A Uniform Analysis of Combinatorial Markov Chains via Hopf Algebras

Amy Pang, LaCIM

slides available at amypang.github.io/2016.pdf

Motivation: a Dynamic Storage Allocation Problem

- You have n files, arranged in a list.
- You request files one-by-one independently, removing one from the list and returning it in a possibly different position.
- You request file i with a fixed, unknown, probability p_i.
- Each time you make a request, you search from the front of the list for the file you need.

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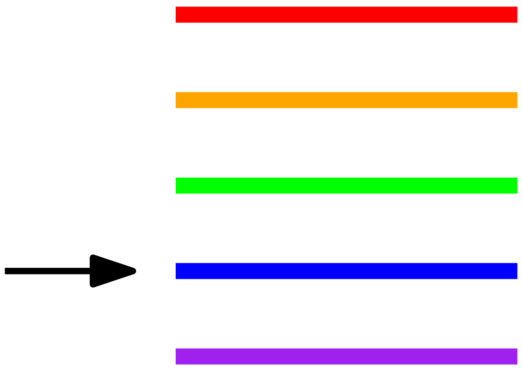
Question: where should you return a file to minimise the average search time?

Two Answers by McCabe (1965)

Starting list



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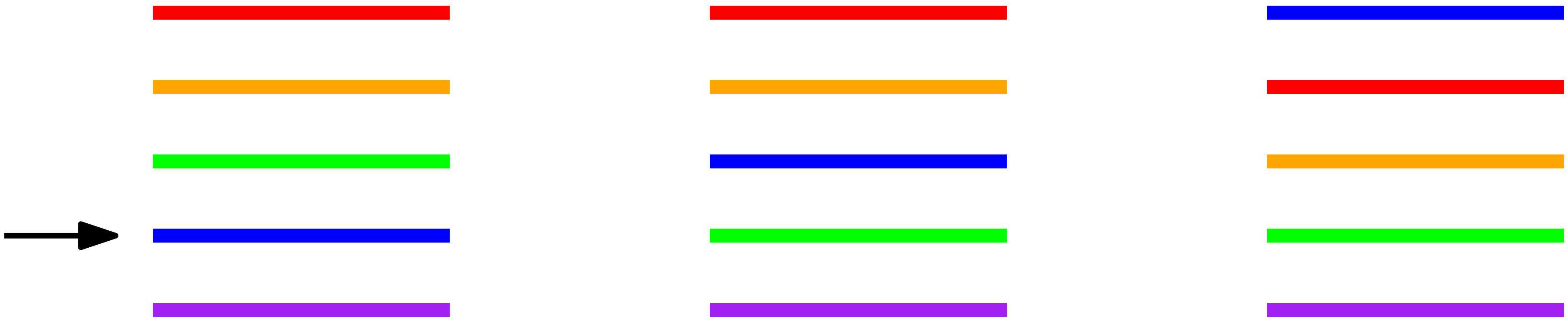

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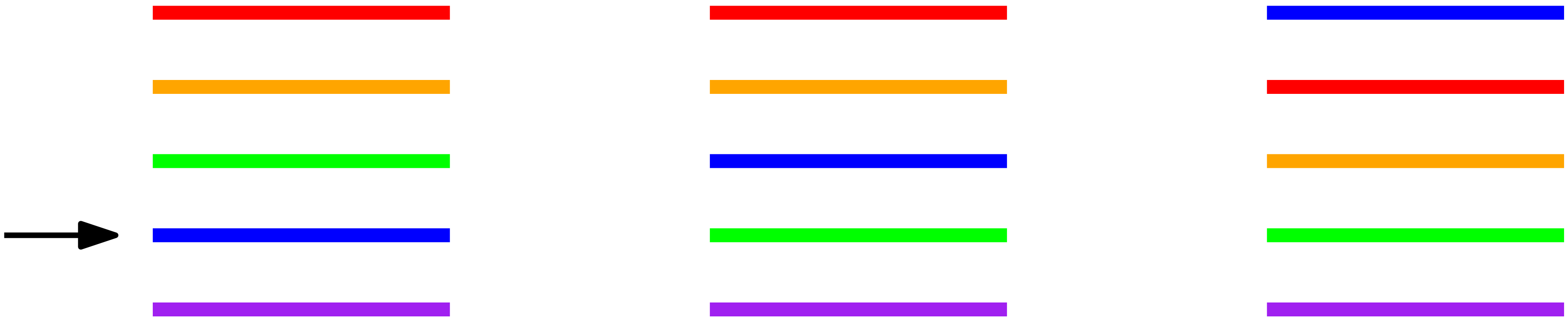

Stationary distribution:

$$p^4 p^3 p^2 p$$

$$\frac{p}{1}\,\frac{p}{1-p}\,\frac{p}{1-p-p}\,\frac{p}{1-p-p-p}$$

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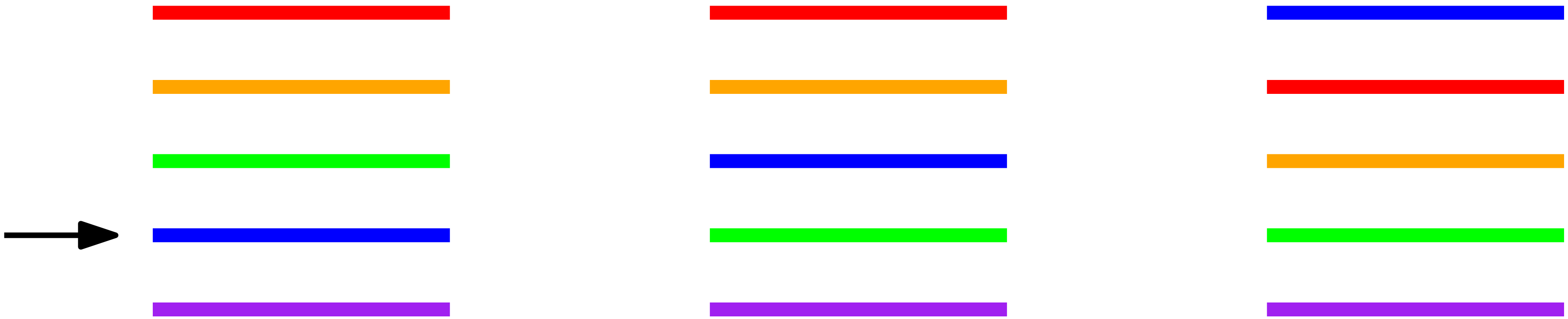

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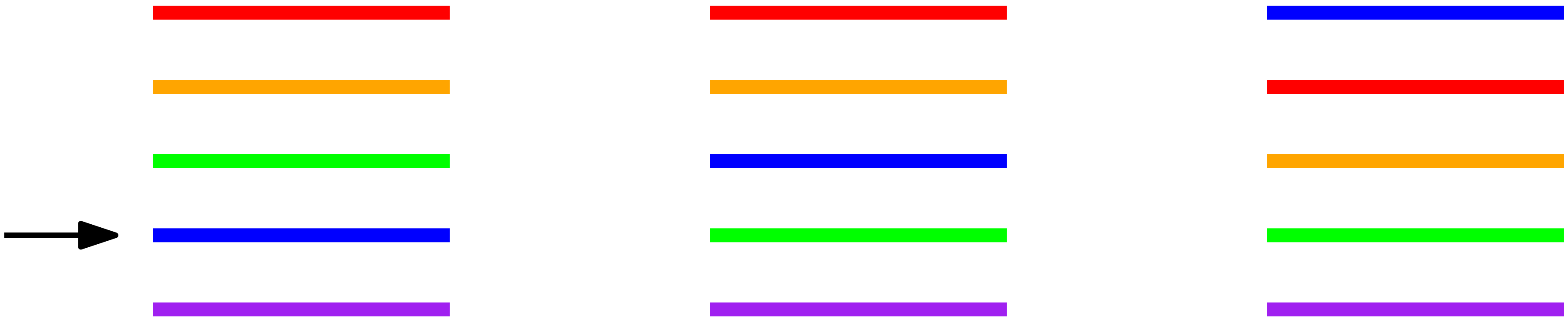

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Rivest (1976): lower average search time when in stationary distribution

Bitner (1979): reaches stationary distribution earlier

Markov Chains

- $\mathcal{X}$ a (finite) state space. all possible orders of n files
- X_t a random variable taking values in $\mathcal{X}$, for each $t \in \mathbb{N}$.

  the order of the files after t requests
- The process $\{X_t\}$ is memoryless, in that $\text{Prob}(X_{t+1} = y | X_t = x) = K(x, y)$, a number independent of $X_1, X_2, ...X_{t-1}$ and of t.

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Important questions:

- Stationary distribution: $\sum_{x \in \mathcal{X}} \pi_x K(x, y) = \pi_y$.

  eigenvector of eigenvalue 1
- Convergence rate. subdominant eigenvalue (spectral gap)

More applications of Markov Chains

To model a process:

- Exclusion process (Quastel 1992, Diaconis, Saloff-Coste 1993)
- DNA sequencing (Ching, Fung, Ng 2004)

To sample from a given distribution:

- Configurations of particles in a liquid (Allen, Tildesley 1989)
- Contingency tables (Hernek 1998)

To obtain good approximations to optimisation problems:

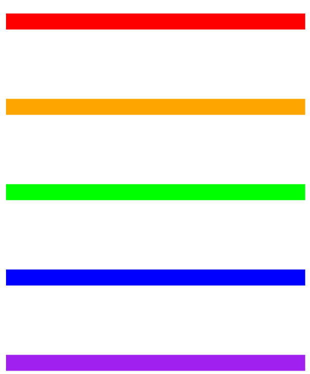
- Data augmentation (Tanner, Wong 1987)
- Decoding prisoner communication (Connor 2003)

The Top-to-Random Shuffle

(time-reversal of move-to-front with equal request probabilities)

- Remove top card
- Reinsert this card at a uniformly chosen position

For example:



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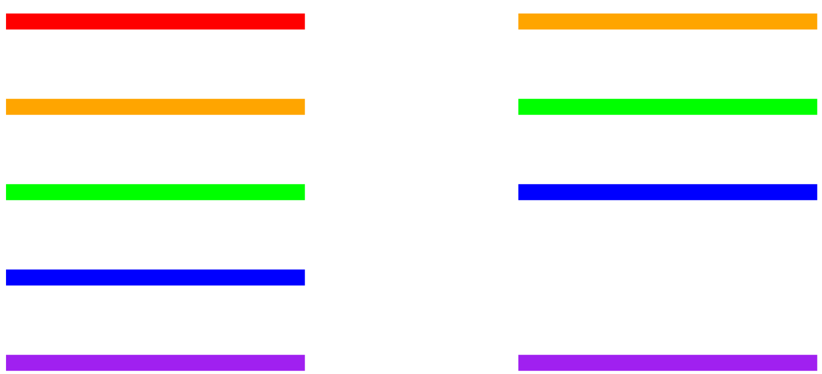

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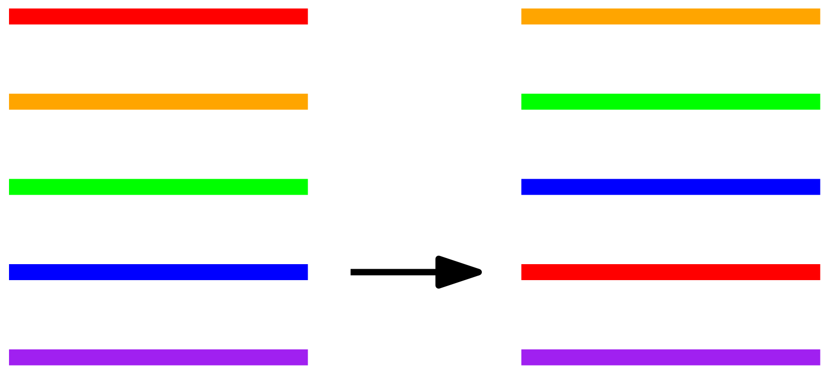

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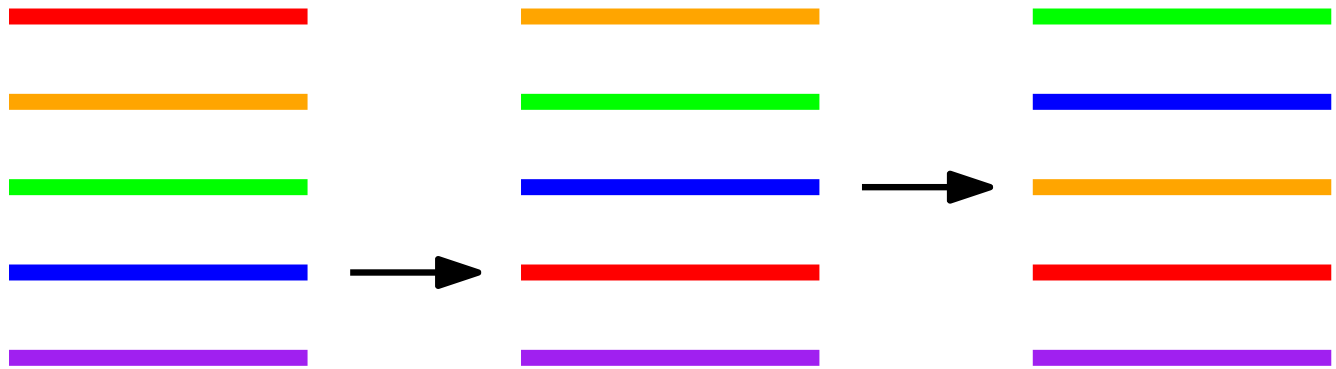

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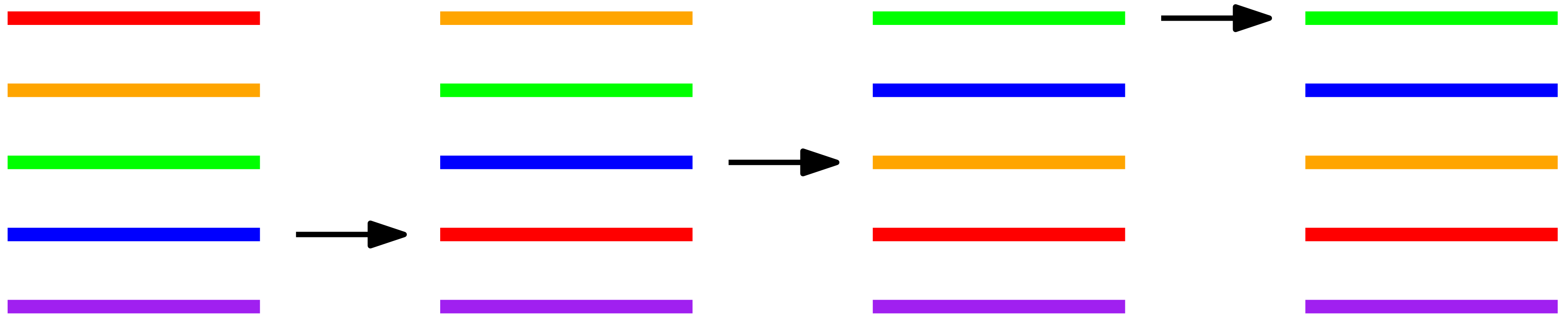

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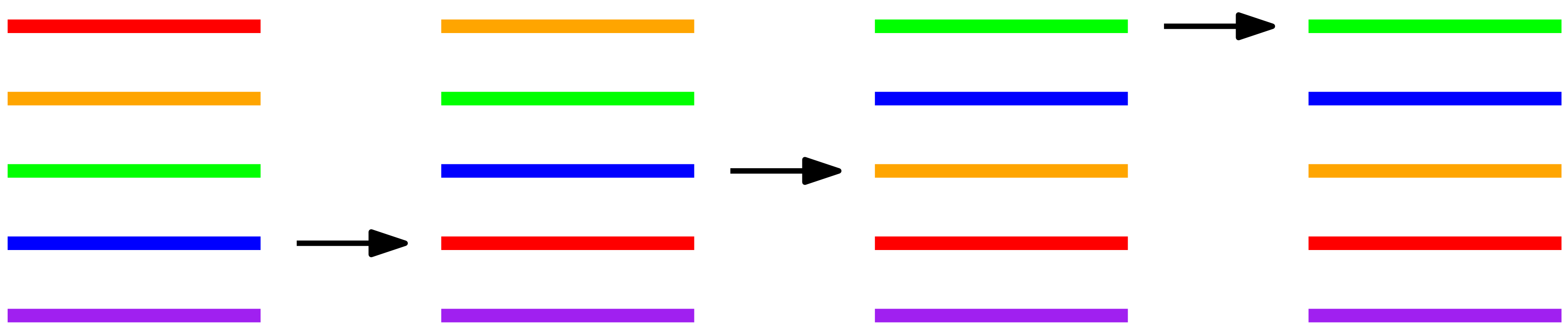
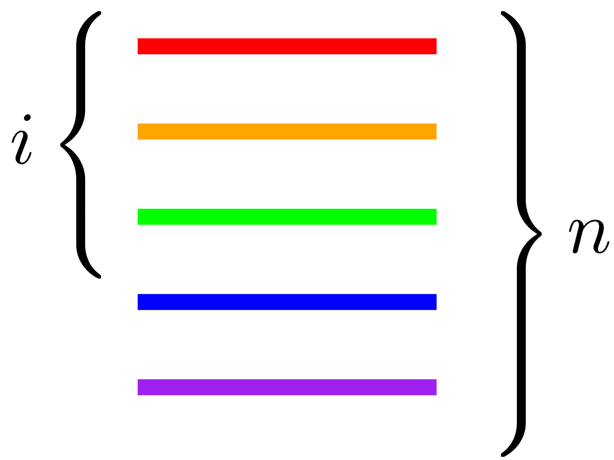


Aldous-Diaconis (1986): convergence rate is $n \log n$.
($\sim$ 205 when $n = 52$).

The Riffle Shuffle

- Cut the deck with symmetric binomial distribution;



$$\text{Prob} = 2^{-n}\binom{n}{i}$$

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Prob $= \frac{3}{5}$

Prob $= \frac{2}{5}$

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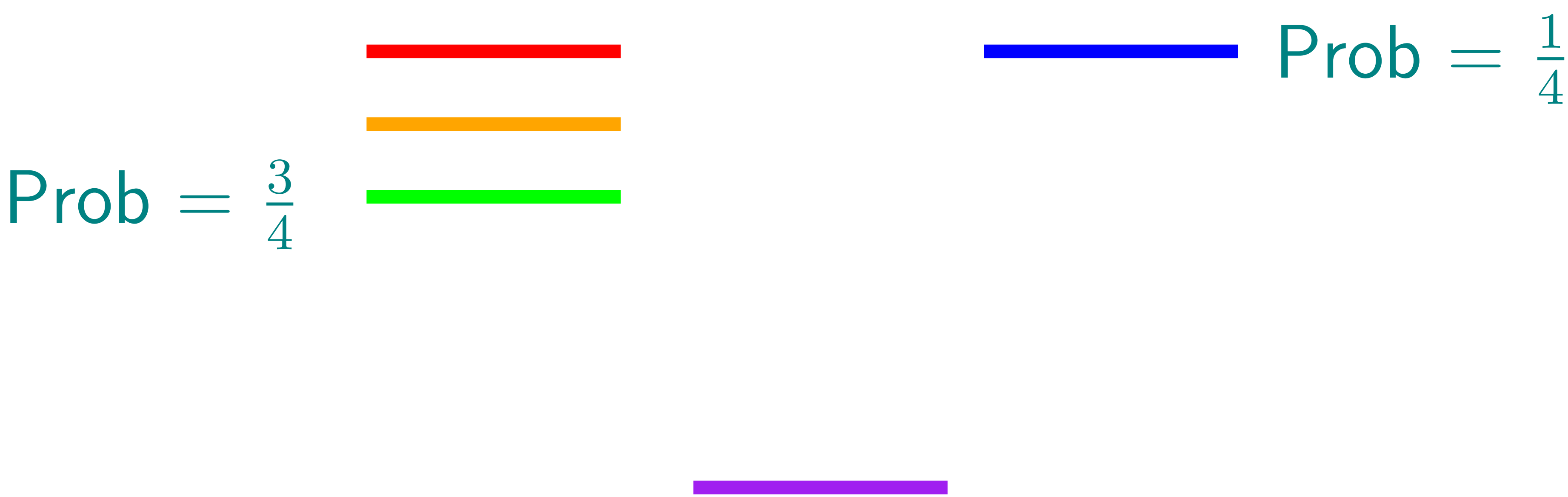

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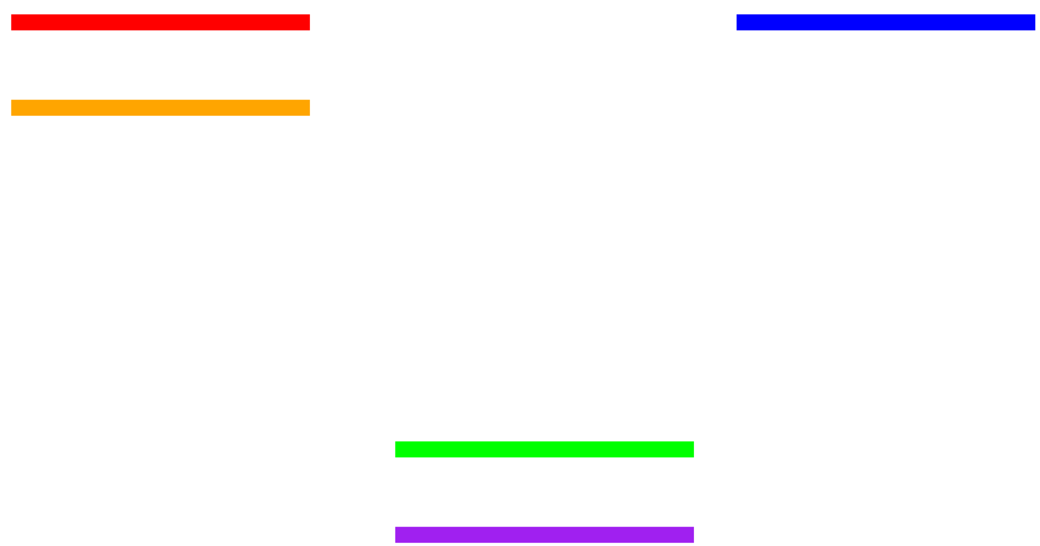

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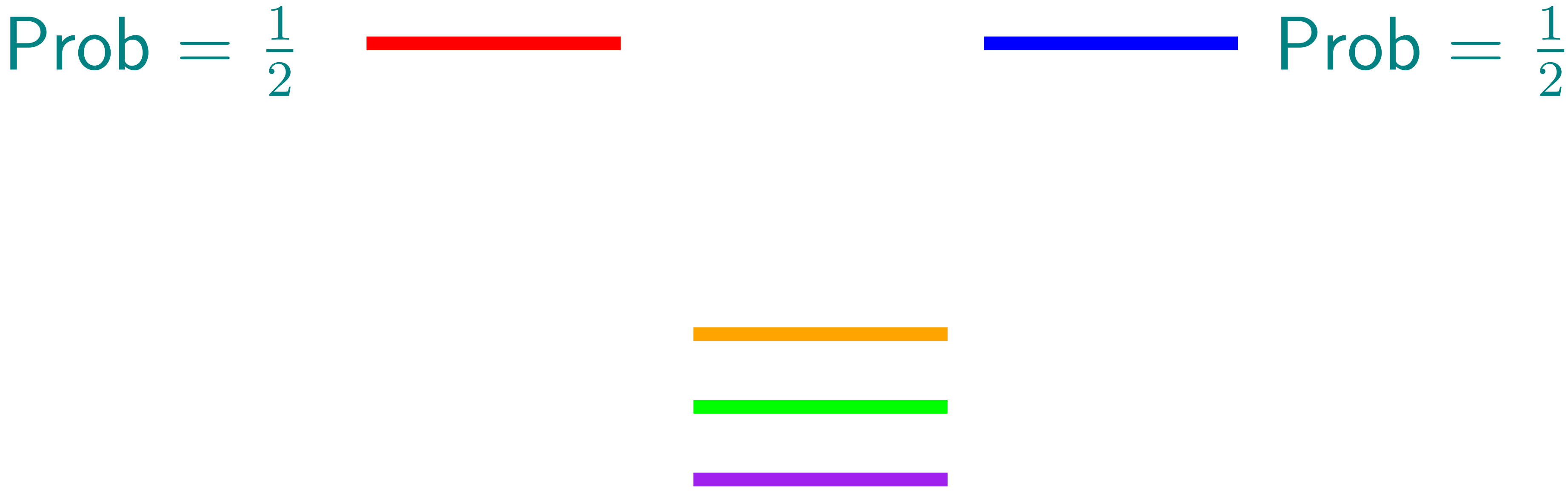

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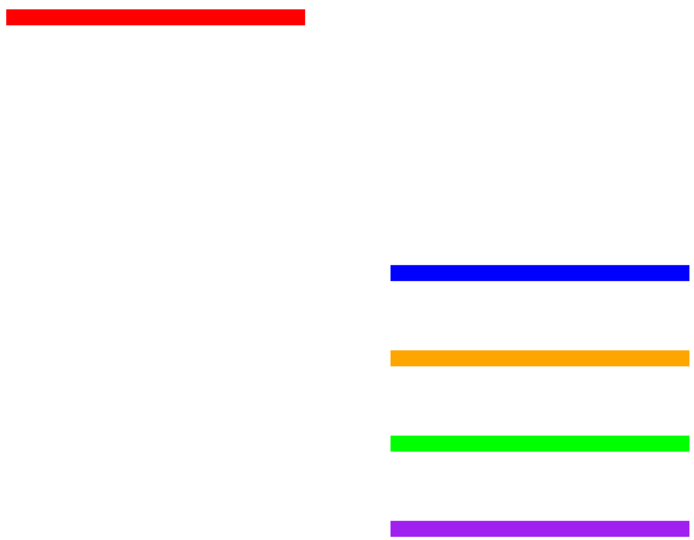
Prob $= \frac{1}{1}$



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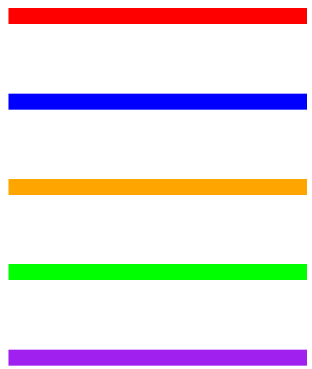

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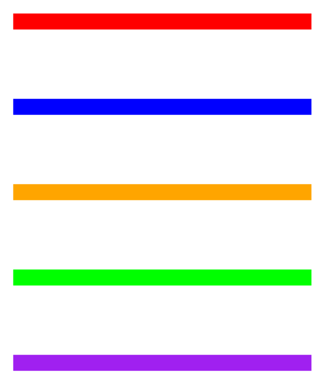

Bayer-Diaconis (1992): convergence rate is $\frac{3}{2}\log n$.

($\sim$ 7 when $n = 52$).

Analysing Shuffles using Maps on Hopf Algebras

	Top-to-random	Riffle
Theorem :	(2015)	(+Diaconis, Ram 2014)
extensions of results by:	Diaconis-Fill-Pitman (1992)	Bayer-Diaconis (1992), Hanlon (1990)

The unique stationary distribution is the uniform distribution.

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Eigenvalues: $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-2}{n}, 1.$ $\quad 2^{-n+1}, \ldots, 2^{-1}, 1.$

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Multiplicities of eigenvalues, for all cards distinct	number of permutations of n cards with j fixed points.	number of permutations of n cards with j cycles.

Analysing Shuffles using Maps on Hopf Algebras

- An algorithm to compute an eigenbasis.

Corollary : Start with n distinct cards in ascending order. After t top-to-random shuffles:

$$\text{Prob (descent at the bottom)} = \left(1 - \left(\frac{n-2}{n}\right)^t\right)\frac{1}{2}.$$

$\uparrow$ big card on small card

$$\text{Prob (peak at the bottom)} = \left(1 - \left(\frac{n-3}{n}\right)^t\right)\frac{1}{3}.$$

$\uparrow$ triple of cards with biggest in middle

Analysing Shuffles using Maps on Hopf Algebras

- An algorithm to compute an eigenbasis.

Corollary : Start with n distinct cards in ascending order. After t riffle shuffles:

$$\text{Expect (number of descents)}=\left(1-\left(\frac{1}{2}\right)^t\right)\frac{n-1}{2}.$$

$$\text{Expect (number of peaks)}=\left(1-\left(\frac{1}{4}\right)^t\right)\frac{n-2}{3}.$$

A new Connection: Ree's Shuffle (Hopf) Algebra

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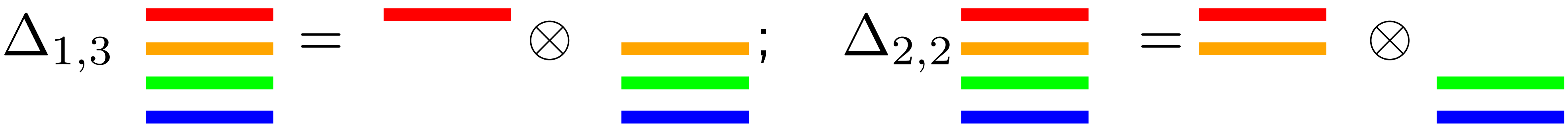

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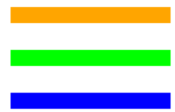
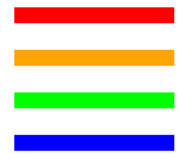
$\Delta_{1,3}$  $=$  $\otimes$ ; $\Delta_{2,2}$  $=$  $\otimes$ 

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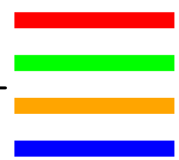
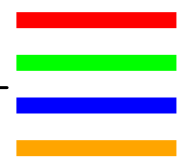
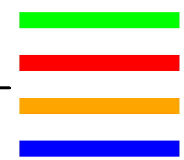
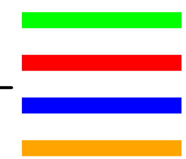
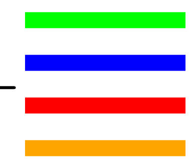
$=$  $+$  $+$  $+$  $+$  $+$ 

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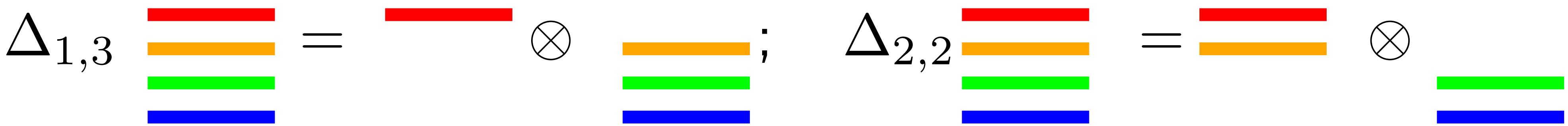

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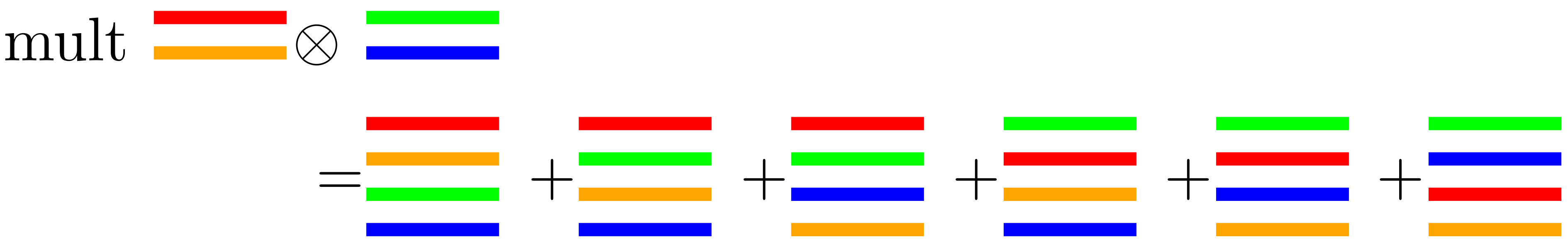

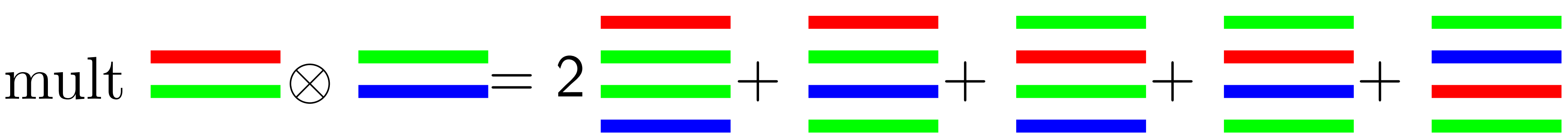

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(x, y are decks of n cards)

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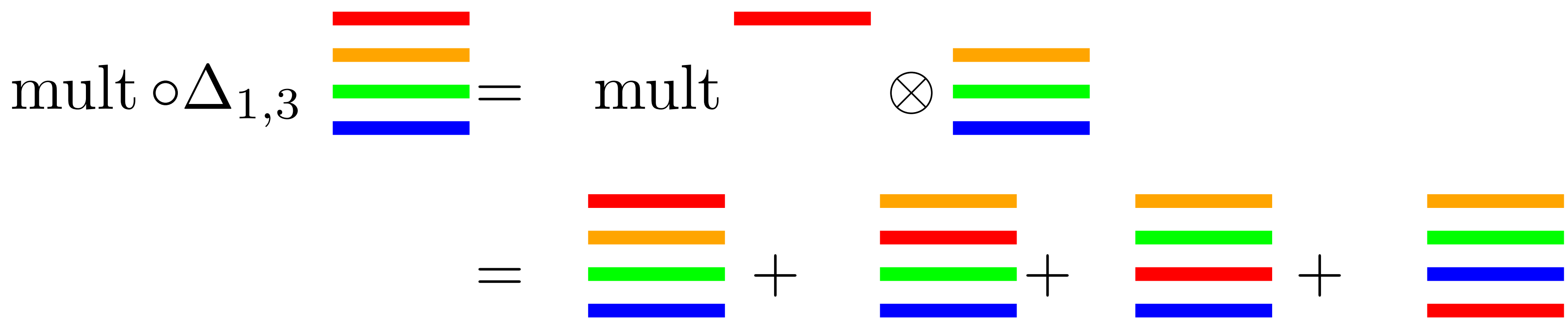

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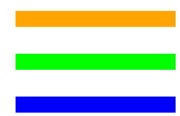
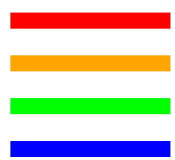
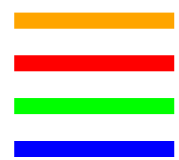
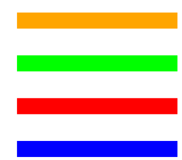
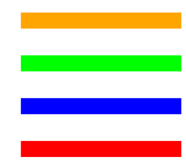
$\frac{1}{4} \operatorname{mult} \circ \Delta_{1,3}$  $= \frac{1}{4} \operatorname{mult}$  $\otimes$ 

$= \frac{1}{4}$  $+ \frac{1}{4}$  $+ \frac{1}{4}$  $+ \frac{1}{4}$ 

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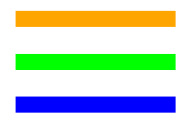
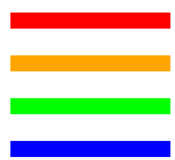
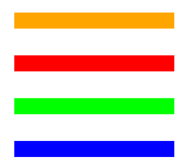
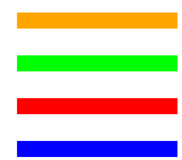
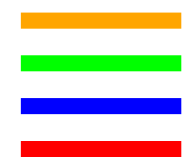
$\frac{1}{4}\,\mathrm{mult} \circ \Delta_{1,3}$  $= \frac{1}{4}\,\mathrm{mult}$  $\otimes$ 

$= \frac{1}{4}$  $+ \frac{1}{4}$  $+ \frac{1}{4}$  $+ \frac{1}{4}$ 

Riffle:

Prob$(x \to y)$= coefficient of y in $\dfrac{1}{2^n}\,\mathrm{mult} \circ \displaystyle\sum_{i=0}^{n} \Delta_{i,n-i}(x)$.

Chains on Other Combinatorial Objects

Markov chain	Hopf al	basis	stationary distribution
shuffling	shuffle	words / decks of cards	uniform
inverse-shuffling	free ass	words / decks of cards	uniform
edge-removal	$\mathscr{G}$	unlabelled graphs	absorbing at empty graph
edge-removal	$\mathscr{G}$	labelled graphs	absorbing at empty graph
restriction-then-induction	represe	irreducible representations	plancherel
rock-breaking	symme	elementary or complete	absorbing at $(1, 1, \ldots, 1)$
tree-pruning	Connes-	rooted forests	absorbing at disconnected fore
descent-set-under-shuffling	quasisy	fundamental (compositions)	proportion of permutations wi
jeu-de-tacquin	Poirer-	standard Young tableaux	proportion of standard tableau
shuffle with standardisation	Malven	fundamental (permutations)	uniform

Chains on Other Combinatorial Objects

Markov chain	Hopf al	basis	stationary distribution
shuffling	shuffle a	words / decks of cards	uniform
inverse-shuffling	free ass	words / decks of cards	uniform
edge-removal	$\mathcal{G}$	unlabelled graphs	absorbing at empty graph
edge-removal	$\mathcal{G}$	labelled graphs	absorbing at empty graph
restriction-then-induction	represe	irreducible representations	plancherel
rock-breaking	symme	elementary or complete	absorbing at $(1, 1, \ldots, 1)$
tree-pruning	Connes-	rooted forests	absorbing at disconnected fore
descent-set-under-shuffling	quasisy	fundamental (compositions)	proportion of permutations wi
jeu-de-tacquin	Poirer-R	standard Young tableaux	proportion of standard tableau
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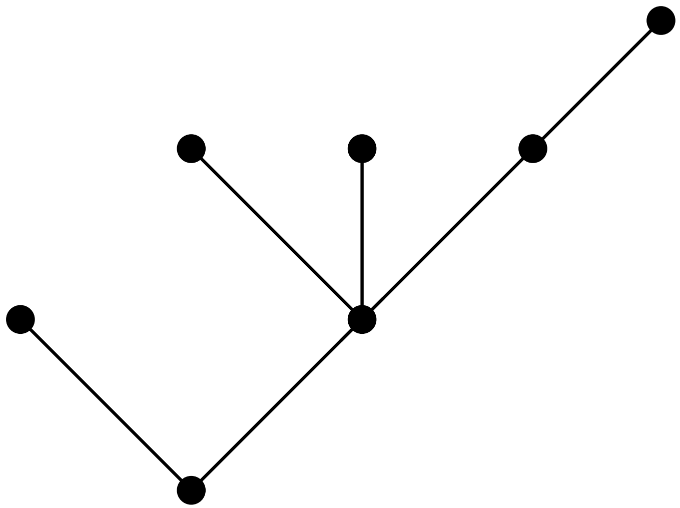

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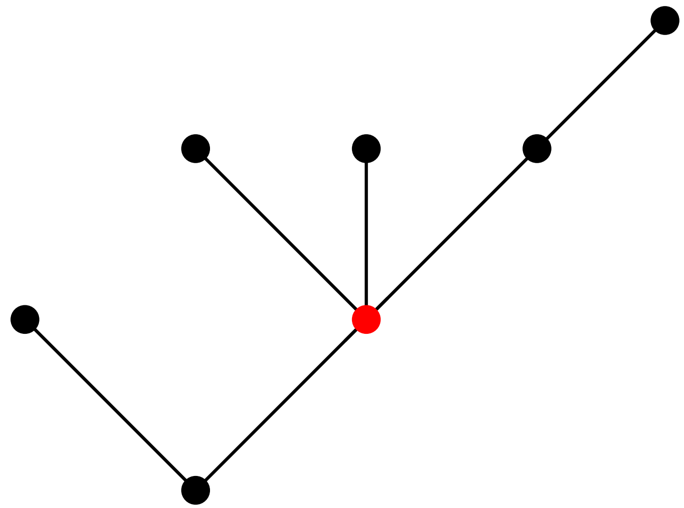

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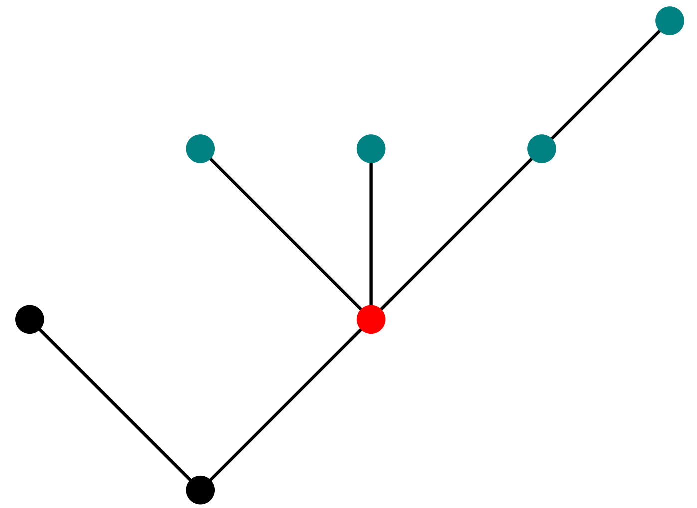

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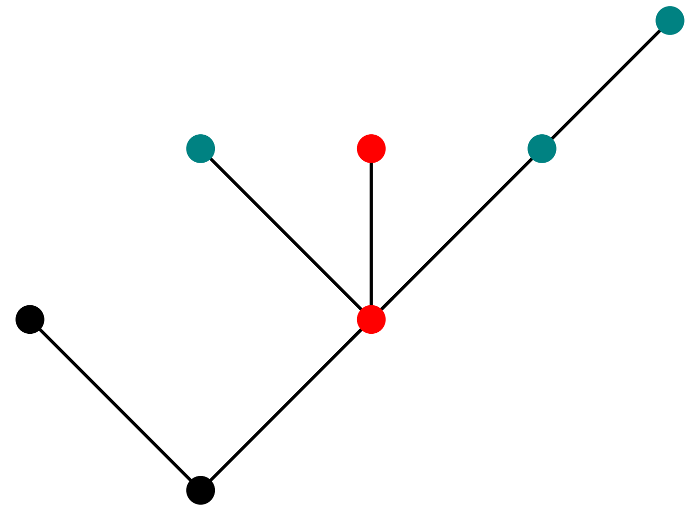

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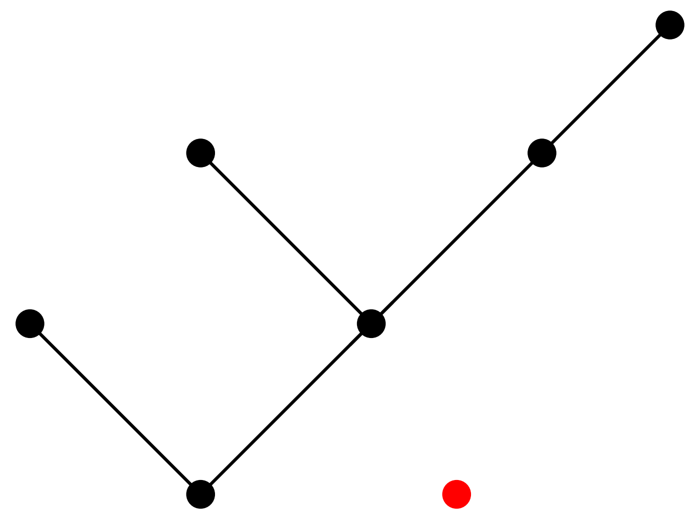

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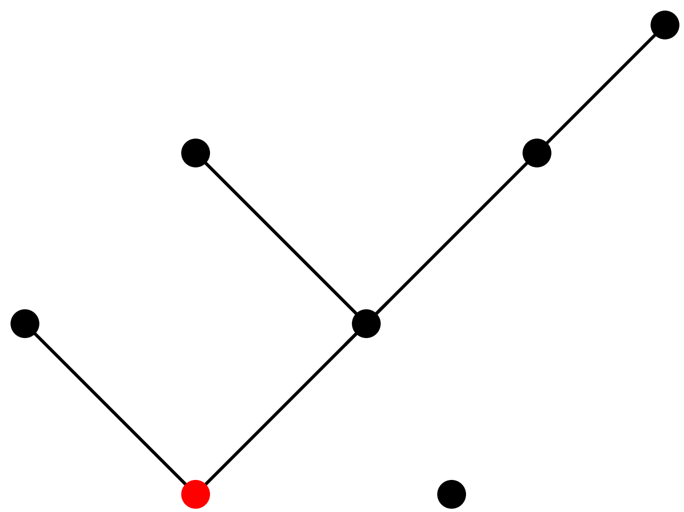

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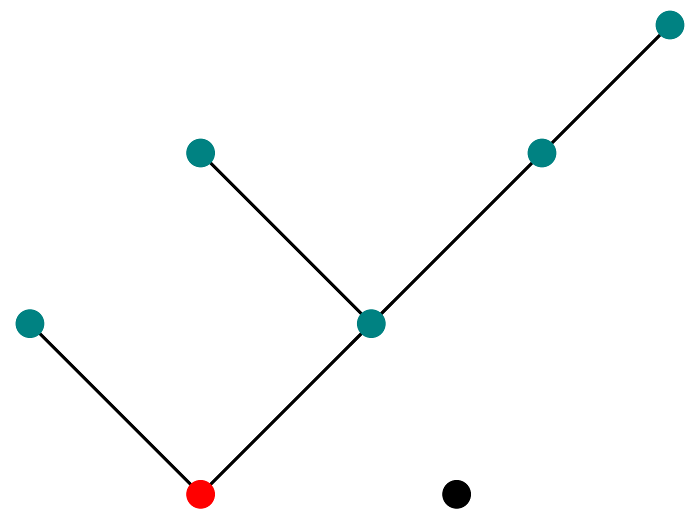

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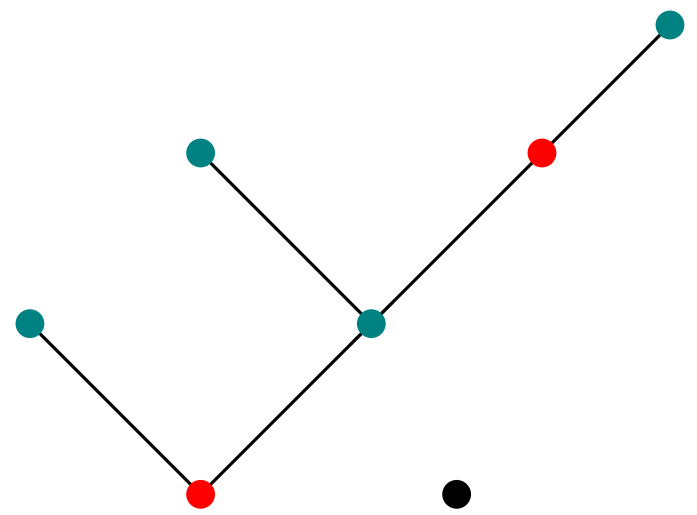

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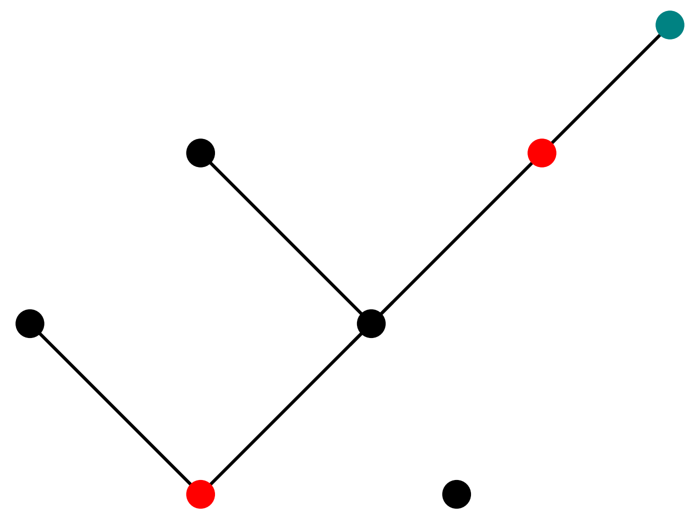

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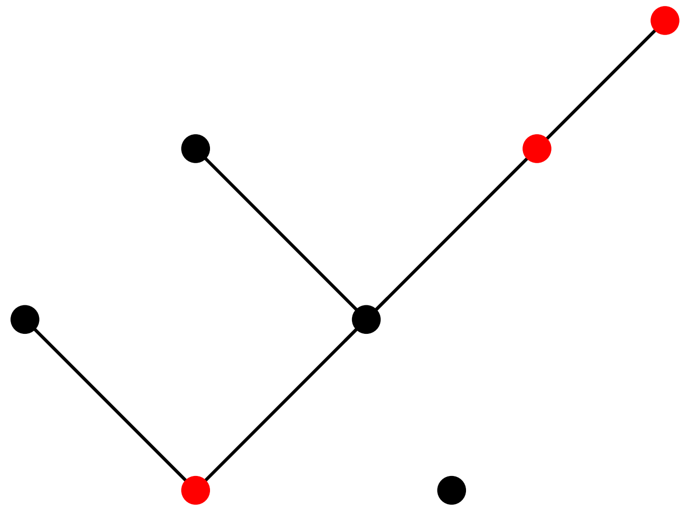

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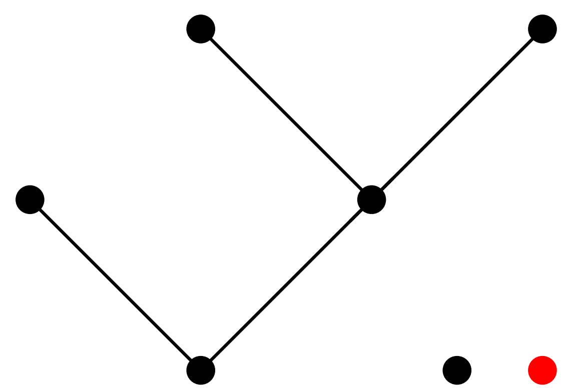

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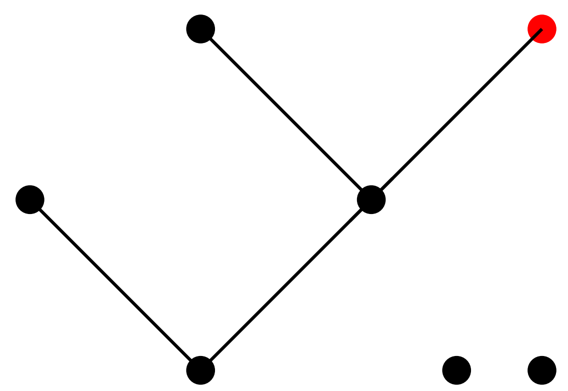

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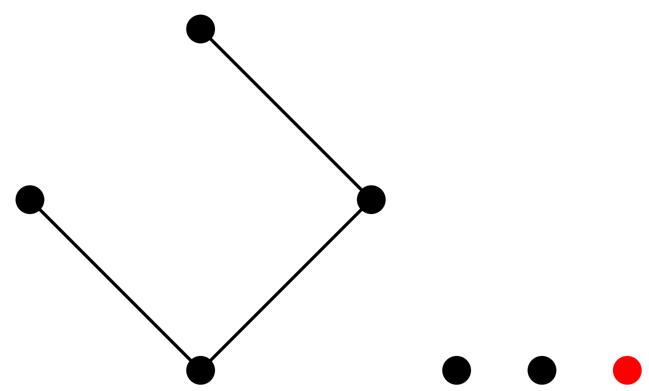

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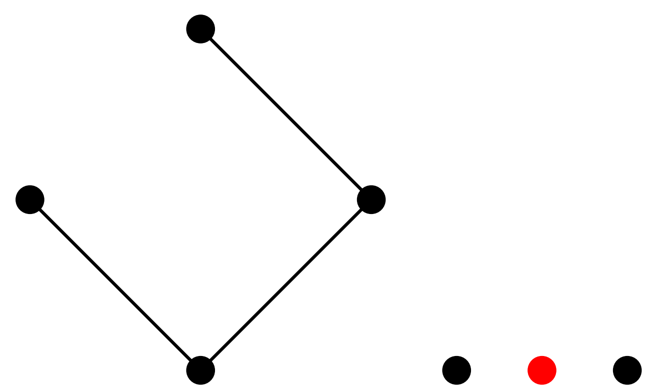

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Theorem (2015, 2016+): The eigenvalues are $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-2}{n}, 1.$

Let f_j be the number of j-tuples of vertices on different "branches". Then

$$\text{Expect}(f_j(X_t)) = \left(\frac{n-j}{n}\right)^t f_j(X_0).$$

The Future

- More combinatorial objects (e.g. phylogenetic trees)
- More linear maps (e.g. move-to-front with arbitrary request probabilities)
- Use probability to understand Hopf algebras (+Josuat-Verges, 2016+)

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Thank you!

Reader-friendly summary: "Card-Shuffling via Convolutions of Projections on Combinatorial Hopf Algebras" arXiv:1503.08368

Beyond eigen-information: "A Hopf-Algebraic Lift of the Down-Up Markov Chain on Partitions to Permutations" arXiv:1508.01570