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Rearrangement:  $(2)x_1 + (6)x_2 = 0$

Another example:  $x_2 = \sqrt{2}(\sqrt{6} - x_1) + x_3$

Rearrangement:  $\sqrt{2}x_1 + (1)x_2 + (-1)x_3 = 2\sqrt{3}$

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In general, a **linear equation** is an equation of the form

$$a_1x_1 + a_2x_2 + \dots a_nx_n = b.$$

$x_1, x_2, \dots x_n$  are the **variables**.  $a_1, a_2, \dots a_n$  are the **coefficients**.

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**Definition:** A **system of linear equations** (or a **linear system**) is a collection of linear equations involving the same set of variables.

Example:

$$\begin{array}{rclcl} x & +y & & = & 3 \\ 3x & & +2z & = & -2 \end{array}$$

This is a system of **2 equations** in **3 variables**,  $x, y, z$ .

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|          |  |   |
|----------|--|---|
| Example: | $\begin{array}{rcl} x & +y & = 3 \\ 3x & & +2z = -2 \end{array}$ | A solution is:<br>$(x, y, z) = (2, 1, -4).$ |
|----------|--|---|

This is a system of **2 equations** in **3 variables**,  $x, y, z$ .

**Definition:** A **solution** of a linear system is a list  $(s_1, s_2, \dots, s_n)$  of numbers that makes each equation a true statement when the values  $s_1, s_2, \dots, s_n$  are substituted for  $x_1, x_2, \dots, x_n$  respectively.

**Definition:** The **solution set** of a linear system is the set of all possible solutions.



**Definition:** A linear system is *consistent* if it has a solution,  
and *inconsistent* if it does not have a solution.

**Fact:** A linear system has either

- |                             |              |
|-----------------------------|--------------|
| • exactly one solution      | consistent   |
| • infinitely many solutions | consistent   |
| • no solutions              | inconsistent |

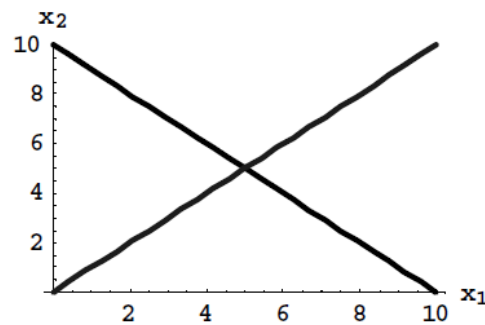
**Definition:** A linear system is *consistent* if it has a solution,  
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- infinitely many solutions              consistent
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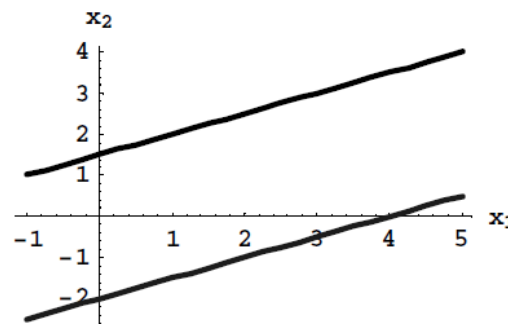
**EXAMPLE** Two equations in two variables:

$$\begin{aligned}x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0\end{aligned}$$



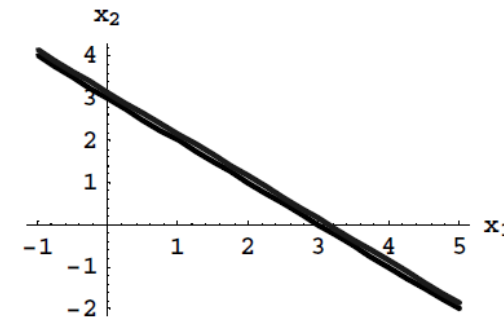
one unique solution  
consistent

$$\begin{aligned}x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8\end{aligned}$$



no solution  
inconsistent

$$\begin{aligned}x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6\end{aligned}$$

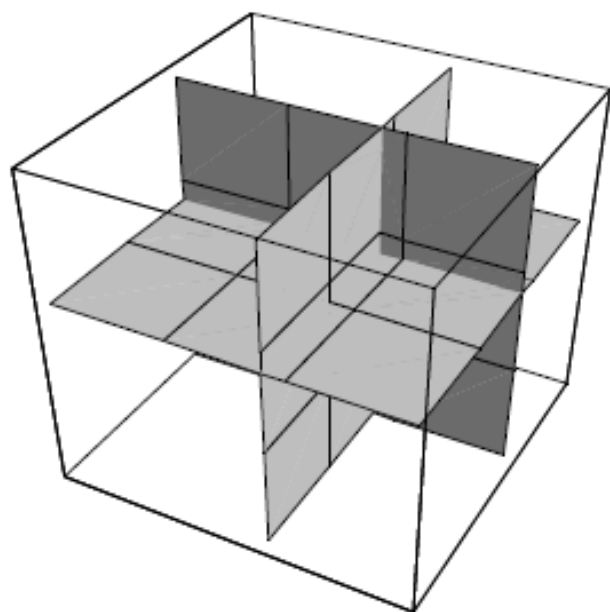


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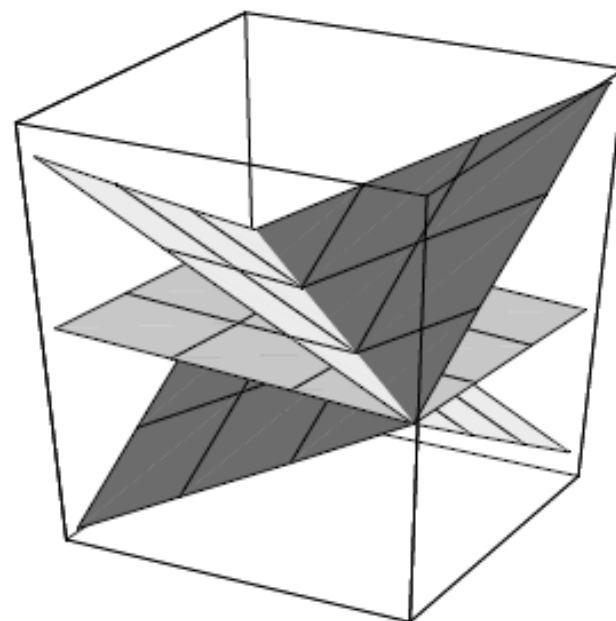
$$ax + by + cz = d, \text{ or } z = a'x + b'y + d'$$

**EXAMPLE:** Three equations in three variables. Each equation determines a plane in 3-space.

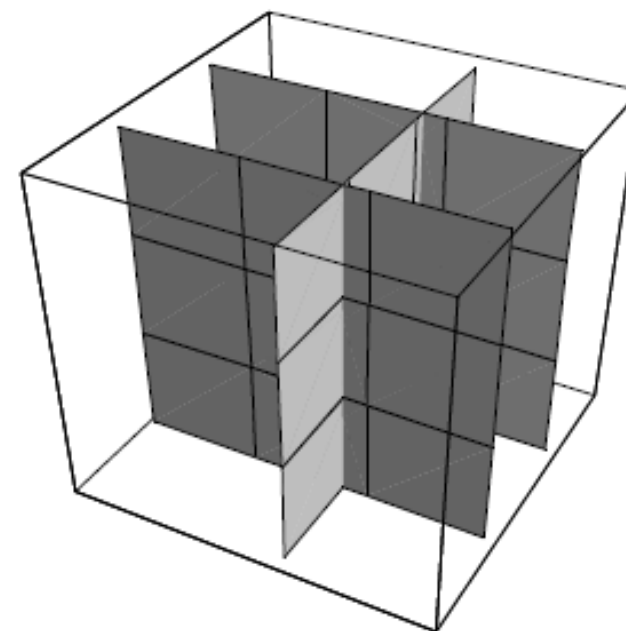
i) The planes intersect in one point. (*one solution*)



ii) The planes intersect in one line. (*infinitely many solutions*)



iii) There is no point in common to all three planes. (*no solution*)

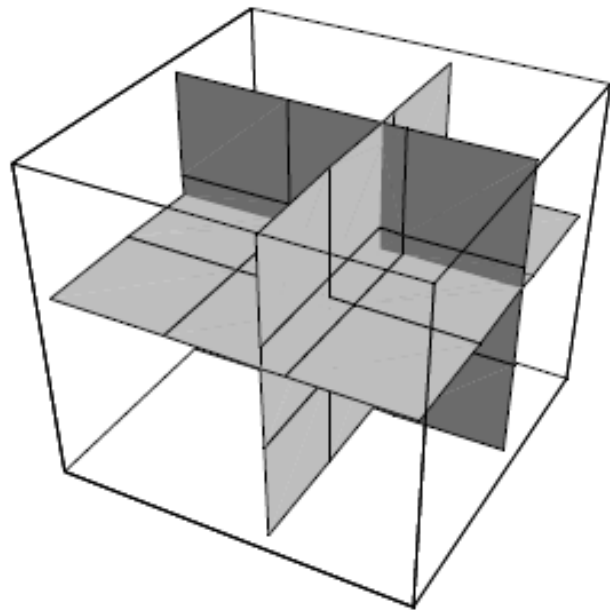


Which of these cases are consistent?

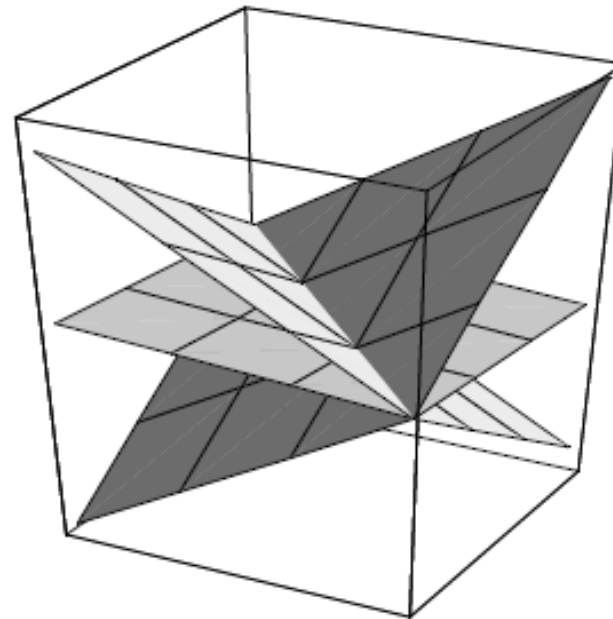
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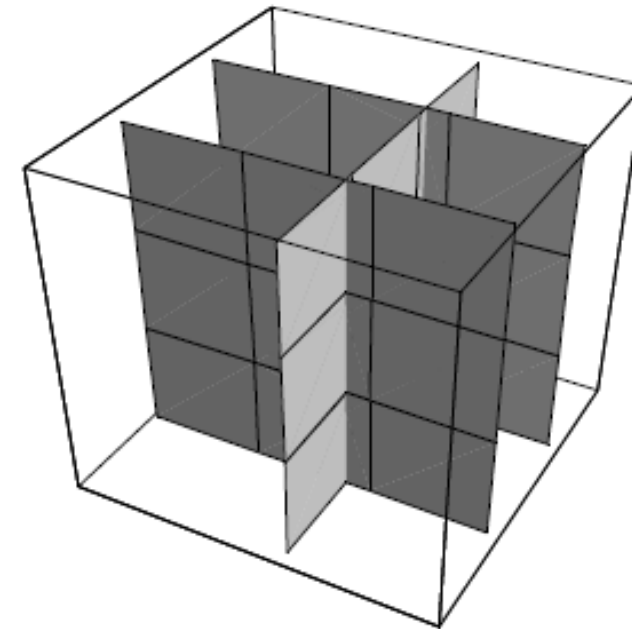
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How to solve a linear system? **Example:**

$$x_1 - 2x_2 = -1$$

$$-x_1 + 3x_2 = 3$$

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$$\begin{array}{rcl} x_1 - 2x_2 & = & -1 \\ -x_1 + 3x_2 & = & 3 \end{array} \quad \rightarrow \quad \begin{array}{rcl} x_1 - 2x_2 & = & -1 \\ & x_2 & = & 2 \end{array} \quad \rightarrow \quad \begin{array}{rcl} x_1 & = & 3 \\ & x_2 & = & 2 \end{array}$$

How to solve a linear system? **Example:**

$$\begin{array}{lcl} R_1 & x_1 - 2x_2 & = -1 \\ R_2 & -x_1 + 3x_2 & = 3 \end{array} \quad \xrightarrow{R_1 + R_2} \quad \begin{array}{lcl} & x_1 - 2x_2 & = -1 \\ & x_2 & = 2 \end{array} \quad \xrightarrow{R_1 + 2R_2} \quad \begin{array}{lcl} & x_1 & = 3 \\ & x_2 & = 2 \end{array}$$

# How to solve a linear system? **Example:**

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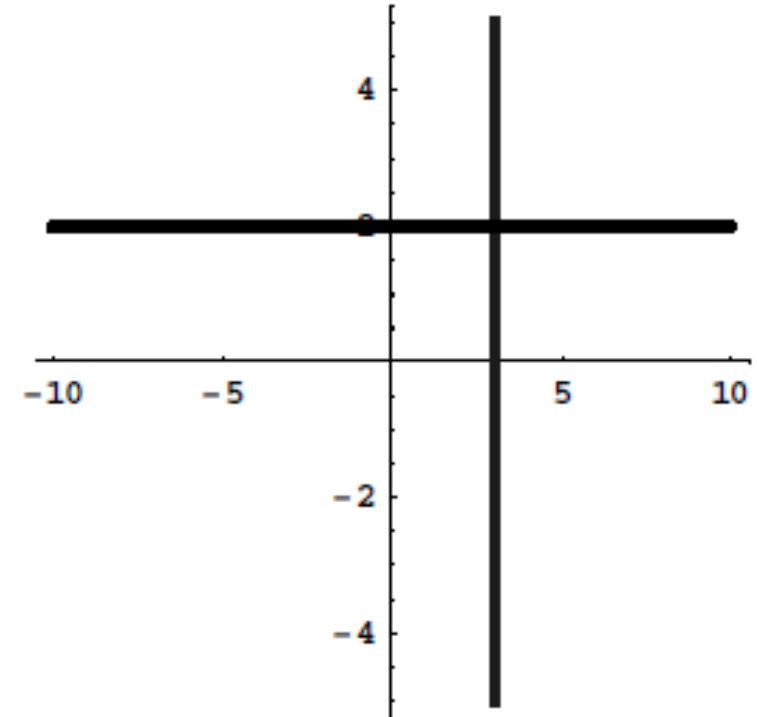
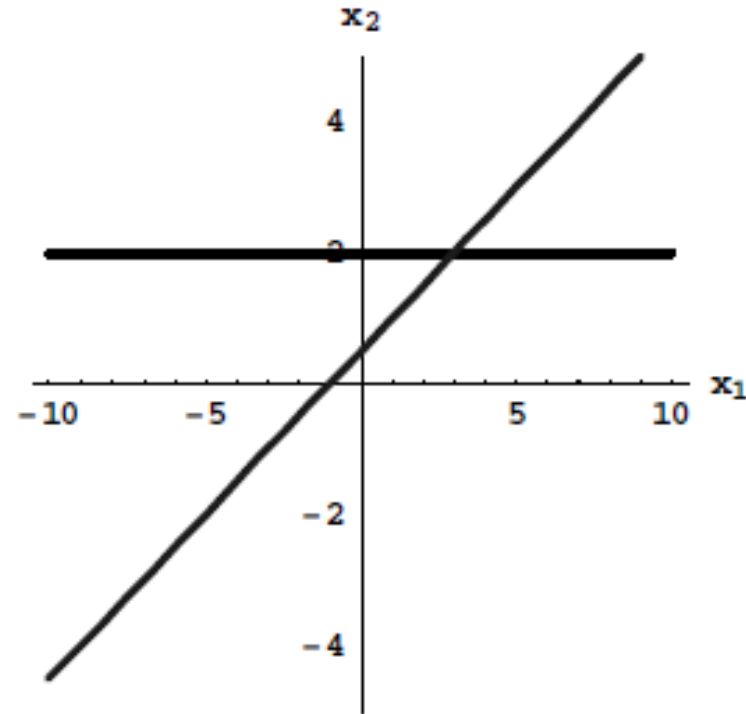
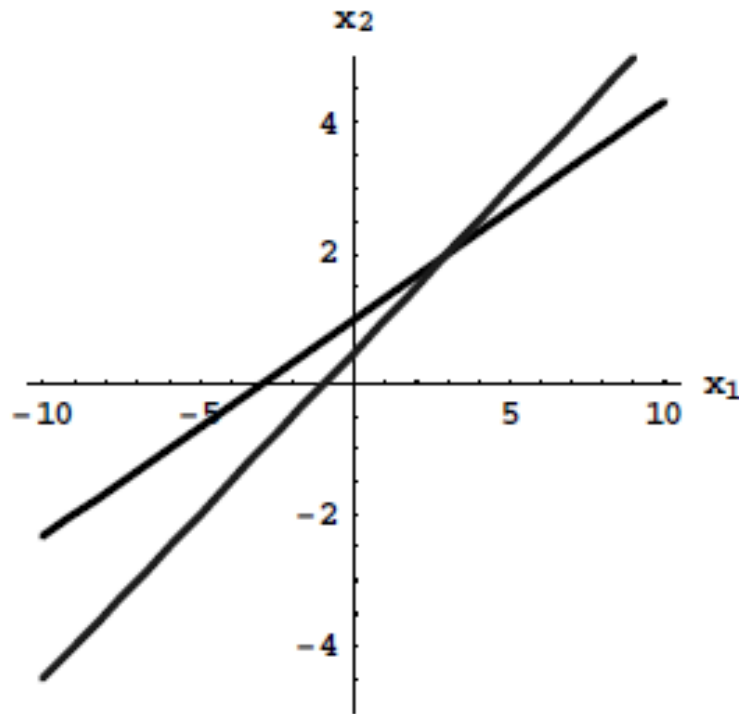
$$R_2 \quad -x_1 + 3x_2 = 3$$

$$\rightarrow \quad x_1 - 2x_2 = -1$$

$$R_1 + R_2 \rightarrow \quad x_2 = 2$$

$$R_1 \rightarrow R_1 + 2R_2 \rightarrow \quad x_1 = 3$$

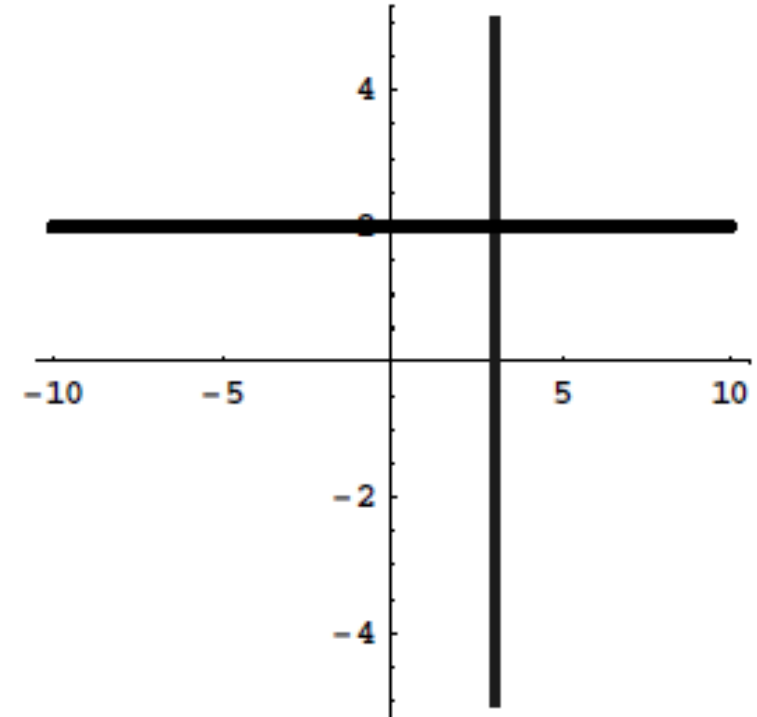
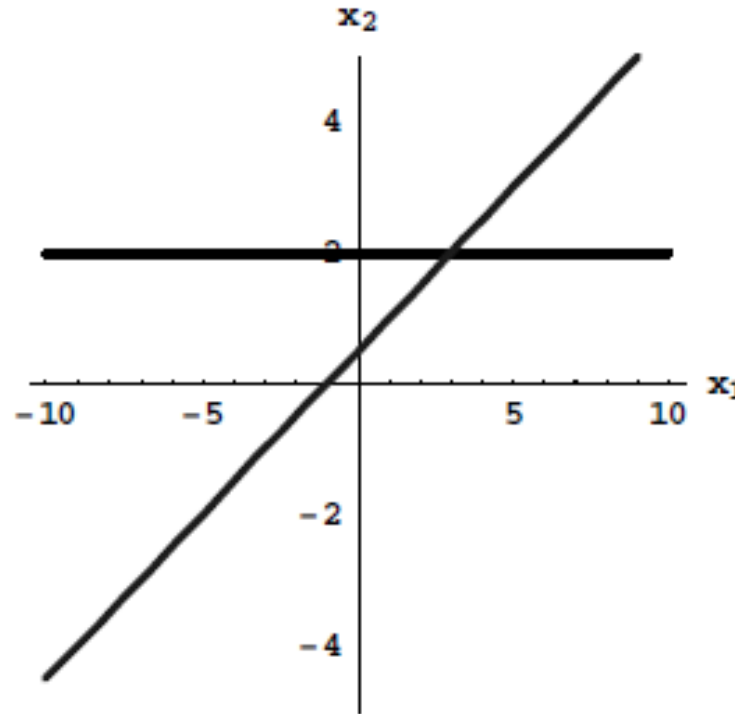
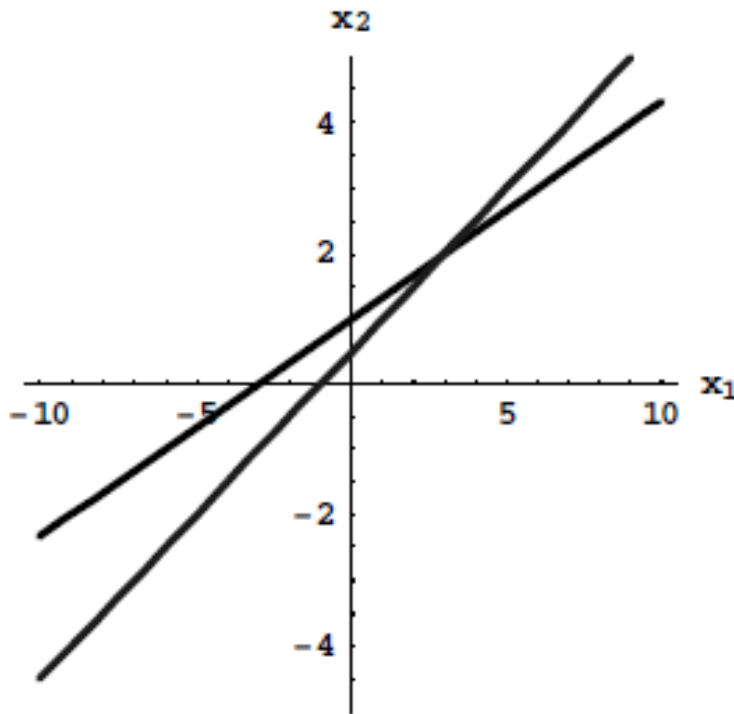
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## How to solve a linear system? **Example:**

$$\begin{array}{lcl} R_1 & x_1 - 2x_2 & = -1 \\ R_2 & -x_1 + 3x_2 & = 3 \end{array} \quad \rightarrow \quad \begin{array}{lcl} & x_1 - 2x_2 & = -1 \\ R_1 + R_2 & \rightarrow & x_2 = 2 \end{array} \quad \begin{array}{lcl} R_1 \rightarrow +2R_2 & \rightarrow & x_1 = 3 \\ & & x_2 = 2 \end{array}$$



**Definition:** Two linear systems are *equivalent* if they have the same solution set.

General strategy for solving a linear system: replace one system with an equivalent system that is easier to solve.

Simplify the writing by using **matrix notation**:

$$\begin{array}{lcl}
 R_1 & x_1 - 2x_2 = -1 & \\
 R_2 & -x_1 + 3x_2 = 3 & 
 \end{array}
 \xrightarrow{R_1 + R_2}
 \begin{array}{lcl}
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$$\left[ \begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array} \right] \longrightarrow \left[ \begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array} \right] \longrightarrow \left[ \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array} \right]$$

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The **augmented matrix** of a linear system contains the right hand side:

$$\left[ \begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array} \right]$$

The **coefficient matrix** of a linear system is the left hand side only:

$$\left[ \begin{array}{cc} 1 & -2 \\ -1 & 3 \end{array} \right]$$

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Elementary row operations:

1. **Replacement**: add a multiple of one row to another row.  $R_i \rightarrow R_i + cR_j$
2. **Interchange**: interchange two rows.  $R_i \rightarrow R_j, R_j \rightarrow R_i$
3. **Scaling**: multiply all entries in a row by a nonzero constant.  $R_i \rightarrow cR_i, c \neq 0$

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**Definition:** Two matrices are *row equivalent* if one can be transformed into the other by a sequence of elementary row operations.

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**Definition:** Two matrices are *row equivalent* if one can be transformed into the other by a sequence of elementary row operations.

**Fact:** If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set, i.e. they are equivalent linear systems.

**Warning:** Do not do multiple elementary row operations at the same time, **except** adding multiples of **the same** row to several rows.

$$\begin{aligned}x_1 - 2x_2 &= 1 \\ -x_1 + 3x_2 &= 3\end{aligned}$$

$$x_2 = 2$$

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These are NOT  
equivalent systems.

$$\left[ \begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 0 & 1 & 2 \\ 0 & 1 & 2 \end{array} \right] \begin{array}{l} \leftarrow R_1 + R_2 \\ \leftarrow R_2 + R_1 \end{array}$$

$$\begin{aligned}x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3\end{aligned} \quad \left[ \begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right] \begin{array}{l} \leftarrow R_1 - R_3 \\ \leftarrow R_2 + 4R_3 \end{array}$$

$$\begin{aligned}x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3\end{aligned} \quad \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

Two fundamental questions:

1. **Existence** of solutions: is the system consistent?
2. **Uniqueness** of solutions: if a solution exists, is it the only one?



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1. **Existence** of solutions: is the system consistent?
2. **Uniqueness** of solutions: if a solution exists, is it the only one?

Answering this requires less work than finding the solution.

**Example:**

$$\begin{array}{rrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \quad \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right]$$

$$\begin{array}{rrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & & -3x_2 & + & 13x_3 & = & -9 \end{array} \quad \left[ \begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

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We can stop here:  
back-substitution shows  
that we can find a unique  
solution.

echelon

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$$

reduced echelon

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

Are these matrices in echelon form, reduced echelon form, or neither?

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

echelon

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$$

reduced echelon

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

Are these matrices in echelon form, reduced echelon form, or neither?

$$\begin{bmatrix} 1 & 0 & 0 \\ \textcolor{red}{1} & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

neither

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

reduced  
echelon

$$\begin{bmatrix} 1 & \textcolor{red}{1} & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

echelon

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

reduced  
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$$\begin{array}{rcl} x_1 & & = 29 \\ & x_2 & = 16 \\ & & x_3 = 3 \end{array}$$

**Theorem:** Any matrix  $A$  is row-equivalent to exactly one reduced echelon matrix, which is called its **reduced echelon form** and written  $\text{rref}(A)$ .

General strategy for solving a linear system: apply row operations to its augmented matrix to obtain its  $\text{rref}$ .

General strategy for determining existence/uniqueness of solutions: apply row operations to its augmented matrix to obtain an **echelon form**, i.e. a row-equivalent echelon matrix.

These processes of row operations (to get to echelon or reduced echelon form) are called **row reduction**.

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Warning: an echelon form is not unique. Its entries depend on the row operations we used. But its pattern of ■ and \* is unique.

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Row reduction:

augmented matrix of linear  
system



echelon  
form



reduced  
echelon  
form

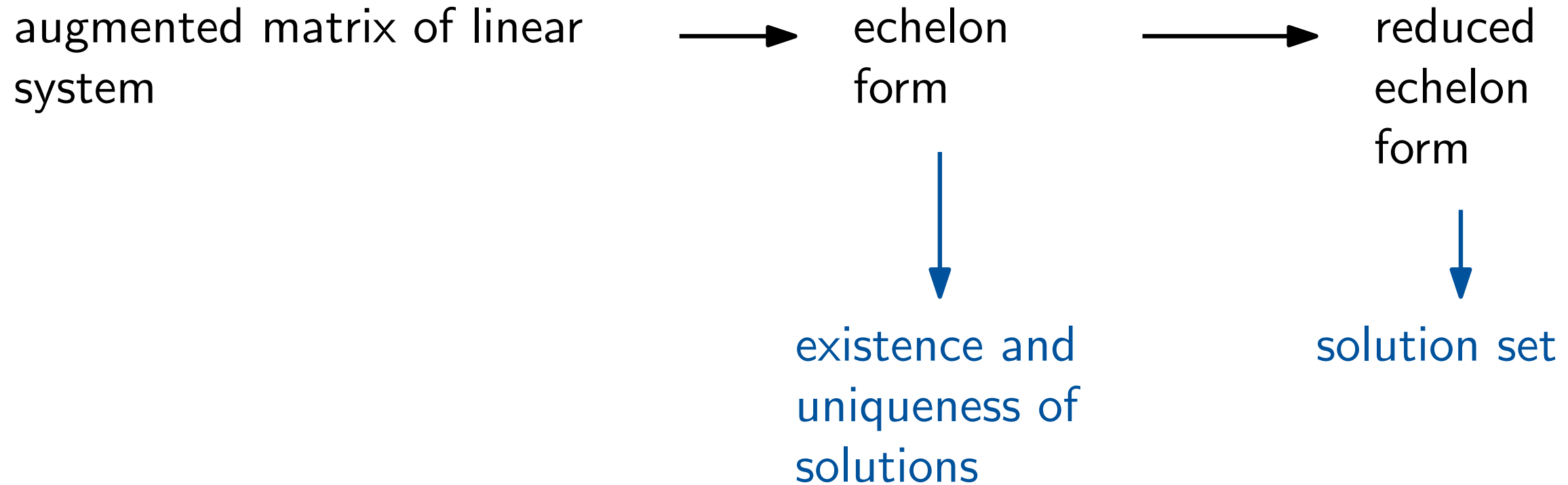


existence and  
uniqueness of  
solutions



solution set

Row reduction:



The rest of this section:

- The row reduction algorithm
- Getting the solution, existence/uniqueness from the (reduced) echelon form



Important terms in the row reduction algorithm:

- **pivot position**: the position of a leading entry in a row-equivalent echelon matrix.
- **pivot**: a nonzero entry of the matrix that is used in a pivot position to create zeroes below it.
- **pivot column**: a column containing a pivot position.

The black squares are the pivot positions.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$$

Check your answer: [www.wolframalpha.com](http://www.wolframalpha.com)



rref{{0 , 3 , -6 , 6 , 4 , -5},{3 , -7 , 8 , -5 , 8 , 9},{1 , -3 , 4 , -3 , 2 , 5}}

☆

≡



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Input:

|            |   |
|------------|---|
| row reduce | $\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{pmatrix}$ |
|------------|---|

Result:

$$\begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

Step-by-step solution

Getting the solution set from the reduced echelon form:

A **basic variable** is a variable corresponding to a pivot column.

All other variables are **free variables**.

**Example:**

$$\left[ \begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right]$$

$$\begin{aligned} x_1 - 2x_3 + 3x_4 &= -24 \\ x_2 - 2x_3 + 2x_4 &= -7 \\ x_5 &= 4 \end{aligned}$$

basic variables:  $x_1, x_2, x_5$ , free variables:  $x_3, x_4$ .

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basic variables:  $x_1, x_2, x_5$ , free variables:  $x_3, x_4$ .

The free variables can take any value. These values then uniquely determine the basic variables.

**Example:**

$$x_1 = -24 + 2x_3 - 3x_4$$

$$x_2 = -7 + 2x_3 - 2x_4$$

$$x_3 = x_3$$

$$x_4 = x_4$$

$$x_5 = 4$$

Getting the solution set from the reduced echelon form:

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**Example:**

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**Example:**

$$\begin{aligned} x_1 &= -24 + 2x_3 - 3x_4 \\ x_2 &= -7 + 2x_3 - 2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \\ x_5 &= 4 \end{aligned}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -24 + 2s - 3t \\ -7 + 2s - 2t \\ s \\ t \\ 4 \end{pmatrix}$$

Getting the solution set from the reduced echelon form:

Another example: reduced echelon form is  $\left[ \begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array} \right]$

Getting the solution set from the reduced echelon form:

Another example: reduced echelon form is  $\left[ \begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array} \right]$

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the system is inconsistent

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$0x_1 + 0x_2 + 0x_3 = 15 \longrightarrow$   
the system is inconsistent

## Theorem 2: Existence and Uniqueness:

A linear system is consistent if and only if an echelon form of its augmented matrix has **no** row of the form  $[0 \dots 0 | *]$  with  $* \neq 0$ .

If a linear system is consistent, then:

- it has a unique solution if there are no free variables;
- it has infinitely many solutions if there are free variables.



Next week: we talk about this:

