§6.2 Linear dependence / independence

Def 6.2.1: β is a $\mathbb{F}$-linear combination of $\alpha_1, \alpha_2, \cdots, \alpha_n$ if there are scalars (weights) $a_1, \cdots, a_n \in \mathbb{F}$ such that $\beta = a_1\alpha_1 + \cdots + a_n\alpha_n$.

(Textbook also says "β is linearly dependent on $\alpha_1, \cdots, \alpha_n$")

Ex: in $\mathbb{C}^3$

$\begin{pmatrix} 2+3i \\ i \\ i \end{pmatrix}$ is a linear combination of

$$\begin{pmatrix} 1 \\ -i \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -i \\ 3-i \\ 4 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2+3i \\ i \\ i \end{pmatrix} = i\begin{pmatrix} 1 \\ -i \\ 1 \end{pmatrix} + (1+i)\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} -i \\ 3-i \\ 4 \end{pmatrix}.$$

In continuous functions over $\mathbb{R}$:

$2+3x$ is a linear combination of x, $\sin^2 x$ and $\cos^2 x$:

$$2+3x = 2\sin^2 x + 2\cos^2 x + 3x$$

Def 6.2.2: (for finite sets:) $\{\alpha_1, \dots, \alpha_n\}$ is <u>linearly dependent (over $\mathbb{F}$)</u> if there exists $a_1, \dots, a_n \in \mathbb{F}$ not all zero such that $a_1\alpha_1 + \dots + a_n\alpha_n = \vec{0}$. This equation is a <u>linear dependence relation</u>.

The opposite: $\{\alpha_1, \cdots, \alpha_n\}$ is <u>linearly independent (over $\mathbb{F}$)</u>

if: whenever $a_1\alpha_1 + \cdots + a_n\alpha_n = \vec{0}$ with $a_1, \cdots, a_n \in \mathbb{F}$,

then $a_1 = \cdots = a_n = 0$.

(i.e. there are no linear dependence relations)

(for finite or infinite sets): S is <u>linearly dependent</u>

if $\exists\, \alpha_1, \cdots, \alpha_n \in S$, $a_1, \cdots, a_n \in \mathbb{F}$ not all zero,

with $a_1\alpha_1 + \cdots + a_n\alpha_n = \vec{0}$.

S is <u>linearly independent</u> if, $\forall\, \alpha_1, \cdots, \alpha_n \in S$

$a_1, \cdots, a_n \in \mathbb{F}$ with $a_1\alpha_1 + \cdots + a_n\alpha_n = \vec{0}$, it means

$a_1 = \cdots = a_n = \vec{0}$.

To show linear dependence: give <u>one</u> example of a linear dependence relation.

To show linear independence: suppose

$a_1 \alpha_1 + \cdots + a_n \alpha_n = \vec{0}$, show $a_1 = \cdots = a_n = 0$.

(This applies to calculations and to proofs.)

Ex: $\{1, e^x + 1, e^x - 1\}$ is linearly dependent

$$-2(1) + 1(e^x + 1) + (-1)(e^x - 1) = 0$$

$\{1 + x, 1, x^2 - 1\}$ is linearly independent if $a_1(1+x) + a_2(1) + a_3(x^2-1) = 0$

$$(a_1 + a_2 - a_3) + a_1 x + a_3 x^2 = 0$$

$$a_1 = 0 \qquad a_3 = 0$$

☆

$$a_1 + a_2 - a_3 = 0 \implies a_2 = 0.$$

The step ☆ implicitly uses:

- $\{1, x, \dots, x^n\}$ is linearly independent (proof by substitution / differentiation 2207 week 8 p11?)
- $\{1, x, x^2, \dots\}$ is linearly independent.

Proof outline: suppose $a_1 x^{i_1} + a_2 x^{i_2} + \cdots + a_n x^{i_n} = \vec{0}$. for some $i_1, \cdots, i_n \in$

now use substitution / differentiation.

Remark 6.2.3 – useful properties and explanation

② if $\vec{0} \in S \subseteq V$, then S is linearly dependent.

$\underset{a_1 \neq 0}{1} \cdot \underset{\alpha_1}{\vec{0}} = \vec{0}$ is a linear dependence relation

$(1 \cdot \vec{0} + 0\alpha_2 + \cdots + 0\alpha_n = \vec{0})$

③ if $\alpha \neq \vec{0}$, then $\{\alpha\}$ is linearly independent.

HW1 Q1b the only solution to $a_1 \alpha = \vec{0}$ is $a_1 = 0$ $\because \alpha \neq \vec{0}$

① the empty set $\emptyset$ is linearly independent.

there are no vectors in the empty set $\therefore$ no linear dependence relations

Consider $S \subseteq T$

④ if S is linearly dependent, then so is T.

$\hookrightarrow \exists a_1, \cdots, a_n$ not all zero, $\alpha_1, \cdots, \alpha_n \in S$ with $a_1\alpha_1 + \cdots + a_n\alpha_n = \vec{0}$. But $\alpha_1, \cdots, \alpha_n \in T$ also

⑤ if T is linearly independent, then so is S

this is the contrapositive of ④

(i.e. ④ gives a proof by contradiction)

Th 6.2.6: A set of non-zero vectors $\{\alpha_1, \cdots, \alpha_n\}$ is linearly dependent $\Leftrightarrow$ some α_k is a linear combination of $\alpha_1, \cdots, \alpha_{\boxed{k-1}}$.

Proof: $\Leftarrow$ $\alpha_k = a_1\alpha_1 + \cdots + a_{k-1}\alpha_{k-1}$ then

$$a_1\alpha_1 + \cdots + a_{k-1}\alpha_{k-1} + \underbrace{(-1)}_{\neq 0}\alpha_k = 0$$

is a linear dependence relation

$\Rightarrow$: Given a linear dependence relation

$$a_1\alpha_1 + \cdots + a_n\alpha_n = \vec{0},$$

let k be the largest with $a_k \neq 0$.

(this k exists $\because$ not all a_i are 0)

so $a_1\alpha_1 + \cdots + a_k\alpha_k = \vec{0}$

$$\alpha_k = \frac{-a_1}{a_k}\alpha_1 + \cdots + \frac{-a_{k-1}}{a_k}\alpha_{k-1}$$