

You must justify your answers to receive full credit.

1. a) (Proposition 6.5.14) Let  $W_1, W_2, \dots, W_n$  be subspaces of a vector space  $V$ . Prove that  $W_1 + W_2 + \dots + W_n$  is a direct sum if and only if,  $\forall \alpha \in W_1 + W_2 + \dots + W_n$ ,  $\alpha$  can be expressed uniquely in the form  $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_n$  with  $\alpha_i \in W_i$ ,  $i = 1, 2, \dots, n$ . (Hint: modify appropriately the proof of Proposition 5.1.10, for  $n = 2$ . You can use induction for one of the directions, but I think it is easier without induction.)
- b) Find non-zero subspaces  $W_1, W_2, W_3$  of  $\mathbb{R}^3$  satisfying

$$W_1 \cap W_2 = \{\mathbf{0}\}, W_1 \cap W_3 = \{\mathbf{0}\}, W_2 \cap W_3 = \{\mathbf{0}\}, \quad (*)$$

and some  $\alpha \in \mathbb{R}^3$  which can be written as  $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ , with  $\alpha_i \in W_i$ , in two different ways. (This explains why  $(*)$  is not the condition for a direct sum of three subspaces.)

2. Consider the following two subspaces of  $M_{2,2}(\mathbb{R})$ :

$$W_1 = \left\{ \begin{pmatrix} x & -x \\ y & z \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}, \quad W_2 = \left\{ \begin{pmatrix} a & b \\ -a & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

Find a basis of  $W_1 \cap W_2$ , and extend it to bases of  $W_1$ ,  $W_2$  and  $W_1 + W_2$ .

3. Let  $U, V, W$  be vector spaces, and take  $\sigma, \sigma' \in L(U, V)$ ,  $\tau \in L(V, W)$ .
  - a) (Proposition 7.1.8, axiom (V1) for  $L(U, V)$ ) Prove that the sum  $\sigma + \sigma'$  is a linear transformation.
  - b) (Proposition 7.1.7) Prove that the composition  $\tau \circ \sigma$  is in  $L(?, ?)$  where the ?s are vector spaces that you should specify.
  - c) (Proposition 7.1.14) Let  $V'$  be a subspace of  $V$ . Show that the preimage  $\sigma^{-1}[V'] = \{\alpha \in U \mid \sigma(\alpha) \in V'\}$  is a subspace of  $U$ .
4. Let  $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ , and define  $\sigma : M_{2,2}(\mathbb{R}) \rightarrow M_{2,2}(\mathbb{R})$  by  $\sigma(X) = AX - XA$ .
  - a) Show that  $\sigma$  is a linear transformation.
  - b) Find  $[\sigma]_{\mathcal{A}}$ , where  $\mathcal{A} = \{E^{1,1}, E^{1,2}, E^{2,1}, E^{2,2}\}$  is the standard basis of  $M_{2,2}(\mathbb{R})$ .
  - c) Find all  $X \in M_{2,2}(\mathbb{R})$  such that  $AX = XA$ .
  - d) Using part c) or otherwise, find the rank of  $\sigma$ .
  - e) Let  $U \subseteq M_{2,2}(\mathbb{R})$  be the subspace of symmetric matrices. Find a basis for the image  $\sigma(U)$ . (Click here for a hint)

5. Fill in the blanks and choose the correct words at the †s to complete the proof of the Rank Nullity Theorem (print this page and hand it in):

**Theorem:** if  $\dim(U) < \infty$  **and**  $\sigma \in L(U, V)$ , **then**  $\text{rank } \sigma + \text{nullity } \sigma = \dim U$ .

**Proof:** Let  $\dim U = n$ . Because  $\ker \sigma$  is a subspace of  $U$ , so  $\ker \sigma$  is finite-dimensional. Let  $\dim \ker \sigma = k$ .

Take a basis  $\{\alpha_1, \dots, \alpha_k\}$  of  $\ker \sigma$ , and a basis  $\{\beta_1, \dots, \beta_{n-k}\}$  of a complement of  $\ker \sigma$  in  $U$ , so  $\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_{n-k}\}$  is a basis for  $U$ .

We show  $\{\sigma(\beta_1), \dots, \sigma(\beta_{n-k})\}$  is a basis for  $\text{range } \sigma$  – that would mean  $\text{rank } \sigma = \dim \text{range } \sigma = n - k = \dim U - \text{nullity } \sigma$ .

- We show  $\{\sigma(\beta_1), \dots, \sigma(\beta_{n-k})\}$  is linearly independent:

Suppose  $b_1\sigma(\beta_1) + \dots + b_{n-k}\sigma(\beta_{n-k}) = \mathbf{0}$  (\*)

We need to show \_\_\_\_\_.

Because  $\sigma$  is linear, we can rewrite (\*) as

$$\sigma\left(\text{_____}\right) = \mathbf{0}.$$

So

$$\text{_____} \in \ker \sigma.$$

Because  $\{\alpha_1, \dots, \alpha_k\}$  is a basis for  $\ker \sigma$ , we have, for some  $a_i \in \mathbb{F}$ ,

$$\text{_____} = a_1\alpha_1 + \dots + a_k\alpha_k.$$

Move all terms to one side:

$$-a_1\alpha_1 \text{_____} = \mathbf{0}.$$

Because  $\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_{n-k}\}$  is a basis for  $U$ , this set is linearly independent, so

$$\text{_____}.$$

- We show  $\text{Span } \{\sigma(\beta_1), \dots, \sigma(\beta_{n-k})\} = \text{range } \sigma$ :

By the Lemma from class,

$$\sigma(\text{Span } \{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_{n-k}\}) = \text{Span } \{\sigma(\alpha_1), \dots, \sigma(\alpha_k), \sigma(\beta_1), \dots, \sigma(\beta_{n-k})\}.$$

The left / right † hand side is  $\text{Span } \{\sigma(\beta_1), \dots, \sigma(\beta_{n-k})\}$  because \_\_\_\_\_

$$\text{_____}.$$

The left / right † hand side is  $\text{range } \sigma$  because \_\_\_\_\_.

6. Let  $P_{<n}(\mathbb{R})$  denotes the vector space of polynomials with degree less than  $n$ . Consider the bases

$$\begin{aligned}\mathcal{A} &= \{1 + x - x^2, 1 + x, 1 + x^2\} \subseteq P_{<3}, \\ \mathcal{B} &= \{1, x\} \subseteq P_{<2}, \\ \mathcal{C} &= \{1 + x, 1 - x\} \subseteq P_{<2}.\end{aligned}$$

Let  $\sigma : P_{<3} \rightarrow P_{<2}$  be the linear transformation given by

$$\sigma(f) = f(2)x + \frac{df}{dx}.$$

- Find  $[2x + 3x^2]_{\mathcal{A}}$ , the  $\mathcal{A}$ -coordinates of  $2x + 3x^2$ .
- Find the polynomial  $f$  whose  $\mathcal{C}$ -coordinates are  $[f]_{\mathcal{C}} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ .
- Find the matrix of transition from  $\mathcal{B}$  to  $\mathcal{C}$ .
- Find the matrix representation of  $\sigma$  relative to  $\mathcal{A}$  and  $\mathcal{C}$  (i.e. the matrix denoted  ${}_{\mathcal{C}}[\sigma]_{\mathcal{A}}$  or  $[\sigma]_{\mathcal{C}}^{\mathcal{A}}$ ).
- Is  $\sigma$  surjective? Explain your answer.

The following two questions are to prepare you for upcoming classes, and is unrelated to the material from recent classes.

7. Find all eigenvalues and eigenvectors of  $A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ -2 & 2 & -1 \end{pmatrix}$ , and hence diagonalise  $A$ , i.e.

find a  $P$  and a diagonal  $D$  such that  $A = PDP^{-1}$ . You do **not** need to compute  $P^{-1}$ . (You may use an online RREF calculator, but remember you only have an ordinary calculator in the exams.)

8. a) Find the eigenvalues of  $\begin{pmatrix} 5 & 2 & 0 & 0 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ .

Suppose  $A$  is a  $3 \times 3$  matrix whose only eigenvalues are 1 and 2.

- b) If  $A$  is diagonalisable, then what are the possibilities for the dimensions of its eigenspaces? (Answer in this way: “ $\dim E_1 = ?$  and  $\dim E_2 = ?$ , or  $\dim E_1 = ?$  and  $\dim E_2 = ?$ , or ...”, and then give your reasons.)
- c) If  $A$  is not diagonalisable, then what are the possibilities for the dimensions of its eigenspaces? (Answer in the same way as in part b.)

**Optional questions** If you attempted seriously all the above questions, then your scores for the following questions may replace any lower scores for two of the above questions.

9. Let  $V$  be a vector space, and take  $\sigma \in L(V, V)$ . Show that, if  $\text{rank}(\sigma) = \text{rank}(\sigma \circ \sigma)$ , then  $\text{range}(\sigma) \cap \ker(\sigma) = \{\mathbf{0}\}$ .  
(Hint: the equation  $\text{rank}(\sigma) = \text{rank}(\sigma \circ \sigma)$  means these numbers are finite. You may assume  $V$  is finite-dimensional if you wish. It may be useful to show that  $\text{rank}(\sigma) \geq \text{rank}(\sigma \circ \sigma)$ .)
10. Let  $U, W_1, W_2$  be subspaces of a vector space  $V$ , such that  $W_1 \cap W_2 = \{\mathbf{0}\}$ . Is  $U \cap (W_1 \oplus W_2) = (U \cap W_1) \oplus (U \cap W_2)$ ? Give a proof or a counterexample.

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