The Eigenvalues of Permuted Descent Operators on Combinatorial Hopf Algebras, and Markov Chains on Permutations and Trees

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Abstract

We introduce a permuted generalisation of Patras's descent operators on combinatorial Hopf algebras, which are a composition of a graded refinement of the coproduct, a permutation of the resulting tensorands, followed by the product. We show that the eigenvalues of any such operator is equal to that of its unpermuted counterpart.

The motivation for our operators is in extending the connection, from the author's previous work, between descent operators and Markov chains which model the breaking then recombining of combinatorial objects. The stationary distributions of the chains involving permutation are the same as their unpermuted counterparts. For a subclass of operators that we term "top or bottom to random", we obtain an expression for their eigenvectors, applicable in all Hopf algebras. We specialise these formulae to the Malvenuto-Reutenauer algebra of permutations and the Loday-Ronco algebra of trees; they give the expected values of certain statistics under respectively a to-do list chain and a remove-and-readd-a-vertex chain.

1 Introduction

There has long been interest in using algebra and combinatorics to study Markov chains [Dia88; BHR99; Bro00; DS05; DS18]. Recently, [DPR14] proposed studying Markov chains through associated operators on combinatorial Hopf algebras. A combinatorial Hopf algebra is a graded vector space $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$ with basis indexed by a family of combinatorial objects, such as trees [LR98; CP17], graphs [Sch93] or permutations [MR95], admitting a graded product $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ and coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ that encode respectively how these objects combine and break apart; much theory on these algebras is developed in [ABS06; BLL12; AM10; AA17]. It was shown in [DPR14] that the coefficients of the composite map $m \circ \Delta : \mathcal{H}_n \to \sum_{i=0}^{n} \mathcal{H}_i \otimes \mathcal{H}_{n-i} \to \mathcal{H}_n$ on different Hopf algebras give the transition probabilities of the riffle-shuffle of a deck of cards and other break-and-recombine chains on different objects. Thus the eigenvalues and eigenvectors of $m \circ \Delta$, as calculated in [DPR14], reflect the mixing time and other long term behaviour of the chains.

This analysis was extended in [Pan18], which connected graded refinements of $m \circ \Delta$, such as $m_{1,n-1} \circ \Delta_{1,n-1} : \mathcal{H}_n \to \mathcal{H}_1 \otimes \mathcal{H}_{n-1} \to \mathcal{H}_n$, to analogues of the top-to-random shuffle. For the Markov chains associated to $m_{1,n-1} \circ \Delta_{1,n-1}$ on graphs and trees, each step removes a single vertex and reattaches it elsewhere. Again the full eigenvalues and selected eigenvectors were calculated (a uniform expression applicable to all combinatorial Hopf algebras), giving bounds for the expectations and probabilities of various statistics and events. More interestingly, [Pan18] proved similar eigendata formulae for any interpolation of $m_{1,n-1} \circ \Delta_{1,n-1}$ with $m_{n-1,1} \circ \Delta_{n-1,1}$ (representing a

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different way to remove and reattach a vertex): given a parameter $q \in [0,1]$, the *top-or-bottom-to-random* operator is

$$\mathrm{T/B2R}_n(q) := \frac{q}{n} m_{1,n-1} \circ \Delta_{1,n-1} + \frac{1-q}{n} m_{n-1,1} \circ \Delta_{n-1,1}.$$

At each step of the associated chain, a coin is flipped: with probability q it lands heads and we carry out $m_{1,n-1} \circ \Delta_{1,n-1}$, with complimentary probability $1-q$ we carry out $m_{n-1,1} \circ \Delta_{n-1,1}$. However, few examples of such interpolated chains were analysed, since the rigidity of pairing $m_{1,n-1}$ with $\Delta_{1,n-1}$ only and $m_{n-1,1}$ with $\Delta_{n-1,1}$ can be awkward for certain algebras.

In the present work, we allow more flexible combinations of coproduct and product operators by introducing a permutation of tensorands in between the breaking and combining step: for example, if $\tau : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is the linear map that exchanges the tensorands, then we may consider

$$m_{1,n-1} \circ \tau \circ \Delta_{n-1,1} : \mathcal{H}_n \to \mathcal{H}_{n-1} \otimes \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_{n-1} \to \mathcal{H}_n,$$

or the more complicated

$$\mathrm{LeftRightT/B2R}_n(q,q') := \left(q' m_{1,n-1} + (1-q') m_{n-1,1} \circ \tau\right) \circ \left(\frac{q}{n} \Delta_{1,n-1} + \frac{1-q}{n} \tau \circ \Delta_{n-1,1}\right)$$

for parameters $q, q' \in [0,1]$. More generally, we define a permuted descent operator to be linear combinations of composites of the form $m_{\sigma(D)} \circ \sigma \circ \Delta_D$ where D is a composition of l parts and σ is a permutation of tensorands on $\mathcal{H}^{\otimes l}$.

Our main algebraic result, Theorem 2.7, is that the eigenvalues and multiplicities of $m_{\sigma(D)} \circ \sigma \circ \Delta_D$ is the same as that of its unpermuted counterpart $m_D \circ \Delta_D$, and similarly for linear combinations. We also prove in Theorem 3.3 many eigenvector formulae for the $\mathrm{LeftRightT/B2R}_n(q,q')$ operator; in particular, for a commutative $\mathcal{H}$ where $\mathrm{LeftRightT/B2R}_n(q,q')$ agrees with $\mathrm{T/B2R}_n(q)$, the new formulae here provide more eigenvectors than in [Pan18].

The remainder of the paper concerns Markov chains associated to permuted descent operators. We have two main results regarding arbitrary operators on arbitrary algebras: Theorem 4.11 shows that, under mild positivity conditions, the matrix for a permuted descent operator on an arbitrary graded Hopf algebra can be rescaled into a matrix of transition probabilities, using the same formula as for unpermuted descent operators. And Theorem 4.12 shows that the stationary distributions of these chains are precisely those of their unpermuted counterpart.

We then specialise our framework to two specific chains. First, the chain driven by $\mathrm{RightT/B2R}_n(q) := \mathrm{LeftRightT/B2R}_n(q,0)$ on the Malvenuto-Reutenauer algebra of permutations model the relative times that tasks spend on a to-do list, where on each day there is probability q of completing the top task on the list, and complementary probability $1-q$ of abandoning the bottom task. The $q=1$ version of this chain is similar to that in [Pan18, Sec. 6]. [....].

Secondly, the chain driven by $\mathrm{LeftT/B2R}_n(q) := \mathrm{LeftRightT/B2R}_n(q,1)$ on the Loday-Ronco Hopf algebra of binary trees removes either the leftmost or rightmost vertex, then re-attaches it to a uniformly-chosen leaf. Its stationary distribution is given by the proportion of increasing trees. By considering "path trees", we express one eigenvector for each eigenvalue in terms of descendant and ancestor relationships. For the case $q=1$, we additionally obtain a recursive expression for the subdominant eigenvector, which gives a upper bound on the probability of the right subtree being empty.

This paper is organised as follows: Section 2 gives the Hopf-algebraic background, defines the permuted descent operators, and proves the eigenvalue result. Section 3 gives the eigenvalues and eigenvectors of $\mathrm{LeftRightT/B2R}_n$ and related operators. This ends the algebraic part. Section 4 constructs Markov chains from arbitrary permuted descent operators on arbitrary algebras, and proves their stationary distribution. Sections ... and ... examine the chains on permutations and trees respectively.

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2 Permuted Descent Operators

Section 2.1 recalls the notation of Hopf algebra operations, that are necessary to define the permuted descent operators in Section 2.3. Section 2.4 then gives the eigenvalues and multiplicities of an arbitrary permuted descent operator.

2.1 Hopf algebra operations

Recall that a Hopf algebra $\mathcal{H}$ is a vector space equipped with a product $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ and a coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, satisfying the compatibility axiom $\Delta(xy) = \Delta(x)\Delta(y)$ (factor-wise multiplication on the right) and some other axioms, see [Swe69]. This work concerns graded Hopf algebras $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$, with a distinguished graded basis $\mathcal{B} = \amalg_{n\in\mathbb{N}}\mathcal{B}_n$. Often, $\mathcal{B}_n$ corresponds to the set of combinatorial objects of size n, e.g. trees with n internal vertices, or permutations of n objects. (We assume $\mathcal{B}_n$ is finite, i.e. $\mathcal{H}_n$ is finite-dimensional.) Then the product and coproduct operations model respectively how two objects combine into a larger one, and how one object breaks into two parts. We require also that $\mathcal{H}$ is *connected*, meaning $\dim \mathcal{H}_0 = 1$, i.e. there is a unique "empty object" of size 0. See [GR14] for more details concerning Hopf algebras in combinatorics.

Example 2.1. The Malvenuto-Reutenauer algebra **FQSym** [MR95, Sec. 3] has a basis $\mathcal{B}_n = \{\text{permutations of } n\}$. (Strictly speaking, we work with the fundamental basis F_u.) We use one-line notation to denote the permutations, adding parentheses to distinguish them from integers, e.g. $\mathcal{B}_3 = \{(123),\ (132),\ (213),\ (231),\ (312),\ (321)\}$.

The product of two permutations is their *shifted-shuffle*: if $u \in \mathcal{B}_i$ and $v \in \mathcal{B}_j$, then add i to each number of v, then sum over all "shuffles" of the result with u. In other words, the product $m(u \otimes v) = uv$ is the sum over all permutations where $1, 2, \dots, i$ are in the same order as in u, and $i+1, i+2, \dots, i+j$ are in the same relative order as in v. For example:

$$\begin{aligned} m((312) \otimes (21)) = (312)(21) &= (312) \sqcup\!\sqcup (54) \\ &= (31254) + (31524) + (31542) + (35124) + (35142) \\ &\quad + (35412) + (53124) + (53142) + (53412) + (54312). \end{aligned}$$

The coproduct of a permutation is "deconcatenate and standardise":

$$\Delta(u_1 \dots u_n) = \sum_{i=0}^{n} \operatorname{std}(u_1 \dots u_i) \otimes \operatorname{std}(u_{i+1} \dots u_n),$$

where the standardisation map std converts an arbitrary string of distinct letters into a permutation by preserving the relative order of the letters. For example:

$$\begin{aligned}
&\Delta(4132)\\
&=()\otimes(4132)+\mathrm{std}(4)\otimes\mathrm{std}(132)+\mathrm{std}(41)\otimes\mathrm{std}(32)+\mathrm{std}(413)\otimes\mathrm{std}(2)+(4132)\otimes()\\
&=()\otimes(4132)+(1)\otimes(132)+(21)\otimes(21)+(312)\otimes(1)+(4132)\otimes().
\end{aligned}\tag{2.2}$$

In any graded connected Hopf algebra $\mathcal{H}$, the product and coproduct respect the grading in a way that allows the following refinements (as in [Pan18]): $m_{i,j}:\mathcal{H}_i\otimes\mathcal{H}_j\to\mathcal{H}_{i+j}$ is the restriction of m to the domain $\mathcal{H}_i\otimes\mathcal{H}_j$, and $\Delta_{i,j}:\mathcal{H}_{i+j}\to\mathcal{H}_i\otimes\mathcal{H}_j$ is Δ followed by a projection $\mathcal{H}\otimes\mathcal{H}\to\mathcal{H}_i\otimes\mathcal{H}_j$. For example, (2.2) shows that $\Delta_{1,3}(4132)=(1)\otimes(132)$ and $\Delta_{2,2}(4132)=(21)\otimes(21)$. Note that, if $x\in\mathcal{H}_n$, then $\Delta(x)=\sum_{i=0}^{n}\Delta_{i,n-i}(x)$. The compatibilty axiom between product and coproduct then translates to

$$\Delta_{i,j}(xy)=\sum_{k=0}^{\deg x}\Delta_{k,\deg x-k}(x)\Delta_{i-k,\deg y-i+k}(y).\tag{2.3}$$

To extend this notation to the product of more than two factors: define a sequence of non-negative integers $D=(d_1,\dots,d_{l(D)})$ to be a *weak-composition of* n if $d_1+\dots+d_{l(D)}=n$. The numbers d_i are the *parts* of D, and $l(D)$ is the *length* of D. Write 1^i to indicate i consecutive parts equal to 1. Then, for each weak-composition D of n, let $m_D:\mathcal{H}_{d_1}\otimes\dots\otimes\mathcal{H}_{d_l}\to\mathcal{H}_n$ denote the iterated product. m_D describes the combining of l combinatorial objects, of sizes $d_1,d_2,\dots,d_l$ respectively, into one object. For example, in **FQSym**

$$\begin{aligned}
m_{(1^2,4,2)}((1)\otimes(1)\otimes(4132)\otimes(12))&=m_{(1,1,4,2)}((1)\otimes(1)\otimes(4132)\otimes(12))\\
&=(1)\sqcup\!\sqcup(2)\sqcup\!\sqcup(6354)\sqcup\!\sqcup(78).
\end{aligned}$$

Similarly $\Delta_D:\mathcal{H}_n\to\mathcal{H}_{d_1}\otimes\dots\otimes\mathcal{H}_{d_l}$ describes the breaking of one object into l pieces, of sizes $d_1,d_2,\dots,d_l$ respectively. We first give an example in **FQSym**, and then the precise definition:

$$\Delta_{(4,2,1)}(4721365)=\mathrm{std}(4721)\otimes\mathrm{std}(36)\otimes\mathrm{std}(5)=3421\otimes12\otimes1.$$

More rigorously, define $\Delta^{[l]}:\mathcal{H}\to\mathcal{H}^{[l]}$ by $\Delta^{[l]}:=(\Delta\otimes\mathrm{id}^{\otimes l-2})\circ\dots\circ(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\circ(\Delta\otimes\mathrm{id})\circ\Delta$ (so $\Delta^{[2]}=\Delta$), and $\mathrm{Proj}_D:\mathcal{H}^{\otimes l(D)}\to\mathcal{H}_{d_1}\otimes\dots\otimes\mathcal{H}_{d_{l(D)}}$ to be the projecction. Then $\Delta_D=\mathrm{Proj}_D\circ\Delta^{[l(D)]}$.

It will be useful to write $\binom{n}{D}$ for the multinomial coefficient $\binom{n}{d_1\dots d_l}$.

2.2 Dual Hopf algebra

Analysing a Markov chain on a Hopf algebra $\mathcal{H}$ requires working in its *graded dual* Hopf algebra $\mathcal{H}^*$. Here $\mathcal{H}^*=\bigoplus_{n\in\mathbb{N}}\mathcal{H}_n^*$, where $\mathcal{H}_n^*$ is the dual vector space to $\mathcal{H}_n$, i.e. the space of linear functions on $\mathcal{H}_n$. For $f\in\mathcal{H}_n^*$ and $x\in\mathcal{H}_n$, use the inner product notation $\langle f,x\rangle$ to denote the evaluation of f at x. Given a graded basis $\mathcal{B}=\amalg_{n\in\mathbb{N}}\mathcal{B}_n$ of $\mathcal{H}$, its dual basis, of $\mathcal{H}^*$ is $\mathcal{B}^*=\amalg_n\mathcal{B}_n^*$, where $\mathcal{B}_n^*=\{x^*|x\in\mathcal{B}_n\}$ satisfies $\langle x^*,x\rangle=1$ and $\langle x^*,y\rangle=0$ for $y\neq x\in\mathcal{B}_n$.

The product and coproduct on $\mathcal{H}^*$ (written using the same symbols m and Δ) are the duals, respectively, of the coproduct and product on $\mathcal{H}$. More explicitly, given $f,g\in\mathcal{H}^*$ and $x,y\in\mathcal{H}$,

$$\langle m(f\otimes g),x\rangle=\langle f\otimes g,\Delta(x)\rangle,\quad\langle\Delta(f),x\otimes y\rangle=\langle f,xy\rangle,\tag{2.4}$$

where $\langle f \otimes g, x \otimes y\rangle$ means $\langle f, x\rangle\langle g, y\rangle$.

It follows that, for all weak-compositions D, the refined operations m_D and Δ_D on $\mathcal{H}^*$ are the duals of Δ_D and m_D on $\mathcal{H}$.

Example 2.5. Recall the Hopf structure on **FQSym** from Example 2.1. The coproduct on **FQSym*** is the dual of the shifted-shuffle product of **FQSym**. Note that each permutation that appears in a shifted-shuffle product has coefficient 1. So (2.4) means that $\Delta_{i,j}(w^*)$ is a sum of $u^* \otimes v^*$, over all $u \in \mathcal{B}_i$, $v \in \mathcal{B}_j$ such that w appears in the product uv. In fact, there is only one such term $u^* \otimes v^*$: $u \in \mathcal{B}_i$ consists of $1, 2, \ldots, i$ in the same order as they appear in w, and $v \in \mathcal{B}_j$ consists of $1, 2, \ldots, j$ in the same order as $i+1, \ldots, i+j$ appear in w. For example, $\Delta_{4,3}((4721365)^*) = (4213)^* \otimes (321)^*$.

Recall that the primary basis of **FQSym** in this work is $\{F_u\}$. Its dual basis in **FQSym*** is usually denoted $\{G_u\}$. As explained in [DHT02, Sec. 3.1-3.2], **FQSym** is in fact self-dual, with $G_u = F_{u^{-1}}$.

2.3 Permuted descent operators

Recall from [Pan18] that the most basic descent operators have the form $m_D \circ \Delta_D$, which models breaking an object into $l(D)$ pieces of sizes $d_1, \ldots, d_l$ respectively and then reassembling the pieces into a single object. To generalise this idea to a permuted descent operator, we insert a step in between these two: after the breaking, the pieces may be permuted into a different order before the reassembling step. To describe the permutation of these pieces rigorously, define an action of the symmetric group $\mathfrak{S}_l$ on weak-compositions of length l: for $D = (d_1, \ldots, d_l)$ and $\sigma \in \mathfrak{S}_l$, define $\sigma(D) = (d_{\sigma^{-1}(1)}, \ldots, d_{\sigma^{-1}(l)})$. So σ moves the ith part into the $\sigma(i)$th position. For example, if $\sigma \in \mathfrak{S}_3$ is the 3-cycle 231, then $\sigma((4,2,1)) = (1,4,2)$.

Definition 2.6. A *permuted descent operator* is a linear combination of composites of the form

$$m_{\sigma(D)} \circ \sigma \circ \Delta_D,$$

where $D = (d_1, \ldots, d_l)$ is a weak-composition and $\sigma \in \mathfrak{S}_l$ is a permutation. Here, $\Delta_D : \mathcal{H}_n \to \mathcal{H}_{d_1} \otimes \cdots \otimes \mathcal{H}_{d_l}$ is an iterated coproduct followed by a projection to the graded subspace, as described in Section 2.1, and $\sigma : \mathcal{H}_{d_1} \otimes \cdots \otimes \mathcal{H}_{d_l} \to \mathcal{H}_{d_{\sigma^{-1}(1)}} \otimes \cdots \otimes \mathcal{H}_{d_{\sigma^{-1}(l)}}$ is a permutation of the tensorands, moving the ith tensorand to the $\sigma(i)$th position. In Sweedler-like notation, if $\sum_{(x)} x_{(1)}x_{(2)} \ldots x_{(l)} = \Delta_D(x)$ (as opposed to the full $\Delta^{[l]}(x)$ as usual),

$$m_{\sigma(D)} \circ \sigma \circ \Delta_D(x) = \sum_{(x)} x_{(\sigma^{-1}(1))} x_{(\sigma^{-1}(2))} \ldots x_{(\sigma^{-1}(l))}.$$

For example, if $\sigma \in \mathfrak{S}_3$ is the 3-cycle 231, then

$$m_{(1,4,2)} \circ \sigma \circ \Delta_{(4,2,1)} : \mathcal{H}_7 \to \mathcal{H}_4 \otimes \mathcal{H}_2 \otimes \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_4 \otimes \mathcal{H}_2 \to \mathcal{H}_7.$$

Permuted descent operators previously appeared briefly in [Gri16, Qu. 7.3] . We recover the descent operators of [Pat94; Pan18] in the case where σ is the identity permutation. Note also that, if $\mathcal{H}$ is commutative (resp. cocommutative), then, for all $\sigma \in \mathfrak{S}_l$, it holds that $m_{\sigma(D)} \circ \sigma \circ \Delta_D = m_D \circ \Delta_D$ (resp. $m_{\sigma(D)} \circ \sigma \circ \Delta_D = m_{\sigma(D)} \circ \Delta_{\sigma(D)}$). So, in both cases, the permuted descent operators are simply descent operators.

Unlike the case of descent operators, the linear algebraic dual map to a permuted descent operator is not the same map on the dual Hopf algebra. Instead,

$$\left(m_{\sigma(D)} \circ \sigma \circ \Delta_D\right)^* = m_D \circ \sigma^{-1} \circ \Delta_{\sigma(D)}.$$

This is important for calculating right-eigenfunctions of the Markov chains in Sections 5 and 6.

Remark. The permuted descent operators are one example of the more general notion of *permuted convolution product* (non-standard nomenclature and notation):

$$*_\sigma(f_1,\dots,f_l) := m^{[l]} \circ \sigma \circ (f_1 \otimes \cdots \otimes f_l) \circ \Delta^{[l]} = \sum_{(x)} f_{\sigma^{-1}(1)}(x_{(\sigma^{-1}(1))}) \dots f_{\sigma^{-1}(l)}(x_{(\sigma^{-1}(l))}),$$

in the case where $f_i = \mathrm{Proj}_{(d_i)}$, the projection to the subspace of degree d_i.

2.4 The spectrum of permuted descent operators

This section proves the following result:

Theorem 2.7. *Fix $n \in \mathbb{N}$. Let $a_i \geq 0$ be constants, D_i be weak-compositions of n having length l_i respectively, and $\sigma_i \in \mathfrak{S}_{l_i}$ be permutations. Then the spectrum of the permuted descent operator $\sum_i a_i m_{D_i} \circ \sigma_i \circ \Delta_{D_i} : \mathcal{H}_n \to \mathcal{H}_n$ is equal to that of its unpermuted counterpart $\sum_i a_i m_{D_i} \circ \Delta_{D_i}$.*

Since the formula for such eigenvalues is complicated, we refer the reader to [Pan18, Sec. 3.2] or Theorem 4.13 below for their exact expressions.

Proof. The key, as in [AL15], is to work in $\mathrm{gr}(\mathcal{H})$, the associated graded Hopf algebra with respect to the coradical filtration of $\mathcal{H}$. The operation $\mathcal{H} \to \mathrm{gr}(\mathcal{H})$ is functorial, so

$$\begin{aligned} \mathrm{gr}(m_{\sigma(D)} \circ \sigma \circ \Delta_D : \mathcal{H} \to \mathcal{H}) &= \mathrm{gr}(m_{\sigma(D)}) \circ \mathrm{gr}(\sigma) \circ \mathrm{gr}(\Delta_D) : \mathrm{gr}(\mathcal{H}) \to \mathrm{gr}(\mathcal{H}) \\ &= m_{\sigma(D)} \circ \sigma \circ \Delta_D : \mathrm{gr}(\mathcal{H}) \to \mathrm{gr}(\mathcal{H}), \end{aligned}$$

by definition of the Hopf structure on $\mathrm{gr}(\mathcal{H})$. Moreover, the spectrum of a coradical-filtration-preserving map $\theta : \mathcal{H} \to \mathcal{H}$ is equal to that of $\mathrm{gr}(\theta)$. Hence, using the commutativity of $\mathrm{gr}(\mathcal{H})$ [Swe69, Th. 11.2.5.a; AS05a, Prop. 1.6] in the second equality:

$$\begin{aligned} \text{spectrum of } \sum_i a_i m_{D_i} \circ \sigma_i \circ \Delta_{D_i} : \mathcal{H} \to \mathcal{H} &= \text{spectrum of } \sum_i a_i m_{D_i} \circ \sigma_i \circ \Delta_{D_i} : \mathrm{gr}(\mathcal{H}) \to \mathrm{gr}(\mathcal{H}) \\ &= \text{spectrum of } \sum_i a_i m_{D_i} \circ \Delta_{D_i} : \mathrm{gr}(\mathcal{H}) \to \mathrm{gr}(\mathcal{H}) \\ &= \text{spectrum of } \sum_i a_i m_{D_i} \circ \Delta_{D_i} : \mathcal{H} \to \mathcal{H}. \end{aligned}$$

(The last line is the argument of the first line backwards.) □

3 Top-or-bottom-to-random operators

The main result of this section is Theorem 3.3, an eigenvector formulae for some simple permuted descent operators. As in [Pan18, Sec. 4], all the weak-compositions involved in these simple operators have all except one part equal to 1. Thus they model removing parts of size 1 from the combinatorial object – such removals are easier to describe than general breaking into larger pieces. Because of a card-shuffling connection, these operators are informally known as top-to-bottom-to-random operators, or T/B2R for short.

3.1 Definitions

The operators of interest are

$$\begin{aligned}
\mathrm{LeftT/B2R}_n(q) &:= \frac{q}{n} m_{1,n-1} \circ \Delta_{1,n-1} + \frac{1-q}{n} m_{1,n-1} \circ \tau \circ \Delta_{n-1,1} \\
&= m_{1,n-1} \circ \left(\frac{q}{n} \Delta_{1,n-1} + \frac{1-q}{n} \tau \circ \Delta_{n-1,1} \right); \\
\mathrm{RightT/B2R}_n(q) &:= \frac{q}{n} m_{n-1,1} \circ \tau \circ \Delta_{1,n-1} + \frac{1-q}{n} m_{n-1} \circ \Delta_{n-1,1} \\
&= m_{n-1,1} \circ \left(\frac{q}{n} \tau \circ \Delta_{1,n-1} + \frac{1-q}{n} \Delta_{n-1,1} \right); \\
\mathrm{LeftRightT/B2R}_n(q,q') &:= q' \, \mathrm{LeftT/B2R}_n(q) + (1-q') \, \mathrm{RightT/B2R}_n(q) \\
&= \left(q' m_{1,n-1} + (1-q') m_{n-1,1} \circ \tau \right) \circ \left(\frac{q}{n} \Delta_{1,n-1} + \frac{1-q}{n} \tau \circ \Delta_{n-1,1} \right).
\end{aligned}$$

where $\tau \in \mathfrak{S}_2$ is the transposition. Note that these all map from $\mathcal{H}_n$ to $\mathcal{H}_n$. As Theorem 4.11 will show, a Markov chain driven by $\mathrm{LeftT/B2R}_n(q)$ has the following general form: toss a q-biased coin; if heads, remove a piece of size one from the left; if tails, remove a piece of size one from the right; in either case, re-attach the removed piece on the left. And $\mathrm{RightT/B2R}_n(q)$ models a similar process, except the final reattachment happens on the right instead of the left. For $\mathrm{LeftRightT/B2R}_n(q,q')$, a second coin with parameter q' is tossed to determine whether this final attachment is to the left or right.

Note that

$$\mathrm{LeftT/B2R}_n(q) = \mathrm{LeftRightT/B2R}_n(q,1), \quad \mathrm{RightT/B2R}_n(q) = \mathrm{LeftRightT/B2R}_n(q,0). \tag{3.1}$$

To find right-eigenfunctions for the Markov chains driven by the above operators, it is necessary to study the dual maps, which are:

$$\begin{aligned}
\mathrm{LeftT/B2R}^*_n(q) &:= \frac{q}{n} m_{1,n-1} \circ \Delta_{1,n-1} + \frac{1-q}{n} m_{n-1,1} \circ \tau \circ \Delta_{1,n-1} \\
&= \left(\frac{q}{n} m_{1,n-1} + \frac{1-q}{n} m_{n-1,1} \circ \tau \right) \circ \Delta_{1,n-1}; \\
\mathrm{RightT/B2R}^*_n(q) &:= \frac{q}{n} m_{1,n-1} \circ \tau \circ \Delta_{n-1,1} + \frac{1-q}{n} m_{1,n-1} \circ \Delta_{n-1,1}, \\
&= \left(\frac{q}{n} m_{1,n-1} \circ \tau + \frac{1-q}{n} m_{n-1,1} \right) \circ \Delta_{n-1,1}; \\
\mathrm{LeftRightT/B2R}^*_n(q,q') &:= q' \, \mathrm{LeftT/B2R}^*_n(q) + (1-q') \, \mathrm{RightT/B2R}^*_n(q) \\
&= \left(\frac{q}{n} m_{1,n-1} + \frac{1-q}{n} m_{n-1,1} \circ \tau \right) \circ \left(q' \Delta_{1,n-1} + (1-q') \tau \circ \Delta_{n-1,1} \right) \\
&= \mathrm{LeftRightT/B2R}_n(q',q).
\end{aligned}$$

3.2 Eigenvalues

Theorem 2.7 above asserts that the spectrums of $\mathrm{LeftT/B2R}_n(q)$, $\mathrm{RightT/B2R}_n(q)$ and $\mathrm{LeftRightT/B2R}_n(q)$ are equal to their unpermuted counterpart, which is

$$\mathrm{T/B2R}_n(q) := \frac{q}{n} m_{1,n-1} \circ \Delta_{1,n-1} + \frac{1-q}{n} m_{n-1,1} \circ \Delta_{n-1,1}$$

(same unpermuted version for all three operators). And the other three operators, being duals, also have this same spectrum. This spectrum of $\mathrm{T/B2R}_n(q)$ is given in [Pan18, Th. 4.4.i].

Theorem 3.2. *Let $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$ be a graded connected Hopf algebra over $\mathbb{R}$ with each $\mathcal{H}_n$ finite-dimensional. Then the eigenvalues of all six* T/B2R *operators on $\mathcal{H}_n$ are $\frac{j}{n}$ for $j \in [0, n-2] \cup \{n\}$, with multiplicity equal to the coefficient of $x^{n-j}y^j$ in $\left(\frac{1-x}{1-y}\right)^{\dim \mathcal{H}_1} \sum_n \dim \mathcal{H}_n x^n$.* □

3.3 Eigenvectors

Below is the permuted version of [Pan18, Th. 4.4.ii].

Theorem 3.3. *Let $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$ be a graded connected Hopf algebra over $\mathbb{R}$ with each $\mathcal{H}_n$ finite-dimensional. Fix $j \in \{0, 1, \ldots, n-2\}$, and $c_1, \ldots, c_j \in \mathcal{H}_1$ (not necessarily distinct).*

i) For any $p \in \ker\left(\frac{q}{n}\Delta_{1,n-j-1} + \frac{1-q}{n}\tau \circ \Delta_{n-j-1,1}\right) \subseteq \mathcal{H}_{n-j}$,

$$\sum_{\sigma\in\mathfrak{S}_j} c_{\sigma(1)} \ldots c_{\sigma(j)} p \text{ is a } \frac{j}{n}\text{-eigenvector for } \mathrm{LeftT/B2R}_n(q);$$

$$\sum_{\sigma\in\mathfrak{S}_j} p c_{\sigma(1)} \ldots c_{\sigma(j)} \text{ is a } \frac{j}{n}\text{-eigenvector for } \mathrm{RightT/B2R}_n(q);$$

$$\sum_{i=0}^{j} \sum_{\sigma\in\mathfrak{S}_j} \binom{j}{i} q'^i (1-q')^{j-i} c_{\sigma(1)} \ldots c_{\sigma(i)} p c_{\sigma(i+1)} \ldots c_{\sigma(j)} \text{ is a } \frac{j}{n}\text{-eigenvector for } \mathrm{LeftRightT/B2R}_n(q, q').$$

ii) For any $p \in \ker(\Delta_{1,n-j-1}) \subseteq \mathcal{H}_{n-j}$,

$$\sum_{i=0}^{j} \sum_{\sigma\in\mathfrak{S}_j} \binom{j}{i} q^i (1-q)^{j-i} c_{\sigma(1)} \ldots c_{\sigma(i)} p c_{\sigma(i+1)} \ldots c_{\sigma(j)} \text{ is a } \frac{j}{n}\text{-eigenvector for } \mathrm{LeftT/B2R}^*_n(q).$$

iii) For any $p \in \ker(\Delta_{n-j-1,1}) \subseteq \mathcal{H}_{n-j}$,

$$\sum_{i=0}^{j} \sum_{\sigma\in\mathfrak{S}_j} \binom{j}{i} q^i (1-q)^{j-i} c_{\sigma(1)} \ldots c_{\sigma(i)} p c_{\sigma(i+1)} \ldots c_{\sigma(j)} \text{ is a } \frac{j}{n}\text{-eigenvector for } \mathrm{RightT/B2R}^*_n(q).$$

Remark. If $\mathcal{H}$ is commutative, so $\mathrm{LeftT/B2R}_n(q)$, $\mathrm{RightT/B2R}_n(q)$, $\mathrm{LeftRightT/B2R}_n(q)$ all reduce to $\mathrm{T/B2R}_n(q)$, then the first point above actually provides more eigenvectors than [Pan18, Th. 4.4.ii].

Proof. We prove the case of $\mathrm{LeftRightT/B2R}_n(q, q')$ only, as all other statements are specialisations of this case by (3.1).

Fix $c_1, \ldots, c_j \in \mathcal{H}_1$ and $p \in \mathcal{H}_{n-j}$. Write $\Delta_{1,n-j-1}(p) = \sum_{(p)} p_{(1)} \otimes p_{(2)}$ and $\Delta_{n-j-1,1}(p) = \sum_{(p')} p'_{(2)} \otimes p'_{(1)}$ in Sweedler-like notation. Because of the compatibility axiom (2.3),

$$\begin{aligned}\Delta_{1,n-1}\left(c_1 \ldots c_i p c_{i+1} \ldots c_j\right) = &\sum_{k=1}^{i} c_k \otimes c_1 \ldots \hat{c_k} \ldots c_i p c_{i+1} \ldots c_j \\ &+ \sum_{(p)} p_{(1)} \otimes c_1 \ldots c_i p_{(2)} c_{i+1} \ldots c_j \\ &+ \sum_{k=i+1}^{j} c_k \otimes c_1 \ldots c_i p c_{i+1} \ldots \hat{c_k} \ldots c_j,\end{aligned}$$

where $\hat{c_k}$ denotes that c_k is omitted from the product. The expression for $\Delta_{n-1,1}\left(c_1 \ldots c_i p c_{i+1} \ldots c_j\right)$ is similar. Hence

$$\begin{aligned}&\left(\frac{q}{n}\Delta_{1,n-1} + \frac{1-q}{n}\tau \circ \Delta_{n-1,1}\right)\left(c_1 \ldots c_i p c_{i+1} \ldots c_j\right) \\ =&\frac{1}{n}\sum_{k=1}^{i} c_k \otimes c_1 \ldots \hat{c_k} \ldots c_i p c_{i+1} \ldots c_j \\ &+ \frac{1}{n}\sum_{(p)} p_{(1)} \otimes c_1 \ldots c_i p_{(2)} c_{i+1} \ldots c_j + \frac{1}{n}\sum_{(p')} p'_{(1)} \otimes c_1 \ldots c_i p'_{(2)} c_{i+1} \ldots c_j \\ &+ \frac{1}{n}\sum_{k=i+1}^{j} c_k \otimes c_1 \ldots c_i p c_{i+1} \ldots \hat{c_k} \ldots c_j.\end{aligned}$$

Now, if $p \in \ker\left(\frac{q}{n}\Delta_{1,n-j-1} + \frac{1-q}{n}\tau \circ \Delta_{n-j-1,1}\right)$, then the middle line above vanishes, and

$$\begin{aligned}&\left(q' m_{1,n-1} + (1-q') m_{n-1,1} \circ \tau\right) \circ \left(\frac{q}{n}\Delta_{1,n-1} + \frac{1-q}{n}\tau \circ \Delta_{n-1,1}\right)\left(c_1 \ldots c_i p c_{i+1} \ldots c_j\right) \\ =&\frac{q'}{n}\sum_{k=1}^{i} c_k c_1 \ldots \hat{c_k} \ldots c_i p c_{i+1} \ldots c_j + \frac{1-q'}{n}\sum_{k=1}^{i} c_1 \ldots \hat{c_k} \ldots c_i p c_{i+1} \ldots c_j c_k \\ &+ \frac{q'}{n}\sum_{k=i+1}^{j} c_k c_1 \ldots c_i p c_{i+1} \ldots \hat{c_k} \ldots c_j + \frac{1-q'}{n}\sum_{k=i+1}^{j} c_1 \ldots c_i p c_{i+1} \ldots \hat{c_k} \ldots c_j c_k.\end{aligned}$$

Next, symmetrise over all orders to take the product of the c_i. To simplify notation, let $\omega_i := \sum_{\sigma \in \mathfrak{S}_j} c_{\sigma(1)} \ldots c_{\sigma(i)} p c_{\sigma(i+1)} \ldots c_{\sigma(j)}$. Then

$$\mathrm{LeftRightT/B2R}_n(q, q')(\omega_i) = \frac{q'}{n} i\omega_i + \frac{1-q'}{n} i\omega_{i-1} + \frac{q'}{n}(j-i)\omega_{i+1} + \frac{1-q'}{n}(j-i)\omega_i.$$

So

$$
\begin{aligned}
&\mathrm{LeftRightT/B2R}_n(q,q')\left(\sum_{i=0}^{j}\binom{j}{i}q'^i(1-q')^{j-i}\omega_i\right)\\
=&\frac{j}{n}\sum_{i=1}^{j}\binom{j-1}{i-1}q'^{i+1}(1-q')^{j-i}\omega_i+\frac{j}{n}\sum_{i=1}^{j}\binom{j-1}{i-1}q'^i(1-q')^{j-(i-1)}\omega_{i-1}\\
&\quad+\frac{j}{n}\sum_{i=0}^{j-1}\binom{j}{i}q'^{i+1}(1-q')^{j-i}\omega_{i+1}+\frac{j}{n}\sum_{i=0}^{j-1}\binom{j-1}{i}q'^i(1-q')^{j-i+1}\omega_i\\
=&\frac{j}{n}\sum_{i=1}^{j}\binom{j-1}{i-1}q'^{i+1}(1-q')^{j-i}\omega_i+\frac{j}{n}\sum_{i=0}^{j-1}\binom{j-1}{i}q'^{i+1}(1-q')^{j-i}\omega_i\\
&\quad+\frac{j}{n}\sum_{i=1}^{j}\binom{j}{i}q'^i(1-q')^{j-(i-1)}\omega_i+\frac{j}{n}\sum_{i=0}^{j-1}\binom{j-1}{i}q'^i(1-q')^{j-i+1}\omega_i\\
=&\frac{j}{n}\sum_{i=1}^{j}\binom{j}{i}q'^{i+1}(1-q')^{j-i}\omega_i+\frac{j}{n}\sum_{i=1}^{j}\binom{j}{i}q'^i(1-q')^{j-i+1}\omega_i\\
=&\frac{j}{n}\sum_{i=1}^{j}\binom{j}{i}q'^i(1-q')^{j-i}\omega_i.
\end{aligned}
$$

(This last step can be replaced by a Markov chain argument involving detailed balance, as in [Pan18, Th. 4.4.ii].) □

4 Markov Chains Driven by Permuted Descent Operators

This section explains how to associate a Markov chain to the action of any permuted descent operator on a graded subspace of a combinatorial Hopf algebra, and derives some basic properties of this chain, most notably its stationary distribution.

4.1 Background on Markov chains

We give the essential definitions here, following the textbook [LPW09]. A (discrete-time) Markov chain is a sequence $\{X_t\}$ of random variables modelling the evolution of an object through discrete units of time. The possible values of X_t are called *states*, and we assume they come from a finite set Ω known as the *state space*. The key feature of a Markov chain is past-independence, i.e. for all states $x_0, x_1, \dots, x_t \in \Omega$

$$\mathrm{Prob}(X_t = x_t|X_0 = x_0, X_1 = x_1, \dots, X_{t-1} = x_{t-1}) = \mathrm{Prob}(X_t = x_t|X_{t-1} = x_{t-1}).$$

(Here, $\mathrm{Prob}(X|Y)$ denotes the probability of event X given event Y.) Thus all information about a Markov chain is summarised in its $|\Omega|$-by-$|\Omega|$ *transition matrix*

$$K(x, y) := \mathrm{Prob}(X_t = y|X_{t-1} = x).$$

Note that we follow the probabilists' convention, that the rows x index the source state, and the columns y index the destination state. (Some combinatorialists, notably [Ayy+15], take the transposed convention.) It is easy to show that powers of the transition matrix give multistep transition probabilities: $K^t(x,y) = \text{Prob}(X_t = y|X_0 = x)$. Note also that a matrix K is a transition matrix for some Markov chain if and only if $K(x,y) \geq 0$ for all $x, y \in \Omega$, and $\sum_{y\in\Omega} K(x,y) = 1$ for each $x \in \Omega$.

A non-negative function $\pi : \Omega \to \mathbb{R}$ is a *stationary distribution* of the Markov chain $\{X_t\}$ if $\sum_{x\in\Omega} \pi(x)K(x,y) = \pi(y)$ for each $y \in \Omega$. Stationary distributions are of interest because, under mild conditions (e.g. if there exists t such that $K^t(x,y) > 0$ whenever $\pi(x), \pi(y) > 0$ [LPW09, Th. 4.9]), they are limiting distributions: $\lim_{t\to\infty} \text{Prob}(X_t = x) = \pi(x)$. The rate of convergence towards the stationary distribution is also important; the sharpest results in this area come from analytic techniques, but, algebraically, eigenfunctions can give some insight.

Definition 4.1. A function $\mathbf{f} : \Omega \to \mathbb{R}$ is a *right-eigenfunction* for the Markov chain $\{X_t\}$, with eigenvalue β, if it satisfies $\sum_{y\in\Omega} K(x,y)\mathbf{f}(y) = \beta\mathbf{f}(x)$ for all $x \in \Omega$, or equivalently

$$K \begin{bmatrix} \mathbf{f}(x_1) \\ \mathbf{f}(x_2) \\ \vdots \\ \mathbf{f}(x_{|\Omega|}) \end{bmatrix} = \beta \begin{bmatrix} \mathbf{f}(x_1) \\ \mathbf{f}(x_2) \\ \vdots \\ \mathbf{f}(x_{|\Omega|}) \end{bmatrix}. \tag{4.2}$$

(Here $x_1, \ldots, x_{|\Omega|}$ are the elements of Ω, listed in the order in which they index the rows and columns of the transition matrix K.)

Our motivation for investigating right-eigenfunctions is the following elementary fact:

Proposition 4.3 (Expectations from right-eigenfunctions)**.** *[DPR14, Sec. 2.1 Use A] If* $\mathbf{f}$ *is a right-eigenfunction of a Markov chain* $\{X_t\}$ *with eigenvalue* β*, and* $\mathbf{f}'$ *is another function on the state space of* $\{X_t\}$ *satisfying* $\mathbf{f}'(x) \leq \mathbf{f}(x)$ *for all states* x*, then the expected value of* $\mathbf{f}$ *after* t *steps of the chain satisfies*

$$\text{Expect}(\mathbf{f}'(X_t)|X_0 = x_0) \leq \text{Expect}(\mathbf{f}(X_t)|X_0 = x_0) = \beta^t\mathbf{f}(x_0).$$

We may interpret this at $\mathbf{f}'(X_t) \to 0$ at the rate given by $O(\beta^t)$ (if $|\beta| < 1$).

Proof.

$$\begin{aligned}
\text{Expect}(\mathbf{f}'(X_t)|X_0 = x_0) &= \sum_y \mathbf{f}'(y)\,\text{Prob}(X_t = y|X_0 = x_0) \\
&\leq \sum_y \mathbf{f}(y)\,\text{Prob}(X_t = y|X_0 = x_0) = \text{Expect}(\mathbf{f}(X_t)|X_0 = x_0) \\
&= \sum_y \mathbf{f}(y)K^t(x_0, y) \\
&= \beta^t\mathbf{f}(x_0).
\end{aligned}$$

(In the last line, we use that an eigenfunction of K is also an eigenfunction of K^t with the powered eigenvalue.) ∎

4.2 Markov chains driven by linear operators

Following [Pan19], a Markov chain $\{X_t\}$ is said to be *driven by* a linear transformation $\mathbf{T} : V \to V$ if its transition matrix K is the transpose of a matrix for $\mathbf{T}$ relative to some basis $\mathcal{B}$ of V. In other words, for $x, y \in \mathcal{B}$, the probability $\text{Prob}(X_t = y|X_{t-1} = x)$, of "moving from x to y", is the coefficient of y in $\mathbf{T}(x)$. Note that the basis $\mathcal{B}$ is the state space of $\{X_t\}$. Note that the same $\mathbf{T}$ may drive different Markov chains by using different bases $\mathcal{B}$; if $\mathcal{B}$ is not clear from the context, we may say "the Markov chain on $\mathcal{B}$ driven by $\mathbf{T}$" to emphasise the choice of basis. The motivation for considering Markov chains driven by linear operators is that their long term behaviour can be inferred from the eigenvectors of $\mathbf{T}$ and of its dual operator $\mathbf{T}^* : V^* \to V^*$, as explained three paragraphs below.

Clearly, the transition matrix conditions $K(x,y) \geq 0$, $\sum_{y\in\Omega} K(x,y) = 1$ mean that not every linear transformation can drive a Markov chain. But this is somewhat less restrictive than it first appears, as the second, "rows sum to 1" condition may be achieved by rescaling the basis according to the Doob transform. (This transform is not required in two specific families of chains in Sections 5 and 6, so the reader may skip the following three paragraphs, and take $\check{K} = K$,$\check{\mathcal{B}} = \mathcal{B}$ and $\eta(x) \equiv 1$ for all $x \in \mathcal{B}$ in the formulas below.) We give a simplified version of the Doob transform below, that will suffice when $\mathbf{T}$ is a permuted descent operator; the general case where η has an eigenvalue different from 1 can be found in the references below (it merely requires rescaling $\mathbf{T}$).

Theorem 4.4 (Doob transform)**.** *[Pan19, Th. 2.3; KSK66, Def. 8.11, 8.12; LPW09, Sec.17.6.1] Let $\mathbf{T} : V \to V$ be a linear transformation, and let K denote the transpose of a matrix for $\mathbf{T}$ relative to some basis $\mathcal{B}$ of V. Suppose $K(x,y) \geq 0$ for all $x, y \in \mathcal{B}$, and that $\eta \in V^*$ is an eigenvector of $\mathbf{T}^* : V^* \to V^*$ with eigenvalue 1, such that $\eta(x) > 0$ for all $x \in \mathcal{B}$. Then*

$$\check{K}(x,y) := \frac{\eta(y)}{\eta(x)} K(x,y)$$

defines a transition matrix. Equivalently, $\check{K}$ is the transpose of the matrix for $\mathbf{T}$ relative to the rescaled basis $\check{\mathcal{B}} := \left\{ \frac{x}{\eta(x)} | x \in \mathcal{B} \right\}$.

Proof. Recall that K is the transpose of the matrix for $\mathbf{T}$ relative $\mathcal{B}$ of V so $\mathbf{T}^*\eta = \eta$ translates to $\sum_{y\in\mathcal{B}} K(x,y)\eta(y) = \eta(x)$, for each $x \in \mathcal{B}$. Since $\eta(x) > 0$ for all x, it is clear that $\check{K}(x,y) \geq 0$. It remains to show that the rows of $\check{K}$ sum to 1:

$$\sum_{y\in\mathcal{B}} \check{K}(x,y) = \frac{\sum_y K(x,y)\eta(y)}{\eta(x)} = \frac{\eta(x)}{\eta(x)} = 1.$$

□

The function $\eta : V \to \mathbb{R}$ above is the *rescaling function*. Different choices of η for the same linear operator $\mathbf{T}$ can lead to different Markov chains, but the notation suppresses the dependence on η because, when $\mathbf{T}$ is a permuted descent operator, there is a canonical choice of η as in (4.8) below.

It follows easily from the definitions that, if a Markov chain is driven by $\mathbf{T}$, then its stationary distributions are the eigenvectors of $\mathbf{T}$ of eigenvalue 1, and its right-eigenfunctions are eigenvectors of the dual operator $\mathbf{T}^*$:

Proposition 4.5 (Eigenfunctions for chains driven by linear transformations)**.** *[Pan14, Prop. 3.2.1; Zho08, Lemma 4.4.1.4; Swa12, Lem. 2.11] Let $\{X_t\}$ be the Markov chain on $\mathcal{B}$ driven by $\mathbf{T} : V \to V$ with rescaling function η.*

i) The stationary distributions of $\{X_t\}$ are precisely the functions of the form

$$\pi(x) := \eta(x) \frac{\xi_x}{\sum_{y\in\mathcal{B}} \xi_y \eta(y)},$$

where $\sum_{x\in\mathcal{B}}\xi_x x \in V$ *is an eigenvector of* $\mathbf{T}$ *with eigenvalue 1, whose coefficients* ξ_x *are all non-negative.*

ii) The right-eigenfunctions $\mathbf{f}:\mathcal{B}\to\mathbb{R}$ *for* $\{X_t\}$ *are in bijection with the eigenvectors* $f\in V^*$ *of the dual map* $\mathbf{T}^*$*, with the same eigenvalue, through the vector space isomorphism*

$$\mathbf{f}(x) := \frac{1}{\eta(x)}\langle f, x\rangle.$$

□

4.3 Markov chains driven by permuted descent operators

The paper [Pan18, Sec. 3.1] set up the theory of Markov chains driven by descent operators, that model the breaking and then reassembling of a combinatorial object. The major results in this theory are the transition matrix, the probabilities of the breaking and reassembling steps separately, its stationary distribution, and eigenvalues and eigenfunctions. The purpose of this section is to give predictable generalisations of all these theorems to permuted descent operators. We follow the notation from [Pan18, Sec. 3.1], namely:

Notation 4.6. Let:

- $\mathcal{H}=\bigoplus_{n\geq 0}\mathcal{H}_n$ be a graded connected Hopf algebra over $\mathbb{R}$, such that each $\mathcal{H}_n$ is finite-dimensional;
- $\mathcal{B}_n$ be a basis of $\mathcal{H}_n$, and $\mathcal{B}=\amalg_{n\geq 0}\mathcal{B}_n$;
- ξ and η denote the product and coproduct *structure constants* with respect to $\mathcal{B}$, i.e. for $x,y,z_1,\dots,z_l\in\mathcal{B}$, let

$$z_1\dots z_l=\sum_{y\in\mathcal{B}}\xi^{y}_{z_1,\dots,z_l}y,\qquad \Delta^{[l]}(x)=\sum_{z_1,\dots,z_l\in\mathcal{B}}\eta_x^{z_1,\dots,z_l}z_1\otimes\dots\otimes z_l, \tag{4.7}$$

- the *rescaling function* $\eta:\mathcal{H}\to\mathbb{R}$ be

$$\eta(x):=\langle\sum_{c\in\mathcal{B}_1}c^*,x\rangle=\text{sum of coefficients (in the } \mathcal{B}_1\otimes\dots\otimes\mathcal{B}_1 \text{ basis) of } \Delta_{1,\dots,1}(x). \tag{4.8}$$

Note that $\eta(x)$ represents the "number of ways to break x into pieces of size 1". Because of coassociativity, $\eta(x)$ can be calculated by applying only refined coproducts of the form $\Delta_{1,i}$ or $\Delta_{i,1}$ for different i, since, on $\mathcal{H}_n$,

$$\begin{aligned}\Delta_{1,\dots,1}&=(\Delta_{1,1}\otimes\mathrm{id}^{\otimes n-2})\circ\dots(\Delta_{1,n-3}\otimes\mathrm{id}\otimes\mathrm{id})\circ(\Delta_{1,n-2}\otimes\mathrm{id})\circ\Delta_{1,n-1}\\&=(\mathrm{id}^{\otimes n-2}\otimes\Delta_{1,1})\circ\dots(\mathrm{id}\otimes\mathrm{id}\otimes\Delta_{n-3,1})\circ(\mathrm{id}\otimes\Delta_{n-2,1})\circ\Delta_{n-1,1}.\end{aligned} \tag{4.9}$$

Recall from Sections 2.3-2.4 that a permuted descent operator has the form $\sum_i a_i m_{D_i}\circ\sigma_i\circ\Delta_{D_i}$. In order for the resulting Markov chain to be easily describable, it is useful to rewrite the coefficients a_i in terms of a probability distribution. For fixed n, let P be a probability distribution on the pairs (D,σ), where D is a weak-composition of n, and $\sigma \in \mathfrak{S}_{l(D)}$. Associate to P the permuted descent operator

$$\mathbf{T}_P := \sum_D \frac{P(D,\sigma)}{\binom{n}{D}} m_{\sigma(D)} \circ \sigma \circ \Delta_D. \tag{4.10}$$

(Recall that $\binom{n}{D} = \binom{n}{d_1 \dots d_l}$.)

Note that $\text{LeftT/B2R}_n(q)$ corresponds to a probability distribution that is supported on two (D,σ) pairs only, namely $P((1,n-1),\text{id}) = q$ and $P((n-1,1),\tau) = 1-q$. Similarly, $\text{RightT/B2R}_n(q)$ corresponds to $P((1,n-1),\tau) = q$ and $P((n-1,1),\text{id}) = 1-q$. And the probability distribution for $\text{LeftRightT/B2R}_n(q,q')$ is supported on four (D,σ) pairs:

$$\begin{aligned}
P((1,n-1),\text{id}) &= qq';\\
P((1,n-1),\tau) &= q(1-q');\\
P((n-1,1),\text{id}) &= (1-q)(1-q')\\
P((n-1,1),\tau) &= q'(1-q).
\end{aligned}$$

As in [Pan18, Def. 3.1], we give below some sufficient conditions on the basis of a combinatorial Hopf algebra, such that the Markov chain driven by a given permuted descent operator is well-defined.

Theorem 4.11. *Work in the setup of Notation 4.6 and (4.10). Suppose that, for all $D = (d_1,\dots,d_l)$ such that $P(D,\sigma) \neq 0$, the structure constants satisfy $\eta_x^{z_1,\dots,z_l} \geq 0$ whenever $z_i \in \mathcal{B}_{d_i}$, and $\xi^y_{z_1,\dots,z_l} \geq 0$ whenever $z_i \in \mathcal{B}_{d_{\sigma^{-1}(i)}}$. Suppose also that the rescaling function satisfies $\eta(x) > 0$ for all $x \in \mathcal{B}_n$. Then the matrix*

$$\check{K}(x,y) := \frac{\eta(y)}{\eta(x)} \text{ coefficient of } y \text{ in } \mathbf{T}_P(x)$$

is a transition matrix. Each step of this Markov chain, starting at $x \in \mathcal{B}_n$, is equivalent to the following three-step process:

1. *Choose a pair (D,σ) according to the distribution P.*
2. *Choose $z_1 \in \mathcal{B}_{d_1}, z_2 \in \mathcal{B}_{d_2}, \dots, z_{l(D)} \in \mathcal{B}_{d_{l(D)}}$ with probability $\frac{1}{\eta(x)}\eta_x^{z_1,\dots,z_{l(D)}}\eta(z_1)\dots\eta(z_{l(D)})$.*
3. *Choose $y \in \mathcal{B}_n$ with probability $\left(\binom{n}{D}\eta(z_{\sigma^{-1}(1)})\dots\eta(z_{\sigma^{-1}(l)})\right)^{-1}\xi^y_{z_{\sigma^{-1}(1)},\dots,z_{\sigma^{-1}(l)}}\eta(y)$.*

Proof. The positivity conditions ensure that all entries of $\check{K}$ are positive. So, by Theorem 4.4, it suffices to show that $\eta = \sum_{c \in \mathcal{B}_1} c^*$ is an eigenvector of $\mathbf{T}_P^*$ with eigenvalue 1. Recall $\mathbf{T}_P^* = \sum_D \frac{P(D,\sigma)}{\binom{n}{D}} \left(m_{\sigma(D)} \circ \sigma \circ \Delta_D\right)^* = \sum_D \frac{P(D,\sigma)}{\binom{n}{D}} m_D \circ \sigma^{-1} \circ \Delta_{\sigma(D)}$.

Consider separately each weak-composition D in this sum, and let l denote its length $l(D)$. As $\deg(\eta) = 1$, the iterated coproduct sends η to $\Delta^{[l]}(\eta) = \eta \otimes 1 \otimes \dots \otimes 1 \ + \ 1 \otimes \eta \otimes 1 \otimes \dots \otimes 1 \ + \ \dots \ + \ 1 \otimes \dots \otimes \eta$, i.e. the sum of l terms, each with l tensorands, one of which is η and all others are 1. Because of the compatibility of product and coproduct,

$$\begin{aligned}
\Delta^{[l]}((\eta)^n) &= (\eta \otimes 1 \otimes \dots \otimes 1 \ + \ 1 \otimes \eta \otimes 1 \otimes \dots \otimes 1 \ + \ \dots \ + \ 1 \otimes \dots \otimes \eta)^n\\
&= \sum_{i_1,\dots,i_l} \binom{n}{i_1 \dots i_l} \eta^{i_1} \otimes \dots \otimes \eta^{i_l}.
\end{aligned}$$

Hence

$$\Delta_{\sigma(D)}(z^n) = \binom{n}{D} \eta^{d_{\sigma^{-1}(1)}} \otimes \cdots \otimes \eta^{d_{\sigma^{-1}(l)}},$$

so $m_D \circ \sigma^{-1} \circ \Delta_{\sigma(D)}(\eta^n) = \binom{n}{D}\eta^n$. Thus $\mathbf{T}_P^*(\eta^n) = \sum_D \frac{P(D,\sigma)}{\binom{n}{D}} \binom{n}{D}\eta^n = \eta^n$.

As for the three-step description, the proof of [Pan18, Th. 3.4] applies verbatim. □

Next, we show that the stationary distributions of descent operator chains, as identified in [Pan18, Th. 3.12], are also the stationary distributions of the permuted chains. Given a multiset $\{c_1, \dots, c_n\}$ in $\mathcal{B}_1$, define $\pi_{c_1,\dots,c_n} : \mathcal{B}_n \to \mathbb{R}$ by

$$\pi_{c_1,\dots,c_n}(x) := \frac{\eta(x)}{n!^2} \sum_{\sigma \in \mathfrak{S}_n} \xi^x_{c_{\sigma(1)},\dots,c_{\sigma(n)}} = \frac{\eta(x)}{n!^2} \sum_{\sigma \in \mathfrak{S}_n} \text{coefficient of } x \text{ in the product } c_{\sigma(1)} \dots c_{\sigma(n)}.$$

Note that, if $\mathcal{B}_1$ consists of a single element that we denote by $\bullet$, then the only such function is

$$\pi(x) := \frac{\eta(x)}{n!} \xi^x_{\bullet,\dots,\bullet} = \frac{\eta(x)}{n!} \text{coefficient of } x \text{ in the product } \bullet^n.$$

Note that both formulae are independent of P; as in the unpermuted case, all permuted descent operator chains share the same stationary distributions.

Theorem 4.12. *Work in the setup of Notation 4.6 and (4.10), and suppose the conditions in Theorem 4.11 are satisfied. Suppose additionally that P is non-zero on some (D, σ) where D has at least two non-zero parts. Then, for the Markov chain driven by $\mathbf{T}_P$:*

i) For any multiset $\{c_1, \dots, c_n\} \subseteq \mathcal{B}_1$ such that $\pi_{c_1,\dots,c_n}(x) \geq 0$ for all $x \in \mathcal{B}_n$, $\pi_{c_1,\dots,c_n}$ is a stationary distribution.

ii) Any stationary distribution can be uniquely written as a linear combination of the functions $\pi_{c_1,\dots,c_n}$.

iii) In the special case where $\mathcal{B}_1 = \{\bullet\}$, π is the unique stationary distribution.

Proof. In view of Proposition 4.5.i, we first show that $\sum_{\sigma \in \mathfrak{S}_n} c_{\sigma(1)} \dots c_{\sigma(n)}$ is an eigenvector of $\mathbf{T}_P$ with eigenvalue 1, then argue the remainder by applying the proof of [Pan18, Th. 3.12] to the nonpermuted counterpart of $\mathbf{T}_P$.

First consider separately each weak-composition D involved in $\mathbf{T}_P$, applied to just one summand in $\sum_{\sigma \in \mathfrak{S}_n} c_{\sigma(1)} \dots c_{\sigma(n)}$:

$$\Delta_D(c_1 \dots c_n) = \sum_{B_1,\dots,B_{l(D)}} \prod_{i \in B_1} c_i \otimes \prod_{i \in B_2} c_i \otimes \cdots \otimes \prod_{i \in B_{l(D)}} c_i,$$

where the sum runs over all set compositions $B_1|\dots|B_{l(D)}$ of $\{1, 2, \dots, n\}$ with d_i elements in B_i. Hence

$$m_{\sigma(D)} \circ \sigma \circ \Delta_D(c_1 \dots c_n) = \sum_{B_1,\dots,B_{l(D)}} \left(\prod_{i \in B_{\sigma^{-1}(1)}} c_i \right) \left(\prod_{i \in B_{\sigma^{-1}(2)}} c_i \right) \cdots \left(\prod_{i \in B_{\sigma^{-1}(l(D))}} c_i \right).$$

Each summand here is a product of $c_1, \dots, c_n$ in some order, and which orders appear (and with what multiplicities) depends only on the starting order $c_1 \dots c_n$, not on the c_i themselves. So the symmetrised product $\sum_{\sigma\in\mathfrak{S}_n} c_{\sigma(1)}\dots c_{\sigma(n)}$ is an eigenvector of $m_{\sigma(D)}\circ\sigma\circ\Delta_D$, with eigenvalue equal to the number of set-compositions in this sum, i.e. $\binom{n}{D}$. This holds for all weak-compositions D, hence

$$\mathbf{T}_P\left(\sum_{\sigma\in\mathfrak{S}_n} c_{\sigma(1)}\dots c_{\sigma(n)}\right)=\sum_D \frac{P(D,\sigma)}{\binom{n}{D}} m_{\sigma(D)}\circ\sigma\circ\Delta_D\left(\sum_{\sigma\in\mathfrak{S}_n} c_{\sigma(1)}\dots c_{\sigma(n)}\right)$$
$$=\sum_D \frac{P(D,\sigma)}{\binom{n}{D}}\binom{n}{D}\sum_{\sigma\in\mathfrak{S}_n} c_{\sigma(1)}\dots c_{\sigma(n)}=\sum_{\sigma\in\mathfrak{S}_n} c_{\sigma(1)}\dots c_{\sigma(n)}.$$

As stated above, the formula for the stationary distribution is independent of P. Hence it is also the stationary distribution for chains driven by the non-permuted counterpart of $\mathbf{T}_P$. So [Pan18, Th. 3.12] already showed that $\pi_{c_1,\dots,c_n}$ is a distribution if it is non-negative on $\mathcal{B}_n$, that $\pi_{c_1,\dots,c_n}$ are linearly independent over choices of multisets $\{c_1,\dots,c_n\}\subseteq\mathcal{B}_1$, and that $\dim\operatorname{span}\{\pi_{c_1,\dots,c_n}\}$ is the algebraic multiplicity of the eigenvalue 1, for the chain driven by the non-permuted counterpart. But the spectrum for $\mathbf{T}_P$ is equal to that of its non-permuted counterpart, so $\dim\operatorname{span}\{\pi_{c_1,\dots,c_n}\}$ is also the algebraic multiplicity of the eigenvalue 1 for the chain driven by $\mathbf{T}_P$. Hence all stationary distributions may be written as a linear combination of $\pi_{c_1,\dots,c_n}$. $\square$

For ease of reference, we record below the eigenvalues of the chain driven by $\mathbf{T}_P$, using the same wording as for the unpermuted counterpart in [Pan18, Th. 3.5].

Theorem 4.13. *Work in the setup of Notation 4.6 and (4.10), and suppose the conditions in Theorem 4.11 are satisfied. Then, for the Markov chain driven by $\mathbf{T}_P$, the eigenvalues are given by*

$$\beta_\lambda^P := \sum_D \frac{1}{\binom{n}{D}}\left(\sum_{\sigma\in\mathfrak{S}_{l(D)}} P(D,\sigma)\right)\beta_\lambda^D,$$

where β_λ^D is the number of set compositions $B_1|\dots|B_{l(D)}$ of $\{1,2,\dots,l(\lambda)\}$ such that, for each i, we have $\sum_{j\in B_i}\lambda_j=d_i$. The multiplicity of the eigenvalue β_λ^P is the coefficient of $x_\lambda := x_{\lambda_1}\dots x_{\lambda_{l(\lambda)}}$ in the generating function $\prod_i(1-x_i)^{-b_i}$, where the numbers b_i satisfy

$$\sum_n \dim\mathcal{H}_n x^n=\prod_i(1-x^i)^{-b_i}.$$

4.4 Right-eigenfunctions for chains driven by $\mathrm{LeftT/B2R}_n(q)$ and $\mathrm{RightT/B2R}_n(q)$

To facilitate the analysis of chains driven by $\mathrm{LeftT/B2R}_n(q)$ and $\mathrm{RightT/B2R}_n(q)$ in Sections 5 and 6, we give below the formulae for their right-eigenfunctions. These result from combining Proposition 4.5.ii with Theorem 3.3.ii, in the simplified case where $\eta\equiv 1$ (no basis rescaling necessary) and $\mathcal{B}_1=\{\bullet\}$ (no symmetrisation of c_is necessary).

Theorem 4.14. *Work in the setup of Notation 4.6, and suppose the conditions in Theorem 4.11 are satisfied for $\mathrm{LeftT/B2R}_n(q)$ and $\mathrm{RightT/B2R}_n(q)$. Assume also that $\eta\equiv 1$ on $\mathcal{B}_n$ and $\mathcal{B}_1=\{\bullet\}$.*

Fix $j\in\{0,1,\dots,n-2\}$. For each $x\in\mathcal{B}_n$, and each $i\in\{0,1,\dots,j\}$, define $x_i\in\mathcal{H}_{n-j}$ by $\Delta_{1^i,n-j,1^{j-i}}(x)=\bullet^{\otimes i}\otimes x_i\otimes\bullet^{\otimes j-i}$. Then *i) For any $p^* \in \ker(\Delta_{1,n-j-1}) \subseteq \mathcal{H}^*_{n-j}$, the function $\mathbf{f}_{p^*} : \mathcal{B}_n \to \mathbb{R}$ defined by*

$$\mathbf{f}_{p^*}(x) = \sum_{i=0}^{j} \binom{j}{i} q^i (1-q)^{j-i} \langle p^*, x_i \rangle$$

is a right-eigenfunction for the chain driven by $\mathrm{LeftT/B2R}_n(q)$ *with eigenvalue* $\frac{j}{n}$.

ii) For any $p^* \in \ker(\Delta_{n-j-1,1}) \subseteq \mathcal{H}^*_{n-j}$, the function $\mathbf{f}_{p^*} : \mathcal{B}_n \to \mathbb{R}$ defined by

$$\mathbf{f}_{p^*}(x) = \sum_{i=0}^{j} \binom{j}{i} q^i (1-q)^{j-i} \langle p^*, x_i \rangle$$

is a right-eigenfunction for the chain driven by $\mathrm{RightT/B2R}_n(q)$ *with eigenvalue* $\frac{j}{n}$. □

5 A Chain on Permutations

This section applies the theory above to RightT/B2R on **FQSym**, the Malvenuto-Reutenauer Hopf algebra [MR95, Sec. 3], to analyse a chain similar to the "to-do list" chain studied in [Pan18, Sec. 6].

5.1 Hopf Operations on Permutations

FQSym is an important algebra in combinatorics - many other combinatorial Hopf algebras, on compositions, trees and tableaux, can phrased as quotients of **FQSym** under a systematic procedure. The notation **FQSym** is from [DHT02] for free quasisymmetric functions; it is also denoted $\mathfrak{S}Sym$ in [AS05b].

To analyse RightT/B2R, it suffices to consider the following refined Hopf operations on **FQSym** and its dual, which come from specialising Examples 2.1 and 2.5:

(5.1) $m_{n-1,1}(u \otimes (1))$ is the sum of all permutations obtained by inserting n into u;

(5.2) $\Delta_{1,n-1}(u) = (1) \otimes \mathrm{std}(u_2 \dots u_n)$;

(5.3) $\Delta_{n-1,1}(u) = \mathrm{std}(u_1 \dots u_{n-1}) \otimes (1)$;

(5.4) $\Delta_{n-j-1,1}(u^*) = (u_1 \dots \widehat{n-j} \dots u_{n-j})^* \otimes (1)^*$ - in the left factor $n-j$ is removed.

It follows from (4.9) and (5.2) that $\eta \equiv 1$ on the basis of **FQSym**. Thus, by Theorem 4.11, the chain driven by $\mathrm{RightT/B2R}_n(q)$ on **FQSym** has the following description, starting at a permutation u of n:

i) With probability q (resp. $1-q$), remove the first (resp. last) number in u and standardise the result;

ii) Insert n in a uniformly chosen position.

We give two interpretations of this process, analogous to the interpretations in [Pan18, Sec. 6] of the $\mathrm{T/B2R}_n(1)$ chain. The first visualisation concerns a to-do list of n tasks, listed vertically in decreasing order of urgency. Each day, there is an independent probability q that conditions are favourable and the most urgent (top) task is completed; in the complementary probability $1-q$,

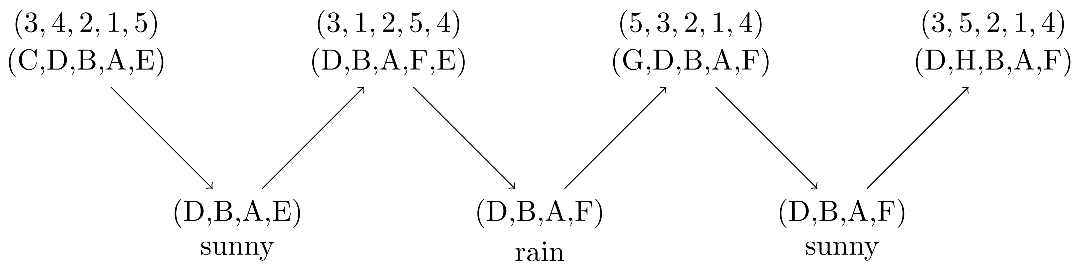


Figure 5.1: Three steps of the Markov chain on permutations (in one-line notation) arising from LeftT/B2R$_n(q)$ on $FQSym$, viewed as the relative time that tasks spend on a to-do list. The letters label the tasks and are not part of the chain.

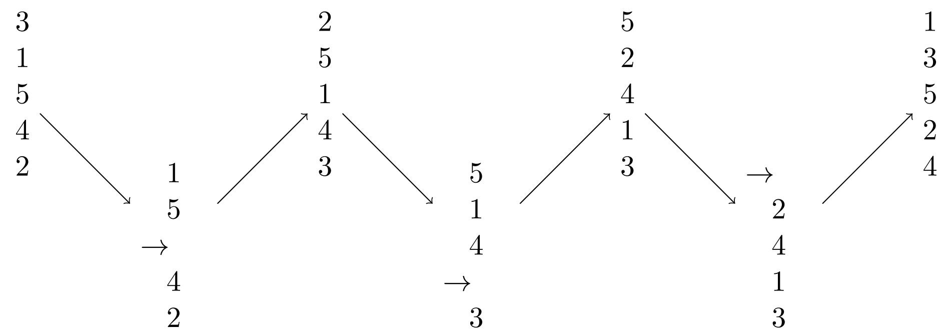


Figure 5.2: A possible three-day trajectory of the Markov chain of "relative time on a to-do list", for $n = 5$. The horizontal arrows indicate the (random) positions of incoming tasks.

no task is completed, but the least urgent (bottom) task is abandoned so the list does not grow indefinitely. At the end of the day, a new task arrives, which is added to the list in a uniformly chosen position (i.e. its urgency relative to the current tasks is uniformly distributed). To produce a chain on permutations from this data, we label each task with the day that it arrived, and then standardise the labels. In other words, 1 indicates the task that spent the most time on list, 2 the next oldest task, and so on, so n denotes the newest addition to the list. Figure **??** illustrates a few steps of this chain.

In the second interpretation,

By Theorem 4.12

6 A Chain on Trees

This section applies the theory above to LeftT/B2R on $YSym$ [AS06], the graded dual of the Loday-Ronco Hopf algebra [LR98], to analyse the remove-and-reattach-a-vertex chain as described in the introduction. It is a vast extension of the undergraduate thesis [Cha18], supervised by the author.

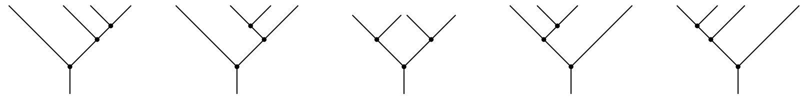


Figure 6.1: The planar binary trees of degree 3.

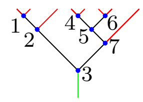


Figure 6.2: A planar binary tree of degree 7, with its vertices labelled in in-order. The leaves are in red, the vertices are in blue, the root is in green.

6.1 Planar Binary Trees

The basis $\mathcal{B}_n$ of $YSym_n$ consists of planar binary trees with n internal vertices. In the present convention, each vertex of a planar binary tree has exactly two edges going up and one edge going down (i.e. a full binary tree). Hence $\mathcal{B}_n$ may also be characterised as the planar binary tree with $n+1$ leaves.

The edges need not have vertices at both ends; an edge with only a vertex at its lower end is a *leaf*, and the sole edge with only a vertex at its upper end is the *root*. See Figure 6.2 below.

If an edge has a vertex at both ends, then the vertex at the upper end is the *child* of the vertex at the lower end; it is a *left-child* or *right-child* depending on the direction of the edge. For example, in Figure 6.2, 4 (resp. 6) is the left-child (resp. right-child) of 5, and 2 (resp. 7) is the left-child (resp. right-child) of 3. Extending this: given vertices x and y, we say x is a *left-descendant* (resp. *right-descendant*) of y if there is a sequence of vertices $x, x_1, \dots x_{k-1}, x_k = y$ such that x_1 is a left-child (resp. right-child) of x, and x_{i-1} is a child of x_i for all $i \in [2, k]$. For example, in Figure 6.2, 6 is a right-descendant of 3, and a left-descendant of 7.

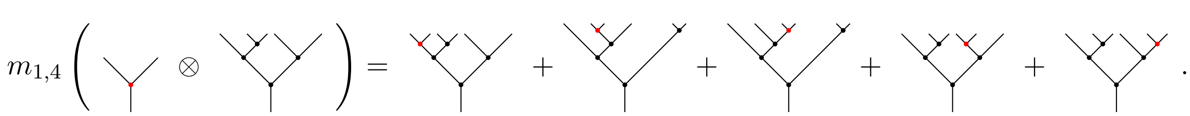
Given a planar binary tree T, the *in-order* is an ordering its vertices such that, for each vertex x, the left-descendants of x are ordered before x, and the right-descendants of x are ordered after x [Cor+09, P. 287]; see Figure 6.2. This will be the order that vertices are removed from T under $\Delta_{1,n-1}$.

A tree is a *path tree* if all but one vertex has exactly one child; the childless vertex is the *top vertex*. An equivalent characterisation is that each vertex has only left descendants or only right descendants.

6.2 Hopf Operations on Planar Binary Trees

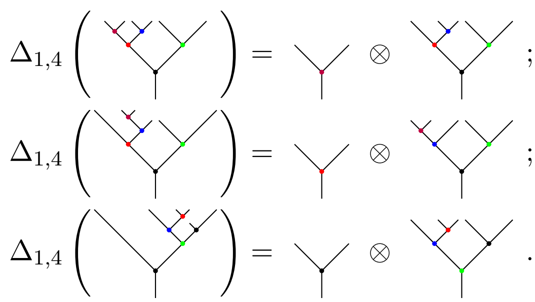
We describe below the particular product and coproduct operations that relevant to the remove-and-reattach Markov chains - these are special cases of the cladogram operations in the chains of Aldous [Ald00] - and refer the reader to [AS06] and [LR98] for the full product and coproduct, which are considerably more complicated.

For a tree $T \in \mathcal{B}_{n-1}$, the product $m_{1,n-1}(\bullet \otimes T)$ is the sum of all ways to attach the root of $\bullet$ to the a leaf of T, i.e. to change that leaf into an internal vertex. For example



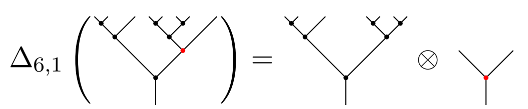
Since each $T \in \mathcal{B}_{n-1}$ has n leaves and the n attachments result in different trees, so $m_{1,n-1}(\bullet \otimes T)$ consists of n distinct terms.

For a tree $T \in \mathcal{B}_n$, its coproduct $\Delta_{1,n-1}(T)$ is computed by removing the leftmost vertex (first vertex in in-order) and "sliding down" any vertices above this removed vertex - that is, fuse the first vertex with its right child if it has one, else simply delete the leftmost vertex and leftmost leaf. For example,



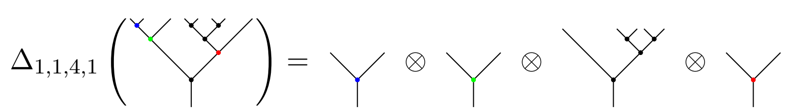
Equivalently, $\Delta_{1,n-1}(T)$ is the tree formed by vertices $2, 3, \ldots, n$ of T (in in-order) that have the same descendant relationships as in T. For example, in the second line above, the purple vertex remains a descendant of the blue vertex, and both of them remain descendants of the black vertex

Similarly, $\Delta_{n-1,1}(T)$ removes the rightmost vertex (last vertex in in-order) then "slides down" any vertices above this removed vertex, i.e. fusing the rightmost vertex with its left child, e.g.



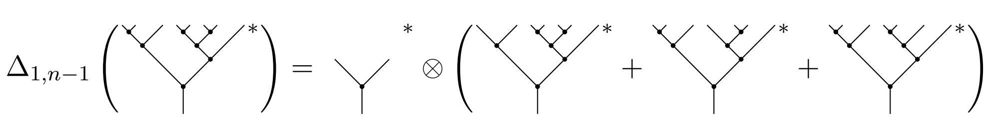
Equivalently, $\Delta_{n-1,1}(T)$ is the tree formed by vertices $1, 2, \ldots, n-1$ of T (in in-order) that have the same descendant relationships as in T.

So the non-trivial tensorand in $\Delta_{1^i,n-j,1^{j-i}}(T)$ is the tree formed by vertices $i+1, i+2, \ldots, i+n-j$ of T (in in-order) that have the same descendant relationships as in T, for example



Note that, in the image of any tree under $\Delta_{1^i,n-j,1^{j-i}}$, the sole non-trivial tensorand is exactly one tree (not a linear combination); in particular $\Delta_{1,\ldots,1}$ applied to any tree results in exactly $Y \otimes \cdots \otimes Y$. So, as for permutations above, the rescaling function η is constant on $\mathcal{B}_n$, i.e. no rescaling is necessary.

The right-eigenfunction formula in Theorem 4.14 also involves coproducts in $\mathcal{H}^*$, the Loday-Ronco Hopf algebra. For $T^* \in \mathcal{B}_n^*$, the coproduct $\Delta_{1,n-1}(T^*)$ is the sum of all ways to remove a vertex with no children from T, i.e. to remove a vertex from the "top" of T. For example:



6.3 Markov Chains on Planar Binary Trees

The chain driven by $\mathrm{LeftT/B2R}_n(q)$ on planar binary trees has the following description: at each step, toss a coin decide which vertex to remove: with probability q the leftmost vertex is removed (and its right child slides down to fuse with it), with probability $1-q$ the rightmost vertex is removed (and its left child slides down to fuse with it). Then choose a leaf with uniform probability to replace with a vertex. Figure 6.3 demonstrates two steps of this chain. It would be good to have

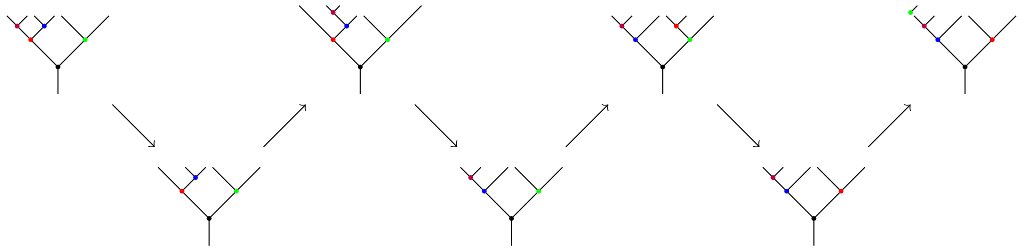


Figure 6.3: Three steps of the Markov chain on binary trees. The removals in the first two steps are from the left, and the removal from the third step is from the right. The colours are not part of the chain; they only serve to clarify the operations.

an interpretation of this chain in terms of phylogenetic trees, data structures or other trees applied in the sciences.

[need to explain Hopf map more. Maybe copy more from LRchain write-up?] By [Pan19, Th. 2.6], the Hopf-morphism $\mathbf{FQSym} \to YSym$ means that the chain driven by any permuted descent operator $\mathbf{T}_P$ on planar binary trees is the image of the $\mathbf{T}_P$-Markov chain on permutations, under taking the decreasing tree. This chain on permutations is analysed in detail in [Pan18, Sec. 6] and [Pan19, Sec. IIC]; for the present work it suffices to know that this chain has a unique stationary distribution, namely the uniform distribution. It follows that

Theorem 6.1 (Stationary Distribution). *The unique stationary distribution of the Markov chain on planar binary trees is given by*

$$\pi(T) = \textit{number of increasing labellings of } T.$$

□

(This can also be obtained directly from Theorem 4.12.)

From the right-eigenfunction formula of Theorem 4.14.i, we obtain the following:

Theorem 6.2. *Fix $j \in \{0, 1, \ldots, n-2\}$, and $S \subseteq \{1, 2, \ldots, n\}$. Define $\mathbf{f}_S : \mathcal{B}_n \to \mathbb{R}$ by*

$$\mathbf{f}_S(T) = \begin{cases} 1 & \textit{if each vertex in } S \textit{ has only left-descendants or only right-descendants within } S, \\ & \textit{ and the unique vertex in } S \textit{ without descendants within } S \\ & \textit{ is a right-descendant of its most recent ancestor within } S; \\ -1 & \textit{if each vertex in } S \textit{ has only left-descendants or only right-descendants within } S, \\ & \textit{ and the unique vertex in } S \textit{ without descendants within } S \\ & \textit{ is a left-descendant of its most recent ancestor within } S; \\ 0 & \textit{otherwise} \\ & \textit{i.e. if some vertex in } S \textit{ has both left-descendants and right-descendants within } S. \end{cases}$$

Then $\mathbf{f}_j : \mathcal{B}_n \to \mathbb{R}$ defined by

$$\mathbf{f}_j(T) := \sum_{i=0}^{j} \binom{j}{i} q^i (1-q)^{j-i} \mathbf{f}_{\{i+1, i+2, \ldots i+n-j\}}(T) \tag{6.3}$$

is a $\frac{j}{n}$-right-eigenfunction of the chain on planar binary trees driven by $\mathrm{LeftT/B2R}_n(q)$.

The following is a straightforward application of Proposition 4.3:

Corollary 6.4. *If* $\mathbf{f}_j : \mathcal{B}_n \to \mathbb{R}$ *is as defined above, and* $\{X_t\}$ *denotes the chain on planar binary trees driven by* $\mathrm{LeftT/B2R}_n(q)$*, then*

$$\text{Expect}\left(\mathbf{f}_j(X_t)|X_0 = x_0\right) = \left(\frac{j}{n}\right)^t \mathbf{f}_j(x_0). \tag{6.5}$$

In particular, for the chain driven by $\mathrm{T2R}_n = \mathrm{LeftT/B2R}_n(1)$*,*

$$\text{Expect}\left(\mathbf{f}_{\{j+1,j+2,\dots n\}}(X_t)|X_0 = x_0\right) = \left(\frac{j}{n}\right)^t \mathbf{f}_{\{j+1,j+2,\dots n\}}(x_0).$$

□

The proof of Theorem 6.2 will be given after an example and a specialisation.

Example 6.6. We calculate $\mathbf{f}_j(x_0)$ when $n = 7$, $j = 4$, and x_0 is the tree in Figure 6.2.

- $\mathbf{f}_{\{1,2,3\}}(x_0) = -1$ as vertex 1 is a left-descendant of vertex 2.
- $\mathbf{f}_{\{2,3,4\}}(x_0) = 0$ as vertex 2 is a left-descendent of vertex 3, and vertex 4 is a right-descendent of vertex 3.
- $\mathbf{f}_{\{3,4,5\}}(x_0) = -1$ as vertex 4 is a left-descendant of vertex 5.
- $\mathbf{f}_{\{4,5,6\}}(x_0) = 0$ as vertex 5 has both a left-child and a right-child within $\{4, 5, 6\}$.
- $\mathbf{f}_{\{5,6,7\}}(x_0) = 1$ as vertex 6 is a right-descendant of vertex 5.

So $\mathbf{f}_j(x_0) = -\binom{4}{0}q^0(1-q)^4 - \binom{4}{2}q^2(1-q)^2 + \binom{4}{4}q^4(1-q)^0$.

Note that, if x_0 is the tree in Figure 6.2, then $\mathbf{f}_{\{3,4\}} = 1$, because vertex 4 is a right-descendant of vertex 3, even though vertex 4 is a left-child.

In the case where $j = n-2$, the $\mathbf{f}_S$ that occur in (6.5) simplify greatly, since such S consists only of two consecutive vertices:

$$\mathbf{f}_{\{i,i+1\}}(T) = \begin{cases} 1 & \text{if } i+1 \text{ is a descendant of } i; \\ -1 & \text{if } i \text{ is a descendant of } i+1. \end{cases}$$

This has the following ...

Proof of Theorem 6.2. By Proposition 4.5, it suffices to show that

$$\mathbf{f}_{\{i+1,i+2,\dots i+n-j\}}(T) = \langle p^*_{n-j}, T_i \rangle$$

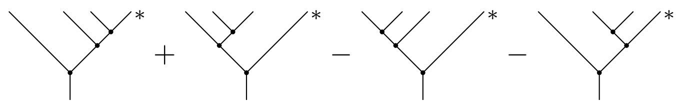
for some $p^*_{n-j} \in \ker(\Delta_{1,n-j-1}) \subseteq \mathcal{H}^*_{n-j}$, where T_i is defined by $\Delta_{1^i,n-j,1^{j-i}}(T) = \bullet^{\otimes i} \otimes T_i \otimes \bullet^{\otimes j-i}$.

These p^*_k are built from path trees, as follows. Given any path tree $T \in \mathcal{B}_{k-1}$, let $T_L \in \mathcal{B}_k$ denote the tree obtained by adding a left child to the topmost vertex, and T_R the tree that adds a right child to this topmost vertex. For example

$$?_L = ?, \qquad ?_R = ?.$$

By construction, T_L and T_R are path trees, and $\Delta_{1,k-1}(T_L^*) = \Delta_{1,k-1}(T_R^*) = T^*$, i.e. $T_R^* - T_L^* \in \ker(\Delta_{1,k-1})$. Note that no other path trees have image T^* under $\Delta_{1,n-1}$. So, let p_k^* be the sum of $T_R^* - T_L^*$ over all path trees $T \in \mathcal{B}_{k-1}$; then $p_k^* \in \ker(\Delta_{1,k-1})$. Equivalently, p_k^* is the sum of all path trees in $\mathcal{H}_k^*$ whose topmost vertex is a right child, subtract the sum of all path trees in $\mathcal{H}_k^*$ whose topmost vertex is a left child. For example:

$p_2^* :=$ 

$p_3^* :=$ 

Thus, for $T_i \in \mathcal{B}_{k-1}$,

$$\langle p_k^*, T\rangle = \begin{cases} 1 & \text{if } T_i \text{ is a path tree whose topmost vertex is a right child;} \\ -1 & \text{if } T_i \text{ is a path tree whose topmost vertex is a left child;} \\ 0 & \text{otherwise.} \end{cases}$$

Recall that "T_i is a path tree" is equivalent to "each vertex of T_i has only left-descendants or only right-descendants", and since the coproduct operations do not change descendant relationships, this is further equivalent to "each vertex of T within $\{i+1, i+2, \dots, i+n-j\}$ has only left-descendants or only right-descendants within this same set". $\square$

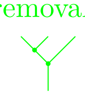
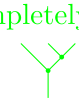
We give an extra result for $j = n-2$ and $q = 1$, i.e. for the T2R$_n$ chain. Let $L(T), R(T)$ denote respectively the left-subtree and right-subtree of T.

Proposition 6.7 (Recursion for right-eigenfunctions).

$$\mathbf{f}_{p_2^*}(T) = \begin{cases} -1 & \textit{if } \deg R(T) = 0; \\ 1 & \textit{if } \deg R(T) = 1; \\ \mathbf{f}_{p_2^*}(R(T)) & \textit{otherwise.} \end{cases}$$

Proof. The key observation is: so long as $L(T)$ is nonempty, $\Delta_{1,n-1}$ removes a vertex from $L(T)$; if $L(T)$ is empty, then $\Delta_{1,n-1}$ removes the bottommost vertex, i.e. $\Delta_{1,n-1}(T) = Y \otimes R(T)$. Hence repeated applications of $\Delta_{1,*}$, as is necessary for calculating $\mathbf{f}_p(T)$, first removes vertices from the left-subtree, then, if more than $\deg L(T)$ vertices must be removed, it will remove the bottommost vertex and then remove from what was previously the right-subtree - i.e. the vertices removed follows in-order.

The case $j = 2$: so we are considering the removal of $n-2$ vertices from T.

- If $\deg R(T) = 0$, then $\deg L(T) = n-1$, so all $n-2$ removals are from $L(T)$, and one vertex will remain in $L(T)$ - thus the remaining tree is , so $\mathbf{f}_{p_2^*}(T) = -1$.
- If $\deg R(T) = 1$, then $\deg L(T) = n-2$, so the $n-2$ removals delete $L(T)$ completely, so only the bottommost vertex and the sole vertex of $R(T)$ remain - these form , so $\mathbf{f}_{p_2^*}(T) = 1$.

- If $\deg R(T) \geq 2$, then $\deg L(T) < n-2$, so removing the $n-2$ "leftmost" vertices of T will remove all vertices of $L(T)$ and the bottommost vertex, and, if $\deg R(T) \geq 3$, also remove some vertices from $R(T)$. Thus the effect is the same as removing $(n-2)-(\deg L(T)-1)$ vertices from $R(T)$, which is the removal required to calculate $\mathbf{f}_{p_2^*}(R(T))$.

$\square$

Theorem 6.8 (Probability bounds for subtree sizes)**.** *Let $\{X_t\}$ denote the Markov chain driven by* T2R$_n$ *on binary trees [more description necessary?? e.g. "of Section??" ??] Then*

$$\mathrm{Prob}\{R(X_t) = \emptyset\} \leq \frac{1}{2}\left(\left(\frac{n-2}{n}\right)^t + 1\right).$$

Remark. This bound is not tight for large t - for example, the event $\deg R(X_t) = 0$ corresponds to the event "the permutation ends with n" on the lift chain, and since the lift chain has uniform stationary distribution, the limit of this probability as $t \to \infty$ is $\frac{1}{n}$, whereas the limit of the above bound as $t \to \infty$ is $\frac{1}{2}$. ... However, computation says useful for small t?

Proof. Consider $\mathbf{f} : \mathcal{B}_n \to \mathbb{R}$ given by

$$\mathbf{f}(T) = \begin{cases} 1 & \text{if } R(T) = \emptyset; \\ -1 & \text{otherwise.} \end{cases}$$

From Proposition 6.7, the right-eigenfunction $(\mathbf{f}_{p_2^*})$ takes value 1 when $R(T) = \emptyset$ and is between 1 and -1 otherwise, so $\mathbf{f}(T) \leq (-\mathbf{f}_{p_2^*})(T)$ for all T. So, applying Proposition 4.3 shows

$$\mathrm{Expect}(\mathbf{f}(X_t)|X_0 = x_0) \leq \mathrm{Expect}(-\mathbf{f}_{p_2^*}(X_t)|X_0 = x_0) = \left(\frac{n-2}{n}\right)^t (-\mathbf{f}_{p_2^*})(x_0).$$

Now observe that the function $\frac{1}{2}(\mathbf{f}+1)$ satisfies

$$\frac{1}{2}(\mathbf{f}+1) = \begin{cases} 1 & \text{if } R(T) = \emptyset; \\ 0 & \text{otherwise,} \end{cases}$$

so $\frac{1}{2}(\mathbf{f}+1)$ is the indicator function of the event $(R(T) = \emptyset)$. Now the probability of an event is the expectation of its indicator function, and hence

$$\begin{aligned}\mathrm{Prob}\{R(X_t) = \emptyset | X_0 = x_0\} &= \mathrm{Expect}\left(\left[\frac{1}{2}(\mathbf{f}+1)\right](X_t)\middle| X_0 = x_0\right) \\ &= \frac{1}{2}\left(\mathrm{Expect}(\mathbf{f}(X_t)|X_0 = x_0) + 1\right) \leq \frac{1}{2}\left(\left(\frac{n-2}{n}\right)^t (-\mathbf{f}_{p_2^*})(x_0) + 1\right).\end{aligned}$$

To obtain the bound as stated in the theorem, that is independent of the starting state x_0, notice that $\mathbf{f}_{p_2^*}$ takes only values 1 and -1, so $-\mathbf{f}_{p_2^*}(x_0) \leq 1$. $\square$

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