


From the beginning of last week:

Remember from calculus the addition and scalar multiplication of polynomials:

$$\text{e.g. } (2t^2 + 1) + (-t^2 + 3t + 2) = t^2 + 3t + 3.$$

$$\text{e.g. } (-3)(-t^2 + 3t + 2) = 3t^2 - 9t - 6.$$



We want to represent abstract vectors as column vectors so we can do calculations (e.g. row-reduction) to study linear systems (e.g. week 7 p21) and linear transformations (e.g. week 7 p44).

Is this really different from  $\mathbb{R}^3$ ?

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}.$$

$$(-3) \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \\ 3 \end{bmatrix}.$$

← coefficient of 1  
← coefficient of  $t$   
← coefficient of  $t^2$

We want to represent abstract vectors as column vectors so we can do calculations (e.g. row-reduction) to study linear systems and linear transformations.

$$\text{In } \mathbb{R}^n, \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \mathbf{e}_1 + \cdots + x_n \mathbf{e}_n.$$

We can copy this idea: in  $V$ , pick a special set of vectors  $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ , write each  $\mathbf{v}$  in  $V$  as  $c_1 \mathbf{b}_1 + \cdots + c_n \mathbf{b}_n$  and represent  $\mathbf{v}$  by the column vector  $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ .

**Example:** In  $\mathbb{P}_2$ , let  $\mathbf{b}_1 = 1, \mathbf{b}_2 = t, \mathbf{b}_3 = t^2$ .

Then we represent  $a_0 + a_1 t + a_2 t^2$  by  $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$  (see previous page).

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must span  $V$



We can copy this idea: in  $V$ , pick a special set of vectors  $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ , write **each**

$\mathbf{x}$  in  $V$  **uniquely** as  $c_1 \mathbf{b}_1 + \cdots + c_n \mathbf{b}_n$  and represent  $\mathbf{x}$  by the column vector  $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ .

$\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  must be  
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## §4.3: Bases

**Definition:** Let  $W$  be a subspace of a vector space  $V$ . An indexed set of vectors  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$  in  $V$  is a *basis for  $W$*  if

- i  $\mathcal{B}$  is a linearly independent set, and
- ii  $\text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\} = W$ .

↑  
The order matters:  
 $\{\mathbf{b}_1, \mathbf{b}_2\}$  and  $\{\mathbf{b}_2, \mathbf{b}_1\}$   
are different bases.

i means: The only solution to  $x_1\mathbf{b}_1 + \dots + x_p\mathbf{b}_p = \mathbf{0}$  is  $x_1 = \dots = x_p = 0$ .

ii means:  $W$  is the set of vectors of the form  $c_1\mathbf{b}_1 + \dots + c_p\mathbf{b}_p$  where  $c_1, \dots, c_p$  can take any value.

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Condition ii implies that  $\mathbf{b}_1, \dots, \mathbf{b}_p$  must be in  $W$ , because  $\text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$  contains each of  $\mathbf{b}_1, \dots, \mathbf{b}_p$ .

Every vector space  $V$  is a subspace of itself, so we can take  $W = V$  in the definition and talk about bases for  $V$ .

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**Example:** The *standard basis* for  $\mathbb{R}^3$  is  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ , where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

To check that this is a basis:  $\begin{bmatrix} | & | & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  is in reduced echelon form.

The matrix has a pivot in every column, so its columns are linearly independent.

The matrix has a pivot in every row, so its columns span  $\mathbb{R}^3$ .

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A basis for  $W$  is **not** unique: (different bases are useful in different situations, see week 9).

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**Example:** Let  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ . Is  $\{\mathbf{v}_1, \mathbf{v}_2\}$  a basis for  $\mathbb{R}^3$ ?

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**Answer:** No, because two vectors cannot span  $\mathbb{R}^3$ :  $\begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$  cannot have a pivot in every row.



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**Answer:** Form the matrix  $A = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ . Because

$\det A = 1 \neq 0$ , the matrix  $A$  is invertible, so (by Invertible Matrix Theorem) its columns are linearly independent and its columns span  $\mathbb{R}^3$ .

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basis for  $\mathbb{R}^3$ ?

**Answer:** No, because four vectors in  $\mathbb{R}^3$  must be linearly dependent:

$\begin{bmatrix} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$  cannot have a pivot in every column.

By the same logic as in the above examples:

**Fact:**  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is a basis for  $\mathbb{R}^n$  if and only if:

- $p = n$  (i.e. the set has exactly  $n$  vectors), and

- $\det \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_n \\ | & & | \end{bmatrix} \neq 0.$

Fewer than  $n$  vectors: not enough vectors, can't span  $\mathbb{R}^n$ .  
More than  $n$  vectors: too many vectors, linearly dependent.

**Example:** The **standard basis** for  $\mathbb{P}_n$  is  $\mathcal{B} = \{1, t, t^2, \dots, t^n\}$ .

To check that this is a basis:

ii By definition of  $\mathbb{P}_n$ , every element of  $\mathbb{P}_n$  has the form

$a_0 + a_1t + a_2t^2 + \dots + a_nt^n$ , so  $\mathcal{B}$  spans  $\mathbb{P}_n$ .

i To see that  $\mathcal{B}$  is linearly independent, we show that  $c_0 = c_1 = \dots = c_n = 0$  is the only solution to

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Substitute  $t = 0$ : we find  $c_0 = 0$ .

Differentiate, then substitute  $t = 0$ : we find  $c_1 = 0$ .

Differentiate again, then substitute  $t = 0$ : we find  $c_2 = 0$ .

Repeating many times, we find  $c_0 = c_1 = \dots = c_n = 0$ .

Once we have the standard basis of  $\mathbb{P}_n$ , it will be easier to check if other sets are bases of  $\mathbb{P}_n$ , using **coordinates** (later, p14).

Advanced exercise: what do you think is the standard basis for  $M_{m \times n}$ ?

One way to make a basis for  $V$  is to start with a set that spans  $V$ .

**Theorem 5: Spanning Set Theorem:** If  $V = \text{Span} \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ , then some subset of  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is a basis for  $V$ .

**Proof:** basically, the idea of the casting-out algorithm (week 7 p29-35) works in abstract vector spaces too.

- If  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is linearly independent, it is a basis for  $V$ .
- If  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is linearly dependent, then one of the  $\mathbf{v}_i$ s is a linear combination of the others. Removing this  $\mathbf{v}_i$  from the set still gives a set that spans  $V$ . Continue removing vectors in this way until the remaining vectors are linearly independent.

**Example:**  $\mathbb{P}_2 = \text{Span} \{5, 3 + t, 1 + 2t^2, 4 + 2t - 4t^2\}$ , but this set is not linearly independent because  $4 + 2t - 4t^2$  is a linear combination of the other polynomials:  $4 + 2t - 4t^2 = 2(3 + t) - 2(1 + 2t^2)$ . So remove  $4 + 2t - 4t^2$  to get the set  $\{5, 3 + t, 1 + 2t^2\}$ , which is in fact a basis (we can show this with coordinates, p14-15).



# PP 234, 238-240 (§4.4), 307-308 (§5.4): Coordinates

Recall (p2) that our motivation for finding a basis is because we want to write each vector  $\mathbf{x}$  as  $c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$  in a unique way. Let's show that this is indeed possible

**Theorem 7: Unique Representation Theorem:** Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  be a basis for a vector space  $V$ . Then for each  $\mathbf{x}$  in  $V$ , there exists a unique set of scalars  $c_1, \dots, c_n$  such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

**Proof:**

Since  $\mathcal{B}$  spans  $V$ , there exists scalars  $c_1, \dots, c_n$  such that the above equation holds.

Suppose  $\mathbf{x}$  has another representation

$$\mathbf{x} = d_1\mathbf{b}_1 + \cdots + d_n\mathbf{b}_n.$$

for some scalars  $d_1, \dots, d_n$ . Then

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \cdots + (c_n - d_n)\mathbf{b}_n.$$

Because  $\mathcal{B}$  is linearly independent, all the weights in this equation must be zero, i.e.  $(c_1 - d_1) = \cdots = (c_n - d_n) = 0$ . So  $c_1 = d_1, \dots, c_n = d_n$ .

Because of the Unique Representation Theorem, we can make the following definition:

**Definition:** Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  be a basis for  $V$ . Then, for any  $\mathbf{x}$  in  $V$ , the *coordinates of  $\mathbf{x}$  relative to  $\mathcal{B}$* , or the  *$\mathcal{B}$ -coordinates of  $\mathbf{x}$* , are the unique weights  $c_1, \dots, c_n$  such that

$$\mathbf{x} = c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n.$$

The vector in  $\mathbb{R}^n$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the *coordinate vector of  $\mathbf{x}$  relative to  $\mathcal{B}$* , or the  *$\mathcal{B}$ -coordinate vector of  $\mathbf{x}$* .

**Example:** Let  $\mathcal{B} = \{1, t, t^2, t^3\}$  be the standard basis for  $\mathbb{P}_3$ . Then the coordinate

vector of an arbitrary polynomial is  $[a_0 + a_1 t + a_2 t^2 + a_3 t^3]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$ .

Why are coordinate vectors useful?

Because of the Unique Representation Theorem, the function  $V$  to  $\mathbb{R}^n$  given by  $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$  (e.g.  $a_0 + a_1t + a_2t^2 + a_3t^3 \mapsto \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$ ) is linear, one-to-one and onto.

**Definition:** A linear transformation  $T : V \rightarrow W$  that is both one-to-one and onto is called an *isomorphism*. We say  $V$  and  $W$  are *isomorphic*.

This means that, although the notation and terminology for  $V$  and  $W$  are different, the two spaces behave the same as vector spaces. *Every vector space calculation in  $V$  is accurately reproduced in  $W$ , and vice versa.*

Important consequence: if  $V$  has a basis of  $n$  vectors, then  $V$  and  $\mathbb{R}^n$  are isomorphic, so we can solve problems about  $V$  (e.g. span, linear independence) by working in  $\mathbb{R}^n$ .

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**Example:** Is the set of polynomials  $\{1, 2 - t, (2 - t)^2, (2 - t)^3\}$  linearly independent?

**Answer:** The coordinates of these polynomials relative to the standard basis of  $\mathbb{P}_3$  are

$$\begin{aligned} [1]_{\mathcal{B}} &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & [(2 - t)^2]_{\mathcal{B}} = [4 - 4t + t^2]_{\mathcal{B}} &= \begin{bmatrix} 4 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \\ [2 - t]_{\mathcal{B}} &= \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, & [(2 - t)^3]_{\mathcal{B}} = [8 - 12t + 6t^2 - t^3]_{\mathcal{B}} &= \begin{bmatrix} 8 \\ -12 \\ 6 \\ -1 \end{bmatrix} \end{aligned}$$

The set of polynomials is linearly independent if and only if their coordinate vectors are linearly independent (continued on next page).

**Example:** Is the set of polynomials  $\{1, 2 - t, (2 - t)^2, (2 - t)^3\}$  linearly independent?

**Answer:** (continued). The matrix 
$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & -1 & -4 & -12 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

is in echelon form and has a pivot in every column, so its columns are linearly independent in  $\mathbb{R}^4$ . So the polynomials are linearly independent.

(Alternative: this matrix is upper triangular so its determinant is the product of the diagonal entries  $1 \cdot -1 \cdot 1 \cdot -1 = 1 \neq 0$ , so the matrix is invertible, and by the Invertible Matrix Theorem, its columns are linearly independent in  $\mathbb{R}^4$ .)

In fact the polynomials form a basis: IMT says that the columns of the above matrix also span  $\mathbb{R}^4$ , so the polynomials span  $\mathbb{P}_3$ .

Advanced exercise: if  $\mathbf{p}_i$  has degree exactly  $i$ , then  $\{\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n\}$  is a basis for  $\mathbb{P}_n$ .  
(This idea is how I usually prove that a set is a basis in my research work.)

**Harder example:** (in preparation for week 9, change of coordinates) Let  $\mathcal{F} = \{1, 2 - t, (2 - t)^2, (2 - t)^3\}$ . We just showed that  $\mathcal{F}$  is a basis. So if the

$\mathcal{F}$ -coordinates of a polynomial  $\mathbf{p}$  is  $[\mathbf{p}]_{\mathcal{F}} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}$ , then what is  $\mathbf{p}$ ?

**Answer:**

$$[\mathbf{p}]_{\mathcal{F}} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix} \text{ means } \mathbf{p} = 2 + 4(2 - t) + 0(2 - t)^2 - 1(2 - t)^3 = 2 + 8t - 6t^2 + t^3.$$

If  $V$  has a basis of  $n$  vectors, then  $V$  and  $\mathbb{R}^n$  are isomorphic, so we can solve problems about  $V$  (e.g. span, linear independence) by working in  $\mathbb{R}^n$ .

What about problems concerning linear transformations  $T : V \rightarrow W$ ?

Remember from week 4 §1.9: Every linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a matrix transformation  $T(\mathbf{x}) = A\mathbf{x}$ , where

$$A = \begin{bmatrix} | & & | \\ T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \\ | & & | \end{bmatrix}$$

apply  $T$  to  $i$ th basis vector, put the coordinates of the result into column  $i$   
(standard matrix of  $T$ ).

The standard matrix is useful because we can solve  $T(\mathbf{x}) = \mathbf{y}$  by row-reducing  $[A|\mathbf{y}]$ .

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**Definition:** If  $V$  is a vector space with basis  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  and  $T : V \rightarrow V$  is a linear transformation, then the *matrix for  $T$  relative to  $\mathcal{B}$*  is

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix} \cdot \quad \text{(so the standard matrix of } T \text{ is the matrix for } T \text{ relative to the standard basis of } \mathbb{R}^n \text{.)}$$



The matrix  $[T]_{\mathcal{B}}$  is useful because

$$[T(\mathbf{x})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}, \quad (*)$$

so we can solve  $T(\mathbf{x}) = \mathbf{y}$  by row-reducing  $\left[ [T]_{\mathcal{B}} \mid [\mathbf{y}]_{\mathcal{B}} \right]$ .

**Example:** Let  $T : \mathbb{P}_2 \rightarrow \mathbb{P}_2$  be the differentiation function  $T(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$  as on the previous page. Here is an example of equation  $(*)$  for  $\mathbf{x} = 2 + 3t - t^2$ .

$$T(2 + 3t - t^2) = \frac{d}{dt}(2 + 3t - t^2) = 3 - 2t$$

$$[T]_{\mathcal{B}} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}.$$

The matrix  $[T]_{\mathcal{B}}$  is useful because

$$[T(\mathbf{x})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}, \quad (*)$$

so we can solve  $T(\mathbf{x}) = \mathbf{y}$  by row-reducing  $\left[ [T]_{\mathcal{B}} \mid [\mathbf{y}]_{\mathcal{B}} \right]$ .

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Some other things about  $T$  that we can learn from the matrix  $[T]_{\mathcal{B}}$ :

- We can solve the differential equation  $\frac{d}{dt}\mathbf{p} = 1 - 3t$  by row-reducing  $\left[ \begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right]$ .
- $[T]_{\mathcal{B}}$  is in echelon form, and it does not have a pivot in every column, so  $T$  is not one-to-one (which you know from calculus - this is why indefinite integrals have  $+C$ ).

Basis and coordinates for subspaces:

**Example:** Let  $W = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ . We showed (week 7 p14) that  $W$  is a subspace of  $\mathbb{R}^3$  because  $W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ . Since  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$  is furthermore linearly independent, it is a basis for  $W$ .

Because  $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ , the coordinate vector of  $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$ , relative to the basis  $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ , is  $\begin{bmatrix} a \\ b \end{bmatrix}$ . So  $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \mapsto \begin{bmatrix} a \\ b \end{bmatrix}$  is an isomorphism from  $W$  to  $\mathbb{R}^2$ .

Coordinates for subspaces (e.g. planes in  $\mathbb{R}^3$ ) are useful as they allow us to represent points in the subspace with fewer numbers (e.g. with 2 numbers instead of 3 numbers).

In this picture (p239 of textbook, example 7 in §4.4),  $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$  is a basis for a plane. The basis allows us to draw a coordinate grid on the plane.

The  $\mathcal{B}$ -coordinate vector of  $\mathbf{x}$  is  $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ . This coordinate vector describes the location of  $\mathbf{x}$  relative to this coordinate grid.

