Can I change the σ in the question, so the numbers are less confusing?

Let's say $\sigma\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x - y \\ 2x - 2z \\ y + z \end{pmatrix}$.

Let $\beta_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\beta_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\beta_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$, $\mathcal{B} = \{\beta_1, \beta_2, \beta_3\}$.

The question asks for $[\sigma]_{\mathcal{B}}$, i.e. ${}_{\mathcal{B} \leftarrow}[\sigma]_{\mathcal{B}}$.

There are two ways to do this:

<u>Method 1</u> (what we were doing on Friday)

$$
{}_{\mathcal{B} \leftarrow}[\sigma]_{\mathcal{B}} = \begin{array}{cccc} \beta_1 & \beta_2 & \beta_3 & \leftarrow \text{inputs (into } \sigma) \\ \begin{array}{|c|}\hline a \\ \hline b \\ \hline c \\ \hline \end{array} & \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} & \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} & \left.\begin{array}{l} \beta_1 \\ \beta_2 \\ \beta_3 \end{array}\right\} \text{outputs} \end{array}
$$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is $[\sigma(\beta_1)]_{\mathcal{B}}$, i.e. find a, b, c so that $\sigma(\beta_1) = a\beta_1 + b\beta_2 + c\beta_3$, then put these a, b, c into the first column of the table.

i.e.: $\sigma(\beta_1) = \sigma\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{matrix} \leftarrow x=1 \\ \leftarrow y=1 \\ \leftarrow z=1 \end{matrix}$

$= \begin{pmatrix} 3-1 \\ 2-2 \\ 1+1 \end{pmatrix}$ using (*)

$= \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

Let $\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = a\beta_1 + b\beta_2 + c\beta_3$

$= a\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

To solve this, row reduce $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 2 & -2 & 0 \\ 1 & 0 & 1 & 2 \end{array}\right) \longmapsto a = 0, b = 2, c = 2$

$$
\therefore {}_{\mathcal{B} \leftarrow}[\sigma]_{\mathcal{B}} = \begin{array}{ccc|l} \beta_1 & \beta_2 & \beta_3 & \\ \hline 0 & & & \beta_1 \\ 2 & & & \beta_2 \\ 2 & & & \beta_3 \end{array}
$$

To find the other two columns, do the same thing with $\sigma(\beta_2)$ and $\sigma(\beta_3)$: you should try.

Method 2 (change of coordinates)

Let $\mathcal{A} = \{e_1, e_2, e_3\} = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$, the standard basis of $\mathbb{R}^3$.

☆ $$[\sigma]_{\mathcal{A} \leftarrow \mathcal{A}} = \begin{array}{c|c|c|c} e_1 & e_2 & e_3 & \\ \hline 3 & -1 & 0 & e_1 \\ \hline 2 & 0 & -2 & e_2 \\ \hline 0 & 1 & 1 & e_3 \end{array}$$ from (*)

$\sigma(e_1)$ $\sigma(e_2)$ $\sigma(e_3)$

$\sigma(e_1) = \sigma\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}x=1\\y=0\\z=0\end{pmatrix}$

$\sigma(e_2) = \sigma\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\begin{pmatrix}x=0\\y=1\\z=0\end{pmatrix}$

$\sigma(e_3) = \sigma\begin{pmatrix}0\\0\\1\end{pmatrix}$, $\begin{pmatrix}x=0\\y=0\\z=1\end{pmatrix}$

Now $[\sigma]_{\mathcal{B} \leftarrow \mathcal{B}} = [\iota \circ \sigma \circ \iota]_{\mathcal{B} \leftarrow \mathcal{B}}$ ι is the identity function, $\iota(\alpha) = \alpha \ \forall \alpha$.

$$= \underbrace{[\iota]_{\mathcal{B} \leftarrow \mathcal{A}}}_{①} \underbrace{[\sigma]_{\mathcal{A} \leftarrow \mathcal{A}}}_{②} \underbrace{[\iota]_{\mathcal{A} \leftarrow \mathcal{B}}}_{③}$$

① is the inverse of ③

② found in ☆

△ $$[\iota]_{\mathcal{A} \leftarrow \mathcal{B}} = \begin{array}{c|c|c|c} \beta_1 & \beta_2 & \beta_3 & \\ \hline 1 & 1 & 0 & e_1 \\ \hline 1 & 2 & -2 & e_2 \\ \hline 1 & 0 & 1 & e_3 \end{array}$$

inputs: $\beta_1, \beta_2, \beta_3$; outputs: e_1, e_2, e_3

first column $= \beta_1$

second column $= \beta_2$.

Now invert this matrix, multiply ① ② ③ as above, and check the answer is the same as via method 1.