - important theoretically, but may be hard to calculate for general examples in $\mathbb{R}^n$.

$[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ means $\mathbf{x} = -2\mathbf{b}_1 + 3\mathbf{b}_2$.

So $[\mathbf{x}]_\mathcal{F} = [-2\mathbf{b}_1 + 3\mathbf{b}_2]_\mathcal{F} = -2[\mathbf{b}_1]_\mathcal{F} + 3[\mathbf{b}_2]_\mathcal{F} = \begin{bmatrix} | & | \\ [\mathbf{b}_1]_\mathcal{F} & [\mathbf{b}_2]_\mathcal{F} \\ | & | \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

because $\mathbf{x} \mapsto [\mathbf{x}]_\mathcal{F}$ is an isomorphism, so every vector space calculation is accurately reproduced using coordinates.

$$\left.\begin{aligned} \mathbf{b}_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{f}_1 - \mathbf{f}_2 \text{ so } [\mathbf{b}_1]_\mathcal{F} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \\ \mathbf{b}_2 &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{f}_1 + \mathbf{f}_2 \text{ so } [\mathbf{b}_2]_\mathcal{F} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}\right\}$$

This step can be hard to calculate if the $\mathbf{b}_i$ are not "easy" linear combinations of the $\mathbf{f}_i$. But if you need to change bases in a practical application, the bases are probably "nicely" related.

So $[\mathbf{x}]_\mathcal{F} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Theorem 15: Change of Basis: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{F} = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ be two bases of a vector space V. Then, for all $\mathbf{x}$ in V,

$$[\mathbf{x}]_\mathcal{F} = \begin{bmatrix} | & & | \\ [\mathbf{b}_1]_\mathcal{F} & \dots & [\mathbf{b}_n]_\mathcal{F} \\ | & & | \end{bmatrix} [\mathbf{x}]_\mathcal{B}.$$

Notation: write $\underset{\mathcal{F}\leftarrow\mathcal{B}}{\mathcal{P}}$ for the matrix $\begin{bmatrix} | & & | \\ [\mathbf{b}_1]_\mathcal{F} & \dots & [\mathbf{b}_n]_\mathcal{F} \\ | & & | \end{bmatrix}$, the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{F}$.

A tip to get the direction correct:

$$[\mathbf{x}]_\mathcal{F} = \underbrace{\underset{\mathcal{F}\leftarrow\mathcal{B}}{\mathcal{P}}[\mathbf{x}]_\mathcal{B}}$$

A $\mathcal{F}$-coordinate vector

a linear combination of columns of $\underset{\mathcal{F}\leftarrow\mathcal{B}}{\mathcal{P}}$, so these columns should be $\mathcal{F}$-coordinate vectors

Theorem 15: Change of Basis: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{F} = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ be two bases of a vector space V. Then, for all $\mathbf{x}$ in V,

$$[\mathbf{x}]_\mathcal{F} = \begin{bmatrix} | & & | \\ [\mathbf{b}_1]_\mathcal{F} & \dots & [\mathbf{b}_n]_\mathcal{F} \\ | & & | \end{bmatrix} [\mathbf{x}]_\mathcal{B}.$$

Properties of the change-of-coordinates matrix $\underset{\mathcal{F}\leftarrow\mathcal{B}}{\mathcal{P}} = \begin{bmatrix} | & & | \\ [\mathbf{b}_1]_\mathcal{F} & \dots & [\mathbf{b}_n]_\mathcal{F} \\ | & & | \end{bmatrix}$:

- $\underset{\mathcal{B}\leftarrow\mathcal{F}}{\mathcal{P}}^{-1} = \underset{\mathcal{F}\leftarrow\mathcal{B}}{\mathcal{P}}$.
- If V is $\mathbb{R}^n$ and $\mathcal{E}$ is the standard basis $\{\mathbf{e}_1, \dots \mathbf{e}_n\}$, then

  $\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}} = \mathcal{P}_\mathcal{B} = \begin{bmatrix} | & & | \\ \mathbf{b}_1 & \dots & \mathbf{b}_n \\ | & & | \end{bmatrix}$, because $[\mathbf{b}_i]_\mathcal{E} = \mathbf{b}_i$. Also $\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}} = \mathcal{P}_\mathcal{B}^{-1}$.
- If V is $\mathbb{R}^n$, then $\underset{\mathcal{F}\leftarrow\mathcal{B}}{\mathcal{P}} = \mathcal{P}_\mathcal{F}^{-1}\mathcal{P}_\mathcal{B}$.

iii: Change of coordinates and linear transformations:

Remember that the matrix for a linear transformation $T : V \to V$ relative to $\mathcal{B}$ is

$[T]_\mathcal{B} = \begin{bmatrix} | & & | \\ [T(\mathbf{b}_1)]_\mathcal{B} & \dots & [T(\mathbf{b}_n)]_\mathcal{B} \\ | & & | \end{bmatrix}$, and this matrix is useful because

$$[T(\mathbf{x})]_\mathcal{B} = [T]_\mathcal{B}[\mathbf{x}]_\mathcal{B} \qquad (*)$$

(i.e. if you're working in $\mathcal{B}$-coordinates, then applying the function T is the same as multiplying by the matrix $[T]_\mathcal{B}$).

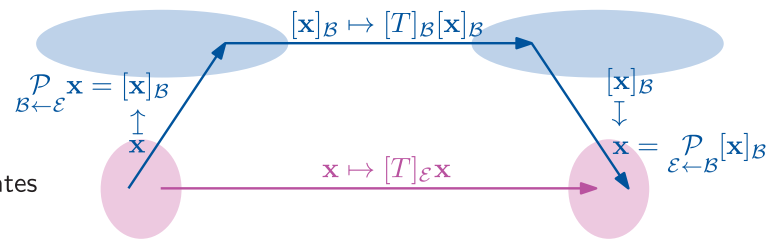
Often, it is easier to find the matrix for T relative to one basis than to another (later, p14). So it's important to know how to find $[T(\mathbf{x})]_\mathcal{F}$ if we know $[T(\mathbf{x})]_\mathcal{B}$. In $\mathbb{R}^n$, the following is true:

$$T(\mathbf{x}) = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T(\mathbf{x})]_\mathcal{B} \overset{(*)}{=} \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_\mathcal{B}[\mathbf{x}]_\mathcal{B} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_\mathcal{B}\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}}\mathbf{x}.$$

Let $[T]_\mathcal{E}$ be the standard matrix of T. Then the equation above shows that $[T]_\mathcal{E}\mathbf{x} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_\mathcal{B}\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}}\mathbf{x}$ for all $\mathbf{x}$. So

$$\boxed{[T]_\mathcal{E} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_\mathcal{B}\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}}.}$$

A picture to illustrate $[T]_\mathcal{E} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_\mathcal{B}\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}}$:



Because $\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}} = \mathcal{P}_\mathcal{B}$ and $\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}} = \mathcal{P}_\mathcal{B}^{-1}$:

$$[T]_\mathcal{E} = \mathcal{P}_\mathcal{B}[T]_\mathcal{B}\mathcal{P}_\mathcal{B}^{-1}.$$

Multiply both sides by $\mathcal{P}_\mathcal{B}^{-1}$ on the left and by $\mathcal{P}_\mathcal{B}$ on the right:

$$\mathcal{P}_\mathcal{B}^{-1}[T]_\mathcal{E}\mathcal{P}_\mathcal{B} = [T]_\mathcal{B}$$

These two equations are hard to remember ("where does the inverse go?"). Instead, remember $\boxed{[T]_\mathcal{E} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_\mathcal{B}\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}}}$ (which works for all vector spaces, not just $\mathbb{R}^n$).

Change of basis is useful for finding matrices of geometric linear transformations:

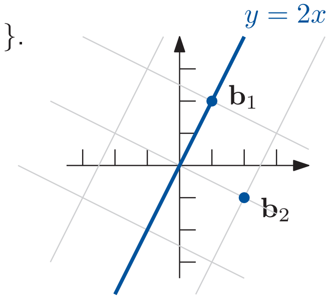
Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection through the line $y = 2x$.

$\begin{bmatrix}1\\2\end{bmatrix}$ is on the line $y = 2x$, so it is unchanged by the reflection: $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$.

$\begin{bmatrix}2\\-1\end{bmatrix}$ is perpendicular to $y = 2x$, so its image is its negative: $T\left(\begin{bmatrix}2\\-1\end{bmatrix}\right) = \begin{bmatrix}-2\\1\end{bmatrix}$.

(You will NOT be required to do the above steps.) Find the standard matrix of T.

Answer: Let $\mathbf{b}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix}2\\-1\end{bmatrix}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$.



By above, $T(\mathbf{b}_1) = 1\mathbf{b}_1 + 0\mathbf{b}_2$ so $[T(\mathbf{b}_1)]_\mathcal{B} = \begin{bmatrix}1\\0\end{bmatrix}$.

And $T(\mathbf{b}_2) = 0\mathbf{b}_1 + (-1)\mathbf{b}_2$ so $[T(\mathbf{b}_2)]_\mathcal{B} = \begin{bmatrix}0\\-1\end{bmatrix}$.

So $[T]_\mathcal{B} = \begin{bmatrix} | & | \\ [T(\mathbf{b}_1)]_\mathcal{B} & [T(\mathbf{b}_2)]_\mathcal{B} \\ | & | \end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$.

Answer: (continued) To get the standard matrix $[T]_\mathcal{E}$ from $[T]_\mathcal{B} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$:

$$[T]_\mathcal{E} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_\mathcal{B}\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}}$$

and

$$[\mathbf{x}]_\mathcal{B} = \begin{bmatrix}c_1\\c_2\end{bmatrix} \quad\text{means}\quad \mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = \begin{bmatrix} | & | \\ \mathbf{b}_1 & \mathbf{b}_2 \\ | & | \end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}[\mathbf{x}]_\mathcal{B}$$

so

$$[T]_\mathcal{E} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_\mathcal{B}\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}} = \begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}^{-1} = \begin{bmatrix}-3/5 & 4/5\\4/5 & 3/5\end{bmatrix}.$$

(To check your arithmetic: check that $[T]_\mathcal{E}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$, as given in the question).

Many alternative methods exist: e.g. see Homework 3 Q3 solution.

Remember

$$[T]_\mathcal{E} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_\mathcal{B}\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_\mathcal{B}\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}.$$

This motivates the following definition:

Definition: Two squares matrices A and D are *similar* if there is an invertible matrix P such that $A = PDP^{-1}$.

Similar matrices represent the same linear transformation in different bases.

Similar matrices have the same determinant and the same rank, because the signed volume scaling factor and the dimension of the image are coordinate-independent properties of the linear transformation. (You can also prove $\det D = \det(PDP^{-1})$ from the multiplicative property of determinants.)

Why is change of basis important?

Example: If x, y are the prices of two stocks on a particular day, then their prices the next day are respectively $\frac{1}{2}y$ and $-x + \frac{3}{2}y$. How are the prices after many days related to the prices today?

Answer: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the function representing the changes in stock prices from one day to the next, i.e. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}y \\ -x + \frac{3}{2}y \end{bmatrix}$. We are interested in T^k for large k. (You will NOT be required to do this step.)

T is a linear transformation; its standard matrix is $[T]_{\mathcal{E}} = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix}$. Calculating $\begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix}^k$ by direct matrix multiplication will take a long time.

Answer: (continued) Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$.

$$T(\mathbf{b}_1) = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2}\mathbf{b}_1, \quad T(\mathbf{b}_2) = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathbf{b}_2,$$

so $[T]_{\mathcal{B}} = \begin{bmatrix} | & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & [T(\mathbf{b}_2)]_{\mathcal{B}} \\ | & | \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$. Use $[T]_{\mathcal{E}} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_{\mathcal{B}}\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_{\mathcal{B}}\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}$:

$$\begin{aligned}
[T]_{\mathcal{E}}^k &= \left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_{\mathcal{B}}\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}\right)^k \\
&= \left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_{\mathcal{B}}\cancel{\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}}\right)\left(\cancel{\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}}[T]_{\mathcal{B}}\cancel{\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}}\right)\cdots\cdots\left(\cancel{\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}}[T]_{\mathcal{B}}\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1}\right) \\
&= \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_{\mathcal{B}}^k\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}^k\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{2}^k & 0 \\ 0 & 1^k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -1 + \frac{1}{2}^{k-1} & 1 - \frac{1}{2}^k \\ -2 + \frac{1}{2}^{k-1} & 2 - \frac{1}{2}^k \end{bmatrix}.
\end{aligned}$$

So $[T]_{\mathcal{E}}^k = \begin{bmatrix} -1 + \frac{1}{2}^{k-1} & 1 - \frac{1}{2}^k \\ -2 + \frac{1}{2}^{k-1} & 2 - \frac{1}{2}^k \end{bmatrix}$. When k is very large, this is very close to $\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$.

So essentially the stock prices after many days is $-x + y$ and $-2x + 2y$, where x, y are the prices today. (In particular, the prices stabilise, which was not clear from $[T]_{\mathcal{E}}$.)

The important points in this example:

- We have a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ and we want to find T^k for large k.
- We find a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ where $T(\mathbf{b}_1) = \lambda_1\mathbf{b}_1$ and $T(\mathbf{b}_2) = \lambda_2\mathbf{b}_2$ for some scalars λ_1, λ_2. (In the example, $\lambda_1 = \frac{1}{2}, \lambda_2 = 1$.)
- Relative to the basis $\mathcal{B}$, the matrix for T is a diagonal matrix $[T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.
- It is easy to compute with $[T]_{\mathcal{B}}$, and we can then use change of coordinates to transfer the result to the standard matrix $[T]_{\mathcal{E}}$.

Next week (§5): does a "magic" basis like this always exist, and how to find it?
(Don't worry: you can do many of the computations in §5 without fully understanding change of coordinates.)