Homework 4, due 16:00 Monday, 15 April 2019 to Dr. Pang's mailbox

You must justify your answers to receive full credit.

1. Consider

$$\mathscr{A} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

You are given that $\mathscr{A}$ is a basis of $\mathbb{R}^3$.

a) Find the dual basis $\hat{\mathscr{A}} = \{\phi_1, \phi_2, \phi_3\} \subseteq \hat{\mathbb{R}^3}$ (i.e. for each element in $\hat{\mathscr{A}}$, write out what it does to $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$).

b) Let $\phi \in \hat{\mathbb{R}^3}$ be defined by $\phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x + 2y$. Express ϕ as a linear combination of the elements in $\hat{\mathscr{A}}$.

c) Now consider $\sigma : P_{<2}(\mathbb{R}) \to \mathbb{R}^3$ given by $\sigma(f) = \begin{pmatrix} f(0) \\ 2f(0) \\ -f(1) \end{pmatrix}$. If $\psi \in \hat{\mathbb{R}^3}$ is given by $\psi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$, then what is $\hat{\sigma}(\psi)$?

2. (Lagrange Interpolation) Let $V = P_{<3}(\mathbb{R})$ be the vector space of polynomials over $\mathbb{R}$ of degree less than 3, with some basis $\mathscr{A} = \{\alpha_1, \alpha_2, \alpha_3\}$. Suppose the dual basis $\hat{\mathscr{A}} = \{\phi_1, \phi_2, \phi_3\}$ of $\hat{V}$ is given by

$$\phi_1(f) = f(1), \quad \phi_2(f) = f(2), \quad \phi_3(f) = f(3).$$

a) Check that $\alpha_1 = \dfrac{(x-2)(x-3)}{(1-2)(1-3)}$.

By the same argument, $\alpha_2 = \dfrac{(x-1)(x-3)}{(2-1)(2-3)}$.

b) Write down, without explanation, α_3.

c) Suppose $f \in V$ satisfies $f(1) = 6, f(2) = 5, f(3) = 10$. By writing f as a linear combination of α_i, find f. (Click here for a hint)

3. Let V be a finite-dimensional vector space over $\mathbb{F}$.

   a) Let W be a subspace of V, and $\alpha \in V, \alpha \notin W$. By considering dual bases, or otherwise, show that there is a $\phi \in \hat{V}$ such that $\phi(\alpha) = 1$ and $\phi(\beta) = 0$ for all $\beta \in W$.

   b) Let α, β be linearly independent vectors in V. Using part a or otherwise, show that there is a $\phi \in \hat{V}$ such that $\phi(\alpha) = 1$ and $\phi(\beta) = 0$.

   c) Now suppose $\alpha, \beta \in V$ has the property that, given any $\phi \in \hat{V}$ with $\phi(\beta) = 0$, then $\phi(\alpha) = 0$. Show that α is a multiple of β. (Hint: you may wish to use a proof by contradition, using parts a and b.)

4. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 10 \end{pmatrix}$$

   a) Find a matrix P and a diagonal matrix D such that $P^T AP = D$.

   b) What is the rank of A?

   c) What is the signature of A?

   d) What is the definiteness of A?

5. Let $V = P_{<3}(\mathbb{R})$ be the vector space of polynomials over $\mathbb{R}$ of degree less than 3. Define a quadratic form on V by

$$q(p) = p(1)p'(1) + \int_0^1 p'(t)^2 \, dt.$$

   a) Find the symmetric bilinear form f such that $q(p) = f(p, p)$.

   b) Consider the basis $\mathscr{A} = \{1, 2 - x, x^2\}$ of V. Find the matrix $\{f\}_{\mathscr{A}}$.

   c) Let $\mathscr{B} = \{3, 2 - x, 4 - 2x + x^2\}$ of V. Find the matrix $\{f\}_{\mathscr{B}}$. **You may give your answer as a product of matrices and/or their inverses.**

6. a) Let V be a vector space over $\mathbb{F}$. Recall from class that, given $\alpha \in V$, we can define a function f on $\hat{V}$ by "evaluation at α", i.e. $f(\phi) = \phi(\alpha)$, for each $\phi \in \hat{V}$. Show that f is linear (in its input in $\hat{V}$).

   b) Let A be a positive definite matrix over $\mathbb{R}$. Show that $\det A > 0$. (Hint: Let $D = P^T AP$, and consider $\det D$.)

The following two questions are to prepare you for upcoming classes, and is unrelated to the material from recent classes.

7. Let $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ -2 \end{pmatrix} \right\} \subseteq \mathbb{R}^4$.

   a) Calculate the orthogonal projection of $\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ onto W.

   b) Find the closest point in W to $\begin{pmatrix} 0 \\ 3 \\ -3 \\ 3 \end{pmatrix}$.

   c) Find the distance from $\begin{pmatrix} 0 \\ 3 \\ -3 \\ 3 \end{pmatrix}$ to W.

8. Consider (over $\mathbb{R}$)

$$A = \begin{pmatrix} 8 & 4 & -1 \\ 4 & -7 & 4 \\ -1 & 4 & 8 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} 1 \\ -4 \\ -17 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}.$$

   a) By stating an appropriate theorem (no calculation!), explain why A is diagonalisable.
   b) It is known that α_1, α_2 are eigenvectors of A, and that A has another eigenvector β whose eigenvalue is different from that of α_1 and α_2. By using an appropriate theorem, and without finding any eigenvalues, find one such β.

Optional questions If you attempted seriously all the above questions, then your scores for the following questions may replace any lower scores for two of the above questions.

9. Let U, V be finite-dimensional vector spaces over $\mathbb{F}$. Suppose $\sigma \in L(U, V)$ is surjective. Show, without using matrices, that its dual map $\hat{\sigma} \in L(\hat{V}, \hat{U})$ is injective.

10. True or false: A is positive definite if and only if all entries of A are positive.

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