Homework 5, due 20:00 Thursday 2 May 2019 to Dr. Pang's mailbox

You must justify your answers to receive full credit.

1. Consider $\mathbb{C}^2$ with the standard dot product. Let

$$\alpha = \begin{pmatrix} 2+3i \\ -1-i \end{pmatrix}, \beta = \begin{pmatrix} -2i \\ 2 \end{pmatrix}, \gamma = \begin{pmatrix} -1+4i \\ 2-i \end{pmatrix}.$$

Calculate the following quantities:

a) $\alpha \cdot \beta$

b) $\beta \cdot \alpha$

c) the length of α

d) the distance between β and γ.

2. Prove the following "parallelogram law":

$$\|\alpha+\beta\|^2 + \|\alpha-\beta\|^2 = 2\|\alpha\|^2 + 2\|\beta\|^2 .$$

(You should use a general inner product $\langle\, , \rangle$.)

3. Consider $P_{<3}(\mathbb{R})$, the vector space of polynomials over $\mathbb{R}$ of degree less than 3, with inner product This is NOT an inner product! It's not positive definite, e.g. $\langle x-1, x-1\rangle < 0$.

$$\langle f, g\rangle = \int_{-1}^{1} (1+3x) f(x) g(x)\, dx.$$

Construct an orthogonal basis for $P_{<3}(\mathbb{R})$. You may use the following:

$$\int_{-1}^{1} x^n\, dx = \begin{cases} 0 & \text{if } n \text{ odd} \\ \dfrac{2}{n+1} & \text{if } n \text{ even.} \end{cases}$$

(Click here for a hint)

4. Consider $M_{2,2}(\mathbb{R})$ with the inner product

$$\langle A, B\rangle = \mathrm{Tr}(A^T B).$$

Consider the subspace W with orthogonal basis $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\}$.

a) Calculate the orthogonal projection of $\begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}$ onto W.

b) Find the closest point in W to $\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$.

5. Recall that, for a square matrix X, its trace $\mathrm{Tr}(X)$ is the sum of the diagonal entries of X.

   a) Show that $\mathrm{Tr}(BC) = \mathrm{Tr}(CB)$ for all $B, C \in M_{2,2}(\mathbb{F})$.

   b) It is true that $\mathrm{Tr}(BC) = \mathrm{Tr}(CB)$ for all $B, C \in M_{n,n}(\mathbb{F})$ (you do not need to prove this). Show that, if $A, J \in M_{n,n}(\mathbb{F})$ are similar matrices, then $\mathrm{Tr}\,A = \mathrm{Tr}\,J$.

   c) Now suppose $\mathbb{F} = \mathbb{C}$, i.e. A is an $n \times n$ complex matrix. Using part b or otherwise, explain why $\mathrm{Tr}\,A$ is the sum (with multiplicity) of the eigenvalues of A.

6. Prove the (simplified) Riesz Representation Theorem: let $\Phi : V \to \hat{V}$ be given by $\Phi(\gamma) = \langle \gamma, - \rangle$. Show that:

   a) Φ is conjugate-linear, i.e. $\Phi(a\gamma + \gamma') = \bar{a}\Phi(\gamma) + \Phi(\gamma')$.

   b) Φ is injective, i.e. $\Phi(\gamma) = \Phi(\gamma')$ means $\gamma = \gamma'$. (Please give full details.)

7. Let $V = P_{<2}(\mathbb{R})$, the vector space of polynomials over $\mathbb{R}$ of degree less than 2, with inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx.$$

   Define $\phi \in \hat{V}$ by $\phi(g) = g(-1)$.

   a) By direct calculation, find $f \in V$ such that $\langle f, g \rangle = \phi(g)$.

   You are given that $\mathcal{A} = \left\{1, \sqrt{3} - 2\sqrt{3}x\right\}$ is an orthonormal basis for V (you do **not** need to check this).

   b) Find the same f as in part a, using the formula for $\Lambda(\phi)$ from class.

8. Let U_1, U_2, W_1, W_2 be subspaces of an inner product space V.

   a) Prove that, if $U_1 \subseteq U_2$, then $U_2^\perp \subseteq U_1^\perp$.

   b) Prove that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$.

Optional questions If you attempted seriously all the above questions, then your scores for the following questions may replace any lower scores for two of the above questions.

9. (Sturm-Liouville theory for PDEs) (Adjoints are NOT on the exam)
Let W be the vector space of continuous functions on $[0,1]$, and let V be the subspace of W defined by $V = \{f \in C^0([0,1]) \mid f'' \text{ is continuous and} f(0) = f(1) = 0\}$. Let $\sigma : V \to W$ be the double-derivative operator, i.e. $\sigma(f) = f''$.

a) Let W have inner product given by
$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$
By integration by parts twice, or otherwise, show that,
$$\langle f, \sigma(g)\rangle = \langle \sigma(f), g\rangle \quad \text{for all } f, g \in V. \tag{$*$}$$
Non-examinable note: in Sturm-Liouville theory and other applications, a linear transformation satisfying $(*)$ is called self-adjoint. But $(*)$ is NOT the definition of self-adjoint that will be shown in the optional class – for us, only a linear transformation with same domain and codomain can be self-adjoint. (This condition of "same domain and codomain" cannot happen with differential operators, that's why they have to change the definition.)

b) Now let W have inner product given by
$$\langle f, g\rangle = \int_0^1 e^x f(x)g(x)\,dx.$$
By integration by parts twice, or otherwise, find the function $\tau : V \to W$ such that, for all $f, g \in V$, $\langle f, \sigma(g)\rangle = \langle \tau(f), g\rangle$.
Non-examinable note: in Sturm-Liouville theory, τ is called the adjoint of σ, but this is NOT the definition of adjoint that will be shown in the optional class – for us, the adjoint $\sigma^* : W \to V$ must satisfy $\langle f, \sigma(g)\rangle = \langle \sigma^*(f), g\rangle$ for all $g \in V$ and <u>all $f \in W$</u>, not just all $f \in V$. (Again, our conditions cannot happen with differential operators, so they change the definition.)

10. Let V be an inner product space, and let $f : V \to V$ be a self-adjoint function, i.e. for all $\alpha, \beta \in V$,
$$\langle f(\alpha), \beta\rangle = \langle \alpha, f(\beta)\rangle\,.$$
Show that f is linear.

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