§10.2 Adjoints — i.e. using inner products to understand dual space

As before, for $\gamma \in V$, consider $\phi_\gamma : V \to \mathbb{F}$ given by

$$\phi_\gamma = \langle \gamma, - \rangle \quad \text{i.e.} \quad \phi_\gamma(\alpha) = \langle \gamma, \alpha \rangle$$

$\phi_\gamma \in L(V, \mathbb{F}) = \hat{V}$

Ex: in $\mathbb{R}^3$ with dot product: let $\gamma = \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix}$ then $\phi_\gamma \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$= 5x + y + 7z$

in general: $\gamma = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then $\phi_\gamma \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$

From §9.1: every $\phi \in \widehat{\mathbb{R}^3}$ is of the form $\phi\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$ for some a, b, c.

so every $\phi \in \widehat{\mathbb{R}^3}$ is ϕ_γ for some $\gamma \in \mathbb{R}^3$.

$\therefore$ any question about $\phi \in \widehat{\mathbb{R}^3}$ can be "translated" into a question about $\gamma \in \mathbb{R}^3$

Th (~10.2.1, 10.2.2) (simplified) Riesz representation theorem:

Let $\Phi : V \to \hat{V}$

$\Phi(y) = \phi_y = \langle y, \cdot \rangle$

Then Φ is an injective conjugate-linear function.

$\Phi(ay+y') = \bar{a}\,\Phi(y) + \Phi(y')$

In particular, if V is finite-dimensional, (so $\dim V = \dim \hat{V}$), then Φ is surjective,

i.e. every $\phi \in \hat{V}$ is ϕ_y for some $y \in V$.

Proof: see homework.

$\boxed{?}$ (when $\dim V < \infty$) Given $\phi \in \hat{V}$, how to find the corresponding y?

i.e. what is the inverse function $\Lambda : \hat{V} \to V$ to $\Phi : V \to \hat{V}$? ($\Lambda = \Phi^{-1}$)

Recall $\mathbb{R}^3$ example:

if $\phi\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$

then $y = \Lambda(\phi) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

i.e. if $\phi = a\phi_1 + b\phi_2 + c\phi_3$

where $\{\phi_1, \phi_2, \phi_3\}$ is dual basis to $\{e_1, e_2, e_3\}$

then $y = ae_1 + be_2 + ce_3$.

e.g. $\phi = \phi_1$ corresponds to $y = e_1$. i.e. $\Lambda(\phi_1) = e_1$, $\Phi(e_1) = \phi$

i.e. $\underbrace{\phi\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\phi_1} = x$ is the same as $\underbrace{\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{e_1} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\Phi(e_1) = \phi_{e_1}} = x$

why? $\phi_1(xe_1 + ye_2 + ze_3)$

$= x\underbrace{\phi_1(e_1)}_{1} + y\cancel{\phi_1(e_2)}^{0} + z\cancel{\phi_1(e_3)}^{0} = x$

$\langle e_1, xe_1 + ye_2 + ze_3 \rangle$

$= x\underbrace{\langle e_1, e_1 \rangle}_{1} + y\cancel{\langle e_1, e_2 \rangle}^{0} + z\cancel{\langle e_1, e_3 \rangle}^{0} = x$

$\because \{e_1, e_2, e_3\}$ is an orthonormal basis.

In general:

<u>Prop</u>: Let $\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\}$ be an orthonormal basis of V,
$\hat{\mathcal{A}} = \{\phi_1, \cdots, \phi_n\}$ be the dual basis of $\hat{V}$.

Let $\Phi : V \to \hat{V}$ be $\Phi(\gamma) = \langle \gamma, - \rangle$

Then $\Phi^{-1} = \Lambda : \hat{V} \to V$ satisfies $\Lambda(\phi_i) = \alpha_i$

i.e. $\phi_i = \langle \alpha_i, - \rangle$

$\because$ Λ is conjugate-linear, so

if $\phi = \underset{\phi(\alpha_1)}{\underset{\parallel}{b_1}} \phi_1 + \cdots + b_n \phi_n$, then $\Lambda(\phi) = \overline{b_1} \alpha_1 + \cdots + \overline{b_n} \alpha_n$.

i.e. $\Lambda(\phi) = \overline{\phi(\alpha_1)} \alpha_1 + \cdots + \overline{\phi(\alpha_n)} \alpha_n$.