

## §12.5: Chain Rule

Recall the chain rule for single-variable functions:

$$\frac{d}{dt}f(x(t)) = f'(x(t))x'(t), \quad \text{i.e.} \quad \frac{df}{dt} = \frac{df}{dx} \frac{dx}{dt}.$$

Here's an informal way to understand the chain rule.

The linearisation of  $f$  says:  $f(x + \delta x) \approx f(x) + f'(x)\delta x. \quad (*)$

Write  $x + \delta x$  for  $x(t + \delta t)$ . Using the linearisation of  $x$ :

$$x + \delta x = x(t + \delta t) \approx x(t) + x'(t)\delta t$$

$$\delta x \approx x'(t)\delta t$$

Substituting into  $(*)$ :

$$f(x(t + \delta t)) \approx f(x(t)) + \boxed{f'(x(t))x'(t)}\delta t.$$

Compare the above to the linearisation of the composite function  $f(x(t))$ :

$$f(x(t + \delta t)) \approx f(x(t)) + \boxed{\frac{d}{dt}f(x(t))}\delta t.$$

So the quantities in the blue rectangles should be the same.

Now we derive a simple example of a multivariate chain rule in the same way.

Imagine that you are walking on  $\mathbb{R}^2$ , and your position at time  $t$  is  $(x(t), y(t))$ . The temperature at the point  $(x, y)$  is  $f(x, y)$ . So the temperature that you feel at time  $t$  is the composite function  $f(x(t), y(t))$ . What is  $\frac{d}{dt}f((x(t), y(t)))$ , the rate of change of temperature that you feel?

The linearisation of the temperature function is

$$f(x + \delta x, y + \delta y) \approx f(x, y) + \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y.$$

And the linearisations of  $x$  and  $y$  tell us that

$$\delta x \approx \frac{dx}{dt} \delta t; \quad \delta y \approx \frac{dy}{dt} \delta t.$$

Substituting into (\*)

$$f(x(t + \delta t), y(t + \delta t)) \approx f(x, y) + \frac{\partial f}{\partial x} \frac{dx}{dt} \delta t + \frac{\partial f}{\partial y} \frac{dy}{dt} \delta t.$$

Comparing with the linearisation of  $f(x(t), y(t))$ :

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

(This is not a rigorous proof because we haven't checked that the errors are small enough. We sketch a rigorous and more general version of this argument on p10. For a different rigorous proof, see the first page of §12.5 in the textbook.)

**Example:** Let  $f(x, y) = xy^2$ , and  $x = \ln t, y = 3t^2$ .  
Find  $\frac{df}{dt}$ .

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

We showed that, if  $f(x, y)$  is a 2-variable function, and  $x$  and  $y$  are functions of  $t$ , then

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

Now suppose  $x, y$  are multivariate functions, e.g.  $x(s, t), y(s, t)$ .

To find  $\frac{\partial f}{\partial t}$ , we treat  $s$  as a constant throughout, so

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t};$$

And similarly:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}.$$

**Example:** Let  $f(x, y) = xy^2$ , and  $x(\textcolor{red}{s}, \textcolor{red}{t}) = \ln(s + t)$ ,  
 $y(\textcolor{red}{s}, \textcolor{red}{t}) = 3t^2 \cos s$ . Find  $\frac{\partial f}{\partial s}$  and  $\frac{\partial f}{\partial s}(0, 1)$ .

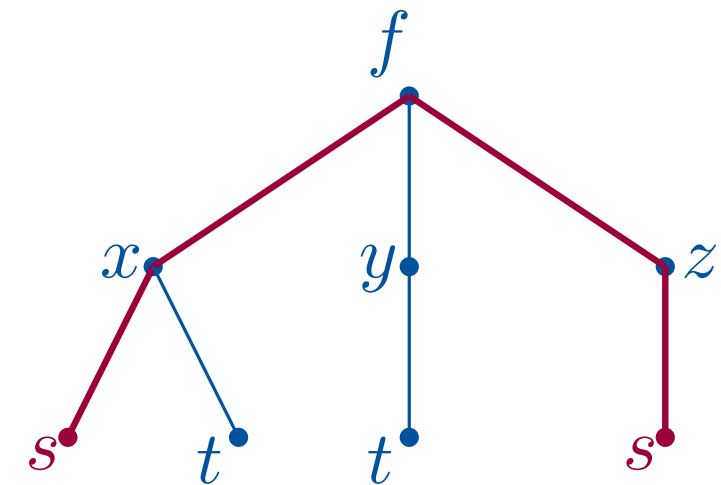
$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}.$$

In ex. sheet #15 Q2, we are given  $f(x, y, z)$  and  $x(s, t) = e^{st}$ ,  $y(s, t) = t^2$ ,  $z(s, t) = s^2 + 1$ . The chain rule says

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial s},$$

but the second term (in  $y$ ) is unnecessary because  $y$  does not depend on  $s$ . To simplify things in such cases, we can draw a **dependency chart** showing which functions depend on which variables. Then the terms in the chain rule for  $\frac{\partial f}{\partial s}$  correspond to all the paths from  $s$  to  $f$ .

Dependency charts can be really useful when there are many variables, or when dealing with a triple composition (e.g. if  $s$  and  $t$  here are functions of  $u, v, w$ ).



As in the 1D case, we can compute higher order derivatives of composite functions by applying the chain rule repeatedly.

**Example:** Let  $f(x, y)$  be a two variable function, and  $x = 2s + 3t, y = st$ . Find an expression for  $\frac{\partial^2}{\partial s \partial t} f(x(s, t), y(s, t))$  in terms of the partial derivatives of  $f$ .

## The chain rule in terms of Jacobian matrices and the derivative linear transformation

Remember from p4 that, for  $f(x, y)$ ,  $x(s, t)$ ,  $y(s, t)$ , we have

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}; \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}.$$

In the notation of Jacobian matrices, we have

$$Df(s, t) = \begin{pmatrix} \frac{\partial f}{\partial s} & \frac{\partial f}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = Df(x(s, t), y(s, t)) D\mathbf{g}(s, t),$$

writing  $\mathbf{g}(s, t)$  for  $(x(s, t), y(s, t))$  (i.e.  $g_1 = x$  and  $g_2 = y$ ).

In general, the Jacobian matrix of a composite function is the matrix product of the Jacobian matrices

$$D(\mathbf{f} \circ \mathbf{g})(\mathbf{t}) = D\mathbf{f}(\mathbf{g}(\mathbf{t})) D\mathbf{g}(\mathbf{t}).$$

Because the product of matrices correspond to the composition of linear transformations, this says that the derivative of a composition is a composition of the derivatives.



**Example:** Let  $\mathbf{g} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a function such that

$$\mathbf{g}(1, 2) = (1, 2, 1) \text{ and } D\mathbf{g}(1, 2) = \begin{pmatrix} 1/2 & 1/2 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let  $\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be given by  $\mathbf{f}(x, y, z) = (x^2 e^y, y^2 z)$ .  
Find  $D(\mathbf{f} \circ \mathbf{g})(1, 2)$ .

$$D(\mathbf{f} \circ \mathbf{g})(\mathbf{t}) = D\mathbf{f}(\mathbf{g}(\mathbf{t}))D\mathbf{g}(\mathbf{t}).$$

## Non-examinable: the proof of the chain rule (different from the textbook)

The main idea is the linearisation argument on pp1-2. We will show carefully that the errors in the linearisation are small compared to  $|\delta \mathbf{t}|$ , as required in the definition of the derivative.

We wish to show that  $D(\mathbf{f} \circ \mathbf{g})(\mathbf{t}) = D\mathbf{f}(\mathbf{g}(\mathbf{t}))D\mathbf{g}(\mathbf{t})$ . So we need to show that  $D\mathbf{f}(\mathbf{g}(\mathbf{t}))D\mathbf{g}(\mathbf{t})$  satisfies the definition of the derivative  $D(\mathbf{f} \circ \mathbf{g})$ , i.e.

$$\frac{(\mathbf{f} \circ \mathbf{g})(\mathbf{t} + \delta \mathbf{t}) - (\mathbf{f} \circ \mathbf{g})(\mathbf{t}) - [D\mathbf{f}(\mathbf{g}(\mathbf{t}))][D\mathbf{g}(\mathbf{t})]\delta \mathbf{t}}{|\delta \mathbf{t}|} \rightarrow 0 \text{ as } \delta \mathbf{t} \rightarrow \mathbf{0}.$$

Let  $\mathbf{x} = \mathbf{g}(\mathbf{t})$  and  $\mathbf{x} + \delta \mathbf{x} = \mathbf{g}(\mathbf{t} + \delta \mathbf{t})$ , and rewrite the expression above as

$$\begin{aligned} & \frac{\mathbf{f}(\mathbf{g}(\mathbf{t} + \delta \mathbf{t})) - \mathbf{f}(\mathbf{g}(\mathbf{t})) - [D\mathbf{f}(\mathbf{g}(\mathbf{t}))]\delta \mathbf{x}}{|\delta \mathbf{t}|} + \frac{[D\mathbf{f}(\mathbf{g}(\mathbf{t}))]\delta \mathbf{x} - [D\mathbf{f}(\mathbf{g}(\mathbf{t}))][D\mathbf{g}(\mathbf{t})]\delta \mathbf{t}}{|\delta \mathbf{t}|} \\ &= \underbrace{\frac{\mathbf{f}(\mathbf{x} + \delta \mathbf{x}) - \mathbf{f}(\mathbf{x}) - [D\mathbf{f}(\mathbf{x})]\delta \mathbf{x}}{|\delta \mathbf{x}|}}_{\text{goes to 0 because } D\mathbf{f} \text{ is the derivative of } \mathbf{f}.} \underbrace{\frac{|\delta \mathbf{x}|}{|\delta \mathbf{t}|}}_{\text{is finite because } \mathbf{x} = \mathbf{g} \text{ is differentiable.}} + [D\mathbf{f}(\mathbf{g}(\mathbf{t}))] \underbrace{\left( \frac{\delta \mathbf{x} - [D\mathbf{g}(\mathbf{t})]\delta \mathbf{t}}{|\delta \mathbf{t}|} \right)}_{\text{goes to 0 because } D\mathbf{g} \text{ is the derivative of } \mathbf{g} = \mathbf{x}.} \end{aligned}$$