

You must justify your answers to receive full credit.

1. (This question requires material from §7.1.) Suppose $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are eigenvectors of a symmetric matrix. Explain why these two eigenvectors correspond to the same eigenvalue.

2. Let

$$\mathbf{y} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ -5 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix},$$

and let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

- Use the Gram-Schmidt process to find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for W .
- Write \mathbf{y} as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ from part a).
- Let $\text{proj}_W : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the orthogonal projection onto W . Find the standard matrix of proj_W .
- Let \mathbf{x}_4 be a nonzero vector in W^\perp , and let $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$. Find the matrix of proj_W relative to \mathcal{B} . (Hint: you do not need to do any calculation.)

3. Let

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -4 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix}.$$

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is an orthogonal basis for \mathbb{R}^4 . Let W be the line $\text{Span}\{\mathbf{v}_4\}$.

- Find the orthogonal projection of \mathbf{y} to W .
 - Using your answer to a) and the Orthogonal Decomposition Theorem, and without calculating any more dot products, write \mathbf{y} as the sum of two vectors $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, such that $\hat{\mathbf{y}}$ is in $\text{Span}\{\mathbf{v}_4\}$ and \mathbf{z} is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. (You do **not** have to express \mathbf{z} as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$).
 - Using your answer to b), and without calculating any more dot products, find the orthogonal projection of \mathbf{y} to $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
4. We would like to model the average monthly temperature T (in Celsius) in Hong Kong in month m (so $m = 1$ is January, $m = 2$ is February, etc.) by

$$T = a + b \sin \frac{\pi m}{6} + c \cos \frac{\pi m}{6}.$$

The Hong Kong Observatory website gives the following data for the year 2014:

m_i	1	3	5	6	8
T_i	16.3	18.7	26.4	29.0	29.0

Write down the observation vector, the design matrix and the parameter vector you would use to estimate a, b, c .

5. Remember from class that the least-squares line through the data points $(x_1, y_1), \dots, (x_n, y_n)$ is given by $y = \hat{\beta}_0 + \hat{\beta}_1 x$, where $(\hat{\beta}_0, \hat{\beta}_1)$ is the least-squares solution to

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

The goal of this problem is to prove, from this linear algebra perspective, the formulas for the regression coefficients $\hat{\beta}_0$ and $\hat{\beta}_1$ from Probability and Statistics class:

$$\hat{\beta}_1 = \frac{SS_{xy}}{SS_{xx}}, \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

where

$$SS_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}, \quad SS_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}, \quad \bar{y} = \frac{\Sigma y}{n}, \quad \bar{x} = \frac{\Sigma x}{n}.$$

- a) By considering the normal equations, find a matrix M such that $M \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \Sigma y \\ \Sigma xy \end{bmatrix}$.
(The entries of M will be in terms of n, x and y .)
 - b) Assume that x_1, \dots, x_n are not all the same. Using a theorem from class, explain why this means $\hat{\beta}_0$ and $\hat{\beta}_1$ are unique.
 - c) The uniqueness of $\hat{\beta}_0$ and $\hat{\beta}_1$ implies that the matrix M is invertible. By inverting M , show that $\hat{\beta}_1$ has the formula given above, then show that $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$.
6. State whether each of the following statements is (always) true or (sometimes) false. If it is true, give a brief justification; if it is false, give a counterexample with an explanation.
- a) If a square matrix A is diagonalisable, then it is invertible.
 - b) If a square matrix A is invertible, then it is diagonalisable.
 - c) Let W be a subspace of \mathbb{R}^n . If $\{\mathbf{w}_1, \dots, \mathbf{w}_p\}$ is an orthogonal set in W and $\{\mathbf{u}_1, \dots, \mathbf{u}_q\}$ is an orthogonal set in W^\perp , then their union $\{\mathbf{w}_1, \dots, \mathbf{w}_p, \mathbf{u}_1, \dots, \mathbf{u}_q\}$ is an orthogonal set.
 - d) Let U be an $n \times n$ orthogonal matrix. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in \mathbb{R}^n , then $\{U\mathbf{v}_1, U\mathbf{v}_2, U\mathbf{v}_3\}$ is also an orthogonal set.
 - e) Let U and W be subspaces of \mathbb{R}^n , and suppose U is a subset of W (i.e. every vector in U is also in W). Then U^\perp is a subset of W^\perp .
 - f) If U is a square matrix with orthonormal columns, then the rows of U are also orthonormal.

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