Trigonometric Identities/Values

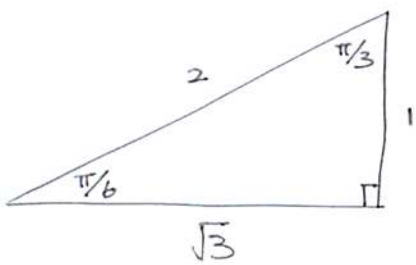
$$\sin^2 x + \cos^2 x = 1$$

$$\div \cos^2 x \hookrightarrow \quad \tan^2 x + 1 = \sec^2 x$$

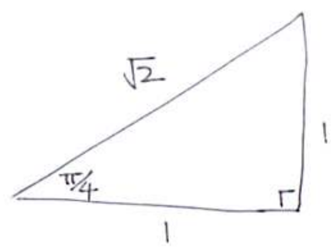
$\sin x$ is odd: $\sin(-x) = -\sin x$

$\cos x$ is even: $\cos(-x) = \cos x$

half of equilateral



isoceles



$\sin \frac{\pi}{6} = \frac{1}{2}$

$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos \frac{\pi}{3} = \frac{1}{2}$

$\tan \frac{\pi}{3} = \sqrt{3}$

$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

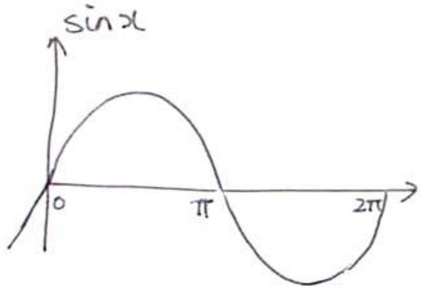
$\tan \frac{\pi}{4} = 1$

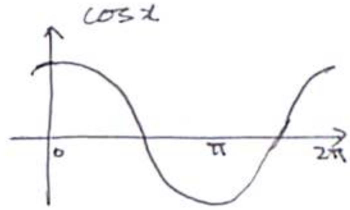
$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$,

$\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$, etc.

$\sin \frac{5\pi}{6} = \frac{1}{2}$

$\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ etc.





Exponential / Logarithmic Identities/Values

$e^{a+b} = e^a e^b$

$\ln(ab) = \ln a + \ln b$

$e^{-a} = \frac{1}{e^a}$

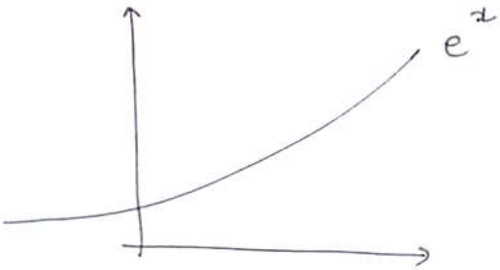
$\ln(a^{-1}) = -\ln a$

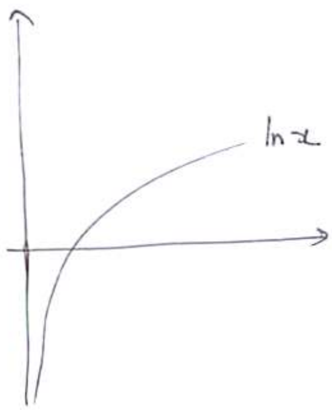
$(e^a)^b = e^{ab}$

$a\ln(b) = \ln(b^a)$

$e^{\ln a} = a$

$\ln(e^a) = a$





$e^0 = 1$

$\lim_{x\to\infty} e^x = \infty$

$\lim_{x\to-\infty} e^x = 0$

$\ln 1 = 0$

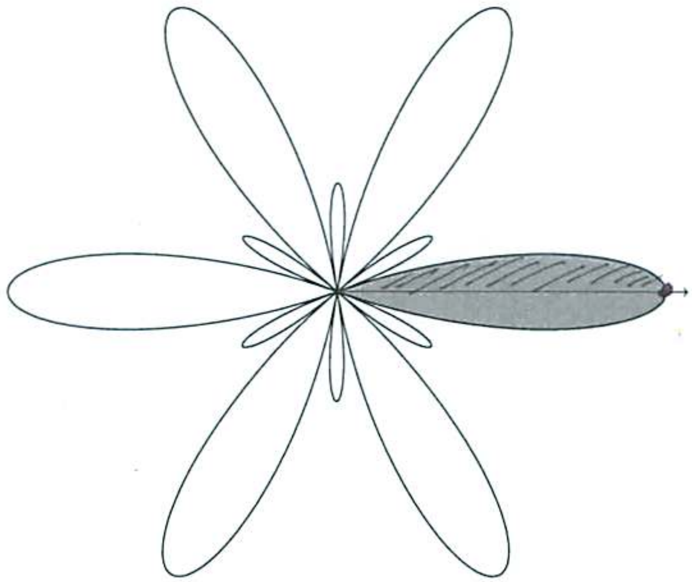
$\lim_{x\to\infty} \ln x = \infty$

$\lim_{x\to 0^+} \ln x = -\infty$

The diagram shows the polar curve

$$r = 1 + 2\cos(6\theta).$$

Find the shaded area. **Simplify your answer as much as possible.**



Want a shape $\theta=b$  $\theta=a$ or  $\theta=a$

(use symmetry)

guess values of a, b ; check $r(a)>0$.
(otherwise, try $a+\pi$, check $r(a+\pi)<0$)

guess value of a, check $r(a)>0$
b is the smallest solution to $r(\theta)=0$ with $b>a$.

By symmetry, shaded area $= 2\times$ area of top half

When $\theta=0$, $r=1+2\cos 0 = 3>0$, so "right hand corner" of is $(3,0)$ (r,θ)

ie. lower limit of integration is 0.

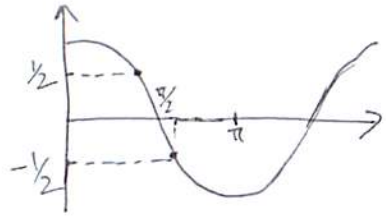
Upper limit of integration is smallest $\theta>0$ with $r(\theta)=0$

$$1+2\cos 6\theta = 0$$

$$\cos 6\theta = -\frac{1}{2} \qquad \theta = \frac{1}{6}\arccos\left(-\frac{1}{2}\right)$$

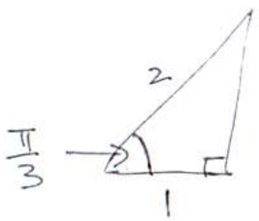
$$6\theta = \frac{2\pi}{3}$$

$$\theta = \frac{\pi}{9}.$$



Area of top half is

$$\int_0^{\frac{\pi}{9}} \frac{1}{2} r^2 \, d\theta$$



$$= \int_0^{\frac{\pi}{9}} \frac{1}{2}\left(1+2\cos(6\theta)\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{9}} 1 + 4\cos(6\theta) + 4\cos^2(6\theta) \; d\theta$$

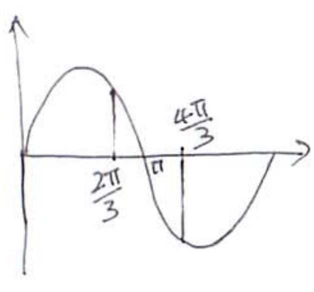
$$= \frac{1}{2}\left[\theta + \frac{4\sin(6\theta)}{6}\right]_0^{\frac{\pi}{9}} + \int_0^{\frac{\pi}{9}} 2\cos^2(6\theta)\, d\theta$$

$$= \frac{1}{2}\left(\frac{\pi}{9} + \frac{4\sin\left(\frac{6\pi}{9}\right)}{6}\right) + \int_0^{\frac{\pi}{9}} 1 + \cos(12\theta)\, d\theta$$

$$= \frac{1}{2}\left(\frac{\pi}{9} + \frac{4\sin\frac{2\pi}{3}}{6}\right) + \left[\theta + \frac{\sin(12\theta)}{12}\right]_0^{\frac{\pi}{9}}$$

$$= \frac{1}{2}\left(\frac{\pi}{9} + \frac{4}{6}\sin\frac{2\pi}{3}\right) + \left(\frac{\pi}{9} + \frac{\sin\frac{12\pi}{9}}{12}\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{9} + \frac{4}{6}\sin\frac{2\pi}{3}\right) + \left(\frac{\pi}{9} + \frac{1}{12}\sin\frac{4\pi}{3}\right)$$



$$= \frac{1}{2}\left(\frac{\pi}{9} + \frac{4}{6}\frac{\sqrt{3}}{2}\right) + \left(\frac{\pi}{9} + \frac{1}{12}\left(\frac{-\sqrt{3}}{2}\right)\right)$$

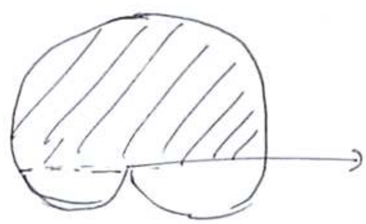
$$= \frac{\pi}{6} + \frac{\sqrt{3}}{8}$$

shaded area $= 2 \times$ area of top half $= \frac{\pi}{3} + \frac{\sqrt{3}}{4}$.

From lecture on arclength in polar coordinates:

Wanted perimeter of

$r = 1 + \sin\theta$



limits are 0 to π.

just have to check

that $r(0) > 0$.

$r(0) = 1 + \sin 0 = 1$. ✓.

Compute the following indefinite integral:

$$\int \sqrt{18x - x^2}\, dx.$$

"contains a square root"

$u = 18x - x^2$ doesn't help

complete the square: $18x - x^2 = -(x+A)^2 + B$

$= -x^2 - 2Ax - A^2 + B$

coeff of x: $18 = -2A \Rightarrow A = -9$

coeff of constant: $0 = -A^2 + B \Rightarrow B = 81$

$$\int \sqrt{18x - x^2}\, dx = \int \sqrt{-(x-9)^2 + 81}\, dx$$

$\sqrt{-(x)^2 + (\;)^2}$

$\|$

$(\;) \sin\theta$

$\boxed{x-9} = \boxed{9}\sin\theta$

$dx = 9\cos\theta\, d\theta$

$$= \int 9\cos\theta\, (9\cos\theta\, d\theta)$$

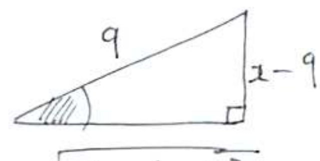
even power of cosine

$$= \int 81 \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{81\theta}{2} + \frac{81}{2}\frac{\sin 2\theta}{2} + C$$

$\sin\theta = \frac{x-9}{9} = \frac{\text{opp}}{\text{hyp}}$

$$= \frac{81\theta}{2} + \frac{81}{2}\frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}} + C$$



$$= \frac{81}{2}\arcsin\left(\frac{x-9}{9}\right)$$

$$+ \frac{81}{2}\,\frac{x-9}{9}\,\frac{\sqrt{81-(x-9)^2}}{9} + C$$

Determine if the following series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{n+3};$$

run the step-by-step: this is not a p-series

not a geometric series

$\lim_{n\to\infty} (-1)^n \frac{\sqrt{n}}{n+3} = 0$ so test for divergence doesn't help.

Check first for absolute convergence:

i.e. does $\sum_{n=1}^{\infty} \left|(-1)^n \frac{\sqrt{n}}{n+3}\right|$ converge?

$$= \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+3}$$

This is "rationalish", so use limit comparison test

set $b_n = \frac{\sqrt{n}}{n} = \frac{1}{n^{1/2}}$, and let $a_n = \frac{\sqrt{n}}{n+3}$.

$\frac{a_n}{b_n} = \frac{\sqrt{n}}{n+3} \frac{n}{\sqrt{n}} = \frac{n}{n+3} = \frac{1}{1+3n^{-1}}$, so $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{1+0} = 1$

and $\sum_{n=1}^{\infty} b_n$ diverges (p-series with $p = \frac{1}{2} < 1$)

so $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+3}$ diverges, so $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+3}$ is not absolutely convergent

so two possibilities left: conditionally convergent or divergent.

Check for conditional convergence: in this class, we only know one way of doing this.

alternating series test: $\lim_{n\to\infty} \frac{\sqrt{n}}{n+3} = \lim_{n\to\infty} \frac{n^{-1/2}}{1+3n^{-1}} = \frac{0}{1+0} = 0$

check $\frac{\sqrt{n}}{n+3}$ decreasing:

$$\frac{d}{dx}\left(\frac{\sqrt{x}}{x+3}\right) = \frac{(x+3)\frac{1}{2\sqrt{x}} - \sqrt{x}}{(x+3)^2} = \frac{(x+3) - \sqrt{x}(2\sqrt{x})}{(x+3)^2 2\sqrt{x}} = \frac{-x+3}{(x+3)^2 2\sqrt{x}}.$$

for $x \geq 1$, $(x+3)^2 2\sqrt{x} > 0$, so $\frac{d}{dx}\left(\frac{\sqrt{x}}{x+3}\right) < 0$ if $-x+3<0$

i.e. $3 < x$

and this is enough, using the "more generally".

Both conditions satisfied, so $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+3}$ converges.

So this series is conditionally convergent.

Determine if the following series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty}(-1)^n \frac{\ln(e^n)}{n^8 \cos(n\pi)}.$$

this isn't simplified: simplifying should make the problem easier.

$$= \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^8 (-1)^n}$$ ← because

$\cos(n\pi) = \begin{cases} -1 & \text{if } n \text{ odd} \\ 1 & \text{if } n \text{ even} \end{cases}$

$$= \sum_{n=1}^{\infty} \frac{1}{n^7}$$

$\therefore \cos(n\pi) = (-1)^n$

This converges (p-series with $p = 7 > 1$)

This series is positive, so convergence is the same as absolute convergence

i.e. this series is absolutely convergent.