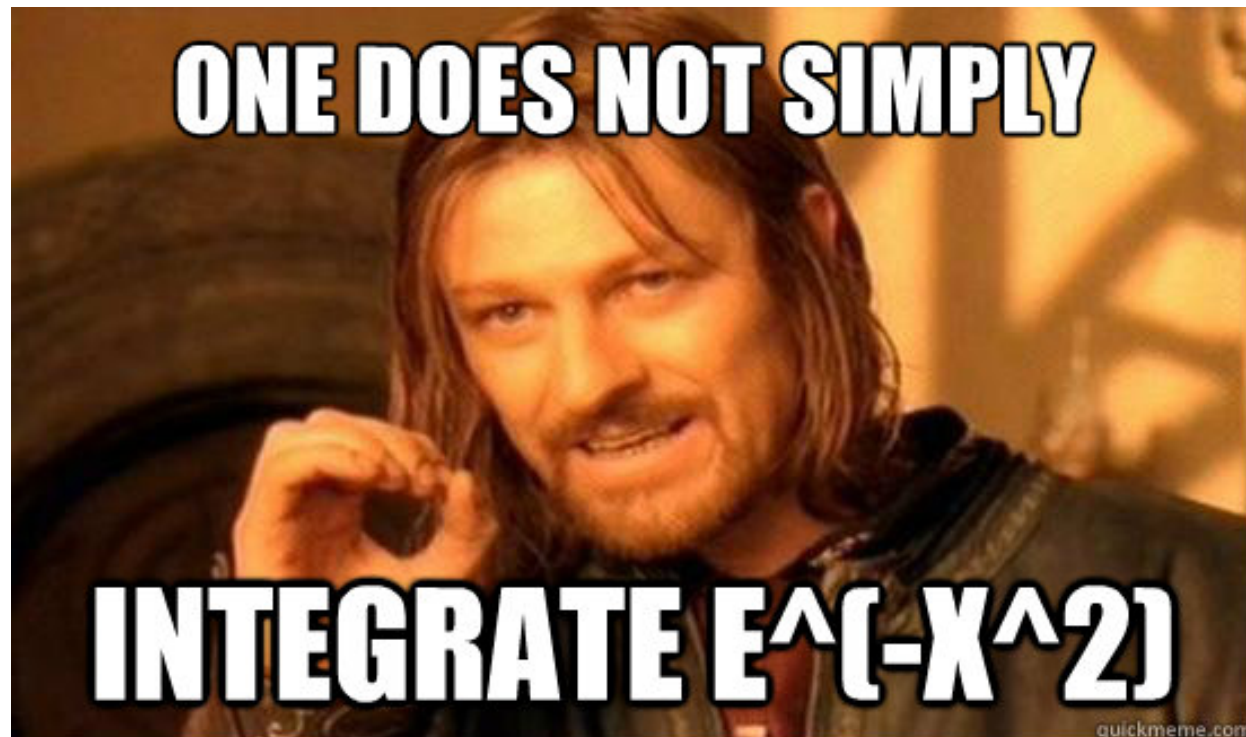
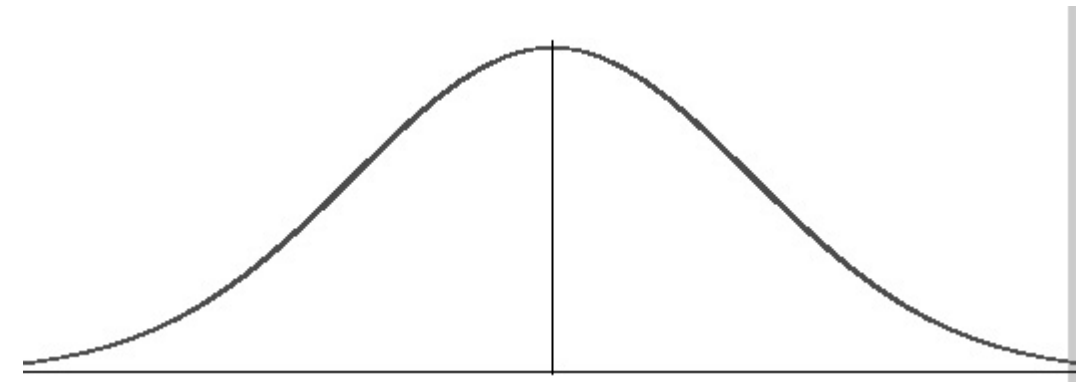


Application 1 of multivariate calculus to probability/statistics: the Gaussian integral

The normal distribution (also called the Gaussian distribution) has probability density function proportional to e^{-x^2} , i.e. $p(x) = \frac{1}{Z}e^{-x^2}$ for some Z . We need to choose the constant of proportionality Z so that the total probability is 1, i.e.

$$Z = \int_{-\infty}^{\infty} e^{-x^2} dx.$$



The problem is, we cannot write down the antiderivative of e^{-x^2} - it is not an elementary function. But multiple integration will help us in a surprising, clever way.

First, we need to show that the improper integral $Z = \int_{-\infty}^{\infty} e^{-x^2} dx$ converges.

It will be enough to show that $Z' = \int_1^{\infty} e^{-x^2} dx$ converges, because then

$$Z = \int_{-\infty}^{-1} e^{-x^2} dx + \int_{-1}^1 e^{-x^2} dx + \int_1^{\infty} e^{-x^2} dx = Z' + \int_{-1}^1 e^{-x^2} dx + Z'.$$

We show that Z' converges using the (non-examinable) technique of integral estimation by inequalities:

e^{-x^2} is a non-negative function, so FTC1 says that $F(R) = \int_1^R e^{-x^2} dx$ is an

increasing function (in R).

So, using some theorems from analysis, we know that, if there is a number M such that

$M \geq F(R) = \int_1^R e^{-x^2} dx$ for every $R > 1$, then $\lim_{R \rightarrow \infty} F(R)$ exists, i.e. Z' converges.

For all $x \geq 1$, we have $e^{-x^2} \leq e^{-x}$, so $\int_1^R e^{-x^2} dx \leq \int_1^R e^{-x} dx = e - e^{-R} \leq e$, so e

is the upper bound M that we want.

Reminder: we wish to evaluate $Z = \int_{-\infty}^{\infty} e^{-x^2} dx$.

Consider the double integral $\iint_{\mathbb{R}^2} e^{-x^2} e^{-y^2} dA$. The integrand is always positive, so we can calculate this improper integral using an iterated integral:

$$\iint_{\mathbb{R}^2} e^{-x^2} e^{-y^2} dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dy dx = \int_{-\infty}^{\infty} e^{-x^2} Z dx = Z^2$$

But we can also calculate this double integral using polar coordinates (yes, you can use change of variables on improper integrals):

$$\begin{aligned} \iint_{\mathbb{R}^2} e^{-x^2} e^{-y^2} dA &= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta \quad \text{inner integral independent of } \theta, \\ &\quad \text{and substitution } u = r^2 \\ &= 2\pi \int_0^{\infty} \frac{e^{-u}}{2} du = \pi \lim_{R \rightarrow \infty} \int_0^R e^{-u} du = \pi \left(\lim_{R \rightarrow \infty} 1 - e^{-R} \right) = \pi. \end{aligned}$$

So $Z^2 = \pi$, i.e. $Z = \sqrt{\pi}$.