Homework 6, due **15:45 Monday 30 April 2018, to Dr. Pang's mailbox on 12F FSC**

You must justify your answers to receive full credit.

1. (This question requires material from §7.1.) Suppose $\begin{bmatrix}-1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\\0\end{bmatrix}$ are eigenvectors of a symmetric matrix. Explain why these two eigenvectors correspond to the same eigenvalue.

2. Let
$$\mathbf{y}=\begin{bmatrix}2\\0\\1\\-5\end{bmatrix},\quad \mathbf{x}_1=\begin{bmatrix}3\\1\\-1\\3\end{bmatrix},\quad \mathbf{x}_2=\begin{bmatrix}-5\\1\\5\\-7\end{bmatrix},\quad \mathbf{x}_3=\begin{bmatrix}1\\1\\-2\\8\end{bmatrix},$$
and let $W=\text{Span}\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\}$.

   a) Use the Gram-Schmidt process to find an orthogonal basis $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ for W.
   b) Write $\mathbf{y}$ as a linear combination of the vectors $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ from part a).
   c) Let $\text{proj}_W:\mathbb{R}^4\to\mathbb{R}^4$ be the orthogonal projection onto W. Find the standard matrix of proj_W.
   d) Let $\mathbf{x}_4$ be a nonzero vector in $W^\perp$, and let $\mathcal{B}=\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\}$. Find, with justification, the matrix of proj_W relative to $\mathcal{B}$. (Hint: you do not need to do any calculation.)

3. Let
$$\mathbf{v}_1=\begin{bmatrix}0\\1\\-4\\-1\end{bmatrix},\quad \mathbf{v}_2=\begin{bmatrix}3\\5\\1\\1\end{bmatrix},\quad \mathbf{v}_3=\begin{bmatrix}1\\0\\1\\-4\end{bmatrix},\quad \mathbf{v}_4=\begin{bmatrix}5\\-3\\-1\\1\end{bmatrix},\quad \mathbf{y}=\begin{bmatrix}10\\-8\\2\\0\end{bmatrix}.$$
$\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$ is an orthogonal basis for $\mathbb{R}^4$. Let W be the line $\text{Span}\{\mathbf{v}_4\}$.

   a) Find the orthogonal projection of $\mathbf{y}$ to W.
   b) Using your answer to a) and the Orthogonal Decomposition Theorem, and without calculating any more dot products, write $\mathbf{y}$ as the sum of two vectors $\mathbf{y}=\hat{\mathbf{y}}+\mathbf{z}$, such that $\hat{\mathbf{y}}$ is in $\text{Span}\{\mathbf{v}_4\}$ and $\mathbf{z}$ is in $\text{Span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$. (You do **not** have to express $\mathbf{z}$ as a linear combination of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$).
   c) Using your answer to b), and without calculating any more dot products, find the orthogonal projection of $\mathbf{y}$ to $\text{Span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$.

4. Suppose radioactive substances A and B have decay constants of 0.02 and 0.07, respectively. If a mixture of these two substances at time $t=0$ contains M_A grams of A and M_B grams of B, then a model for the total mass in grams y of the mixture at time t is
$$y=M_Ae^{-0.02t}+M_Be^{-0.07t}.$$
Suppose the initial amounts M_A and M_B are unknown, but a scientist is able to measure the total mass at several times t_i to obtain the following data points:

t_i	10	11	12	14	15
y_i	21.34	20.68	20.05	18.87	18.30

Write down the observation vector, the design matrix and the parameter vector. (Do **not** evaluate the exponentials, just write $e^{-0.2}$ etc.) You do **not** need to solve for M_A and M_B.

5. Remember from class that the least-squares line through the data points $(x_1, y_1), \ldots, (x_n, y_n)$ is given by $y = \hat{\beta}_0 + \hat{\beta}_1 x$, where $(\hat{\beta}_0, \hat{\beta}_1)$ is the least-squares solution to

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

The goal of this problem is to prove, from this linear algebra perspective, the formulas for the regression coefficients $\hat{\beta}_0$ and $\hat{\beta}_1$ from Probability and Statistics class:

$$\hat{\beta}_1 = \frac{\mathrm{SS}_{xy}}{\mathrm{SS}_{xx}}, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \qquad (*)$$

where

$$\mathrm{SS}_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}, \quad \mathrm{SS}_{xx} = \Sigma xx - \frac{(\Sigma x)^2}{n}, \quad \bar{y} = \frac{(\Sigma y)}{n}, \quad \bar{x} = \frac{(\Sigma x)}{n}.$$

a) By considering the normal equations, find a matrix M such that $M \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \Sigma y \\ \Sigma xy \end{bmatrix}$. (The entries of M will be in terms of n, x and y.)

b) Assume that $x_1, \ldots, x_n$ are not all the same. Using a theorem from class, explain why this means $\hat{\beta}_0$ and $\hat{\beta}_1$ are unique.

c) Explain why the uniqueness of $\hat{\beta}_0$ and $\hat{\beta}_1$ implies that the matrix M is invertible, and that SS_{xx} is nonzero.

d) Use row reduction to solve for $\hat{\beta}_1$ and $\hat{\beta}_0$ (be careful not to divide by zero!), and show they have the formula given in $(*)$.

6. State whether each of the following statements is (always) true or (sometimes) false. If it is true, give a brief justification; if it is false, give a counterexample with an explanation.

a) If a square matrix A is diagonalisable, then it is invertible.

b) If a square matrix A is invertible, then it is diagonalisable.

c) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be vectors in $\mathbb{R}^n$. Then $(\mathbf{a} \cdot \mathbf{b})\mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$.

d) Let W be a subspace of $\mathbb{R}^n$. If $\{\mathbf{w}_1, \ldots \mathbf{w}_p\}$ is an orthogonal set in W and $\{\mathbf{u}_1, \ldots \mathbf{u}_q\}$ is an orthogonal set in $W^\perp$, then their union $\{\mathbf{w}_1, \ldots \mathbf{w}_p, \mathbf{u}_1, \ldots \mathbf{u}_q\}$ is an orthogonal set.

e) If U is an orthogonal matrix, then $\det U$ is 1 or -1.

f) If U is a square matrix with orthonormal columns, then the rows of U are also orthonormal.

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