1. (7 points) Compute the following two improper integrals, or explain why they do not converge. **Simplify your answer as much as possible.**

(a)

$$\int_2^\infty \frac{e^x}{\sqrt{2e^x-2}}dx.$$

$$= \lim_{t\to\infty} \int_2^t \frac{e^x}{\sqrt{2e^x-2}}\,dx$$

substitution $u = 2e^x - 1$
$du = 2e^x\,dx$

$$= \lim_{t\to\infty} \left[\frac{\sqrt{2e^x-2}}{2\ \tfrac{1}{2}} \right]_2^t$$

$$= \lim_{t\to\infty} \left(\sqrt{2e^t-2} - \sqrt{2e^2-2} \right)$$

which diverges, because, as $t\to\infty$,
$e^t \to \infty$
so $\sqrt{2e^t-2} \to \infty$.

(b)

$$\int_0^1 \frac{e^x}{\sqrt{2e^x-2}}dx.$$

$$= \lim_{t\to 0^+} \int_t^1 \frac{e^x}{\sqrt{2e^x-2}}\,dx$$

$$= \lim_{t\to 0^+} \left[\sqrt{2e^x-2} \right]_t^1$$

$$= \lim_{t\to 0^+} \sqrt{2e-2} - \sqrt{2e^t-2}$$

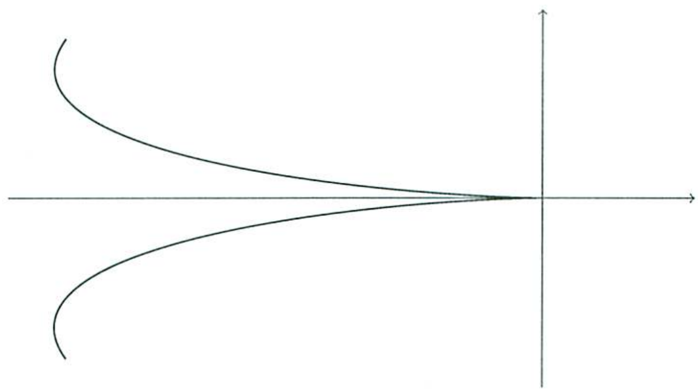
$$= \sqrt{2e-2} - \sqrt{2e^0-2}$$

$$= \sqrt{2e-2}$$

2. (14 points) Let C be the parametrised curve with equation

$$x = \frac{t^4}{4} - 8t^2, \quad y = \frac{8t^3}{3},$$

as shown in the diagram below.



(a) Find the point(s) where C has a vertical tangent. **Simplify your answer as much as possible.**

C has a vertical tangent when $\frac{dx}{dt} = 0$, $\frac{dy}{dt} \neq 0$.

$\frac{dx}{dt} = 0$ when $t^3 - 16t = 0$

$t(t^2 - 16) = 0$

$t = 0$ or $t = 4$ or $t = -4$

When $t = 0$: $\frac{dy}{dt} = 8(0)^2 = 0$ $t=0$ corresponds to $(x,y) = (0,0)$, and the diagram above shows that there is no vertical tangent there.

When $t = 4$: $\frac{dy}{dt} = 8(4)^2 \neq 0$

When $t = -4$: $\frac{dy}{dt} = 8(-4)^2 \neq 0$

$\Big\}$ $\therefore$ these do give vertical tangents

$t = 4$ corresponds to $x = \frac{4^4}{4} - 8(4)^2 = 64 - 16 \times 8 = -64$, $y = \frac{8(4)^3}{3} = \frac{512}{3}$

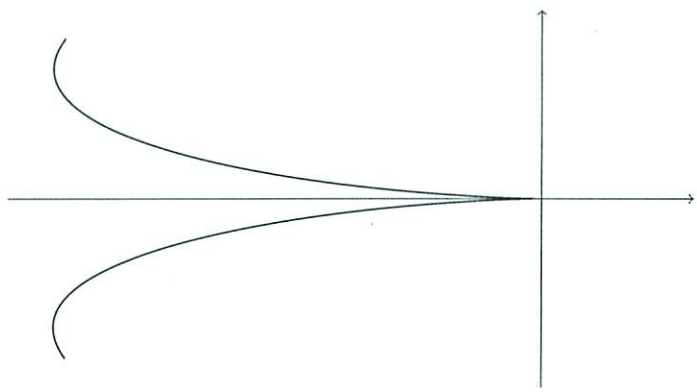
$t = -4$ corresponds to $x = \frac{(-4)^4}{4} - 8(-4)^2 = -64$ $y = \frac{8(-4)^3}{3} = \frac{-512}{3}$

$\therefore$ C has vertical tangents at $(-64, \frac{512}{3})$ and $(-64, \frac{-512}{3})$.

(Question 2 continues on the next page.)

(b) For your convenience, here again is the information about the parametrised curve C:

$$x = \frac{t^4}{4} - 8t^2, \quad y = \frac{8t^3}{3}.$$



Find the length of the part of C with $-3 \le t \le -1$. **Simplify your answer as much as possible.**

$$\text{length} = \int_{-3}^{-1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_{-3}^{-1} \sqrt{(t^3-16t)^2 + (8t^2)^2}\, dt$$

$$= \int_{-3}^{-1} \sqrt{t^6 - 32t^4 + 16^2t^2 + 64t^4}\, dt$$

$$= \int_{-3}^{-1} \sqrt{t^6 + 32t^4 + 16^2t^2}\, dt$$

$$= \int_{-3}^{-1} \sqrt{t^2(t^2+16)^2}\, dt$$

$$= \int_{-3}^{-1} |t|\,|t^2+16|\, dt$$

on the interval $-3 \le t \le -1$, $t < 0$, $t^2+16 > 0$.

$$= \int_{-3}^{-1} -t(t^2+16)\, dt$$

$$= \left[-\frac{t^4}{4} - \frac{16t^2}{2}\right]_{-3}^{-1}$$

$$= \left(-\frac{1}{4} - 8\right) - \left(-\frac{81}{4} - 72\right) = 84$$