Difficulty levels:

easy: 1abcefg; 2a,b; 3a,b; 4a; 6a,c

medium: 1d; 2c; 4b; 5a,b; 7

hard: 4c; 6b

4c, 6b, 7 are "think on your feet" questions – you are not supposed to "know" how to solve them.

PART A: ANSWER ONLY

Enter **only your final answer** in the boxes provided; if there is any ambiguity, you will not receive credit.

1. (a) (2 points) Calculate the indefinite integral

$$\int \frac{x^3 + x^5}{\sqrt{x}} dx.$$

$$= \int x^{5/2} + x^{9/2} \, dx$$

$$= \frac{x^{7/2}}{7/2} + \frac{x^{11/2}}{11/2} + C$$

Answer only

$$\frac{x^{7/2}}{7/2} + \frac{x^{11/2}}{11/2} + C$$

(b) (2 points) Calculate the indefinite integral

$$\int \sin(2x) + \cos(2x) dx.$$

$$= \frac{-\cos 2x}{2} + \frac{\sin 2x}{2} + C$$

Answer only

$$\frac{-\cos 2x}{2} + \frac{\sin 2x}{2} + C$$

(Question 1 continues on the next page.)

(c) (2 points) Calculate the definite integral

$$\int_0^2 \frac{5x^4 - 8}{\sqrt{x^5 - 8x + 9}} dx.$$

Simplify your answer as much as possible. (You may wish to double-check your arithmetic, since there is no partial credit available on this question.)

$$= \left[\frac{\sqrt{x^5 - 8x + 9}}{1/2} \right]_0^2$$

substitution $u = x^5 - 8x + 9$

$du = 5x^4 - 8$

$$= 2\left(\sqrt{32 - 16 + 9} - \sqrt{9} \right)$$

$$= 2(5-3)$$

$$= 4$$

Answer only

4

(Question 1 continues on the next page.)

(d) (4 points) Calculate the indefinite integral

$$\int \tan^3 x \sec x \, dx.$$

$$= \int \tan^2 x \, (\tan x \sec x) \, dx$$

$$= \int (\sec^2 x - 1)(\tan x \sec x \, dx)$$

substitution $u = \sec x$.

$$= \frac{\sec^3 x}{3} - \sec x + C$$

Answer only

$$\frac{\sec^3 x}{3} - \sec x + C$$

(Question 1 continues on the next page.)

(e) (4 points) Approximate the integral

$$\int_{-1}^{3} \arctan\left(\frac{1}{x+3}\right) dx$$

by a right Riemann sum with 2 subintervals.

$\Delta x = \frac{3-(-1)}{2} = 2$

$\therefore \quad x_0 = -1$
$x_1 = 1$
$x_2 = 3$

Answer only

$2\arctan\left(\frac{1}{1+3}\right) + 2\arctan\left(\frac{1}{3+3}\right)$

(f) (4 points) Consider the function

$$g(x) = \frac{d}{dx}\int_{3x}^{x^3} \sin(e^{2t})\, dt.$$

Give an expression for $g(x)$ without integral signs.

let $F(x)$ be an antiderivative of $\sin(e^{2x})$

$\therefore \ g(x) = \frac{d}{dx}\left(F(x^3) - F(3x)\right)$
$= \sin(e^{2x^3})\frac{d}{dx}(x^3) - \sin(e^{6x})\frac{d}{dx}(3x)$
$= 3x^2\sin(e^{2x^3}) - 3\sin(e^{6x})$

Answer only

$3x^2\sin(e^{2x^3}) - 3\sin(e^{6x})$

(Question 1 continues on the next page.)

(g) (4 points) The velocity of a particle at time t is given by

$$v(t) = t^3 - 1.$$

Calculate the **total distance** travelled by the particle in the time interval [-1,3]. **Simplify your answer as much as possible.** (You may wish to double-check your arithmetic, since there is no partial credit available on this question.)

Total distance $= \int_{-1}^{3} |v(t)|\, dt$

$v(t)>0$ when $t^3>1 \Rightarrow t>1$

$v(t)<0$ when $t^3<1 \Rightarrow t<1$

$$= \int_{-1}^{1} -t^3+1 \; dt + \int_{1}^{3} t^3-1 \; dt$$

$$= \left[-\frac{t^4}{4}+t\right]_{-1}^{1} + \left[\frac{t^4}{4}-t\right]_{1}^{3}$$

$$= -\frac{1}{4}+1+\frac{1}{4}-(-1)+\frac{81}{4}-3-\frac{1}{4}+1$$

$$= 20$$

Answer only

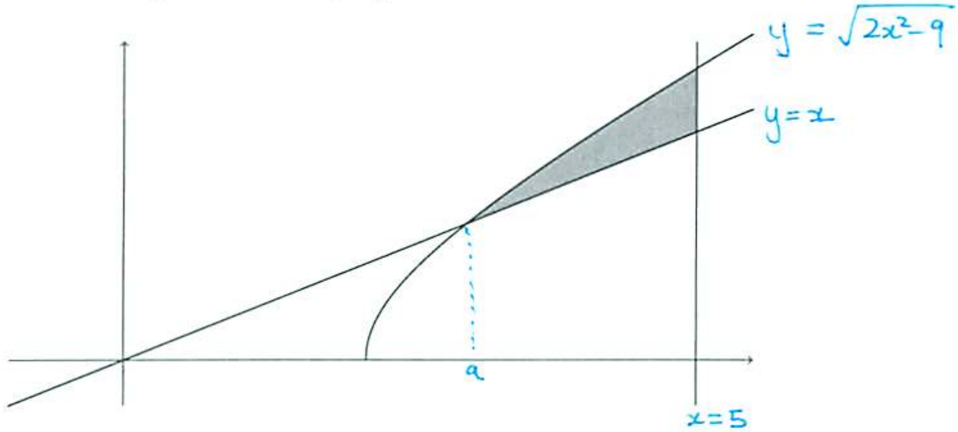
20

PART B: FULL SOLUTIONS REQUIRED

In order to receive full credit, please show all of your work and justify your answers.

2. (13 points) The shaded region in the diagram below is bounded by

$$y = \sqrt{2x^2 - 9}, \quad y = x \quad \text{and } x = 5.$$



(a) Calculate the volume of the solid formed by rotating this region **about the x-axis**.

a is the solution to $\sqrt{2x^2-9} = x$

$2x^2 - 9 = x^2$

$x^2 = 9 \Rightarrow x = 3$ or -3.

a is clearly positive so $a = 3$.

$\therefore$ volume $= \pi \int_3^5 \left(\sqrt{2x^2-9}\right)^2 - x^2 \, dx$ (by slicing)

$= \pi \int_3^5 2x^2 - 9 - x^2 \, dx$

$= \pi \int_3^5 x^2 - 9 \, dx$

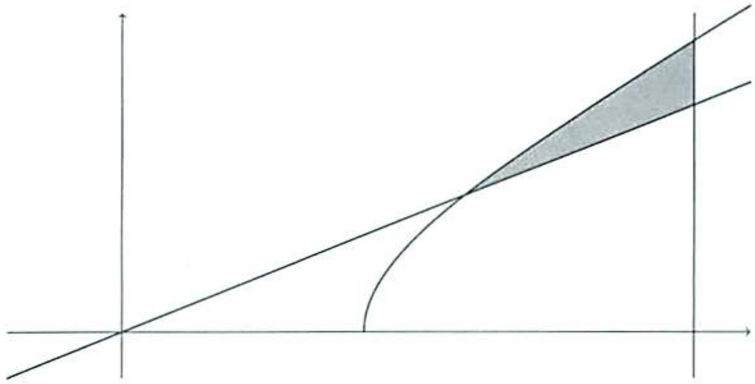
$= \pi \left[\frac{x^3}{3} - 9x \right]_3^5 = \pi \left(\frac{5^3}{3} - 9(5) - \frac{3^3}{3} + 9(3) \right)$ ← this line is fine as a final answer

$= \frac{44\pi}{3}$

(Question 2 continues on the next page.)

(b) For your convenience, here again is the diagram of the region bounded by

$$y = \sqrt{2x^2 - 9}, \quad y = x \quad \text{and } x = 5.$$



Calculate the volume of the solid formed by rotating this region **about the y-axis.**

$$\text{volume} = 2\pi \int_3^5 x\left(\sqrt{2x^2-9} - x\right)\,dx$$

$$= 2\pi \int_3^5 x\sqrt{2x^2-9} - x^2 \;dx$$

$$= 2\pi \int_3^5 x\sqrt{2x^2-9}\;dx - 2\pi\int_3^5 x^2\,dx$$

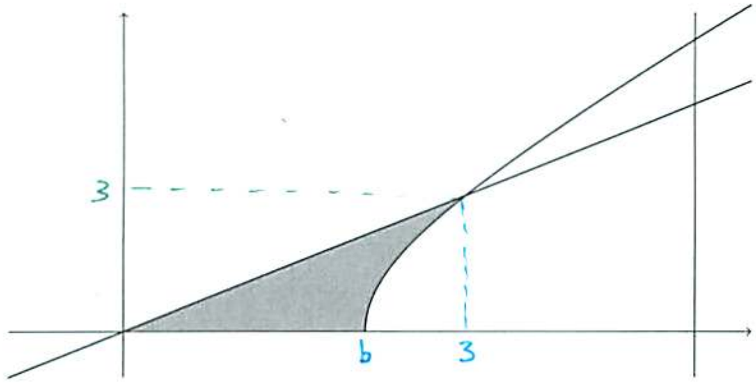
$$= 2\pi\left[\frac{(2x^2-9)^{3/2}}{3/2 \; 4}\right]_3^5 - 2\pi\left[\frac{x^3}{3}\right]_3^5 \qquad \begin{array}{l} u = 2x^2-9 \\ du = 4x \end{array}$$

$$= \frac{2\pi}{3}\left(\frac{41^{3/2}}{2} - \frac{9^{3/2}}{2} - 125 + 27\right) \qquad \leftarrow \text{this line is fine as final answer}$$

$$= \frac{\pi}{3}\left((41)^{3/2} - 223\right)$$

Here are the same curves and a **different** region, bounded by

$$y = \sqrt{2x^2 - 9}, \quad y = x \quad \text{and } y = 0.$$



Find an expression, in terms of integrals, for the area of the shaded region. You do **not** need to evaluate your expression.

b is the solution to $\sqrt{2x^2-9} = 0$

$$2x^2 - 9 = 0$$
$$2x^2 = 9$$
$$x = \frac{3}{\sqrt{2}} \text{ or } \frac{-3}{\sqrt{2}}$$ — but b is clearly positive so $b = \frac{3}{\sqrt{2}}$.

Area of shaded region $= \int_0^{3/\sqrt{2}} x \, dx + \int_{3/\sqrt{2}}^{3} x - \sqrt{2x^2-9} \, dx$.

Alternative:
$$y = \sqrt{2x^2-9}$$
$$y^2 = 2x^2 - 9$$
$$y^2 + 9 = 2x^2$$
$$\frac{y^2+9}{2} = x^2$$

$\therefore$ area of shaded region $= \int_0^3 \sqrt{\frac{y^2+9}{2}} - y \, dy$

3. (11 points) (a) Compute the following indefinite integral:

$$\int \frac{x^2}{(x+1)(x^2+1)}dx.$$

Partial fractions: $\dfrac{x^2}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$

$$x^2 = A(x^2+1) + (Bx+C)(x+1)$$

$x = -1$ $\quad 1 = A2 \quad \therefore A = \frac{1}{2}$

coefficient of x^2: $\quad 1 = A + B \quad \Rightarrow B = \frac{1}{2}$

coefficient of x: $\quad 0 = C + B \quad \Rightarrow C = -\frac{1}{2}$

$$\therefore \int \frac{x^2}{(x+1)(x^2+1)} dx = \int \frac{1}{2(x+1)} + \frac{x}{2(x^2+1)} - \frac{1}{2(x^2+1)} dx$$

$$= \frac{1}{2}\ln|x+1| + \frac{1}{4}\ln|x^2+1| - \frac{1}{2}\arctan(x) + C$$

(Question 3 continues on the next page.)

(b) Using part a or otherwise, compute the following improper integral, or explain why it diverges:

$$\int_0^\infty \frac{x^2}{(x+1)(x^2+1)}dx.$$

$$= \lim_{t\to\infty} \int_0^t \frac{x^2}{(x+1)(x^2+1)}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{1}{2}\ln|x+1| + \frac{1}{4}\ln|x^2+1| - \frac{1}{2}\arctan(x) \right]_0^t \quad \text{from part a.}$$

$$= \lim_{t\to\infty} \frac{1}{2}\ln|t+1| + \frac{1}{4}\ln|t^2+1| - \frac{1}{2}\arctan t - \frac{1}{2}\ln 1 - \frac{1}{4}\ln 1 - \frac{1}{2}\arctan 0$$

$$= \lim_{t\to\infty} \underbrace{\frac{1}{2}\ln|t+1|}_{\substack{\downarrow \\ \infty \\ \text{as } t\to\infty}} + \underbrace{\frac{1}{4}\ln|t^2+1|}_{\substack{\downarrow \\ \infty \\ \text{as } t\to\infty}} - \underbrace{\frac{1}{2}\arctan t}_{\frac{1}{2}\frac{\pi}{2} \text{ as } t\to\infty}$$

so this integral is divergent.

4. (20 points) (a) Compute the following improper integral, or explain why it diverges:

$$\int_{-4}^{0} \frac{1}{\sqrt{4+x}} dx.$$

$$= \lim_{t \to -4^+} \int_t^0 \frac{1}{\sqrt{4+x}} dx$$

substitution $u = 4+x$

$$= \lim_{t \to -4^+} \left[\frac{\sqrt{4+x}}{1/2} \right]_t^0$$

$$= 2\sqrt{4} - 2\sqrt{0} = 4$$

(Question 4 continues on the next page.)

(b) Compute the following indefinite integral:

$$\int \frac{1}{x^2\sqrt{4+x^2}}\,dx.$$

(Hint: You may find it easier to calculate this **without** using part a.)

$x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$

$\sqrt{4+x^2} = \sqrt{4+4\tan^2\theta}$

$= 2\sec\theta$

$$= \int \frac{1}{(2\tan\theta)^2(2\sec\theta)} 2\sec^2\theta \, d\theta$$

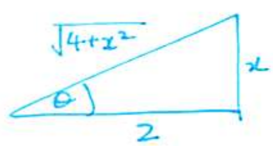
$$= \int \frac{\sec\theta}{4\tan^2\theta}\, d\theta$$

$$= \int \frac{\cos\theta}{4\sin^2\theta}\, d\theta$$

substitution $u = \sin\theta$

$$= \frac{1}{4}\,\frac{\sin^{-1}\theta}{-1} + C$$

$$= -\frac{1}{4}\,\frac{\sqrt{4+x^2}}{x} + C$$



$\tan\theta = \frac{x}{2}$

$\therefore \sin\theta = \frac{x}{\sqrt{4+x^2}}$

(Question 4 continues on the next page.)

(c) Compute the following indefinite integral:

$$\int \frac{1}{x\sqrt{4+x}}dx.$$

(Hint: You may wish to start with the substitution $u = \sqrt{4+x}$. You may find it easier to calculate this **without** using parts a and b.)

$u = \sqrt{4+x}$

$u^2 = 4+x$

$2u\,du = dx$

$x = u^2 - 4$

$$= \int \frac{1}{(u^2-4)u} 2u\,du$$

$$= \int \frac{2}{u^2-4}\,du$$

Partial fractions: $\frac{2}{u^2-4} = \frac{A}{u-2} + \frac{B}{u+2}$

$$2 = A(u+2) + B(u-2)$$

$u=2$: $\quad 2 = A4 \Rightarrow A = \frac{1}{2}$

$u=-2$: $\quad 2 = B(-4) \Rightarrow B = \frac{-1}{2}$

$$\therefore \int \frac{2}{u^2-4}\,du = \frac{1}{2}\int \frac{1}{(u-2)}\,du - \frac{1}{2}\int \frac{1}{u+2}\,du$$

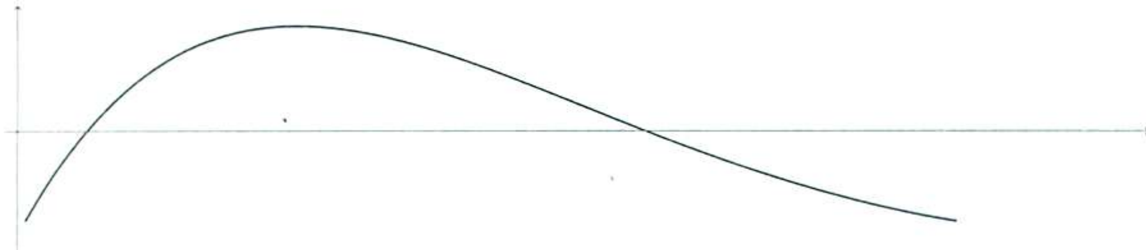
$$= \frac{1}{2}\ln|u-2| - \frac{1}{2}\ln|u+2| + C$$

$$= \frac{1}{2}\ln\left|\sqrt{4+x} - 2\right| - \frac{1}{2}\ln\left|\sqrt{4+x}+2\right| + C.$$

5. (14 points) Let C be the parametrised curve with equation

$$x = t^2, \quad y = -\cos t, \quad \frac{\pi}{6} \le t \le \frac{11\pi}{6},$$

as shown in the diagram below.



(a) Find the point(s) where C has a horizontal tangent.

C has a horizontal tangent when $\frac{dy}{dt} = 0$, $\frac{dx}{dt} \neq 0$

$\frac{dy}{dt} = \sin t$. On $\left[\frac{\pi}{6}, \frac{11\pi}{6}\right]$, $\sin t = 0$ has solutions $t = \pi$.

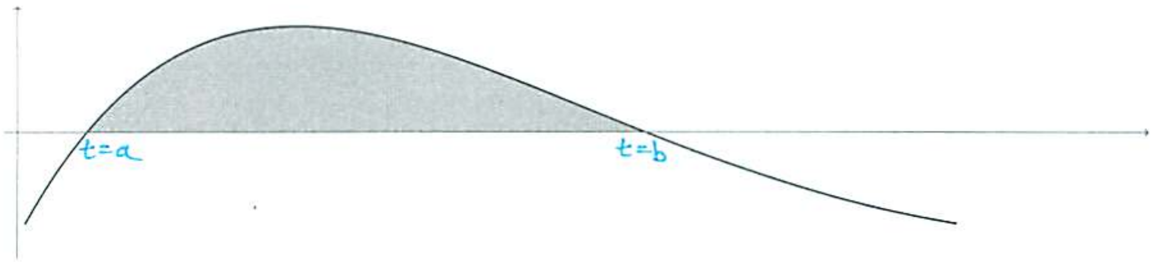
At $t = \pi$, $\frac{dx}{dt} = 2t = 2\pi \neq 0$.

$\therefore$ C has a horizontal tangent at $t = \pi$ i.e. $(\pi^2, 1)$.

(Question 5 continues on the next page.)

(b) For your convenience, here again is the information about the parametrised curve C:

$$x = t^2, \quad y = -\cos t, \quad \frac{\pi}{6} \le t \le \frac{11\pi}{6}.$$



Find the area of the shaded region. (Hint: if you cannot determine the limits of the associated integral, you can earn partial credit by computing the indefinite integral.)

(Note that $\frac{dx}{dt} = 2t > 0$, so the curve is being drawn from left to right – i.e. $a<b$)

$$\text{Area} = \int_a^b y \, \frac{dx}{dt}\, dt$$

where $t=a$, $t=b$ are the solutions to $y=0$

i.e. $\cos t = 0$

$\Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -\cos t\,(2t)\, dt$$

$$= \left[-2t\sin t\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -2\sin t\, dt$$

integration by parts:

$u = 2t \qquad dv = -\cos t\, dt$

$du = 2\, dt \qquad v = -\sin t$

$$= -3\pi(-1) + \pi(1) - \left[2\cos t\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= 3\pi + \pi - \left(2(0) - 2(0)\right) = 4\pi$$

6. (12 points)

(a) Determine whether the following series is convergent or divergent. Clearly state which tests you are using.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}.$$

Use integral test: $\frac{1}{n(\ln n)^3} = f(n)$ with $f(x) = \frac{1}{x(\ln x)^3}$.

for $x \geqslant 2$, $x > 0$ and $\ln x > 0$ $\therefore f(x) > 0$.

x, $\ln x$ are increasing functions $\therefore f(x)$ is decreasing.

$$\int_2^\infty \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^3}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{1}{(-2)(\ln x)^2}\right]_2^t \quad \text{substitution } u = \ln x.$$

$$= \lim_{t\to\infty} \frac{-1}{2(\ln t)^2} + \frac{1}{2(\ln 2)^2} = \frac{1}{2(\ln 2)^2}$$

because $\ln t \to \infty$ as $t \to \infty$.

so $\int_2^\infty \frac{1}{x(\ln x)^3}\,dx$ is convergent, so $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ is convergent.

(Question 6 continues on the next page.)

(b) Determine whether the following series is convergent or divergent. Clearly state which tests you are using.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3 + 1}.$$

Use comparison test with the series in part a:

$$\frac{1}{n(\ln n)^3+1} \leq \frac{1}{n(\ln n)^3} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3} \text{ converges,}$$

$$\text{so } \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3+1} \text{ converges.}$$

(Question 6 continues on the next page.)

(c) Determine whether the following series is convergent or divergent. Clearly state which tests you are using.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 + n}.$$

Use alternating series test: n^3+n is increasing in n,

so $\frac{1}{n^3+n}$ is decreasing in n.

also, $\lim_{n\to\infty} \frac{1}{n^3+n} = 0$

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+n}$ converges.

Alternative: To show that this series converges, it's enough to show that the series is absolutely convergent – that is, it's enough to show that $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^3+n}\right| = \sum_{n=1}^{\infty} \frac{1}{n^3+n}$ converges.

Use comparison test:

$\frac{1}{n^3+n} < \frac{1}{n^3}$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series with $p=3$)

so $\sum_{n=1}^{\infty} \frac{1}{n^3+n}$ converges.

(limit comparison test also works)

7. (8 points) Let Q be the polar curve with equation

$$r = \theta^2.$$

Compute the arc length of the part of Q between $(0, 0)$ and $(4, 2)$. **Simplify your answer as much as possible.**

(Hint: you may wish to use the identity $\sqrt{x^2 + x^k} = |x|\sqrt{1 + x^{k-2}}$, for $k \geq 2$.)

$$\text{arc length} = \int_0^2 \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

$$= \int_0^2 \sqrt{(2\theta)^2 + (\theta^2)^2}\, d\theta \qquad \frac{dr}{d\theta} = 2\theta$$

$$= \int_0^2 \sqrt{4\theta^2 + \theta^4}\, d\theta$$

$$= \int_0^2 |\theta| \sqrt{4 + \theta^2}\, d\theta \qquad \text{using identity in the hint}$$

$$= \int_0^2 \theta\sqrt{4 + \theta^2}\, d\theta \qquad \text{as } \theta > 0 \text{ for } 0 \leq \theta \leq 2$$

$$= \left[\frac{(4 + \theta^2)^{3/2}}{2 \cdot 3/2}\right]_0^2 \qquad \text{substitution } u = 4 + \theta^2,\ du = 2\theta\, d\theta$$

$$= \left(\frac{(4 + 4)^{3/2}}{3} - \frac{4^{3/2}}{3}\right)$$

$$= \frac{1}{3}\left(16\sqrt{2} - 8\right)$$

$$= \frac{16\sqrt{2}}{3} - \frac{8}{3}$$