Recall from last week:

FACT: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

i) if $ad - bc \neq 0$, then A is invertible and $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$,

ii) if $ad - bc = 0$, then A is not invertible,

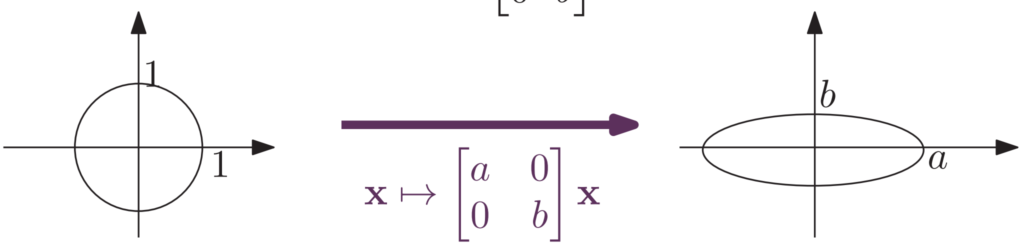
What is the mysterious quantity $ad - bc$?

§3.1-3.3: Determinants

Conceptually, the determinant $\det A$ of a square $n \times n$ matrix A is the signed area/volume scaling factor of the linear transformation $T(\mathbf{x}) = A\mathbf{x}$, i.e.:

- For any region S in $\mathbb{R}^n$, the volume of its image $T(S)$ is $|\det A|$ multiplied by the original volume of S,
- If $\det A > 0$, then T does not change "orientation". If $\det A < 0$, then T changes "orientation".

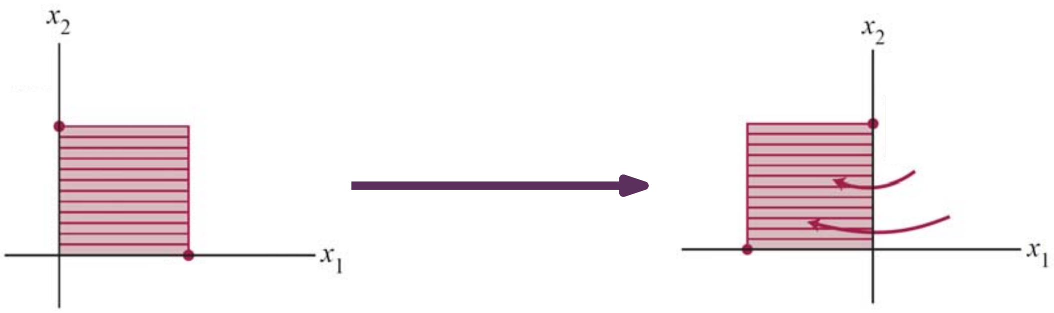
Example: Area of ellipse $= \det \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \times$ area of unit circle $= ab\pi$.



This idea is useful in multivariate calculus.

Formula for 2×2 matrix: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.

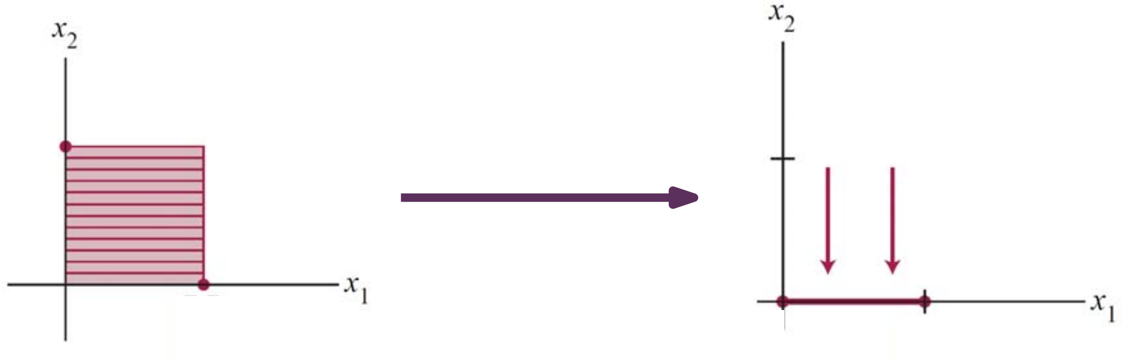
Example: The standard matrix for reflection through the x_2-axis is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.
Its determinant is $-1 \cdot 1 - 0 \cdot 0 = -1$: reflection does not change area, but changes orientation.



Exercise: Guess what the determinant of a rotation matrix is, and check your answer.

Formula for 2×2 matrix: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.

Example: The standard matrix of projection onto the x_1-axis is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Its determinant is $1 \cdot 0 - 0 \cdot 0 = 0$. Projection sends the unit square to a line, which has zero area.



Theorem: A is invertible if and only if $\det A \neq 0$.

Calculating Determinants

Notation: A_{ij} is the submatrix obtained from matrix A by deleting the ith row and jth column of A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \\ & & \end{bmatrix}$$

Recall that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and we let $\det[a] = a$.

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

EXAMPLE: Compute the determinant of $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

THEOREM 1 The determinant of an $n \times n$ matrix A can be computed by expanding across any row or down any column:

$$\det A = (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} + \cdots + (-1)^{i+n} a_{in} \det A_{in}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$ (expansion across row i)

$$\det A = (-1)^{1+j} a_{1j} \det A_{1j} + (-1)^{2+j} a_{2j} \det A_{2j} + \cdots + (-1)^{n+j} a_{nj} \det A_{nj}$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$ (expansion down column j)

Use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

EXAMPLE: An easier way to compute the determinant of $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

EXAMPLE:

$$\begin{vmatrix} 4 & 3 & 1 & 8 \\ 5 & 0 & 3 & -1 \\ 0 & 0 & -3 & 0 \\ 7 & 0 & 2 & 4 \end{vmatrix} =$$

It's easy to compute the determinant of a triangular matrix:

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$

(upper triangular) (lower triangular)

EXAMPLE:

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{vmatrix} =$$

THEOREM 2: If A is a triangular matrix, then $\det A$ is the product of the diagonal entries of A.

How the determinant changes under row operations:

1. Replacement: add a multiple of one row to another row. $R_i \to R_i + cR_j$
   determinant does not change.
2. Interchange: interchange two rows. $R_i \to R_j,\ R_j \to R_i$
   determinant changes sign.
3. Scaling: multiply all entries in a row by a nonzero constant. $R_i \to cR_i,\ c \neq 0$
   determinant scales by a factor of c.

To help you remember:

original	after replacement	after interchange	after scaling
$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$	$\begin{vmatrix} 1 & c \\ 0 & 1 \end{vmatrix} = 1,$	$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1,$	$\begin{vmatrix} c & 0 \\ 0 & 1 \end{vmatrix} = c.$

Because we can compute the determinant by expanding down columns instead of across rows, the same changes hold for "column operations".

1. Replacement: $R_i \to R_i + cR_j$ determinant does not change.
2. Interchange: $R_i \to R_j,\ R_j \to R_i$ determinant changes sign.
3. Scaling: $R_i \to cR_i,\ c \neq 0$ determinant scales by a factor of c.

Usually we compute determinants using a mixture of "expanding across a row or down a column with many zeroes" and "row reducing to a triangular matrix".

Example:

$$\begin{vmatrix} 2 & 3 & 4 & 6 \\ 0 & 5 & 0 & 0 \\ 5 & 5 & 6 & 7 \\ 7 & 9 & 6 & 10 \end{vmatrix} = 5 \begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} \overset{\text{factor out 2 from } R_1}{=} 5 \cdot 2 \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} \overset{\substack{R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 7R_1}}{=} 5 \cdot 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix}$$

$$\overset{\text{factor out -4 from } R_2}{=} 5 \cdot 2 \cdot -4 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} \overset{R_3 \to R_3 + 8R_2}{=} 5 \cdot 2 \cdot -4 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix} = 5 \cdot 2 \cdot -4 \cdot 1 \cdot 1 \cdot 5 = -200.$$

1. Replacement: $R_i \to R_i + cR_j$ determinant does not change.
2. Interchange: $R_i \to R_j$, $R_j \to R_i$ determinant changes sign.
3. Scaling: $R_i \to cR_i$, $c \neq 0$ determinant scales by a factor of c.

Useful fact: If two rows of A are multiples of each other, then $\det A = 0$.

Proof: Use a replacement row operation to make one of the rows into a row of zeroes, then expand along that row.

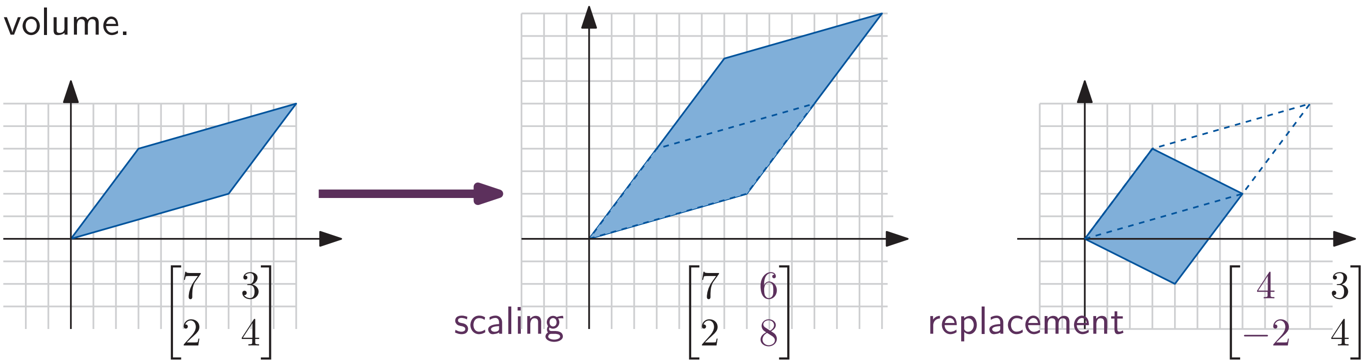
Example:

$$\begin{vmatrix} 1 & 3 & 4 \\ 5 & 9 & 3 \\ 2 & 6 & 8 \end{vmatrix} \overset{R_3 \to R_3 - 2R_1}{=} \begin{vmatrix} 1 & 3 & 4 \\ 5 & 9 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0 \begin{vmatrix} 3 & 4 \\ 9 & 3 \end{vmatrix} - 0 \begin{vmatrix} 1 & 4 \\ 5 & 3 \end{vmatrix} + 0 \begin{vmatrix} 1 & 3 \\ 5 & 9 \end{vmatrix} = 0.$$

Why does the determinant change like this under row and column operations? Two views:

Either: It is a consequence of the expansion formula in Theorem 1;

Or: By thinking about the signed volume of the image of the unit cube under the associated linear transformation:
2. Interchanging columns changes the orientation of the image of the unit cube.
3. Scaling a column applies an expansion to one side of the image of the unit cube.
1. Column replacement rearranges the image of the unit cube without changing its volume.



Properties of the determinant:

$\det(A^T) = \det A$.

Theorem 6: Determinants are multiplicative:

$$\det(AB) = \det A \det B.$$

In particular:

$$\det(A^{-1}) = \frac{\det I_n}{\det A} = \frac{1}{\det A}, \quad \det(cA) = \det \begin{bmatrix} c & & 0 \\ & \ddots & \\ 0 & & c \end{bmatrix} \det A = c^n \det A.$$

Properties of the determinant:

Theorem 4: Invertibility and determinants: A square matrix A is invertible if and only if $\det A \neq 0$.

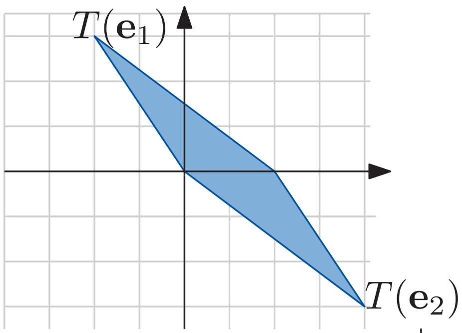
Proof 1: By the Invertible Matrix Theorem, A is invertible if and only if rref(A) has n pivots. Row operations multiply the determinant by nonzero numbers. So $\det A = 0$ if and only if $\det(\text{rref}(A)) = 0$, which happens precisely when rref(A) has fewer than n pivots.

Proof 2: By the Invertible Matrix Theorem, A is invertible if and only if its columns span $\mathbb{R}^n$. Since the image of the unit cube is a subset of the span of the columns, this image has zero volume if the columns do not span $\mathbb{R}^n$.

So we can use determinants to test whether $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in $\mathbb{R}^n$ is linearly independent, or if it spans $\mathbb{R}^n$: it does when $\det\left(\begin{bmatrix} | & & | \\ \mathbf{v}_1 & \ldots & \mathbf{v}_n \\ | & & | \end{bmatrix}\right) \neq 0$.

Other applications: finding volumes of regions with determinants

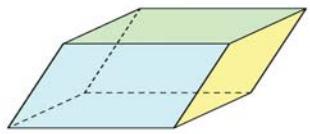
Example: Find the area of the parallelogram with vertices $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-2\\3\end{bmatrix}, \begin{bmatrix}4\\-3\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}$.



Answer: This parallelogram is the image of the unit square under a linear transformation T with $T(\mathbf{e}_1) = \begin{bmatrix}-2\\3\end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix}4\\-3\end{bmatrix}$.

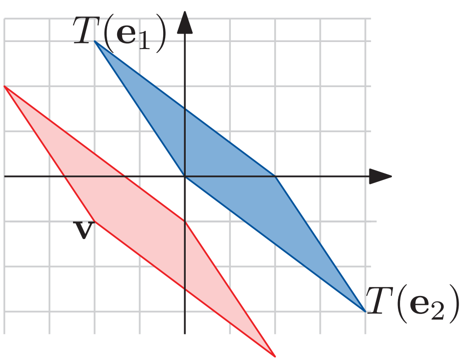
So area of parallelogram $= \left|\det \begin{bmatrix}-2 & 4\\3 & -3\end{bmatrix}\right| \times$ area of unit square $= |-6| \cdot 1 = 6.$

This works for any parallelogram where the origin is one of the vertices (and also in $\mathbb{R}^3$, for parallelopipeds).



Other applications: finding volumes of regions with determinants

Example: Find the area of the parallelogram with vertices $\begin{bmatrix}-2\\-1\end{bmatrix}, \begin{bmatrix}-4\\2\end{bmatrix}, \begin{bmatrix}2\\-4\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}$.



Answer: Use a translation to move one of the vertices of the parallelogram to the origin - this does not change the area.

The formula for this translation function is $\mathbf{x} \mapsto \mathbf{x} - \mathbf{v}$, where $\mathbf{v}$ is one of the vertices of the parallelogram.

Here, the vertices of the translated parallelogram are $\begin{bmatrix}-2\\-1\end{bmatrix} - \begin{bmatrix}-2\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}-4\\2\end{bmatrix} - \begin{bmatrix}-2\\-1\end{bmatrix} = \begin{bmatrix}-2\\3\end{bmatrix}, \quad \begin{bmatrix}2\\-4\end{bmatrix} - \begin{bmatrix}-2\\-1\end{bmatrix} = \begin{bmatrix}4\\-3\end{bmatrix}, \quad \begin{bmatrix}0\\-1\end{bmatrix} - \begin{bmatrix}-2\\-1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}.$

So, by the previous example, the area of the parallelogram is 6.

Other applications: solving linear systems using determinants

Cramer's rule: Let A be an invertible $n \times n$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{\det \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \ldots & \mathbf{b} & \ldots & \mathbf{a}_n \\ | & | & | & | & | \end{bmatrix}}{\det A}.$$

put $\mathbf{b}$ in the ith column instead of $\mathbf{a}_i$

Proof:

$$A \begin{bmatrix} | & | & | & | & | \\ \mathbf{e}_1 & \ldots & \mathbf{x} & \ldots & \mathbf{e}_n \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ A\mathbf{e}_1 & \ldots & A\mathbf{x} & \ldots & A\mathbf{e}_n \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \ldots & \mathbf{b} & \ldots & \mathbf{a}_n \\ | & | & | & | & | \end{bmatrix}.$$

So

$$\det A \det \begin{bmatrix} | & | & | & | & | \\ \mathbf{e}_1 & \ldots & \mathbf{x} & \ldots & \mathbf{e}_n \\ | & | & | & | & | \end{bmatrix} = \det \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \ldots & \mathbf{b} & \ldots & \mathbf{a}_n \\ | & | & | & | & | \end{bmatrix}.$$

this is x_i - expand along ith row

Applying Cramer's rule to $\mathbf{b} = \mathbf{e}_i$ gives a formula for each entry of A^{-1} (see Theorem 8 in textbook; this formula is called the adjugate or classical adjoint).

The 2×2 case of this formula is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Cramer's rule is much slower than row-reduction for linear systems with actual numbers, but is useful for obtaining theoretical results.

Example: If every entry of A is an integer and $\det A = 1$ or -1, then every entry of A^{-1} is an integer.

Proof: Cramer's rule tells us that every entry of A^{-1} is the determinant of an integer matrix divided by $\det A$. And the determinant of an integer matrix is an integer.

Exercise: using the fact $\det AB = \det A \det B$, prove the converse (if every entry of A and of A^{-1} is an integer, then $\det A = 1$ or -1).