You must justify your answers to receive full credit.

1. Let A be the matrix
$$A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix},$$
and T be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.
   a) What is the domain of T?
   b) What is the codomain of T?
   c) Is T onto? (Hint: you can use a theorem.)
   d) Find the kernel of T.
   e) Is T one-to-one?
   f) Find the image of $\mathbf{e_1} + \mathbf{e_2}$ under T.

2. For each of the transformations below:
(i) decide whether it is linear, and explain your answer,
(ii) if it is linear, find its standard matrix.
   a) $f : \mathbb{R}^4 \to \mathbb{R}^2$ given by $f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{2}x_1 + x_2 - x_3 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
   b) $g : \mathbb{R}^2 \to \mathbb{R}^3$ given by $g(\mathbf{x}) = \mathbf{0}$.
   c) $h : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation through an angle of $\frac{\pi}{2}$ counterclockwise about the point (-1,1).

3. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation satisfying
$$F\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 2 \\ 8 \end{bmatrix}, \quad F\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \\ 12 \end{bmatrix}.$$
Find the standard matrix of F.

4. Let
$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$
Calculate the following matrices, or explain why they are not defined.

a) AB b) BAB^T c) $B+I_3$ d) x^TB
e) B^3 f) A^3 g) A^{30}

5. Suppose A is an invertible matrix such that

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & -2 & 3 & 5 \\ 1 & 0 & 1 & 3 \end{bmatrix}.$$

Find A.

6. State whether each of the following statements is (always) true or (sometimes) false. If it is true, give a brief justification; if it is false, give a counterexample with an explanation.

   a) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and $\{T(\mathbf{v_1}), T(\mathbf{v_2}), T(\mathbf{v_3})\}$ span $\mathbb{R}^m$, then T is onto.
   b) If $S : \mathbb{R}^3 \to \mathbb{R}^3$ is onto and $T : \mathbb{R}^3 \to \mathbb{R}^3$ is one-to-one, then $S \circ T$ is one-to-one.
   c) If A is a square matrix, then $(A^2)^T = (A^T)^2$.
   d) Each column of the matrix product AB is a linear combination of the columns of B.
   e) For all 2×2 matrices A, it is true that $(A + I_2)(A - I_2) = A^2 - I_2$.
   f) For all 2×2 matrices A, P, it is true that $A = PAP^{-1}$. (Hint: Try $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.)

7. **Optional Problem**: Consider a group of 5 students. Student 1 is friends on Facebook with each of the other four students. Also, Student 4 is Facebook friends with Student 3 and Student 5. There are no other Facebook friendships among the 5 students.

   Let A be a 5×5 matrix, where the entry in row i and column j is 1 if Student i and Student j are Facebook friends, and 0 if Student i and Student j are not Facebook friends. So A is a symmetric matrix. (We assume Facebook does not allow a student to be friends with himself or herself, so all diagonal entries of A are zero.)

   a) Write down the matrix A.
   b) Let u be the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Calculate $A\mathbf{u}$. What is the meaning of the ith entry of $A\mathbf{u}$?
   c) Calculate A^2. What is the meaning of the (i, j)-entry of A^2, when $i \neq j$?
   This is the beginnings of the subject of "algebraic graph theory".

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