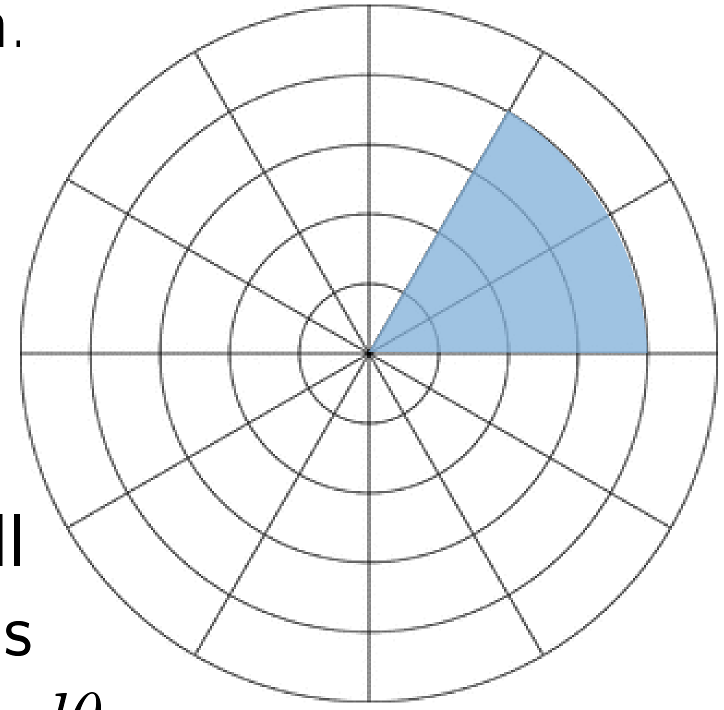
§14.4: Change of Variables in Multiple Integration

In this final week, we see how the Jacobian determinant (week 13 p14) is useful for one technique for multiple integration.

We have already computed some integrals using a change of variables: when the domain of integration is a disk or a sector, we used polar coordinates $[r, \theta]$, because its gridlines run along the boundary of the domain.



As we saw in week 5, writing a 2D integral as an iterated integral in polar coordinates requires 3 steps:

$$\boxed{\int_\alpha^\beta \int_0^b} \boxed{f(r\cos\theta, r\sin\theta)} \boxed{r\,dr d\theta}$$

1. Express the domain in terms of r and θ, to find the limits of integration

2. The area of a small piece in the r, θ-grid is $r \Delta r \Delta\theta$, so $dA = r\,dr\,d\theta$.

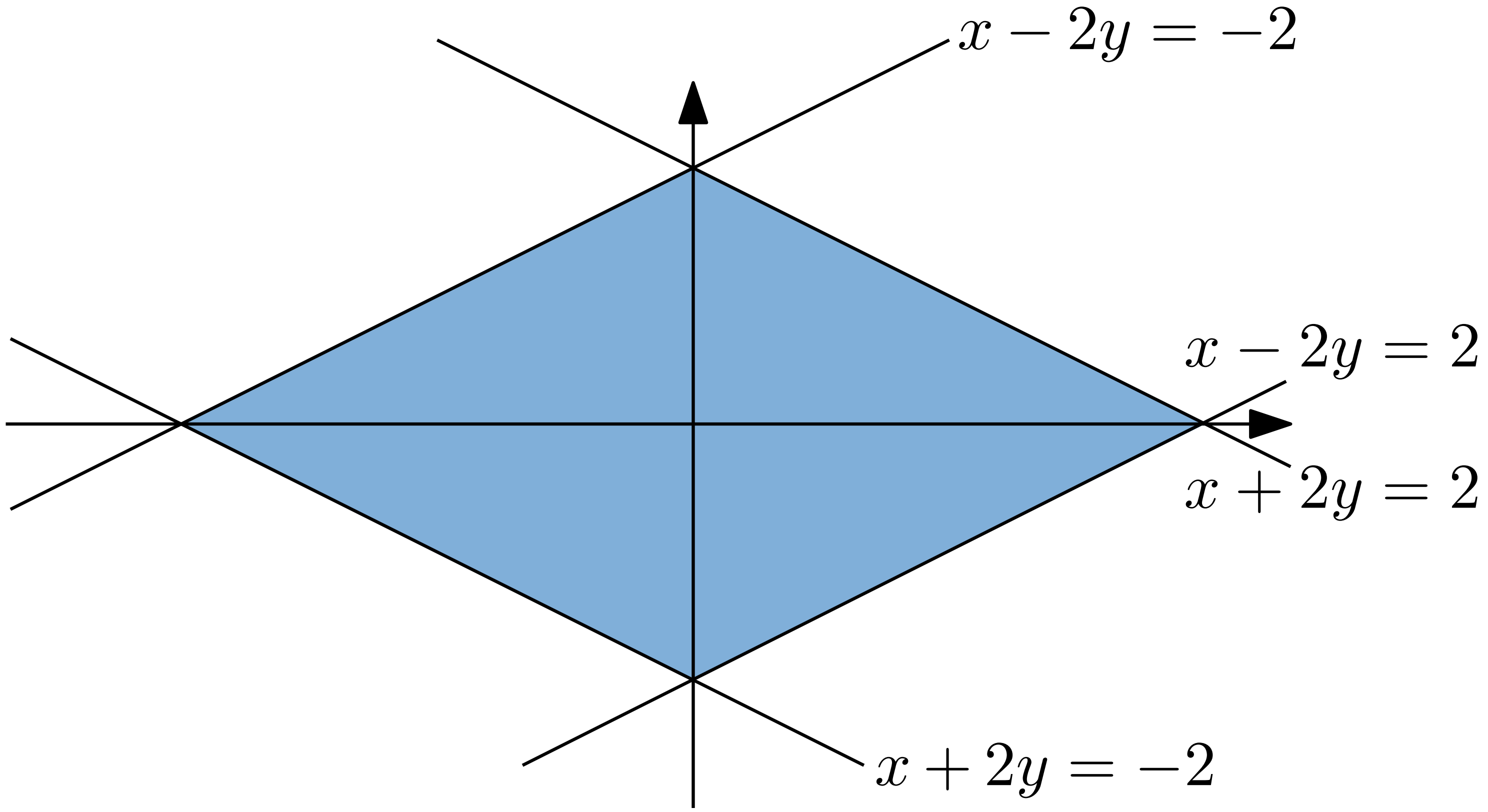
3. Use $x = r\cos\theta$, $y = r\sin\theta$ to write the "integrand" in terms of r, θ.

We can apply the same process to other domains.

Example: Evaluate $\iint_D (3x+6y)^2\,dA$, where D is the region bounded by $x+2y=2$, $x+2y=-2$, $x-2y=2$ and $x-2y=-2$.

To compute this using x, y-coordinates, we need to use two integrals, because the "top" and "bottom" of D are each defined piecewise (and same for the "left side" and "right side").

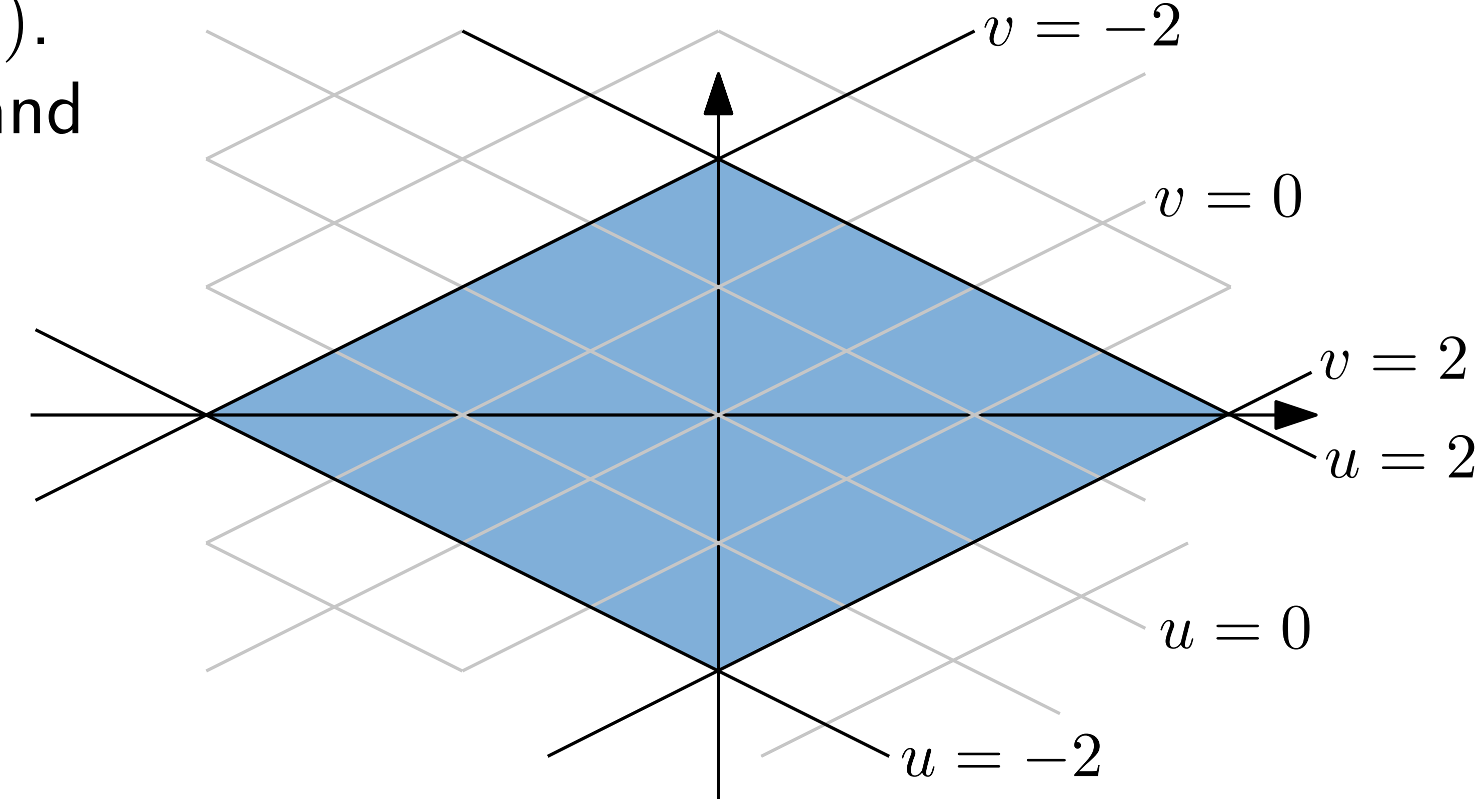
$$\iint_D (3x+6y)^2\,dA$$
$$= \int_{-2}^{0}\int_{1/2(-x-2)}^{1/2(x+2)} (3x+6y)^2\,dy\,dx$$
$$+ \int_{0}^{2}\int_{1/2(x-2)}^{1/2(-x+2)} (3x+6y)^2\,dy\,dx$$



Alternatively, we can work in a different coordinate system, whose grid lines are parallel to the sides of the parallelogram D: i.e. set $u = x + 2y$ and $v = x - 2y$. (The grid lines are the level curves of u and v, i.e. $x + 2y = C$ and $x - 2y = C$.)

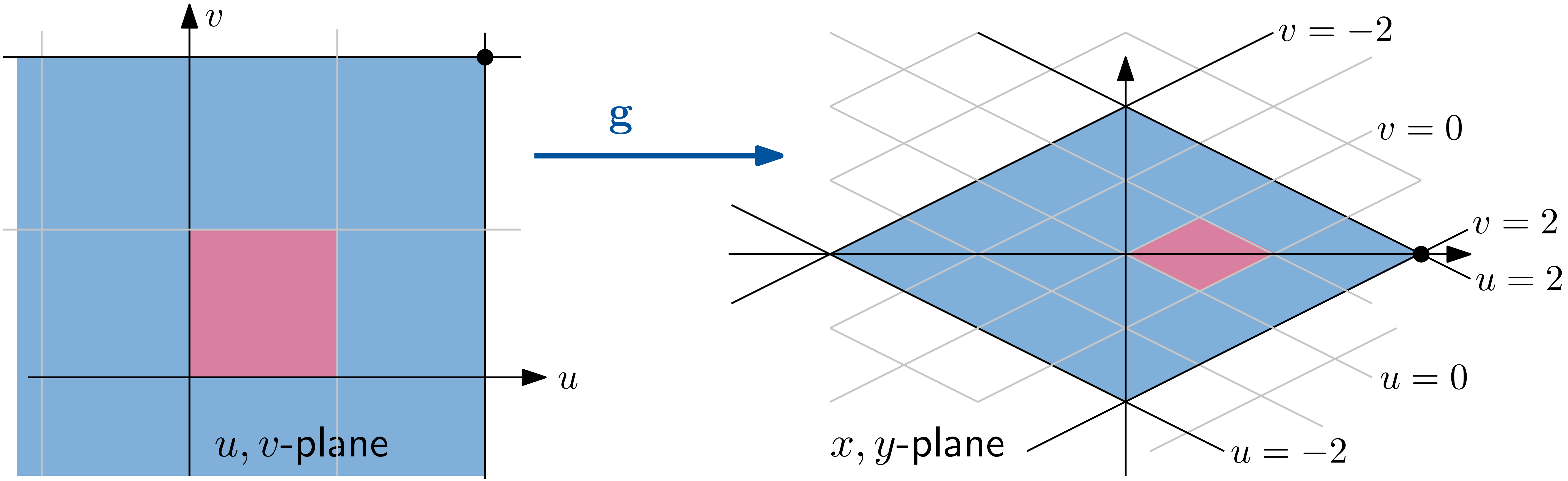
Then D is the region $-2 \leq u \leq 2$, $-2 \leq v \leq 2$. (Step 1 is completed.)
Step 2: we need the area ΔA of a small parallelogram in the u, v-grid.

To find ΔA, consider the function $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathbf{g}(u, v) = (x, y)$. That is, $\mathbf{g}$ takes two numbers (a, b) and outputs the point in the x, y-plane labelled by $u = a$ and $v = b$.
For example, $\mathbf{g}(2, 2)$ is the rightmost corner of D (i.e. $\mathbf{g}(2, 2) = (2, 0)$), and $\mathbf{g}$ takes the square with corners $(-2, -2)$, $(-2, 2)$, $(2, 2)$ and $(2, -2)$ to D.



In particular, the grid piece $0 \leq u \leq \Delta u, 0 \leq v \leq \Delta v$ (in red) is the image under $\mathbf{g}$ of the square with corners $(0,0),(0,\Delta v),(\Delta u,\Delta v),(\Delta u,0)$. This square has area $\Delta u \Delta v$. To find ΔA, we need to know how $\mathbf{g}$ changes areas.

In this example, we can solve $u = x + 2y$ and $v = x - 2y$ to find $\mathbf{g}(u,v) = (x,y)$ $= \left(\frac{u+v}{2}, \frac{u-v}{4}\right)$. So $\mathbf{g}$ is a linear transformation: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & -1/4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$, and $\mathbf{g}$ scales areas by the absolute value of the determinant of this matrix, which is $1/4$.



So we found that D is the region $-2 \leq u \leq 2$, $-2 \leq v \leq 2$, and the area of the small piece of the u, v-grid (in the x, y-plane) with $0 \leq u \leq \Delta u$, $0 \leq v \leq \Delta v$ is $\Delta A = \frac{1}{4}\Delta u \Delta v$. And all the pieces in the u, v grid have the same area.

So our integral is

$$\iint_D (3x+6y)^2 \, dA$$

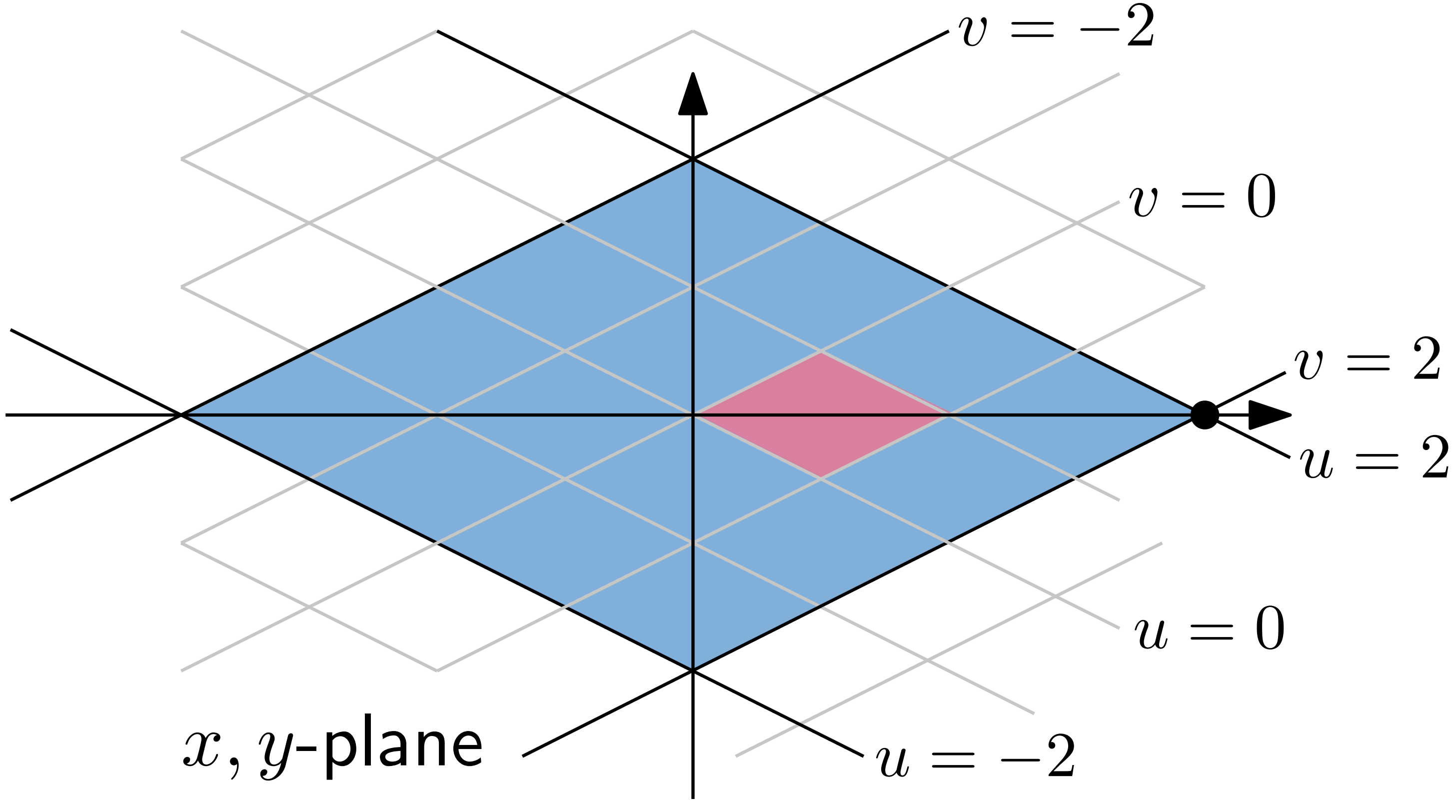
$$= \int_{-2}^{2}\int_{-2}^{2} (3x+6y)^2 \frac{1}{4} du \, dv$$

$$= \int_{-2}^{2}\int_{-2}^{2} (3u)^2 \frac{1}{4} du \, dv$$

$$= \int_{-2}^{2} \left. \frac{3}{4}u^3 \right|_{u=-2}^{u=2} dv$$

$$= \int_{-2}^{2} 12 dv = 12v\big|_{-2}^{2} = 48.$$

Step 3: change the integrand to u, v

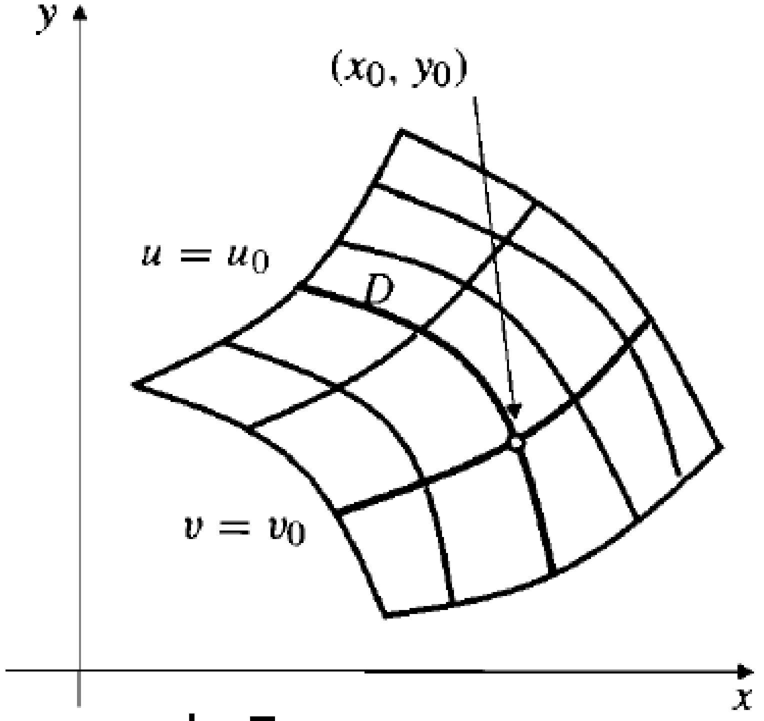


In the previous example, the change-of-coordinates function $\mathbf{g}(u, v) = (x, y)$ is a linear function, and $\Delta A = |\det(\text{matrix for } \mathbf{g})| \, \Delta u \Delta v$.
How can we find ΔA when the u, v-grid (in the x, y-plane) does not consist of straight lines (e.g. for polar coordinates), so $\mathbf{g}$ is not a linear function? Now the area of each grid piece may be different: the area ΔA_{ij} of the piece $u_i \leq u \leq u_i + \Delta u$, $v_j \leq v \leq v_j + \Delta v$ will depend on i and j.

The idea is to approximate $\mathbf{g}$ by its derivative: we will show (p13-14) that

$$\Delta A_{ij} \approx |\det D\mathbf{g}(u_i, v_j)| \, \Delta u \Delta v = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v,$$

evaluating the Jacobian determinant at (u_i, v_j).



For example, for polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$,

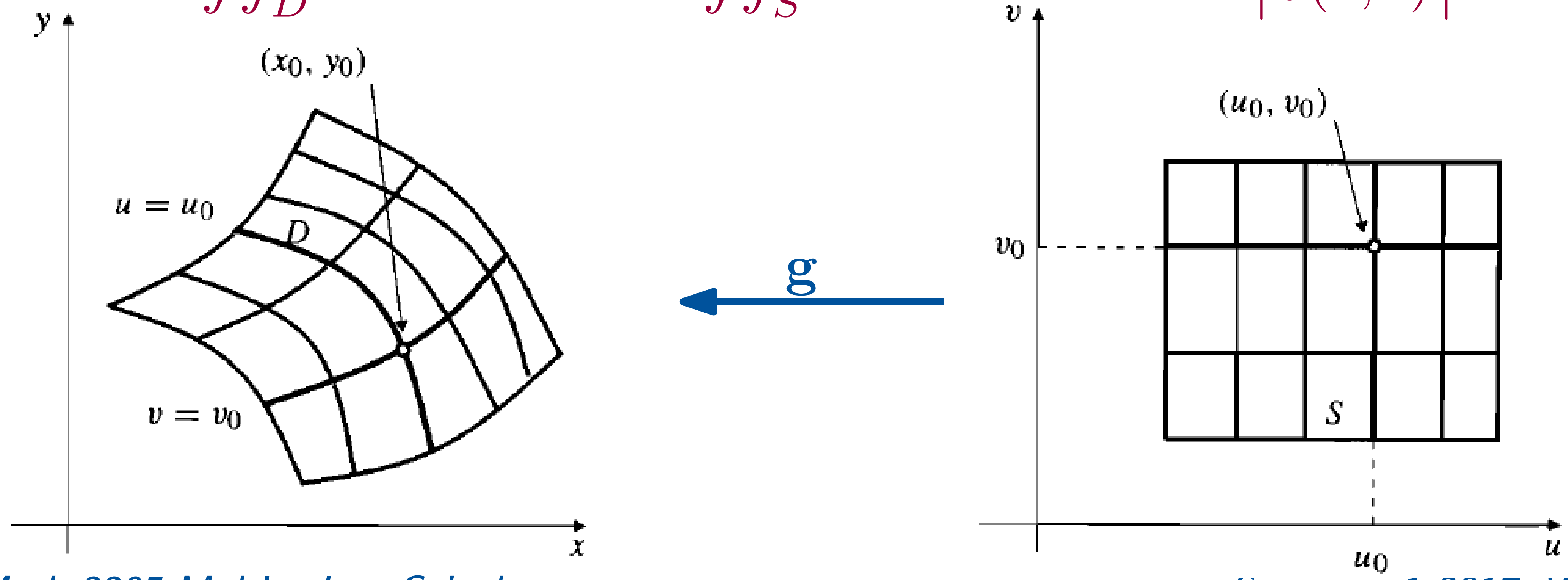
so $$\left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| = \left| \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \right| = \left| \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \right|$$

$= \left| r(\cos^2\theta + \sin^2\theta) \right| = |r|$, so $\Delta A_{ij} \approx r \Delta r \Delta \theta$, as we saw in week 5.

Since $\Delta A_{ij} \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v$, we have

Theorem 4: Change of variables for double integrals: Let S be a domain in the u,v-plane, and $\mathbf{g}: S \to \mathbb{R}^2$, $\mathbf{g}(u,v) = (x,y)$ be a function whose range in the x,y-plane is a domain D, and which is one-to-one on the interior of S. Suppose that $\mathbf{g}$ is continuous, and has continuous first-order partial derivatives. Then

$$\iint_D f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$



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$$\iint_D f(x, y)\, dx\, dy = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv.$$

As shown in our first example, and in the pictures on the previous page, a main reason for using change of variables is to change an irregularly-shaped domain D into a rectangle S (or another easy shape). We usually do this by setting $u(x, y)$ and $v(x, y)$ to be the functions defining the boundary of D. Then it may be hard to invert the functions and write $x(u, v), y(u, v)$. So it is often more convenient to find the required Jacobian determinant using the inverse function theorem:

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}}$$

(Polar coordinates is an unusual case where $x = r \cos\theta, y = r \sin\theta$ is easier than writing r, θ in terms of x, y.)

Example: Evaluate $\iint_D \frac{1}{x^2}\, dA$, where D is the region bounded by $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $y = 0$ and $y = \frac{x}{2}$.

Another common use of change-of-variables is to change an ellipse into a circle.

Example: Evaluate $\displaystyle\iint_D y^2\,dA$, where $D = \left\{(x,y) \in \mathbb{R}^2 \,|\, 4x^2 + y^2 \leq 4\right\}$.

The previous two examples showed two standard situations where change-of-variables simplifies the domain:

1. When the domain has the form $a \leq u(x,y) \leq b$, $c \leq v(x,y) \leq d$, working in u,v-coordinates allows us to simply integrate over a rectangle (p2-5, p9);
2. When the domain is an ellipse: e.g. for $a^2x^2 + b^2y^2 = r^2$, we can use $u = ax, v = by$ to turn the ellipse in the x,y-plane into a circle in the u,v-plane, and then use polar coordinates (p10).
   (Change-of-variables works in the expected way for domains in $\mathbb{R}^3$, so we can similarly turn ellipsoids into spheres and then use spherical coordinates.)

We can view change of variables as a generalisation of the method of substitution for 1D integrals (see also week 4 p14-15, week 5 p35):

$$\text{2D} \qquad \iint_D f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv,$$

$$\text{1D} \qquad \int_{d_1}^{d_2} f(x)\,dx = \int_{s_1}^{s_2} f(x(u))\frac{dx}{du}\,du.$$

(There is no absolute value sign around $\frac{dx}{du}$ in the 1D version because, if $\frac{dx}{du} < 0$, then x is a decreasing function of u, so $s_1 > s_2$, but the $\int_S$ notation always puts the smaller number as the lower endpoint. In other words, $\int_{s_1}^{s_2} f(x(u))\frac{dx}{du}\,du = -\int_{s_1}^{s_2} f(x(u))\left|\frac{dx}{du}\right| du = \int_{s_2}^{s_1} f(x(u))\left|\frac{dx}{du}\right| du = \int_S f(x(u))\left|\frac{dx}{du}\right| du.$)

In the theorem for change of variables, there are two conditions on the change of variables function $\mathbf{g} : S \to \mathbb{R}^2$ given by $\mathbf{g}(u, v) = (x, y)$: the range of $\mathbf{g}$ must be the x, y-domain D, and $\mathbf{g}$ must be one-to-one on the interior of S. Informally, this says every point (x, y) in D corresponds to some point (u, v) in S, and at most one point in the interior of S. These conditions are necessary to ensure that we "count" each piece of the u, v grid in D exactly once.

An example where $\mathbf{g}$ is not one-to-one: the unit disk $D = \left\{x^2 + y^2 \leq 1\right\}$ is the image of $S = \{0 \leq r \leq 1,\, 0 \leq \theta \leq 4\pi\}$ under the polar coordinates transformation $x = r\cos\theta$, $y = r\sin\theta$. But $\iint_D f(x, y)\, dx\, dy = \frac{1}{2}\iint_S f(x(r, \theta), y(r, \theta)) \left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| dr d\theta$, because S maps onto D twice.

Because of an advanced inverse function theorem (due to Hadamard), $\mathbf{g}$ will be one-to-one if: S is closed and bounded, $\frac{\partial(x, y)}{\partial(u, v)} \neq 0$ on all of S, and D has "no holes". If these three conditions do not all hold (e.g. for polar coordinates where $\frac{\partial(x, y)}{\partial(r, \theta)} = 0$ when $r = 0$), then we have to carefully check that $\mathbf{g}$ is one-to-one.

Why is ΔA_{ij}, the area of the grid piece $u_i \le u \le u_i + \Delta u$, $v_j \le v \le v_j + \Delta v$ in the x, y-plane, approximately equal to $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v$?

Use the linearisation of x and y around $P = (u_i, v_j)$:

$$x(u_i + h, v_j + k) \approx x(u_i, v_j) + \left.\frac{\partial x}{\partial u}\right|_P h + \left.\frac{\partial x}{\partial v}\right|_P k$$

$$y(u_i + h, v_j + k) \approx y(u_i, v_j) + \left.\frac{\partial y}{\partial u}\right|_P h + \left.\frac{\partial y}{\partial v}\right|_P k$$

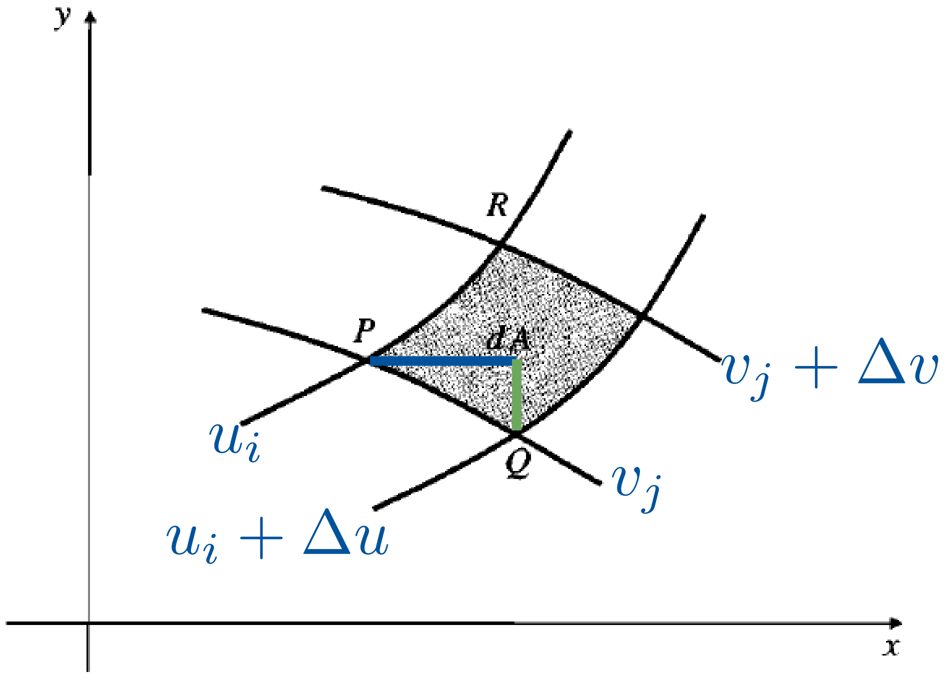
So, at Q: $x(u_i + \Delta u, v_j) \approx x(u_i, v_j) + \left.\frac{\partial x}{\partial u}\right|_P \Delta u$

$$y(u_i + \Delta u, v_j) \approx y(u_i, v_j) + \left.\frac{\partial y}{\partial u}\right|_P \Delta u$$

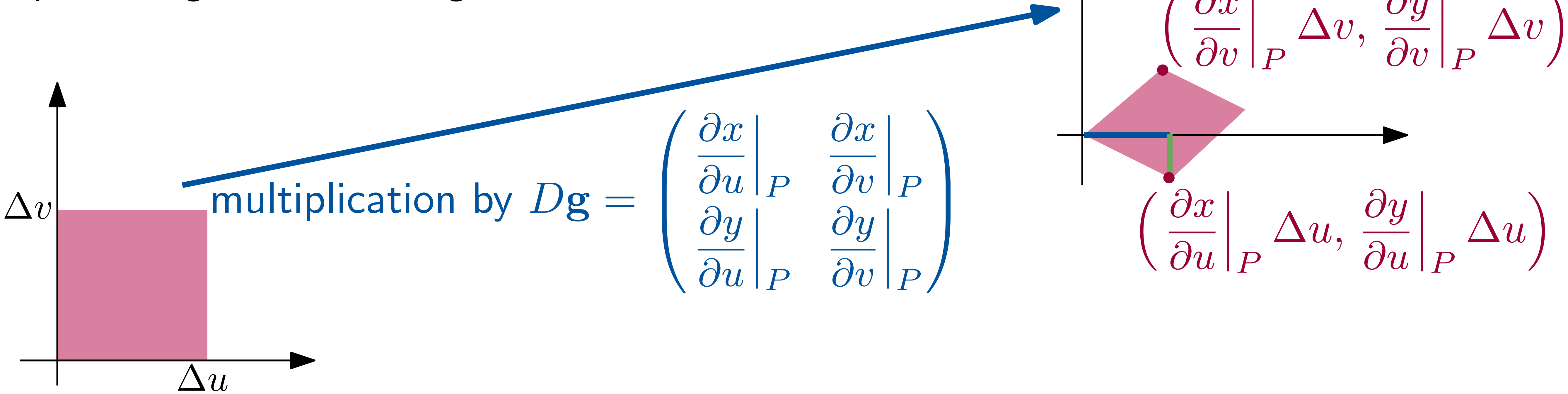
at R:

$$x(u_i, v_j + \Delta v) \approx x(u_i, v_j) + \left.\frac{\partial x}{\partial v}\right|_P \Delta v$$

$$y(u_i, v_j + \Delta v) \approx y(u_i, v_j) + \left.\frac{\partial y}{\partial v}\right|_P \Delta v$$



So ΔA_{ij} is approximately the area of the parallelogram on the right.



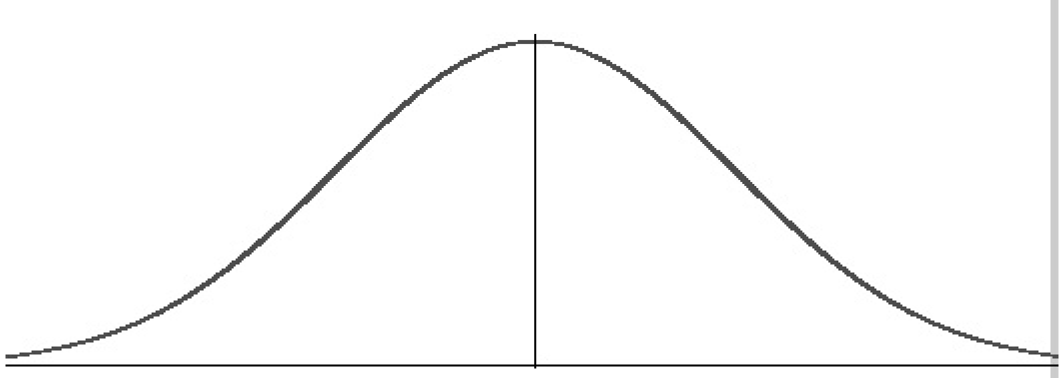
This parallelogram is the image of the square $0 \leq h \leq \Delta u, 0 \leq k \leq \Delta v$ under the linear transformation $\begin{pmatrix} h \\ k \end{pmatrix} \mapsto \begin{pmatrix} \left.\frac{\partial x}{\partial u}\right|_P & \left.\frac{\partial x}{\partial v}\right|_P \\ \left.\frac{\partial y}{\partial u}\right|_P & \left.\frac{\partial y}{\partial v}\right|_P \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix}$.

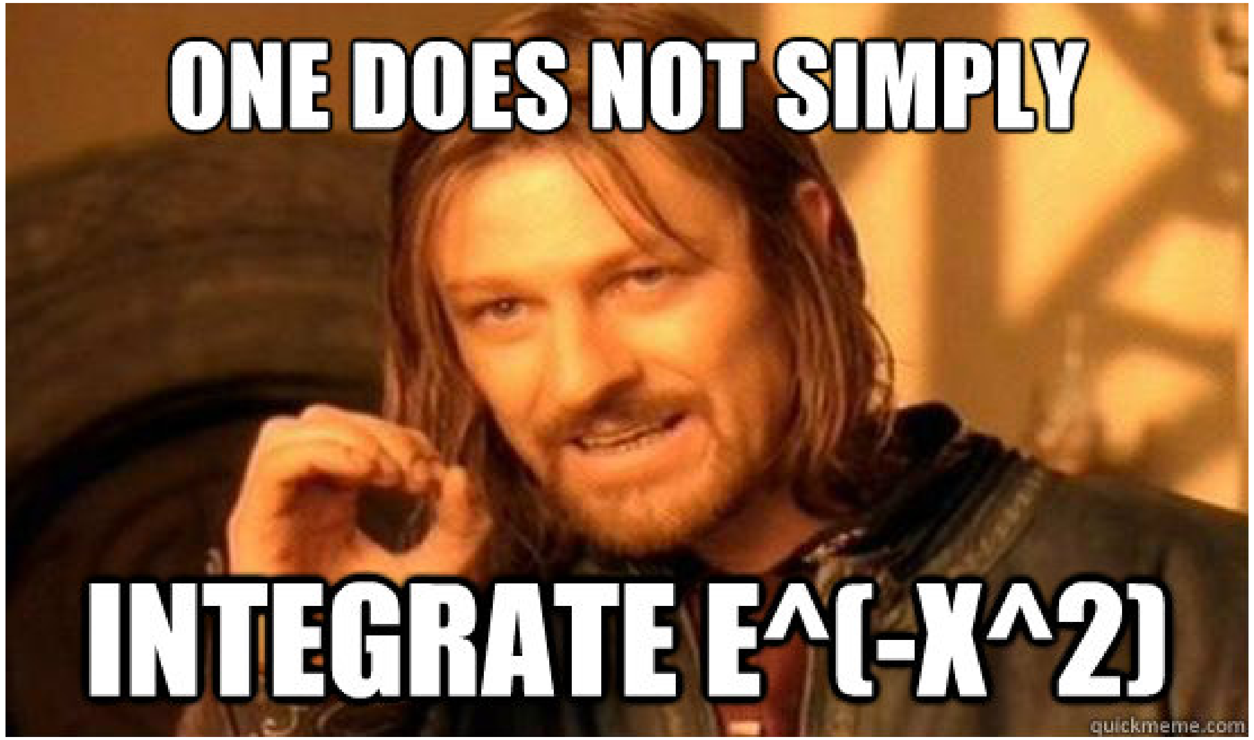
So the area of the parallelogram is $|\det(\text{the matrix above})| \, \Delta u \Delta v = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$.

Non-examinable: using change-of-variables to evaluate the Gaussian integral

The normal distribution (also called the Gaussian distribution) has probability density function proportional to e^{-x^2}, i.e. $p(x) = \frac{1}{Z}e^{-x^2}$ for some Z. We need to choose the constant of proportionality Z so that the total probability is 1, i.e.

$$Z = \int_{-\infty}^{\infty} e^{-x^2}\,dx.$$





The problem is, we cannot write down the antiderivative of e^{-x^2} - it is not an elementary function. But multiple integration will help us in a surprising, clever way.

First, we need to show that the improper integral $Z = \int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges.

It will be enough to show that $Z' = \int_{1}^{\infty} e^{-x^2}\,dx$ converges, because then

$$Z = \int_{-\infty}^{-1} e^{-x^2}\,dx + \int_{-1}^{1} e^{-x^2}\,dx + \int_{1}^{\infty} e^{-x^2}\,dx = Z' + \int_{-1}^{1} e^{-x^2}\,dx + Z'.$$

We show that Z' converges using the (non-examinable) technique of integral estimation by inequalities:

e^{-x^2} is a non-negative function, so FTC1 says that $F(R) = \int_{1}^{R} e^{-x^2}\,dx$ is an increasing function (in R).

So, using some theorems from analysis, we know that, if there is a number M such that $M \geq F(R) = \int_{1}^{R} e^{-x^2}\,dx$ for every $R > 1$, then $\lim\limits_{R\to\infty} F(R)$ exists, i.e. Z' converges.

For all $x \geq 1$, we have $e^{-x^2} \leq e^{-x}$, so $\int_{1}^{R} e^{-x^2}\,dx \leq \int_{1}^{R} e^{-x}\,dx = e^{-1} - e^{-R} \leq e^{-1}$,

so e^{-1} is the upper bound M that we want.

Reminder: we wish to evaluate $Z = \int_{-\infty}^{\infty} e^{-x^2}\,dx$.

Consider the double integral $\iint_{\mathbb{R}^2} e^{-x^2} e^{-y^2}\,dA$. The integrand is always positive, so we can calculate this improper integral using an iterated integral:

$$\iint_{\mathbb{R}^2} e^{-x^2} e^{-y^2}\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\,dy\,dx = \int_{-\infty}^{\infty} e^{-x^2} Z\,dx = Z^2$$

But we can also calculate this double integral using polar coordinates (yes, you can use change of variables on improper integrals):

$$\iint_{\mathbb{R}^2} e^{-x^2} e^{-y^2}\,dA = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta$$

inner integral independent of θ, and substitution $u = r^2$

$$= 2\pi \int_0^{\infty} \frac{e^{-u}}{2}\,du = \pi \lim_{R\to\infty} \int_0^{R} e^{-u}\,du = \pi\left(\lim_{R\to\infty} 1 - e^{-R}\right) = \pi.$$

So $Z^2 = \pi$, i.e. $Z = \sqrt{\pi}$.