§7.2 Coordinates for vectors

Def 7.2.1 If $\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\}$ is a basis of V, then each $\alpha \in V$ can be written uniquely as

i.e. $a_1, \cdots, a_n$ unique.

$$\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n$$

(2207 Week 8 p11, Prop. 6.4.5)

and the $\mathcal{A}$-coordinates of α is $[\alpha]_{\mathcal{A}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n$

(The order of $\alpha_1, \cdots, \alpha_n$ is important — whenever we use coordinates, we assume the order of the basis vectors is fixed.)

Not in textbook:

$$\mathrm{Coord}_{\mathcal{A}} : V \longrightarrow \mathbb{F}^{\dim V}$$

$$\mathrm{Coord}_{\mathcal{A}}(\alpha) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

can write $\mathrm{Coord}_{\mathcal{A}}(W) \subseteq \mathbb{F}^{\dim V}$

$$\mathrm{Decoord}_{\mathcal{A}} : \mathbb{F}^{\dim V} \longrightarrow V$$

$$\mathrm{Decoord}_{\mathcal{A}}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = a_1\alpha_1 + \cdots + a_n\alpha_n = \alpha$$

e.g. $V = M_{2,2}(\mathbb{R})$, $\mathcal{A} = \{E^{1,1}, E^{1,2}, E^{2,1}, E^{2,2}\}$

standard basis

$$\alpha = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow [\alpha]_{\mathcal{A}} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}.$$

Using a different basis gives different coordinates:

e.g. $$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

$$\alpha = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightarrow [\alpha]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \end{pmatrix}$$

Coordinates allow us to use RREF methods in any (finite-dimensional) V.

e.g. to find a basis of

$$W = \operatorname{Span}\{\underset{\alpha_1}{-x+x^2},\ \underset{\alpha_2}{1+x+x^2},\ \underset{\alpha_3}{1+2x}\} \subseteq P_{<3}(\mathbb{R})$$

with casting-out algorithm:

1 Choose a basis $\mathcal{A}$ of $P_{<3}(\mathbb{R})$, and apply $\mathrm{Coord}_{\mathcal{A}}$:

$\mathcal{A} = \{1, x, x^2\}$:

$$[\alpha_1]_{\mathcal{A}} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},\quad [\alpha_2]_{\mathcal{A}} = \begin{array}{cl} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} & \begin{array}{l} \leftarrow 1 \\ \leftarrow x \\ \leftarrow x^2 \end{array} \end{array}$$

$$[\alpha_3]_{\mathcal{A}} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

(2) Compute with coordinate vectors in $\mathbb{R}^3$:

$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathrm{Coord}_{\mathcal{A}}(W) \leq \mathbb{R}^3$.

(3) Apply $\mathrm{Decoord}_{\mathcal{A}}$:

$\{-x+x^2, 1+x+x^2\}$ is a basis for W.

Matrix representations of linear transformations:

Def 7.2.5: Let $\sigma \in L(U,V)$,

$\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$ a basis of U

$\mathcal{B} = \{\beta_1, \dots, \beta_m\}$ a basis of V.

The matrix representation of σ, relative to $\mathcal{A}$ and $\mathcal{B}$ is

$$\underset{\mathcal{B} \leftarrow \mathcal{A}}{[\sigma]} = \begin{pmatrix} | & & | \\ [\sigma(\alpha_1)]_{\mathcal{B}} & \cdots & [\sigma(\alpha_m)]_{\mathcal{B}} \\ | & & | \end{pmatrix}$$

$[\sigma]_{\mathcal{B}}^{\mathcal{A}}$ in textbook

If $U=V$, then we usualy want $\mathcal{A}=\mathcal{B}$,

then ${}_{\mathcal{A}\leftarrow\mathcal{A}}[\sigma] = \left([\sigma(\alpha_1)]_{\mathcal{A}} \cdots [\sigma(\alpha_n)]_{\mathcal{A}} \right)$

also written $[\sigma]_{\mathcal{A}}$.

Warning: $[\alpha]_{\mathcal{A}}$ is a vector $\in \mathbb{F}^{?}$

$[\sigma]_{\mathcal{A}}$ is a matrix

Ex: multiplication by $2+x^2$

$\sigma : P_{<2}(\mathbb{R}) \longrightarrow P_{<4}(\mathbb{R})$

$[\sigma(f)](x) = f(x)(2+x^2)$

$\mathcal{A}=\{1,x\}, \ \mathcal{B}=\{1,x,x^2,x^3\}$

$\sigma(1) = 1\,(2+x^2)$

$= 2\cdot 1 + 0\cdot x + 1\cdot x^2 + 0\cdot x^3$

$\sigma(x) = x\,(2+x^2)$

$= 0\cdot 1 + 2\cdot x + 0\cdot x^2 + 1\cdot x^3$

$$\begin{array}{cc} & \overbrace{\begin{array}{cc}1 & x\end{array}}^{\mathcal{A}} \\ \mathcal{B}\left\{\begin{array}{c}1\\ x\\ x^2\\ x^3\end{array}\right. & \begin{pmatrix}2 & 0\\ 0 & 2\\ 1 & 0\\ 0 & 1\end{pmatrix}\end{array}$$

Exercise: what is the matrix for "evaluation at 2": $P_{<3}(\mathbb{R}) \to \mathbb{R}$.

$\sigma(f) = f(2)$, relative to standard bases

$\mathcal{A}=\{1,x,x^2\}, \ \mathcal{B}=\{1\}$.