Here is a simplified example of how to use second order Taylor polynomials to classify critical points of multivariate functions.

Example: Find and classify the critical points of $f(x, y) = y^2 - x^3 + x$.

Find At a critical point: $f_x(x,y) = 0$ and $f_y(x,y) = 0$

$-3x^2 + 1 = 0$ $\qquad$ $2y = 0$

$1 = 3x^2$ $\qquad$ $y = 0$

$x = \frac{1}{\sqrt{3}}$, or $x = \frac{-1}{\sqrt{3}}$ $\qquad$ critical points are $(\frac{1}{\sqrt{3}}, 0)$ and $(\frac{-1}{\sqrt{3}}, 0)$.

Classify $f_{xx}(x,y) = -6x$ $\qquad$ $f_{xy}(x,y) = 0$ $\qquad$ $f_{yy}(x,y) = 2$

Second-order Taylor polynomial at $(\frac{-1}{\sqrt{3}}, 0)$:

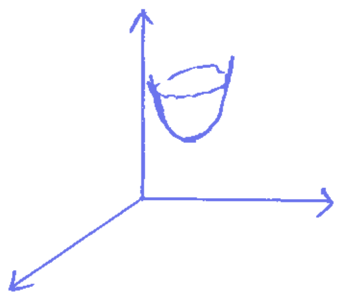
$$f(\tfrac{-1}{\sqrt{3}} + h, 0 + k) \approx f(\tfrac{-1}{\sqrt{3}}, 0) + \cancel{f_x(\tfrac{-1}{\sqrt{3}}, 0)h + f_y(\tfrac{-1}{\sqrt{3}}, 0)k} \to 0$$

$$+ \frac{1}{2!}\left(f_{xx}(\tfrac{-1}{\sqrt{3}}, 0)h^2 + 2f_{xy}(\tfrac{-1}{\sqrt{3}}, 0)hk + f_{yy}(\tfrac{-1}{\sqrt{3}}, 0)k^2\right)$$

$$= f(\tfrac{-1}{\sqrt{3}}, 0) + \frac{1}{2!}\left(-6\left(\tfrac{-1}{\sqrt{3}}\right)h^2 + 2\cdot 0\cdot hk + 2k^2\right) = f(\tfrac{-1}{\sqrt{3}}, 0) + \underbrace{\sqrt{3}h^2 + 2k^2}_{>0 \text{ if } h,k \text{ not both } 0} > f(\tfrac{-1}{\sqrt{3}}, 0)$$

so $(\frac{-1}{\sqrt{3}}, 0)$ is a minimum

Graph of f near $\left(\frac{-1}{\sqrt{3}}, 0\right)$ is like an elliptic paraboloid.



Second-order Taylor polynomial at $\left(\frac{1}{\sqrt{3}}, 0\right)$:

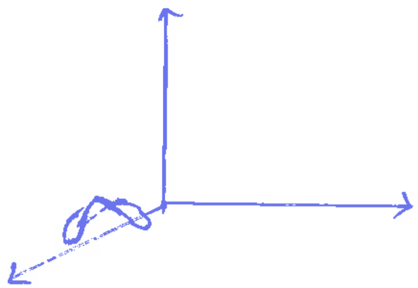
$$f\left(\tfrac{1}{\sqrt{3}}+h, 0+k\right) \approx f\left(\tfrac{1}{\sqrt{3}},0\right) + \frac{1}{2!}\left(f_{xx}\left(\tfrac{1}{\sqrt{3}},0\right)h^2 + 2f_{xy}\left(\tfrac{1}{\sqrt{3}},0\right)hk + f_{yy}\left(\tfrac{1}{\sqrt{3}},0\right)k^2\right)$$

$$= f\left(\tfrac{1}{\sqrt{3}},0\right) + \frac{1}{2!}\left(\frac{-6}{\sqrt{3}}h^2 + 0\,hk + 2k^2\right)$$

$$= f\left(\tfrac{1}{\sqrt{3}},0\right) - \sqrt{3}h^2 + 2k^2 \begin{cases} < f\left(\tfrac{1}{\sqrt{3}},0\right) & \text{if } h \neq 0,\ k=0 \\ > f\left(\tfrac{1}{\sqrt{3}},0\right) & \text{if } h=0,\ k\neq 0 \end{cases}$$

$\left(\frac{1}{\sqrt{3}}, 0\right)$ is not a maximum or minimum

Graph of f near $\left(\frac{1}{\sqrt{3}}, 0\right)$ is like hyperbolic paraboloid.



Example: Show that $(0,0)$ is a critical point of $f(x,y) = xy + y^2e^x - 3x^2$, and determine if it is a minimum, maximum or neither.

$f_x(0,0) = y + y^2e^x - 6x \big|_{(0,0)} = 0$

$f_y(0,0) = x + 2ye^x \big|_{(0,0)} = 0$

so $(0,0)$ is a critical point.

$f_{xx}(0,0) = y^2e^x - 6 \big|_{(0,0)} = -6$

$f_{xy}(0,0) = 1 + 2ye^x \big|_{(0,0)} = 1$

$f_{yy}(0,0) = 2e^x \big|_{(0,0)} = 2$.

$H(0,0) = \begin{pmatrix} -6 & 1 \\ 1 & 2 \end{pmatrix}$

$D_1(0,0) = -6 < 0$

$D_2(0,0) = \begin{vmatrix} -6 & 1 \\ 1 & 2 \end{vmatrix} = -6 \cdot 2 - 1 \cdot 1 = -13 < 0$

this means determinant

$\therefore (0,0)$ is a saddle point

Alternatives to the second derivative test

If $D_n(\mathbf{a}) = 0$, so the second derivative test is inconclusive, or if it is inconvenient to calculate second derivatives, then:

- We can show $\mathbf{a}$ is a saddle point by finding a direction (not a point!) where $f(\mathbf{x}) > f(\mathbf{a})$ and another direction where $f(\mathbf{x}) < f(\mathbf{a})$ (below, ex. sheet #18 Q2).
- We can show $\mathbf{a}$ is a $\begin{array}{l}\text{maximum}\\ \text{minimum}\end{array}$ by showing that $\begin{array}{l} f(\mathbf{x}) \cancel{\geq} \leq f(\mathbf{a}) \\ f(\mathbf{x}) \cancel{\leq} \geq f(\mathbf{a}) \end{array}$ for all $\mathbf{x}$ close to $\mathbf{a}$ (p13).

Example: Classify the critical point $(0,0)$ of $f(x,y) = x^2 + y^5$.

$$\underbrace{H(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \text{ so } D_2(0,0) = 0 \text{ so the second derivative test is inconclusive}}$$

$f(0,0) = 0$

along $x=0$ $\quad f(x,y) = 0^2 + y^5 \begin{cases} > 0 = f(0,0) & \text{if } y > 0 \\ < 0 = f(0,0) & \text{if } y < 0 \end{cases}$

so $(0,0)$ is a saddle point

WRONG answer:

$f(0,1) = 1 > 0$

$f(0,-1) = -1 < 0$.

These points are not arbitrarily close to $(0,0)$.

Example: Classify the critical point $(0,0)$ of $f(x,y) = x^2y^2 + x^3y^2$.

near $(0,0)$, lower powers dominate $\therefore f(x,y) \approx x^2y^2 > 0 = f(0,0) \to$ probably a minimum

$f(x,y) = x^2y^2(1+x) \geq 0$ if $x \geq -1$

$\therefore$ close to $(0,0)$, $x \geq -1$ is true, so $f(x,y) \geq 0 = f(0,0)$.

so $(0,0)$ is a minimum.

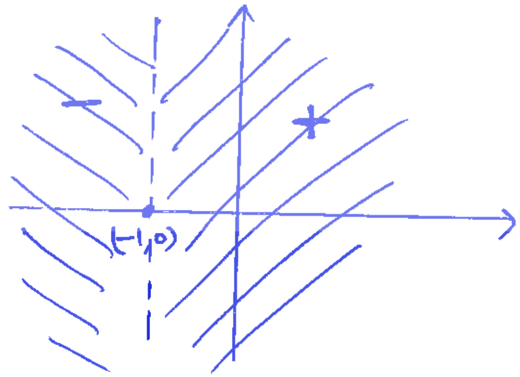
Algebraic manipulation such as factoring can also show that a certain point is a saddle point.

Example: Classify the critical point $(-1, 0)$ of $f(x, y) = x^2y^2 + x^3y^2$.

$\therefore$ No obvious path to make f increase or decrease : $f \equiv 0$ on $x=-1$, and $y=0$ but obvious factoring.

$f(-1,0) = 0$

$f(x,y) = x^2y^2(1+x) \begin{cases} >0 & \text{if } x>-1,\ y\neq 0 \\ <0 & \text{if } x<-1,\ y\neq 0 \end{cases}$



points of both types exist arbitrarily close to $(-1,0)$, so $(-1,0)$ is a saddle point