§7.2 Coordinates for vectors

Def 7.2.1 If $\mathcal{A}=\{\alpha_1,\cdots,\alpha_n\}$ is a basis of V, then each $\alpha\in V$ can be written uniquely as

$$\alpha = a_1\alpha_1+\cdots+a_n\alpha_n$$

i.e. $a_1,\cdots,a_n$ unique.

(2207 Week 8 p11, Prop. 6.4.5)

and the $\mathcal{A}$-coordinates of α is

$$[\alpha]_{\mathcal{A}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n$$

(The order of $\alpha_1,\cdots,\alpha_n$ is important — whenever we use coordinates, we assume the order of the basis vectors is fixed.)

Not in textbook:

$$\mathrm{Coord}_{\mathcal{A}} : V \longrightarrow \mathbb{F}^{\dim V}$$

$$\mathrm{Coord}_{\mathcal{A}}(\alpha) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

can write $\mathrm{Coord}_{\mathcal{A}}(W) \subseteq \mathbb{F}^{\dim V}$

$$\mathrm{Decoord}_{\mathcal{A}} : \mathbb{F}^{\dim V} \longrightarrow V$$

$$\mathrm{Decoord}_{\mathcal{A}}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = a_1\alpha_1+\cdots+a_n\alpha_n = \alpha$$

e.g. $V = M_{2,2}(\mathbb{R})$, $\mathcal{A} = \{E^{1,1}, E^{1,2}, E^{2,1}, E^{2,2}\}$

standard basis

$$\alpha = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow [\alpha]_{\mathcal{A}} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}.$$

Using a different basis gives different coordinates:

e.g. $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

$$\alpha = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightarrow [\alpha]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \end{pmatrix}$$

Coordinates allow us to use RREF methods in any (finite-dimensional) V.

e.g. to find a basis of

$$W = \operatorname{Span}\{\underbrace{-x+x^2}_{\alpha_1},\ \underbrace{1+x+x^2}_{\alpha_2},\ \underbrace{1+2x}_{\alpha_3}\} \subseteq P_{<3}(\mathbb{R})$$

with casting-out algorithm:

1. Choose a basis $\mathcal{A}$ of $P_{<3}(\mathbb{R})$, and apply $\mathrm{Coord}_{\mathcal{A}}$:

$\mathcal{A} = \{1, x, x^2\}$:

$$[\alpha_1]_{\mathcal{A}} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad [\alpha_2]_{\mathcal{A}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{matrix} \leftarrow 1 \\ \leftarrow x \\ \leftarrow x^2 \end{matrix}$$

$$[\alpha_3]_{\mathcal{A}} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

② Compute with coordinate vectors in $\mathbb{R}^3$:

$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathrm{Coord}_{\mathcal{A}}(W) \subseteq \mathbb{R}^3$.

③ Apply $\mathrm{Decoord}_{\mathcal{A}}$:

$\{-x+x^2, 1+x+x^2\}$ is a basis for W.

Matrix representations of linear transformations:

Def 7.2.5: Let $\sigma \in L(U,V)$,

$\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\}$ a basis of U

$\mathcal{B} = \{\beta_1, \cdots, \beta_m\}$ a basis of V.

The matrix representation of σ, relative to $\mathcal{A}$ and $\mathcal{B}$ is

$$[\sigma]_{\mathcal{B} \leftarrow \mathcal{A}} = \begin{pmatrix} | & & | \\ [\sigma(\alpha_1)]_{\mathcal{B}} & \cdots & [\sigma(\alpha_m)]_{\mathcal{B}} \\ | & & | \end{pmatrix}$$

$[\sigma]^{\mathcal{A}}_{\mathcal{B}}$ in textbook

If $U=V$, then we usualy want $\mathcal{A}=\mathcal{B}$,

then ${}_{\mathcal{A}\leftarrow\mathcal{A}}[\sigma] = \begin{pmatrix} | & & | \\ [\sigma(\alpha_1)]_{\mathcal{A}} & \cdots & [\sigma(\alpha_n)]_{\mathcal{A}} \\ | & & | \end{pmatrix}$

also written $[\sigma]_{\mathcal{A}}$.

Warning: $[\alpha]_{\mathcal{A}}$ is a vector $\in \mathbb{F}^{?}$

$[\sigma]_{\mathcal{A}}$ is a matrix

Ex: multiplication by $2+x^2$

$\sigma : P_{<2}(\mathbb{R}) \longrightarrow P_{<4}(\mathbb{R})$

$[\sigma(f)](x) = f(x)(2+x^2)$

$\mathcal{A}=\{1,x\}$, $\mathcal{B}=\{1,x,x^2,x^3\}$

$$\sigma(1) = 1(2+x^2)$$
$$= 2\cdot 1 + 0\cdot x + 1\cdot x^2 + 0\cdot x^3$$

$$\sigma(x) = x(2+x^2)$$
$$= 0\cdot 1 + 2\cdot x + 0\cdot x^2 + 1\cdot x^3$$

$$\overbrace{\begin{array}{cc} 1 & x \end{array}}^{\mathcal{A}}$$
$$\mathcal{B}\left\{\begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \end{array}\right. \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Exercise: what is the matrix for "evaluation at 2": $P_{<3}(\mathbb{R}) \to \mathbb{R}$.

$\sigma(f) = f(2)$, relative to standard bases

$\mathcal{A}=\{1,x,x^2\}$, $\mathcal{B}=\{1\}$.