In this week's notes, we are interested in finding the *local maxima* / *local minima* of a multivariate function f, i.e. the points $(a_1, \ldots, a_n)$ such that

$$\begin{array}{l} f(a_1, \ldots, a_n) \geq f(x_1, \ldots, x_n) \\ f(a_1, \ldots, a_n) \leq f(x_1, \ldots, x_n) \end{array}$$

for all $(x_1, \ldots, x_n)$ close to $f(a_1, \ldots, a_n)$.

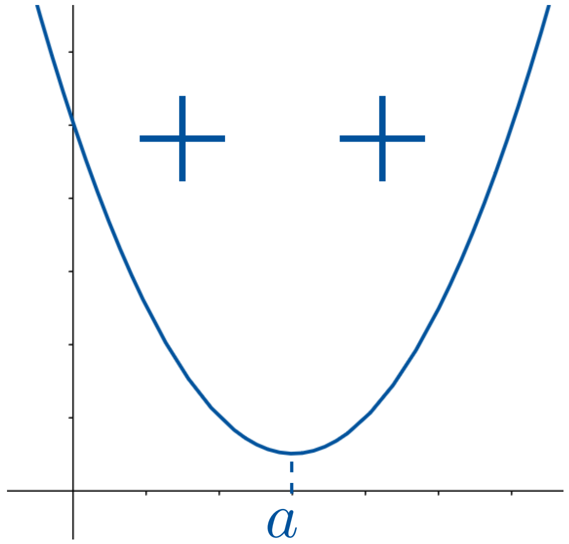
If $\nabla f(a_1, \ldots, a_n) \neq \mathbf{0}$, then f is increasing in the direction $\nabla f(a_1, \ldots, a_n)$ and decreasing in the direction $-\nabla f(a_1, \ldots, a_n)$, so $(a_1, \ldots, a_n)$ cannot be a local maximum or minimum. So a local maximum or minimum must be a critical point.

Definition: A point $(a_1, \ldots, a_n)$ is a *critical point* of $f : \mathbb{R}^n \to \mathbb{R}$ if $\nabla f(a_1, \ldots, a_n) = \mathbf{0}$, i.e. if all its partial derivatives are 0.

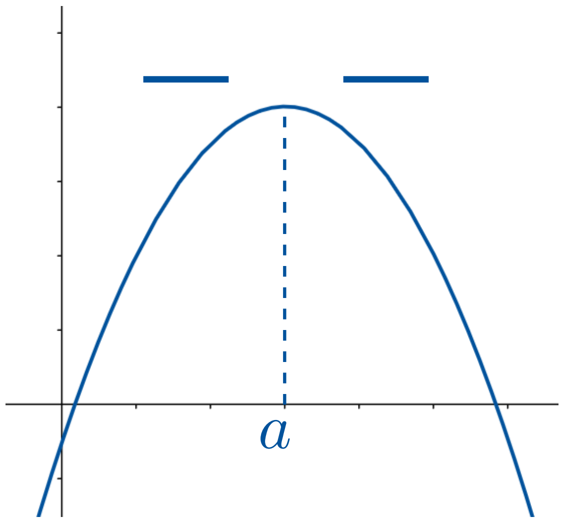
But not every critical point is a local maximum or minimum as we will see.

§13.1: Classifying Critical Points

Recall that a critical point of a single-variable function f is where the derivative f' is zero. A standard way to determine whether it is a local maximum, a local minimum, or neither, is the second derivative test:



if $f''(a) > 0$ then a is a local minimum



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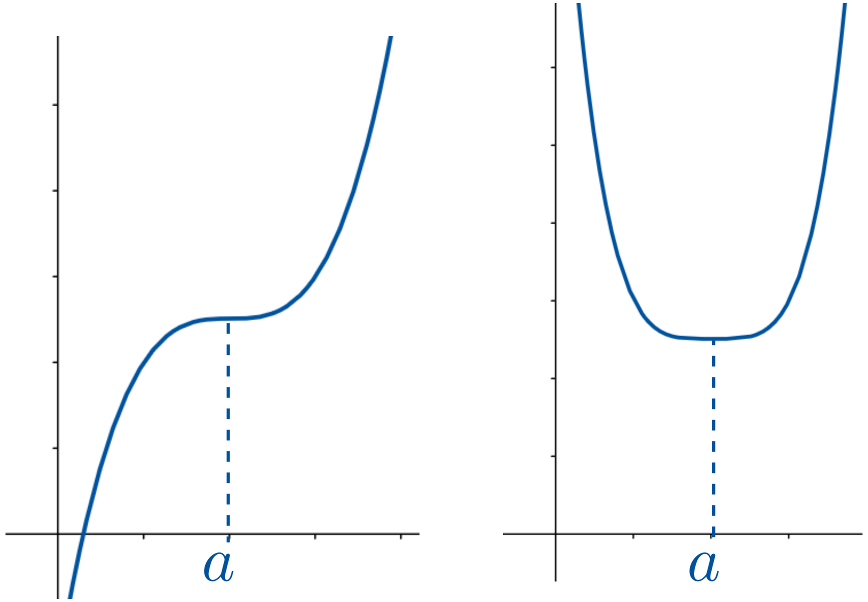

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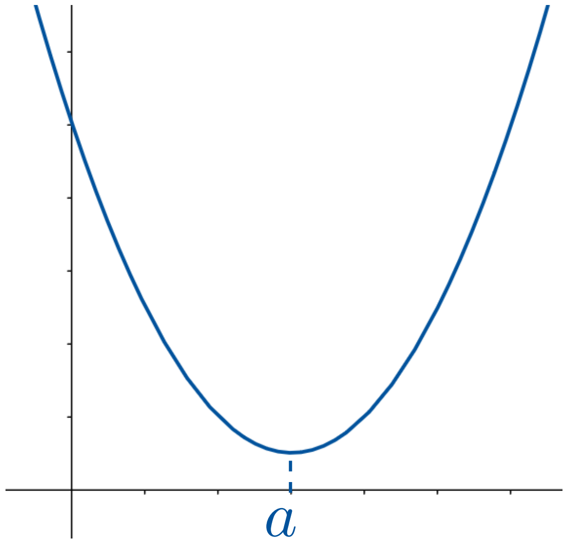
The reason is clear from considering the change in the slope of the graph, but because graphs of multivariate functions are hard to visualise, we give a different justification on the next page.



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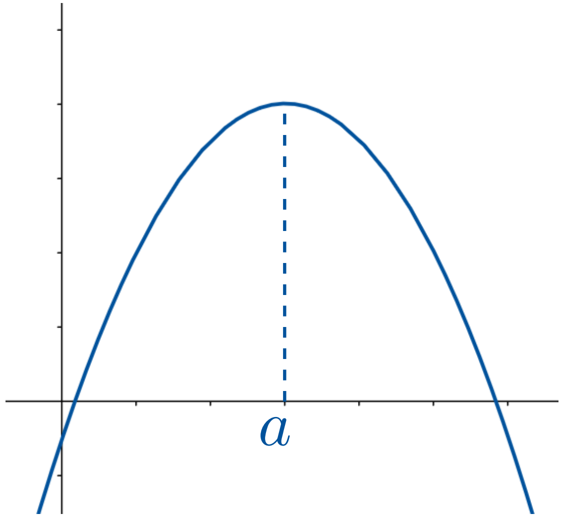

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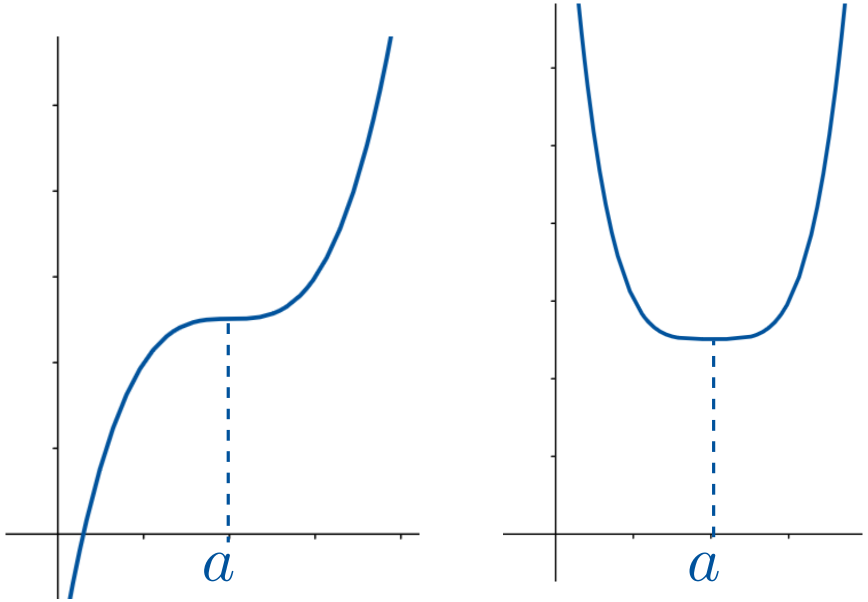

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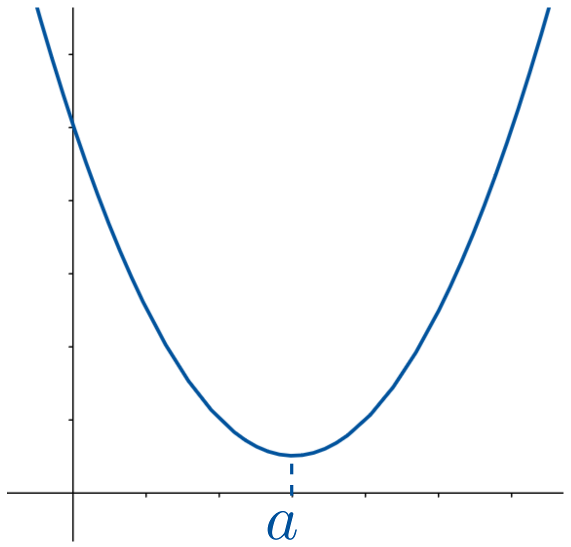
The second-order Taylor polynomial of f at a is

$$f(a+h) \approx f(a) + f'(a)\,h + \frac{f''(a)}{2!}h^2$$



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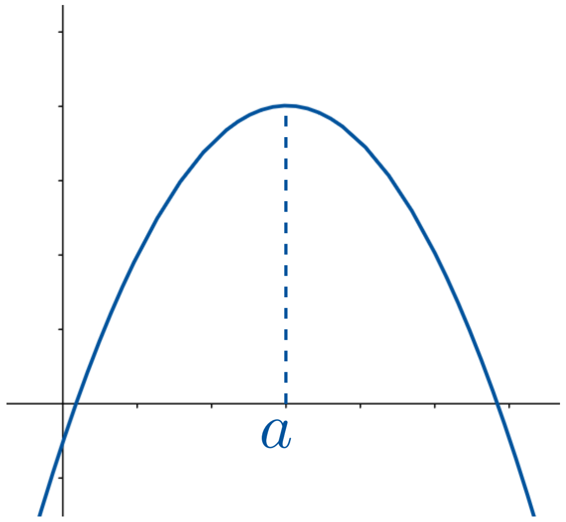

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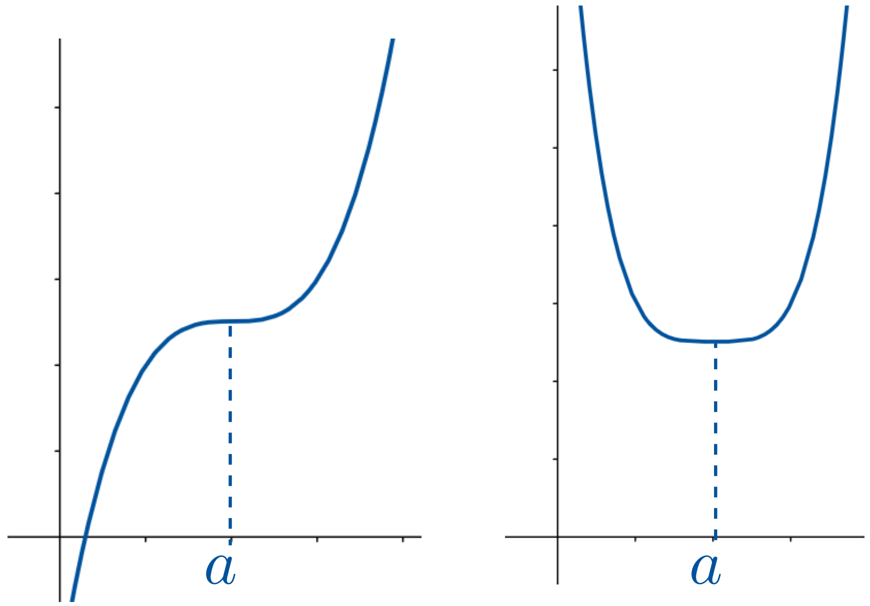

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The second-order Taylor polynomial of f at a is

is 0 if a is a critical point

$$f(a+h) \approx f(a) + f'(a)h + \frac{f''(a)}{2!}h^2$$

is positive if $h \neq 0$, i.e. $x \neq a$

$$= f(a) + \frac{f''(a)}{2!}h^2 \qquad \begin{cases} > f(a) & \text{if } f''(a) > 0 \text{ and } h \neq 0 \\ < f(a) & \text{if } f''(a) < 0 \text{ and } h \neq 0 \end{cases}$$

Here is a simplified example of how to use second order Taylor polynomials to classify critical points of multivariate functions.

Example: Find and classify the critical points of $f(x, y) = y^2 - x^3 + x$.

Now we develop a multivariate second derivative test by copying the previous example's argument in general.

The second-order Taylor polynomial of f about (a, b) is

$$f(a+h, b+k) \approx f(a,b)+f_x(a,b)h+f_y(a,b)k+\frac{f_{xx}(a,b)h^2+2f_{xy}(a,b)hk+f_{yy}(a,b)k^2}{2!},$$

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Definition: A function $Q : \mathbb{R}^n \to \mathbb{R}$ is a *quadratic form* if it is homogeneous of degree two i.e. a linear combination of $x_i x_j$. A quadratic form Q is:

- *positive definite* if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$; $\implies$ maximum
- *negative definite* if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$; $\implies$ minimum
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A quadratic form can be at most one of the three types. But it is possible to be none of the three types, e.g. $Q(h, k) = h^2$. (see later)

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Let's start with the 2-variable case: any 2-variable quadratic form has the form $Ah^2 + 2Bhk + Ck^2$. (We are interested in $f_{xx}(a,b)h^2 + 2f_{xy}(a,b)hk + f_{yy}(a,b)k^2$.)

In the previous example where $B = 0$, we can quickly tell the definiteness from the signs of A and C. In the general case, we will have to complete the square:

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So $Q(x)$ is positive definite

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To phrase this in a way that will extend to functions of more than 2 variables, we consider a matrix containing the second-order partial derivatives:

Definition: The *Hessian matrix* $\mathcal{H}(\mathbf{a})$ of $f : \mathbb{R}^n \to \mathbb{R}$ at a point $\mathbf{a}$ in $\mathbb{R}^n$ is the $n \times n$ matrix with $\dfrac{\partial^2 f}{\partial x_j x_i}(\mathbf{a})$ in row i and column j.

Example: For a 2-variable function f, the Hessian matrix is $\mathcal{H}(a,b) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$.

Let $D_i(\mathbf{a})$ denote the determinant of the $i \times i$ matrix containing only the first i rows and the first i columns of $\mathcal{H}(\mathbf{a})$.

Theorem: Second Derivative Test for 2-variable functions: Let (a,b) be a critical point of $f : \mathbb{R}^2 \to \mathbb{R}$.

If $D_1(a,b), D_2(a,b) > 0$, then $\mathcal{H}(a,b)$ is positive definite and (a,b) is a minimum.

If $D_1(a,b), D_2(a,b) > 0$, then $\mathcal{H}(a,b)$ is negative definite and (a,b) is a maximum.

If $D_2(a,b) \neq 0$ and the above conditions do not hold, then $\mathcal{H}(a,b)$ is indefinite and (a,b) is not a minimum or maximum.

If $D_2(a,b) = 0$, then the second derivative test is inconclusive, and we need more information.

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An n-variable quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$ can encoded in a $n \times n$ matrix $\mathcal{H}$ that is symmetric (i.e. $\mathcal{H}_{ij} = \mathcal{H}_{ji}$): let $Q(\mathbf{x}) = \mathcal{H}_{11}x_1^2 + \ldots$ Let D_i be the determinant of the

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We make the same definition for functions of more variables.

Definition: A point $(a_1, \ldots, a_n)$ is a *critical point* of $f : \mathbb{R}^n \to \mathbb{R}$ if $\nabla f(a_1, \ldots, a_n) = \mathbf{0}$, i.e. if all its partial derivatives are 0.

For a 2-variable function, a critical point is the (x, y) coordinates of a point where the tangent plane to the graph is horizontal.