Previously, we integrated a single-variable function over an interval (i.e. a subset of $\mathbb{R}^1$).

$$\int_a^b f(x)\,dx$$

This week's notes will focus on multiple integration, of a function of two or three variables, over a domain D, which is a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$:

- Integrals over rectangular domains (p3-10, §14.1-14.2)
- Integrals over other 2D domains (p11-22, §14.1-14.2)
- Integrals over discs and sectors (p25-35, §14.4)

$$\iint_D f(x,y)\,dA$$

- Integrals over 3D domains (p36-42, §14.5)
- Integrals over cylinders (p44-46, §10.6,14.6)
- Integrals over balls and cones (p47-53, §10.6,14.6)

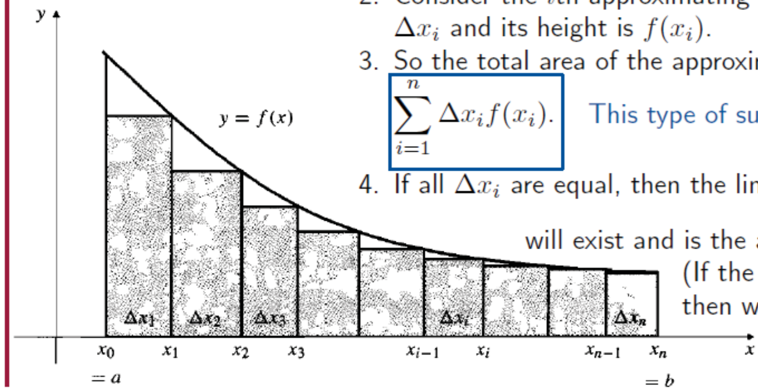
$$\iiint_D f(x,y,z)\,dV$$

Note that these are all definite integrals - we will not consider indefinite integrals in higher dimensions. The symbol $\iint$, without a D attached, has no meaning.

Remember from Week 3 slides:

In general, to find the area under the graph of a continuous, positive function $f:[a,b]\to\mathbb{R}$:

1. Divide $[a,b]$ into n subintervals by choosing x_i satisfying $a=x_0<x_1<\cdots<x_n=b$. Let $\Delta x_i = x_i - x_{i-1}$.
2. Consider the ith approximating rectangle: its width is Δx_i and its height is $f(x_i)$.
3. So the total area of the approximating rectangles is $\boxed{\sum_{i=1}^{n}\Delta x_i f(x_i).}$ This type of sum is a *Riemann sum*
4. If all Δx_i are equal, then the limit $\lim_{n\to\infty}\sum_{i=1}^{n}\Delta x_i f(x_i)$ will exist and is the area under the graph. (If the Δx_i are not all equal, then we have to choose x_i carefully.)

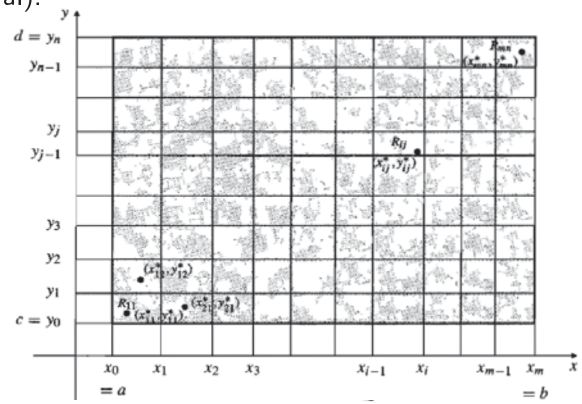


§14.1-14.2: Double Integrals

define later, §12.2

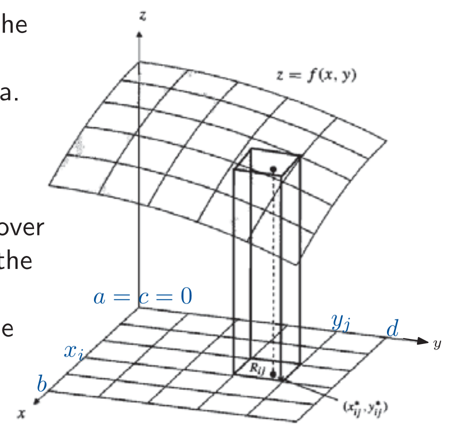
Suppose we wish to find the volume under the graph of a continuous, positive 2-variable function $f(x,y)$, whose domain is a rectangle $a\le x\le b$, $c\le y\le d$ (a 2-dimensional version of an interval).

1. Divide the domain into mn smaller rectangles by choosing x_i and y_j with $a=x_0<x_1<\cdots<x_m=b$ and $c=y_0<y_1<\cdots<y_n=d$. Let R_{ij} be the small rectangle with $x_{i-1}<x<x_i$ and $y_{j-1}<y<y_j$, and write ΔA_{ij} for its area.



We wish to find the volume under the graph of a continuous, positive 2-variable function $f(x,y)$, whose domain is a rectangle $a\le x\le b$, $c\le y\le d$.

1. Divide the domain into mn smaller rectangles by choosing x_i and y_j with $a=x_0<x_1<\cdots<x_m=b$ and $c=y_0<y_1<\cdots<y_n=d$. Let R_{ij} be the small rectangle with $x_{i-1}<x<x_i$ and $y_{j-1}<y<y_j$, and write ΔA_{ij} for its area.
2. Choose a point (x_{ij}^*,y_{ij}^*) in each small rectangle R_{ij}. Make a rectangular box above each R_{ij} with height $f(x_{ij}^*,y_{ij}^*)$.
3. The collection of such rectangular boxes, over all the small rectangles R_{ij}, approximate the region under the graph surface. The total volume of these approximating boxes is the Riemann sum $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*,y_{ij}^*)\Delta A_{ij}$.



4. Letting $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$, the total approximate volume is $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x_i \Delta y_j$.

If all Δx_i are equal and all Δy_j are equal, (or if x_i, y_j are chosen in some other careful way), then the limit $\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x_i \Delta y_j$ will exist, and is the volume under the graph surface.

To calculate this limit, note that we can calculate the Riemann sum in two stages:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x_i \Delta y_j = \sum_{i=1}^{m}\left(\underbrace{\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta y_j}\right)\Delta x_i.$$

2. Sum the approximating volumes from part 1. for the different x_i.

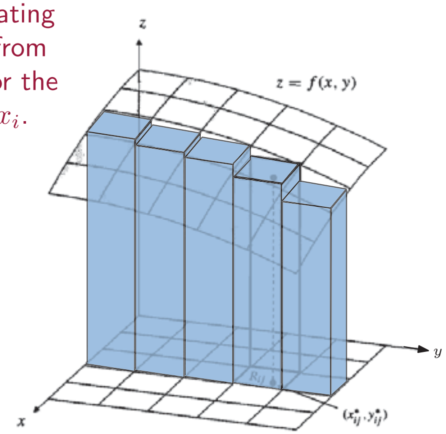
1. Fix x_i and sum the volumes of the approximating boxes for the different y_j.

To calculate this 2-dimensional integral, note that we can calculate the Riemann sum in two stages:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*))\Delta x_i \Delta y_j$$
$$= \sum_{i=1}^{m}\left(\underbrace{\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta y_j}\right)\Delta x_i.$$

2. Sum the approximating volumes from part 1. for the different x_i.

1. Fix x_i and sum the volumes of the approximating boxes for the different y_j.



To continue, take the special case where $x_{ij}^* = x_i^*$ for all j and $y_{ij}^* = y_j^*$ for all i.

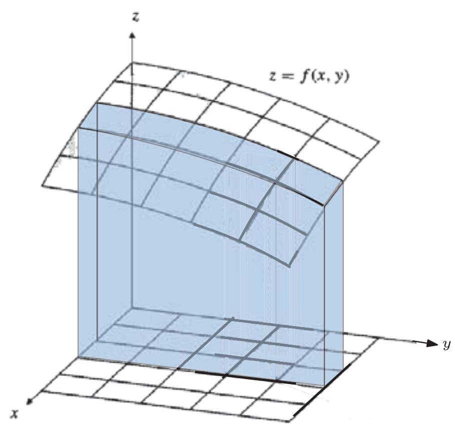
So
$$\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta x_i \Delta y_j$$
$$= \lim_{m\to\infty}\sum_{i=1}^{m}\left(\lim_{n\to\infty}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta y_j\right)\Delta x_i$$

cross-sectional area of a vertical slice at $x = x_i^*$

$$= \lim_{m\to\infty}\sum_{i=1}^{m}\left(\int_c^d f(x_i^*, y)\, dy\right)\Delta x_i$$

Treat x_i^* as a constant when computing this integral; the result is a function in x_i^*.

$$= \int_a^b\left(\int_c^d f(x,y)\, dy\right) dx.$$ This is called an iterated integral.



Example: Find the volume lying under the surface $z = 2x^2y + 3y^2$ and above the region $0 \le x \le 3$, $1 \le y \le 2$.

$$\int_a^b\left(\int_c^d f(x,y)\, dy\right) dx$$

Previously (p7) we said

$$\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*,y_j^*)\Delta x_i \Delta y_j = \lim_{m\to\infty}\sum_{i=1}^{m}\left(\lim_{n\to\infty}\sum_{j=1}^{n} f(x_i^*,y_j^*)\Delta y_j\right)\Delta x_i$$
$$= \lim_{m\to\infty}\sum_{i=1}^{m}\left(\int_c^d f(x_i^*,y)\,dy\right)\Delta x_i = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx.$$

But we could instead have chosen to sum first in the x-direction:

$$\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*,y_j^*)\Delta x_i \Delta y_j = \lim_{n\to\infty}\sum_{j=1}^{n}\left(\lim_{m\to\infty}\sum_{i=1}^{m} f(x_i^*,y_j^*)\Delta x_i\right)\Delta y_j$$
$$= \lim_{n\to\infty}\sum_{j=1}^{n}\left(\int_a^b f(x,y_j^*)\,dy\right)\Delta y_j = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy.$$

For a continuous function f, the two iterated integrals give the same answer.

Redo Example: (p8) Find, by first integrating in x, the volume lying under the surface $z = 2x^2y + 3y^2$ and above the region $0 \le x \le 3$, $1 \le y \le 2$.

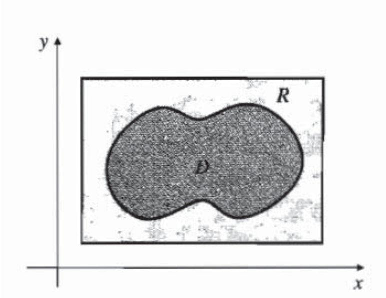
$$\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy.$$

So we know how to find the volume under the graph of $f(x,y)$ over a rectangle:

$$\int_a^b\int_c^d f(x,y)\,dy\,dx \quad\text{or}\quad \int_c^d\int_a^b f(x,y)\,dx\,dy.$$

It would be useful to find volumes over domains D of other shapes.

Theoretically, the idea is simple: draw a large rectangle R around D, then extend the domain of the function to R by defining $f(x,y) = 0$ on points outside of D. Now f is defined on a rectangle, so we can use the previous Riemann sum formula. (The extended function is not continuous, because there is a jump on the boundary of D, but the Riemann sum will have a limit if D is "well-behaved", see p13).



Putting the above all together into a rigorous definition:

Definition: Suppose $f : D \to \mathbb{R}$ is a 2-variable function. Choose a rectangle $R = \{(x,y) \in \mathbb{R}^2 | a \le x \le b, c \le y \le d\}$ such that $D \subseteq R$. Define the function $\hat{f} : R \to \mathbb{R}$ by $\hat{f} = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is not in } D. \end{cases}$

Let $a = x_0 < x_1 < \cdots < x_m = b$ be a division of $[a,b]$ into m subintervals of equal width, and let $c = y_0 < y_1 < \cdots < y_n = d$ be a division of $[c,d]$ into n subintervals of equal width. Let A_{ij} be the area of the small rectangle with $x_{i-1} < x < x_i$ and $y_{j-1} < y < y_j$, and (x_{ij}^*, y_{ij}^*) be any point in this rectangle.

Then f is *integrable* on D if $\displaystyle\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\hat{f}(x_{ij}^*,y_{ij}^*)\Delta A_{ij}$ exists and is independent of the choice of (x_{ij}^*, y_{ij}^*). The value of this limit is the *integral of f on D*:

$$\iint_D f(x)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\hat{f}(x_{ij}^*,y_{ij}^*)\Delta A_{ij}.$$

In the single-variable case, we have a theorem that says a continuous function on an interval $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$ is integrable.

As you might expect, there is a 2-dimensional version of this theorem, which says continuous functions on "reasonable" domains are integrable:

Theorem 1: Continuous functions on closed and bounded sets are integrable:
If $f : D \to \mathbb{R}$ is a continuous function and the domain D is a closed and bounded set whose boundary consists of finitely many curves of finite length, then f is integrable on D.

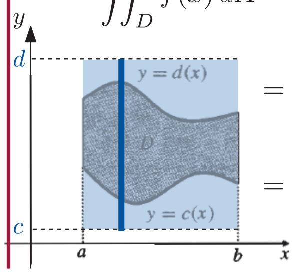
We haven't yet defined "continuous" (§12.2), or "closed", or "bounded" (§13.2), but:

- Any elementary function (i.e. sums, products and compositions of $x^n, e^x, \ln x, \sin x, \cos x$) is continuous;
- A set that is contained in a large rectangle (i.e. not "going to infinity") is bounded;
- A set defined by a finite number of weak inequalities (i.e. $\leq$ or $\geq$) of elementary functions is closed, and its boundary is finitely many curves.

(e.g. closed: $\left\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1, x \geq 0\right\}$; not closed: $\left\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \neq 1\right\}$.)
Almost all our examples will satisfy these stronger conditions.

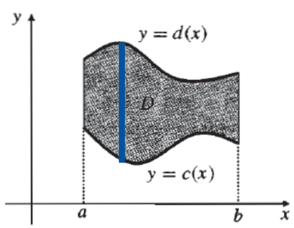
We have defined
$$\iint_D f(x)\,dx = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \hat{f}(x^*_{ij}, y^*_{ij})\Delta A_{ij}, \quad \hat{f} = \begin{cases} f & \text{on } D \\ 0 & \text{outside } D. \end{cases}$$
What does this mean for our computation using iterated integrals?

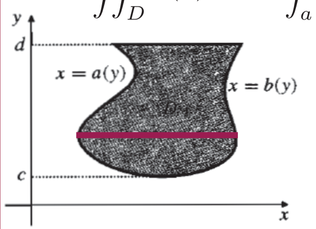
Suppose $D = \left\{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c(x) \leq y \leq d(x)\right\}$ as in the picture.
We put D in a rectangle by choosing $c < c(x)$ and $d > d(x)$ for all x. Then
$$\iint_D f(x)\,dA = \int_a^b \left(\int_c^d \hat{f}(x, y)\,dy \right) dx$$
$$= \int_a^b \left(\int_c^{c(x)} \underbrace{\hat{f}(x, y)}_{=0}\,dy + \int_{c(x)}^{d(x)} \underbrace{\hat{f}(x, y)}_{=f}\,dy + \int_{d(x)}^{d} \underbrace{\hat{f}(x, y)}_{=0}\,dy \right) dx$$
$$= \int_a^b \left(\int_{c(x)}^{d(x)} f(x, y)\,dy \right) dx$$



The shape of D is encoded in the limits of the inner (first) integral.

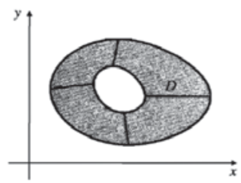
From the previous slide:
If $D = \left\{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c(x) \leq y \leq d(x)\right\}$,
then $\displaystyle\iint_D f(x)\,dA = \int_a^b \int_{c(x)}^{d(x)} f(x, y)\,dy\,dx.$



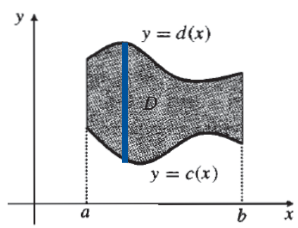


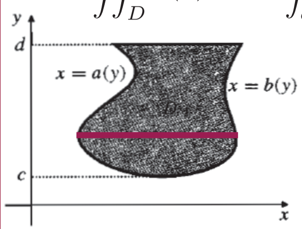
Similarly, if $D = \left\{(x, y) \in \mathbb{R}^2 | a(y) \leq x \leq b(y), c \leq y \leq d\right\}$,
then $\displaystyle\iint_D f(x)\,dA = \int_c^d \int_{a(y)}^{b(y)} f(x, y)\,dx\,dy.$

Many domains (rectangles, triangles) can be written in both the above ways, so both formulas work. (We already saw this when we calculated areas of plane regions (Week 4 p23-25) - indeed, the area of D is $\iint_D 1\,dA$.)
If a domain cannot be written in either way, then we must split it into regions which are of these forms.



From the previous slide:
If $D = \left\{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c(x) \leq y \leq d(x)\right\}$,
then $\displaystyle\iint_D f(x)\,dA = \int_a^b \int_{c(x)}^{d(x)} f(x, y)\,dy\,dx.$





Similarly, if $D = \left\{(x, y) \in \mathbb{R}^2 | a(y) \leq x \leq b(y), c \leq y \leq d\right\}$,
then $\displaystyle\iint_D f(x)\,dA = \int_c^d \int_{a(y)}^{b(y)} f(x, y)\,dx\,dy.$

Warning: In both cases, only the inner (first) integral may have limits that contain variables, and those must be the variables of the outer (second) integral.

Some wrong examples: $\cancel{\displaystyle\int_{c(x)}^{d(x)} \int_a^b f(x, y)\,dx\,dy}$, $\cancel{\displaystyle\int_a^b \int_{c(y)}^{d(y)} f(x, y)\,dy\,dx}$

Example: Find the volume lying under the surface $z = x^2 + y^2$ and above the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$.

As we saw above, we get the same answer whether we integrate first in x and then in y, or first in y and then in x. Sometimes one order is much easier than the other - changing the order is called reiterating the integral.

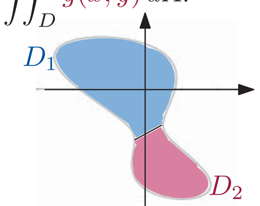
Example: Evaluate $\int_0^1 \int_{\sqrt{x}}^1 e^{-y^3}\,dy\,dx$.

The integral $\int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx$ is sometimes written $\int_a^b dx \int_{c(x)}^{d(x)} f(x,y)\,dy$, to emphasise that the limits a, b refer to the variable x. Similarly, $\int_c^d dy \int_{a(y)}^{b(y)} f(x,y)\,dx$ means $\int_c^d \int_{a(y)}^{b(y)} f(x,y)\,dx\,dy$.

Some properties of multiple integrals, analogous to properties for 1D definite integrals (same labelling as in Week 3 p19-20):

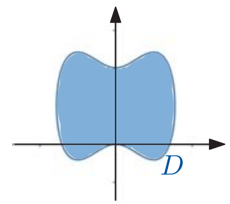
c. An integral depends linearly on the integrand: if L and M are constants, then
$$\iint_D Lf(x,y) + Mg(x,y)\,dA = L\iint_D f(x,y)\,dA + M\iint_D g(x,y)\,dA.$$

d. An integral depends additively on the domain of integration: if D_1 and D_2 are non-overlapping domains (except possibly on their boundaries), then
$$\int_{D_1} f(x,y)\,dA + \int_{D_2} f(x,y)\,dA = \int_{D_1\cup D_2} f(x,y)\,dx.$$



Some properties of multiple integrals, analogous to properties for 1D definite integrals (same labelling as in Week 3 p19-20):

g. If $f(x,y)$ is an odd function in x (i.e. $f(x,y) = -f(-x,y)$) and D is symmetric about the y-axis (i.e. replacing x by $-x$ in the definition of D doesn't change D), then $\int_D f(x,y)\,dA = 0$ (and similarly for an odd function in y and a domain symmetric about the x-axis).



Example: Find $\iint_D y + \sin x \cos y\,dA$, where $D = \{(x,y) \in \mathbb{R}^2 | 4x^2 + y^2 \le 4\}$.

There is a more interesting symmetry property of 2D integrals that we don't see in 1D definite integrals. This comes from the fact that the variable of integration is a dummy variable, so $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$, or, in 2D,

$$\int_c^d \int_{a(y)}^{b(y)} f(x,y)\,dx\,dy = \int_c^d \int_{a(t)}^{b(t)} f(s,t)\,ds\,dt.$$

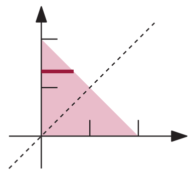
In particular,

$$\int_c^d \int_{a(y)}^{b(y)} f(x,y)\,dx\,dy = \int_c^d \int_{a(x)}^{b(x)} f(y,x)\,dy\,dx,$$

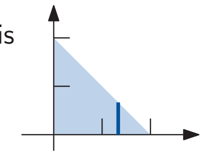
and if the domains on the two sides $\left\{(x,y)\in\mathbb{R}^2 | a(y)\le x\le b(y), c\le y\le d\right\}$ and $\left\{(x,y)\in\mathbb{R}^2 | a(x)\le y\le b(x), c\le x\le d\right\}$ are equal, (i.e. D is symmetric in x and y) then this is often helpful.

$$\int_c^d \int_{a(y)}^{b(y)} f(x,y)\,dx\,dy = \int_c^d \int_{a(x)}^{b(x)} f(y,x)\,dy\,dx.$$

Example: (ex sheet #8 q2) Let D be the region bounded by the lines $x=0$, $y=0$ and $x+y=2$. There are two ways to see that D is symmetric in x and y: the set of three lines do not change when we exchange x and y in the equations; from the diagram, D is unaffected by reflection in the line $y=x$.



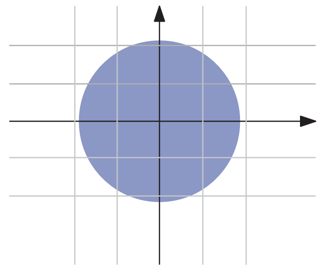
Consider $\iint_D x^2\,dA$. One way to write it as an interated integral is $\int_0^2 \int_0^{2-y} x^2\,dx\,dy$. Renaming the variables of integration, this is the same as $\int_0^2 \int_0^{2-x} y^2\,dy\,dx$, and the domain of this double integral is also D. Hence $\iint_D x^2\,dA = \iint_D y^2\,dA$, i.e. $\iint_D x^2 - y^2\,dA = 0$.



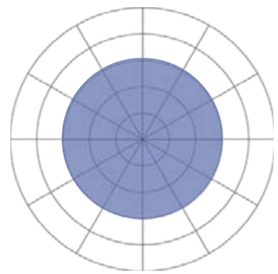
The following example leads to a very complicated integral that we will redo (p31) in a much easier way.

Example: Find the volume of the region bounded by $z=1-x^2-y^2$ and $z=0$.

The integral in the previous example was very complicated because the domain was circular, but we were using a method based on Riemann sums over rectangular subdomains.



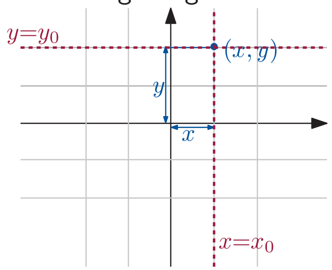
In the next section, we derive a second method for evaluating double integrals that use Riemann sums over the subdomains in the grid at the bottom.



(This second method is one special case of the method of substitution for multiple integrals - we will discuss this in general in the final week of class.)

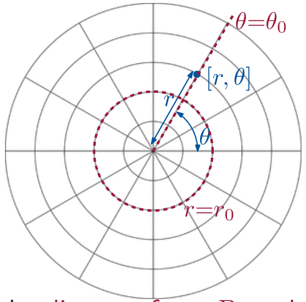
§14.4: Double Integrals in Polar Coordinates

Cartesian coordinates (x, y) specify the location of a point P relative to a rectangular grid:



- x is the distance of P to the right of the y-axis (i.e. horizontal distance);
- y is the distance of P above the x-axis (i.e. vertical distance).

Polar coordinates $[r, \theta]$ specify the location of P relative to the polar grid below:



- r is the distance from P to the origin;
- θ is the counterclockwise angle from the positive x-axis to the vector $\overrightarrow{OP}$ and (i.e. from $\mathbf{i}$ to $\overrightarrow{OP}$).

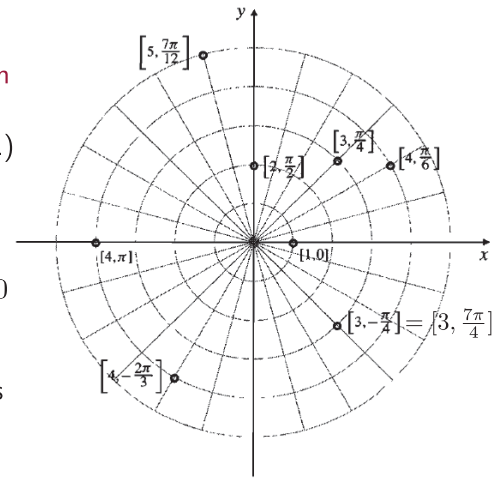
Polar coordinates $[r, \theta]$ specify the location of a point P relative to the polar grid:

- r is the distance from P to the origin;
- θ is the counterclockwise angle between the vector $\overrightarrow{OP}$ and the positive x-axis.

(See the first page of §8.5 in the textbook.)

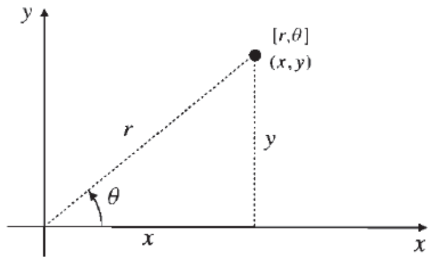
Conventions:

- We will consider only $r \geq 0$. (Different conventions exist regarding $r < 0$.)
- On the upper half plane, θ is between 0 and π. On the lower half plane, we will sometimes take $\theta \in (\pi, 2\pi)$ (a large counterclockwise angle) and sometimes take $\theta \in (-\pi, 0)$ (a small clockwise angle).

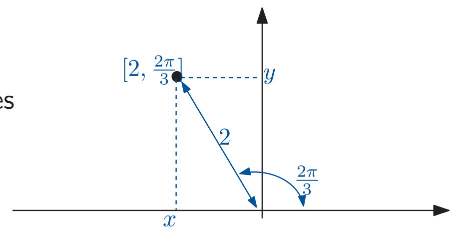


This diagram illustrates how to change between Cartesian and polar coordinates.

- To change from $[r, \theta]$ to (x, y), take $x = r\cos\theta$ and $y = r\sin\theta$;

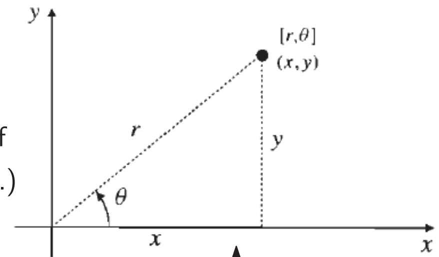


Example: Let P be $[2, \dfrac{2\pi}{3}]$ in polar coordinates. Then its Cartesian coordinates are $x = 2\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $y = 2\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, i.e. $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$.



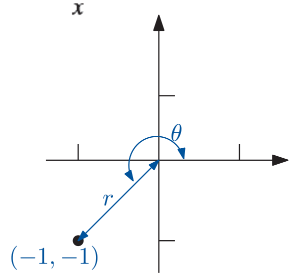
This diagram illustrates how to change between Cartesian and polar coordinates.

- To change from $[r, \theta]$ to (x, y), take $x = r\cos\theta$ and $y = r\sin\theta$;
- To change from (x, y) to $[r, \theta]$, take $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$. (Be careful: if $x < 0$, then $\theta \neq \tan^{-1}\frac{y}{x}$, see example below.)

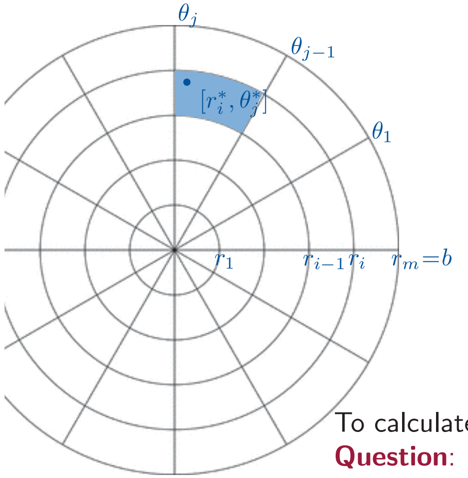


Example: Let P be $(-1, -1)$ in Cartesian coordinates. Then its distance to the origin is $\sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$, and the angle between $-\mathbf{i} - \mathbf{j}$ and $\mathbf{i}$ is $\frac{5\pi}{4}$, so its polar coordinates are $[\sqrt{2}, \frac{5\pi}{4}]$. Another correct answer is $[\sqrt{2}, -\frac{3\pi}{4}]$.

Note that $\tan^{-1}\frac{y}{x} = \tan^{-1} 1 = \frac{\pi}{4} \neq \frac{5\pi}{4}$ nor $\frac{-3\pi}{4}$. It is safest to find θ using a diagram.



Back to integration:
Suppose our domain D is a disk of radius b, centred at the origin.



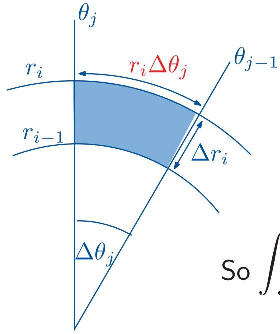
Choose r_i with $0 = r_0 < r_1 < \cdots < r_m = b$ and θ_j with $0 = \theta_0 < \theta_1 < \cdots < \theta_n = 2\pi$, and divide the disk into small pieces along the circles $r = r_i$ and the lines $\theta = \theta_j$.
Write A_{ij} for the area of the piece with $r_{i-1} < r < r_i$ and $\theta_{j-1} < \theta < \theta_j$, and choose a point $[r_i^*, \theta_j^*]$ in this piece.

Then $\displaystyle\iint_D f(x,y)\,dA =$

$$\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\Delta A_{ij}.$$

To calculate this using iterated integrals, we need to know:
Question: What is ΔA_{ij}, the area of the small pieces?

$\displaystyle\iint_D f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\Delta A_{ij}$, where ΔA_{ij} is the area of the small pieces of the domain (e.g. the shaded piece in the diagram).
(Note that the units of ΔA_{ij} is (unit of length)^2, so ΔA_{ij} cannot simply be $\Delta r_i \Delta\theta_j$, whose units is (unit of length), because θ is an angle with no units.)



Approximate each piece by a rectangle:
- the length of a straight side is Δr_i;
- the outer curved side is an arc of a circle, with angle $\Delta\theta_j$ and radius r_i, so its length is $r_i\Delta\theta_j$.

So the area of each piece is approximately $\Delta A_{ij} = r_i \Delta r_i \Delta\theta_j$ (it can be proved rigorously that the error in this approximation goes to zero when we take the limit in the Riemann sum).

So $\displaystyle\iint_D f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*) r_i \Delta r_i \Delta\theta_j$

$$= \int_0^{2\pi}\int_0^{b} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

Redo Example: (p23) Find the volume of the region bounded by $z = 1 - x^2 - y^2$ and $z = 0$.

$$\int_0^{2\pi}\int_0^{b} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

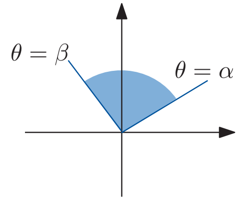
Example: Find the volume of the smaller region bounded by $z = \sqrt{3x^2 + 3y^2}$ and $x^2 + y^2 + z^2 = 1$.

$$\int_0^{2\pi}\int_0^{b} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

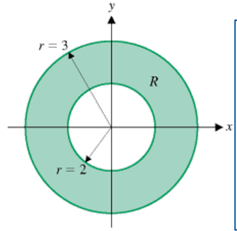
Polar coordinates are also useful for integrating over sectors:
Example: Find the volume of the region bounded by $z = x^2$, $x^2 + y^2 = 4$ and $z = 0$ with $x \geq 0$.

$$\int_\alpha^\beta \int_0^b f(r\cos\theta, r\sin\theta) r \, dr d\theta$$

Some domains where polar coordinates are useful:

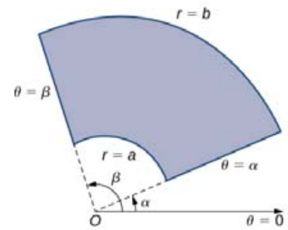


A sector:
$0 < r < b$, $\alpha \leq \theta \leq \beta$



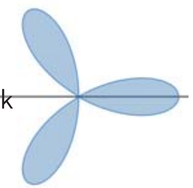
An annulus:
$a \leq r \leq b$, $0 \leq \theta \leq 2\pi$

e.g volume bounded by $z = x^2$, $z = 0$, $x^2 + y^2 = 4$, $x^2 + y^2 = 9$.



An "annulus-sector":
$a \leq r \leq b$, $\alpha \leq \theta \leq \beta$

In this class, we will mainly focus on the domains above, where both sets of limits of integration are constant, but integrals of the form $\displaystyle\int_\alpha^\beta \int_{b(\theta)}^{a(\theta)} f(r\cos\theta, r\sin\theta) r \, dr d\theta$ are also useful - indeed, it can work with domains that are almost impossible to describe in Cartesian coordinates, e.g. $r \leq \cos(3\theta)$.



(pictures from Calculus by Smith and Minton, archive.cnx.org)

Notice the similarity between the formula for double integrals in polar coordinates
$$\int_{c'}^{d'} \int_{a'}^{b'} f(x,y) \, dx \, dy = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r \, dr d\theta$$
and the method of substitution
$$\int_{a'}^{b'} f(u) \, du = \int_a^b f(u(t)) \frac{du}{dt} \, dt.$$

We will see, in the final week, that the factor of r in the polar double integral formula is a 2-dimensional "derivative", and that similar formulas exist for other substitutions (e.g. for elliptical domains).

§14.5: Triple Integrals

We can define the integral of a 3-variable function f in a similar way:
$$\iiint_D f(x,y,z) \, dV = \lim_{m,n,p\to\infty} \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p \hat{f}(x_i^*, y_j^*, z_k^*) \Delta V_{ijk},$$
where $\hat{f}$ is the extension of f to a rectangular box containing D, and x_i, y_j, z_k divide the sides of the box equally, and ΔV_{ijk} is the volume of the smaller boxes.

In practice, we calculate triple integrals by iterated integrals (see p41)
$$\int_a^b \int_{c(x)}^{d(x)} \int_{p(x,y)}^{q(x,y)} f(x,y,z) \, dz \, dy \, dx.$$

The graph of a 3-variable function is in $\mathbb{R}^4$, so the "hypervolume" under such a graph is not a naturally interesting quantity. However, there are many good reasons to consider Riemann sums (and therefore integrals) of a 3-variable function.
(The applications on the next four pages also apply to 1D and 2D integrals.)

$$\iiint_D f(x,y,z)dV = \lim_{m,n,p\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} \hat{f}(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}, \quad \hat{f} = \begin{cases} f & \text{on } D \\ 0 & \text{outside } D. \end{cases}$$

There are many good reasons to consider Riemann sums of a 3-variable function:

1. (Geometry) The volume of D is $\iiint_D 1\,dV$.

This is the "primary school method" of calculating areas/volumes: put D in a rectangular grid, count the number of rectangles inside D, and multiply this number by the area/volume of each rectangle. (Remember that $\hat{1}(x_i^*, y_j^*, z_k^*) = 1$ if x_i^*, y_j^*, z_k^* is in D, and 0 otherwise.)

$$\iiint_D f(x,y,z)dV = \lim_{m,n,p\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} \hat{f}(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}, \quad \hat{f} = \begin{cases} f & \text{on } D \\ 0 & \text{outside } D. \end{cases}$$

There are many good reasons to consider Riemann sums of a 3-variable function:

2. (Probability) The average value of f on D is $\bar{f} = \dfrac{\iiint_D f(x,y,z)\,dV}{\iiint_D 1\,dV}$.

   (Don't be confused by the notation: $\bar{f}$ is a number, not a function.)

To understand where this formula comes from: to approximate the average value of f, we can take the value of f at many points throughout D and take the average of those values - this gives us $\frac{1}{N}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\hat{f}(x_i^*, y_j^*, z_k^*)$, where N is the number of points (x_i^*, y_j^*, z_k^*) lying inside D. Now multiply the numerator and denominator by ΔV_{ijk} and take the limit.

$$\iiint_D f(x,y,z)dV = \lim_{m,n,p\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} \hat{f}(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}, \quad \hat{f} = \begin{cases} f & \text{on } D \\ 0 & \text{outside } D. \end{cases}$$

There are many good reasons to consider Riemann sums of a 3-variable function:

3. (Physics) The mass of an object occupying the region D is $\iiint_D \delta(x,y,z)\,dV$, where $\delta(x,y,z)$ is the density function.

To understand where this formula comes from: the density of an object is its mass per unit volume. If an object has constant density, then its mass is this constant multiplied by its volume. Hence the mass of the small rectangular box $x_{i-1} < x < x_i, y_{j-1} < y < y_j, z_{k-1} < z < z_k$ is approximately $\delta(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}$.

(This formula also works for 2D objects, if $\delta(x,y)$ is a 2D-density function, i.e. mass per unit area.)

$$\iiint_D f(x,y,z)dV = \lim_{m,n,p\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} \hat{f}(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}, \quad \hat{f} = \begin{cases} f & \text{on } D \\ 0 & \text{outside } D. \end{cases}$$

There are many good reasons to consider Riemann sums of a 3-variable function:

4. (Physics) The centre of mass of an object occupying the region D, with density function $\delta(x,y,z)$ is the point with coordinates
   $$\left(\frac{\iiint_D x\delta(x,y,z)\,dV}{\iiint_D \delta(x,y,z)\,dV}, \frac{\iiint_D y\delta(x,y,z)\,dV}{\iiint_D \delta(x,y,z)\,dV}, \frac{\iiint_D z\delta(x,y,z)\,dV}{\iiint_D \delta(x,y,z)\,dV}\right).$$
   (This formula will be given to you in exams if necessary.)

If the density is constant, then the centre of mass is the "average position" $(\bar{x}, \bar{y}, \bar{z})$, and is called the centroid.

Example: Find the mass of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 1$ and $x + y - 2z = 0$, whose density function is $\rho(x, y, z) = x$.

As with 2D integrals (p18), it is sometimes useful to reiterate 3D integrals for easier computation.

Example: Evaluate $\displaystyle\int_0^1 \int_x^1 \int_0^{y-x} \frac{\sin z}{y - z}\, dz\, dy\, dx$.

§14.6: Triple Integrals in Polar Coordinates

Our current method to evaluate triple integrals uses Riemann sums over rectangular subdomains, i.e. we divide our domain along a Cartesian grid, by cutting along the vertical planes $x = x_i, y = y_j$ and the horizontal planes $z = z_k$.

As we saw in the 2D case, if our domain is "circular", this method may lead to very complicated integrals (because of square roots). We would like alternative methods that divide the domain in a "circular" way.

There are two common coordinate systems (or "subdivision methods") for this purpose: cylindrical coordinates $[r, \theta, z]$ and spherical coordinates $[R, \theta, \phi]$.

In cylindrical coordinates, we keep the z-coordinate, and use 2D polar coordinates $[r, \theta]$ instead of (x, y). Spherical coordinates are less similar to 2D polar coordinates.

The *cylindrical coordinates* $[r, \theta, z]$ of a point P are:

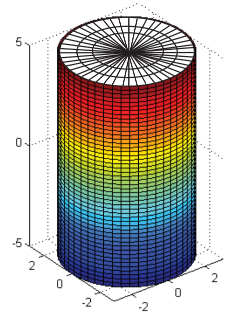
- r is the distance from P to the z-axis (i.e. horizontal distance);
- θ is the counterclockwise angle from the x, z-plane to the plane containing P and the z-axis;
- z is the distance from P to the x, y-plane (i.e. vertical distance).

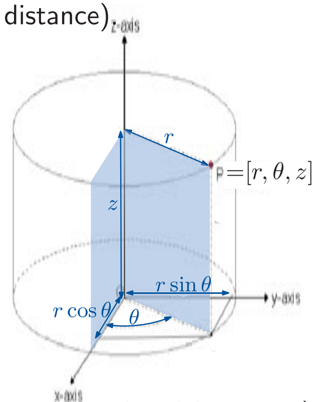
So the distance from P to the origin is $\sqrt{r^2 + z^2}$.

Informally, in terms of dividing our domain, we are slicing horizontally along the planes $z = z_k$, and then dividing each slice according to 2D polar coordinates.

To change to Cartesian coordinates:

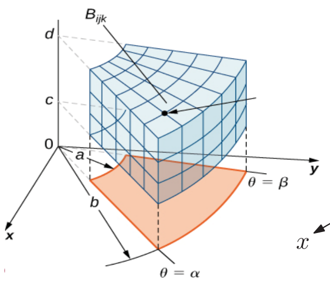
$x = r\cos\theta$,
$y = r\sin\theta$,
$z = z$.

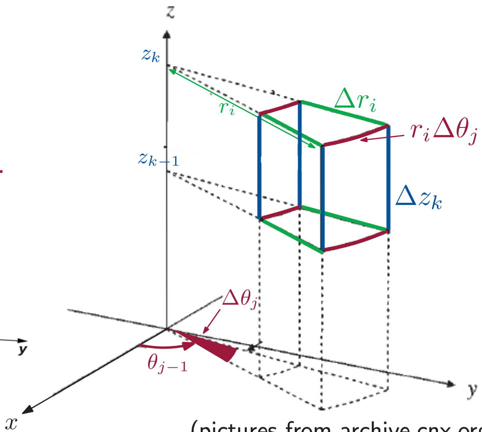




(pictures from santoshlinkha.wordpress.com, astarmathsandphysics.com)

To compute iterated integrals using cylindrical coordinates, we need to know the volume ΔV_{ijk} of each small piece B_{ijk} in the cylindrical coordinate grid.
The base of B_{ijk} is a piece of a 2D polar grid, so its area is $\Delta A_{ij} \approx r_i \Delta r_i \Delta\theta_j$ (p30).
The height of B_{ijk} is Δz_k.
So $\Delta V_{ijk} \approx r_i \Delta r_i \Delta\theta_j \Delta z_k$.
So $\iiint_D f(x,y,z)dV$
$= \int_c^d \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta, z) r\, dr\, d\theta\, dz.$





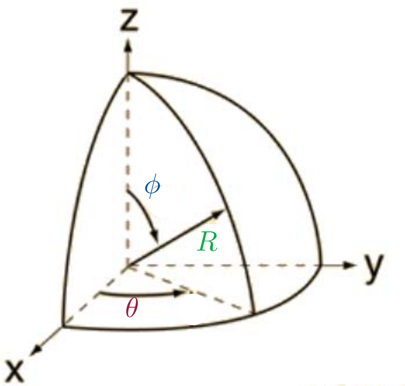
(pictures from archive.cnx.org)

Example: Find the mass of the smaller region bounded by $z = \sqrt{3x^2 + 3y^2}$ and $x^2 + y^2 + z^2 = 1$, with density function $\delta(x,y,z) = x^2 z$.

$$\int_c^d \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta, z) r\, dr\, d\theta\, dz$$

The disadvantage of cylindrical coordinates is that the distance from $[r,\theta,z]$ to the origin is $\sqrt{r^2+z^2}$, which still involves a squareroot. This may be inconvenient in problems involving spheres.
A possibly better choice is *spherical coordinates* $[R,\theta,\phi]$ (see next page for the coordinate grid):

- R is the distance from P to the origin;
- θ is the counterclockwise angle from the x,z-plane to the plane containing P and the z-axis;
- ϕ is the angle from the positive z-axis to the vector $\overrightarrow{OP}$.

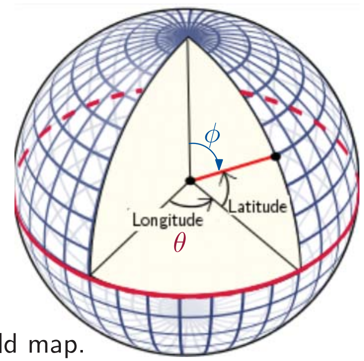


Warnings:

- R in spherical coordinates is not the same as r in cylindrical coordinates (but θ is the same in both coordinates);
- θ goes from 0 to 2π, but ϕ goes from 0 to π.

(picture from Hyperphysics)

- R is the distance from P to the origin;
- θ is the counterclockwise angle from the x,z-plane to the plane containing P and the z-axis;
- ϕ is the angle from the positive z-axis to the vector $\overrightarrow{OP}$.



Informally, in terms of dividing our domain, the spherical coordinate grid first separates the domain into spheres of different radii centred at the origin (R), then slices the sphere along the "horizontal", latitude lines (ϕ) and vertical, longitude lines (θ), like on a world map.

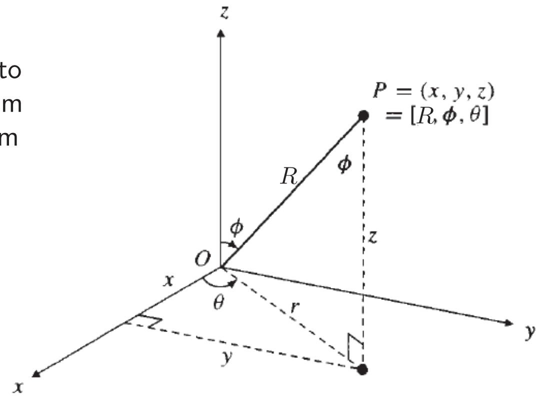
(A small difference between geographic and mathematical conventions: latitude is measured from the equator (so the equator is 0, the north pole is $\frac{\pi}{2}$, the south pole is $\frac{-\pi}{2}$); ϕ is measured from the north pole (so the north pole is $\phi = 0$, the equator is $\phi = \frac{\pi}{2}$, the south pole is $\phi = \pi$. So ϕ is called the "colatitude".)

Different authors use different symbols for the angles in spherical coordinates - outside of this class, you should say "colatitude" or "longitude".

(picture from vvvv.org)

- R is the distance from P to the origin;
- θ is the counterclockwise angle from the x, z-plane to the plane containing P and the z-axis;
- ϕ is the angle from the positive z-axis to the vector $\overrightarrow{OP}$.

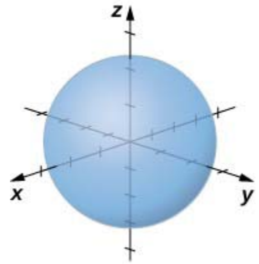
To change from spherical coordinates to Cartesian coordinates, first observe from the right-angled triangle in this diagram that $z = R\cos\phi$ and $r = R\sin\phi$. So
$x = r\cos\theta = R\sin\phi\cos\theta$,
$y = r\sin\theta = R\sin\phi\sin\theta$,
$z = R\cos\phi$.



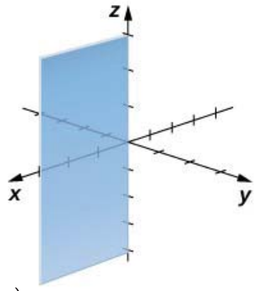
To understand spherical coordinates, it may help to consider the surfaces where one coordinate is fixed and the other two change.

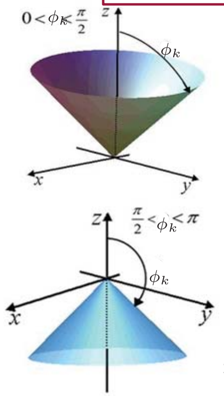
$r = R\sin\phi$,
$x = r\cos\theta = R\sin\phi\cos\theta$,
$y = r\sin\theta = R\sin\phi\sin\theta$,
$z = R\cos\phi$.

The surfaces $R = R_i$ are spheres centred at the origin.



The surfaces $\theta = \theta_j$ are half-planes which include the z-axis.



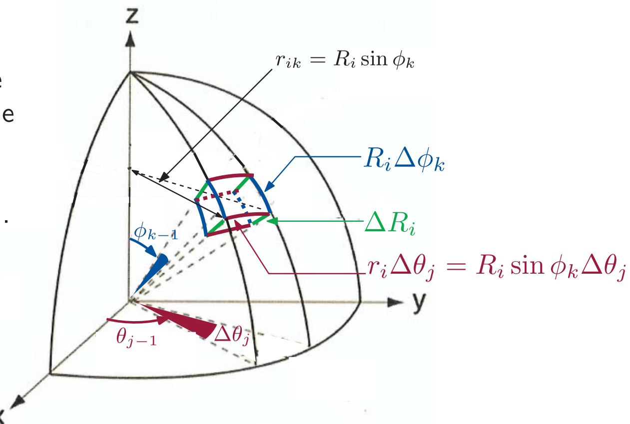


The surfaces $\phi = \phi_k$ are half-cones opening in the z-direction. To see this, remember that a half-cone has equation $z = c\sqrt{x^2+y^2} = cr$, so $\frac{1}{c} = \frac{r}{z} = \tan\phi$.

(picture from archive.cnx.org)

To compute iterated integrals using spherical coordinates, we need to know the volume ΔV_{ijk} of each small piece in the spherical coordinate grid.

We approximate each small piece by a rectangular box. The diagram shows the lengths of the sides of this box. So its volume is the product of these lengths: $\Delta V_{ijk} = R_i^2 \sin\phi_k \Delta R_i \Delta\theta_j \Delta\phi_k$.



$$\iiint_D f(x,y,z)dV = \int_\gamma^\delta \int_\alpha^\beta \int_a^b f(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)R^2\sin\phi\, dR\, d\theta\, d\phi$$

(picture from Hyperphysics)

$$\int_\gamma^\delta \int_\alpha^\beta \int_a^b f(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)R^2\sin\phi\, dR\, d\theta\, d\phi$$

Redo Example: (p46) Find the mass of the smaller region bounded by $z = \sqrt{3x^2+3y^2}$ and $x^2+y^2+z^2 = 1$, with density function $\delta(x,y,z) = x^2 z$.

$$\int_{\gamma}^{\delta}\int_{\alpha}^{\beta}\int_{a}^{b} f(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)R^2\sin\phi\,dR\,d\theta\,d\phi$$

Example: A chocolate occupies the region between $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=9$. Its density function is $\delta(x,y,z)=x^2$. Find the mass of the chocolate.

Given a integral over a 3D domain that is "circular" in some way, it may not be obvious whether to use cylindrical or spherical coordinates. In many examples, both methods work, and even for the same situation different people may have different opinions about which method is easier. The only way to find out what's easiest for you is to do many examples.

Here are my preferences (you may disagree):

- If the region involves cylinders or paraboloids (which are hard to describe in spherical coordinates), I try cylindrical coordinates first. This applies even if spheres, cones and other shapes are involved.
- If the region involves spheres and cones only, I look at the integrand:
  - If the integrand involves complicated functions of x^2+y^2, I try cylindrical coordinates first.
  - Otherwise, I try spherical coordinates first.

If, after changing into my chosen coordinates, the integral looks very complicated, I will rewrite it in the other coordinates before trying to evaluate it.