

For the next few weeks, we focus on differentiation of multivariate functions (in a different order from the textbook):

$\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ (i.e. **vector-valued** functions)

- This week: differentiating a multivariate function (§12.2, 12.3 first two pages, 12.4 first two pages, 12.6 first two pages, fifth and sixth pages)
- Week 9: the chain rule, for differentiating compositions (§12.5, and the matrix version in §12.6)

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ (i.e. **scalar-valued** functions)

- Week 10: direction of greatest increase, tangent planes, Taylor polynomials (§12.7, 12.9 first four pages)
- Week 11: classifying critical points (§13.1, the subsection “Classifying Critical Points” until Example 7; the rest is in Week 11)
- Week 12: finding maxima and minima (§13.1-13.5 8E, §13.1-13.4 7E)

Notation: we call the m “outputs” of \mathbf{f} by f_1, f_2, \dots, f_m , these are the **coordinate functions**. e.g. $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is denoted $\mathbf{f}(x, y) = (f_1(x, y), f_2(x, y), f_3(x, y))$. We will often analyze \mathbf{f} by analysing its coordinate functions separately.

For one-variable functions, the derivative is the limit of a difference quotient:

$$\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

To discuss the differentiability of multivariate functions, we must first define the limit of a multivariate function $\lim_{(x,y) \rightarrow (a,b)} g(x,y)$.

Unlike the limits of 2-variable Riemann sums that we saw in multiple integration, the limit of a 2-variable function **cannot** be calculated by taking 1D limits separately in the x and y directions. It requires a more careful analysis.

On the next page we give an informal definition of a limit for functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, but we will concentrate on when the domain is \mathbb{R}^2 and the codomain is \mathbb{R} .

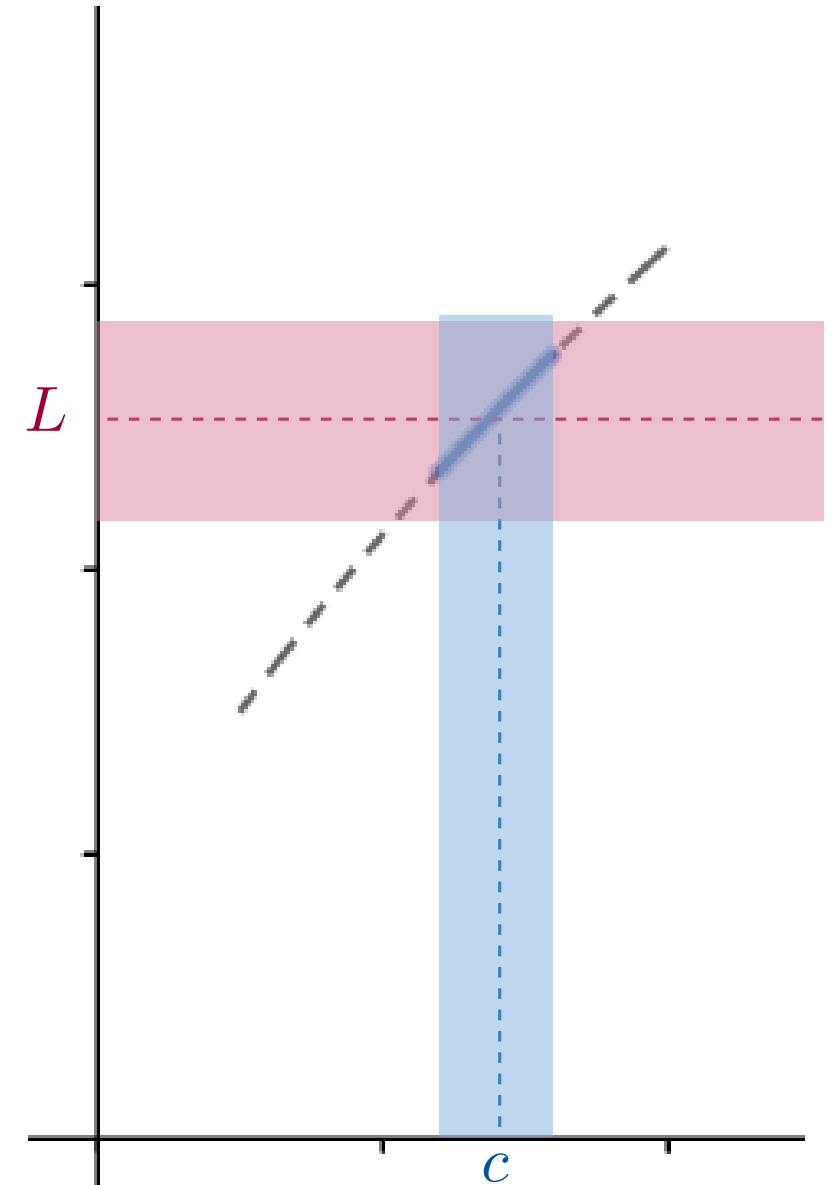
We will first discuss ways to show that a limit does not exist (p5-8, ex. sheet #14 q1), and then ways to evaluate a limit that does exist (p9-12).

§12.2: Limits and Continuity

Remember the informal definition of a single-variable limit:

Definition: Given a function $f(x)$ defined near a point c , the statement $\lim_{x \rightarrow c} f(x) = L$ means: we can ensure that $f(x)$ is as close as we want to L by taking x close enough (but not equal) to c .

In other words: given any small interval around L (the height of the red rectangle), we can find a small interval around c (the width of the blue rectangle) so the values of $f(x)$ when x is in this small “blue” interval all lie in the “red” interval around L (i.e. the part of the graph of f in the blue rectangle is also in the red rectangle).

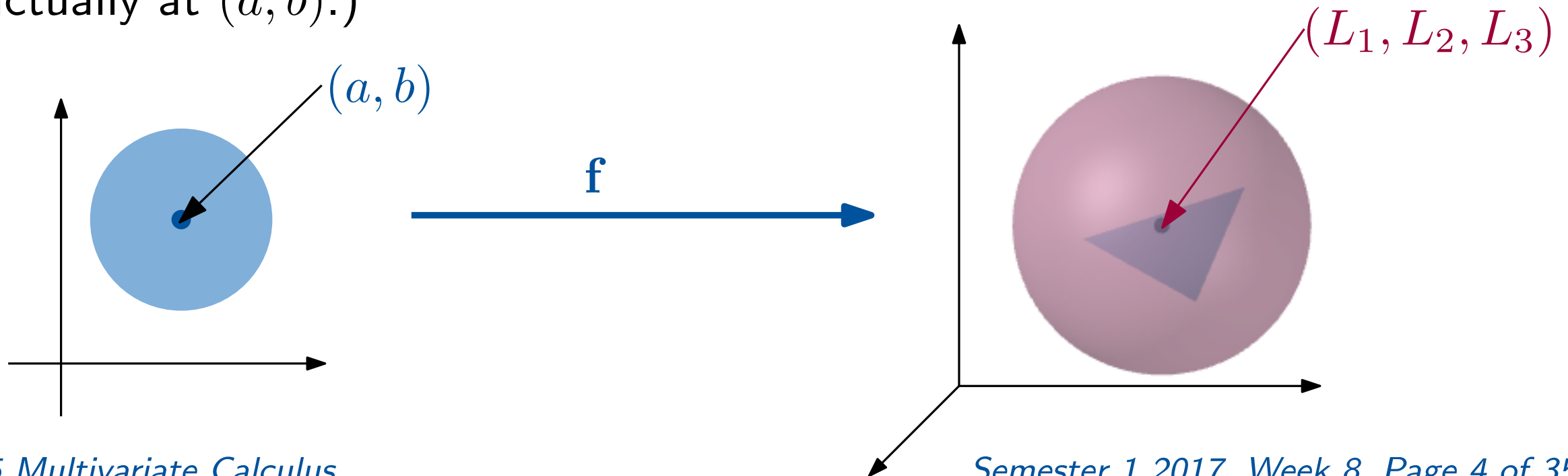


A limit of a multivariate function is the same idea, using balls and spheres instead of intervals:

Example: Given a 2-variable function $\mathbf{f} : \mathcal{D} \rightarrow \mathbb{R}^3$ whose domain \mathcal{D} contains points close to (a, b) , the statement $\lim_{(x,y) \rightarrow (a,b)} \mathbf{f}(x, y) = (L_1, L_2, L_3)$ means:

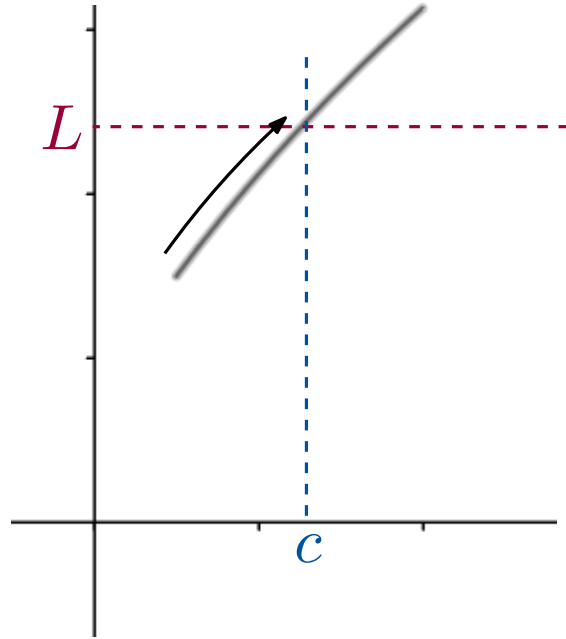
given a small sphere around (L_1, L_2, L_3) , we can find a small disk around (a, b) such that the image of this disk under \mathbf{f} is entirely contained in the sphere.

(Strictly speaking, $\mathbf{f}(a, b)$ does not need to be in the sphere for the limit statement to hold, because a limit is about how a function behaves **around** (a, b) but not actually at (a, b) .)

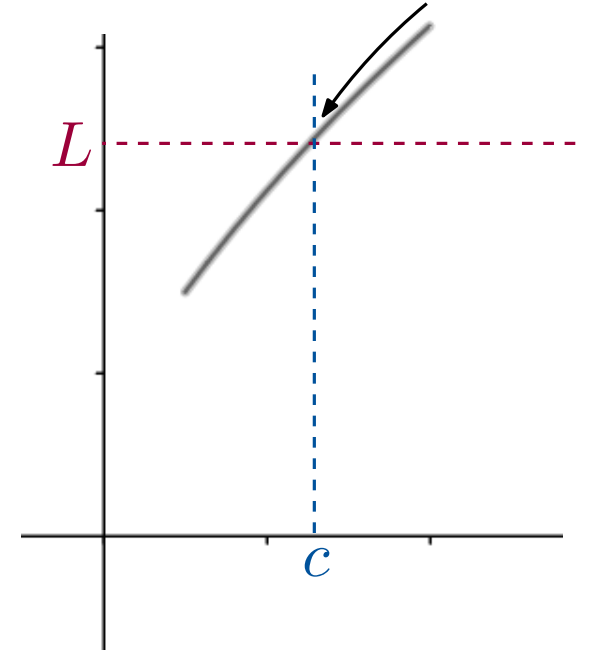


Another informal way to think about single-variable limits is: how does $f(x)$ behave as x moves towards c ? This is the **one-sided limit**:

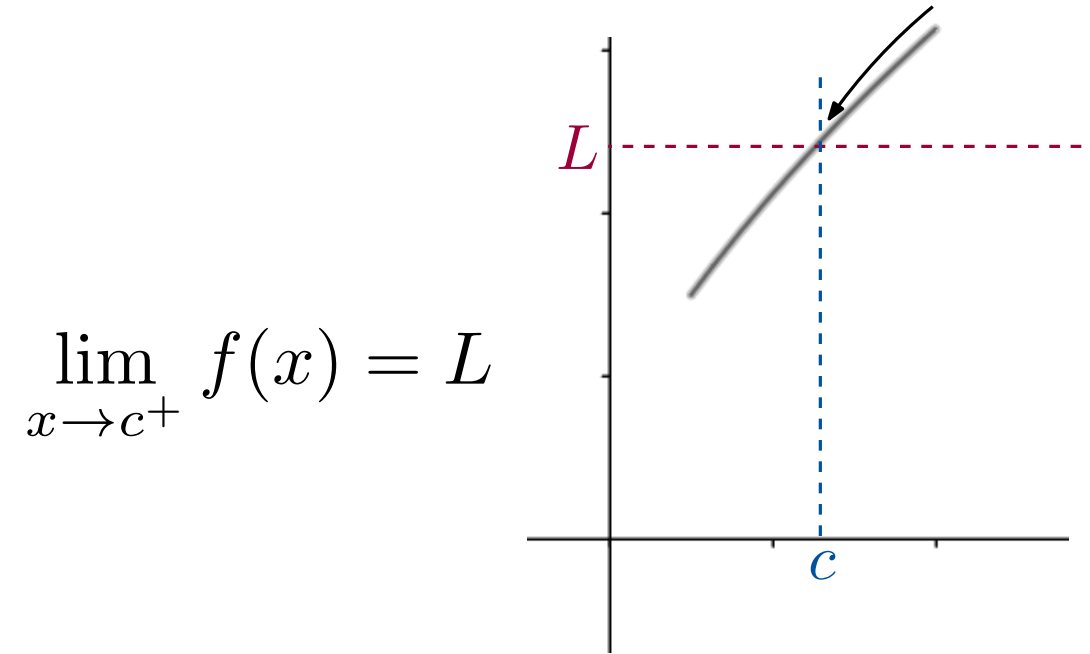
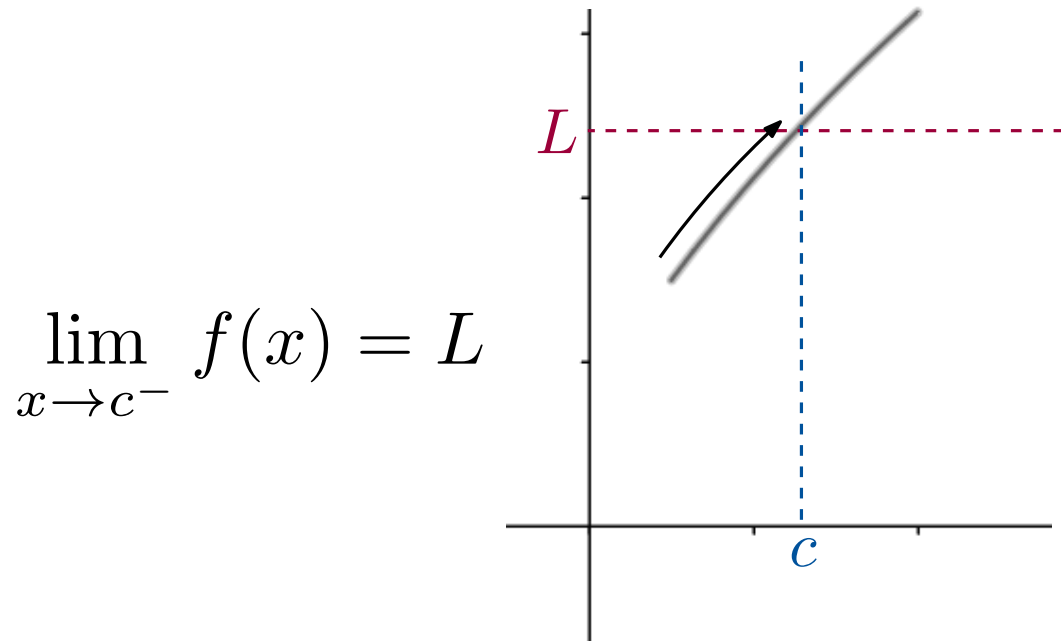
$$\lim_{x \rightarrow c^-} f(x) = L$$



$$\lim_{x \rightarrow c^+} f(x) = L$$



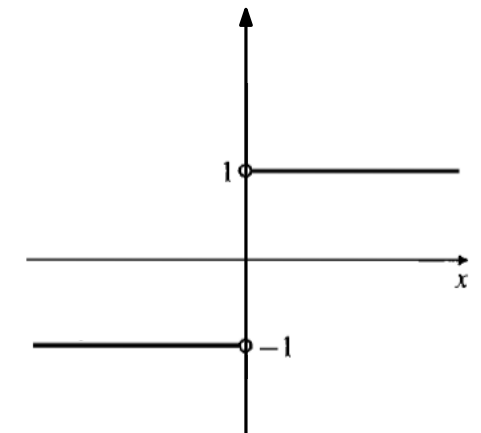
Another informal way to think about single-variable limits is: how does $f(x)$ behave as x moves towards c ? This is the **one-sided limit**:



It is a theorem that $\lim_{x \rightarrow c} f(x)$ exists if and only if the two one-sided limits exist and are equal - i.e. $f(x)$ “goes towards” the same number no matter how we move towards c .

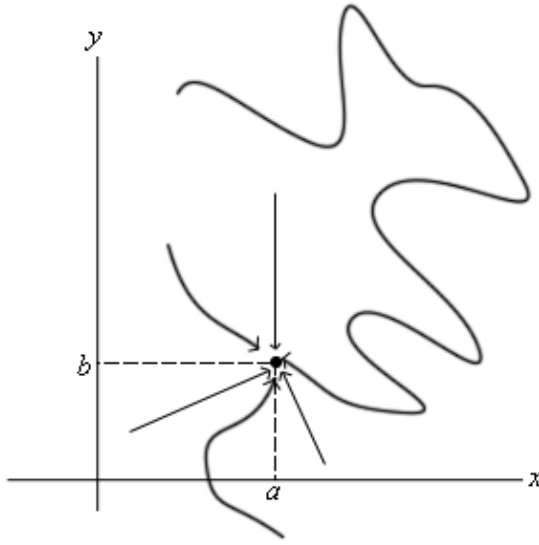
Example: $\lim_{x \rightarrow 0} \frac{x}{|x|}$ does not exist because $\lim_{x \rightarrow 0^-} \frac{x}{|x|} = -1$,

$\lim_{x \rightarrow 0^+} \frac{x}{|x|} = 1$, and these limits are not equal.



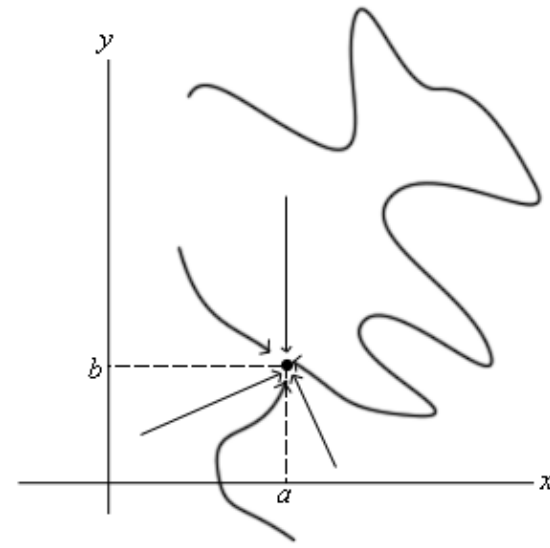
The same is true for multivariate limits, but there are now many more ways for (x, y) to approach (a, b) .

Each way of approach can be formalised as a **path**, i.e. a function $t \mapsto (x(t), y(t))$ such that $x(c) = a, y(c) = b$. (Imagine drawing one of the paths in the diagram, and recording the position of your pen at time t . Write c for the time that your pen reaches the point of interest (a, b) .) We then study f by considering the values that f takes along the path, i.e. by considering the composition $f(x(t), y(t))$.



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Theorem: Multivariate Limits and Paths: Let $f : \mathcal{D} \rightarrow \mathbb{R}$ be a two-variable function, and suppose \mathcal{D} contains points arbitrarily close to (a, b) . We have

$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$ if and only if, for **all paths** $t \mapsto (x(t), y(t))$ such that $x(c) = a, y(c) = b$, the limits $\lim_{t \rightarrow c} f(x(t), y(t))$ **all exist and are equal to L** .

Because the existence of the 2D limit is equivalent to the existence of 1D limits along **infinitely many** paths, it is not practical to use this theorem to prove the existence of a 2D limit. However, the theorem is useful for showing a 2D limit

HK. doesn't exist: simply find two paths along which the limits are different.

Example: Show that the limit $\lim_{(x,y) \rightarrow (-1,2)} \frac{x^2 - 1}{4x^2 - y^2}$ does not exist.

Example: Show that the limit $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^3 + y^3}$ does not exist.

Now we give some strategies for showing that a 2-variable limit does exist. This will use the concept of continuity, which has the same definition as the 1D case.

Definition: An n -variable function $\mathbf{f} : \mathcal{D} \rightarrow \mathbb{R}^m$ is *continuous* at a point (a_1, \dots, a_n) in the domain \mathcal{D} if

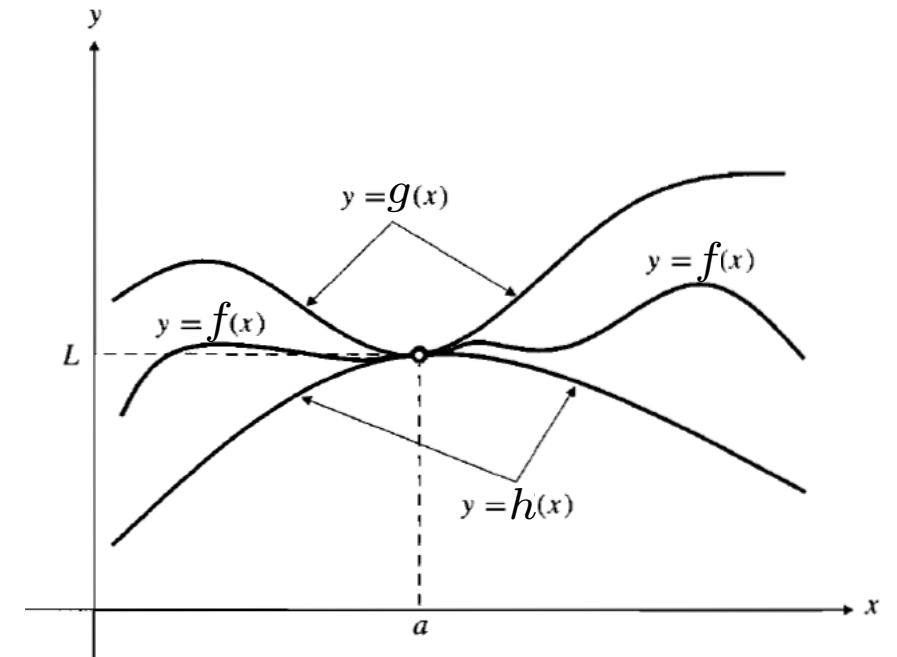
$$\lim_{(x_1, \dots, x_n) \rightarrow (a_1, \dots, a_n)} \mathbf{f}(x_1, \dots, x_n) = \mathbf{f}(a_1, \dots, a_n).$$

As in the 1D case, elementary functions (i.e. sums, products and compositions of x^n , e^x , $\ln x$, $\sin x$, $\cos x$) are continuous. So the following example is easy:

Example: Evaluate the limit $\lim_{(x,y) \rightarrow (-1,2)} \frac{x^2 - 2}{y^2 - 1}$, or prove that it does not exist.

In more complicated examples, our main tool for evaluating limits is the squeeze theorem. The multivariate squeeze theorem is a very simple extension of the 1D statement. (The diagram is in 1D, but we can easily imagine a 2D version.)

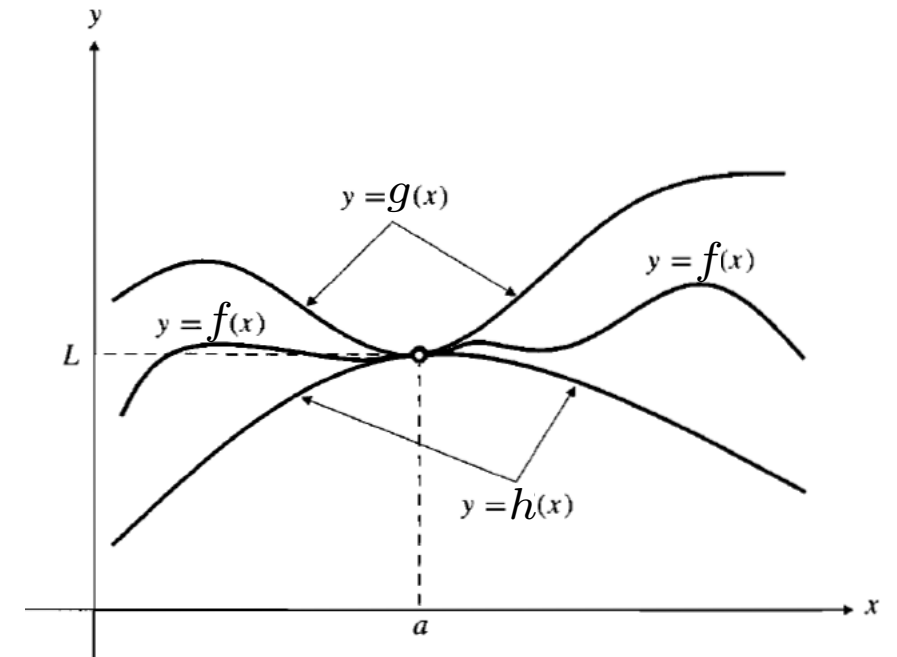
Squeeze Theorem: Suppose there are functions $g(x, y)$ and $h(x, y)$ such that, for all points (x, y) in the domain of f that are near (a, b) , we have the inequality $h(x, y) \leq f(x, y) \leq g(x, y)$. Suppose also that $\lim_{(x,y) \rightarrow (a,b)} g(x, y) = \lim_{(x,y) \rightarrow (a,b)} h(x, y) = L$. Then $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$ exists and is also equal to L .



We will often choose our squeezing functions $g(x, y)$ and $h(x, y)$ to be elementary functions, so their limits are easy to calculate, using continuity.

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In the special case where we want to show $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = 0$, it is enough to find one squeezing function $g(x, y)$ such that $\lim_{(x,y) \rightarrow (a,b)} g(x, y) = 0$ and

$-g(x, y) \leq f(x, y) \leq g(x, y)$ - this inequality is equivalent to $|f(x, y)| \leq g(x, y)$.

Here's a simple 2-dimensional version of the standard 1D squeeze theorem example (see Homework 3 final question).

Example: Evaluate $\lim_{(x,y) \rightarrow (0,0)} xy^2 \sin\left(\frac{1}{y}\right)$, or prove that the limit does not exist.

Most applications of the Squeeze Theorem in 2D don't involve trigonometric functions, and look more like this next example:

Example: Evaluate the limit $\lim_{(x,y) \rightarrow (0,0)} \frac{yx^2}{x^2 + y^2}$, or prove that it does not exist.

Summary on analysing the limit $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$:

Step 1 If f is a continuous function that is defined at (a,b) , then the limit is $f(a,b)$ (see p9).

Step 2 Evaluate the limit of f along straight line paths: $\lim_{x \rightarrow a} f(x,b)$, $\lim_{y \rightarrow b} f(a,y)$, $\lim_{x \rightarrow 0} f(x, mx)$ (if $(a,b) = (0,0)$). If you find two different limits, then f does not have a limit at (a,b) (see p8).

Step 3 Evaluate the limit of f along paths of the form $y = x^n$ or $(x,y) = (t^i, t^j)$ (if $(a,b) = (0,0)$). If these give different limits from the paths in Step 2, then f does not have a limit at (a,b) (see ex. sheet #14 Q1).

Step 4 Try to prove that $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$ is the value of the limits found in Steps 2 and 3, by using the Squeeze Theorem (see p12).

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(Note that the squeeze theorem only applies to scalar-valued functions, because there is no concept of $<$ or $>$ in \mathbb{R}^m for $m > 1$. But we can apply the squeeze theorem to the coordinate functions of a vector-valued function - \mathbf{f} has a limit if and only if each coordinate function f_i has a limit.)

§12.3-4: Partial Derivatives

Remember that the derivative of a single-variable function is the limit of a difference quotient, measuring the rate of change of f as we change the input variable x :

$$f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

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The derivative of a 2-variable function will be a more complicated concept because there are many different ways to change the input variables (x, y) . We start with the simplest way, where we fix one variable and change the other:

Definition: The *first-order partial derivatives* of the function $f(x, y)$, with respect to the variables x and y respectively, are given by:

$$f_x(x, y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k}$$

These are **1D limits**, not the 2D limits of the previous section.

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It is clear how to define first-order partial derivatives for an n -variable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ - there will be n of them, one for each variable (which will change that variable and keep the other $n - 1$ variables fixed).

And for a vector-valued function $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$, we can take partial derivatives of each coordinate function, so there will be nm partial derivatives altogether (see p31).

(The textbook writes $f_1(x, y)$ to mean “differentiate with respect to the first variable”, i.e. what we are calling $f_x(x, y)$. But this causes problems when \mathbf{f} is vector-valued, because f_i is also the i th coordinate function of \mathbf{f} .)

If f is an elementary function, then we can use our single-variable differentiation rules to calculate $\frac{\partial f}{\partial x}$, by treating y as a constant (and similarly for $\frac{\partial f}{\partial y}$).

Example: $\frac{\partial}{\partial x} x^3 y^2 = 3x^2 y^2$; $\frac{\partial}{\partial y} x^3 y^2 = 2x^3 y$.

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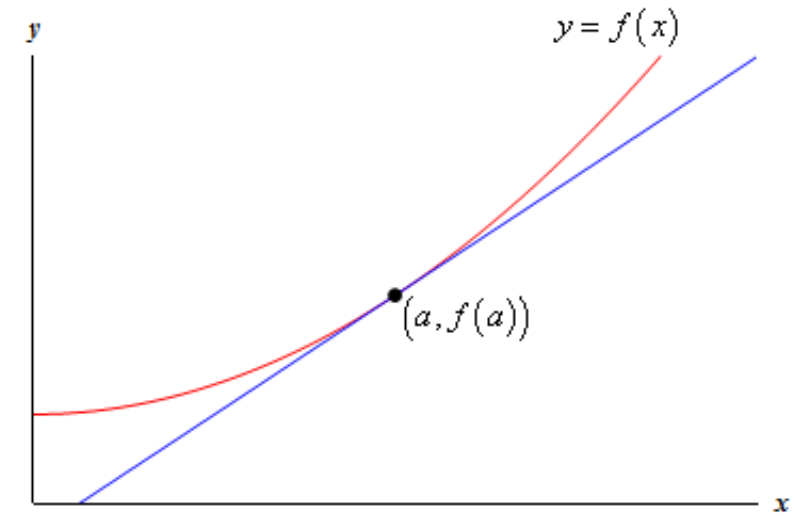
Example: Find the first-order partial derivatives of $f(x, y) = \frac{xy}{x+1}$ at $(1, 2)$.

When f is defined by different formulae around (a, b) , we need to use the limit definition to calculate the partial derivatives at (a, b) .

Example: Let $f(x, y) = \begin{cases} \frac{y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$. Find $f_y(0, 0)$.

The geometric meaning of partial derivatives

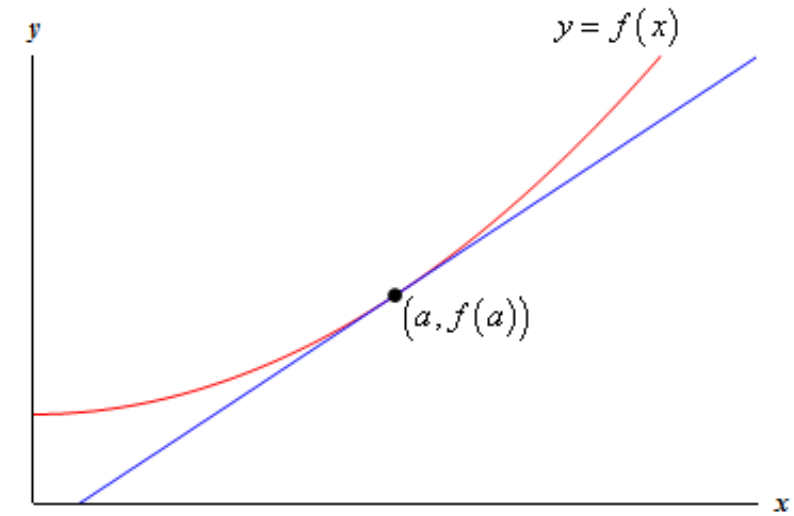
Recall that the derivative $f'(a)$ of a single-variable function f is the slope of the tangent line to the graph of f at the point $(a, f(a))$.



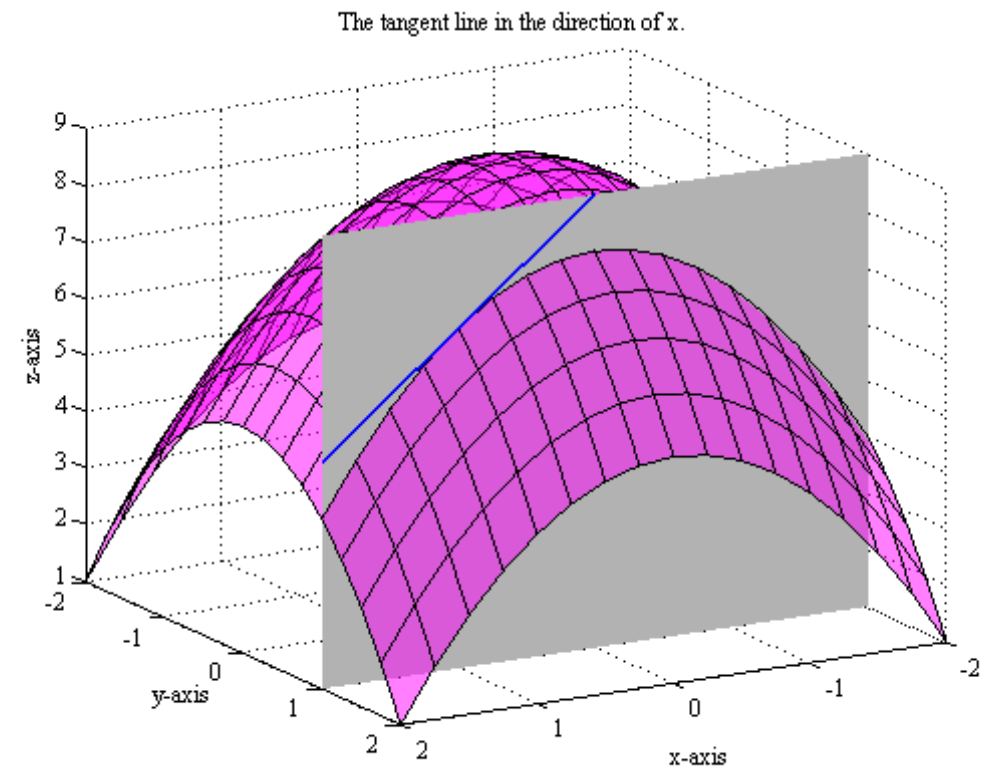
(pictures from Paul's Online Math Notes and WikiHow)

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The partial derivatives $f_x(a, b)$ and $f_y(a, b)$ are also the slopes of tangent lines to certain curves: these curves are the graphs of $f(x, b)$ and $f(a, y)$, which are the intersections of the graph of f with the planes $y = b$ and $x = a$ respectively.



(pictures from Paul's Online Math Notes and WikiHow)

Higher order partial derivatives

Remember that the second derivative of a single-variable function comes from taking the derivative twice: $f''(x) = \frac{d^2 f}{dx^2} = \frac{d^2}{dx^2} f(x) = \frac{d}{dx} \left(\frac{df}{dx} \right)$.

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Similarly, we can take the derivative twice of a 2-variable function $f(x, y)$ - now there are many combinations depending on which variable we differentiate with respect to, as they can be different variables for the first time and the second time:

Definition: The *second-order partial derivatives* of the function $f(x, y)$ are:

$$\begin{aligned} f_{xx}(x, y) &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2}{\partial x^2} f(x, y) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) \\ \text{mixed} \quad \left\{ \begin{aligned} f_{xy}(x, y) &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2}{\partial y \partial x} f(x, y) = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \\ f_{yx}(x, y) &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y} f(x, y) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \end{aligned} \right. \\ \text{partial} \quad \left\{ \begin{aligned} f_{yy}(x, y) &= \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2}{\partial y^2} f(x, y) = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \end{aligned} \right. \\ \text{derivatives} \end{aligned}$$

It is clear how to define third-order partial derivatives, by repeated partial differentiation.

Example: Find the second-order partial derivatives of $f(x, y) = \frac{xy}{x+1}$.

In the previous example, we found that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. This is a general fact for well-behaved functions:

Theorem 1: Equality of Mixed Partial Derivatives: (Clairaut's Theorem)
Suppose p, q are two k th-order partial derivatives of f obtained by differentiating with respect to the same set of k variables (possibly with repetition) but in different orders. Suppose also that p, q are continuous at (a, b) , and all $k - 1$ th-order partial derivatives of f are continuous around (a, b) . Then $p(a, b) = q(a, b)$.

Example: Clairaut's Theorem says that, if $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ has third-order partial derivatives that are continuous everywhere, then there are only four different third-order partial derivatives:

$$\begin{aligned} &f_{xxx}; \\ &f_{xxxy} = f_{xyxx} = f_{yxxx}; \\ &f_{xyyy} = f_{yyxy} = f_{yyyx}; \\ &f_{yyy}. \end{aligned}$$

The proof of Clairaut's Theorem uses the 1D mean value theorem separately in the x and y directions - see the textbook.

§12.6: Linear Approximation and Differentiability

Remember that a single-variable function f is said to be differentiable at a point a if the derivative $f'(a)$ exists.

It is not a good idea to say that a multivariate function is differentiable if all its partial derivatives exist, since there are discontinuous functions that have partial

derivatives. One example is $f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$ It is easy to

show that f does not have a limit at $(0, 0)$ (exercise), so f is not continuous, but because f is always 0 on the x and y axes, its partial derivatives do exist and are 0. (You will prove this carefully on Homework 4.)

The existence of partial derivatives means that f is well-behaved in the x and y directions, but f can still be horrible in other directions. A good definition of differentiability at (a, b) must be a statement about all points around (a, b) .

We will say that f is differentiable if it is **locally well-approximated by a linear function**.

Let's first understand what this means for a single-variable function $f : \mathbb{R} \rightarrow \mathbb{R}$. Remember that the **linearisation** of f at a is

$$L(x) = f(a) + f'(a)(x - a).$$

This linear function is important because we can use it to approximate $f(x)$ when x is near a , i.e. when $x = a + h$ and h is small. This approximation is good because the error satisfies

$$\begin{aligned} f(x) - L(x) &= f(x) - f(a) - f'(a)(x - a) \\ f(a + h) - L(a + h) &= f(a + h) - f(a) - f'(a)h \\ \frac{f(a + h) - L(a + h)}{h} &= \frac{f(a + h) - f(a)}{h} - f'(a) \\ \lim_{h \rightarrow 0} \frac{f(a + h) - L(a + h)}{h} &= \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} - f'(a) = 0 \end{aligned}$$

In other words, the error is small compared to h , the distance from a to x .

Remember again that the linearisation of $f : \mathbb{R} \rightarrow \mathbb{R}$ at a is $L(x) = f(a) + f'(a)(x - a)$.
A straightforward multivariate generalisation is:

Definition: The *linearisation* of a function $f(x, y)$ at (a, b) is

$$L(x, y) = f(a, b) + \underset{\substack{\text{rate of change of } f \\ \text{with respect to } x}}{\underset{\uparrow}{f_x(a, b)}} \underset{\substack{\text{change} \\ \text{in } x}}{\underset{\uparrow}{(x - a)}} + \underset{\substack{\text{rate of change of } f \\ \text{with respect to } y}}{\underset{\uparrow}{f_y(a, b)}} \underset{\substack{\text{change} \\ \text{in } y}}{\underset{\uparrow}{(y - b)}}.$$

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And then f is differentiable if the error from using $L(x, y)$ to approximate $f(x, y)$ is small compared to the distance from (x, y) to (a, b) :

Definition: A function $f(x, y)$ is *differentiable* at (a, b) if

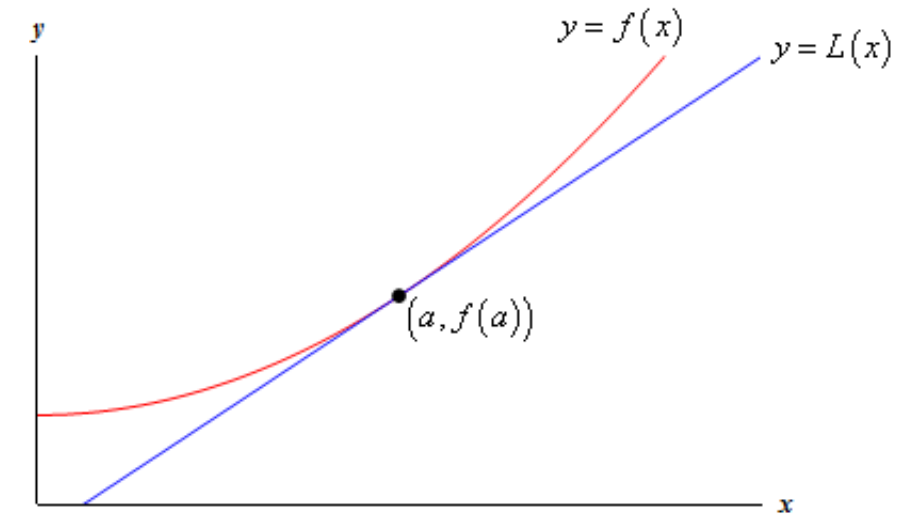
$$\lim_{(h, k) \rightarrow (0, 0)} \frac{f(a + h, b + k) - f(a, b) - f_x(a, b)h - f_y(a, b)k}{\sqrt{h^2 + k^2}} = 0.$$

It is clear how to generalise these definitions for n -variable scalar-valued functions. They also make sense for vector-valued functions, using addition and scalar-multiplication in \mathbb{R}^m (see p34).

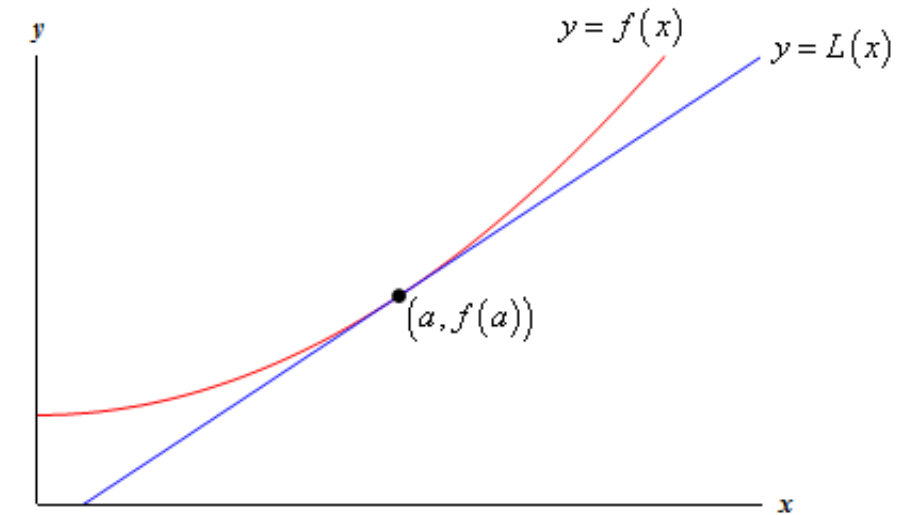
Before continuing with the theory of differentiability, let us make sure we understand the linearisation and its applications:

Example: Calculate the linearisation of $f(x, y) = x^2y$ at $(1, 2)$, and use it to estimate $f(1.1, 1.8)$.

To motivate the second main application of linearisations, recall that the graph of a single-variable linearisation $y = f(a) + f'(a)(x - a)$ is the tangent line to the graph of f at $(a, f(a))$.

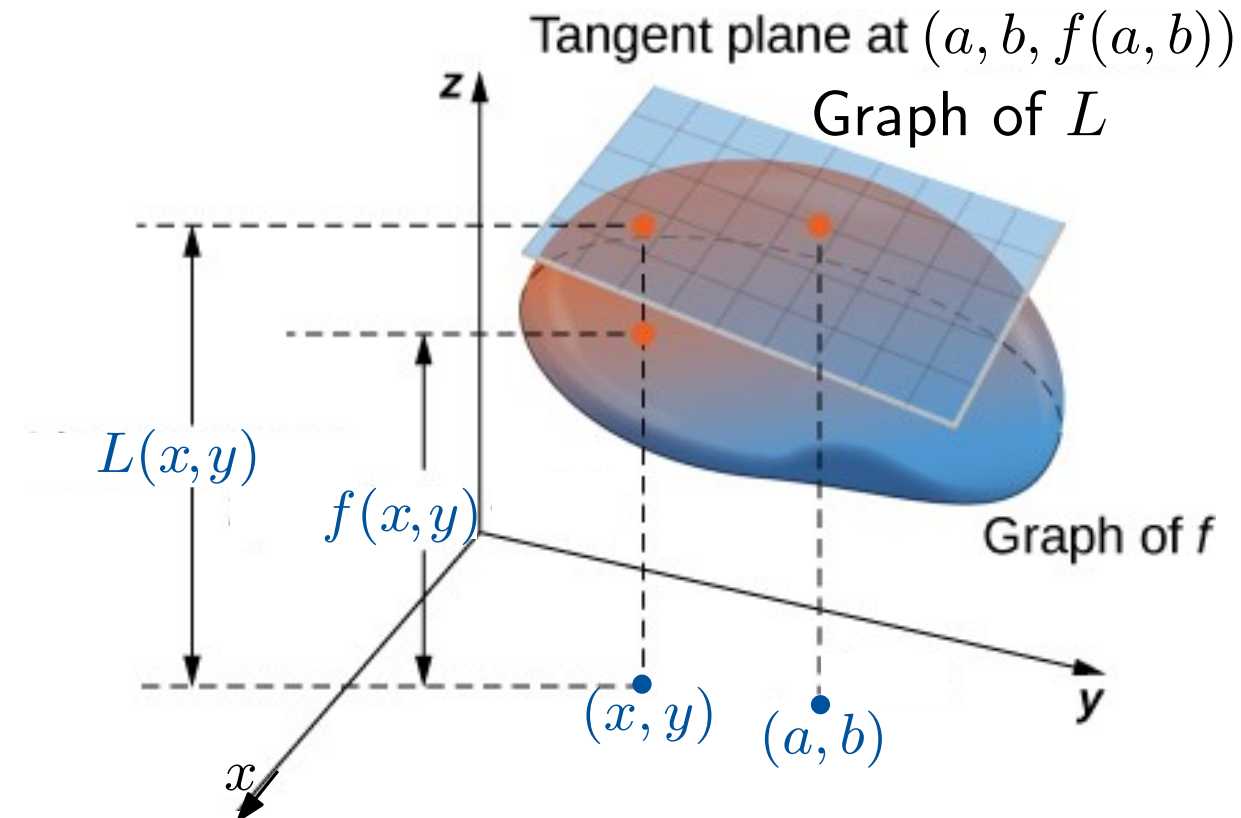


To motivate the second main application of linearisations, recall that the graph of a single-variable linearisation $y = f(a) + f'(a)(x - a)$ is the tangent line to the graph of f at $(a, f(a))$.



Consider the linearisation of a 2-variable function $f(x, y)$ at (a, b) . Its graph is $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$, and this is the tangent plane to the graph of f at $(a, b, f(a, b))$.

In Week 10 (§12.7) we will see a more general method that computes tangent planes to any surface, not just to a graph.



(pictures from Paul's Online Math Notes and archive.cnx.org)

Back to differentiability: remember that

Definition: A function $f(x, y)$ is *differentiable* at (a, b) if

$$\lim_{(h,k) \rightarrow (0,0)} \frac{f(a+h, b+k) - f(a, b) - f_x(a, b)h - f_y(a, b)k}{\sqrt{h^2 + k^2}} = 0.$$

We will rarely need to use this definition to check if functions are differentiable, thanks to the following theorem:

Theorem 4: Continuous Partial Derivatives guarantee Differentiability:

Given a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, if all its partial derivatives $\frac{\partial f}{\partial x_i}$ are continuous around (a, b) , then f is differentiable at (a, b) .

The main idea of the proof (p29) is to write $f(a+h, b+k) - f(a, b)$ in the definition of differentiable in terms of partial derivatives, using a multivariate mean value theorem. On the next page we state precisely the 2D mean value theorem, you can imagine the analogous statement when the domain is \mathbb{R}^n .

We are using the 1D MVT phrased as follows: if g is differentiable on $[a, a + h]$, then there is a point between a and $a + h$, i.e. a point $a + \theta h$ for θ between 0 and 1, such that $f(a + h) - f(a) = hf'(a + \theta h)$.

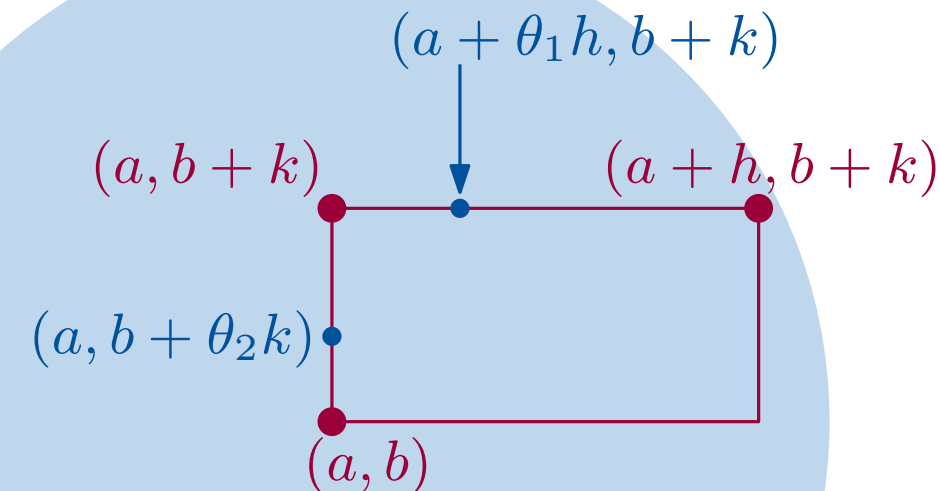
Theorem 3: Mean Value Theorem (MVT): Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ has continuous partial derivatives f_x and f_y on a small disk around (a, b) . If h, k are small enough that $(a + h, b + k)$ are in this disk, then there exist numbers θ_1, θ_2 between 0 and 1 such that

$$f(a + h, b + k) - f(a, b) = hf_x(a + \theta_1 h, b + k) + kf_y(a, b + \theta_2 k).$$

Proof:

$$\begin{aligned} f(a + h, b + k) - f(a, b) &= \underbrace{f(a + h, b + k) - f(a, b + k)}_{\substack{\text{1D MVT in} \\ x\text{-direction}}} + \underbrace{f(a, b + k) - f(a, b)}_{\substack{\text{1D MVT in} \\ y\text{-direction}}} \\ &= hf_x(a + \theta_1 h, b + k) + kf_y(a, b + \theta_2 k). \end{aligned}$$

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Proof: (of Theorem 4, sketch):

Recall that MVT says there are numbers θ_1, θ_2 between 0 and 1 with $f(a + h, b + k) - f(a, b) = hf_x(a + \theta_1 h, b + k) + kf_y(a, b + \theta_2 k)$.

We now use this to show that, if $f(x, y)$ has continuous partial derivatives, then f is differentiable. So we need to show that

$$\frac{f(a + h, b + k) - f(a, b) - f_x(a, b)h - f_y(a, b)k}{\sqrt{h^2 + k^2}} \rightarrow 0 \text{ as } (h, k) \rightarrow (0, 0).$$

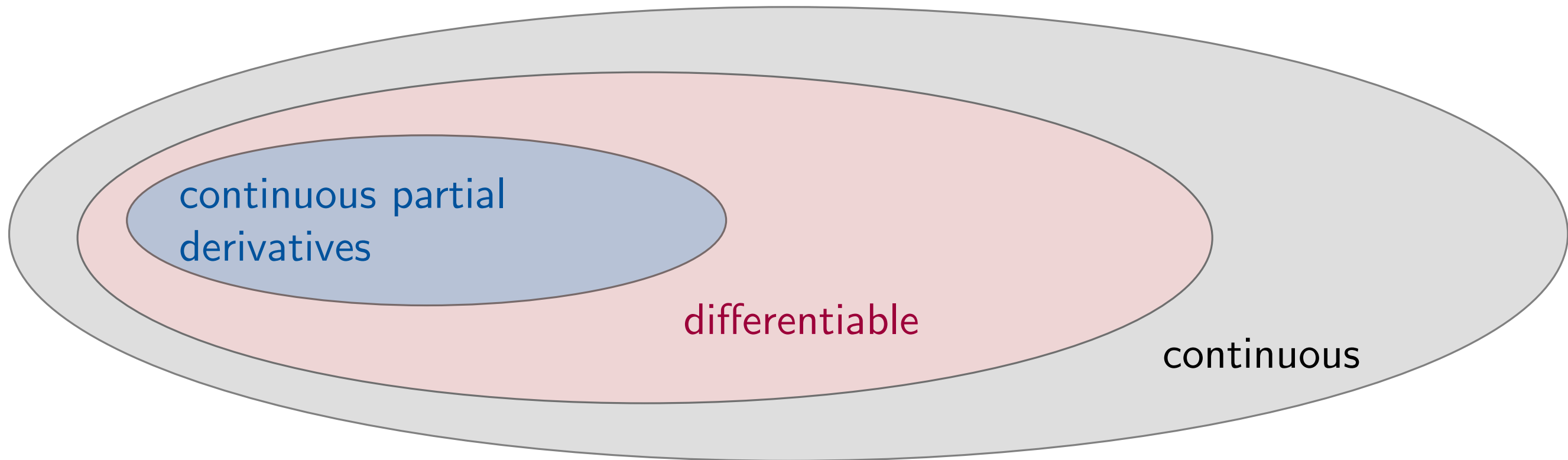
Using MVT to replace the first two terms in the numerator:

$$\begin{aligned} & \frac{hf_x(a + \theta_1 h, b + k) + kf_y(a, b + \theta_2 k) - f_x(a, b)h - f_y(a, b)k}{\sqrt{h^2 + k^2}} \\ &= \underbrace{\frac{h}{\sqrt{h^2 + k^2}}}_{\text{is finite.}} \underbrace{(f_x(a + \theta_1 h, b + k) - f_x(a, b))}_{\text{goes to 0 because } f_x \text{ is continuous}} + \underbrace{\frac{k}{\sqrt{h^2 + k^2}}}_{\text{is finite.}} \underbrace{(f_y(a, b + \theta_2 k) - f_y(a, b))}_{\text{goes to 0 because } f_y \text{ is continuous}} \end{aligned}$$

There is one more important result about differentiability - for simplicity we state it below for 2-variable functions, but it holds for any $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$:

Theorem: Differentiable Functions are Continuous: If $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable at (a, b) , then it is continuous at (a, b) .

So the hierarchy of functions is as follows:



Theorem: Differentiable Functions are Continuous: If $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable at (a, b) , then it is continuous at (a, b) .

Proof: (sketch, same as the 1D proof):

We show that $\lim_{(h,k) \rightarrow (0,0)} f(a+h, b+k) - f(a, b) = 0$.

$$\begin{aligned}
 & f(a+h, b+k) - f(a, b) \\
 = & f(a+h, b+k) - f(a, b) - f_x(a, b)h - f_y(a, b)k + f_x(a, b)h + f_y(a, b)k \\
 = & \underbrace{\sqrt{h^2 + k^2}}_{\text{goes to 0 because it is a continuous function of } (h, k)} \underbrace{\frac{f(a+h, b+k) - f(a, b) - f_x(a, b)h - f_y(a, b)k}{\sqrt{h^2 + k^2}}}_{\text{goes to 0 because } f \text{ is differentiable}} + \underbrace{f_x(a, b)h + f_y(a, b)k}_{\text{goes to 0 because it is a continuous function of } (h, k)}
 \end{aligned}$$

Derivatives of vector-valued functions

Consider a function $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$. We can compute the linearisation of its coordinate functions separately:

$$L_1(x, y) = f_1(a, b) + \left. \frac{\partial f_1}{\partial x} \right|_{(a, b)} (x - a) + \left. \frac{\partial f_1}{\partial y} \right|_{(a, b)} (y - b)$$

$$L_2(x, y) = f_2(a, b) + \left. \frac{\partial f_2}{\partial x} \right|_{(a, b)} (x - a) + \left. \frac{\partial f_2}{\partial y} \right|_{(a, b)} (y - b)$$

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This is matrix multiplication

$$\begin{pmatrix} \left. \frac{\partial f_1}{\partial x} \right|_{(a,b)} & \left. \frac{\partial f_1}{\partial y} \right|_{(a,b)} \\ \left. \frac{\partial f_2}{\partial x} \right|_{(a,b)} & \left. \frac{\partial f_2}{\partial y} \right|_{(a,b)} \\ \left. \frac{\partial f_3}{\partial x} \right|_{(a,b)} & \left. \frac{\partial f_3}{\partial y} \right|_{(a,b)} \end{pmatrix} \begin{pmatrix} x - a \\ y - b \end{pmatrix}$$

So one way to organise the mn partial derivatives of a function $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$, that makes sense with linear algebra, is with an $m \times n$ matrix:

Definition: The *Jacobian matrix* $D\mathbf{f}(\mathbf{x})$ of a function $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the $m \times n$ matrix with $\frac{\partial f_i}{\partial x_j}$ in row i and column j :

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

As observed on the previous page, we can write the linearisation of a vector-valued function using the Jacobian matrix:

$$\mathbf{L}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + \underbrace{D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a})}_{\text{matrix-vector multiplication}}.$$

matrix-vector multiplication

Example: Calculate the Jacobian matrix of $\mathbf{f}(x, y) = \left(\frac{xy}{x+1}, x^2y, x \right)$ at $(1, 2)$, and use it to estimate $\mathbf{f}(1.1, 2.3)$.

Non-examinable: the derivative as a linear transformation

Recall that the Jacobian matrix is

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix},$$

and the linearisation is:

$$\mathbf{L}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a}).$$

The linear transformation represented by the Jacobian matrix is called *the derivative* of \mathbf{f} . It allows a definition of differentiability without reference to coordinates: \mathbf{f} is differentiable at \mathbf{a} if there is a linear transformation $D\mathbf{f}(\mathbf{a})$ such that

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a})}{|\mathbf{x} - \mathbf{a}|} = \mathbf{0}.$$