MATH 2207: Linear Algebra
Homework 4, due 13:45 Monday, October 31

You must justify your answers to receive full credit.

1. Suppose
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$
Find the following determinants, and explain your answers:

a) $\begin{vmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{vmatrix}$ b) $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$

c) $\begin{vmatrix} a & b & c \\ d & e & f \\ g+3a & h+3b & i+3c \end{vmatrix}$ d) $\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}$ e) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$

2. Which of the following sets are subspaces of $\mathbb{R}^3$? Explain your answer (if it is a subspace, use a theorem or check all three closure axioms directly; if it is not a subspace, give a counterexample to one of the closure axioms).

a) W is the set of vectors of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where $x, y, z \geq 0$.

b) U is the set of vectors of the form $\begin{bmatrix} 5b+2c \\ b \\ c \end{bmatrix}$ where b, c can take any value.

c) X is the set of vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfying $a - 2b = 4c$ and $2a = c + 3b$.

d) Z is the set of vectors of the form $\begin{bmatrix} a \\ 0 \\ b+1 \end{bmatrix}$, where a, b can take any value. (Hint: this is hard.)

3. Suppose
$$A = \begin{bmatrix} -2 & 2 & 6 & 4 \\ 2 & -3 & -11 & -7 \\ -3 & 4 & 14 & 9 \end{bmatrix}, \quad \operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

a) Find a basis for the null space of A.

b) Find a basis for the column space of A.

c) Find a basis for the row space of A.

d) Show that your answers for parts a) and c), together, form a basis for $\mathbb{R}^4$.

4. Let $\mathbb{P}_3$ denote the set of polynomials of degree at most 3, with the standard basis $\mathcal{B} = \{1, t, t^2, t^3\}$.

   a) Use coordinate vectors to determine if the set of polynomials
   $$\left\{1 + 2t^3, 2 + t - 3t^2, -t + 3t^2 + 4t^3\right\}$$
   is linearly independent.

   Let $T : \mathbb{P}_3 \to \mathbb{P}_3$ be the function given by $T\mathbf{p} = t\frac{d}{dt}\mathbf{p}$, i.e.
   $$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = t(a_1 + 2a_2 t + 3a_3 t^2).$$

   b) Show that T is a linear transformation.

   c) Find the matrix of T relative to the standard basis $\mathcal{B}$.

5. Let D be the determinant
   $$D = \begin{vmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \\ 1 & x & x^2 & x^3 \end{vmatrix}$$
   (so D is a function of x).

   a) Without computation, find three solutions to $D(x) = 0$. Explain your answer.

   b) **Optional**: Explain why these three are the only solutions to $D(x) = 0$. (Hint: $D(x)$ is a polynomial in x, of what degree?)
   This is an example of a "Vandermonde determinant".

6. State whether each of the following statements is (always) true or (sometimes) false. If it is true, give a brief justification; if it is false, give a counterexample with an explanation.

   a) The determinant of a square matrix is the product of its diagonal entries.

   b) If A is a square matrix, then $\det(-A) = -\det(A)$.

   Let $\mathbb{P}_3$ be the set of polynomials of degree at most 3.

   c) The set of polynomials in $\mathbb{P}_3$ satisfying $\mathbf{p}(1) = 1$ is a subspace of $\mathbb{P}_3$.

   d) The set of polynomials of the form $\mathbf{p}(t) = at^2 + bt + a$ is a subspace of $\mathbb{P}_3$.

   Let $M_{2\times 2}$ be the set of 2×2 matrices.

   e) The set of 2×2 symmetric matrices (i.e. all 2×2 matrices A with $A = A^T$) is a subspace of $M_{2\times 2}$.

   f) The determinant function $\det : M_{2\times 2} \to \mathbb{R}$ is a linear transformation.

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