Homework 2, due 17:00 Thursday 21 February 2019 to Dr. Pang's mailbox)

You must justify your answers to receive full credit.

1. a) (Proposition 5.1.14) Let $W_1, W_2, \ldots, W_n$ be subspaces of a vector space V. Prove that $W_1 + W_2 + \cdots + W_n$ is a direct sum if and only if, $\forall \alpha \in W_1 + W_2 + \cdots + W_n$, α can be expressed uniquely in the form $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ with $\alpha_i \in W_i$, $i = 1, 2, \ldots, n$. (Hint: modify appropriately the proof of Proposition 5.1.10, for $n = 2$. You can use induction for one of the directions, but I think it is easier without induction.)

   b) Find subspaces W_1, W_2, W_3 of $\mathbb{R}^3$ satisfying

$$W_1 \cap W_2 = \{\mathbf{0}\}, W_1 \cap W_3 = \{\mathbf{0}\}, W_2 \cap W_3 = \{\mathbf{0}\}, \qquad (*)$$

   and some $\alpha \in \mathbb{R}^3$ which can be written as $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, with $\alpha_i \in W_i$, in two different ways. (This explains why $(*)$ is not the definition of a direct sum of three subspaces.)

2. Consider the following two subspaces of $M_{2,2}(\mathbb{R})$:

$$W_1 = \left\{ \begin{pmatrix} x & -x \\ y & z \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}, \quad W_2 = \left\{ \begin{pmatrix} a & b \\ -a & c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}.$$

   Find a basis of $W_1 \cap W_2$, and extend it to bases of W_1, W_2 and $W_1 + W_2$.

3. to be released later

4. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$, and define $\sigma : M_{2,2}(\mathbb{R}) \to M_{2,2}(\mathbb{R})$ by $\sigma(X) = AX - XA$.

   a) Show that σ is a linear transformation.

   b) Find $[\sigma]_{\mathscr{A}}$, where $\mathscr{A} = \{E^{1,1}, E^{1,2}, E^{2,1}, E^{2,2}\}$ is the standard basis of $M_{2,2}(\mathbb{R})$.

   c) Find all $X \in M_{2,2}(\mathbb{R})$ such that $AX = XA$.

   d) Using part c) or otherwise, find the rank of σ.

   e) Let $U \subseteq M_{2,2}(\mathbb{R})$ be the subspace of symmetric matrices. Find a basis for the image $\sigma(U)$. (Click here for a hint)

5. to be released later

6. to be released later

The following two questions are to prepare you for upcoming classes, and is unrelated to the material from recent classes.

7. Find all eigenvalues and eigenvectors of $A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ -2 & 2 & -1 \end{pmatrix}$, and hence diagonalise A, i.e. find a P and a diagonal D such that $A = PDP^{-1}$. You do **not** need to compute P^{-1}. (You may use an online RREF calculator, but remember you only have an ordinary calculator in the exams.)

8. a) Find the eigenvalues of $\begin{pmatrix} 5 & 2 & 0 & 0 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

Suppose A is a 3×3 matrix whose only eigenvalues are 1 and 2.

b) If A is diagonalisable, then what are the possibilities for the dimensions of its eigenspaces? (Answer in this way: "$\dim E_1 =?$ and $\dim E_2 =?$, or $\dim E_1 =?$ and $\dim E_2 =?$, or ...", and then give your reasons.)

c) If A is not diagonalisable, then what are the possibilities for the dimensions of its eigenspaces? (Answer in the same way as in part b.)

Optional questions If you attempted seriously all the above questions, then your scores for the following questions may replace any lower scores for two of the above questions.

9. Take $\sigma \in L(V, V)$. Show that, if $\operatorname{rank}(\sigma) = \operatorname{rank}(\sigma \circ \sigma)$, then $\operatorname{range}(\sigma) \cap \ker(\sigma) = \{\mathbf{0}\}$. (Hint: first prove that $\operatorname{rank}(\sigma) \geq \operatorname{rank}(\sigma \circ \sigma)$.)

10. Let U, W_1, W_2 be subspaces of a vector space V, such that $W_1 \cap W_2 = \{\mathbf{0}\}$. Is $U \cap (W_1 \oplus W_2) = (U \cap W_1) \oplus (U \cap W_2)$? Give a proof or a counterexample.

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