Remember from calculus the addition and scalar multiplication of polynomials:

e.g $(2t^2+1)+(-t^2+3t+2)=t^2+3t+3.$

e.g $(-3)(-t^2+3t+2)=3t^2-9t-6.$

We want to represent abstract vectors as column vectors so we can do calculations (e.g. row-reduction) to study linear systems (e.g. week 7 p15) and linear transformations (e.g. week 7 p28).

Is this really different from $\mathbb{R}^3$?

$$\begin{bmatrix}2\\0\\1\end{bmatrix}+\begin{bmatrix}-1\\3\\2\end{bmatrix}=\begin{bmatrix}1\\3\\3\end{bmatrix}. \qquad (-3)\begin{bmatrix}-1\\3\\2\end{bmatrix}=\begin{bmatrix}3\\-9\\-6\end{bmatrix}.$$

We want to represent abstract vectors as column vectors so we can do calculations (e.g. row-reduction) to study linear systems and linear transformations.

In $\mathbb{R}^n$, $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$.

We can copy this idea: in V, pick a special set of vectors $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, write each $\mathbf{v}$ in V as $c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ and represent $\mathbf{v}$ by the column vector $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

Example: In $\mathbb{P}_2$, let $\mathbf{b}_1 = 1, \mathbf{b}_2 = t, \mathbf{b}_3 = t^2$.

Then we represent $a_0 + a_1t + a_2t^2$ by $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$ (slightly different from previous page; see p9, p12).

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$\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ must span V

We can copy this idea: in V, pick a special set of vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, write each $\mathbf{x}$ in V uniquely as $c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ and represent $\mathbf{x}$ by the column vector $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

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§4.3: Bases

Definition: Let W be a subspace of a vector space V. An $\boxed{\text{indexed set}}$ of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a *basis for W* if

i $\mathcal{B}$ is a linearly independent set, and

ii Span $\{\mathbf{b}_1, \dots, \mathbf{b}_p\} = W$.

The order matters: $\{\mathbf{b}_1, \mathbf{b}_2\}$ and $\{\mathbf{b}_2, \mathbf{b}_1\}$ are different bases.

i means: The only solution to $x_1\mathbf{b}_1 + \dots + x_p\mathbf{b}_p = \mathbf{0}$ is $x_1 = \dots = x_p = 0$.

ii means: W is the set of vectors of the form $c_1\mathbf{b}_1 + \dots + c_p\mathbf{b}_p$ where $c_1, \dots, c_p$ can take any value.

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Condition ii implies that $\mathbf{b}_1, \ldots, \mathbf{b}_p$ must be in W, because $\text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ contains each of $\mathbf{b}_1, \ldots, \mathbf{b}_p$.

Every vector space V is a subspace of itself, so we can take $W = V$ in the definition and talk about bases for V.

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Example: The standard basis for $\mathbb{R}^3$ is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

To check that this is a basis: $\begin{bmatrix} | & | & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is in reduced echelon form.

The matrix has a pivot in every column, so its columns are linearly independent.
The matrix has a pivot in every row, so its columns span $\mathbb{R}^3$.

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A basis for W is not unique: (different bases are useful in different situations, more later).

Let's look for a different basis for $\mathbb{R}^3$.

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for $\mathbb{R}^3$?

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Answer: No, because two vectors cannot span $\mathbb{R}^3$: $\begin{bmatrix}| & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & |\end{bmatrix} = \begin{bmatrix}1 & 0\\2 & 1\\0 & 1\end{bmatrix}$ cannot have a pivot in every row.

Definition: Let W be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a *basis for W* if

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Answer: Form the matrix $A = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. Because $\det A = 1 \neq 0$, the matrix A is invertible, so (by Invertible Matrix Theorem) its columns are linearly independent and its columns span $\mathbb{R}^3$.

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Let's look for a different basis for $\mathbb{R}^3$.

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ a basis for $\mathbb{R}^3$?

Definition: Let W be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a *basis for W* if

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Let's look for a different basis for $\mathbb{R}^3$.

Example: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ a basis for $\mathbb{R}^3$?

Answer: No, because four vectors in $\mathbb{R}^3$ must be linearly dependent:

$\begin{bmatrix} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ cannot have a pivot in every column.

By the same logic as in the above examples:

Fact: $\{\mathbf{v}_1, \dots \mathbf{v}_p\}$ is a basis for $\mathbb{R}^n$ if and only if:

- $p = n$ (i.e. $\boxed{\text{the set has exactly } n \text{ vectors}}$), and
- $\det \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_n \\ | & & | \end{bmatrix} \neq 0.$

Fewer than n vectors: not enough vectors, can't span $\mathbb{R}^n$.
More than n vectors: too many vectors, linearly dependent.

Example: The standard basis for $\mathbb{P}_n$ is $\mathcal{B} = \{1, t, t^2, \ldots, t^n\}$.

To check that this is a basis:

ii By definition of $\mathbb{P}_n$, every element of $\mathbb{P}_n$ has the form $a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$, so $\mathcal{B}$ spans $\mathbb{P}_n$.

i To see that $\mathcal{B}$ is linearly independent, we show that $c_0 = c_1 = \cdots = c_n = 0$ is the only solution to

$$c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n = 0. \text{ (the zero function)}$$

Substitute $t = 0$: we find $c_0 = 0$.

Differentiate, then substitute $t = 0$: we find $c_1 = 0$.

Differentiate again, then substitute $t = 0$: we find $c_2 = 0$.

Repeating many times, we find $c_0 = c_1 = \cdots = c_n = 0$.

Once we have the standard basis of $\mathbb{P}_n$, it will be easier to check if other sets are bases of $\mathbb{P}_n$, using coordinates (later, p14).

Advanced exercise: what do you think is the standard basis for $M_{m \times n}$?

One way to make a basis for V is to start with a set that spans V.

Theorem 5: Spanning Set Theorem: If $V = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, then some subset of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a basis for V.

Proof: (essentially the casting-out algorithm - see week 3)

- If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent, it is a basis for V.
- If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent, then one of the $\mathbf{v}_i$s is a linear combination of the others. Removing this $\mathbf{v}_i$ from the set still gives a set that spans V. Continue removing vectors in this way until the remaining vectors are linearly independent.

Example: $\mathbb{R}^2 = \text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$, but this set is not linearly independent because $\begin{bmatrix}2\\2\end{bmatrix}$ is a linear combination of the others: $\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$. So remove $\begin{bmatrix}2\\2\end{bmatrix}$ to get the basis $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$.

PP 234, 238-240 (§4.4), 307-308 (§5.4): Coordinates

Recall (p2) that our motivation for finding a basis is because we want to write each vector $\mathbf{x}$ as $c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$ in a unique way. Let's show that this is indeed possible

Theorem 7: Unique Representation Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

Proof:

Since $\mathcal{B}$ spans V, there exists scalars $c_1, \ldots, c_n$ such that the above equation holds.

Suppose $\mathbf{x}$ has another representation

$$\mathbf{x} = d_1\mathbf{b}_1 + \cdots + d_n\mathbf{b}_n.$$

for some scalars $d_1, \ldots, d_n$. Then

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \cdots + (c_n - d_n)\mathbf{b}_n.$$

Because $\mathcal{B}$ is linearly independent, all the weights in this equation must be zero, i.e. $(c_1 - d_1) = \cdots = (c_n - d_n) = 0$. So $c_1 = d_1, \ldots, c_n = d_n$.

Because of the Unique Representation Theorem, we can make the following definition:

Definition: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for V. Then, for any $\mathbf{x}$ in V, the *coordinates of* $\mathbf{x}$ *relative to* $\mathcal{B}$, or the $\mathcal{B}$*-coordinates of* $\mathbf{x}$, are the unique weights $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

The vector in $\mathbb{R}^n$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the *coordinate vector of* $\mathbf{x}$ *relative to* $\mathcal{B}$, or the $\mathcal{B}$*-coordinate vector of* $\mathbf{x}$.

Example: Let $\mathcal{B} = \left\{1, t, t^2, t^3\right\}$ be the standard basis for $\mathbb{P}_3$. Then the coordinate vector of an arbitrary polynomial is $[a_0 + a_1 t + a_2 t^2 + a_3 t^3]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

Why are coordinate vectors useful?

Because of the Unique Representation Theorem, the function V to $\mathbb{R}^n$ given by

$\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ (e.g. $a_0 + a_1 t + a_2 t^2 + a_3 t^3 \mapsto \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$) is linear, one-to-one and onto.

Definition: A linear transformation $T : V \to W$ that is both one-to-one and onto is called an *isomorphism*. We say V and W are isomorphic.

This means that, although the notation and terminology for V and W are different, the two spaces behave the same as vector spaces. Every vector space calculation in V is accurately reproduced in W, and vice versa.

Important consequence: if V has a basis of n vectors, then V and $\mathbb{R}^n$ are isomorphic, so we can solve problems about V (e.g. span, linear independence) by working in $\mathbb{R}^n$.

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Example: Is the set of polynomials $\{1, 2-t, (2-t)^2, (2-t)^3\}$ linearly independent?

Answer: The coordinates of these polynomials relative to the standard basis of $\mathbb{P}_3$ are

$$[1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad [(2-t)^2]_{\mathcal{B}} = [4-4t+t^2]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -4 \\ 1 \\ 0 \end{bmatrix},$$

$$[2-t]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad [(2-t)^3]_{\mathcal{B}} = [(8-12t+6t^2-t^3]_{\mathcal{B}} = \begin{bmatrix} 8 \\ -12 \\ 6 \\ -1 \end{bmatrix}$$

The set of polynomials is linearly independent if and only if their coordinate vectors are linearly independent (continued on next page).

Example: Is the set of polynomials $\{1, 2-t, (2-t)^2, (2-t)^3\}$ linearly independent?

Answer: (continued). The matrix

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & -1 & -4 & -12 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

has determinant $1 \neq 0$ (it is upper triangular so its determinant is the product of the diagonal entries), so it is invertible, and by the Invertible Matrix Theorem, its columns are linearly independent in $\mathbb{R}^4$. So the polynomials are linearly independent. (In fact they form a basis, because IMT says they also span $\mathbb{R}^4$.)

(Because we have a set of four vectors in $\mathbb{R}^4$, we can use the det+IMT. If we had fewer than four vectors, we would have to row reduce: free variable $\implies$ dependent; no free variables / pivot in each column $\implies$ independent.)

Advanced exericse: if $\mathbf{p}_i$ has degree exactly i, then $\{\mathbf{p}_0, \mathbf{p}_1, \ldots \mathbf{p}_n\}$ is a basis for $\mathbb{P}_n$. (This idea is how I usually prove that a set is a basis in my research work.)

If V has a basis of n vectors, then V and $\mathbb{R}^n$ are isomorphic, so we can solve problems about V (e.g. span, linear independence) by working in $\mathbb{R}^n$.

What about problems concerning linear transformations $T : V \to W$?

Remember from week 4: Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation $T(\mathbf{x}) = A\mathbf{x}$, where

apply T to ith basis vector, put the coordinates of the result into column i

$$A = \begin{bmatrix} | & & | \\ T(\mathbf{e}_1) & \dots & T(\mathbf{e}_n) \\ | & & | \end{bmatrix} \quad \text{(standard matrix of } T\text{)}.$$

The standard matrix is useful because we can solve $T(\mathbf{x}) = \mathbf{y}$ by row-reducing $[A|\mathbf{y}]$.

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Definition: If V is a vector space with basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $T : V \to V$ is a linear transformation, then the *matrix for T relative to $\mathcal{B}$* is

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(\mathbf{b}_1)]_{\mathcal{B}} & \dots & [T(\mathbf{b}_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix}.$$

(so the standard matrix of T is the matrix for T relative to the standard basis of $\mathbb{R}^n$.)

The matrix $[T]_{\mathcal{B}}$ is useful because

$$[T(\mathbf{x})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}, \qquad (*)$$

so we can solve $T(\mathbf{x}) = \mathbf{y}$ by row-reducing $\left[[T]_{\mathcal{B}} \middle| [\mathbf{x}]_{\mathcal{B}} \right]$.

Example: Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the differentiation function $T(\mathbf{p}) = \frac{d}{dt}\mathbf{p}$ as on the previous page. Here is an example of equation $(*)$ for $\mathbf{x} = 2 + 3t - t^2$.

$$T(2 + 3t - t^2) = \frac{d}{dt}(2 + 3t - t^2) \qquad = 3 - 2t$$

$$[T]_{\mathcal{B}} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}.$$

Some other things about T that we can learn from the matrix $[T]_{\mathcal{B}}$:

- We can solve the differential equation $\frac{d}{dt}\mathbf{p} = 1 - 3t$ by row-reducing $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$.
- $[T]_{\mathcal{B}}$ is in echelon form, and it does not have a pivot in every row, so T is not onto.

Basis and coordinates for subspaces:

Example: Let W be the set of vectors of the form $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$, where a, b can take any value.

We showed (week 7 p13) that W is a subspace of $\mathbb{R}^3$ because $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(because $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.) Since $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is furthermore linearly independent (the vectors are not multiples of each other), it is a basis for W.

Because $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, the coordinate vector of $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$, relative to the basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, is $\begin{bmatrix} a \\ b \end{bmatrix}$. So $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \mapsto \begin{bmatrix} a \\ b \end{bmatrix}$ is an ismorphism from W to $\mathbb{R}^2$.

Coordinates for subspaces (e.g. planes in $\mathbb{R}^3$) are useful as they allow us to represent points in the subspace with fewer numbers (e.g. with 2 numbers instead of 3 numbers).

In this picture (p239 of textbook, example 7 in §4.4), $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for a plane. The basis allows us to draw a coordinate grid on the plane.

The $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. This coordinate vector describes the location of x relative to this coordinate grid.