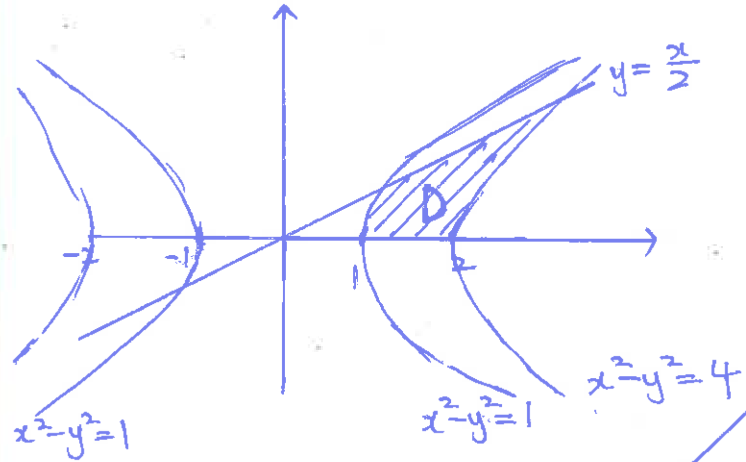
Example: Evaluate $\iint_D \frac{1}{x^2}\,dA$, where D is the region bounded by $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $y = 0$ and $y = \frac{x}{2}$, with $x > 0$

hyperbola

i.e. $y - \frac{x}{2} = 0$ — but what to do with $y=0$?

better: $\frac{y}{x} = \frac{1}{2}$



Let $u = x^2 - y^2$, $v = \frac{y}{x}$

① So D corresponds to (S)

$1 \le u \le 4$, $0 \le v \le \frac{1}{2}$.

② $\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix} = \det\begin{pmatrix} 2x & -2y \\ -\frac{y}{x^2} & \frac{1}{x}\end{pmatrix}$

$= 2 - \frac{2y^2}{x^2}$.

$= \frac{2}{x^2}(x^2 - y^2)$

Is this positive or negative?

and $x^2 - y^2 \ge 1 > 0$

and $\frac{2}{x^2} > 0$

so this is positive

$$\iint_D \frac{1}{x^2}\, dA = \int_0^{\frac{1}{2}}\int_1^4 \frac{1}{x^2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \, du\, dv$$

$$= \int_0^{\frac{1}{2}}\int_1^4 \frac{1}{x^2} \frac{1}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|}\, du\, dv$$

$$= \int_0^{\frac{1}{2}}\int_1^4 \frac{1}{x^2} \frac{1}{2-\frac{2y^2}{x^2}}\, du\, dv$$

$$= \int_0^{\frac{1}{2}}\int_1^4 \frac{1}{2x^2-2y^2}\, du\, dv$$

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$$= \int_0^{\frac{1}{2}}\int_1^4 \frac{1}{2}\, \frac{1}{u}\, du\, dv$$

$$= \int_0^{\frac{1}{2}} \frac{1}{2} \ln u \Big|_{u=1}^{u=4} dv = \int_0^{\frac{1}{2}} \frac{1}{2}\ln 4\, dv = \frac{1}{2}\ln 4\, v\Big|_0^{\frac{1}{2}} = \frac{1}{4}\ln 4.$$