Recall from last week:

FACT: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

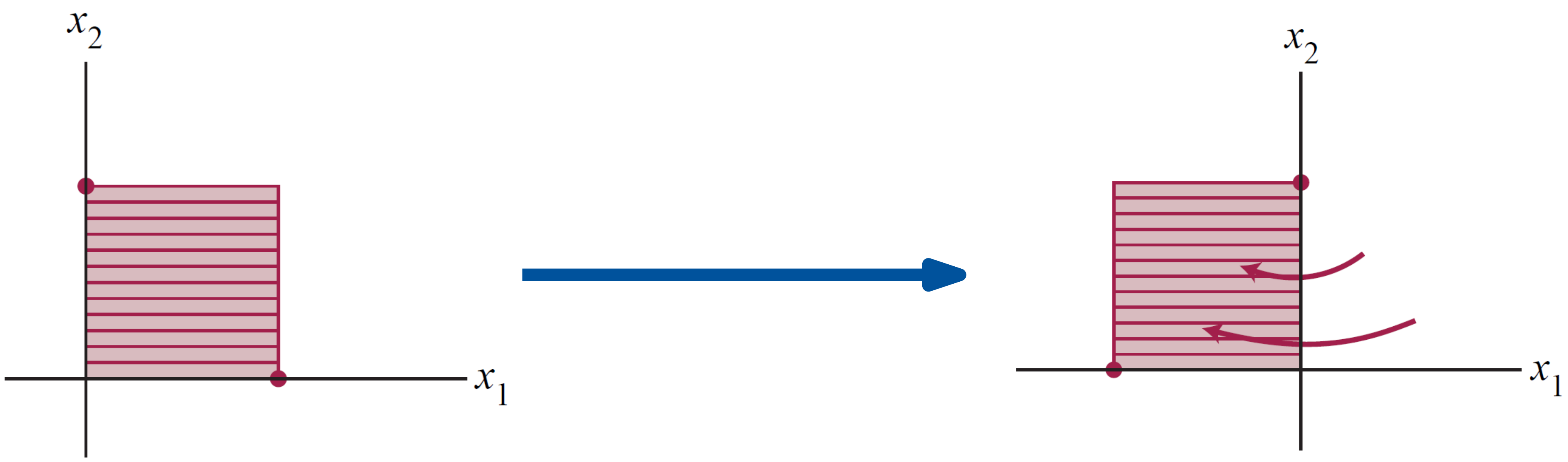
i) if $ad - bc \neq 0$, then A is invertible and $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$,

ii) if $ad - bc = 0$, then A is not invertible,

What is the mysterious quantity $ad - bc$?

Formula for 2×2 matrix: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.

Example: The standard matrix for reflection through the x_2-axis is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.
Its determinant is -$1 \cdot 1 - 0 \cdot 0 = -1$: reflection does not change area, but changes orientation.



Exercise: Guess what the determinant of a rotation matrix is, and check your answer.

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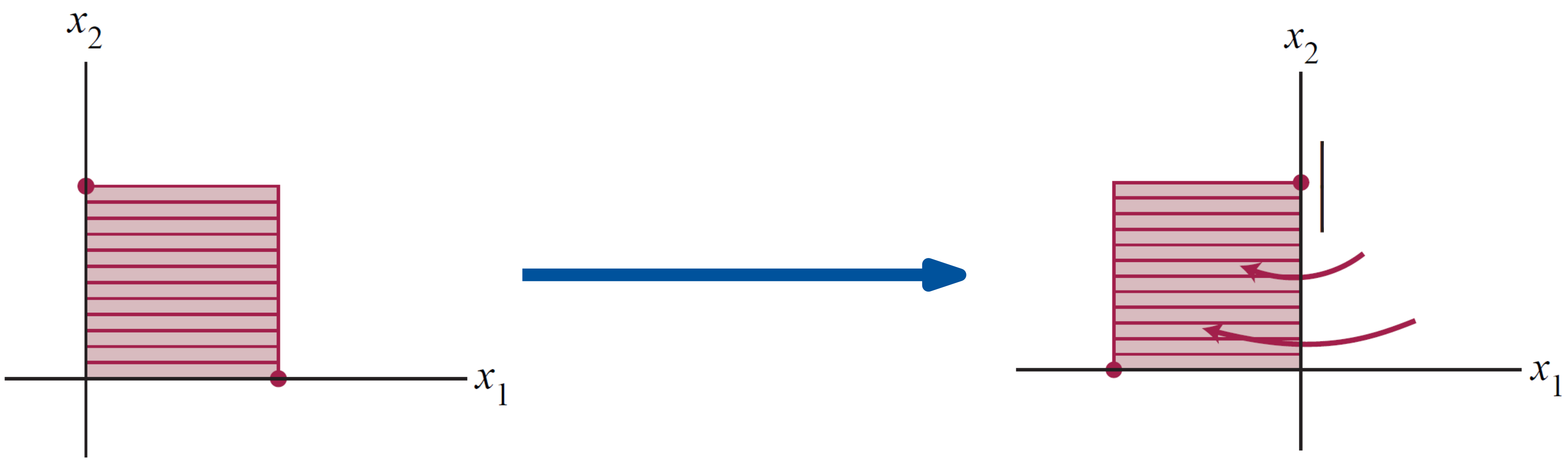
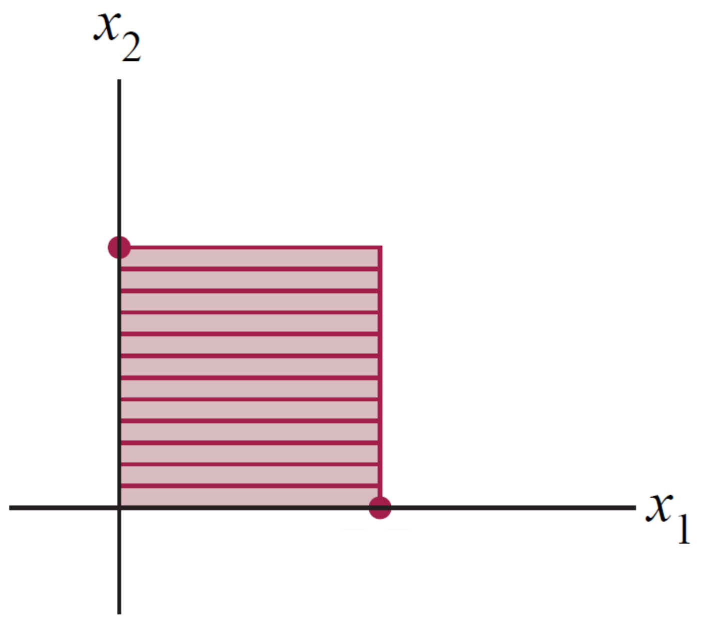
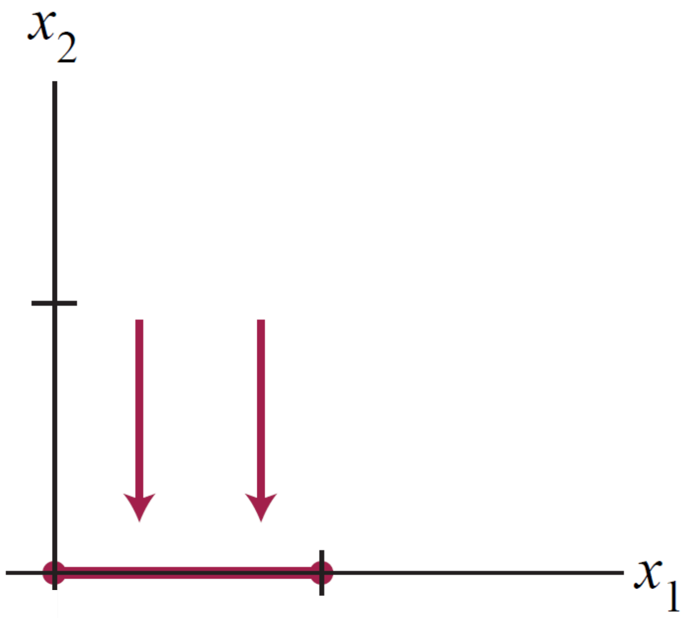


Exercise: Guess what the determinant of a rotation matrix is, and check your answer.

Formula for 2×2 matrix: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.

Example: The standard matrix of projection onto the x_1-axis is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Its determinant is $1 \cdot 0 - 0 \cdot 0 = 0$. Projection sends the unit square to a line, which has zero area.





Theorem: A is invertible if and only if $\det A \neq 0$.

Calculating Determinants

Notation: A_{ij} is the submatrix obtained from matrix A by deleting the ith row and jth column of A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Recall that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and we let $\det[a] = a$.

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

EXAMPLE: Compute the determinant of $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

THEOREM 1 The determinant of an $n \times n$ matrix A can be computed by expanding across any row or down any column:

$$\det A = (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} + \cdots + (-1)^{i+n} a_{in} \det A_{in}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \qquad \text{(expansion across row } i\text{)}$$

$$\det A = (-1)^{1+j} a_{1j} \det A_{1j} + (-1)^{2+j} a_{i2} \det A_{2j} + \cdots + (-1)^{n+j} a_{nj} \det A_{nj}$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \qquad \text{(expansion down column } j\text{)}$$

Use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

EXAMPLE: An easier way to compute the determinant of $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

EXAMPLE:

$$\begin{vmatrix} 4 & 3 & 1 & 8 \\ 5 & 0 & 3 & -1 \\ 0 & 0 & -3 & 0 \\ 7 & 0 & 2 & 4 \end{vmatrix} =$$

It's easy to compute the determinant of a triangular matrix:

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$

(upper triangular) (lower triangular)

EXAMPLE:

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{vmatrix} =$$

THEOREM 2: If A is a triangular matrix, then $\det A$ is the product of the diagonal entries of A.

How the determinant changes under row operations:

1. Replacement: add a multiple of one row to another row. $R_i \to R_i + cR_j$
   determinant does not change.
2. Interchange: interchange two rows. $R_i \to R_j$, $R_j \to R_i$
   determinant changes sign.
3. Scaling: multiply all entries in a row by a nonzero constant. $R_i \to cR_i$, $c \neq 0$
   determinant scales by a factor of c.

To help you remember:

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad \begin{vmatrix} 1 & c \\ 0 & 1 \end{vmatrix} = 1, \quad \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad \begin{vmatrix} c & 0 \\ 0 & 1 \end{vmatrix} = c.$$

Because we can compute the determinant by expanding down columns instead of across rows, the same changes hold for "column operations".

1. Replacement: $R_i \to R_i + cR_j$ determinant does not change.
2. Interchange: $R_i \to R_j$, $R_j \to R_i$ determinant changes sign.
3. Scaling: $R_i \to cR_i$, $c \neq 0$ determinant scales by a factor of c.

Usually we compute determinants using a mixture of "expanding across a row or down a column with many zeroes" and "row reducing to a triangular matrix".

Example:

$$\begin{vmatrix} 2 & 3 & 4 & 6 \\ 0 & 5 & 0 & 0 \\ 5 & 5 & 6 & 7 \\ 7 & 9 & 6 & 10 \end{vmatrix} = 5 \begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} \overset{\text{factor out 2 from } R_1}{=} 5 \cdot 2 \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} \overset{\substack{R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 7R_1}}{=} 5 \cdot 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix}$$

$$\overset{\text{factor out -4 from } R_2}{=} 5 \cdot 2 \cdot -4 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} \overset{R_3 \to R_3 + 8R_2}{=} 5 \cdot 2 \cdot -4 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix} = 5 \cdot 2 \cdot -4 \cdot 1 \cdot 1 \cdot 5 = -200.$$

1. Replacement: $R_i \to R_i + cR_j$ determinant does not change.
2. Interchange: $R_i \to R_j$, $R_j \to R_i$ determinant changes sign.
3. Scaling: $R_i \to cR_i$, $c \neq 0$ determinant scales by a factor of c.

Useful fact: If two rows of A are multiples of each other, then $\det A = 0$.

Proof: Use a replacement row operation to make one of the rows into a row of zeroes, then expand along that row.

Example:

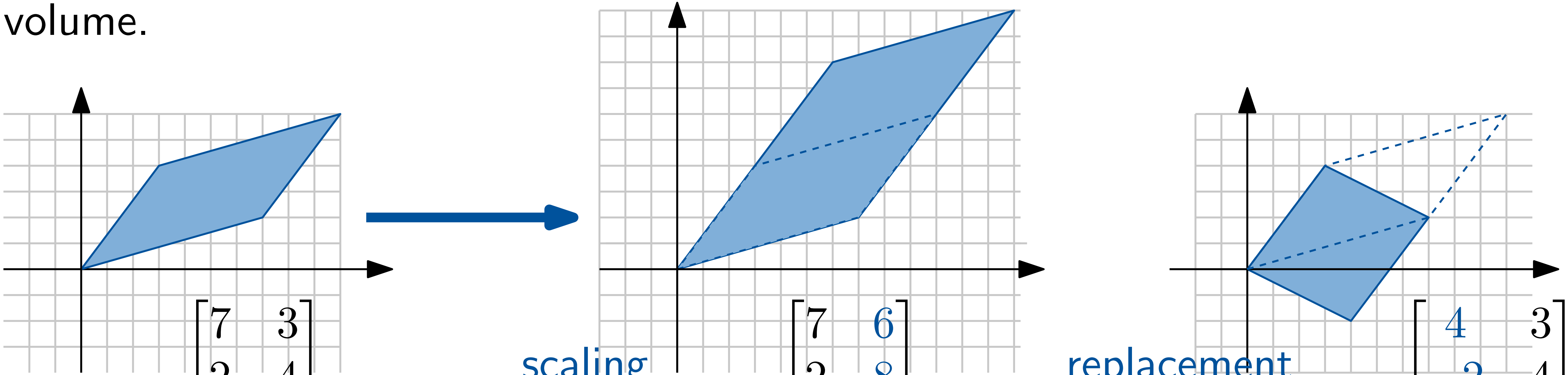
$$\begin{vmatrix} 1 & 3 & 4 \\ 5 & 9 & 3 \\ 2 & 6 & 8 \end{vmatrix} \overset{R_3 \to R_3 - 2R_1}{=} \begin{vmatrix} 1 & 3 & 4 \\ 5 & 9 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0 \begin{vmatrix} 3 & 4 \\ 9 & 3 \end{vmatrix} - 0 \begin{vmatrix} 1 & 4 \\ 5 & 3 \end{vmatrix} + 0 \begin{vmatrix} 1 & 3 \\ 5 & 9 \end{vmatrix} = 0.$$

Why does the determinant change like this under row and column operations? Two views:

Either: It is a consequence of the expansion formula in Theorem 1;

Or: By thinking about the signed volume of the image of the unit cube under the associated linear transformation:

2. Interchanging columns changes the orientation of the image of the unit cube.
3. Scaling a column applies an expansion to one side of the image of the unit cube.
1. Column replacement rearranges the image of the unit cube without changing its volume.



$$\begin{bmatrix} 7 & 3 \\ 2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 7 & 6 \\ 2 & 8 \end{bmatrix} \qquad \begin{bmatrix} 4 & 3 \\ -2 & 4 \end{bmatrix}$$

Properties of the determinant:

$\det(A^T) = \det A.$

Theorem 6: Determinants are multiplicatve:

$$\det(AB) = \det A \det B.$$

In particular:

$\det(A^{-1}) =$ $\qquad\qquad\qquad$ $\det(cA) =$

Properties of the determinant:

Theorem 4: Invertibility and determinants: A square matrix A is invertible if and only if $\det A \neq 0$.

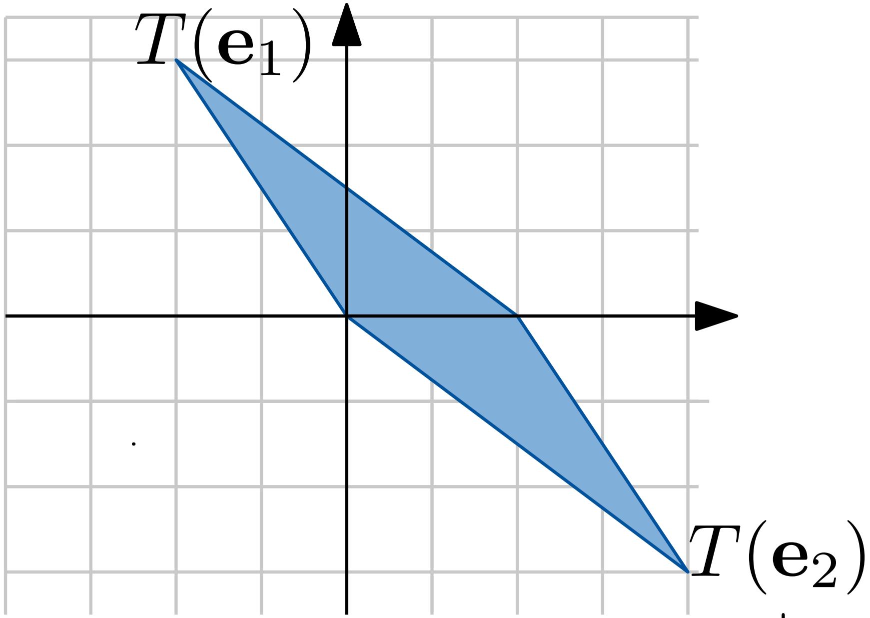
Proof 1: By the Invertible Matrix Theorem, A is invertible if and only if $\text{rref}(A)$ has n pivots. Row operations multiply the determinant by nonzero numbers. So $\det A = 0$ if and only if $\det(\text{rref}(A)) = 0$, which happens precisely when $\text{rref}(A)$ has fewer than n pivots.

Proof 2: By the Invertible Matrix Theorem, A is invertible if and only if its columns span $\mathbb{R}^n$. Since the image of the unit cube is a subset of the span of the columns, this image has zero volume if the columns do not span $\mathbb{R}^n$.

So we can use determinants to test whether $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in $\mathbb{R}^n$ is linearly independent, or if it spans $\mathbb{R}^n$: it does when $\det\left(\begin{bmatrix} | & & | \\ \mathbf{v}_1 & \ldots & \mathbf{v}_n \\ | & & | \end{bmatrix}\right) \neq 0$.

Other applications: finding volumes of regions with determinants

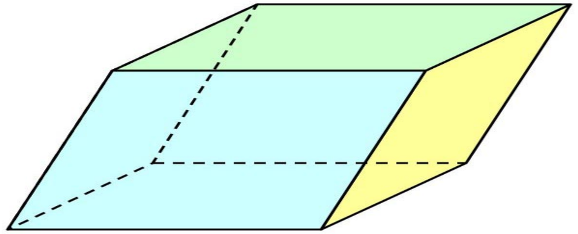
Example: Find the area of the parallelogram with vertices $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-2\\3\end{bmatrix}, \begin{bmatrix}4\\-3\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}$.



Answer: This parallelogram is the image of the unit square under a linear transformation T with $T(\mathbf{e}_1) = \begin{bmatrix}-2\\3\end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix}4\\-3\end{bmatrix}$.

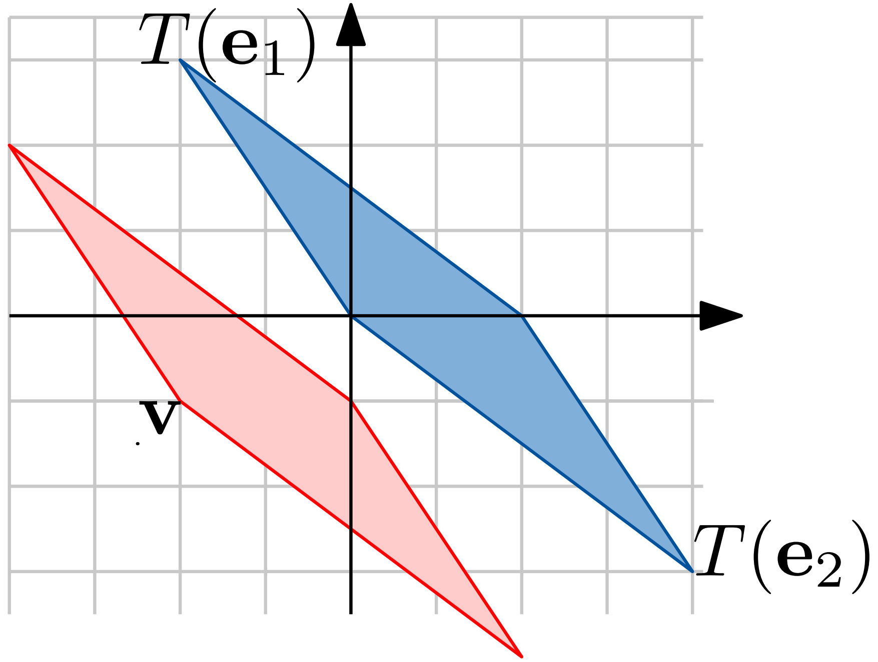
So area of parallelogram $= \left|\det \begin{bmatrix}-2 & 4\\3 & -3\end{bmatrix}\right| \times$ area of unit square $= |-12| \cdot 1 = 12$.

This works for any parallelogram where the origin is one of the vertices (and also in $\mathbb{R}^3$, for parallelopipeds).



Other applications: finding volumes of regions with determinants

Example: Find the area of the parallelogram with vertices $\begin{bmatrix}-2\\-1\end{bmatrix}, \begin{bmatrix}-4\\2\end{bmatrix}, \begin{bmatrix}2\\-4\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}$.



Answer: Use a translation to move one of the vertices of the parallelogram to the origin - this does not change the area.

The formula for this translation function is $\mathbf{x} \mapsto \mathbf{x} - \mathbf{v}$, where $\mathbf{v}$ is one of the vertices of the parallelogram.

Here, the vertices of the translated parallelogram are $\begin{bmatrix}-2\\-1\end{bmatrix} - \begin{bmatrix}-2\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}-4\\2\end{bmatrix} - \begin{bmatrix}-2\\-1\end{bmatrix} = \begin{bmatrix}-2\\3\end{bmatrix}, \quad \begin{bmatrix}2\\-4\end{bmatrix} - \begin{bmatrix}-2\\-1\end{bmatrix} = \begin{bmatrix}4\\-3\end{bmatrix}, \quad \begin{bmatrix}0\\-1\end{bmatrix} - \begin{bmatrix}-2\\-1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}.$

So, by the previous example, the area of the parallelogram is 12.

Other applications: solving linear systems using determinants

Cramer's rule: Let A be an invertible $n \times n$ matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{\det \begin{bmatrix} | & & | & & | \\ \mathbf{a}_1 & \dots & \mathbf{b} & \dots & \mathbf{a}_n \\ | & & | & & | \end{bmatrix}}{\det A}.$$

put $\mathbf{b}$ in the ith column instead of $\mathbf{a}_i$

Proof:

$$A \begin{bmatrix} | & & | & & | \\ \mathbf{e}_1 & \dots & \mathbf{x} & \dots & \mathbf{e}_n \\ | & & | & & | \end{bmatrix} = \begin{bmatrix} | & & | & & | \\ A\mathbf{e}_1 & \dots & A\mathbf{x} & \dots & A\mathbf{e}_n \\ | & & | & & | \end{bmatrix} = \begin{bmatrix} | & & | & & | \\ \mathbf{a}_1 & \dots & \mathbf{b} & \dots & \mathbf{a}_n \\ | & & | & & | \end{bmatrix}.$$

So

$$\det A \det \begin{bmatrix} | & & | & & | \\ \mathbf{e}_1 & \dots & \mathbf{x} & \dots & \mathbf{e}_n \\ | & & | & & | \end{bmatrix} = \det \begin{bmatrix} | & & | & & | \\ \mathbf{a}_1 & \dots & \mathbf{b} & \dots & \mathbf{a}_n \\ | & & | & & | \end{bmatrix}.$$

this is x_i - expand along ith row

Applying Cramer's rule to $\mathbf{b} = \mathbf{e}_i$ gives a formula for each entry of A^{-1} (see Theorem 8 in textbook; this formula is called the adjugate or classical adjoint).

The 2×2 case of this formula is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Cramer's rule is much slower than row-reduction for linear systems with actual numbers, but is useful for obtaining theoretical results.

Example: If every entry of A is an integer and $\det A = 1$ or -1, then every entry of A^{-1} is an integer.

Proof: Cramer's rule tells us that every entry of A^{-1} is the determinant of an integer matrix divided by $\det A$. And the determinant of an integer matrix is an integer.

Exercise: using the fact $\det AB = \det A \det B$, prove the converse (if every entry of A and of A^{-1} is an integer, then $\det A = 1$ or -1).