Example: Let $f(x,y) = xy^2$, and $x = \ln t, y = 3t^2$.
Find $\dfrac{df}{dt}$.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Using multivariate chain rule: $\frac{\partial f}{\partial x} = y^2$, $\frac{\partial f}{\partial y} = 2xy$, $\frac{dx}{dt} = \frac{1}{t}$, $\frac{dy}{dt} = 6t$

$= (3t^2)^2$ $\quad$ $= 2(\ln t)(3t^2)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\therefore \quad = (3t^2)^2 \frac{1}{t} + 2(\ln t)(3t^2)\, 6t = 9t^3(1+4\ln t)$$

Alternative: in single variable calculus: $f(x(t), y(t)) = (\ln t)(3t^2)^2 = 9t^4 \ln t$

$$\frac{df}{dt} = 9(4t^3)\ln t + 9t^4\left(\frac{1}{t}\right) = 9t^3(1+4\ln t)$$

Example: Let $f(x,y) = xy^2$, and $x(s,t) = \ln(s+t)$, $y(s,t) = 3t^2 \cos s$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial s}(0,1)$.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

when $(s,t)=(0,1)$.

$$= y^2 \frac{1}{s+t} + 2xy\, 3t^2(-\sin s) = (3t^2\cos s)^2 \frac{1}{s+t} + 2\ln(s+t)\, 3t^2\cos s\, 3t^2(-\sin s)$$

To find $\frac{\partial f}{\partial s}(0,1)$:

$x(0,1) = \ln(0+1) = 0$

$y(0,1) = 3 \cdot 1^2 \cos 0 = 3$

$$\left.\frac{\partial f}{\partial s}\right|_{(0,1)} = \left. y^2\frac{1}{s+t} + 2xy\, 3t^2(-\sin s)\right|_{(0,1)} = 3^2\frac{1}{0+1} + 2\cdot 0\cdot 3\cdot 3\cdot 1^2(-\sin 0) = 9$$

As in the 1D case, we can compute higher order derivatives of composite functions by applying the chain rule repeatedly.

Example: Let $f(x, y)$ be a two variable function, and $x = 2s + 3t, y = st$. Find an expression for $\dfrac{\partial^2}{\partial s \partial t} f(x(s,t), y(s,t))$ in terms of the partial derivatives of f.

$$\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\right)$$

$$= \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x} 3 + \frac{\partial f}{\partial y} s\right)$$

product rule (in second term)

$$= 3\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial f}{\partial y}\frac{\partial}{\partial s}(s) + \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial y}\right) s$$

multivariate chain rule

$$= 3\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{\partial y}{\partial s}\right) + \frac{\partial f}{\partial y} 1 + \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial s}\right) s$$

$$= 3\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial s}\right) + \frac{\partial f}{\partial y} + \left(\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial s} + \frac{\partial^2 f}{\partial^2 y}\frac{\partial y}{\partial s}\right) s$$

$$D(\mathbf{f}\circ\mathbf{g})(\mathbf{t}) = D\mathbf{f}(\mathbf{g}(\mathbf{t}))D\mathbf{g}(\mathbf{t}).$$

Example: Let $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^3$ be a function such that

$$\mathbf{g}(1,2) = (1,2,1) \text{ and } D\mathbf{g}(1,2) = \begin{pmatrix} 1/2 & 1/2 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $\mathbf{f}(x,y,z) = (x^2e^y, y^2z)$.
Find $D(\mathbf{f}\circ\mathbf{g})(1,2)$.

$$D\vec{f}(\vec{g}(1,2)) = D\vec{f}(1,2,1) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix}\Bigg|_{(1,2,1)} = \begin{pmatrix} 2xe^y & x^2e^y & 0 \\ 0 & 2yz & y^2 \end{pmatrix}\Bigg|_{(1,2,1)}$$

$$= \begin{pmatrix} 2e^2 & e^2 & 0 \\ 0 & 4 & 4 \end{pmatrix}$$

so $$D(\vec{f}\circ\vec{g})(1,2) = D\vec{f}(\vec{g}(1,2))\,D\vec{g}(1,2) = \begin{pmatrix} 2e^2 & e^2 & 0 \\ 0 & 4 & 4 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2e^2\cdot\frac{1}{2} + e^2\cdot 2 & 2e^2\cdot\frac{1}{2}+e^2 \\ 4\cdot 2 + 4\cdot 1 & 4\cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3e^2 & 2e^2 \\ 12 & 4 \end{pmatrix}$$