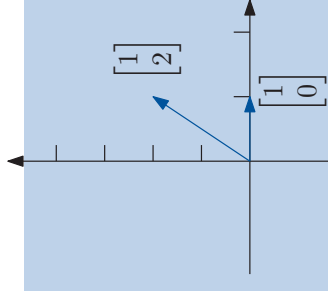


$\text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\} = \text{a line}$



$\text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = \mathbb{R}^2$

§1.7: Linear Independence

Definition: A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is *linearly independent* if the only solution to the vector equation

$$x_1 \mathbf{v}_1 + \dots + x_p \mathbf{v}_p = \mathbf{0}$$

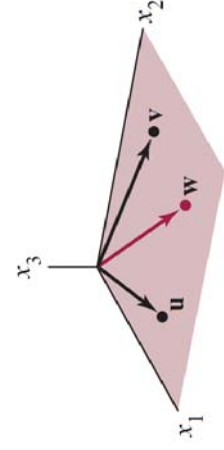
is the **trivial solution** ($x_1 = \dots = x_p = 0$).

The opposite of linearly independent is linearly dependent:

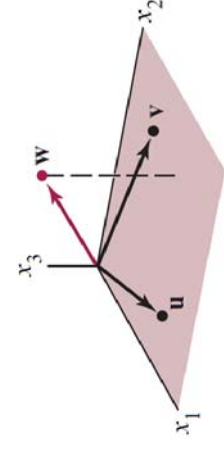
Definition: A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is *linearly dependent* if there are weights c_1, \dots, c_p , **not all zero**, such that

$$c_1 \mathbf{v}_1 + \dots + c_p \mathbf{v}_p = \mathbf{0}.$$

The equation $c_1 \mathbf{v}_1 + \dots + c_p \mathbf{v}_p = \mathbf{0}$ is a **linear dependence relation**.



$\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \text{Span}\{\mathbf{u}, \mathbf{v}\} = \text{a plane}$



$\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathbb{R}^3$

When do n vectors span \mathbb{R}^n ?

When they are a **linearly independent** set.

How to find an efficient spanning set?

The **casting out** algorithm.

$$x_1 \mathbf{v}_1 + \dots + x_p \mathbf{v}_p = \mathbf{0}$$

Only solution is $x_1 = \dots = x_p = 0$
 \rightarrow **linearly independent**

There is a solution with some $x_i \neq 0$
 \rightarrow **linearly dependent**

Example: The set $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$ is linearly dependent because

$$2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Example: The set $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is linearly independent because

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{matrix} x_1 + x_2 = 0 \\ 2x_1 = 0 \end{matrix} \implies x_1 = 0, x_2 = 0.$$

Some easy cases:

- Sets containing the zero vector $\{0, v_2, \dots, v_p\}$:

$$(1)0 + (0)v_2 + \dots + (0)v_p = 0 \quad \text{linearly dependent}$$

- Sets containing one vector $\{v\}$:

$$xv = 0$$

linearly independent if $v \neq 0$

$$\begin{bmatrix} xv_1 \\ \vdots \\ xv_n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

. If some $v_i \neq 0$, then $x = 0$ is the only solution.

Some easy cases:

- Sets containing two vectors $\{u, v\}$:

$$x_1u + x_2v = 0$$

if $x_1 \neq 0$, then $u = (-x_2/x_1)v$. if $x_2 \neq 0$, then $v = (-x_1/x_2)u$.

So $\{u, v\}$ is linearly dependent if and only if one of the vectors is a multiple of the other.

- Sets containing more vectors:

A set of vectors is linearly dependent if and only if one of the vectors is a linear combination of the others. (Any vector with nonzero weight in the linear dependency relation will work.)

EXAMPLE Let $v_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$, $v_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$.

- Determine if $\{v_1, v_2, v_3\}$ is linearly independent.
- If possible, find a linear dependence relation among v_1, v_2, v_3 .

Solution. (a)

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix:

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{bmatrix} \quad \text{row reduces to} \quad \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is a free variable \Rightarrow there are nontrivial solutions.

$\{v_1, v_2, v_3\}$ is a linearly dependent set

(b) Reduced echelon form: $\begin{bmatrix} 1 & 0 & 33 & 0 \\ 0 & 1 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Let $x_3 = \dots$ (any nonzero number). Then $x_1 = \dots$ and $x_2 = \dots$.

$$-\frac{\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}}{3} + \frac{\begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}}{5} + \frac{\begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}}{9} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\dots v_1 + \dots v_2 + \dots v_3 = 0$$

(one possible linear dependence relation)

A non-trivial solution to $Ax = 0$ is a linear dependence relation between the columns of A .

Theorem: Uniqueness of solutions for linear systems: For a matrix A , the following are equivalent:

- $Ax = 0$ has no non-trivial solution.
- If $Ax = b$ is consistent, then it has a unique solution.
- The columns of A are linearly independent.
- $\text{rref}(A)$ has a pivot in every column (i.e. all variables are basic).

In particular: the row reduction algorithm produces at most one pivot in each row of $\text{rref}(A)$. So, if A has more columns than rows (a “fat” matrix), then $\text{rref}(A)$ cannot have a pivot in every column.

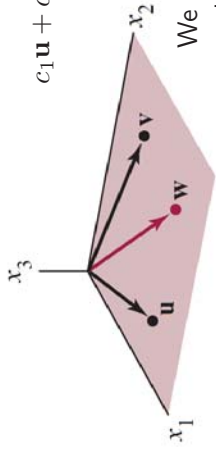
So a set of more than n vectors in \mathbb{R}^n is always linearly dependent.

Exercise: Combine this with Theorem 4 to show that a set of n linearly independent vectors span \mathbb{R}^n .

Problem: if $\{v_1, \dots, v_p\}$ is linearly dependent, then $\text{Span}\{v_1, \dots, v_p\}$ is the span of fewer vectors.

E.g. if $w = au + bv$, then $\text{Span}\{u, v, w\} = \text{Span}\{u, v\}$:

$$\begin{aligned} c_1u + c_2v + c_3w &= c_1u + c_2v + c_3(au + bv) \\ &= (c_1 + c_3a)u + (c_2 + c_3b)v. \end{aligned}$$



We want to remove from $\{v_1, \dots, v_p\}$ some vectors that are linear combinations of other v_i s.

One answer (casting-out algorithm):

Row reduce $\begin{bmatrix} | & | & | & | \\ v_1 & v_2 & \dots & v_p \\ | & | & | & | \end{bmatrix}$ and keep the vectors in the pivot columns.

The casting-out algorithm:

Example: Let

$$S = \left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ 8 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} \right\}.$$

Find a linearly independent subset R of S such that $\text{Span}R = \text{Span}S$.

Answer: $\begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to echelon form}]{\text{row reduction}} \begin{bmatrix} 1 & -3 & 4 & 3 & 2 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

The pivot columns are 1, 2 and 5, so $R = \left\{ \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} \right\}$ is one answer.
(The answer from the casting out algorithm is not the only answer.)

Why the casting-out algorithm works:

Example:

$$\begin{bmatrix} | & | & | & | & | \\ v_1 & v_2 & v_3 & v_4 & v_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to rref}]{\text{row reduction}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\text{rref} \left(\begin{bmatrix} | & | \\ v_1 \\ | \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a pivot in every column, so $\{v_1\}$ is linearly independent.

$\text{rref} \left(\begin{bmatrix} | & | \\ v_1 & v_2 \\ | \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a pivot in every column, so $\{v_1, v_2\}$ is linearly independent.

Why the casting-out algorithm works:

Example:

$$\begin{bmatrix} | & | & | & | & | \\ v_1 & v_2 & v_3 & v_4 & v_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 1 & -3 & 4 & -3 & 2 \end{bmatrix} \xrightarrow[\text{to rref}]{\text{row reduction}} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\text{rref} \left(\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ does not have a pivot in every column.

The solution set to $\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} x = 0$ is $x = s \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ where s can take any value.

Take $s = 1$: $\begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = 0$. So $2v_1 + 2v_2 + v_3 = 0$, so $v_3 = -2v_1 - 2v_2$, a linear combination of v_1 and v_2 . So we don't need v_3 to get the same span.

Why the casting-out algorithm works:

Example:

$$\left[\begin{array}{c|c|c|c|c|c} | & | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \\ \hline | & | & | & | & | & | \end{array} \right] = \left[\begin{array}{c|c|c|c|c|c} 0 & 3 & -6 & 6 & 4 & \\ 3 & -7 & 8 & -5 & 8 & \\ 1 & -3 & 4 & -3 & 2 & \\ \hline | & | & | & | & | & | \end{array} \right] \xrightarrow[\text{to rref}]{\text{row reduction}} \left[\begin{array}{c|c|c|c|c|c} 1 & 0 & -2 & 3 & 0 & \\ 0 & 1 & -2 & 2 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \\ \hline | & | & | & | & | & | \end{array} \right]$$

The solution set to $\left[\begin{array}{c|c|c|c|c|c} | & | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & & \\ \hline | & | & | & | & | & | \end{array} \right] \mathbf{x} = \mathbf{0}$ is $\mathbf{x} = s \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ where s, t can take any value.

Take $s = 0, t = 1$: $\left[\begin{array}{c|c|c|c|c|c} | & | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & & \\ \hline | & | & | & | & | & | \end{array} \right] \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$. So $-3\mathbf{v}_1 - 2\mathbf{v}_2 + 0\mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$, so $\mathbf{v}_4 = 3\mathbf{v}_1 + 2\mathbf{v}_2$, a linear combination of the **pivot columns**.

Why the casting-out algorithm works:

The row reduction algorithm writes the solution set of

$$\left[\begin{array}{c|c|c|c|c|c} | & | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_p & & \\ \hline | & | & | & | & | & | \end{array} \right] \mathbf{x} = \mathbf{0}$$

in the form $s_i \mathbf{w}_i + s_j \mathbf{w}_j + \dots$, where x_i, x_j, \dots are the free variables.

For each column \mathbf{v}_i corresponding to a free variable, the solution $A\mathbf{w}_i = \mathbf{0}$ allows you to write \mathbf{v}_i as a linear combination of the earlier pivot columns.

So $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is the same as the span of the pivot columns.

The casting-out algorithm is a “greedy algorithm”: it prefers vectors that are earlier in the set.

E.g. if you want a linearly independent subset of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ with the same span, and you want \mathbf{w} to be in this set, you should row-reduce $\left[\begin{array}{c|c|c} \mathbf{w} & \mathbf{u} & \mathbf{v} \end{array} \right]$.