Remember from last week:

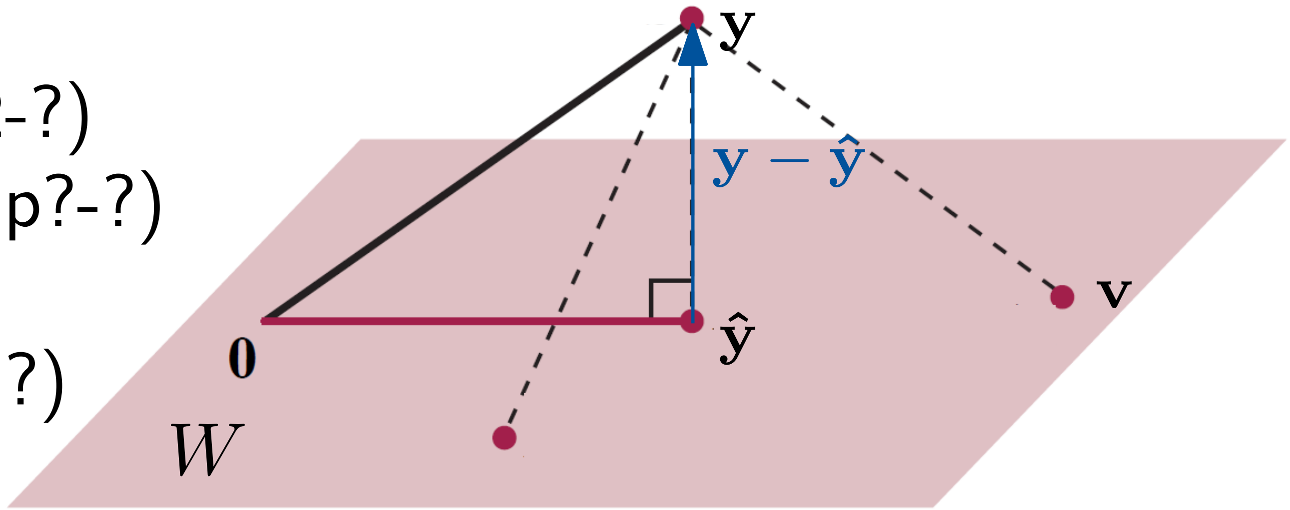
Theorem 9: Best Approximation Thoerem: Let W be a subspace of $\mathbb{R}^n$, and $\mathbf{y}$ a vector in $\mathbb{R}^n$. Then the closest point in W to $\mathbf{y}$ is the unique point $\hat{\mathbf{y}}$ in W such that $\mathbf{y}-\hat{\mathbf{y}}$ is in $W^\perp$. In other words, $\|\mathbf{y}-\hat{\mathbf{y}}\| < \|\mathbf{y}-\mathbf{v}\|$ for all $\mathbf{v}$ in W with $\mathbf{v} \neq \hat{\mathbf{y}}$.

We proved last week that, if $\hat{\mathbf{y}}$ is in W, and $\mathbf{y}-\hat{\mathbf{y}}$ is in $W^\perp$, then $\hat{\mathbf{y}}$ is the unique closest point in W to $\mathbf{y}$. But we did not prove that a $\hat{\mathbf{y}}$ satisfying these conditions always exist.

We will show that the function $\mathbf{y} \mapsto \hat{\mathbf{y}}$ is a linear transformation, called the orthogonal projection to W, and calculate it using a special orthogonal basis for W.

Our remaining goals:

§6.2 The properties of orthogonal bases (p2-?)
§6.3 Calculating the orthogonal projection (p?-?)
§6.4 Constructing orthogonal bases (p?-?)
§6.2 Matrices with orthogonal columns (p?-?)



§6.2: Orthogonal Bases

Definition:
- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an *orthogonal set* if each pair of distinct vectors from the set is orthogonal, i.e. if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$.
- A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an *orthonormal set* if it is an orthogonal set and each $\mathbf{u}_i$ is a unit vector.

Example: $\left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is an orthogonal set, because

$$\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} = -3+0+3 = 0, \quad \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 3+0-3 = 0, \quad \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = -1+10-9 = 0.$$

To obtain an orthonormal set, we normalise each vector in the set:

$$\left\{ \begin{bmatrix} 3/\sqrt{10} \\ 0 \\ -1/\sqrt{10} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{35} \\ 5/\sqrt{35} \\ -3/\sqrt{35} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} \right\} \text{ is an orthonormal set.}$$

Example: In $\mathbb{R}^6$, the set $\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5, \mathbf{e}_6, \mathbf{0}\}$ is an orthogonal set, because $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ for all $i \neq j$, and $\mathbf{e}_i \cdot \mathbf{0} = 0$.

So an orthogonal set may contain the zero vector. But when it doesn't:

Theorem 4: Nonzero Orthogonal sets are Linearly Independent: If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal set of nonzero vectors, then it is linearly independent.

Proof: We need to show that $c_1 = \cdots = c_p = 0$ is the only solution to

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}. \qquad (*)$$

Take the dot product of both sides with $\mathbf{v}_1$:

$$(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) \cdot \mathbf{v}_1 = \mathbf{0} \cdot \mathbf{v}_1$$

$$c_1\mathbf{v}_1 \cdot \mathbf{v}_1 + c_2\mathbf{v}_2 \cdot \mathbf{v}_1 + \cdots + c_p\mathbf{v}_p \cdot \mathbf{v}_1 = 0.$$

Using that $\mathbf{v}_j \cdot \mathbf{v}_1 = 0$ whenever $j \neq 1$:

$$c_1\mathbf{v}_1 \cdot \mathbf{v}_1 + c_2 0 \qquad + \cdots + c_p 0 \qquad = 0.$$

Since $\mathbf{v}_1$ is nonzero, $\mathbf{v}_1 \cdot \mathbf{v}_1$ is nonzero, so it must be that $c_1 = 0$.
By taking the dot product of $(*)$ with each of the other $\mathbf{v}_i$s and using this argument, each c_i must be 0.