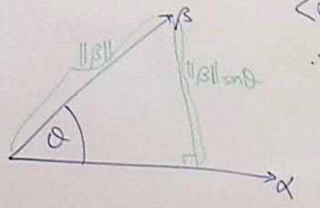
Th 10.1.4: Cauchy-Schwarz inequality:

$$\forall \alpha,\beta \in V, \quad |\langle \alpha,\beta\rangle| \leq \|\alpha\|\,\|\beta\| \quad (*)$$

This is very useful e.g. for error estimation:

$$\left| \int_a^b f(x)g(x)\,dx \right| \leq \left(\int_a^b f(x)^2\,dx \right)^{1/2} \left(\int_a^b g(x)^2\,dx \right)^{1/2}$$

Idea: in $\mathbb{R}^2$ with dot product,



$$\langle \alpha,\beta\rangle = \|\alpha\|\,\|\beta\| \cos\theta$$

$\therefore$ the theorem says

$$-1 \leq \cos\theta \leq 1$$

i.e. $\cos^2\theta \leq 1 = \sin^2\theta + \cos^2\theta$

i.e. $0 \leq \sin^2\theta$

Proof (idea: green side length ≥ 0)

- If $\alpha = \vec{0}$, then both sides of $(*)$ are 0, so inequality holds.
- If $\alpha \neq \vec{0}$:

$$\left\| \beta - \mathrm{Proj}_{\mathrm{Span}\{\alpha\}}(\beta) \right\|^2 \geq 0$$

i.e. $\left\| \beta - \dfrac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}\alpha \right\|^2 \geq 0$

i.e. $\left\langle \beta - \dfrac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}\alpha,\ \beta - \dfrac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}\alpha \right\rangle \geq 0$

i.e. $\langle \beta,\beta\rangle - \left\langle \beta, \dfrac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}\alpha \right\rangle - \left\langle \dfrac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}\alpha, \beta \right\rangle + \left\langle \dfrac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}\alpha, \dfrac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}\alpha \right\rangle \geq 0$

i.e. $\langle \beta,\beta\rangle - \dfrac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}\underbrace{\langle \beta,\alpha\rangle}_{=\overline{\langle \alpha,\beta\rangle}} - \dfrac{\overline{\langle \alpha,\beta\rangle}}{\langle \alpha,\alpha\rangle}\langle \alpha,\beta\rangle + \dfrac{\overline{\langle \alpha,\beta\rangle}}{\langle \alpha,\alpha\rangle}\dfrac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}\langle \alpha,\alpha\rangle \geq 0$

the same

$$\langle \beta,\beta\rangle \geq \frac{\overline{\langle \alpha,\beta\rangle}\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}$$

$$\langle \beta,\beta\rangle\langle \alpha,\alpha\rangle \geq \overline{\langle \alpha,\beta\rangle}\langle \alpha,\beta\rangle \quad \because \langle \alpha,\alpha\rangle > 0$$

$$\|\beta\|^2\|\alpha\|^2 \geq |\langle \alpha,\beta\rangle|^2$$

Square root both sides:

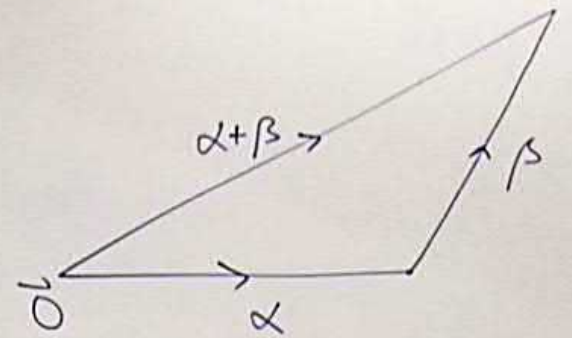
$$\|\beta\|\,\|\alpha\| \geq |\langle \alpha,\beta\rangle|$$

Rem 10.1.5: From the proof, it's clear that $\|\beta\|\,\|\alpha\| = |\langle \alpha,\beta\rangle|$ if and only if $\alpha = \vec{0}$, or $\left\| \beta - \mathrm{Proj}_{\mathrm{Span}\{\alpha\}}\beta \right\|^2 = 0$

i.e. $\beta - \mathrm{Proj}_{\mathrm{Span}\{\alpha\}}\beta = \vec{0}$

i.e. $\beta \in \mathrm{Span}\{\alpha\}$.

Th 10.1.6: Triangle inequality: $\forall \alpha, \beta \in V$, $\|\alpha+\beta\| \leq \|\alpha\| + \|\beta\|$



Proof: $\|\alpha+\beta\|^2$

$$
\begin{aligned}
&= \langle \alpha+\beta, \alpha+\beta \rangle \\
&= \langle \alpha, \alpha \rangle + \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle + \langle \beta, \beta \rangle \\
&= \langle \alpha, \alpha \rangle + 2\,\mathrm{Re}\langle \alpha, \beta \rangle + \langle \beta, \beta \rangle && [z+\bar{z} = 2\,\mathrm{Re}\, z] \\
&\leq \langle \alpha, \alpha \rangle + 2|\langle \alpha, \beta \rangle| + \langle \beta, \beta \rangle && [\mathrm{Re}\, z \leq |z|] \\
&\leq \|\alpha\|^2 + 2\|\alpha\|\|\beta\| + \|\beta\|^2 && [\text{Cauchy-Schwarz}] \\
&= (\|\alpha\| + \|\beta\|)^2
\end{aligned}
$$

Now squareroot both sides.

§10.2 Orthogonal complement

Def: If W is a subspace of V, then the orthogonal complement of W is:

$$W^{\perp} = \{\beta \in V \mid \langle \alpha, \beta \rangle = 0 \ \forall \alpha \in W\}$$

Th 10.2.9: $W^{\perp}$ is a subspace ① and, if $\dim W < \infty$, then $W^{\perp}$ is a complement of W.

i.e. $W \cap W^{\perp} = \{\vec{0}\}$ ②, and $V = W \oplus W^{\perp}$ ③.

so $\dim W^{\perp} = \dim V - \dim W$.

Proof

① check axiom (exercise)

or: for each $\alpha \in W$, consider $\phi_{\alpha} : V \longrightarrow \mathbb{F}$ given by "take inner product with α"

i.e. $\phi_{\alpha} = \langle \alpha, - \rangle$ i.e. $\phi_{\alpha}(\beta) = \langle \alpha, \beta \rangle$

$\because \langle\ ,\ \rangle$ is linear in the second input,

so ϕ_{α} is linear.

and $W^{\perp} = \bigcap_{\alpha \in W} \ker \phi_{\alpha}$

and $\ker \phi_{\alpha}$ is a subspace for each α, and the intersection of subspaces is a subspace.

② if $\alpha \in W \cap W^{\perp}$ then $\langle \underset{W}{\underset{\cap}{\alpha}}, \underset{W^{\perp}}{\underset{\cap}{\alpha}} \rangle = 0$ so $\|\alpha\|^2 = 0 \Rightarrow \alpha = \vec{0}$

③ every $\alpha \in V$ can be written as

$$\alpha = \underbrace{\mathrm{Proj}_W(\alpha)}_{\in W} + \underbrace{(\alpha - \mathrm{Proj}_W(\alpha))}_{\in W^{\perp}}$$