Descent algebras

related topics: noncommutative symmetric functions / quasisymmetric functions
hyperplane arrangements
random walks on a group
reflection / Coxeter groups

Work in the symmetric group algebra $\mathbb{R}S_n$
(the permutations are a basis, add pointwise, multiply by linear extension of compositions of permutations.)
To interpret the multiplication in $\mathbb{R}S_n$: suppose $\alpha = \sum_{\sigma \in S_n} a_\sigma \sigma$; $\beta = \sum_{\sigma \in S_n} b_\sigma \sigma$.
then the coefficient of σ in $\alpha\beta$ is the probability of going from the identity to σ by multiplying by a random permutation with probability a_σ, then multiplying by another random permutation with probability b_σ.
e.g. if $\pi_\sigma^{(k)}$ is the chance of moving from the identity to σ in k repeats of the same process, and $\rho = \sum_{\sigma \in S_n} \pi_\sigma^{(1)} \sigma$, then $\rho^k = \sum_{\sigma \in S_n} \pi_\sigma^{(k)} \sigma$

Suppose ρ models a card-shuffle. Then a question of interest is, how many shuffles are necessary to mix the deck?
The deck is well-mixed if it is equally likely to be in any of the $n!$ orders
$\therefore$ define U to be $\sum_{\sigma \in S_n} \frac{1}{n!} \sigma$
Then the rigorous mathematical question is, given $\varepsilon > 0$, what value of k ensures $\|\rho^k - U\| < \varepsilon$, for some metric $\| \ \|$ on $\mathbb{R}S_n$?

The study of riffle-shuffles uses the following "magic" formula in $\mathbb{R}(\mathfrak{S}_n)[t]$:

$$\sum_{i=1}^{n} e_i t^i = \sum_{\sigma\in\mathfrak{S}_n} \frac{(t-d(\sigma))^{(n)}}{n!}\sigma$$

here, e_i are particular elements of $\mathbb{R}(\mathfrak{S}_n)$

$d(\sigma)$ is the number of descents in σ — i.e. $|\{i\in\{1,\dots,n\} \mid \sigma_{i+1}>\sigma_i\}|$

$x^{(n)}$ denotes the increasing factorial $x(x+1)\cdots(x+n-1)$.

e.g. in $\mathfrak{S}_3$, the right hand side is

$$\frac{t(t+1)(t+2)}{6}[\![123]\!] + \frac{(t-1)t(t+1)}{6}\left([\![132]\!]+[\![213]\!]+[\![231]\!]+[\![312]\!]\right) + \frac{(t-2)(t-1)t}{6}[\![321]\!]$$

so, collecting the powers of t, we get $e_1 = \frac{1}{3}[\![123]\!] - \frac{1}{6}$

$e_2 = \frac{1}{2}[\![123]\!] - \frac{1}{2}[\![321]\!]$

$e_3 = \frac{1}{6}\left([\![123]\!]+[\![132]\!]+[\![213]\!]+[\![231]\!]+[\![312]\!]+[\![321]\!]\right)$

It is always the case that $e_n = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sigma$.

The e_i turn out to be orthogonal idempotents — i.e. $e_i^2 = e_i$, and $e_ie_j = 0$ if $i\neq j$.
Taking $t=1$ in the magic formula: $e_1+e_2+\dots+e_n = \mathrm{id}$

Taking $t=2$ in the magic formula:
because $x^{(n)}=0$ if $-n+1\le x\le 0$, the right hand side is $(n+1)\,\mathrm{id} + \sum_{\sigma\in\mathfrak{S}_n,\, d(\sigma)=1}\sigma$.
and, dividing this by 2^n gives the probabilities of the riffle-shuffle.
(see the paper by Bayer-Diaconis.)

So we are looking for the minimal k such that $\left(\frac{1}{2^n}\sum_{i=1}^{n} e_i 2^i\right)^k$ is close to uniform.

But, because the e_i are orthogonal idempotents, this is simply $\frac{1}{2^{nk}}\sum_{i=1}^{n} e_i 2^{ik}$.
Using the right-hand-side of the magic formula, we take the coefficients of each permutation, and discover that the k-step transition probabilities are

$$\Pi_\sigma^{(k)} = \frac{(2^k-d(\sigma))^{(n)}}{n!}$$

Note that this is non-zero only when $2^k > d(\sigma)$ $\therefore$ this is non-zero for all σ when $k\sim\log_2 n$. And indeed, Bayer-Diaconis shows that $\frac{3}{2}\log_2 n$ riffle shuffles are necessary to mix a deck.
(The idempotent e_1 is originally due to Michael Barr, from Hochschild cohomology)

The descent algebra is originally due to Solomon:

let $D(\sigma) := \{ i \mid 1 \le i \le n-1,\ \sigma_i > \sigma_{i+1} \} \subseteq \{1, 2, \cdots, n-1\}$

in $\mathbb{Q}S_n$, let $y_T := \sum_{D(\sigma)=T} \sigma$, where T is any of the 2^{n-1} subsets of $\{1, 2, \cdots, n-1\}$

e.g. for $n=3$: $y_\phi = [\![123]\!]$ (y_ϕ = identity, always)

$$y_{\{1\}} = [\![213]\!] + [\![312]\!]$$
$$y_{\{2\}} = [\![132]\!] + [\![231]\!]$$
$$y_{\{1,2\}} = [\![321]\!]$$

Theorem of Solomon (which holds for all finite Coxeter groups)

the y_T span a subalgebra of $\mathbb{Q}S_n$: this is $\Sigma[S_n]$, the descent algebra

Equivalently, for two subsets T, R of $\{1, 2, \cdots, n-1\}$, then $y_T y_R = \sum a^K_{TR} y_K$.

Indeed, the a^K_{TR} are integral, and Solomon gives an expression for them explicitly.

Since y_ϕ is the identity permutation, the descent algebra is an algebra with identity.

e.g. for $n=3$: y_ϕ acts as the identity.

$$y_{\{1,2\}} y_{\{1,2\}} = y_\phi$$
$$y_{\{1,2\}} y_{\{1\}} = y_{\{2\}}$$
$$y_{\{1,2\}} y_{\{2\}} = y_{\{1\}}$$
$$y_{\{1\}} \ y_{\{1\}} = y_\phi + y_{\{2\}} + y_{\{1,2\}}.$$

Coxeter groups: these are pairs (W, S) such that W is a finite group, and S generates W

$s, t \in S$ satisfy the relations $s^2 = \mathrm{id}$, $(st)^{m_{st}} = \mathrm{id}$ for some $m_{st} \in \mathbb{N}$, with $m_{st} \ge 2$.

if $m_{s,t} = 2$, then $(st)(st) = \mathrm{id} \Rightarrow \cancel{s}ststs\cancel{t} = st \Rightarrow ts = st$. i.e. s, t commute.

If S is the disjoint union $S_1 \amalg S_2$, and $m_{st} = 2$ for all $s \in S_1$, $t \in S_2$

then the group W generated by S is a cartesian product $W_1 \times W_2$, where S_i generates W_i

$\therefore$ it is enough to understand irreducible Coxeter groups – these have been classified by Coxeter.

If we draw a graph whose vertices are S, and draw $\underset{s}{\bullet} \quad \underset{t}{\bullet}$ if $m_{st} = 2$

$\underset{s}{\bullet}\!\!-\!\!-\!\!-\!\!\underset{t}{\bullet}$ if $m_{st} = 3$

$\underset{s}{\bullet}\!\!=\!\!=\!\!=\!\!\underset{t}{\bullet}$ if $m_{st} = 4$,

then the irreducible Coxeter groups correspond to connected graphs.

Example: the symmetric group S_n corresponds to the graph $\bullet\!-\!-\!\bullet\!-\!-\!\bullet \cdots \bullet\!-\!-\!\bullet$ with $n-1$ vertices

the generator s_i is the transposition $(i, i+1)$
the relations are $(s_i s_j)^2 = id$ if $|j-i| \geq 2$
$(s_i s_{i+1})^3 = id$.

For any $w \in W$, let $l(w)$ be the length of a minimal-length expression of w as a product of generators in S.
Then, for $w \in W$, its descent set is $D(w) = \{i \mid l(ws_i) < l(w)\} \subseteq S$
In the case of the symmetric group: right-multiplication by s_i exchanges the images of i and $i+1$ — i.e. it sends $\sigma_1 \sigma_2 \cdots \sigma_n$ to $\sigma_1 \sigma_2 \cdots \sigma_{i-1} \sigma_{i+1} \sigma_i \sigma_{i+2} \cdots \sigma_n$.
The length of a permutation is its number of inversions: $|\{(i,j) \mid i<j, \sigma_i > \sigma_j\}|$
And exchanging the images of i and $i+1$ reduces the number of inversions precisely when $\sigma_{i+1} > \sigma_i$ $\therefore$ this more general definition of descents agree with our previous definition.

The general version of Solomon's theorem:
for any subset T of S, set $y_T := \sum_{D(w)=T} w$.
then the y_T span a subalgebra of $\mathbb{Q}W$ of dimension $2^{|S|}$, and there is an explicit description of the coefficients in the products $y_T y_R$.

In the case of the symmetric group, the basis elements y_T can also be indexed by compositions c of n (written $c \models n$), since there is a classical bijection between subsets of $n-1$ and compositions of n: $\{i_1, \ldots, i_l\} \to (i_1, i_2-i_1, i_3-i_2, \cdots)$

There is a second basis for $\Sigma[S_n]$: let $B_T := \sum_{D(\sigma) \subseteq T} \sigma$ — i.e. the permutations who might have a descent at T. The change from B_T to y_T is upper-unitriangular, so B_T is a basis. Its multiplication table is much easier, and proves that $\Sigma[S_n]$ is an algebra: $B_C B_D = \sum_M B_{C(M)}$ where the sum is over all matrices M with non-negative integral entries whose j^{th} column sums to c_j (j^{th} part of the composition C), and whose i^{th} row sums to d_i.
Then $C(M)$ is the composition formed by reading the matrix entries along the rows from left to right, from the top row to the bottom row, and deleting zeroes.

In $\mathbb{Q}[S_n]$, there are two anti-isomorphic algebras: the descent algebra $\Sigma[S_n]$, and l'algèbre de battage $\mathcal{B}_n$.

the antiisomorphism is given by $\sigma \to \sigma^{-1}$ (so $\alpha\beta \to \beta^{-1}\alpha^{-1}$, hence an anti-isomorphism). The image of B_T under this antiisomorphism (in $\mathcal{B}_n$) is ($T=\{t_1, t_2, \cdots, t_k\}$)

$12\cdots t_1 \sqcup\!\sqcup\, t_1{+}1\, t_1{+}2 \cdots t_1{+}t_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, t_k{+}1\, t_k{+}2 \cdots n$.

(e.g. the image of $B_{\{3,5\}}$ is $123 \sqcup\!\sqcup\, 45 \sqcup\!\sqcup\, 678$)

(Here, $\sqcup\!\sqcup$ denotes the shuffle product: for α, β who between them have distinct letters, $\alpha \sqcup\!\sqcup\, \beta := \sum_\gamma \gamma$ over all γ which contain α, β as complimentary substrings — i.e. $\alpha = \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k}$, $\beta = \gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_l}$, with $i_1 < i_2 < \cdots < i_k$; $j_1 < j_2 < \cdots < j_l$, $\{i_1, i_2, \cdots, i_k\} \sqcup \{j_1, j_2, \cdots, j_l\} = \{1, 2, \cdots\}$. This is an associative product.)

These images of B_T multiply by an analogous matrix rule, except the matrix composition is read down the columns from the left column to the right column. (This is easy to prove, and the multiplication of B_T may be deduced from this.)

The same multiplication rule describes the Kronecker product (also called internal product) of complete symmetric functions of the same degree. (One definition of this product is that the power sums satisfy $p_\lambda \cdot p_\mu = 0$ if $\lambda \neq \mu$, and $p_\mu \cdot p_\mu = z_\mu p_\mu$, where z_μ is such that $\frac{n!}{z_\mu}$ is the number of permutations of cycle type μ. Equivalently, p_λ / z_λ are orthogonal idempotents.)

As a result, we can define a surjective algebra morphism $\Phi: \Sigma[S_n] \to \Lambda_n$, $\Phi(B_c) = h_c$. This surjection is in fact split — i.e. $\Sigma[S_n]$ is a direct sum of the kernel of S_n and a subspace which is isomorphic to Λ_n via Φ. So we can take the preimages of p_λ / z_λ in this subspace — these are the idempotents E_λ of Garsia and Reutenauer, and summing over all λ of the same number of parts give the e_i from before. More details to follow.

It's easy to verify that the sum of orthogonal idempotents is also an idempotent. We are interested in primitive idempotents, which cannot be written as the sum of orthogonal idempotents.

In a finite-dimensional algebra, orthogonal idempotents are necessarily linearly independent. So the number of mutually orthogonal idempotents is at most the dimension of the algebra. Any set of mutually orthogonal idempotents $\{e_1, e_2, \cdots, e_r\}$ can be extended so that $\sum e_i$ = unit of algebra: this is because, if e is an idempotent, then unit$-e$ is an idempotent orthogonal

to e. If $\sum e_i = 1$, then $\{e_1, \cdots, e_r\}$ is a complete family of idempotents.
Note that complete families of primitive orthogonal idempotents
for example, for the algebra of 2x2 upper-triangular matrices, $\{\begin{pmatrix}1&0\\0&0\end{pmatrix}; \begin{pmatrix}0&0\\0&1\end{pmatrix}\}$
and $\{\begin{pmatrix}1&1\\0&0\end{pmatrix}, \begin{pmatrix}0&-1\\0&1\end{pmatrix}\}$ are both such families.
Idempotents are important for studying the representations of an algebra.
Expanding an algebra element in terms of idempotents is akin to a Fourier transform.

Let $p(x)$ be a polynomial $(x-a_1)(x-a_2)\cdots(x-a_n)$, where a_i are pairwise distinct.
Then, in the algebra, $\mathbb{C}[x]/_{\langle p(x)\rangle}$, a complete family of primitive orthogonal idempotents are the Lagrange interpolation formulae: $e_k(x) = \dfrac{(x-a_1)\cdots(x-a_{k-1})(x-a_{k+1})\cdots(x-a_n)}{(a_k-a_1)\cdots(a_k-a_{k-1})(a_k-a_{k+1})\cdots(a_k-a_n)}$
This is because every element of $\mathbb{C}[x]/_{\langle p(x)\rangle}$ is determined by its value on $a_1, a_2, \dots, a_n$, and e_k is the indicator function on a_k.
So idempotents of algebras are often interesting.

Because $\{p_\lambda/z_\lambda\}$ gives a basis of Λ_n, they must be a complete family of primitive orthogonal idempotents. The unit of the Kronecker product of Λ_n is $\sum_{\lambda\vdash n} p_\lambda/z_\lambda = h_n$.
It happens that their preimages E_λ is a complete family of primitive orthogonal idempotents for $\Sigma[\mathfrak{S}_n]$.
Analogous families exist for other Coxeter groups - but what plays the role of partitions? what is the indexing set?

Let $H(z)$, $E(z)$, $P(z)$ be generating functions for the complete, elementary and power sum symmetric functions: e.g. $H(z) = \sum_{n=0}^{\infty} h_n z^n$.
Then the following relations hold: $H(z) = e^{P(z)}$, $E(z) = e^{-P(-z)}$, $H(z)E(-z) = 1$

$$P(z) = \log(H(z)) = -\log(E(-z))$$
$$P'(z) = H'(z)/H(z)$$

These allow us to express one basis in terms of another.

To define a noncommutative analogue of the symmetric functions, we take S_i to be generators analogous to h_i, and define other bases via noncommutative interpretations of a relation from a basis to h_i. For instance, one analogue Φ_k of

p_k is defined via $\sum \Phi_k z^k := (\sum_{n\geq 1} n S_n z^n)(\sum_{k\geq 1} (-1)^k (\sum_{n\geq 1} S_n z^n)^k)$, where multiplication of S_i on the right hand side is noncommutative. Using a different relation between p_λ and h_λ may give a different basis.

The S_i are non-commutative generators of this algebra SYM, so a basis of SYM is indexed by compositions. The basis elements indexed by compositions of n span the subspace of degree n.

There is a graded projection $\pi: \mathrm{SYM} \to \Lambda$ sending S_i to h_i, which preserves degree.

The map $\Psi: \Sigma[\mathfrak{S}_n] \to \Lambda_n$ lifts to a map $\Theta: \Sigma[\mathfrak{S}_n] \to \mathrm{SYM}_n$ by sending B_T to S_T. Θ is a (vector space) isomorphism. Hence Θ allows the definition of an internal ($\mathrm{SYM}_n \times \mathrm{SYM}_n \to \mathrm{SYM}_n$) product on SYM_n (although it is more useful to define it to be the opposite of the product on $\Sigma[\mathfrak{S}_n]$). Similarly, Θ equips $\bigoplus_n \Sigma[\mathfrak{S}_n]$ with an external ($\Sigma[\mathfrak{S}_i] \times \Sigma[\mathfrak{S}_j] \to \Sigma[\mathfrak{S}_{i+j}]$) product. This uniting of $\Sigma[\mathfrak{S}_n]$ for all n is useful for studying the idempotents of $\Sigma[\mathfrak{S}_n]$. (Indeed, the study of descent algebras was much of the motivation for the definition of SYM.) Expressing the idempotents of $\Sigma[\mathfrak{S}_n]$ in terms of the basis B_T is analogous to changing from noncommutative power sums to the basis S_T — and this involves the external product.

Now we look more closely at the idempotents E_λ. Recall that they do not form a basis. However it is possible to get a basis I_α such that $I_\alpha I_\beta = \begin{cases} w(\mu)\, I_\beta & \text{if } \lambda(\alpha)=\lambda(\beta)=\mu \\ 0 & \text{otherwise} \end{cases}$

Here, $\lambda(\alpha)$ is the partition obtained by putting the parts of α in decreasing order, and $w(\mu)$ is the number of compositions whose partition is μ.

So $I_\alpha / w(\lambda(\alpha))$ is an idempotent, but these are not orthogonal.

Indeed, $I_\alpha := \sum_{\alpha \preceq \beta} \frac{(-1)^{l(\beta)-l(\alpha)}}{f(\alpha,\beta)} B_\beta$ (so the sum is over all refinements β of α — i.e. β is the concatenation $\beta^{(1)}\beta^{(2)}\dots\beta^{(l(\alpha))}$ with $\beta^{(i)} \models \alpha_i$)

where $f(\alpha,\beta)$ is the product of the lengths of the $\beta^{(i)}$

Inversing this relation shows $B_\alpha = \sum_{\alpha \preceq \beta} \frac{1}{g(\alpha,\beta)} I_\beta$,

where $g(\alpha,\beta)$ is the product of the factorials of the lengths of the $\beta^{(i)}$.

It is a theorem of Garsia/Reutenauer that the I_α multiply as described above, and that the orthogonal idempotents E_λ are sums of the I_α over rearrangements of λ:

$$E_\mu = \frac{1}{l(\mu)!} \sum_{\lambda(\alpha)=\mu} I_\alpha$$

To understand this, work in SYM. Define elements F_m via the generating function equality

$$\sum_{n=1}^{\infty} F_n x^n := \log\left(1+\sum_{k=1}^{\infty} S_k x^k\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\sum_{k=1}^{\infty} S_k x^k\right)^n$$ (i.e. use Taylor series for $\log(1+x)$)

For example: $F_1 = S_1$

$F_2 = S_2 - \frac{1}{2} S_{11}$

$F_3 = S_3 - \frac{1}{2}(S_{12}+S_{21}) + \frac{1}{3} S_{111}$.

Also, define $\mathcal{E}_\lambda$ by

$\sum_\lambda \mathcal{E}_\lambda x_\lambda := \exp\left(\sum_{n\geq 1} F_n x_n\right)$

Note that this involves infinitely-many variables x_n, and x_λ denotes $x_{\lambda_1} x_{\lambda_2} \cdots x_{\lambda_{l(\lambda)}}$.

Finally, let E_λ and I_α be the images of $\mathcal{E}_\lambda$ and of $F_{\alpha_1}\cdots F_{\alpha_l}$ under the identification Θ^{-1} of SYM with $\Sigma[S_n]$.

(These calculations are in the paper of Garsia/Reutenauer, though not in the language of SYM, as the paper predates the definition of SYM)

For example, $F_3F_1F_2 = S_{312} - \frac{1}{2}S_{1212} - \frac{1}{2}S_{2112} + \frac{1}{3}S_{11112} - \frac{1}{2}S_{3111} + \frac{1}{4}S_{12111} + \frac{1}{4}S_{21111} - \frac{1}{6}S_{111111}$

so $I_{312} = B_{312} - \frac{1}{2}B_{1212} - \frac{1}{2}B_{2112} + \frac{1}{3}B_{11112} - \frac{1}{2}B_{3111} + \frac{1}{4}B_{12111} + \frac{1}{4}B_{21111} - \frac{1}{6}B_{111111}$

which agrees with the formula at the start of this section, in terms of summing over refinements. (This explains why those formulae resemble the series for log and exp).

We have analogous idempotents E_λ for the descent algebras of all finite Coxeter groups. Details are in the paper by Bergeron/Bergeron/Howlett/Taylor.

For example, take the dihedral group with presentation $\langle s,r \mid s^2=r^2=1, (sr)^p=1\rangle$ for some $p\geq 3$. This has $2p$ elements: $1, s, r, sr, rs, srs, rsr, \ldots$

In the notation of Solomon, for a subset A of the generating reflections, $x_A := \sum_{D(w)\cap A=\phi} w$, the sum of all elements which lengthen when multiplied by each element of A.

So we find $x_{r,s} = 1$

$x_r = 1+s+rs+\cdots$

$x_s = 1+r+sr+\cdots$

$x_\phi = \sum_{w\in W} w$

When p is even, the multiplication table of these elements read:

	1	x_r	x_s	x_ϕ
1	1	x_r	x_s	x_ϕ
x_r	x_r	$2x_r+\frac{p-2}{2}x_\phi$	$\frac{p}{2}x_\phi$	px_ϕ
x_s	x_s	$\frac{p}{2}x_\phi$	$2x_s+\frac{p-2}{2}x_\phi$	px_ϕ
x_ϕ	x_ϕ	px_ϕ	px_ϕ	$2px_\phi$

And from this we find a complete family of primitive orthogonal idempotents:

$$e_{r,s} = 1-\tfrac{1}{2}x_r-\tfrac{1}{2}x_s+\tfrac{p-1}{2p}x_\phi$$
$$e_r = \tfrac{1}{2}\left(x_r-\tfrac{1}{2}x_\phi\right)$$
$$e_s = \tfrac{1}{2}\left(x_s-\tfrac{1}{2}x_\phi\right)$$
$$e_\phi = \tfrac{1}{2p}x_\phi$$

The case for p odd is similar, but in that case the descent algebra is not commutative, so the analogous formulae result in e_r, e_s which are not orthogonal. Hence there are only three primitive orthogonal idempotents: $e_{r,s}$, e_r+e_s, e_ϕ.

The analysis for type B, the hyperoctahedral group, was less immediate. (This is the group of $n\times n$ matrices with a single non-zero entry in each row and column, and each such entry is 1 or -1 e.g. $\begin{pmatrix}0&-1&0\\1&0&0\\0&0&-1\end{pmatrix}$. Its cardinality is $2^n n!$)

It took a while to find an analogue of the symmetric group formula $I_\alpha = \sum_{\alpha\le\beta}\frac{(-1)^{l(\beta)-l(\alpha)}}{f(\alpha,\beta)}B_\beta$,

because the coefficients here are in fact simplified from $\prod_{i=1}^{l(\alpha)}\frac{(-1)(-2)\cdots(-l(\beta^{(i)})+1)}{(2)(3)\cdots(l(\beta^{(i)}))}$,

and $1,2,\cdots,n-1$ are the exponents of the group S_n, being one less than the degrees $2,3,\ldots,n$.
These two sequences are defined for any Coxeter group, and for B_n they are $1,3,5,\cdots,2n-1$ and $2,4,6,\cdots,2n$.
Thus one advantage of generalising away from S_n to other Coxeter groups is that it "explains" the results in S_n.

It is sometimes useful to view the symmetric functions as operators on the variables $x_1, x_2, \cdots$. Denote this set of variables by $X := x_1+x_2+\cdots$
Now impose these formal calculation rules, for $f,g\in\Lambda$, and a constant c:
$(f\cdot g)[X]=f[X]\,g[X]$; $(f+g)[X]=f[X]+g[X]$; $c[X]=c$

Consequently, to calculate the result of any symmetric function acting on any set of variables, it suffices to calculate the actions of the power sums p_k. (explicitly, expand $f \in \Lambda$ as $f = \sum_{\lambda \vdash n} a_\lambda p_\lambda$; then $f[X] = \sum_\lambda a_\lambda \prod_{i=1}^{\ell(\lambda)} (p_{\lambda_i}[X])$)
These actions are defined by, for "expressions" A and B:

$p_k[A+B] = p_k[A] + p_k[B]$ $\quad$ $p_k[A-B] = p_k[A] - p_k[B]$

$p_k[AB] = p_k[A]\, p_k[B]$ $\quad$ $p_k\left[\frac{A}{B}\right] = p_k[A] / p_k[B]$

$p_k[x] = x^k$ if x is a variable, $\quad$ $p_k[c] = c$ if c is a constant

(so one must specify in an expression A what is a variable and what is a constant.)

Since the symmetric functions are operators, they need not commute with other operators e.g. if ev_{-1} denotes evaluation at $x=-1$, then $p_k\, ev_{-1}[x] = p_k[-1] = -1$, but $ev_{-1}\, p_k[x] = ev_{-1}[x^k] = (-1)^k$.

One application of this operator viewpoint is to define the coproduct on Λ:
$\Delta(f) = \sum_i g_i \otimes h_i$ if $f[X+Y] = \sum_i g_i[X]\, h_i[Y]$.