1. (7 points) Compute the following two improper integrals, or explain why they do not converge. **Simplify your answer as much as possible.**

(a)

$$\int_{-\infty}^{-1} \frac{1}{(x-1)^2} - \frac{1}{(x-1)^3}\, dx.$$

$$= \lim_{t\to-\infty} \int_t^{-1} \frac{1}{(x-1)^2} - \frac{1}{(x-1)^3}\, dx$$

substitution
$u = x-1$
$du = dx$

$$= \lim_{t\to-\infty} \left[ \frac{1}{(-1)(x-1)} - \frac{1}{(-2)(x-1)^2} \right]_t^{-1}$$

$$= \lim_{t\to-\infty} \left( \frac{1}{2} + \frac{1}{8} + \frac{1}{t-1} - \frac{1}{2(t-1)^2} \right)$$

$$= \frac{1}{2} + \frac{1}{8}$$

$$= \frac{5}{8}$$

because, as $t\to-\infty$, $t-1\to-\infty$, so $\frac{1}{t-1}\to 0$ and $\frac{1}{2(t-1)^2}\to 0$.

(b)

$$\int_0^1 \frac{1}{(x-1)^2} - \frac{1}{(x-1)^3}\, dx.$$

$$= \lim_{t\to1^-} \int_0^t \frac{1}{(x-1)^2} - \frac{1}{(x-1)^3}\, dx$$

$$= \lim_{t\to1^-} \left[ \frac{-1}{x-1} + \frac{1}{2(x-1)^2} \right]_0^t$$

$$= \lim_{t\to1^-} \left( \frac{-1}{t-1} + \frac{1}{2(t-1)^2} \right) - \left( \frac{-1}{-1} + \frac{1}{2} \right)$$

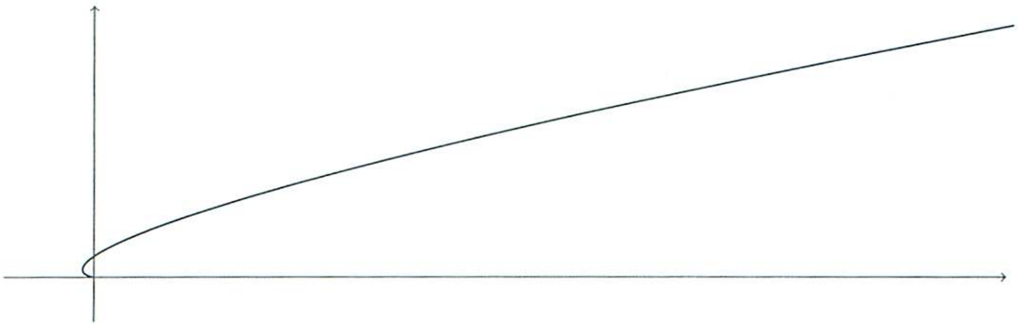
$$= \lim_{t\to1^-} \frac{-2(t-1)+1}{2(t-1)^2} - \frac{3}{2}$$

as $t\to1^-$, $-2(t-1)+1 \to 1$, $2(t-1)^2\to 0$ so $\frac{-2(t-1)+1}{2(t-1)^2} \to \infty$. So this integral is divergent.

2. (14 points) Let $C$ be the parametrised curve with equation

$$x = \frac{3}{4}t^4 - t^3, \quad y = \frac{12}{7}t^{\frac{7}{2}}, \quad t \geq 0,$$

as shown in the diagram below.



(a) Find the point(s) where $C$ has a vertical tangent. **Simplify your answer as much as possible.**

$C$ has a vertical tangent when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

$$3t^3 - 3t^2 = 0$$
$$3t^2(t-1) = 0$$
$$\Rightarrow t = 0 \text{ or } t = 1.$$

When $t=0$, $\frac{dy}{dt} = 6t^{5/2} = 0$ — but this is the point $(0,0)$, and from the picture we see that there isn't a vertical tangent at this point.

When $t=1$, $\frac{dy}{dt} = 6(1)^{5/2} \neq 0$ $\therefore$ this is indeed a vertical tangent.

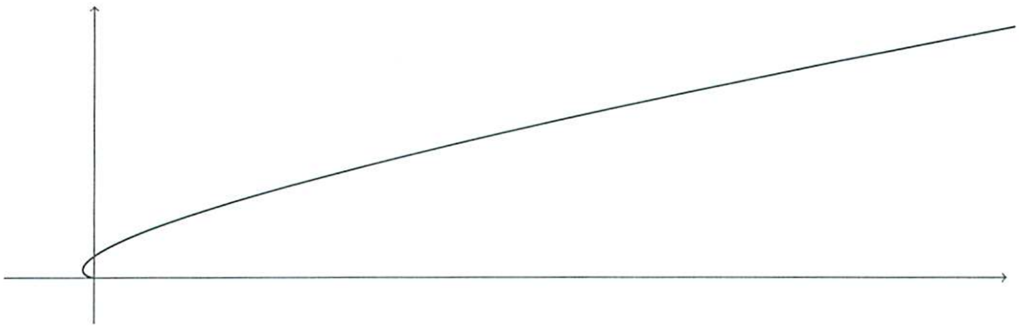
$$x = \frac{3}{4}(1) - 1 = -\frac{1}{4}$$
$$y = \frac{12}{7}(1)$$

$\therefore$ $C$ has a vertical tangent at $\left(-\frac{1}{4}, \frac{12}{7}\right)$

(Question 2 continues on the next page.)

(b) For your convenience, here again is the information about the parametrised curve $C$:

$$x = \frac{3}{4}t^4 - t^3, \quad y = \frac{12}{7}t^{\frac{7}{2}}, \quad t \geq 0.$$



Find the length of the part of $C$ with $2 \leq t \leq 3$. **Simplify your answer as much as possible.**

$$\text{length} = \int_2^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_2^3 \sqrt{(3t^3 - 3t^2)^2 + (6t^{5/2})^2}\, dt$$

$$= \int_2^3 \sqrt{9t^6 - 18t^5 + 9t^4 + 36t^5}\, dt$$

$$= \int_2^3 \sqrt{9t^6 + 18t^5 + 9t^4}\, dt$$

$$= \int_2^3 \sqrt{9t^4(t+1)^2}\, dt$$

$$= \int_2^3 |3t^2|\,|t+1|\, dt$$

for $2 \leq t \leq 3$: $3t^2 > 0$, $t+1 > 0$

$$= \int_2^3 3t^2(t+1)\, dt$$

$$= \left[\frac{3t^4}{4} + \frac{3t^3}{3}\right]_2^3 = \left(\frac{243}{4} + 27\right) - \left(\frac{48}{4} + 8\right) = \frac{271}{4}$$