You must justify your answers to receive full credit.

1. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis of $\mathbb{R}^3$, and suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation with

$$T(\mathbf{v_1}) = \mathbf{v_2}, \quad T(\mathbf{v_2}) = \mathbf{v_1}, \quad T(\mathbf{v_3}) = \mathbf{v_1} + \mathbf{v_3}.$$

   a) Show that $\mathbf{v_1} + \mathbf{v_2}$ is an eigenvector of T, and find its corresponding eigenvalue.
   b) Find the matrix for T relative to $\mathcal{B}$.

   Now let

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ -5 \\ 2 \end{bmatrix}.$$

   c) Find the change-of-coordinates matrix from the standard basis in $\mathbb{R}^3$ to $\mathcal{B}$.
   d) Find the standard matrix for T.

2. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be bases for a vector space V, and suppose

$$\mathbf{f}_1 = 2\mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3, \qquad \mathbf{f}_2 = 3\mathbf{b}_2 + \mathbf{b}_3, \qquad \mathbf{f}_3 = -3\mathbf{b}_1 + 2\mathbf{b}_3$$

   a) What is the dimension of V?
   b) Find the change-of-coordinates matrix from $\mathcal{F}$ to $\mathcal{B}$.
   c) Find $[\mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3]_{\mathcal{B}}$.

3. Let A be the matrix

$$A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}.$$

   a) Diagonalise A, i.e find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.
   b) Using your answer to part a), find a matrix B such that $B^2 = A$. **You may give your answer as a product of matrices and/or their inverses.** (Hint: first find a matrix C such that $C^2 = D$.)

4. Are the following matrices diagonalisable? Calculate as little as possible, and explain your answers.

a) $A = \begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 2 \\ -2 & 0 & 2 \end{bmatrix}$ b) $B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & -1 \\ 0 & 1 & 5 \end{bmatrix}$ c) $C = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

5. Let A be a 2 x 2 matrix.

   a) Explain why there is a non-zero polynomial p of degree at most 4 (i.e. $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$) such that $p(A) = 0$. (Hint: think about linear independence of the set $\{I, A, A^2, \dots\}$ in the vector space of 2 x 2 matrices.)

   b) **Optional** (challenging, within syllabus): Show that, if $a_0 \neq 0$, then A is invertible and A^{-1} is a polynomial in A.

6. State whether each of the following statements is (always) true or (sometimes) false. If it is true, give a brief justification; if it is false, give a counterexample with an explanation.

   a) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ spans a vector space V, then any 7 vectors in V are linearly dependent.

   b) If A is 4×7 matrix and $\text{rank}A = 4$, then $\text{Col}A = \mathbb{R}^4$.

   c) If A is 4×7 matrix and $\text{rank}A = 4$, then $\text{Nul}A = \mathbb{R}^3$.

   d) Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be a linear transformation that is onto. Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors such that $T(\mathbf{u}) = T(\mathbf{v}) = T(\mathbf{w}) = \mathbf{0}$. Then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.

   e) If A is a diagonal matrix, then A is diagonalisable.

   f) Let A and B be square matrices. If A is similar to B, and B is diagonalisable, then A is diagonalisable.

7. **Optional** (challenging, within syllabus): The goal of this exercise is to give an alternate proof of the Rank-Nullity Theorem for general vector spaces, i.e., without using row reduction. For this exercise, let V and W be vector spaces with V finite dimensional, and let $T : V \to W$ be a linear transformation. The equality we would like to prove is

$$\dim(\text{kernel}(T)) + \dim(\text{range}(T)) = \dim(V). \tag{$*$}$$

   a) Explain why this coincides with the Rank-Nullity theorem in the lecture notes when $T(\mathbf{x}) = A\mathbf{x}$.

   Prove $(*)$ by completing the following steps.

   b) By citing an appropriate theorem, explain why $\dim(\text{kernel}(T))$ is finite. Let $\{\mathbf{z}_1, \dots, \mathbf{z}_k\}$ be a basis of $\text{kernel}(T)$, so that $\dim(\text{kernel}(T)) = k$.

   c) Show that $\text{range}(T)$ is finite dimensional by finding a finite spanning set. (Hint: start with a basis for V and look at what the linear transformation T does to it.)

   d) Let $\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$ be a basis for $\text{range}(T)$, so that $\dim(\text{range}(T)) = r$. Explain why there are vectors $\mathbf{x}_1, \dots, \mathbf{x}_r$ in V such that $T(\mathbf{x}_i) = \mathbf{w}_i$ for $i = 1, \dots, r$.

e) We now show that the set $\mathcal{B} = \{\mathbf{x}_1, \ldots, \mathbf{x}_r, \mathbf{z}_1, \ldots, \mathbf{z}_k\}$ forms a basis for V. Suppose there are weights $c_1, \ldots, c_r, d_1, \ldots, d_k \in \mathbb{R}$ such that

$$c_1\mathbf{x}_1 + \cdots + c_r\mathbf{x}_r + d_1\mathbf{z}_1 + \cdots + d_k\mathbf{z}_k = 0. \tag{$\dagger$}$$

Show that this implies (i) $c_1 = \cdots = c_r = 0$, and (ii) $d_1 = \cdots = d_k = 0$, so $\mathcal{B}$ is linearly independent. (Hint: First show (i) by applying the linear transformation T to ($\dagger$) and using the fact that the $\mathbf{w}_i$ form a basis.)

f) Show that $\mathcal{B}$ spans V. (Hint: Let $\mathbf{v}$ be an arbitrary vector in V. By considering $T(\mathbf{v})$, show that $\mathbf{v}$ can be written as a linear combination of the $\mathbf{x}_i$, plus something in the kernel of T.)

g) Conclude that $\mathcal{B}$ is a basis for V. What is $\dim(V)$? Why does this prove the equality $(*)$?

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