Example: Evaluate $\int \cos(x^3)\, 3x^2\, dx$.

$$\int f(u)\frac{du}{dx}\, dx = \int f(u)\, du.$$

$$= \int \cos(u) \frac{du}{dx}\, dx$$

$u = x^3$

$\frac{du}{dx} = 3x^2$

$$= \int \cos(u)\, du$$

$$= \sin u + C$$

$$= \sin(x^3) + C$$

Example: Evaluate $\int e^{3x}\,dx.$

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du.$$

$u = 3x$

$\frac{du}{dx} = 3$

$$= \int \frac{1}{3} e^{3x} (3\,dx)$$

$$= \int \frac{1}{3} e^{3x} \frac{du}{dx}\,dx$$

$$= \int \frac{1}{3} e^{u}\,du$$

$$= \frac{1}{3} e^{u} + C$$

$$= \frac{1}{3} e^{3x} + C$$

With some practice, you won't need to write this out:

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C$$

$$\int \cos(ax)\,dx = \frac{\sin ax}{a} + C \quad \text{etc.}$$

Example: Evaluate $\int x\sqrt{1+x^2}\,dx$.

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du.$$

No obvious algebraic reorganisation

$u = x^2$

$du = 2x\,dx$ differentiate both sides

$\frac{du}{2x} = dx$

$$= \int \cancel{x}\sqrt{1+x^2}\,\frac{du}{2\cancel{x}}$$

$$= \int \frac{\sqrt{1+u}}{2}\,du$$

The antiderivative is not obvious.
We can either: start again with a different substitution (p11)
or: use another technique

$t = 1+u$

$dt = du$

$$= \int \frac{t^{1/2}}{2}\,dt$$

$$= \frac{t^{3/2}}{2\,3/2} + C = \frac{1}{3}(1+u)^{3/2} + C = \frac{1}{3}(1+x^2)^{3/2} + C.$$

The substitution $u = 1+x^2$ would've been faster (next page). But there's no unique correct method