From homework 5 from 2015, by Dr. Felix Kwok, edited by Dr. Amy Pang

8. The goal of this exercise is to give an alternate proof of the Rank-Nullity Theorem for general vector spaces, i.e., without using row reduction. For this exercise, let V and W be vector spaces with V finite dimensional, and let $T : V \to W$ be a linear transformation. The equality we would like to prove is

$$\dim(\text{kernel}(T)) + \dim(\text{range}(T)) = \dim(V). \qquad (*)$$

(a) Explain why this coincides with the rank-nullity theorem in the lecture notes when $T(\mathbf{x}) = A\mathbf{x}$.

Prove $(*)$ by completing the following steps.

(b) By citing an appropriate theorem, explain why $\dim(\text{kernel}(T))$ is finite. Let $\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ be a basis of $\ker(T)$, so that $\dim(\text{kernel}(T)) = k$.

(c) Show that range(T) is finite dimensional by finding a finite spanning set. (Hint: start with a basis for V and look at what the linear transformation T does to it.)

(d) Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_r\}$ be a basis for range(T), so that $\dim(\text{range}(T)) = r$. Explain why there are vectors $\mathbf{x}_1, \ldots, \mathbf{x}_r$ in V such that $T(\mathbf{x}_i) = \mathbf{w}_i$ for $i = 1, \ldots, r$.

(e) We now show that the set $\mathcal{B} = \{\mathbf{x}_1, \ldots, \mathbf{x}_r, \mathbf{z}_1, \ldots, \mathbf{z}_k\}$ forms a basis for V. Suppose there are weights $c_1, \ldots, c_r, d_1, \ldots, d_k \in \mathbb{R}$ such that

$$c_1\mathbf{x}_1 + \cdots + c_r\mathbf{x}_r + d_1\mathbf{z}_1 + \cdots + d_k\mathbf{z}_k = 0. \qquad (\dagger)$$

Show that this implies (i) $c_1 = \cdots = c_r = 0$, and (ii) $d_1 = \cdots = d_k = 0$, so $\mathcal{B}$ is linearly independent. (Hint: First show (i) by applying the linear transformation T to $(\dagger)$ and using the fact that the $\mathbf{w}_i$ form a basis.)

(f) Show that $\mathcal{B}$ spans V. (Hint: Let $\mathbf{v}$ be an arbitrary vector in V. By considering T(v), show that v can be written as a linear combination of the $\mathbf{x}_i$, plus something in the kernel of T.)

(g) Conclude that $\mathcal{B}$ is a basis for V. What is $\dim(V)$? Why does this prove the equality $(*)$?