In an earlier extremisation example (week 11 p13), we parametrised the unit circle $x^2 + y^2 = 1$ by $x = \pm\sqrt{1 - y^2}$, $y = y$ - in other words, we solved for x in terms of y.

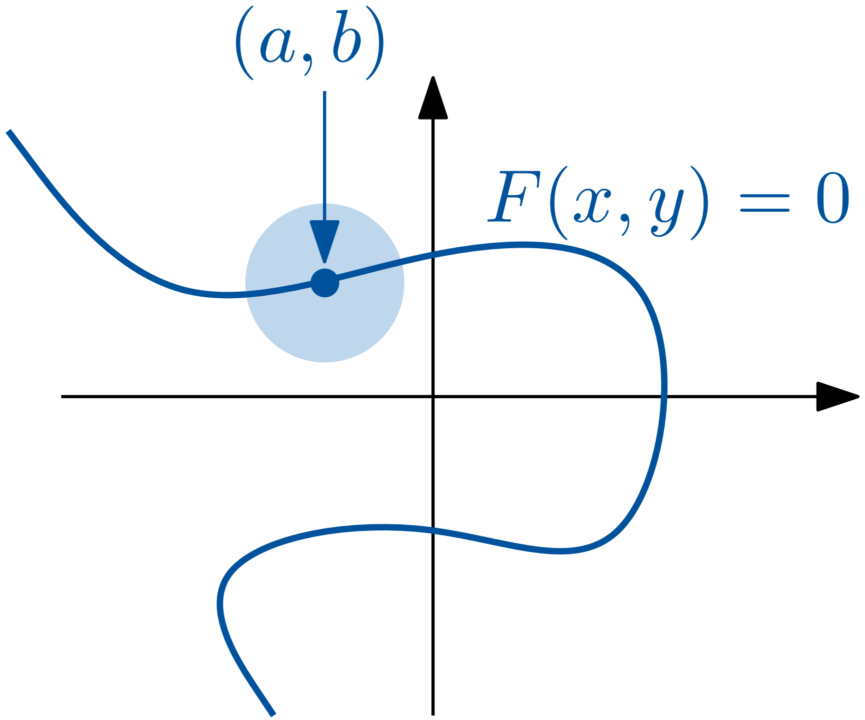
This week we investigate the question: given a relationship between x and y, when can we write x as a function of y, or y as a function of x? Or, given multiple equations relating multiple variables, when can we write some of the variables as functions of the others?

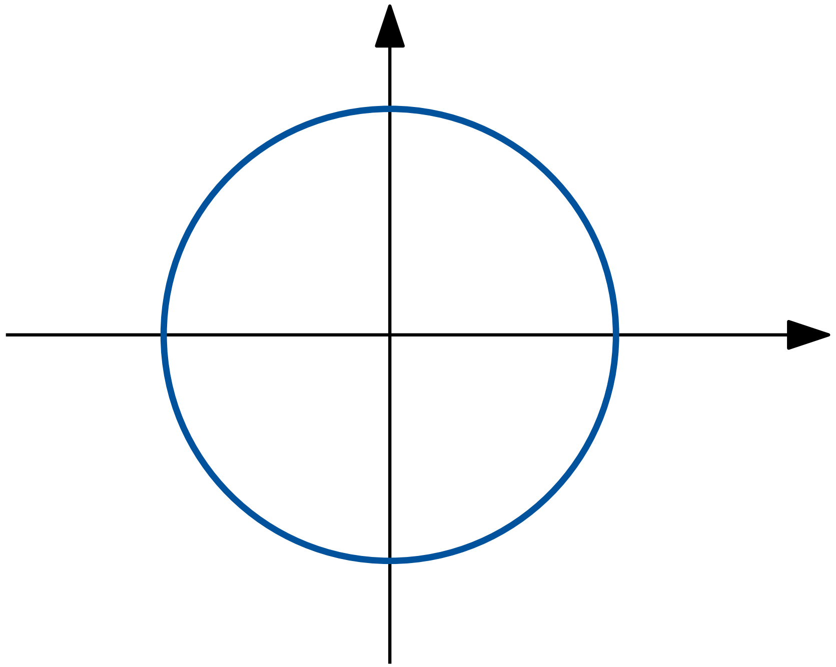
§12.8: Implicit Functions

Suppose we have a relationship between x and y, expressed as $F(x, y) = 0$. (e.g. the unit circle is the case $F(x, y) = x^2 + y^2 - 1$.)

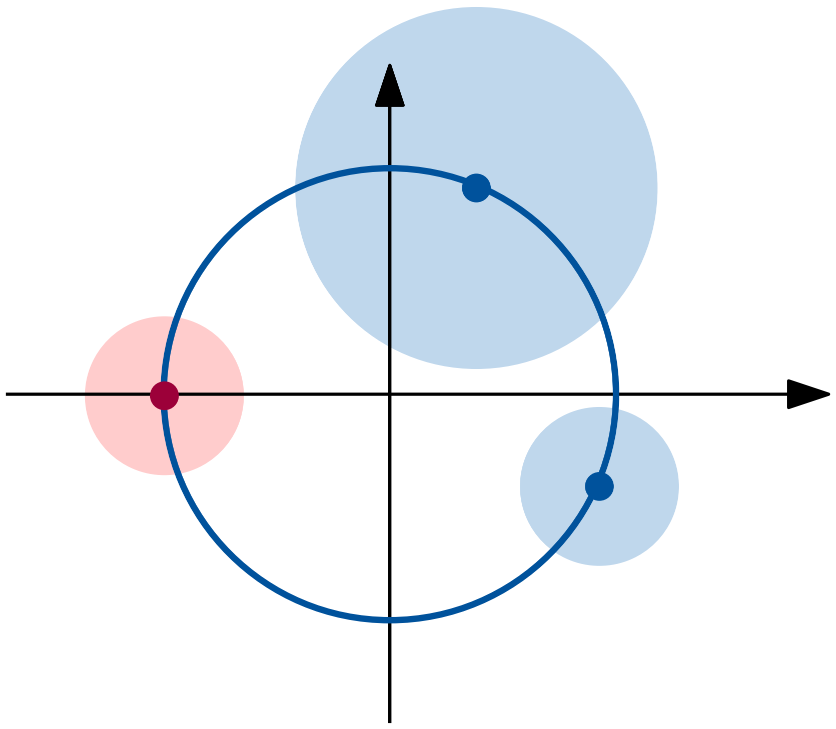
Given a point (a, b) satisfying this relationship, can we rewrite the relationship as $y = y(x)$ for a small ball around (a, b) (i.e. does $F(x, y) = 0$ implicitly define y as a function of x)?

Geometrically: is there a small part of the curve $F(x, y) = 0$, near (a, b), that looks like the graph of a function $y = y(x)$?

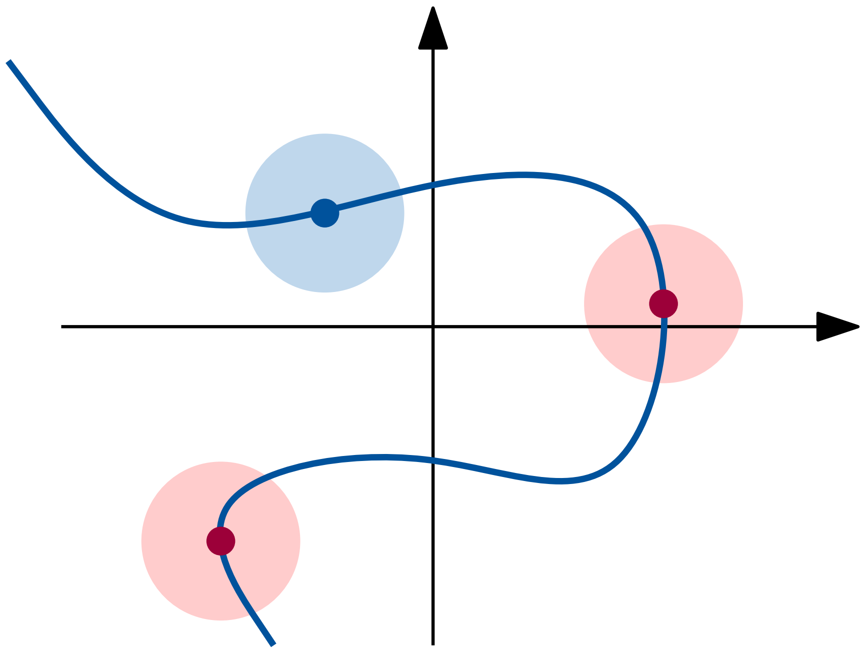




Geometrically: is there a small part of the curve $F(x, y) = 0$, near (a, b), that looks like the graph of a function $y = y(x)$?



For the unit circle:
If $b > 0$, then yes: $y = \sqrt{1 - x^2}$.
If $b < 0$, then yes: $y = -\sqrt{1 - x^2}$.
If $b = 0$ then no: e.g. at $(-1, 0)$, there are values of x arbitrarily close to -1 for which there is more than 1 value of y close to 0 (and a function is not allowed to have more than 1 output). This is because the unit circle has a vertical tangent at $(-1, 0)$.



For the curve in the bottom diagram, there are two points with vertical tangents, and near these two points, we cannot write y as a function of x. Near all other points on the curve, we can write $y = y(x)$.

How can we tell algebraically that the unit circle has a vertical tangent whenever $y = 0$, and so we can't write $y = y(x)$ there?

One answer: if we can write y as a differentiable function of x near (a, b), then we should be able to find $\left.\frac{dy}{dx}\right|_{x=a}$ using implicit differentiation:

$$x^2 + y^2 = 1$$

Differentiate both sides with respect to x, considering y as a function of x:

$$2x + 2y\frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$ in terms of x and y:

$$\frac{dy}{dx} = -\frac{x}{y}$$

This makes sense when $y \neq 0$.

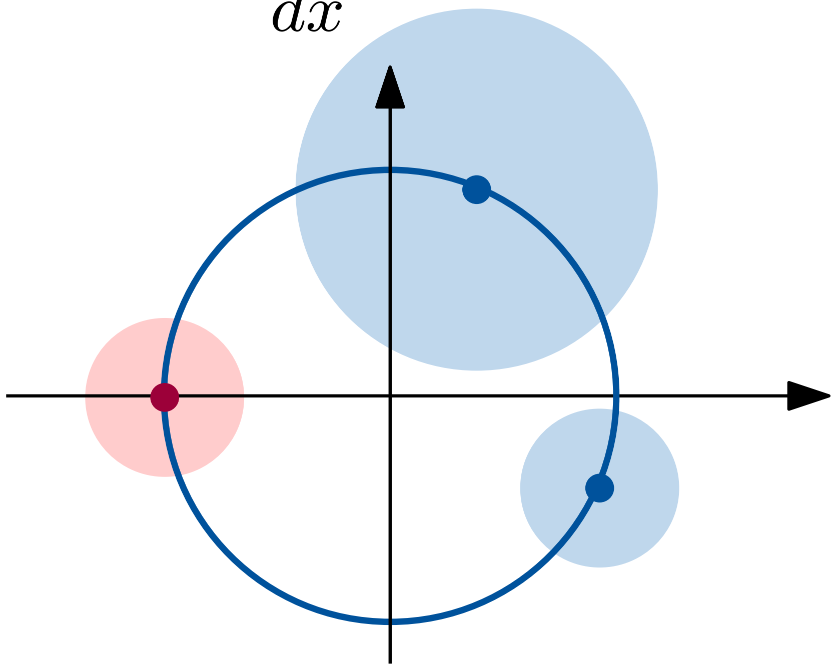
In general:

$$F(x, y) = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

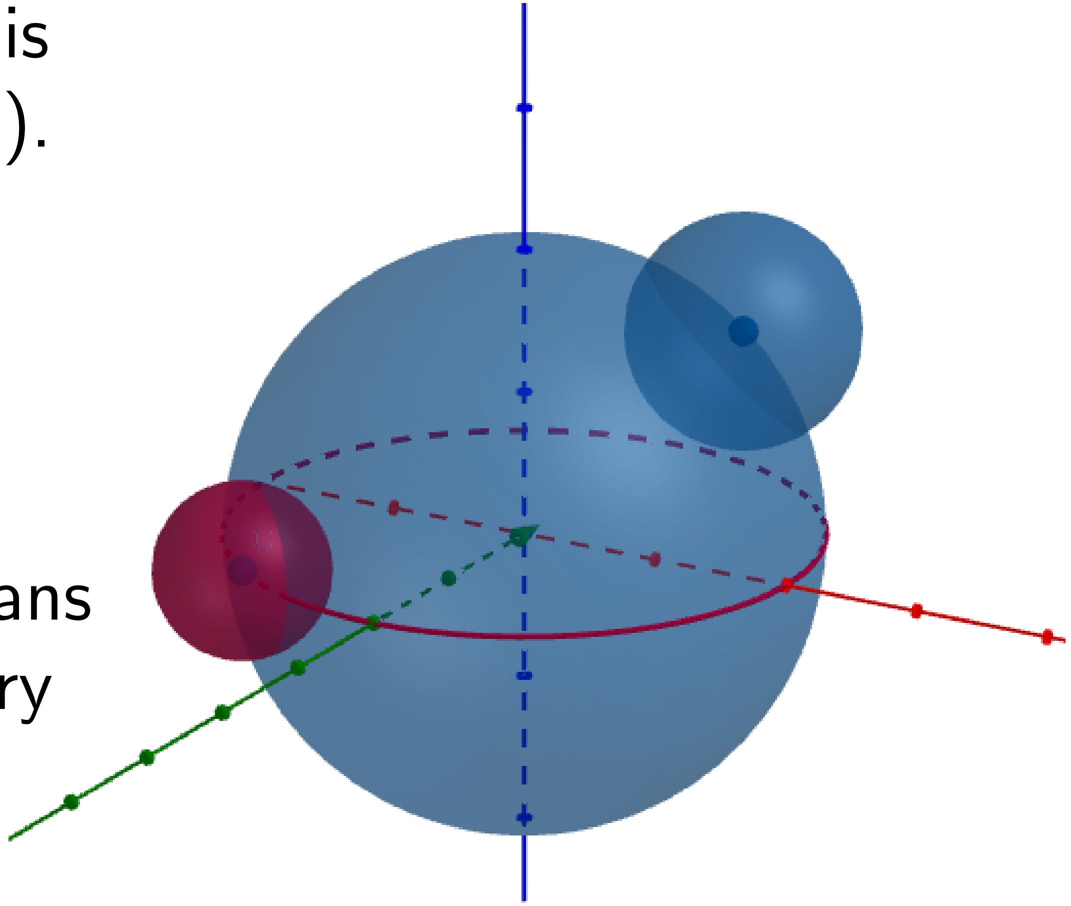
... when $F_y \neq 0$.



Theorem: Implicit Function Theorem 1.0: Suppose (a, b) satisfies $F(x, y) = 0$ and F has continuous first-order partial derivatives near (a, b). If $\frac{\partial F}{\partial y}(a, b) \neq 0$, then y can be expressed as a function of x near (a, b).

Now suppose we have a relationship between 3 variables $F(x, y, z) = 0$. This draws a surface in $\mathbb{R}^3$.
Near what points (a, b, c) on this surface can we write z as a function of x and y?

Example: Suppose $F(x, y, z) = x^2 + y^2 + z^2 - 4$, so our surface is $x^2 + y^2 + z^2 = 4$.
Then we want to say $z = \pm\sqrt{4 - x^2 - y^2}$, but $\pm$ is not allowed in a function (only one output allowed).
Near (a, b, c) with $c > 0$: all the z near c satisfy $z = \sqrt{4 - x^2 - y^2}$
Near (a, b, c) with $c < 0$: all the z near c satisfy $z = -\sqrt{4 - x^2 - y^2}$
Near (a, b, c) with $c = 0$: $z = \pm\sqrt{4 - x^2 - y^2}$ means there are two values of z, both near $c = 0$, for every value of x, y near a, b, so we cannot write z as a function of x, y here.



On the equator, we cannot write $z = z(x, y)$ because the tangent plane to the sphere is vertical there (p8).

As in the 1D case (p4), if z is a differentiable function of x and y near (a,b,c), then we must be able to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ by implicit differentiation:

$$\begin{aligned} x^2 + y^2 + \quad z(x,y)^2 &= 4 \\ 2x + 0 + 2z(x,y)\frac{\partial z}{\partial x} &= 0 \\ \frac{\partial z}{\partial x} &= -\frac{x}{z(x,y)} \end{aligned}$$

$$\begin{aligned} x^2 + y^2 + \quad z(x,y)^2 &= 4 \\ 0 + 2y + 2z(x,y)\frac{\partial z}{\partial y} &= 0 \\ \frac{\partial z}{\partial y} &= -\frac{y}{z(x,y)} \end{aligned}$$

This means the z-coordinate c must be nonzero. It turns out the converse is true: if $c \neq 0$, then the points near (a,b,c) satisfying $x^2+y^2+z^2=4$ can be described by $z=z(x,y)$ (see p8 for a geometric explanation).

In general:

$$\begin{aligned} F(x,y,z) &= 0 \\ \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} &= 0 \\ \frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} \end{aligned}$$

$$\begin{aligned} F(x,y,z) &= 0 \\ \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} &= 0 \\ \frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} \end{aligned}$$

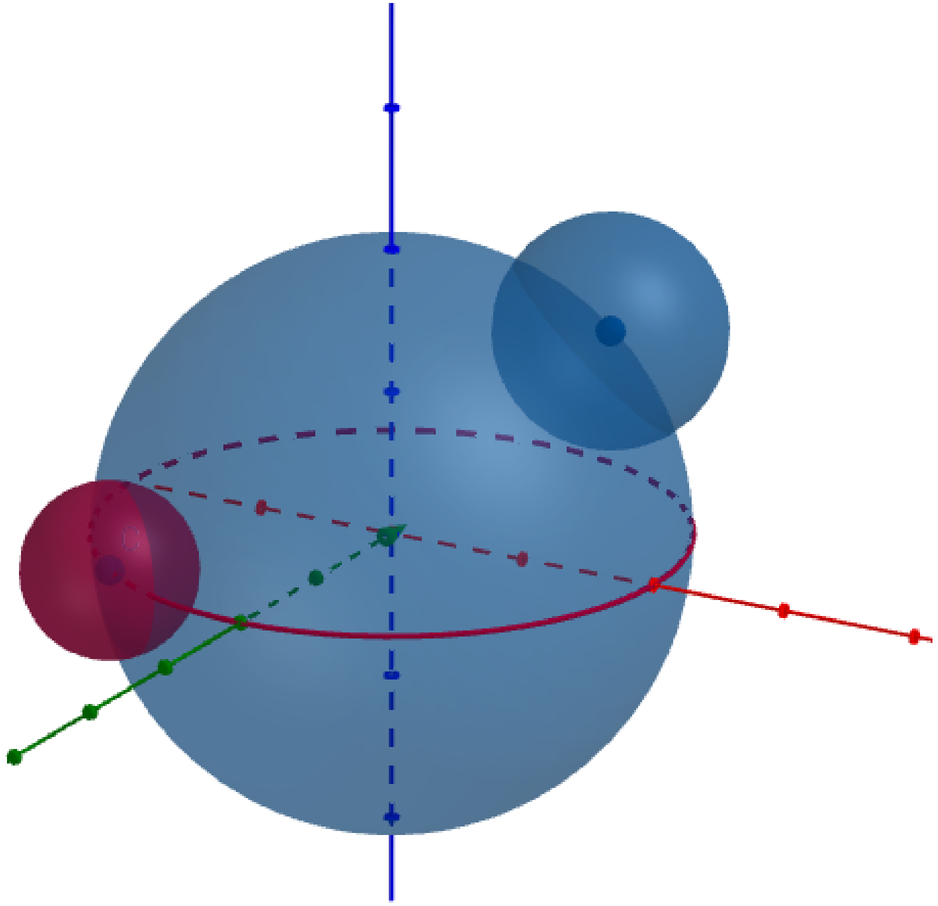
Again it turns out that, as long as $F_z \neq 0$, then we can write $z = z(x,y)$.

Theorem: Implicit Function Theorem 2.0: Suppose (a, b, c) satisfies $F(x, y, z) = 0$ and F has continuous first-order partial derivatives near (a, b, c). If $\frac{\partial F}{\partial z}(a, b, c) \neq 0$, then z can be expressed as a function of x and y near (a, b, c).

In this view, x, y are *independent variables* and z is a *dependent variable* (z depends on x and y).

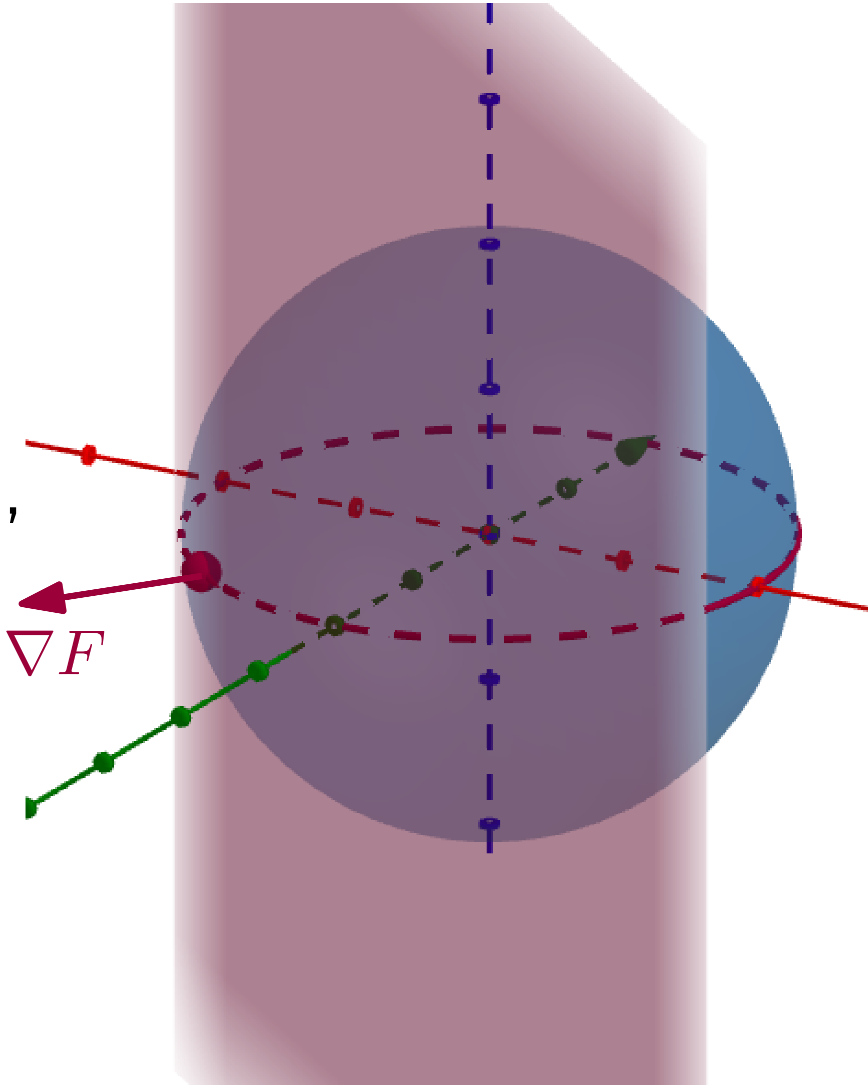
There is nothing special about z in this theorem: we can similarly say that x can be expressed as a function of y and z near (a, b, c) if $F_x(a, b, c) \neq 0$.

In the example of the sphere, $F_x = 2x$, $F_y = 2y$, $F_z = 2z$, and these three partial derivatives cannot all be zero (because $(0, 0, 0)$ is not on the sphere), so at any point on the sphere, it is possible to write some variable as a function of the other two. Indeed, on most points on the equator, we can write $x = x(y, z)$ and $y = y(x, z)$.



Theorem: Implicit Function Theorem 2.0: Suppose (a, b, c) satisfies $F(x, y, z) = 0$ and F has continuous first-order partial derivatives near (a, b, c). If $\dfrac{\partial F}{\partial z}(a, b, c) \neq 0$, then z can be expressed as a function of x and y near (a, b, c).

It is clear how to generalise this theorem to hypersurfaces $F(x_1, \ldots, x_n)$ in $\mathbb{R}^n$. But in $\mathbb{R}^3$ there is a geometric way to see why it is true: remember that $\nabla F(a, b, c) = F_x(a, b, c)\mathbf{i} + F_y(a, b, c)\mathbf{j} + F_z(a, b, c)\mathbf{k}$ is normal to the level set $F(x, y, z) = 0$ at (a, b, c). So, if $F_z(a, b, c) \neq 0$, then the normal is not horizontal direction, so the tangent plane is not vertical.



More examples of implicit differentiation:

Example: Suppose $2x + 2\ln(2y) = 5 - z^2 + xz$. Find $\dfrac{\partial z}{\partial x}$ when $(x, y, z) = (4, \frac{1}{2}, 1)$.

While we're talking about implicit differentiation: we can use implicit differentiation to obtain second-order partial derivatives.

Example: Suppose $2x + 2\ln(2y) = 5 - z^2 + xz$. Find $\dfrac{\partial^2 z}{\partial x^2}$ when $(x, y, z) = (4, \frac{1}{2}, 1)$.

Now back to implicit functions. Suppose we have 2 relationships between 4 variables,

$$F(x, y, z, w) = 0, \quad \text{and} \quad G(x, y, z, w) = 0$$

When can we write 2 of the variables (e.g. z, w) as functions of the other 2 variables (e.g. x, y)? What is the analogue of the condition in the 1-equation case that "the partial derivative of the relationship with respect to the dependent variable is nonzero"? (Notice that the number of dependent variables is the number of equations.)

Before we answer this: we have a notation problem. If we write $\frac{\partial z}{\partial x}$, then z is a function of x and which other variable(s)? $z = z(x, y)$? $z = z(x, w)$? $z = z(x, y, w)$? To make this clear, we indicate the other independent variables with a subscript: $\left(\frac{\partial z}{\partial x}\right)_y$ means we are considering z as a function of x and y.

We follow the same idea as before: if z, w can be written as differentiable functions of x, y, then we can find $\left(\frac{\partial z}{\partial x}\right)_y$, $\left(\frac{\partial z}{\partial y}\right)_x$, $\left(\frac{\partial w}{\partial x}\right)_y$, $\left(\frac{\partial w}{\partial y}\right)_x$ by implicit differentiation.

Example: Suppose $xy + y^2z + zw = 3$ and $w^2x + 3yz = 4$. Calculate $\left(\frac{\partial z}{\partial x}\right)_y$ and $\left(\frac{\partial w}{\partial x}\right)_y$ at the point P where $(x, y, z, w) = (1, 1, 1, 1)$.

From the example, it appears that calculating partial derivatives of implicit functions (i.e. functions defined by relationships between dependent and independent variables) requires solving a linear system. This is indeed true in general:

Differentiate $F(x,y,z,w)=0$ with respect to x: $$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\left(\frac{\partial z}{\partial x}\right)_y+\frac{\partial F}{\partial w}\left(\frac{\partial w}{\partial x}\right)_y=0$$

Differentiate $G(x,y,z,w)=0$ with respect to x: $$\frac{\partial G}{\partial x}+\frac{\partial G}{\partial z}\left(\frac{\partial z}{\partial x}\right)_y+\frac{\partial G}{\partial w}\left(\frac{\partial w}{\partial x}\right)_y=0$$

So, to find $\left(\frac{\partial z}{\partial x}\right)_y, \left(\frac{\partial w}{\partial x}\right)_y$ at a point P, we need to solve the linear system (in matrix notation)

$$\begin{pmatrix} F_z(P) & F_w(P) \\ G_z(P) & G_w(P) \end{pmatrix} \begin{pmatrix} \left(\frac{\partial z}{\partial x}\right)_y \\ \left(\frac{\partial w}{\partial x}\right)_y \end{pmatrix} = -\begin{pmatrix} F_x(P) \\ G_x(P) \end{pmatrix}$$

If the matrix on the left hand side is invertible, then we can multiply both sides by the inverse to solve for $\left(\frac{\partial z}{\partial x}\right)_y, \left(\frac{\partial w}{\partial x}\right)_y$, as in the previous example.

The matrix will be invertible if its determinant is nonzero.

Definition: The *Jacobian determinant* of two functions $F(x, y, \dots)$ and $G(x, y, \dots)$, *with respect to the variables* x, y, is the determinant

$$\frac{\partial(F,G)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{vmatrix} = \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}.$$

Note that the Jacobian determinant is usually not the determinant of a Jacobian matrix: F and G may depend on more variables than just x and y, but the partial derivatives with respect to those additional variables do not appear in the Jacobian determinant (one important exception: p22).

We can similarly define the Jacobian determinant of 3 functions with respect to 3 variables (see p 18), or n functions with respect to n variables: the number of functions must equal the number of variables to obtain a square matrix.

Reminder: differentiating $F(x,y,z,w)=0$ and $G(x,y,z,w)=0$ with respect to x showed that $$\begin{pmatrix} F_z(P) & F_w(P) \\ G_z(P) & G_w(P) \end{pmatrix} \begin{pmatrix} \left(\frac{\partial z}{\partial x}\right)_y \\ \left(\frac{\partial w}{\partial x}\right)_y \end{pmatrix} = -\begin{pmatrix} F_x(P) \\ G_x(P) \end{pmatrix}.$$

We can solve for $\left(\frac{\partial z}{\partial x}\right)_y, \left(\frac{\partial w}{\partial x}\right)_y$ at P if the matrix on the left hand side is invertible, i.e. if $\frac{\partial(F,G)}{\partial(z,w)} \neq 0$ at P. Actually, we can get a formula for $\left(\frac{\partial z}{\partial x}\right)_y, \left(\frac{\partial w}{\partial x}\right)_y$ using Cramer's rule from linear algebra: if A is an invertible square matrix, then the unique solution to $A\mathbf{x}=\mathbf{b}$ is $x_i = \frac{\det A_i}{\det A}$, where A_i denote the matrix obtained from A by replacing the ith column by $\mathbf{b}$ (§10.7, Theorem 6).

So, at P, $$\left(\frac{\partial z}{\partial x}\right)_y = \frac{\begin{vmatrix} -F_x(P) & F_w(P) \\ -G_x(P) & G_w(P) \end{vmatrix}}{\begin{vmatrix} F_z(P) & F_w(P) \\ G_z(P) & G_w(P) \end{vmatrix}} = -\frac{\frac{\partial(F,G)}{\partial(x,w)}}{\frac{\partial(F,G)}{\partial(z,w)}}, \quad \left(\frac{\partial w}{\partial x}\right)_y = \frac{\begin{vmatrix} F_z(P) & -F_x(P) \\ G_z(P) & -G_x(P) \end{vmatrix}}{\begin{vmatrix} F_z(P) & F_w(P) \\ G_z(P) & G_w(P) \end{vmatrix}} = -\frac{\frac{\partial(F,G)}{\partial(z,x)}}{\frac{\partial(F,G)}{\partial(z,w)}}.$$

We have shown that, if w, z are differentiable functions of x, y near P, then we probably have $\frac{\partial(F,G)}{\partial(z,w)} \neq 0$ at P.

So the question "can we write w, z as differentiable functions of x, y near P" is only a sensible question if $\frac{\partial(F,G)}{\partial(z,w)} \neq 0$ at P.

The implicit function theorem (next page) says that the answer is "yes": as long as $\frac{\partial(F,G)}{\partial(z,w)} \neq 0$ at P, then we can write w, z as differentiable functions of x, y near P. ("The obvious condition is actually the only condition" is a common story in mathematical theorems.)

Observe that we did not prove any part of the implicit function theorem (except in the case of one dependent variable and two independent variables, see p8). We informally explained where the condition $\frac{\partial(F,G)}{\partial(z,w)} \neq 0$ comes from.

Theorem 8: Implicit Function Theorem: Consider a set of n equations in $n+m$ variables:

$$\begin{aligned}
F_1(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) &= 0 \\
F_2(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) &= 0 \\
&\vdots \\
F_n(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) &= 0,
\end{aligned}$$

and a point $P = (a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n)$ that satisfies all the equations. Suppose each function F_i (for $i = 1, 2, \ldots n$) has continuous first-order partial derivatives with respect to each of the variables x_j and y_k (for $j = 1, 2, \ldots m$ and $k = 1, 2, \ldots n$) near P. Then, if the Jacobian determinant with respect to the dependent variables $\dfrac{\partial(F_1, \ldots, F_n)}{\partial(y_1, \ldots, y_n)}$ is not 0 at P, then the equations can be solved for $y_1, y_2, \ldots, y_m$ as functions of $x_1, x_2, \ldots, x_m$. Furthermore,

$$\left(\frac{\partial y_i}{\partial x_j}\right)_{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m} = -\frac{\dfrac{\partial(F_1, \ldots, F_n)}{\partial(y_1, \ldots, y_{i-1}, x_j, y_{i+1}, \ldots, y_n)}}{\dfrac{\partial(F_1, \ldots, F_n)}{\partial(y_1, \ldots, y_n)}}.$$

Example: Suppose
$$\begin{aligned} 3x + 2y + u - v^2 &= 0 \\ 4x + 3y + u^2 + vw &= 2 \\ xu^2 + w &= 1 \end{aligned}$$
Show that u, v, w can be written as functions of x, y when $u = 0$ and $v = 1$.

A special case of the implicit function theorem: the inverse function theorem

To simplify things, let's first work in 1D: consider the function $f(x) = x^2$.
f is not invertible, because $f(x) = f(-x)$.
But, near $x = 2$, f is "locally invertible": if $y = x^2$ and x is near 2, then $x = \sqrt{y}$.
f is also locally invertible near $x = -2$: if $y = x^2$ and x is near -2, then $x = -\sqrt{y}$.
f is not locally invertible near $x = 0$: if $y = x^2$ and x is near 0, then x can be $\sqrt{y}$ or $-\sqrt{y}$. We cannot write x as a single function of y.

We can use the implicit function theorem to show that $f(x) = x^2$ is locally invertible near $x = a$ if $a \neq 0$. Rewrite the relationship $y = x^2$ as $F(x, y) = x^2 - y = 0$. The implicit function theorem then says that we can write x as a function of y near $(x, y) = (a, b)$ as long as $\dfrac{\partial F}{\partial x}(a, b) \neq 0$, which is the condition $2a \neq 0$, i.e. $a \neq 0$.

The general 1D problem is as follows: given a function $f : \mathbb{R} \to \mathbb{R}$ differentiable at $x = a$, we seek a condition on a so that there is a differentiable function g that "locally inverts" f, in the sense that, for all x near a with $f(x) = y$, we have $g(y) = x$.

Rewrite $y = f(x)$ as $F(x, y) = 0$, where F is the function $F(x, y) = f(x) - y$. Let $b = f(a)$. Then the implicit function theorem (v1.0, p4) says that we can write x as a function of y near $(x, y) = (a, b)$ as long as $\frac{\partial F}{\partial x}(a, b) \neq 0$, i.e. when $\frac{df}{dx}(a) \neq 0$.

Under this condition, the implicit function theorem also tells us how to compute the derivative of the local inverse g:

$$\left.\frac{dg}{dy}\right|_{y=b} = \left.\frac{dx}{dy}\right|_{y=b} = -\left.\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}}\right|_{x=a,y=b} = -\frac{-1}{\left.\frac{df}{dx}\right|_{x=a}} = \frac{1}{\left.\frac{dy}{dx}\right|_{x=a}}.$$

An alternative way to obtain the derivative of the local inverse is using the chain rule: we have $y = f(x)$ and $x = g(y)$ for all x near a, so $g(f(x)) = x$. Differentiate both sides, using the chain rule on the left:

$$g'(f(x))f'(x) = 1$$

$$\left.\frac{dx}{dy}\right|_{y=b} = g'(f(x)) = \frac{1}{f'(x)} = \left.\frac{dy}{dx}\right|_{x=a}$$

The same argument holds for vector-valued functions $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^n$. If we write $\mathbf{y}=\mathbf{f}(\mathbf{x})$, then a local inverse $\mathbf{g}$ near $\mathbf{x}=\mathbf{a}$ must satisfy $\mathbf{g}(\mathbf{y})=\mathbf{x}$. So finding a local inverse is the same as expressing $x_1,\dots,x_n$ as functions of $y_1,\dots,y_n$.

As in the 1D case, rewrite each coordinate of the equation $\mathbf{y}=\mathbf{f}(\mathbf{x})$ as $F_i(x_1,\dots,x_n,y_1,\dots,y_n)=f_i(x_1,\dots,x_n)-y_i=0$, for $i=1,2,\dots,n$. Then the condition from the implicit theorem is $\dfrac{\partial(F_1,\dots,F_n)}{\partial(x_1,\dots,x_n)}\neq 0$, i.e. $\dfrac{\partial(f_1,\dots,f_n)}{\partial(x_1,\dots,x_n)}\neq 0$ (because $F_i=f_i-y_i$ means $\dfrac{\partial F_i}{\partial x_j}=\dfrac{\partial f_i}{\partial x_j}$ for all i,j), or equivalently $\det D\mathbf{f}\neq 0$.

To find the derivative of the local inverse: we have $\mathbf{g}(\mathbf{f}(\mathbf{x}))=\mathbf{g}(\mathbf{y})=\mathbf{x}$.

Take Jacobian matrix of both sides: $D(\mathbf{g}\circ\mathbf{f})(\mathbf{a})=$ identity matrix

Using the matrix version of the chain rule on the left hand side: $[D\mathbf{g}(\mathbf{f}(\mathbf{a}))][D\mathbf{f}(\mathbf{a})]=$ identity matrix

Multiple both sides by the inverse matrix to $D\mathbf{f}(\mathbf{a})$: $D\mathbf{g}(\mathbf{f}(\mathbf{a}))=[D\mathbf{f}(\mathbf{a})]^{-1}$

Theorem: Inverse Function Theorem: Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be a function. Suppose all first-order partial derivatives are continuous at a point $\mathbf{a}$. Then, if $\det D\mathbf{f}(\mathbf{a}) \neq 0$, then $\mathbf{f}$ is locally invertible around $\mathbf{x} = \mathbf{a}$ - i.e. there is a function $\mathbf{g}$ defined on a small ball around $\mathbf{b} = \mathbf{f}(\mathbf{a})$ such that $\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}$. Furthermore, we have $D\mathbf{g}(\mathbf{b}) = [D\mathbf{f}(\mathbf{a})]^{-1}$, and $\dfrac{\partial(x_1, \dots, x_n)}{\partial(y_1, \dots, y_n)} = \dfrac{1}{\dfrac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)}}$ (writing $\mathbf{y}$ for $\mathbf{f}(\mathbf{x})$).

The last part, about the Jacobain determinant, comes from taking determinants of both sides of the equation $D\mathbf{g}(\mathbf{b}) = [D\mathbf{f}(\mathbf{a})^{-1}]$, remembering that $\det A^{-1} = \frac{1}{\det A}$. This is the generalisation of $\left.\dfrac{dx}{dy}\right|_{y=b} = \dfrac{1}{\left.\dfrac{dy}{dx}\right|_{x=a}}$ in the 1D case (p20) - it requires all the partial derivatives. Generally, $\dfrac{\partial x_i}{\partial y_j} \neq \dfrac{1}{\dfrac{\partial y_j}{\partial x_i}}$.

A particularly interesting type of functions $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ are change of coordinates (ex. sheet #21 Q2).