

Rewriting our example on p2 in this terminology:

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{R \rightarrow \infty} \int_1^R \frac{1}{x^2} dx = \lim_{R \rightarrow \infty} \left. \frac{-1}{x} \right|_1^R = \lim_{R \rightarrow \infty} 1 - \frac{1}{R} = 1.$$

Example: Evaluate $\int_1^{\infty} \frac{1}{x} dx$.

Handwritten work on a whiteboard showing the evaluation of the integral $\int_1^{\infty} \frac{1}{x} dx$.

$$\begin{aligned} & \int_1^{\infty} \frac{1}{x} dx \\ &= \lim_{R \rightarrow \infty} \int_1^R \frac{1}{x} dx \\ &= \lim_{R \rightarrow \infty} \left. \ln |x| \right|_1^R = \lim_{R \rightarrow \infty} \ln R - \ln 1 = +\infty \end{aligned}$$

So this integral diverges.

A small graph of the function $y = \ln x$ is shown on the right, illustrating the behavior of the antiderivative as x increases.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$$

$$= \lim_{R \rightarrow -\infty} \int_R^0 \frac{1}{1+x^2} dx + \lim_{R \rightarrow \infty} \int_0^R \frac{1}{1+x^2} dx$$

$$= \lim_{R \rightarrow -\infty} \tan^{-1}(x) \Big|_R^0 + \lim_{R \rightarrow \infty} \tan^{-1}(x) \Big|_0^R$$

$$= 0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0 = \pi$$

$$\int_2^6 (x-3)^{-1/3} dx \quad \text{integrand is not defined at } x=3.$$

$$= \lim_{c \rightarrow 3^-} \int_2^c (x-3)^{-1/3} dx + \lim_{c \rightarrow 3^+} \int_c^6 (x-3)^{-1/3} dx$$

$$= \lim_{c \rightarrow 3^-} \left. \frac{(x-3)^{2/3}}{2/3} \right|_2^c + \lim_{c \rightarrow 3^+} \left. \frac{(x-3)^{2/3}}{2/3} \right|_c^6$$

substitution
 $u = x-3$
 $du = dx$

$$= 0 - \frac{(-1)^{2/3}}{2/3} + \frac{3^{2/3}}{2/3} - 0 = \frac{3}{2} (3^{2/3} - 1)$$

Alternative: we can do substitution before writing the integral as a limit, if we use one substitution on the whole integrand

$$u = x-3$$

$$du = dx$$

$$u=2 \rightarrow x=-1$$

$$u=6 \rightarrow x=9$$

$$\int_2^6 (x-3)^{-1/3} dx = \int_{-1}^9 u^{-1/3} du = \lim_{c \rightarrow 0^-} \int_{-1}^c u^{-1/3} du + \lim_{c \rightarrow 0^+} \int_c^9 u^{-1/3} du$$

not defined
at $u=0$.

$$= \lim_{c \rightarrow 0^-} \left. \frac{u^{2/3}}{2/3} \right|_{-1}^c + \lim_{c \rightarrow 0^+} \left. \frac{u^{2/3}}{2/3} \right|_c^9$$

$$= 0 - \frac{(-1)^{2/3}}{2/3} + \frac{9^{2/3}}{2/3} - 0 = \frac{3}{2} (3^{2/3} - 1)$$