§4.5: Dimension

From last week:

- Given a vector space V, a basis for V is a linearly independent set that spans V.
- If $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V, then the $\mathcal{B}$-coordinates of $\mathbf{x}$ are the weights c_i in the linear combination $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$.
- Coordinate vectors allow us to test for spanning / linear independence, to solve linear systems, and to test for one-to-one / onto by working in $\mathbb{R}^n$.

Another example of this idea:

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V.

i Any set in V containing more than n vectors must be linearly dependent (theorem 9 in textbook).

ii Any set in V containing fewer than n vectors cannot span V.

We prove this (next page) using coordinate vectors, and the fact that we already know it is true for $V = \mathbb{R}^n$.

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V.

i Any set in V containing more than n vectors must be linearly dependent.

ii Any set in V containing fewer than n vectors cannot span V.

Proof: Let our set of vectors in V be $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$, and consider the matrix

$$A = \begin{bmatrix} | & & | \\ [\mathbf{u}_1]_\mathcal{B} & \dots & [\mathbf{u}_p]_\mathcal{B} \\ | & & | \end{bmatrix},$$

which has p columns and n rows.

i If $p > n$, then rref(A) cannot have a pivot in every column, so $\{[\mathbf{u}_1]_\mathcal{B}, \dots, [\mathbf{u}_p]_\mathcal{B}\}$ is linearly dependent in $\mathbb{R}^n$, so $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is linearly dependent in V.

ii If $p < n$, then rref(A) cannot have a pivot in every row, so the set of coordinate vectors $\{[\mathbf{u}_1]_\mathcal{B}, \dots, [\mathbf{u}_p]_\mathcal{B}\}$ cannot span $\mathbb{R}^n$, so $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ cannot span V.

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V.

i Any set in V containing more than n vectors must be linearly dependent.

ii Any set in V containing fewer than n vectors cannot span V.

Warning: the theorem does not say that "any set of more than n vectors must span V" - this is false, e.g. $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}\right\}$ is a set of 3 vectors in $\mathbb{R}^2$ that does not span $\mathbb{R}^2$. What the theorem says is:

- Fewer than n vectors: cannot span V.
- n or more vectors: has a chance of spanning V, depending on the set.

Similarly, any set of fewer than n vectors may be linearly independent or dependent (think about $\mathbf{0}$).

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V.

i Any set in V containing more than n vectors must be linearly dependent.

ii Any set in V containing fewer than n vectors cannot span V.

As a consequence:

Theorem 10: Every basis has the same size: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

So the following definition makes sense:

Definition: Let V be a vector space.

- If V is spanned by a finite set, then V is *finite-dimensional*.
  The *dimension* of V, written $\dim V$, is the number of vectors in a basis for V.
  (This number is finite because of the spanning set theorem.)
- If V is not spanned by a finite set, then V is *infinite-dimensional*.

Note that the definition does not involve "infinite sets".

Definition: (or convention) The dimension of the zero vector space $\{\mathbf{0}\}$ is 0.

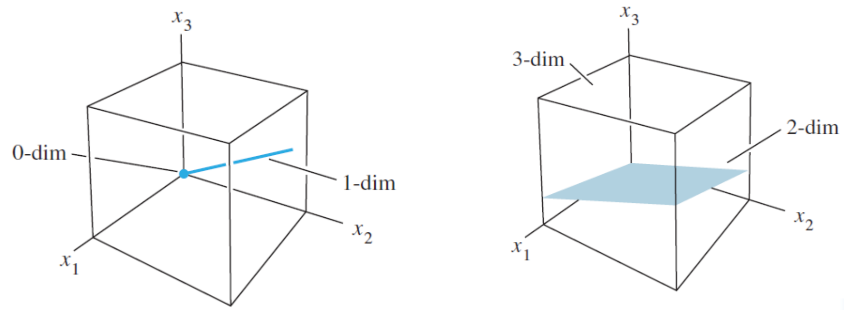
Definition: The *dimension* of V is the number of vectors in a basis for V.

Examples:
- The standard basis for $\mathbb{R}^n$ is $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, so $\dim \mathbb{R}^n = n$.
- The standard basis for $\mathbb{P}_n$ is $\{1, t, \dots, t^n\}$, so $\dim \mathbb{P}_n = n + 1$.
- Exercise: Show that $\dim M_{m\times n} = mn$.

Example: Let W be the set of vectors of the form $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$, where a, b can take any value. We showed (week 8 p20) that a basis for W is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. So $\dim W$=2.

From the theorem on p2, we know that any set of 3 vectors in W must be linearly dependent, because $3 > \dim W$.

Example: We classify the subspaces of $\mathbb{R}^3$ by dimension:
- 0-dimensional: only the zero subspace $\{\mathbf{0}\}$.
- 1-dimensional, i.e. Span $\{\mathbf{v}\}$: lines through the origin.
- 2-dimensional, i.e. Span $\{\mathbf{u}, \mathbf{v}\}$ where $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent: planes through the origin.
- 3-dimensional: by Invertible Matrix Theorem, 3 linearly independent vectors in $\mathbb{R}^3$ spans $\mathbb{R}^3$, so the only 3-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.



Here is a counterpart to the spanning set theorem (week 7 p10):

Theorem 11: Linearly Independent Set Theorem: Let W be a subspace of a finite-dimensional vector space V. If $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a linearly independent set in W, we can find $\mathbf{v}_{p+1}, \dots, \mathbf{v}_n$ so that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for W.

Proof:
- If Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = W$, then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis for W.
- Otherwise $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ does not span W, so there is a vector $\mathbf{v}_{p+1}$ in W that is not in Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$. Adding $\mathbf{v}_{p+1}$ to the set still gives a linearly independent set. Continue adding vectors in this way until the set spans W. This process must stop after at most $\dim V - p$ additions, because a set of more than $\dim V$ elements must be linearly dependent.

The above logic proves something stronger:
Theorem 11 part 2: Subspaces of Finite-Dimensional Spaces: If W is a subspace of a finite-dimensional vector space V, then W is also finite-dimensional and $\dim W \leq \dim V$.

Because of the spanning set theorem and linearly independent set theorem:

Theorem 12: Basis Theorem: If V is a p-dimensional vector space, then
i Any linearly independent set of exactly p elements in V is a basis for V.
ii Any set of exactly p elements that span V is a basis for V.

In other words, to prove that $\mathcal{B}$ is a basis of a p-dimensional vector space V, we only need to show two of the following three things (the third will be automatic):
- $\mathcal{B}$ contains exactly p vectors;
- $\mathcal{B}$ is linearly independent;
- Span$\mathcal{B} = V$.

$\left.\right\}$ If V is a subspace of U, these two statements are usually easier to check because we can work in the big space U (see p10 and p14).

Proof:
i By the linearly independent set theorem, we can add elements to any linearly independent set to obtain a basis for V. But that larger set must contain exactly $\dim V = p$ elements. So our starting set must already be a basis.
ii By the spanning set theorem, we can remove elements from any set that spans V to obtain a basis for V. But that smaller set must contain exactly $\dim V = p$ elements. So our starting set must already be a basis.

Summary:

- If V is spanned by a finite set, then V is finite-dimensional and $\dim V$ is the number of vectors in any basis for V.
- If V is not spanned by a finite set, then V is infinite-dimensional.
- If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ spans V, then some subset is a basis for V (week 7 p10).
- If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent and V is finite-dimensional, then it can be expanded to a basis for V (p5).

If $\dim V = p$ (so V and $\mathbb{R}^p$ are isomorphic):

- Any set of more than p vectors in V is linearly dependent (p2).
- Any set of fewer than p vectors in V cannot span V (p2).
- Any linearly independent set of exactly p elements in V is a basis for V (p6).
- Any set of exactly p elements that span V is a basis for V (p6).

To prove that $\mathcal{B}$ is a basis of V, show two of the following three things:

- $\mathcal{B}$ contains exactly p vectors;
- $\mathcal{B}$ is linearly independent;
- Span$\mathcal{B} = V$.

The basis theorem is useful for finding bases of subspaces:

Example:

Let $W = \text{Span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$. Is $\mathcal{B} = \left\{\begin{bmatrix}1\\0\\0\\3\end{bmatrix}, \begin{bmatrix}2\\0\\0\\5\end{bmatrix}, \begin{bmatrix}3\\0\\1\\2\end{bmatrix}\right\}$ a basis for W?

Answer: We are given that $W = \text{Span}\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4\}$ and $\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4\}$ is a linearly independent set, so $\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4\}$ is a basis for W, and so $\dim W = 3$.
The vectors in $\mathcal{B}$ are all in W, and $\mathcal{B}$ consists of exactly 3 vectors, so it's enough to check whether $\mathcal{B}$ is linearly independent.

Row reduction: $\begin{bmatrix}1&2&3\\0&0&0\\0&0&1\\3&5&2\end{bmatrix} \xrightarrow{R_4 - 3R_1} \begin{bmatrix}1&2&3\\0&0&0\\0&0&1\\0&-1&-7\end{bmatrix} \xrightarrow[R_2]{R_4} \begin{bmatrix}1&2&3\\0&-1&-7\\0&0&1\\0&0&0\end{bmatrix}$ has a pivot in each column, so $\mathcal{B}$ is linearly independent, and is therefore a basis.
Note that we never had to work in W, only in $\mathbb{R}^4$.

§4.6: Rank

Next we look at how the idea of dimension can help us answer questions about existence and uniqueness of solutions to linear systems.

Definition: The *rank* of a matrix A is the dimension of its column space.
The *nullity* of a matrix A is the dimension of its null space.

Example: Let $A = \begin{bmatrix}5&-3&10\\7&2&14\end{bmatrix}$, rref$(A) = \begin{bmatrix}1&0&1/2\\0&1&0\end{bmatrix}$.

A basis for Col$A$ is $\left\{\begin{bmatrix}5\\7\end{bmatrix}, \begin{bmatrix}-3\\2\end{bmatrix}\right\}$ $\longleftarrow$ one vector per pivot

A basis for Nul$A$ is $\left\{\begin{bmatrix}-1/2\\0\\1\end{bmatrix}\right\}$. $\longleftarrow$ one vector per free variable

A basis for Row$A$ is $\{(1,0,1/2),(0,1,0)\}$. $\longleftarrow$ one vector per pivot

So rank$A = 2$, nullity$A = 1$. So rank$A$ + nullity$A =?$

Theorem 14:
Rank Theorem: rank$A = \dim$ Col$A = \dim$ Row$A$ = number of pivots in rref(A).

Rank-Nullity Theorem: For an $m \times n$ matrix A,

$$\text{rank}A + \text{nullity}A = n.$$

Proof: From our algorithms for bases of Col$A$ and Nul$A$ (see week 7 slides):
rank$A$ = number of pivots in rref(A) = number of basic variables,
nullity$A$ = number of free variables.
Each variable is either basic or free, and the total number of variables is n, the number of columns.

An application of the Rank-Nullity theorem:
Example: Suppose a homogeneous system of 10 equations in 12 variables has a solution set that is spanned by two linearly independent vectors. Then the nullity of this system is 2, so the rank is $12 - 2 = 10$. So this system has 10 pivots. Since there are ten equations, there must be a pivot in every row, so any nonhomogeneous system with the same coefficients always has a solution.

Theorem 8 (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The the following statements are equivalent (i.e., for a given A, they are either all true or all false).

a. A is an invertible matrix.
b. A is row equivalent to I_n.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbf{R}^n$.
h. The columns of A span $\mathbf{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I_n$.
k. There is an $n \times n$ matrix D such that $AD = I_n$.
l. A^T is an invertible matrix.

m. The columns of A form a basis for $\mathbf{R}^n$
n. Col $A = \mathbf{R}^n$
o. dim Col $A = n$
p. rank $A = n$
q. Nul $A = \{\mathbf{0}\}$
r. dim Nul $A = 0$

new (rephrasing the previous statements in the new terminology)

Advanced application of the Rank-Nullity Theorem and the Basis Theorem:

Redo Example: (p11) Let $A = \begin{bmatrix} 5 & -3 & 10 \\ 7 & 2 & 14 \end{bmatrix}$. Find a basis for Nul$A$ and Col$A$.

Answer: (a clever trick without any row-reduction)

- Observe that $2\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$, so $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{0}$. So nullity$A \geq 1$.
- The first two columns of A are linearly independent (not multiples of each other), so $\left\{\begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}\right\}$ is a linearly independent set in Col$A$, so rank$A \geq 2$.
- But rank$A$ + nullity$A = 3$, so in fact rank$A = 2$ and nullity$A = 1$, and, by the Basis Theorem, the linearly independent sets we found above are bases: so $\left\{\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\right\}$ is a basis for Nul$A$, $\left\{\begin{bmatrix} 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}\right\}$ is a basis for Col$A$.

So for a general $m \times n$ matrix, it's enough to find k linearly independent vectors in Nul$A$ and $n - k$ linearly independent vectors in Col$A$.

The Rank-Nullity theorem also holds for linear transformations $T : V \to W$ whenever V is finite-dimensional (to prove it yourself, work through q8 of homework 5 from 2015):

$$\dim \text{range of } T + \dim \text{kernel of } T = \dim V.$$

Advanced application:
Example: Find a basis for Q, the set of polynomials $\mathbf{p}(t)$ of degree at most 3 satisfying $\mathbf{p}(2) = 0$.
Answer: Remember (week 6 p43) that Q is the kernel of the evaluation-at-2 function $E_2 : \mathbb{P}_3 \to \mathbb{R}$ given by $E_2(\mathbf{p}) = \mathbf{p}(2)$,

$$E_2(a_0 + a_1t + a_2t^2 + a_3t^3) = a_0 + a_1 2 + a_2 2^2 + a_3 2^3.$$

E_2 is onto, so its range has dimension 1. So $\dim Q = \dim \mathbb{P}_3 - 1 = 4 - 1 = 3$.
Now $\mathcal{B} = \left\{(2-t), (2-t)^2, (2-t)^3\right\}$ is a subset of Q, and is linearly independent (check with coordinate vectors relative to the standard basis of $\mathbb{P}_3$, or because these three polynomials have different degrees - see week 7 p14-15). Since $\mathcal{B}$ contains exactly 3 vectors, it is a basis for Q.