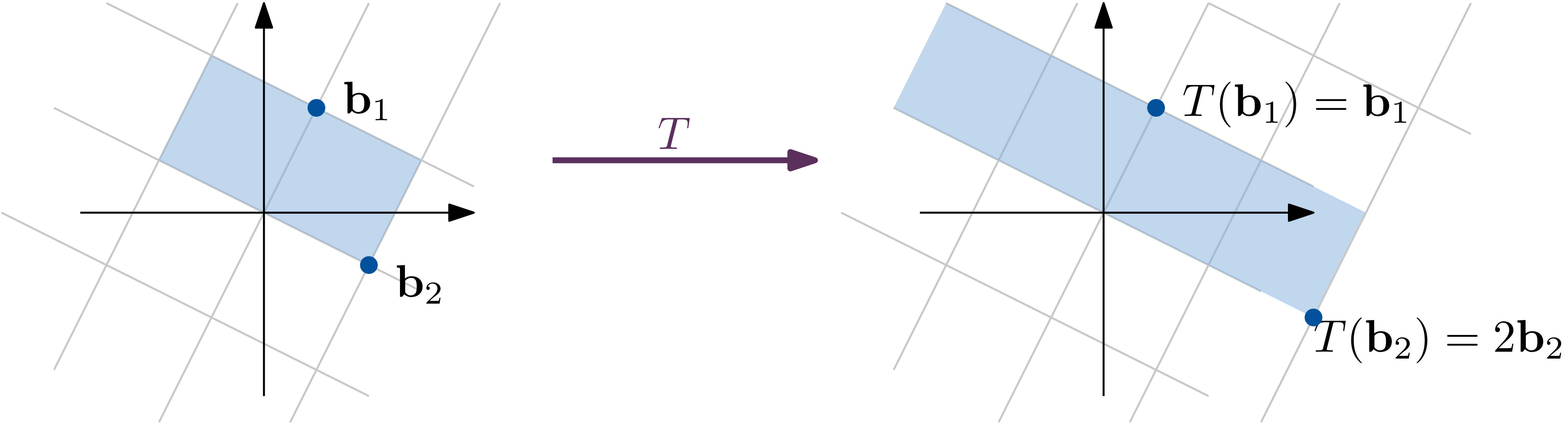
Remember from last week (week 9 p20):
Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, the “right” basis to work in is $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ where $T(\mathbf{b}_i) = \lambda_i \mathbf{b}_i$ for some scalars λ_i. Then the matrix for T relative to $\mathcal{B}$ is a diagonal matrix:

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & \dots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}.$$

Computers are much faster and more accurate when they work with diagonal matrices, because many entries are 0.

Also, it's much easier to understand the linear transformation T from a diagonal matrix, e.g. if $T(\mathbf{b}_1) = \mathbf{b}_1$ and $T(\mathbf{b}_2) = 2\mathbf{b}_2$, so $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, then T is an expansion by a factor of 2 in the $\mathbf{b}_2$ direction.



So it is important to study the equation $T(\mathbf{x}) = \lambda\mathbf{x}$.

(It's also very useful in ODEs - see MATH3405.)

§5.1-5.2: Eigenvectors and Eigenvalues

Definition: Let A be a square matrix.
An *eigenvector* of A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.
Then we call $\mathbf{x}$ an *eigenvector corresponding to λ* (or a λ-eigenvector).
An *eigenvalue* of A is a scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}$.

If $\mathbf{x}$ is an eigenvector of A, then $\mathbf{x}$ and its image $A\mathbf{x}$ are in the same (or opposite, if $\lambda < 0$) direction. Multiplication by A stretches $\mathbf{x}$ by a factor of λ.
If $\mathbf{x}$ is not an eigenvector, then $\mathbf{x}$ and $A\mathbf{x}$ are not geometrically related in any obvious way.

Warning: eigenvalues and eigenvectors exist for square matrices only. If A is not a square matrix, then $\mathbf{x}$ and $A\mathbf{x}$ are in different vector spaces (they are column vectors with a different number of rows), so it doesn't make sense to ask whether $A\mathbf{x}$ is a multiple of $\mathbf{x}$.

$$A\mathbf{x} = \lambda\mathbf{x}$$

eigenvector: cannot be $\mathbf{0}$.
$A\mathbf{0} = \lambda\mathbf{0}$ is always true, so it holds no information about A.

eigenvalue: can be 0.
$A\mathbf{x} = 0\mathbf{x}$ for a nonzero vector $\mathbf{x}$ does hold information about A - it tells you that A is not invertible. In fact, A is invertible if and only if 0 is not an eigenvalue (add to IMT!).

Important computations:

i given an eigenvalue, how to find the corresponding eigenvectors (p5-9, §5.1);

ii how to find the eigenvalues (p10-13, §5.2);

iii how to determine if there is a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of $\mathbb{R}^n$ where each $\mathbf{b}_i$ is an eigenvector (p15-31, §5.3).

Warm up:

Example: Let $A = \begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix}$. Determine whether $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are eigenvectors of A.

Answer:

$\begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ -3 \end{bmatrix}$ is not a multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (because its entries are not equal), so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not an eigenvector of A.

$\begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, so $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue 6.

i: Given the eigenvalues, find the corresponding eigenvectors:

i.e. we know λ, and we want to solve $\quad A\mathbf{x} = \lambda\mathbf{x}.$

This equation is equivalent to $\quad A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0},$

which is equivalent to $\quad (A - \lambda I)\mathbf{x} = \mathbf{0}.$

So the eigenvectors of A corresponding to the eigenvalue λ are the nonzero solutions to $(A - \lambda I)\mathbf{x} = \mathbf{0}$, which we can find by row-reducing $A - \lambda I$.

Because it is sometimes convenient to talk about the eigenvectors and $\mathbf{0}$ together:

Definition: The *eigenspace* of A corresponding to the eigenvalue λ (or the λ-eigenspace of A, sometimes written $E_\lambda(A)$) is the solution set to $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Because λ-eigenspace of A is the null space of $A - \lambda I$, eigenspaces are subspaces.
In the previous example, the eigenspace is a line, but there can also be two-dimensional eigenspaces:

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Because λ-eigenspace of A is the null space of $A - \lambda I$, eigenspaces are subspaces. In the previous example, the eigenspace is a line, but there can also be two-dimensional eigenspaces:

Example: Let $B = \begin{bmatrix} -3 & 0 & 0 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}$. Find a basis for the eigenspace corresponding to the eigenvalue -3.

Answer:

$$B - (-3)I_3 = \begin{bmatrix} -3+3 & 0 & 0 \\ -1 & -2+3 & 1 \\ -1 & 1 & -2+3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{row-reduction}} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So solutions are $x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, for all values of x_2, x_3. So a basis is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Be careful how you write your answer, depending on what the question asks for:

The eigenvectors: $\left\{ s\begin{bmatrix}1\\1\\0\end{bmatrix} + t\begin{bmatrix}1\\0\\1\end{bmatrix} \middle| \boxed{s, t \text{ not both zero}} \right\}.$

DON'T write $s, t \neq 0$, because that's confusing: do you mean $s \neq 0$ AND $t \neq 0$?

The eigenspace: $\left\{ s\begin{bmatrix}1\\1\\0\end{bmatrix} + t\begin{bmatrix}1\\0\\1\end{bmatrix} \middle| s, t \in \mathbb{R} \right\}.$

A basis for the eigenspace: $\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$

Tip: if you found that $(B - \lambda I)\mathbf{x} = \mathbf{0}$ has no nonzero solutions, then you've made an arithmetic error. Please do not write that the eigenvector is $\mathbf{0}$!

ii: Given a matrix, find its eigenvalues:

λ is an eigenvalue of A if $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has non-trivial solutions.
By the Invertible Matrix Theorem, this happens precisely when $A - \lambda I$ is not invertible.
So we must have $\det(A - \lambda I) = 0$.

$\det(A - \lambda I)$ is the characteristic polynomial of A (sometimes written χ_A). If A is $n \times n$, then this is a polynomial of degree n. So A has at most n different eigenvalues.
$\det(A - \lambda I) = 0$ is the characteristic equation of A.
We find the eigenvalues by solving the characteristic equation.

We find the eigenvalues by solving the characteristic equation $\det(A - \lambda I) = 0$.

Example: Find the eigenvalues of $B = \begin{bmatrix} -3 & 0 & 0 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}$.

Answer:

(expand along top row)

$$\det(B - \lambda I) = \begin{vmatrix} -3-\lambda & 0 & 0 \\ -1 & -2-\lambda & 1 \\ -1 & 1 & -2-\lambda \end{vmatrix} = (-3-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix}$$

$$= (-3-\lambda)[(-2-\lambda)(-2-\lambda) - 1]$$

Tip: if you already have a factor, don't expand it

$$= (-3-\lambda)[\lambda^2 + 4\lambda + 3]$$

$$= (-3-\lambda)(\lambda+3)(\lambda+1).$$

So the eigenvalues are the solutions to $(-3-\lambda)(\lambda+3)(\lambda+1) = 0$,

which are $-3, \quad -3, \quad -1$.

Tips:

- Because of the variable λ, it is easier to find $\det(A - \lambda I)$ by expanding across rows or down columns than by using row operations.
- If you already have a factor, do not expand it (e.g. previous page)
- Do not "cancel" λ in the characteristic equation: remember that $\lambda = 0$ can be an eigenvalue (see below).
- The eigenvalues of A are usually not related to the eigenvalues of rref(A).

Tips:

- Because of the variable λ, it is easier to find $\det(A-\lambda I)$ by expanding across rows or down columns than by using row operations.
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- The eigenvalues of A are usually not related to the eigenvalues of rref(A).

Example: Find the eigenvalues of $C = \begin{bmatrix} 3 & 6 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 6 \end{bmatrix}$.

Answer: $C - \lambda I = \begin{bmatrix} 3-\lambda & 6 & -2 \\ 0 & -\lambda & 2 \\ 0 & 0 & 6-\lambda \end{bmatrix}$ is upper triangular, so its determinant is the product of its diagonal entries: $\det(C-\lambda I) = (3-\lambda)(-\lambda)(6-\lambda)$, whose solutions are 3, 0, 6.

By a similar argument (for upper or lower triangular matrices):

Fact: The eigenvalues of a triangular matrix are the diagonal entries.

Summary: To find the eigenvalues and eigenvectors of a square matrix A:
Step 1 Solve the characteristic equation $\det(A - \lambda I) = 0$ to find the eigenvalues;
Step 2 For each eigenvalue λ, solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ to find the eigenvectors.

Summary: To find the eigenvalues and eigenvectors of a square matrix A:

Step 1 Solve the characteristic equation $\det(A - \lambda I) = 0$ to find the eigenvalues;

Step 2 For each eigenvalue λ, solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ to find the eigenvectors.

Thinking about eigenvectors conceptually:

Suppose $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue λ.
Then

$$A^2(\mathbf{v}) = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda A\mathbf{v} = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}.$$

So any eigenvector of A is also an eigenvector of A^2, corresponding to the square of the previous eigenvalue.

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We can also define eigenvalues and eigenvectors for a linear transformation $T : V \to V$ on an abstract vector space V: a nonzero vector $\mathbf{v}$ in V is an eigenvector of T with corresponding eigenvalue λ if $T(\mathbf{v}) = \lambda\mathbf{v}$.

Example: Consider $T : \mathbb{P}_3 \to \mathbb{P}_3$ given by $T(\mathbf{p}) = x\frac{d}{dx}\mathbf{p}$. Then $\mathbf{p}(x) = x^2$ is an eigenvector of T corresponding to the eigenvalue 2, because
$T(x^2) = x\frac{d}{dx}x^2 = x2x = 2x^2$.

§5.3: Diagonalisation

Remember that our motivation for finding eigenvectors is to find a basis relative to which a linear transformation is represented by a diagonal matrix.

Definition: (week 9 p17) Two square matrices A and B are *similar* if there is an invertible matrix P such that $A = PBP^{-1}$.

From the change-of-coordinates formula (week 9 p14)

$$[T]_{\mathcal{E}} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_{\mathcal{B}}\underset{\mathcal{B}\leftarrow\mathcal{E}}{\mathcal{P}} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}[T]_{\mathcal{B}}\underset{\mathcal{E}\leftarrow\mathcal{B}}{\mathcal{P}}^{-1},$$

similar matrices represent the same linear transformation relative to different bases.

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similar matrices represent the same linear transformation relative to different bases.

Definition: A square matrix A is *diagonalisable* if it is similar to a diagonal matrix, i.e. if there is an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Theorem 5: Diagonalisation Theorem: An $n \times n$ matrix A is diagonalisable (i.e. $A = PDP^{-1}$) if and only if A has n linearly independent eigenvectors.

Proof: we prove a stronger theorem: An $n \times n$ matrix A satisfies $AP = PD$ for a $n \times k$ matrix P and a diagonal $k \times k$ matrix D if and only if the ith column of P is an eigenvector of A with eigenvalue d_{ii}, or is the zero vector. This comes from equating column by column the right hand sides of the following equations:

$$AP = A \begin{bmatrix} | & & | \\ \mathbf{p}_1 & \dots & \mathbf{p}_k \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ A\mathbf{p}_1 & \dots & A\mathbf{p}_k \\ | & & | \end{bmatrix}$$

$$PD = \begin{bmatrix} | & & | \\ \mathbf{p}_1 & \dots & \mathbf{p}_k \\ | & & | \end{bmatrix} \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & & \\ \vdots & & \ddots & \\ 0 & & & d_{kk} \end{bmatrix} = \begin{bmatrix} | & & | \\ d_{11}\mathbf{p}_1 & \dots & d_{kk}\mathbf{p}_k \\ | & & | \end{bmatrix}$$

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To deduce The Diagonalisation Theorem, note that $A = PDP^{-1}$ if and only if $AP = PD$ and P is invertible, i.e. (using Invertible Matrix Theorem) if and only if $AP = PD$ and the n columns of P are linearly independent.

iii.i: Diagonalise a matrix i.e. given A, find P and D with $A = PDP^{-1}$:

Example: Diagonalise $A = \begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix}$.

Answer:

Step 1 Solve the characteristic equation $\det(A - \lambda I) = 0$ to find the eigenvalues.

From p11, $\det(A - \lambda I) = \lambda^2 - 8\lambda + 12$, eigenvalues are 2 and 6.

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Step 2 For each eigenvalue λ, solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ to find a basis for the λ-eigenspace.

From p7, $\left\{ \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ is a basis for the 2-eigenspace,

You can check that $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ is a basis for the 6-eigenspace.

Notice that these two eigenvectors are linearly independent (this is automatic, p22).

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If Step 2 gives fewer than n vectors, A is not diagonalisable (p26). Otherwise, continue:

Step 3 Put the eigenvectors from Step 2 as the columns of P. $P = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix}$.

Step 4 Put the corresponding eigenvalues as the diagonal entries of D. $D = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$.

Checking our answer: $PDP^{-1} = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 & 2 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix} = A.$

The matrices P and D are not unique:

- In Step 2, we can choose a different basis for the eigenspaces: e.g. using $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ instead of $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ as a basis for the 2-eigenspace, we can take $P = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$, and then $PDP^{-1} = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix} = A.$

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- In Step 3, we can choose a different order for the columns of P, as long as we put the entries of D in the corresponding order: e.g. $P = \begin{bmatrix} -2 & -2 \\ 1 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$ then $PDP^{-1} = \begin{bmatrix} -2 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 3 & 2 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ -3 & 0 \end{bmatrix} = A.$

Example: Diagonalise $B = \begin{bmatrix} -3 & 0 & 0 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}$.

Answer:

Step 1 Solve the characteristic equation $\det(B - \lambda I) = 0$ to find the eigenvalues.

From p12, $\det(B - \lambda I) = (-3 - \lambda)(\lambda + 3)(\lambda + 1)$, so the eigenvalues are -3 and -1.

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Step 2 For each eigenvalue λ, solve $(B - \lambda I)\mathbf{x} = \mathbf{0}$ to find a basis for the λ-eigenspace.

From p8, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the -3-eigenspace; you can check that $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the -1-eigenspace. You can check that these three eigenvectors are linearly independent (this is automatic, see p22).

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Step 4 Put the corresponding eigenvalues as the diagonal entries of D.

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Remember that $B = PDP^{-1}$ if and only if $BP = PD$ and P is invertible. This allows us to check our answer without inverting P:

$$BP = \begin{bmatrix} -3 & 0 & 0 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 \\ -3 & 0 & -1 \\ 0 & -3 & -1 \end{bmatrix},$$

$$PD = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 \\ -3 & 0 & -1 \\ 0 & -3 & -1 \end{bmatrix} = BP, \text{ and}$$

$$\det P = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -2 \neq 0.$$

We can use the matrices P and D to quickly calculate powers of B (see also week 9 p19):

$$\begin{aligned}
B^3 &= (PDP^{-1})^3 \\
&= (PDP^{-1})(PDP^{-1})(PDP^{-1}) \\
&= PD^3P^{-1} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix}^3 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -27 & 0 & 0 \\ 0 & -27 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -27 & 0 & 0 \\ -13 & -14 & 13 \\ -13 & 13 & -14 \end{bmatrix}.
\end{aligned}$$

(This sometimes works for“ fractional” and negative powers too.)

At the end of Step 2, after finding a basis for each eigenspace, it is unnecessary to explicitly check that the eigenvectors in the different bases, together, are linearly independent:

Theorem 7c: Linear Independence of Eigenvectors: If $\mathcal{B}_1, \ldots, \mathcal{B}_p$ are linearly independent sets of eigenvectors of a matrix A, corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_p$, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ is linearly independent. (Proof idea: see practice problem 3 in §5.1 of textbook.)

Example: In the previous example, $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a linearly independent set in the -3-eigenspace, $\mathcal{B}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a linearly independent set in the -1-eigenspace, so the theorem says that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent.

iii.ii: Determine if a matrix is diagonalisable

From the Diagonalisation Theorem, we know that A is diagonalisable if and only if A has n linearly independent eigenvectors. Can we determine if A has enough eigenvectors without finding all those eigenvectors?

To do so, we need an extra idea:
Definition: The (algebraic) multiplicity of an eigenvalue λ_k is its multiplicity as a root of the characteristic equation, i.e. it is the number of times the linear factor $(\lambda - \lambda_k)$ occurs in $\det(A - \lambda I)$.

Example: Consider $B = \begin{bmatrix} -3 & 0 & 0 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}$. From p12, the characteristic polynomial of B is $\det(B - \lambda I) = (-3 - \lambda)(\lambda + 3)(\lambda + 1) = -(\lambda + 3)(\lambda + 3)(\lambda + 1)$. So -3 has multiplicity 2, and -1 has multiplicity 1.

Theorem 7b: Diagonalisability Criteria: An $n \times n$ matrix A is diagonalisable if and only if both the following conditions are true:

i the characteristic polynomial $\det(A - \lambda I)$ factors completely into linear factors (i.e. it has n solutions counting with multiplicity);

ii for each eigenvalue λ_k, the dimension of the λ_k-eigenspace is equal to the multiplicity of λ_k.

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Example: (failure of i) Consider $\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$, the standard matrix for rotation through $\frac{\pi}{6}$. Its characteristic polynomial is $\begin{vmatrix} \sqrt{3}/2 - \lambda & -1/2 \\ 1/2 & \sqrt{3}/2 - \lambda \end{vmatrix} = (\frac{\sqrt{3}}{2} - \lambda)^2 + \frac{1}{4}$.

This polynomial cannot be written in the form $(\lambda - a)(\lambda - b)$ because it has no solutions, as its value is always $\geq \frac{1}{4}$. So this rotation matrix is not diagonalisable. (This makes sense because, after a rotation through $\frac{\pi}{6}$, no vector is in the same or opposite direction.)

The failure of i can be "fixed" by allowing eigenvalues to be complex numbers, so we concentrate on condition ii.

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i. the characteristic polynomial $\det(A - \lambda I)$ factors completely into linear factors;
ii. for each eigenvalue λ_k, the dimension of the λ_k-eigenspace is equal to the multiplicity of λ_k.

Example: (failure of ii) Consider $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. It is upper triangular, so its eigenvalues are its diagonal entries (with the same multiplicities), i.e. 0 with multiplicity 2. The eigenspace of eigenvalue 0 is the set of solutions to $\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - 0I_2\right)\mathbf{x} = \mathbf{0}$, which is Span $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. So the eigenspace has dimension $1 < 2$, and therefore $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalisable.

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Fact: (theorem 7a in textbook): the dimension of the λ_k-eigenspace is at most the multiplicity of λ_k. (Proof on p30.) So failure of ii in the Diagonalisability Criteria happens only when the eigenspaces are "too small".

Example: Determine if $B = \begin{bmatrix} -3 & 0 & 0 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}$ is diagonalisable.

Answer:

Step 1 Solve $\det(B - \lambda I) = 0$ to find the eigenvalues and multiplicities.

From p12, $\det(B - \lambda I) = (-3 - \lambda)(\lambda + 3)(\lambda + 1)$, so the eigenvalues are -3 (with multiplicity 2) and -1 (with multiplicity 1).

Step 2 For each eigenvalue λ of multiplicity more than 1, find the dimension of the λ-eigenspace (e.g. by row-reducing $(B - \lambda I)$ to echelon form):

The dimensions of all eigenspaces are equal to their multiplicities $\rightarrow$ diagonalisable

The dimension of one eigenspace is less than its multiplicity $\rightarrow$ not diagonalisable

$\lambda = -1$ has multiplicity 1, so we don't need to study it (see p29 for the reason).

$\lambda = -3$ has multiplicity 2, so we need to examine it more closely:

$$B - (-3)I_3 = \begin{bmatrix} -3+3 & 0 & 0 \\ -1 & -2+3 & 1 \\ -1 & 1 & -2+3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{row-reduction}} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This has two free variables (x_2, x_3), so the dimension of the -3-eigenspace is two, which is equal to its multiplicity. So B is diagonalisable.

Example: Let $K = \begin{bmatrix} 6 & -4 & 4 & 9 \\ -9 & 9 & 8 & -17 \\ 0 & 0 & 5 & 0 \\ -5 & 4 & -4 & -8 \end{bmatrix}$. Given that $\det(K - \lambda I) = (\lambda - 1)^2(\lambda - 5)^2$, determine if K is diagonalisable.

Answer:

Step 1 Solve $\det(K - \lambda I) = 0$ to find the eigenvalues and multiplicities.

The eigenvalues are 1 (with multiplicity 2) and 5 (with multiplicity 2).

Step 2 For each eigenvalue λ of multiplicity more than 1, find the dimension of the λ-eigenspace (e.g. by row-reducing $(K - \lambda I)$ to echelon form):

The dimensions of all eigenspaces are equal to their multiplicities $\rightarrow$ diagonalisable

The dimension of one eigenspace is less than its multiplicity $\rightarrow$ not diagonalisable

$$\lambda = 1\text{:}\ K - 1I_4 = \begin{bmatrix} 5 & -4 & 4 & 9 \\ -9 & 8 & 8 & -17 \\ 0 & 0 & 4 & 0 \\ -5 & 4 & -4 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & -4 & 4 & 9 \\ 0 & 4 & * & * \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad \begin{matrix} 5R_2 - 9R_1 \\ \\ \\ R_4 - R_1 \end{matrix}$$

x_4 is the only one free variable, so the dimension of the 1-eigenspace is one, which is less than its multiplicity. So K is not diagonalisable. (We don't need to also check $\lambda = 5$.)

In Step 2, why don't we need to look at eigenvalues with multiplicity 1?

In Step 2, why don't we need to look at eigenvalues with multiplicity 1?
Answer: because the dimension of an eigenspace is always at least 1. So if an eigenvalue has multiplicity 1, then the dimension of its eigenspace must be exactly 1.

In particular: suppose an $n \times n$ matrix has n different eigenvalues. The multiplicity of each eigenvalue is at least 1, and if any eigenvalue has multiplicity > 1, then χ will have more than n factors. So each eigenvalue must have multiplicity exactly 1.

Theorem 6: Distinct eigenvalues implies diagonalisable: If an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalisable.

Example: Is $C = \begin{bmatrix} 3 & 6 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 6 \end{bmatrix}$ diagonalisable?

Answer: C is upper triangular, so its eigenvalues are its diagonal entries (with the same multiplicities), i.e 3, 0 and 6. Since C is 3×3 and it has 3 different eigenvalues, C is diagonalisable.

Warning: an $n \times n$ matrix with fewer than n eigenvalues can still be diagonalisable!

To prove the Diagonalisability Criteria, we first need to prove

Fact: (theorem 7a in textbook): the dimension of the λ_k-eigenspace is at most the multiplicity of λ_k.

Proof: (sketch, hard) Let $\lambda_k = r$, and let d be the dimension of the r-eigenspace. We want to show that $(\lambda - r)^d$ divides $\det(A - \lambda I)$.

For simplicity, I show the case $d = 3$. This means there are 3 linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ corresponding to the eigenvalue r.

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For simplicity, I show the case $d = 3$. This means there are 3 linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ corresponding to the eigenvalue r.
By the Linearly Independent Set theorem, we can extend this to a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_4, \dots, \mathbf{w}_n\}$ of $\mathbb{R}^n$.
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation whose standard matrix is A.
Because $T(\mathbf{v}_i) = r\mathbf{v}_i$, we have

$$\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} = \begin{bmatrix} r & 0 & 0 & * & \dots & * \\ 0 & r & 0 & * & \dots & * \\ 0 & 0 & r & * & \dots & * \\ 0 & 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & * & \dots & * \end{bmatrix}.$$

So $\det(\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} - \lambda I_n)$ is: (expanding down the first column each time)

$$\begin{vmatrix} r-\lambda & 0 & 0 & * & \dots & * \\ 0 & r-\lambda & 0 & * & \dots & * \\ 0 & 0 & r-\lambda & * & \dots & * \\ 0 & 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & * & \dots & * \end{vmatrix} = r-\lambda \begin{vmatrix} r-\lambda & 0 & * & \dots & * \\ 0 & r-\lambda & * & \dots & * \\ 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & * & \dots & * \end{vmatrix} = (r-\lambda)^2 \begin{vmatrix} r-\lambda & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & * & \dots & * \end{vmatrix}$$

$$= (r-\lambda)^3 \times \text{ some polynomial}$$

So $(r-\lambda)^3$ divides $\det(\underset{\mathcal{B}\leftarrow\mathcal{B}}{[T]} - \lambda I_n)$, which is the same as $\det(A - \lambda I_n)$ because similar matrices have the same characteristic polynomial (and therefore the same eigenvalues):

$$\begin{aligned} \det(PBP^{-1} - \lambda I) &= \det(PBP^{-1} - \lambda PP^{-1}) \\ &= \det(P(B-\lambda I)P^{-1}) \\ &= \det P \det(B-\lambda I) \det(P^{-1}) \\ &= \det P \det(B-\lambda I) \frac{1}{\det P} = \det(B-\lambda I). \end{aligned}$$

Write m_k for the multiplicity of λ_k. We proved on the previous page:
Fact: $\dim E_{\lambda_k} \leq m_k$.
We now use it to show **Theorem 7b: Diagonalisability Criteria**: An $n \times n$ matrix A is diagonalisable (i.e. A has n linearly independent eigenvectors) if and only if both the following conditions are true:

i the characteristic polynomial $\det(A - \lambda I)$ factors completely into linear factors, i.e. $m_1 + m_2 + \cdots = n$;

ii for each eigenvalue λ_k, we have $\dim E_{\lambda_k} = m_k$.

Proof: (sketch)
"if" part: This is the diagonalisation algorithm: if ii holds, then we can find m_k linearly independent eigenvectors corresponding to the eigenvalue λ_k. Putting these together gives $m_1 + m_2 + \cdots = n$ eigenvectors, which are linearly independent by the Linear Independence of Eigenvectors Theorem (p22).
"only if" part: Given a set of n linearly independent eigenvectors, suppose b_k of them correspond to λ_k (so $b_1 + b_2 + \cdots = n$). Since we have b_k linearly independent vectors in the λ_k-eigenspace, it must be that $b_k \leq \dim E_{\lambda_k}$, and $\dim E_{\lambda_k} \leq m_k$ from the Fact. So $n = b_1 + b_2 + \cdots \leq m_1 + m_2 + \cdots \leq n$, so all our $\leq$ must be $=$.

Non-examinable: what to do when A is not diagonalisable:

We can still write A as PJP^{-1}, where J is "easy to understand and to compute with". Such a J is called a Jordan form.

For example, all non-diagonalisable 2×2 matrices are similar to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, where λ is the only eigenvalue (allowing complex eigenvalues).

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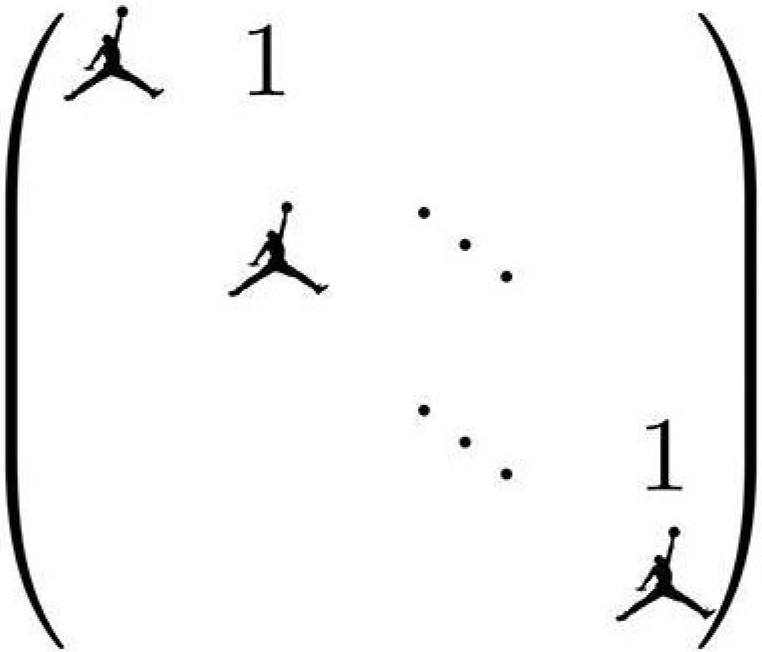

A Jordan block of size-n with eigenvalue ,

(A Jordan form may contain more than one Jordan block, e.g. $\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$ contains two 2×2 Jordan blocks.)

Non-examinable: rectangular matrices (see §7.4 of textbook):

Any $m \times n$ matrix A can be decomposed as $A = QDP^{-1}$ where:

P is an invertible $n \times n$ matrix with columns $\mathbf{p}_i$;

Q is an invertible $m \times m$ matrix with columns $\mathbf{q}_i$;

D is a "diagonal" $m \times n$ matrix with diagonal entries d_{ii}:

e.g. $\begin{bmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \\ 0 & 0 \end{bmatrix}$. So the maximal number of nonzero entries is the smaller of m and n.

Instead of $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$, this decomposition satisfies $A\mathbf{p}_i = d_{ii}\mathbf{q}_i$ for all $i \leq m, n$.

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An important example is the singular value decomposition $A = U\Sigma V^T$. Each diagonal entry of Σ is a singular value of A, which is the squareroot of an eigenvalue of $A^T A$ (a diagonalisable $n \times n$ matrix with non-negative eigenvalues). The singular values contain a lot of information about A, e.g. the largest singular value is the "maximal length scaling factor" of A. (Even for a square matrix, this is in general not true with the eigenvalues of A, so depending on the problem the SVD may be more useful than the diagonalisation of A.)