Example: Find the second-order Taylor polynomial of $f(x,y)=\dfrac{\sin x}{y}$ about $(x,y)=(0,1)$.

$$P_2(x,y) = f(0,1) + f_x(0,1)(x-0) + f_y(0,1)(y-1)$$

$$+\frac{1}{2!}\left(f_{xx}(0,1)(x-0)^2 + 2f_{xy}(0,1)(x-0)(y-1) + f_{yy}(0,1)(y-1)^2\right)$$

$$= \frac{\sin 0}{1} + \overbrace{\left.\frac{\cos x}{y}\right|_{(0,1)}}^{=f_x}(x-0) + \overbrace{\left.\frac{\sin x}{-y^2}\right|_{(0,1)}}^{f_y}(y-1)$$

$$+\frac{1}{2!}\left(\left.\frac{-\sin x}{y}\right|_{(0,1)}(x-0)^2 + 2\left.\frac{\cos x}{-y^2}\right|_{(0,1)}(x-0)(y-1) + \left.\frac{2\sin x}{y^3}\right|_{(0,1)}(y-1)^2\right)$$

$$= 0 + 1(x-0) + 0(y-1) + \frac{1}{2!}\left(0(x-0)^2 - 2(x-0)(y-1) + 0(y-1)^2\right)$$

$$= x - x(y-1)$$

Example: (compare p19) Find the fourth-order Taylor polynomial of $f(x,y) = \dfrac{\sin x}{y}$ about $(x,y) = (0,1)$.

Idea: multiply the 1D Taylor polynomials for $\sin x$ about $x=0$ and for $\frac{1}{y}$ about $y=1$

$$\frac{1}{y} = \frac{1}{1+(y-1)}$$
$$= 1-(y-1)+(y-1)^2-(y-1)^3+(y-1)^4-\cdots$$

from Taylor polynomial for $\dfrac{1}{1+?}$

So
$$\frac{\sin x}{y} = \left(x-\frac{x^3}{3!}+\cdots\right)\left(1-(y-1)+(y-1)^2-(y-1)^3+(y-1)^4-\cdots\right)$$
$$= \underbrace{}_{\text{no constant term}} \underbrace{x}_{\text{degree 1}} \underbrace{-\,x(y-1)}_{\text{degree 2}} \underbrace{+\,x(y-1)^2-\frac{x^3}{3!}}_{\text{degree 3}} \underbrace{-\,x(y-1)^3+\frac{x^3}{3!}(y-1)}_{\text{degree 4}} + \cdots$$

throw away terms of degree >4 in x, y

Example: Find the third-order Taylor polynomial of $\ln(2+2x+2y^2)$ about $(0,0)$.

$$\ln(2+2x+2y^2) = \ln(2(1+x+y^2))$$
$$= \ln 2 + \ln(1+x+y^2)$$
$$= \ln 2 + \left((x+y^2) - \frac{(x+y^2)^2}{2} + \frac{(x+y^2)^3}{3} - \cdots \right)$$
$$= \ln 2 + \left(x + y^2 - \frac{x^2+2xy^2+\cdots}{2} + \frac{x^3+\cdots}{3} - \cdots \right) \quad \text{ignore terms of degree} > 3$$
$$= \ln 2 + x + \left(y^2 - \frac{x^2}{2} \right) + \left(-xy^2 + \frac{x^3}{3} \right) + \cdots$$