A second proof of the Rank-Nullity Theorem

The goal of this exercise is to give a second proof of the Rank-Nullity Theorem, where we start with a basis of the range instead of of the complement of the kernel, as in class. (This is similar to an optional homework question from 2207.)

Theorem: if $\dim(U) < \infty$ **and** $\sigma \in L(U, V)$**, then** $\operatorname{rank}\sigma + \operatorname{nullity}\sigma = \dim U$**.**

We start as in class: Because $\ker\sigma$ is a subspace of U, so $\ker\sigma$ is finite-dimensional. Let $\dim\ker\sigma = k$, and take a basis $\mathscr{A} = \{\alpha_1, \ldots, \alpha_k\}$ of $\ker\sigma$.

c. Show that $\operatorname{range}\sigma$ is finite dimensional by finding a finite spanning set. (Hint: start with a basis for V and look at what the linear transformation T does to it.)

d. Let $\mathscr{B} = \{\gamma_1, \ldots, \gamma_r\}$ be a basis for $\operatorname{range}\sigma$, so that $\dim(\operatorname{range}\sigma) = r$. Explain why there are vectors $\beta_1, \ldots, \beta_r$ in U such that $\sigma(\beta_i) = \gamma_i$ for $i = 1, \ldots, r$.

We now show that the set $\mathscr{D} = \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_r\}$ forms a basis for U.

e. We show $\mathcal{D}$ is linearly independent. Suppose there are weights $a_1, \ldots, a_k, b_1, \ldots, b_r \in \mathbb{F}$ such that

$$a_1\alpha_1 + \cdots + a_k\alpha_k + b_1\beta_1 + \cdots + b_r\beta_r = \mathbf{0}. \qquad (\dagger)$$

Apply σ to $(\dagger)$ and use the fact that $\mathscr{B}$ is a basis of $\operatorname{range}\sigma$ to show that $b_1 = \cdots = b_r = 0$. Then show $a_1 = \cdots = a_k = 0$.

f. We show $\mathcal{D}$ spans U. Let α be an arbitrary vector in U. Explain why we can write $\sigma(\alpha)$ as a linear combination of $\mathscr{B}$, and use this linear combination to write α as a linear combination of $\mathcal{D}$.

Now suppose U is infinite-dimensional. Modify the above proof to show that

Theorem: if $\sigma \in L(U, V)$**, and** $\mathscr{A}$**,** $\mathscr{B}$ **are bases respectively for** $\ker\sigma$**,** $\operatorname{range}\sigma$**, then there is set** $\mathscr{C}$ **such that** $\mathscr{A} \cup \mathscr{C}$ **is a basis for** U**, and** $\{\sigma(\gamma) | \gamma \in \mathscr{C}\} = \mathscr{B}$**.**