

This week's notes are about the **theory** of integration; the notation and details will be complicated, but we will NOT be using most of it for computation (we will compute integrals in week 4 notes). The important thing to understand is this overall "story":

- Informally, the definite integral is the **area under a graph** (p5-11, §5.2 in textbook).
- The definite integral is defined to be a **limit** of something called a **Riemann sum**, and is painfully hard to compute by hand (p12, §5.3-5.4 in textbook).
- The **Fundamental Theorem of Calculus** (FTC) says that a definite integral of f can be **calculated using its antiderivative** (i.e. by finding a function F with $f = \frac{dF}{dx}$). This is much easier than using the definition (p , §5.5 in textbook).
- Many interesting geometric quantities are limits of Riemann sums. By rewriting these as **multiple integrals** and using FTC, we can evaluate some of them using antiderivatives (week 5 notes, §14 in textbook).

This story is extremely important because **only a tiny proportion of elementary functions have elementary antiderivatives**. (An elementary function is a function that is "built out of" $x^n, e^x, \ln x, \sin x, \cos x$.) In other words, the integral of most familiar functions is something that we do not have a name for. So, in almost all applications, functions are **integrated numerically using Riemann sums**.

Sigma notation for sums (§5.1)

Integration is about adding many things together, so it's useful to have some notation for sums.

Definition: If m and n are integers with $m \leq n$, and f is a function defined at $m, m+1, \dots, n$, then

$$\sum_{i=m}^n f(i) = f(m) + f(m+1) + \dots + f(n).$$

In this formula, i is the **index of summation**, m is the **lower limit** and n is the **upper limit**. Note that the index of summation i is a "dummy variable" and can be changed without changing the value of the sum, i.e. $\sum_{i=m}^n f(i) = \sum_{j=m}^n f(j)$.

Examples:

$$\sum_{i=2}^5 i^2 = 2^2 + 3^2 + 4^2 + 5^2 = \sum_{j=5}^n jx^j = 5x^5 + 6x^6 + \dots + (n-1)x^{n-1} + nx^n.$$

Definition: If m and n are integers with $m \leq n$, and f is a function defined at

$m, m+1, \dots, n$, then

$$\sum_{i=m}^n f(i) = f(m) + f(m+1) + \dots + f(n).$$

The function $f(i)$ can itself be a sum (with a different index of summation) - in the example below, $f(i) = \sum_{j=2}^4 \frac{x^i}{i+j}$.

Example:

$$\begin{aligned} \sum_{i=3}^4 \sum_{j=2}^4 \frac{x^i}{i+j} &= \sum_{i=3}^4 \left(\frac{x^i}{i+2} + \frac{x^i}{i+3} + \frac{x^i}{i+4} \right) \\ &= \frac{x^3}{3+2} + \frac{x^3}{3+3} + \frac{x^3}{3+4} + \frac{x^4}{4+2} + \frac{x^4}{4+3} + \frac{x^4}{4+4} \\ &\quad \begin{matrix} i=3 & i=3 & i=3 & i=3 & i=4 & i=4 \\ j=2 & j=3 & j=4 & j=2 & j=3 & j=4 \end{matrix} \end{aligned}$$

Some properties of sums:

- If A and B are constants, then $\sum_{i=m}^n (Af(i) + Bg(i)) = A \sum_{i=m}^n f(i) + B \sum_{i=m}^n g(i)$;

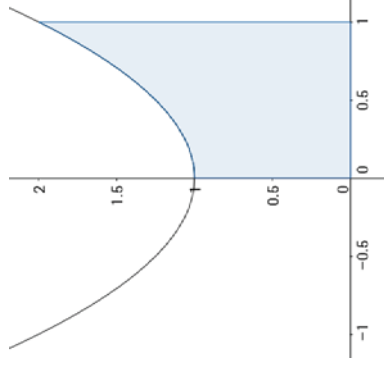
Example: $\sum_{i=1}^n \frac{i^2 + i}{3} = \frac{1}{3} \sum_{i=1}^n i^2 + \frac{1}{3} \sum_{i=1}^n i$ and $\sum_{i=1}^n \frac{i^2 + i}{n} = \frac{1}{n} \sum_{i=1}^n i^2 + \frac{1}{n} \sum_{i=1}^n i$

• $\sum_{i=1}^n 1 = \underbrace{1 + 1 + \dots + 1}_{n \text{ times}} = n$.

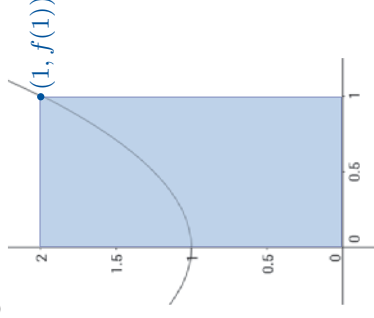
Example: Combining the two properties, $\sum_{i=1}^n \frac{1}{n} = \frac{1}{n} \sum_{i=1}^n 1 = \frac{1}{n} n = 1$.

§5.2: Area under a graph

Suppose we want to find the area of the region bounded by the lines $x = 0$, $x = 1$, $y = 0$ and the graph of $f(x) = x^2 + 1$.



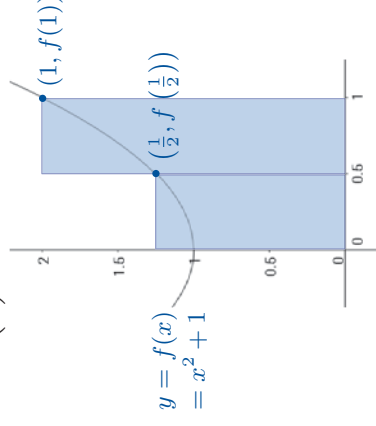
A first step might be to approximate the region by this rectangle:



$$\begin{aligned} \text{Approximate area} \\ &= \text{width} \times \text{height} = 1 \cdot f(1) = 2. \end{aligned}$$

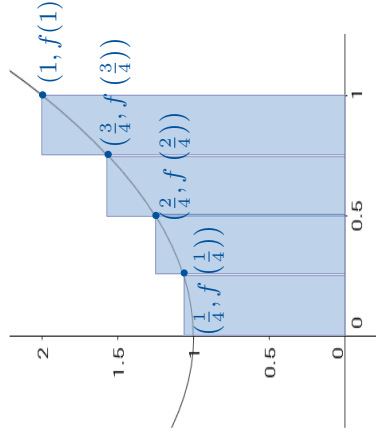
We obtain a better approximation by using two rectangles:

$$\begin{aligned} \text{Approximate area} \\ &= \frac{1}{2} f\left(\frac{1}{2}\right) + \frac{1}{2} f(1) = \frac{1}{2} \cdot \frac{5}{4} + \frac{1}{2} \cdot 2 = 1.625. \end{aligned}$$



We have an even better approximation using four rectangles:

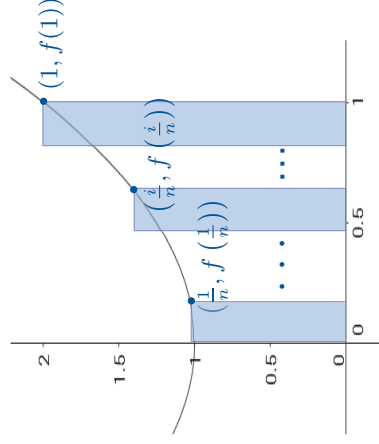
$$\begin{aligned} \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{2}{4}\right) + \frac{1}{4} f\left(\frac{3}{4}\right) + \frac{1}{4} f(1) \\ = 1.46875. \end{aligned}$$



The approximate area using n rectangles is

$$\frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \dots + \frac{1}{n} f\left(\frac{i}{n}\right) + \dots + \frac{1}{n} f(1) = \sum_{i=1}^n \frac{1}{n} f\left(\frac{i}{n}\right),$$

because the i th rectangle has width $\frac{1}{n}$ and height $f\left(\frac{i}{n}\right)$.



Remembering $f(x) = x^2 + 1$, this approximate area is:

$$\begin{aligned} \sum_{i=1}^n \frac{1}{n} \left(\left(\frac{i}{n}\right)^2 + 1 \right) &= \sum_{i=1}^n \left(\frac{i^2}{n^3} + \frac{1}{n} \right) \\ &= \sum_{i=1}^n \frac{i^2}{n^3} + \sum_{i=1}^n \frac{1}{n} \\ &= \frac{1}{n^3} \left(\sum_{i=1}^n i^2 \right) + 1. \end{aligned}$$

because of
the properties
of sums (p4)

From the last page: the approximate area using n rectangles is $\left(\frac{1}{n^3} \sum_{i=1}^n i^2 \right) + 1$.

Fact:
$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

(This formula is unimportant for the rest of the class so we will not prove it, see §5.1 Theorem 1c in textbook.)

So the approximate area using n rectangles is

$$\frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} + 1 = \frac{4}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

Because our approximation becomes more and more accurate as we use more and more rectangles, the true area must be the limit

$$\lim_{n \rightarrow \infty} \frac{4}{3} + \frac{1}{2n} + \frac{1}{6n^2} = \frac{4}{3}$$

(This type of computation is important theoretically, but we will rarely compute like this.)

In general, to find the area under the graph of a continuous, positive function $f : [a, b] \rightarrow \mathbb{R}$:

1. Divide $[a, b]$ into n subintervals by choosing x_i satisfying $a = x_0 < x_1 < \dots < x_n = b$. Let $\Delta x_i = x_i - x_{i-1}$.

2. Consider the i th approximating rectangle: its width is Δx_i and its height is $f(x_i)$.

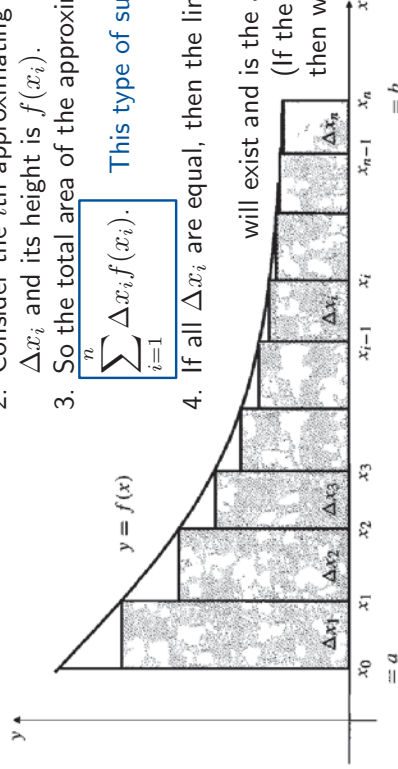
3. So the total area of the approximating rectangles is

$$\sum_{i=1}^n \Delta x_i f(x_i).$$

This type of sum is a **Riemann sum**

4. If all Δx_i are equal, then the limit $\lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x_i f(x_i)$ will exist and is the area under the graph.

(If the Δx_i are not all equal, then we have to choose x_i carefully.)

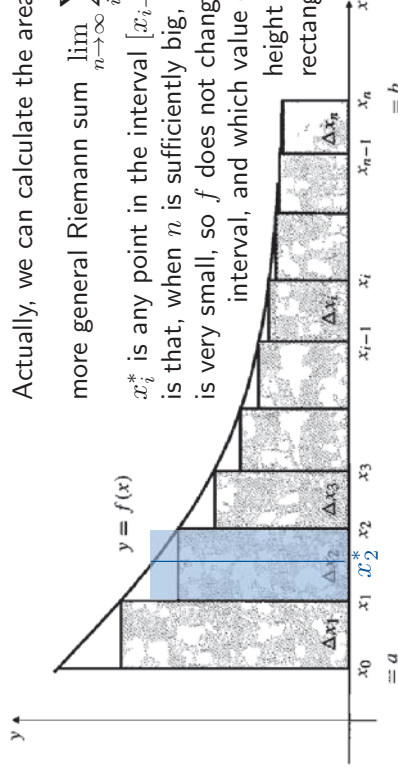


Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous, positive function, and $a = x_0 < x_1 < \dots < x_n = b$ a division of $[a, b]$ into n subintervals of equal width Δx_i . We saw (p9) that the area under the graph of f is $\lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x_i f(x_i)$.

Actually, we can calculate the area as the limit of the

more general Riemann sum $\lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x_i f(x_i^*)$, where

x_i^* is any point in the interval $[x_{i-1}, x_i]$. The intuition is that, when n is sufficiently big, the interval $[x_{i-1}, x_i]$ is very small, so f does not change much within the interval, and which value of f we use as the height of the approximating rectangles will not make much difference.



Example: Consider the function $f : [0, 2] \rightarrow \mathbb{R}$ given by $f(x) = 2 + \cos x$.

- a. Use a Riemann sum with 3 subintervals of equal width to approximate the area under the graph of f .
- b. Express the exact area under the graph of f as a limit of a Riemann sum.

Answer:

- a. To divide $[0, 2]$ into 3 subintervals of equal width, take $\Delta x_i = \frac{2}{3}$, so

$$x_0 = a = 0, x_1 = \frac{2}{3}, x_2 = \frac{4}{3}, x_3 = b = 2. \text{ So the Riemann sum is}$$

$$\sum_{i=1}^3 \Delta x_i f(x_i) = \frac{2}{3} \left(2 + \cos \frac{2}{3} \right) + \frac{2}{3} \left(2 + \cos \frac{4}{3} \right) + \frac{2}{3} \left(2 + \cos 2 \right).$$

- b. To divide $[0, 2]$ into n subintervals of equal width, take $\Delta x_i = \frac{2}{n}$, so $x_i = \frac{2i}{n}$.

So the area under the graph is $\lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x_i f(x_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \left(2 + \cos \frac{2i}{n} \right)$.

§5.3-5.4: The Definite Integral

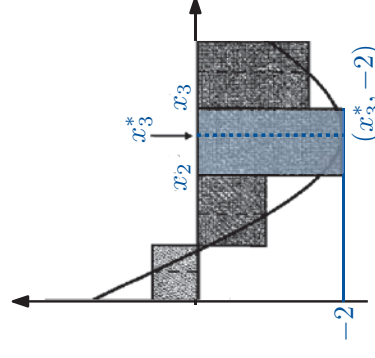
For functions $f : [a, b] \rightarrow \mathbb{R}$ taking both positive and negative values, the Riemann sum $\sum_{i=1}^n \Delta x_i f(x_i^*)$ is still defined. But what does this mean when f is negative?

To answer this, suppose $f(x_3^*) = -2$ in the diagrammed example.

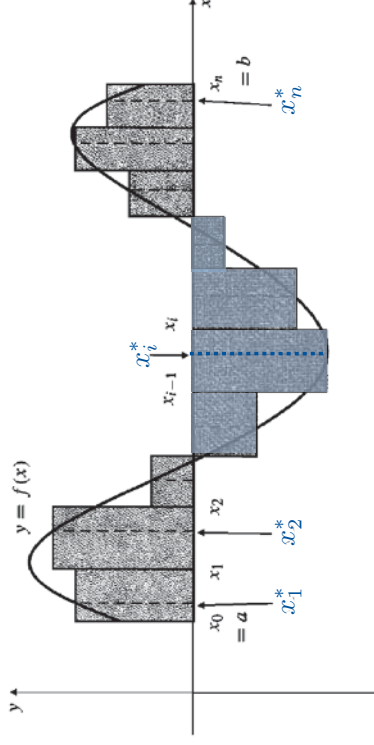
Then the 3rd term in the Riemann sum is

$$\Delta x_3(-2).$$

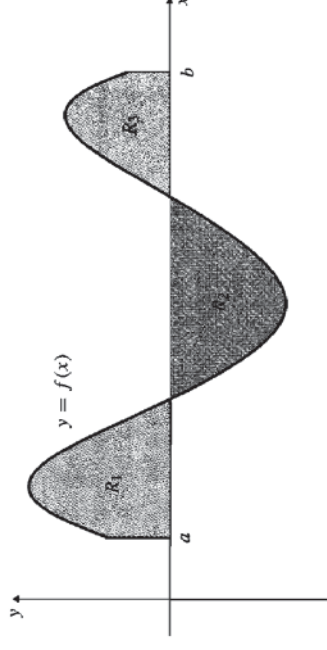
The height of the 3rd (blue) rectangle in the diagram is 2. So its area is $\Delta x_3 2$, the negative of the 3rd term in the Riemann sum.



So the Riemann sum $\sum_{i=1}^n \Delta x_i f(x_i^*)$ is the area of the grey rectangles, which are above the x -axis and below the graph, minus the area of the blue rectangles, which are below the x -axis and above the graph.



So the limit $\lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x_i f(x_i^*)$ is the **signed area**: the total area below the graph and above the x -axis, minus the total area above the graph and below the x -axis.



The signed area is an interesting quantity: for example, if f is velocity, then the signed area is the change in position. So let's define this to be the integral.

Definition: Let $a = x_0 < x_1 < \dots < x_n = b$ be a division of $[a, b]$ into n subintervals of equal width Δx_i , and let x_i^* be a point in $[x_{i-1}, x_i]$. A function $f : [a, b] \rightarrow \mathbb{R}$ is **integrable** if $\lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x_i f(x_i^*)$ exists and is independent of the choice of x_i^* in $[x_{i-1}, x_i]$. The value of this limit is the **integral of f on $[a, b]$** (or the integral of f from a to b):

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x_i f(x_i^*).$$

It is hard to use this definition to prove that a function is integrable. Luckily, we have the following theorem:

Theorem 2: Continuous functions are integrable: If f is (piecewise) continuous on $[a, b]$, then f is integrable on $[a, b]$.

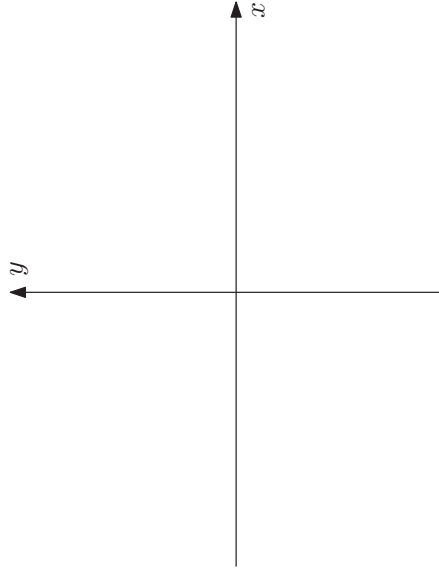
Terminology of the various parts of the integral symbol $\int_a^b f(x) dx$:

- \int is the **integral sign** - it is a long S for "sum".
- a is the **lower limit of integration** and b is the **upper limit of integration**.
- f is the **integrand**, the function that is being integrated.
- dx tells us that the **variable of integration** is x . The variable of integration is a dummy variable like the index of summation (p_2), we can change it without changing the value of the definite integral, e.g. $\int_a^b f(x) dx = \int_a^b f(t) dt$.

Important:

- The definite integral is a **number**, not a function.
- The symbol $\int f(x) dx$, without any limits of integration, is the **indefinite integral** or antiderivative. It is a **function** of x , whose derivative is f . At the moment we do not know that it is related to the definite integral.

Example: By drawing a graph and using geometry, determine $\int_1^2 2 - x \, dx$.



It will be useful to define $\int_a^b f(x) \, dx$ when $a > b$, so we can put variables in the limits of the integral without worrying about which limit is bigger (e.g. p21). The convention which makes all our later theorems work is

$$\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx,$$

i.e. reversing the limits of integration changes the sign of the integral.

Important properties of the definite integral (the labelling follows §5.4 Theorem 3 in textbook):

c. An integral depends **linearly on the integrand**: if A and B are constants, then

$$\int_a^b Af(x) + Bg(x) \, dx = A \int_a^b f(x) \, dx + B \int_a^b g(x) \, dx. \text{ This comes from the corresponding property of Riemann sums (p4).}$$

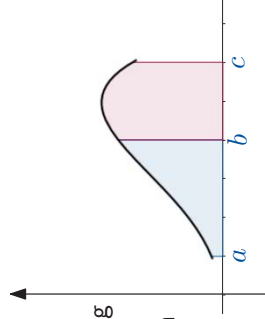
d. An integral depends **additively on the interval of integration**:

$$\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx.$$

For the case $a < b < c$, this is believable from thinking about integrals as signed areas. When a, b, c are in another order, we need to use identity/definition from the previous page.

We can deduce from d. that

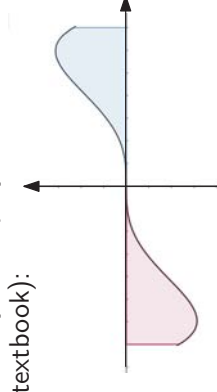
a. $\int_a^a f(x) \, dx = 0.$



The following two properties show how to use symmetry to simplify some integrals (the labelling follows §5.4 Theorem 3 in textbook):

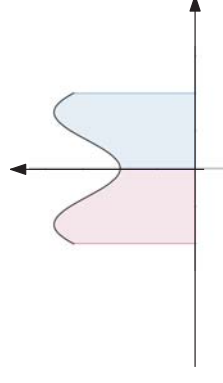
g. If f is an **odd function** ($f(-x) = -f(x)$),

then $\int_{-a}^a f(x) \, dx = 0.$



h. If f is an **even function** ($f(-x) = f(x)$),

then $\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx.$



§5.5: The Fundamental Theorem of Calculus

This important theorem is in two parts:

Theorem 5: Fundamental Theorem of Calculus (FTC): Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function.

FTC1. The cumulative area function $F : [a, b] \rightarrow \mathbb{R}$ defined by $F(x) = \int_a^x f(t) dt$ is differentiable, and is an antiderivative of f , i.e. $F'(x) = f(x)$.

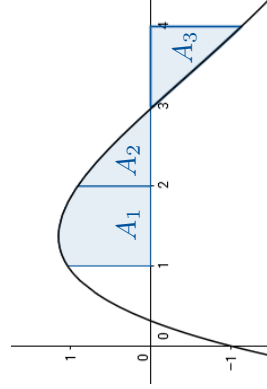
FTC2. If $G : [a, b] \rightarrow \mathbb{R}$ is any antiderivative of f (i.e. $G'(x) = f(x)$), then

$$\int_a^b f(x) dx = G(b) - G(a).$$

FTC1 explains how to differentiate a cumulative area function, and is mainly for theoretical use.

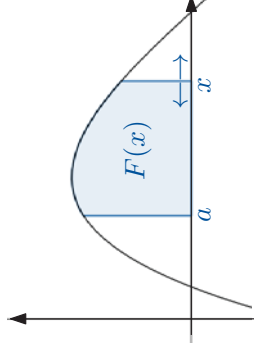
FTC2 explains how to compute a definite integral if you can find the antiderivative of the integrand - this will be very useful to us.

Suppose this is the graph of $f : [1, 4] \rightarrow \mathbb{R}$:



- $F(1) = \int_1^1 f(t) dt = 0$ by the properties of definite integrals.
- $F(2) = \int_1^2 f(t) dt = A_1$, which is a positive number.
- $F(3) = \int_1^3 f(t) dt = A_1 + A_2$. Since $A_2 > 0$, we must have $F(3) > F(2)$, but $A_2 < A_1$ so the increase in F between 2 and 3 is less than it was between 1 and 2
- $F(4) = \int_1^4 f(t) dt = A_1 + A_2 - A_3$, so $F(4) < F(3)$.

FTC1 will be “obvious” if we understand the cumulative area function $F(x) = \int_a^x f(t) dt$.



First note that such a function is defined whether $x \geq a$ or $x < a$, because of our definition / identity (p18) that reversing the limits of an integral changes its sign.

Despite the slightly scary formula, cumulative area functions are very natural: for example, if $f(t)$ is the rate that a company is earning money at time t , then $F(x)$ is the total money earned from time a to time x . (Cumulative area functions are also very important in probability.)

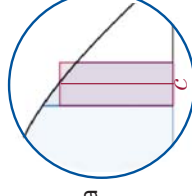
Observe that we were sketching $F(x)$ by considering the increase or decrease of F , i.e. the derivative of F . This derivative is:

$$\begin{aligned} F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^{x+h} f(t) dt - \int_a^x f(t) dt \right] \quad \text{definition of derivative} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\left(\int_a^x f(t) dt + \int_x^{x+h} f(t) dt \right) - \int_a^x f(t) dt \right] \quad \text{definition of } F \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt. \end{aligned}$$

additive dependence on the domain of integration (d, p19)

By the Mean Value Theorem for Integrals (later, §5.4), there is a number $c \in [x, x+h]$ such that $\int_x^{x+h} f(t) dt = hf(c)$. So

$$F'(x) = \lim_{h \rightarrow 0} \frac{1}{h} hf(c) = \lim_{h \rightarrow 0} f(c) = f(x).$$



The previous page proved FTC1: $F(x) = \int_a^x f(t) dt$ is an antiderivative of f .

Now we use FTC1 to prove FTC2: $\int_a^b f(t) dt = G(b) - G(a)$ for any antiderivative G of f .

Because G and F are both antiderivatives of f , we must have $F(x) = G(x) + C$ for some constant C .

$$\text{So } \int_a^b f(t) dt = F(b)$$

definition of F

$$= F(b) - F(a) \quad \text{because } F(a) = \int_a^a f(t) dt = 0$$

$$= (G(b) + C) - (G(a) + C) \quad \text{using } F(x) = G(x) + C$$

$$= G(b) - G(a).$$

To simplify the notation when using FTC2, we write $F(x)|_a^b$ to mean $F(b) - F(a)$. (The alternative notation $[F(x)]_a^b$ will also be accepted.)

Recall that the symbol $\int f(x) dx$ means the general antiderivative of f . So FTC2

$$\text{says } \int_a^b f(x) dx = \left(\int f(x) dx \right) \Big|_a^b.$$

Redo Example: (Q1 ex. sheet #5) Compute $\int_{-3}^1 2x dx$ using FTC2.

Redo Example: (p5-8) Compute $\int_0^1 x^2 + 1 dx$ using FTC2.

Redo Example: (p10) Compute $\int_0^2 2 + \cos x dx$ using FTC2.

As the previous examples showed, it's useful to know some common, simple antiderivatives:

$$\int x^r dx = \frac{x^{r+1}}{r+1} + C \text{ if } r \neq -1.$$

$$\int \sin x dx = -\cos x + C.$$

$$\int \cos x dx = \sin x + C.$$

$$\int e^x dx = e^x + C.$$

$$\int \frac{1}{x} dx = \ln |x| + C.$$

These can be proved by differentiating the right hand side, e.g. for the last line:

if $x > 0$, then $\ln |x| = \ln x$, and $\frac{d}{dx} \ln x = \frac{1}{x}$.

if $x < 0$, then $\ln |x| = \ln(-x)$, and $\frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}$.

Some other useful antiderivatives that will be provided to you in exams:

$$\int \sec^2 x dx = \tan x + C,$$

$$\int \sec x \tan x dx = \sec x + C,$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C,$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C.$$

These can be proved by differentiating the right hand sides: the first four use the quotient rule (see §3.2 of textbook), the last two use implicit differentiation (see §3.5 of textbook).

Warning: FTC2 only works for **continuous** integrands. For example, it cannot be applied to $\frac{1}{x^2}$ on an interval containing 0, where the function is not defined.

$\int_{-1}^1 \frac{1}{x^2} dx \neq \left(\frac{-1}{x} \right) \Big|_{-1}^1 = -2$ - we will see (§6.5) that the associated area is in fact infinite.

(Integrals like these, on an interval containing points where the integrand is not defined, are called **improper integrals**. These regions do sometimes have finite area - we will explore this later.)

