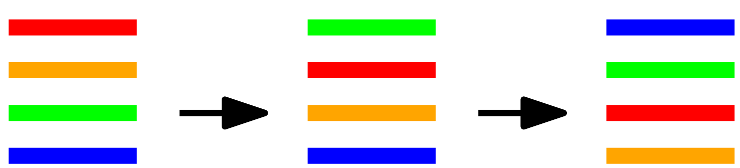
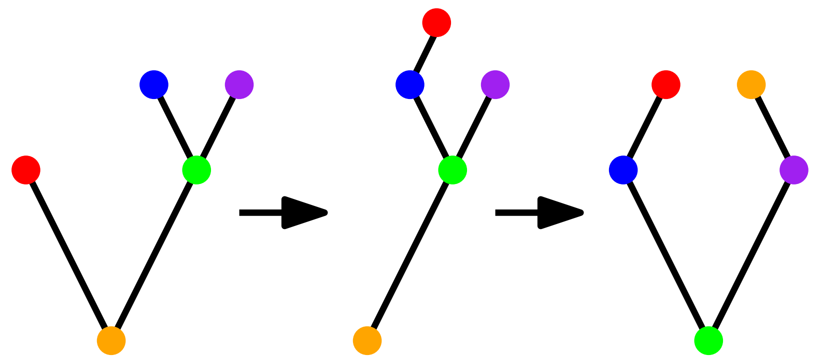
Markov Chains and Descent Subalgebras





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2018 HKBU-NCKU Joint Workshop on Mathematical Sciences

The Tsetlin library Markov chain

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- remove a file at random;
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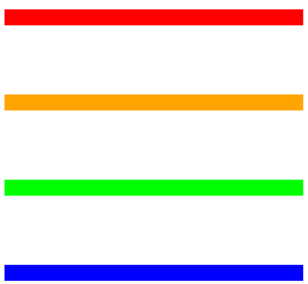
An example of one possibility ($n = 4$):



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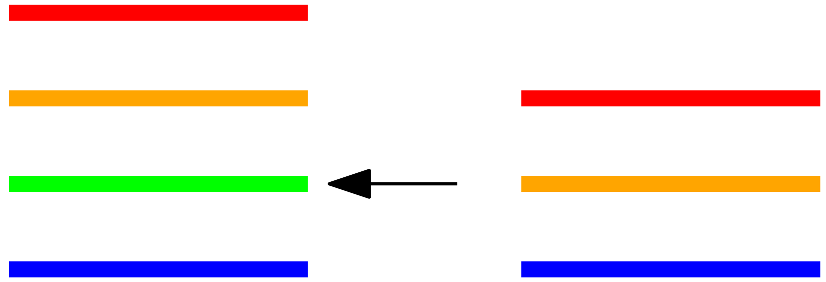

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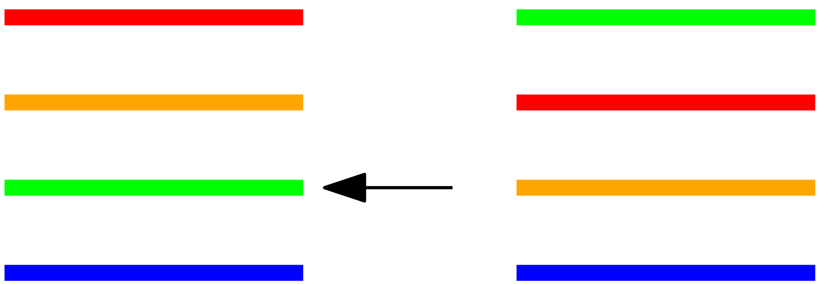

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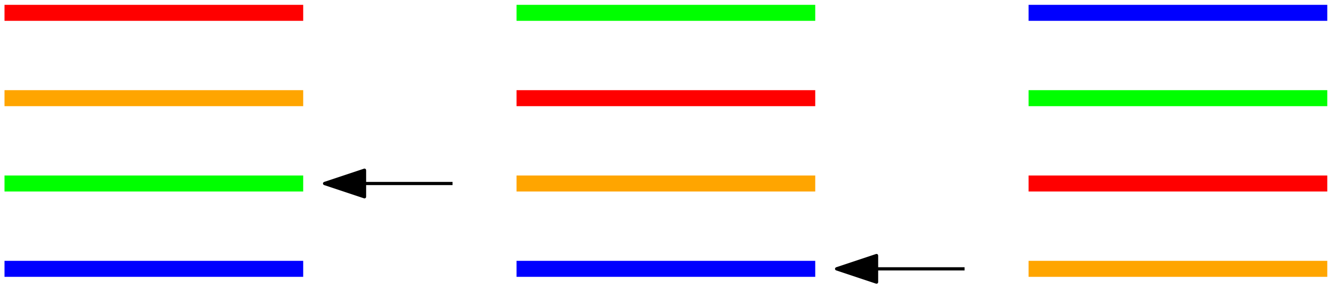

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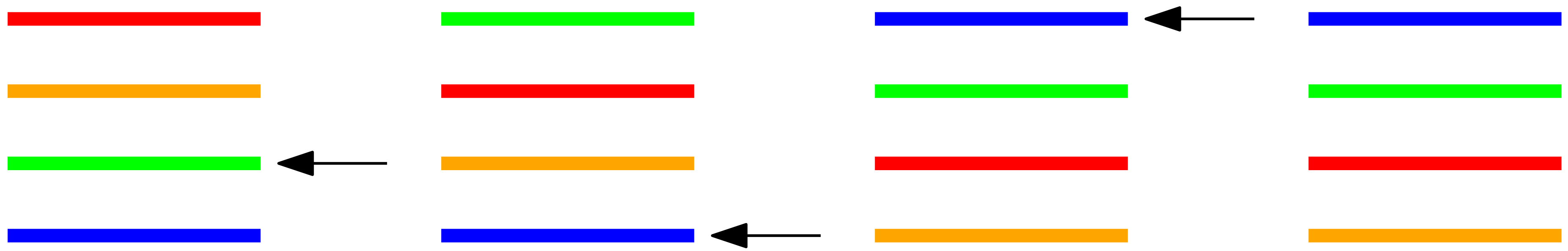

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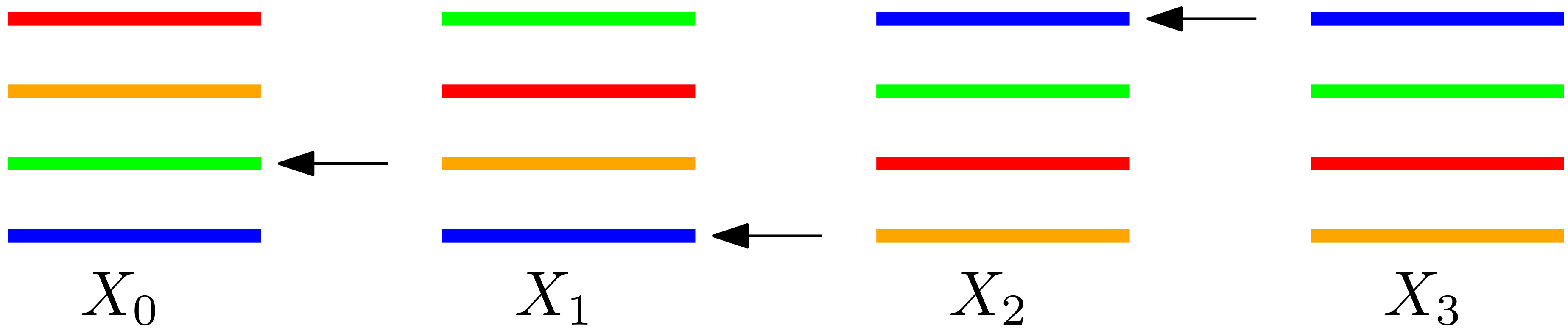

Let X_t = the order of the files after t days (a random variable). $\{X_t\}$ is a Markov chain - i.e. it is memoryless, X_{t+1} depends only on X_t, not on $X_1, \ldots, X_{t-1}$.

As $t \to \infty$, the distribution of $X_t \to$??

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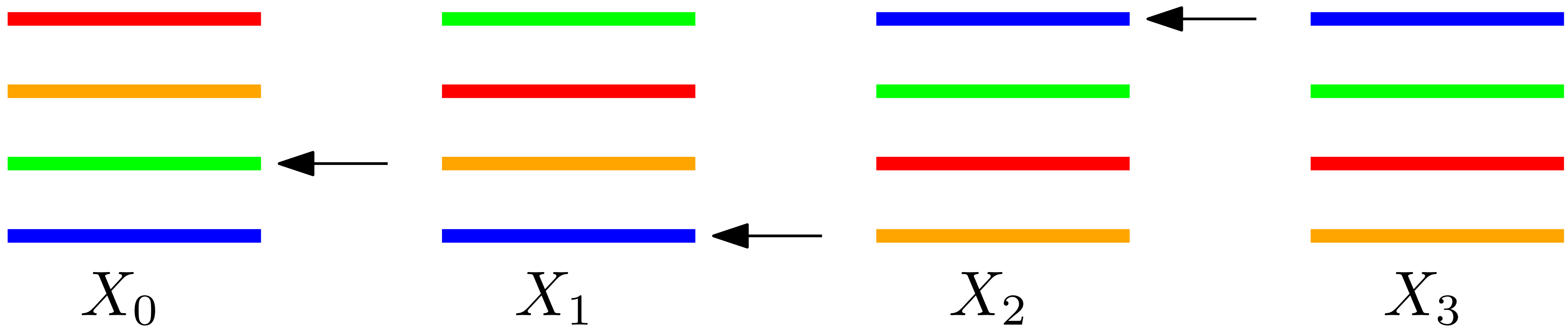

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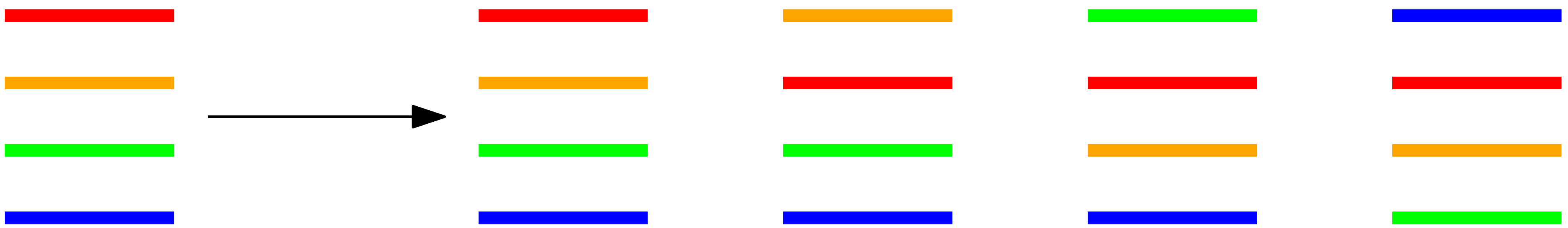
As $t \to \infty$, $X_t \to$ uniform distribution over the $n!$ possible orders. Aldous-Diaconis (1986): convergence rate $\sim n \log n$.

Analysing the Tsetlin library algebraically

Let the symmetric group S_n act on $\{$the $n!$ possible orders of files$\}$ in this way: $\sigma \in S_n$ moves the file in position $\sigma(i)$ to position i.

E.g. for Tsetlin library:



$\sigma(1) = 1$

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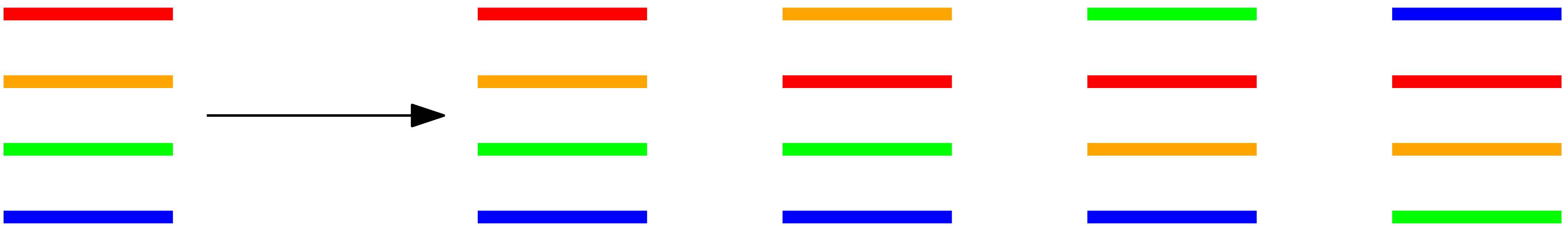

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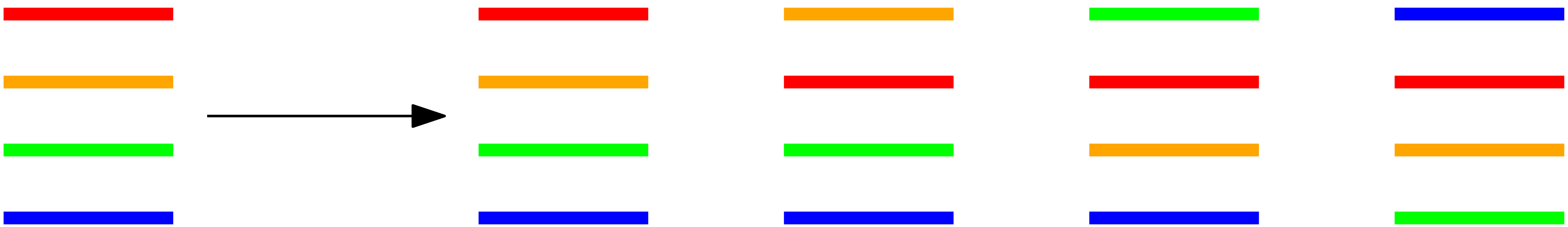

$$
\begin{array}{ll}
\sigma(1) = 1 & \sigma(1) = 2 \\
\sigma(2) = 2 & \sigma(2) = 1 \\
\sigma(3) = 3 & \sigma(3) = 3 \\
\sigma(4) = 4 & \sigma(4) = 4
\end{array}
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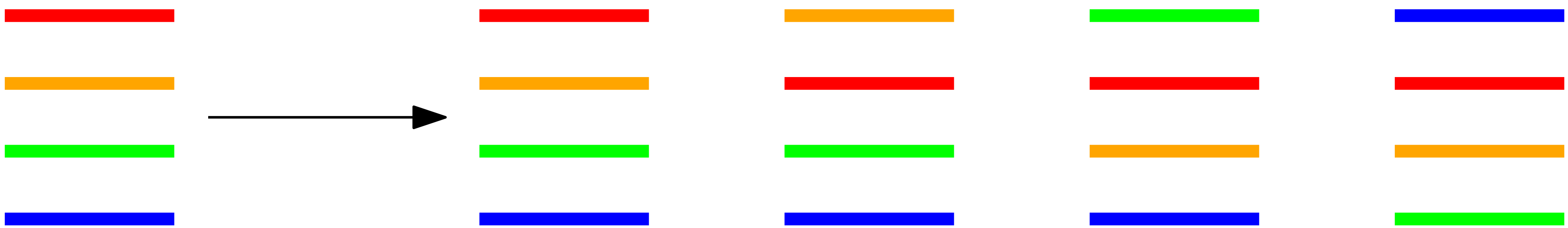

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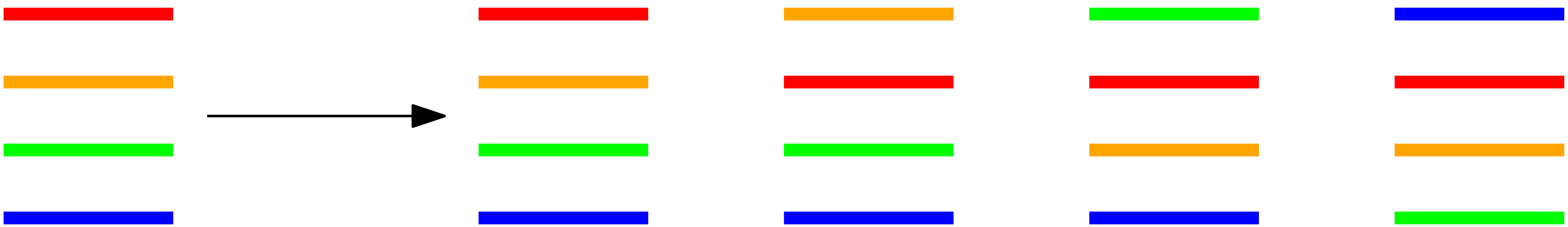

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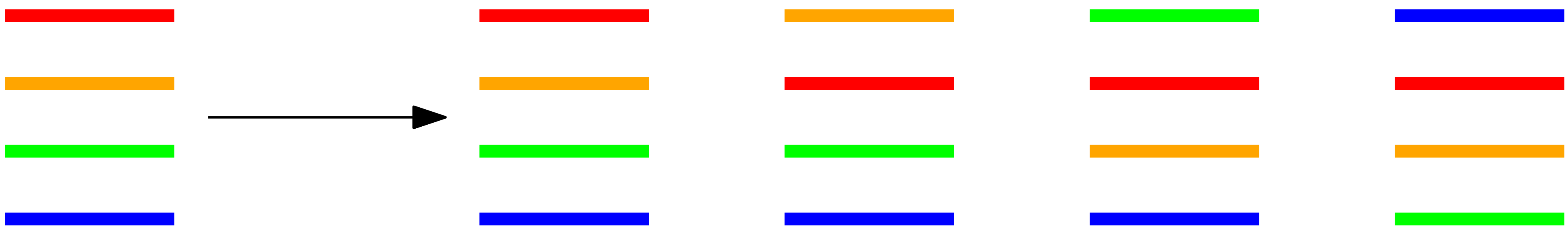

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The permutations describing the Tsetlin library moves are precisely the n permutations satisfying $\sigma(2) < \sigma(3) < \cdots < \sigma(n)$.

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Definition : For $\sigma \in S_n$ and $i \in \{1, \ldots, n-1\}$, we say that i is a descent of σ if $\sigma(i) > \sigma(i+1)$.
Let $\mathrm{Des}(\sigma) \subseteq \{1, \ldots n-1\}$ be the set of descents of σ.

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Since these n moves occur with equal probability, take the sum of these n permutations with equal weights, i.e. in the group algebra $\mathbb{R}S_n$, define $Y_1 := \sum_{\sigma:\mathrm{Des}(\sigma)\subseteq\{1\}} \sigma$.

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By analysing the subalgebra of $\mathbb{R}S_n$ generated by Y_1 (e.g. idempotents, Diaconis-Fill-Pitman, 1992)
Theorem : Prob(— is above — in X_t) $= \left(1 + \left(\frac{n-2}{n}\right)^t\right)\frac{1}{2}$.

Two directions to extend

More generally, for any $I \subseteq \{1, \ldots, n-1\}$, let

$$Y_I := \sum_{\sigma : \mathrm{Des}(\sigma) \subseteq I} \sigma.$$

Theorem (Solomon, 1976): The Y_I span a subalgebra of $\mathbb{R}S_n$. This is known as Solomon's descent algebra (in Coxeter type A), and is isomorphic to noncommutative symmetric functions. Any subalgebra of it is called a descent subalgebra.

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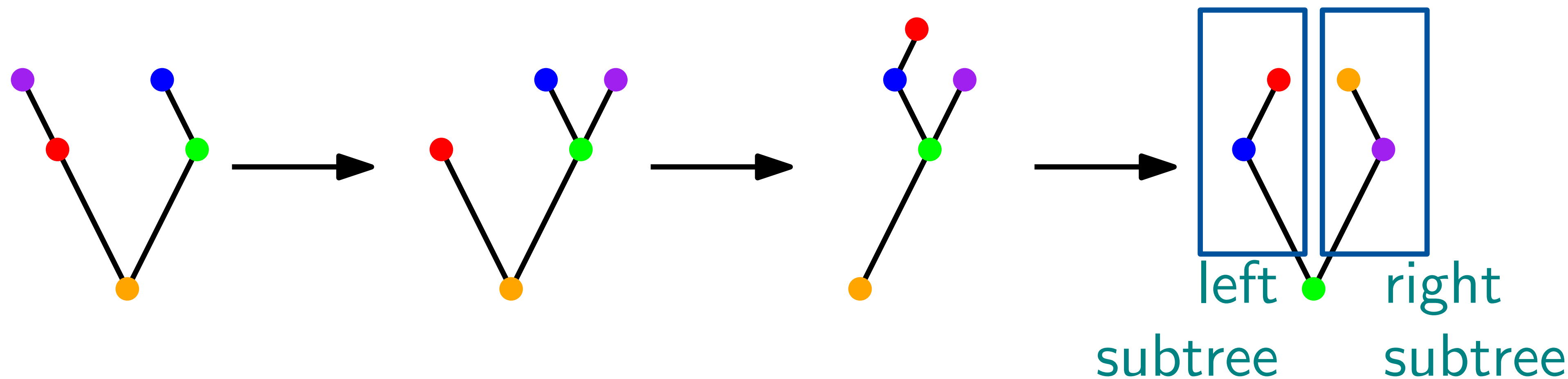
e.g. for the Markov chain corresponding to $< Y_1 >$ on binary trees (Loday-Ronco algebra):



Theorem :

Prob(right subtree of X_t is empty) $\leq \left(1+\left(\frac{n-2}{n}\right)^t\right)\frac{1}{2}$.

Thank you

I am looking for collaborators, PhD students, and postdocs.

My interests:

- Markov chains from algebra actions / linear transformations
- new subalgebras of Solmon's descent algebra / Solomon-Tits algebra
- combinatorial Hopf algebras / monoids

My papers:

- *Card-Shuffling via Convolutions of Projections on Combinatorial Hopf Algebras*, FPSAC conference abstract, 2015
- *Markov Chains from Descent Operators on Combinatorial Hopf Algebras*, arXiv, 2016
- (with M. Josuat-Verges) *Subalgebras of Solomon's descent algebra based on alternating runs*, J. Combin. Theory. Ser. A., 2018

amypang.github.io/research.html amypang@hkbu.edu.hk

Connecting Markov chains with descent algebra elements

For the Tsetlin library and other file-rearrangement chains:
Let $\mathbb{R}\Omega$ be the set of formal linear combinations of "states" (possible values of X_t.)
Represent a probability distribution $\pi : \Omega \to \mathbb{R}$ on states by $\displaystyle\sum_{x\in\Omega} \pi(x)x \in \mathbb{R}\Omega$.
Let $\mathbb{R}S_n$ act on $\mathbb{R}\Omega$ by linearity:

$$\left(\sum_{\sigma\in S_n} a_\sigma \sigma\right)\left(\sum_{x\in\Omega} \pi(x)x\right) = \sum_{\sigma,x} a_\sigma \pi(x)\sigma(x).$$

Then $\frac{1}{n}Y_1(X_t) = X_{t+1}$.

For chains on other combinatorial objects:
If $I = \{i_1, i_2, \ldots, i_r\}$, then Y_I breaks an object randomly into $r+1$pieces of sizes $i_1, i_2 - i_1, i_3 - i_2, \ldots, i_r - i_{r-1}, n - i_r$ respectively, then reassembles these pieces randomly.