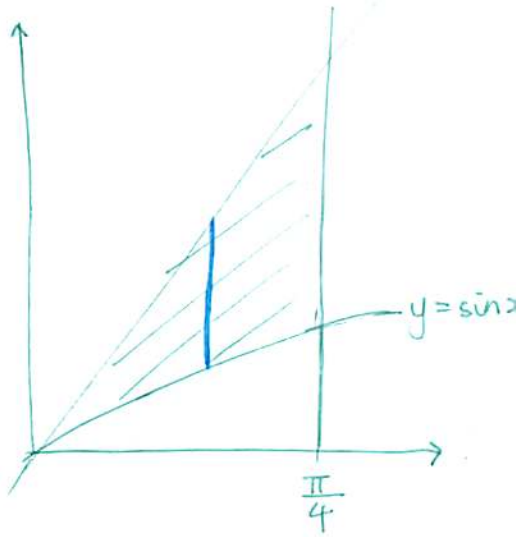
1. (7 points) Let R be the region bounded by the curves

$$y = \sin x, \quad y = 16x, \quad x = \frac{\pi}{4}.$$

Find the volume of the solid obtained by rotating R about the y-axis. **Simplify your answer as much as possible.**



Use cylindrical shells:

$$\text{Volume} = \int_0^{\frac{\pi}{4}} 2\pi x\,(16x - \sin x)\,dx$$

$$= \int_0^{\frac{\pi}{4}} 32\pi x^2 - 2\pi x \sin x \, dx$$

$$= \left[32\pi \frac{x^3}{3}\right]_0^{\frac{\pi}{4}} - 2\pi \int_0^{\frac{\pi}{4}} x \sin x \, dx$$

$$= \frac{32\pi}{3}\left(\frac{\pi}{4}\right)^3 - 2\pi \int_0^{\frac{\pi}{4}} x \sin x \, dx$$

Integration by parts:

$u = x \quad dv = \sin x\, dx$

$du = dx \quad v = -\cos x$

$$\int_0^{\frac{\pi}{4}} x \sin x \, dx = \Big[x(-\cos x)\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} -\cos x \, dx$$

$$= \frac{\pi}{4}\left(-\frac{1}{\sqrt{2}}\right) - 0 - \Big[-\sin x\Big]_0^{\frac{\pi}{4}}$$

$$= \frac{-\pi}{4\sqrt{2}} - \left(-\frac{1}{\sqrt{2}} - 0\right) = \frac{-\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}$$

So volume $= \frac{32\pi}{3}\left(\frac{\pi}{4}\right)^3 - 2\pi\left(\frac{-\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$

$$= \frac{\pi^4}{6} + \frac{\pi^2}{2\sqrt{2}} - \sqrt{2}\,\pi$$

2. (7 points) Compute the following integral:

$$\int x^2 \left(25 - x^2\right)^{-\frac{3}{2}}\, dx.$$

substitution $x = 5\sin\theta$

$dx = 5\cos\theta\, d\theta$

so $25 - x^2 = 25 - 25\sin^2\theta$

$= 25\cos^2\theta$

$$= \int (5\sin\theta)^2 (5\cos\theta)^{-3/2} (5\cos\theta\, d\theta)$$

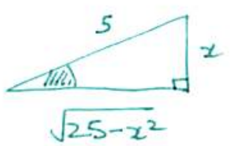
$$= \int \frac{\sin^2\theta}{\cos^2\theta}\, d\theta$$

$$= \int \tan^2\theta\, d\theta$$

$$= \int \sec^2\theta - 1\, d\theta$$

$$= \tan\theta - \theta + C$$

$$= \frac{x}{\sqrt{25-x^2}} - \arcsin\left(\frac{x}{5}\right) + C$$



$\frac{x}{5} = \sin\theta = \frac{\text{opp.}}{\text{hyp}}$

$\tan\theta = \frac{\text{opp.}}{\text{adj}} = \frac{x}{\sqrt{25-x^2}}$

3. (7 points) Compute the following integral:

$$\int \frac{9x}{(x-2)(x+1)^2}dx.$$

Partial fractions:

$$\frac{9x}{(x-2)(x+1)^2} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$9x = A(x+1)^2 + B(x-2)(x+1) + C(x-2)$$

$x=2$: $\quad 18 = A9 \Rightarrow A = 2$

$x=-1$: $\quad -9 = C(-3) \Rightarrow C=3$

coeff of x^2: $\quad 0 = A + B \Rightarrow B = -2$

So $$\int \frac{9x}{(x-2)(x+1)^2}dx = \int \frac{2}{x-2} - \frac{2}{x+1} + \frac{3}{(x+1)^2}\,dx$$

$$= 2\ln|x-2| - 2\ln|x+1| + 3\frac{(x+1)^{-1}}{-1} + C$$

$$= 2\ln|x-2| - 2\ln|x+1| - \frac{3}{x+1} + C$$