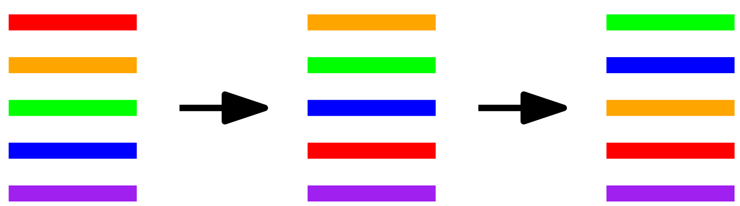
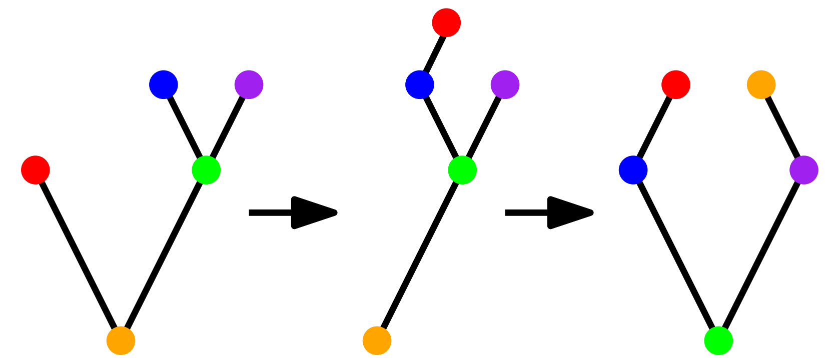
A Uniform Analysis of Combinatorial Markov Chains via Hopf Algebras





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Statistics Seminar, Chinese University of Hong Kong, 4 Oct 2016

slides available from amypang.github.io/cuhk2016.pdf

Motivation: a Dynamic Storage Allocation Problem

- You have n files, arranged in a list.
- You request files one-by-one independently, removing one from the list and returning it in a possibly different position.
- You request file i with a fixed, unknown, probability p_i.
- Each time you make a request, you search from the front of the list for the file you need.

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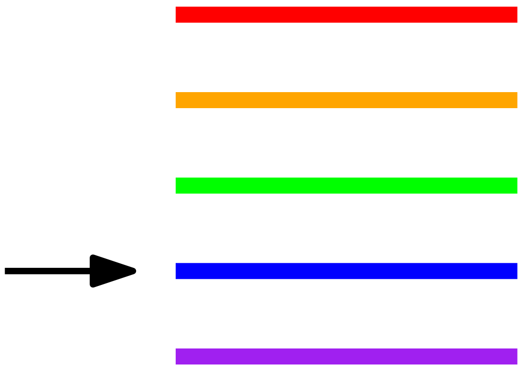
Application to data compression (Bentley-Sleator-Tarjan-Wei, 1986): each word is coded by the binary representation of its position in the list.

Two Answers by McCabe (1965)

Starting list



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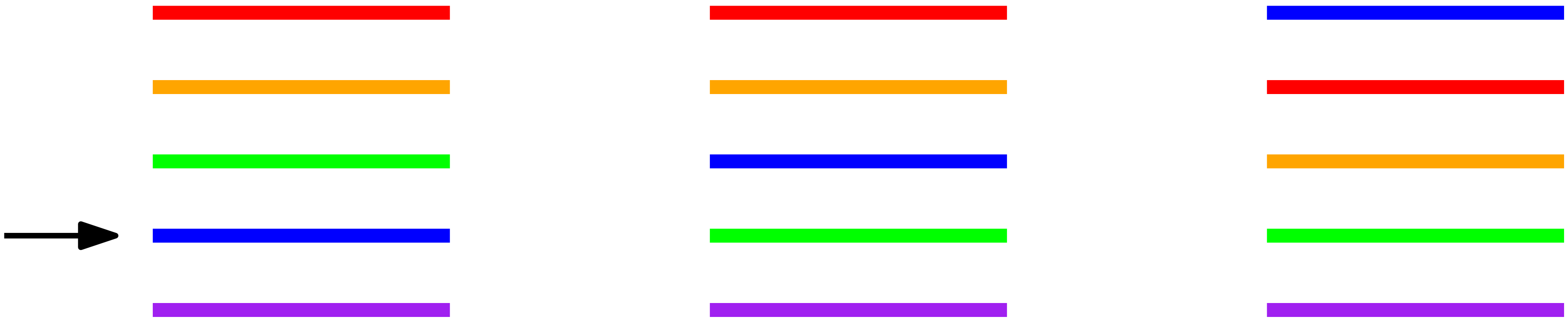
Starting list Transposition Move-to-front



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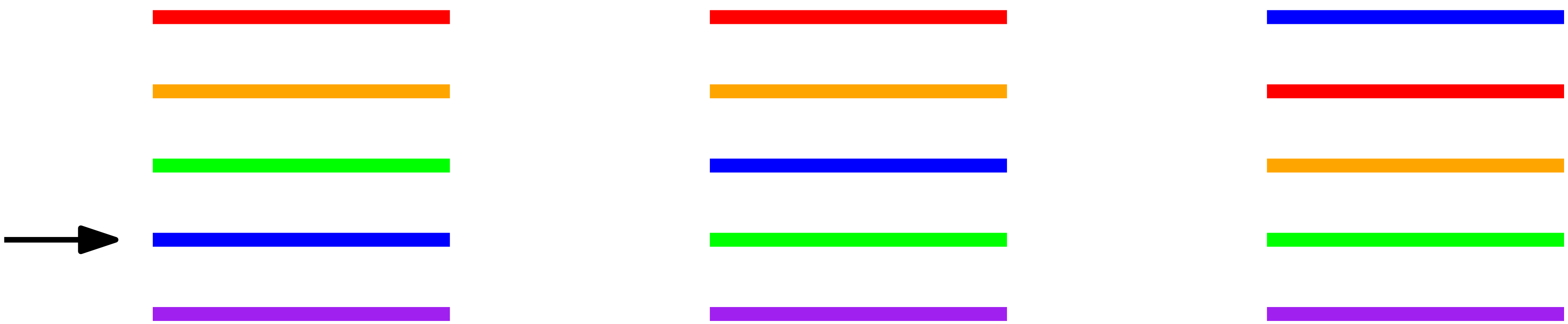

Stationary distribution: limiting probability of being in rainbow order

$$p_1{}^4 p_2{}^3 p_3{}^2 p_4$$

$$\frac{p_1}{1}\,\frac{p_2}{1-p_1}\,\frac{p_3}{1-p_1-p_2}\,\frac{p_4}{1-p_1-p_2-p_3}$$

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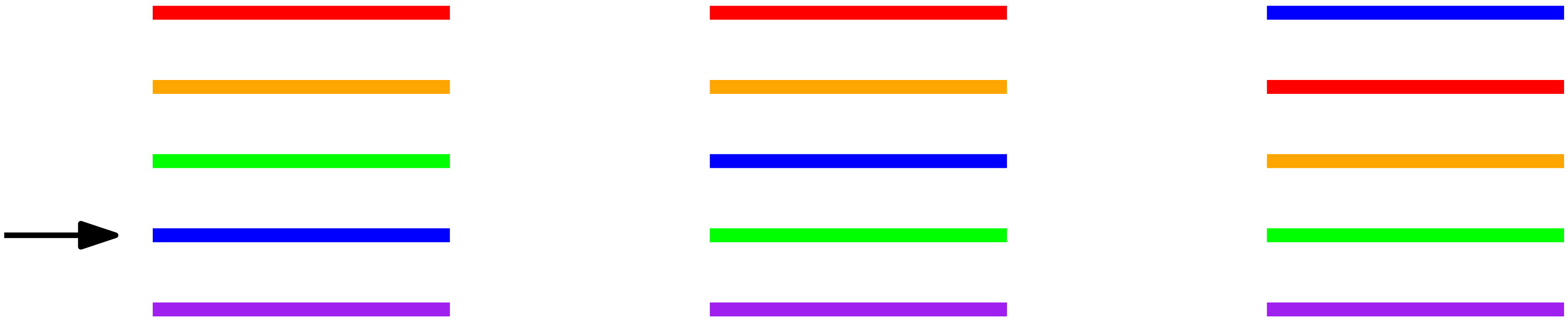

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Rivest (1976): lower average search time when in stationary distribution

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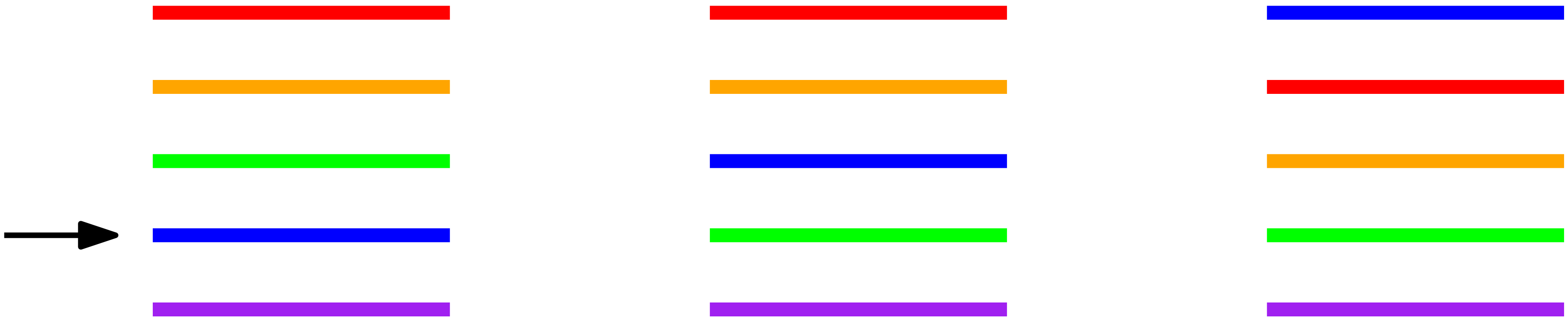

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Rivest (1976): lower average search time when in stationary distribution

Bitner (1979): reaches stationary distribution earlier

Markov Chains

- $\mathcal{X}$ a (finite) state space. all possible orders of n files
- X_t a random variable taking values in $\mathcal{X}$, for each $t \in \mathbb{N}$.

  the order of the files after t requests
- The process $\{X_t\}$ is memoryless, in that Prob$(X_{t+1} = y | X_t = x)$ is a number $K(x, y)$ independent of $X_1, X_2, ...X_{t-1}$ and of t.

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Important questions:

- Stationary distribution: $\sum_{x \in \mathcal{X}} \pi_x K(x, y) = \pi_y$.

  eigenvector of eigenvalue 1
- Convergence rate: $||X_t - \pi|| \leq \epsilon$.

  subdominant eigenvalue (spectral gap)

Markov Chains in Statistics

Generation of uniform contingency tables with given row and column sums (Hernek 1998)

we show that the number of walk steps required to obtain a random $2 \times n$ array is at most quadratic in n and N. It is an open question to determine whether the chain mixes in polynomial time for arrays with more than two rows.

	Favor	Indifferent	Opposed	TOTAL
Democrat	138	83	64	285
Republican	64	67	84	215
TOTAL	202	150	148	500

Fig. 1. Contingency table for political affiliation and opinion.

The data augmentation method to compute a posterior distribution from incomplete data (Tanner, Wong 1987)

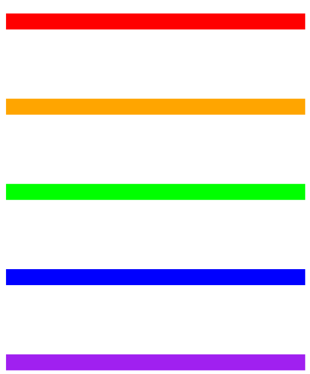
- Generate the missing data using the current estimate of the parameter,
- Make new parameter estimate using the "completed" data.

The Top-to-Random Shuffle

(time-reversal of move-to-front with equal request probabilities)

- Remove top card
- Reinsert this card at a uniformly chosen position

For example:



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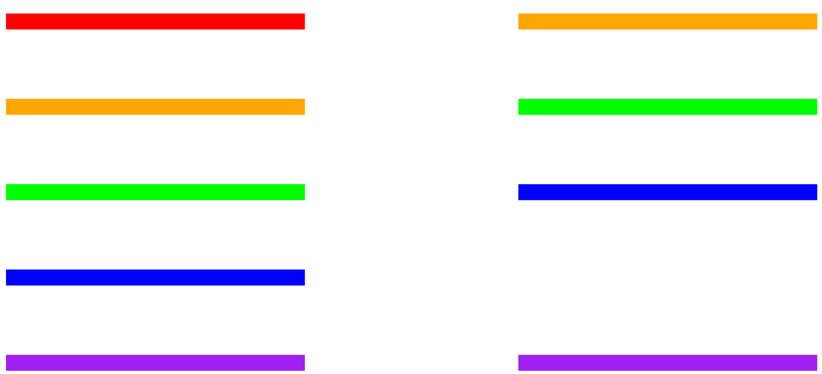

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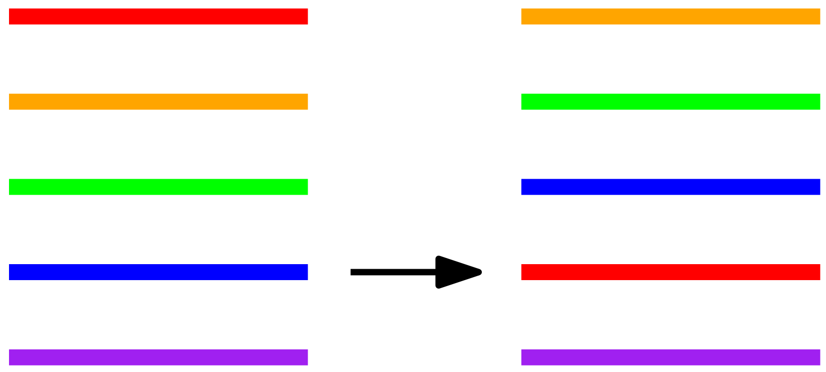

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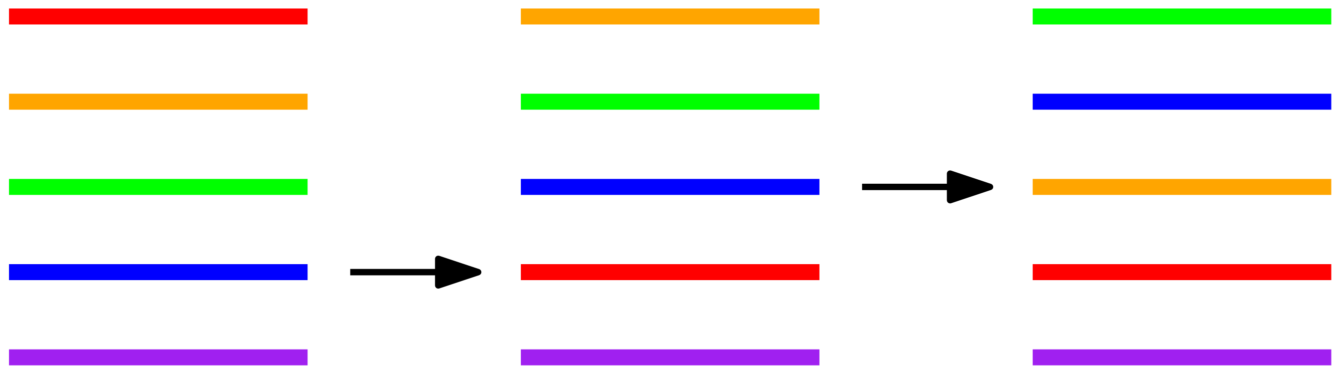

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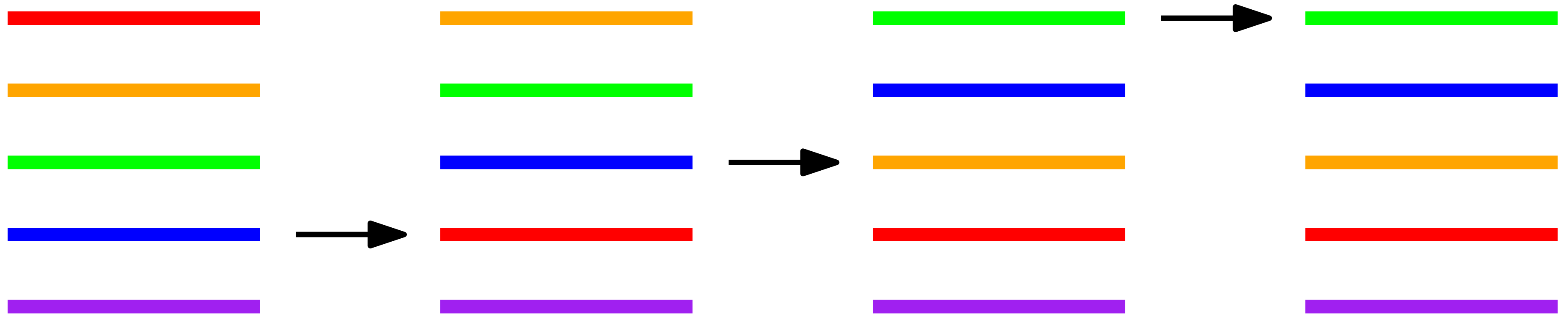

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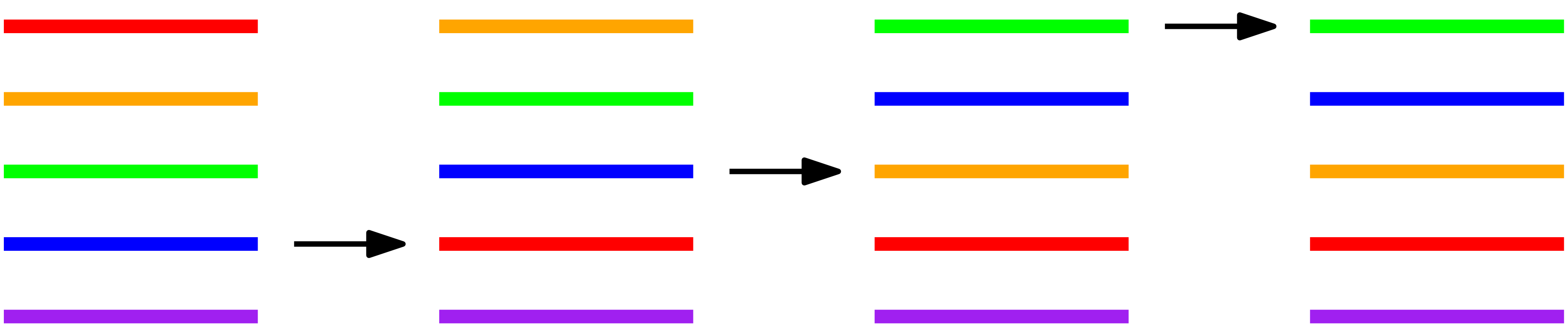
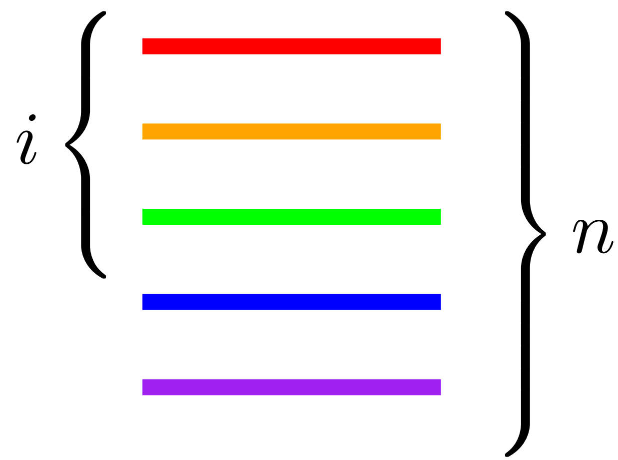


Aldous-Diaconis (1986): convergence rate $\sim n \log n$.
(asymptotically in n; 205 when $n = 52$).

The Riffle Shuffle

- Cut the deck with symmetric binomial distribution;



$$\text{Prob} = 2^{-n}\binom{n}{i}$$

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The Riffle Shuffle

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Prob $= \frac{3}{5}$

Prob $= \frac{2}{5}$

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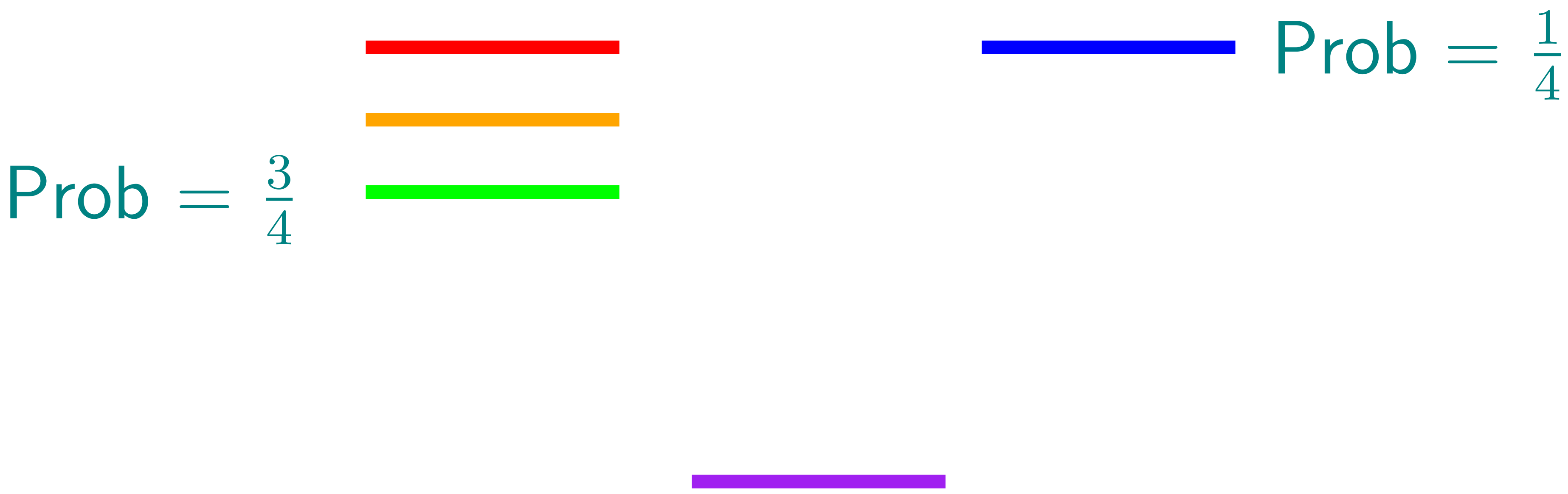

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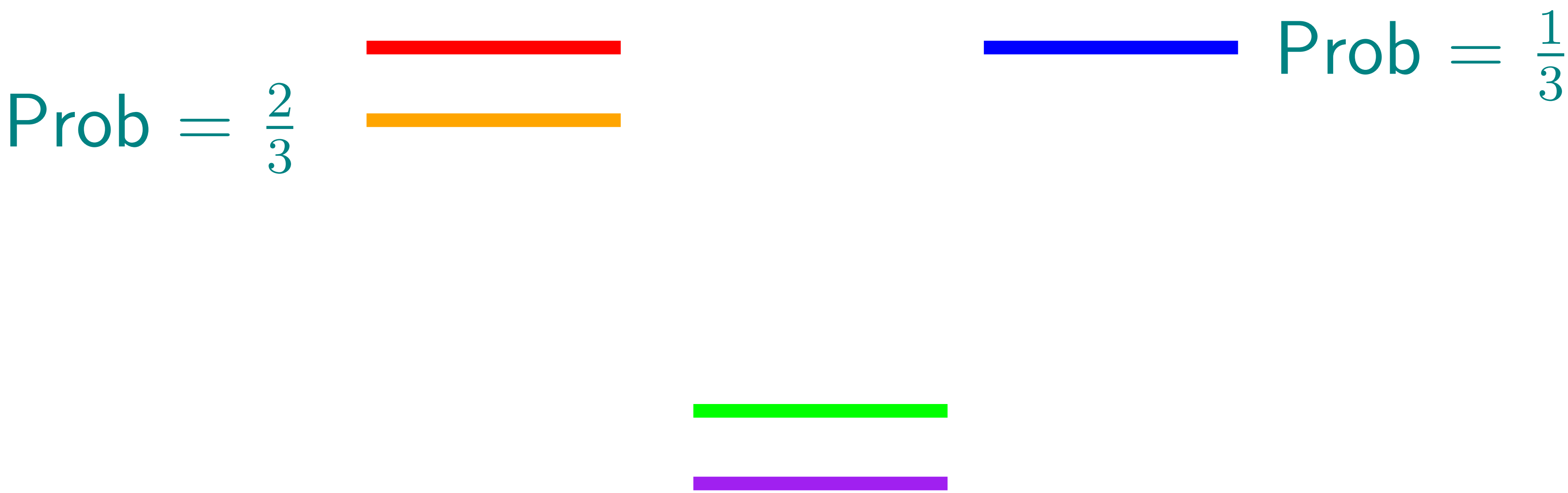

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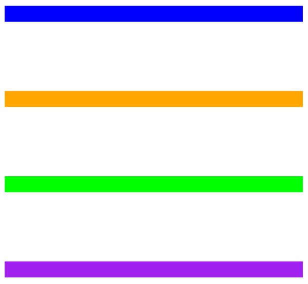
$\text{Prob} = \frac{1}{1}$



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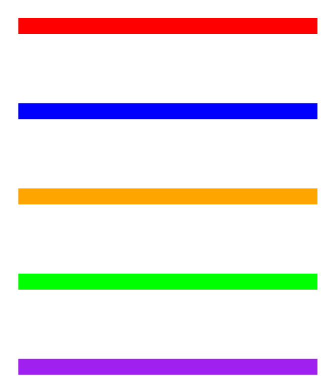

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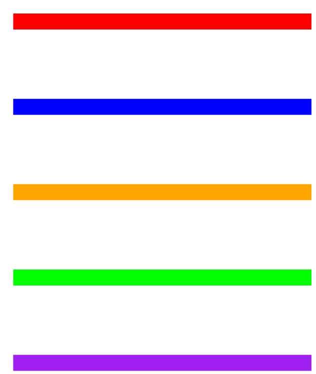

Bayer-Diaconis (1992): convergence rate $\sim \frac{3}{2} \log_2 n$.

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Analysing Shuffles using Maps on Hopf Algebras

	Top-to-random	Riffle
Theorem :	(2015)	(+Diaconis, Ram 2014)
extensions of results by:	Diaconis-Fill-Pitman (1992)	Bayer-Diaconis (1992), Hanlon (1990)

The unique stationary distribution is the uniform distribution.

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Eigenvalues:	$0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-2}{n}, 1.$	$2^{-n+1}, \dots, 2^{-1}, 1.$

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Multiplicities of eigenvalues, for all cards distinct	number of permutations of n cards with j fixed points.	number of permutations of n cards with j cycles.

Analysing Shuffles using Maps on Hopf Algebras

- An explicit formula for an eigenbasis.

ii) For any $p \in \mathcal{H}_{n-j}$ satisfying $\Delta_{1,n-j-1}(p) = 0$, and any $c_1, \ldots, c_j \in \mathcal{H}_1$ (not necessarily distinct),

$$\sum_{\sigma \in \mathfrak{S}_j} c_{\sigma(1)} \cdots c_{\sigma(j)} p$$

is an eigenvector for:

- $\mathrm{T2R}_n$ *and* B

Theorem 3.16 *Let $\mathcal{H}$ be a cocommutative Hopf algebra (over a field of characteristic zero) that is a free associative algebra with word basis $\mathcal{B}$. For $b \in \mathcal{B}$ with factorization into generators $b = c_1 c_2 \cdots c_l$, set g_b to be the polynomial* sym(b) *evaluated at $(e(c_1), e(c_2), \ldots, e(c_l))$. In other words, in the terminology of Sect. 2.3,*

- *for c a generator, set $g_c := e(c)$;*
- *for b a Lyndon word, inductively define $g_b := [g_{b_1}, g_{b_2}]$ where $b = b_1 b_2$ is the standard factorization of b;*
- *for b with Lyndon factorization $b = b_1 \cdots b_k$, set $g_b := \sum_{\sigma \in S_k} g_{b_{\sigma(1)}} g_{b_{\sigma(2)}} \cdots g_{b_{\sigma(k)}}$.*

Then g_b is an eigenvector of Ψ^a of eigenvalue a^k (k the number of Lyndon factors

Related to wavelet bases of a new "multiresolution analysis" of incomplete ranking data (Sibony, Clémençon, Jakubowicz 2016)

Analysing Shuffles using Maps on Hopf Algebras

- An explicit formula for an eigenbasis.

Corollary (2015): Start with n distinct cards in ascending order. After t top-to-random shuffles:

$$\text{Prob (descent at the bottom)} = \left(1 - \left(\frac{n-2}{n}\right)^t\right)\frac{1}{2}.$$

big card on small card

$$\text{Prob (peak at the bottom)} = \left(1 - \left(\frac{n-3}{n}\right)^t\right)\frac{1}{3}.$$

triple of cards with biggest in middle

Analysing Shuffles using Maps on Hopf Algebras

- An explicit formula for an eigenbasis.

Corollary (+Diaconis, Ram, 2014): Start with n distinct cards in ascending order. After t riffle shuffles:

Expect (number of descents)$=\left(1-\left(\frac{1}{2}\right)^t\right)\frac{n-1}{2}$.

Expect (number of peaks)$=\left(1-\left(\frac{1}{4}\right)^t\right)\frac{n-2}{3}$.

A New Connection: Ree's Shuffle (Hopf) Algebra

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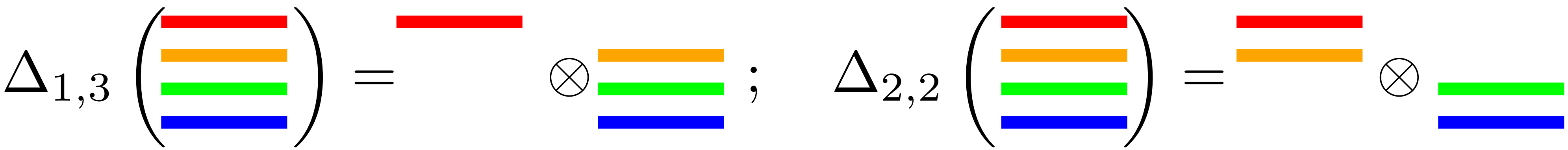

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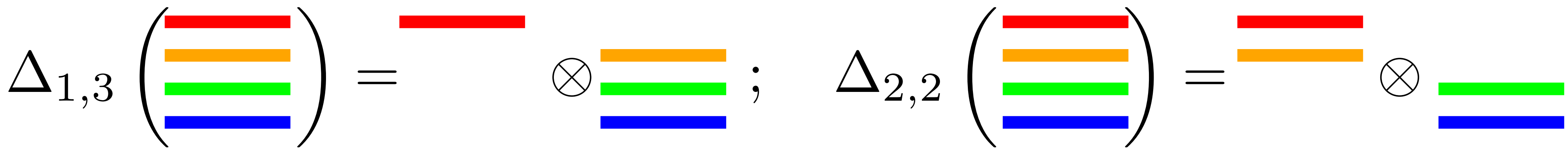
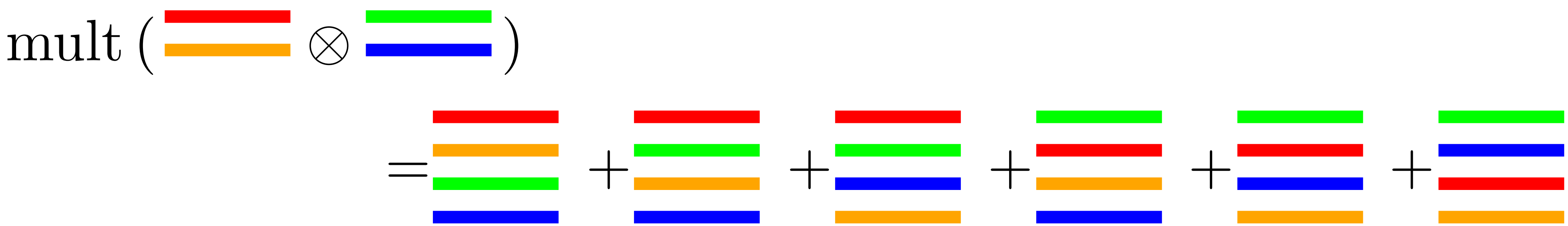


- "interleaving" product $\mathrm{mult} : \mathcal{H}_i \otimes \mathcal{H}_j \to \mathcal{H}_{i+j}$



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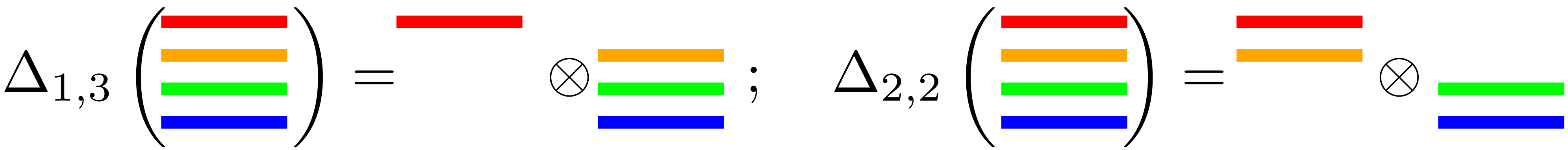

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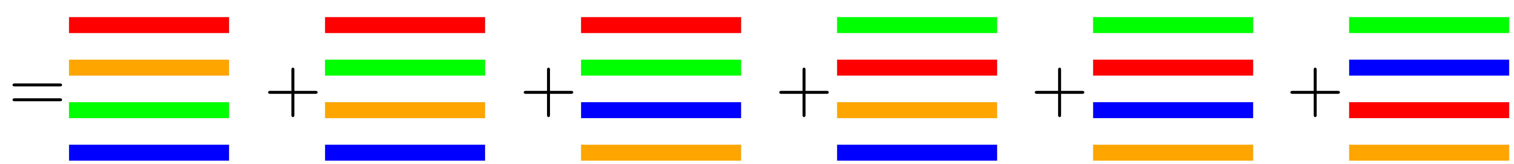

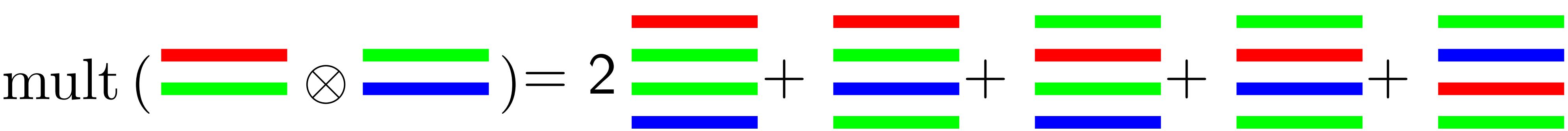

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(x, y are decks of n cards)

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Prob$(x \to y)$= coefficient of y in $\frac{1}{n}\,\mathrm{mult} \circ \Delta_{1,n-1}(x)$.

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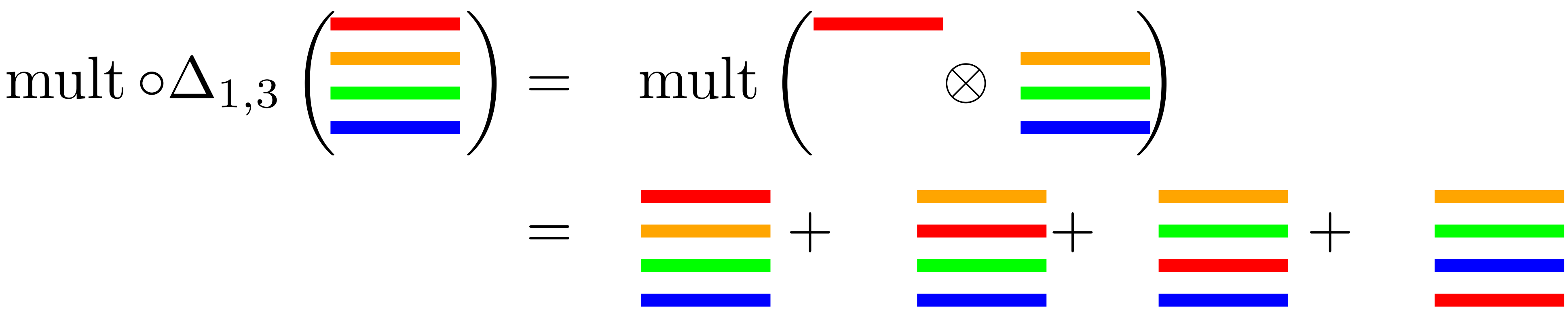

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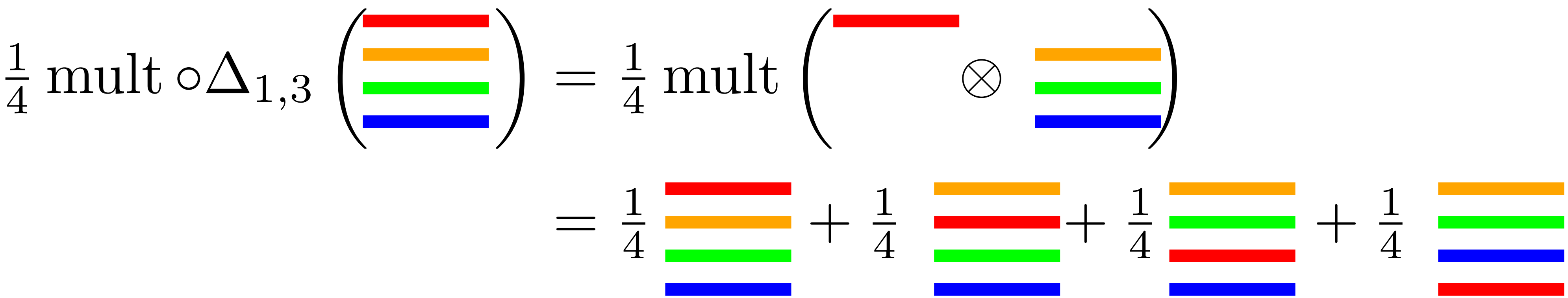

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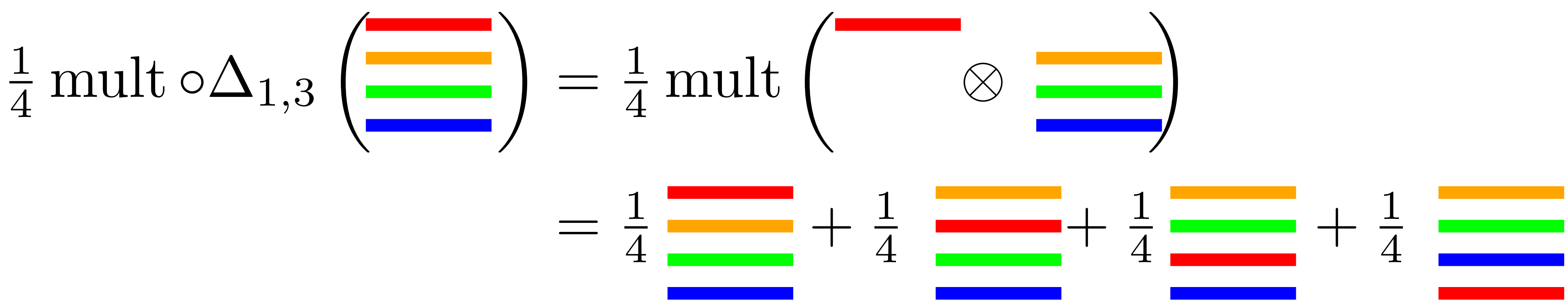

Riffle:

Prob$(x \to y)$= coefficient of y in $\dfrac{1}{2^n}\,\mathrm{mult} \circ \sum_{i=0}^{n} \Delta_{i,n-i}(x)$.

Chains on Other Combinatorial Objects

Markov chain	Hopf algebra / Hopf		basis		stationary distribution
	free?	cofree?		free-comm	
shuffling	shuffle algebra $\mathcal{S}$	x	words / decks of cards		uniform
inverse-shuffling	free associative alge		words / decks of cards		uniform
edge-removal	$\mathcal{G}$		unlabelled graphs	x	absorbing at empty graph
edge-removal	$\mathcal{G}$ x		labelled graphs		absorbing at empty graph
restriction-then-induction	representations of sy	x	irreducible representations		plancherel
rock-breaking	symmetric functi	x	elementary or complete	x	absorbing at $(1,1,\ldots,1)$
tree-pruning	Connes-Kreimer		rooted forests	x	absorbing at disconnected forest
descent-set-under-shuffling	quasisymmetric func	x	fundamental (compositions)		proportion of permutations with this de
jeu-de-taquin	Poirier-Reutenauer		standard Young tableaux	sion of shape	proportion of standard tableaux with th
shuffle with standardisation	Malvenuto-Reut	x	fundamental (permutations)		uniform

also Servando Pineda's masters thesis (2015), "Hopf Random Walks on the Faces of Permutohedra",

and Bergeron-Ceballos (2015), "A Hopf Algebra of Subword Complexes":

The Markov process induced by the top-to-random shuffle for fixed n, k on $\mathcal{Y}$ can be interpreted as follows. We imagine that the elements (W, Q, π, I) are types of rocks. More precisely, we think that if $W = W_1 \times W_2 \times \cdots W_k$ and the W_i are indecomposable, then (W, Q, π, I) is exactly k rocks of certain types. The operator $m_{1,n-1} \circ \Delta_{1,n-1}$ can be though off as a small hammer hitting on the rocks. The expansion in Equation (11) describes the types of rocks (W', Q', π', I') we can get

Chains on Other Combinatorial Objects

Markov chain	Hopf algebra / Hopf	free?	cofree?	basis	free-comm	stationary distribution
shuffling	shuffle algebra		x	words / decks of cards		uniform
inverse-shuffling	free associative alge			words / decks of cards		uniform
edge-removal	$\mathscr{G}$			unlabelled graphs	x	absorbing at empty graph
edge-removal	$\mathscr{G}$	x		labelled graphs		absorbing at empty graph
restriction-then-induction	representations of sy		x	irreducible representations		plancherel
rock-breaking	symmetric functi		x	elementary or complete	x	absorbing at $(1,1,\ldots,1)$
tree-pruning	Connes-Kreimer			rooted forests	x	absorbing at disconnected forest
descent-set-under-shuffling	quasisymmetric func		x	fundamental (compositions)		proportion of permutations with this de
jeu-de-taquin	Poirier-Reutenauer			standard Young tableaux		proportion of standard tableaux with th
shuffle with standardisation	Malvenuto-Reutenauer		x	fundamental (permutations)		uniform

also Servando Pineda's masters thesis (2015), "Hopf Random Walks on the Faces of Permutohedra",

and Bergeron-Ceballos (2015), "A Hopf Algebra of Subword Complexes":

The Markov process induced by the top-to-random shuffle for fixed n, k on $\mathcal{Y}$ can be interpreted as follows. We imagine that the elements (W, Q, π, I) are types of rocks. More precisely, we think that if $W = W_1 \times W_2 \times \cdots W_k$ and the W_i are indecomposable, then (W, Q, π, I) is exactly k rocks of certain types. The operator $m_{1,n-1} \circ \Delta_{1,n-1}$ can be though off as a small hammer hitting on the rocks. The expansion in Equation (11) describes the types of rocks (W', Q', π', I') we can get

The Top-to-Random Chain on Trees

Hopf algebra of trees: Butcher (1972), Connes-Kreimer (1998)

Transition probabilities from $\frac{1}{n}\,\mathrm{mult} \circ \Delta_{1,n-1}$

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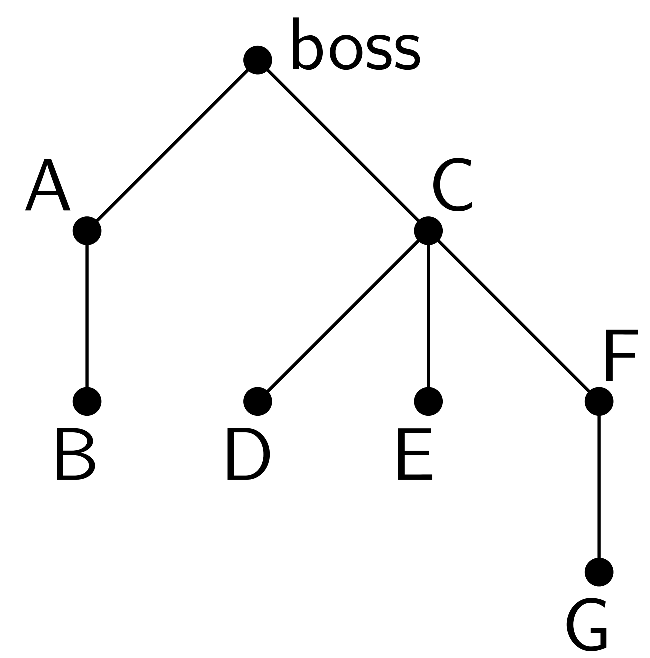

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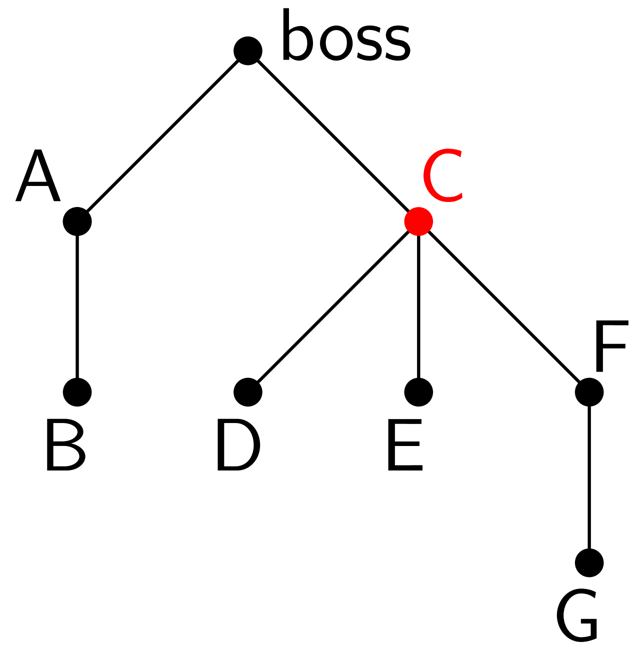

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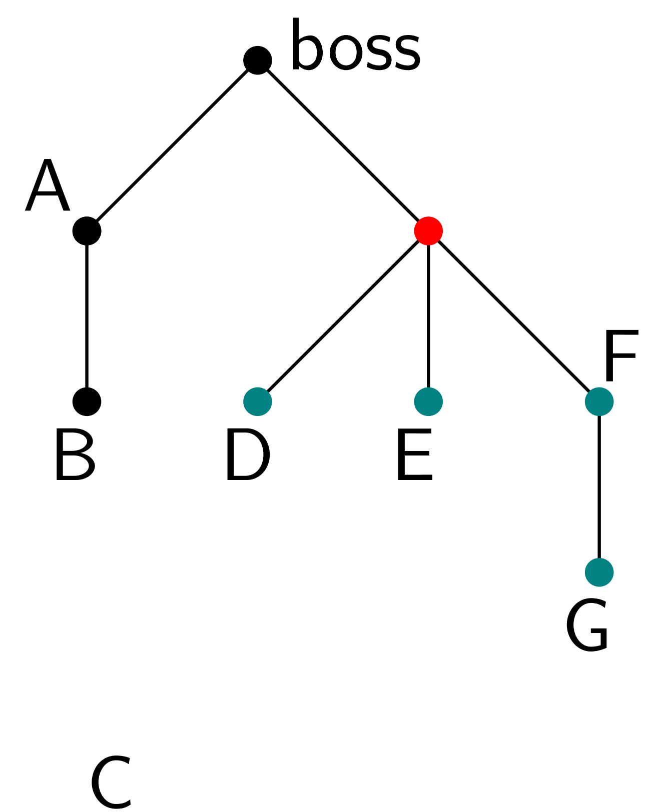

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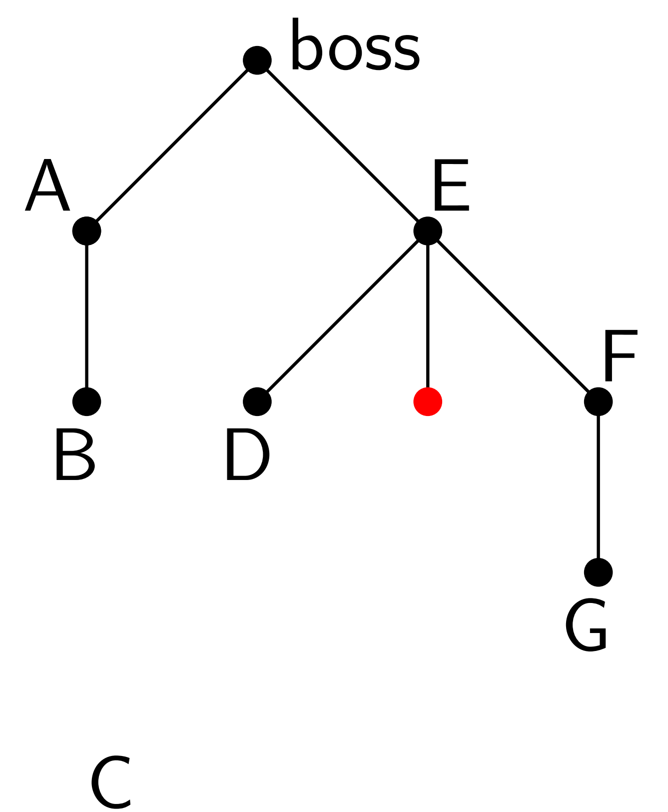

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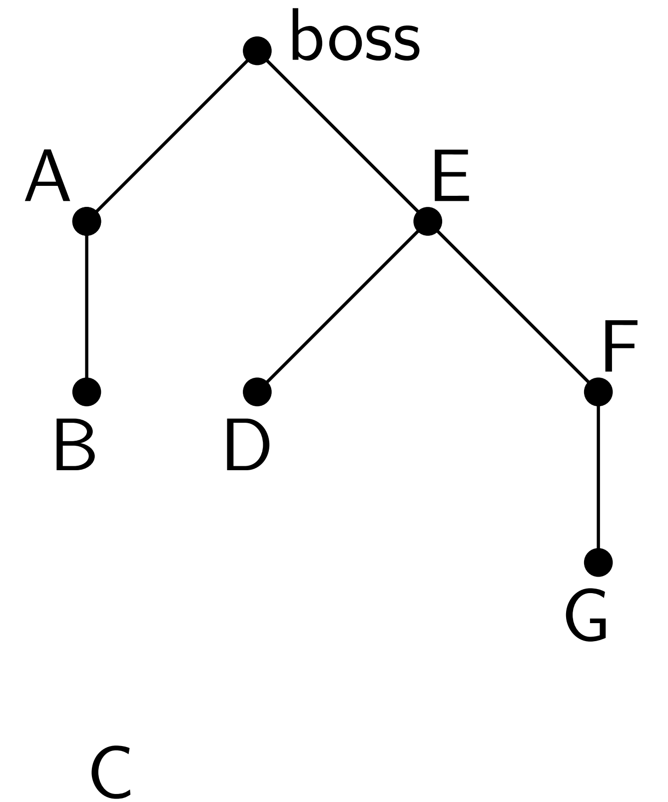

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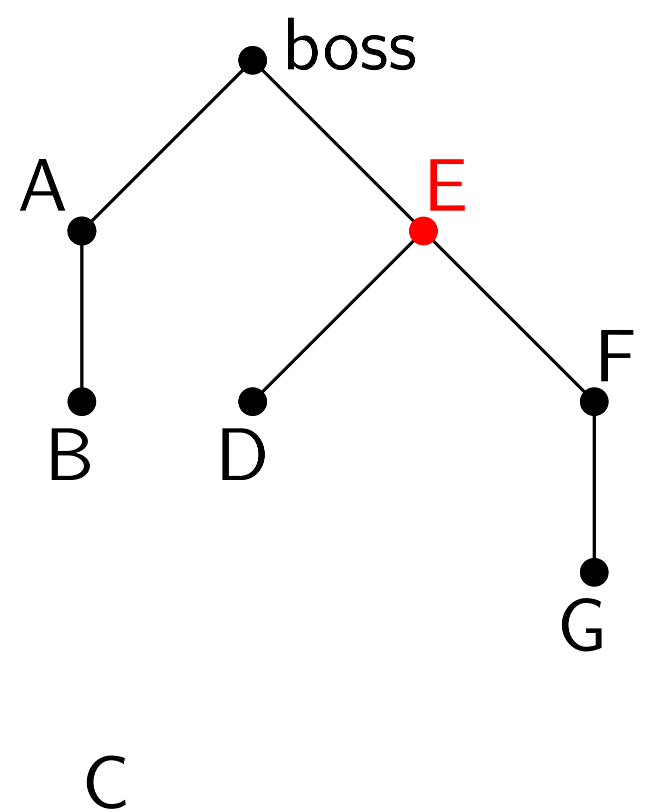

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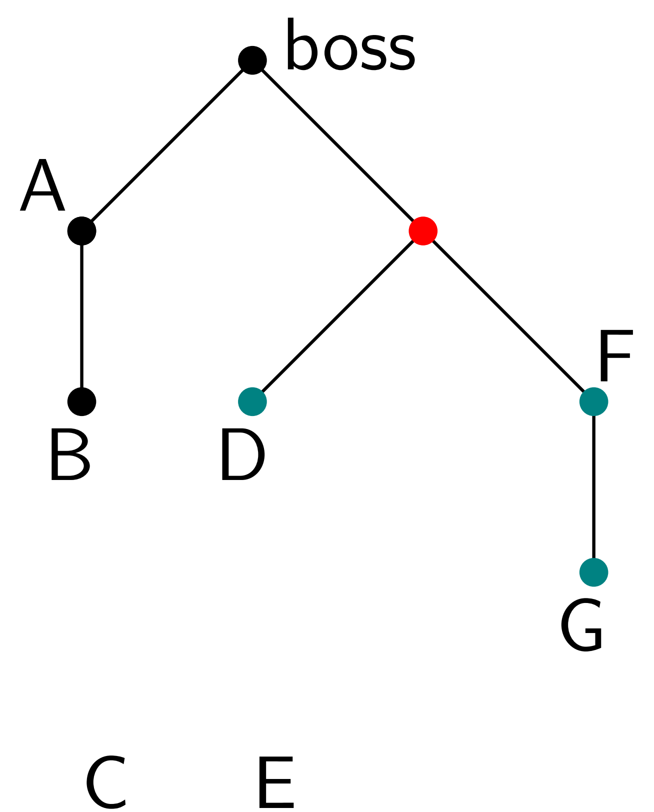

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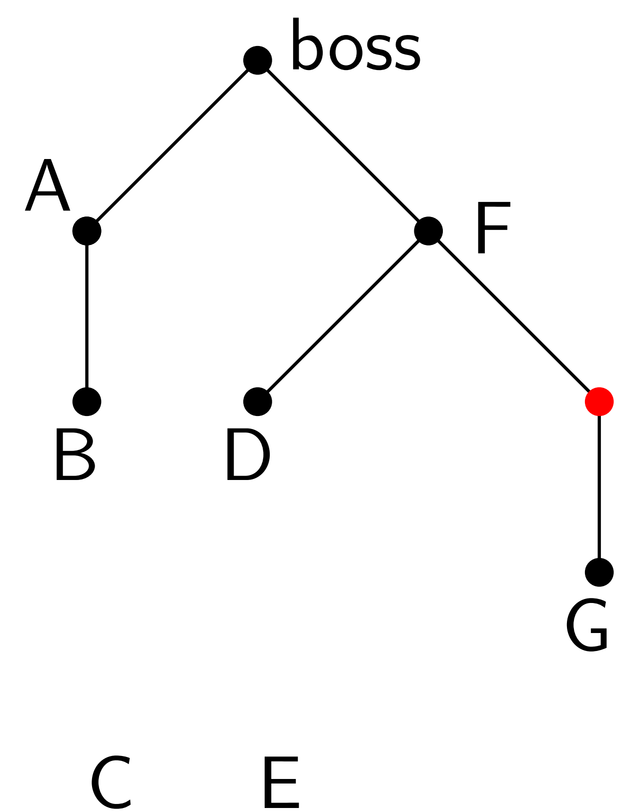

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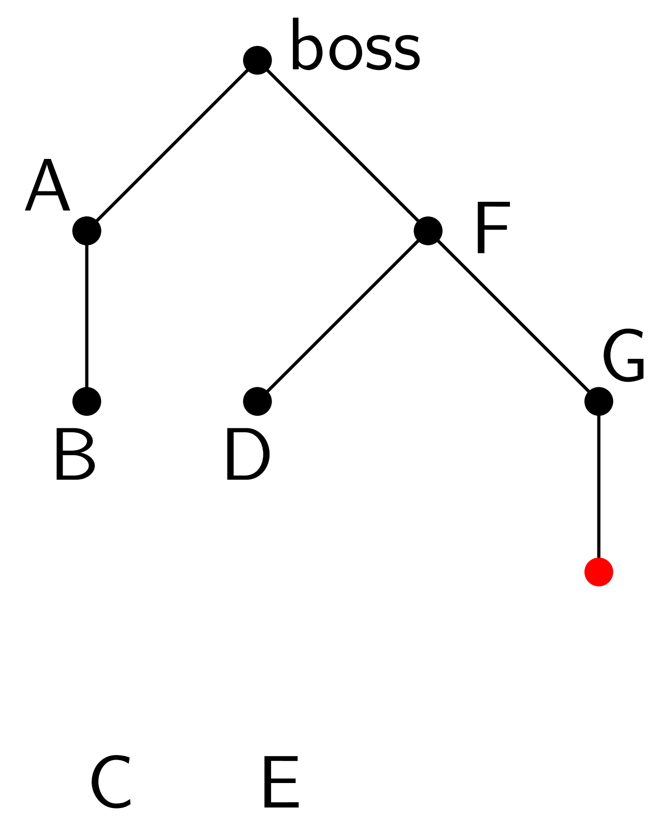

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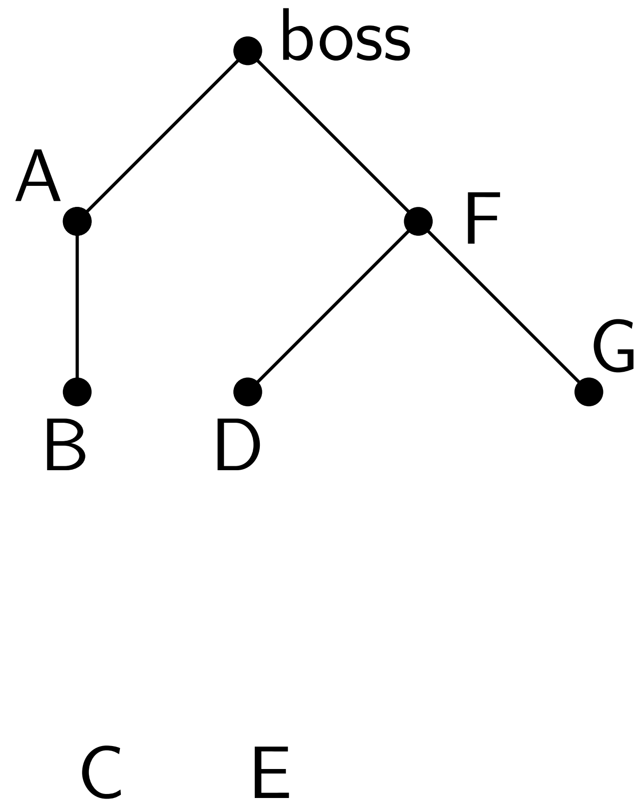

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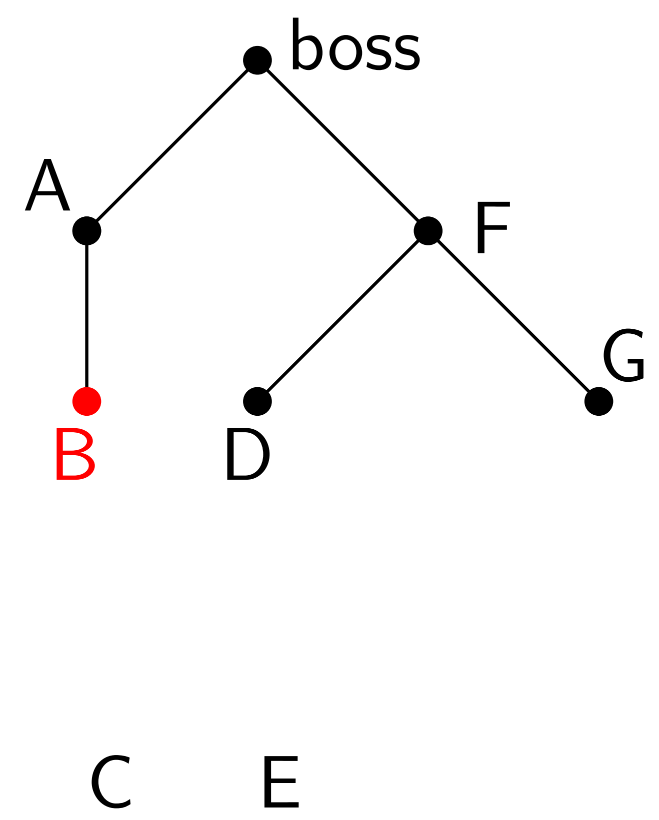

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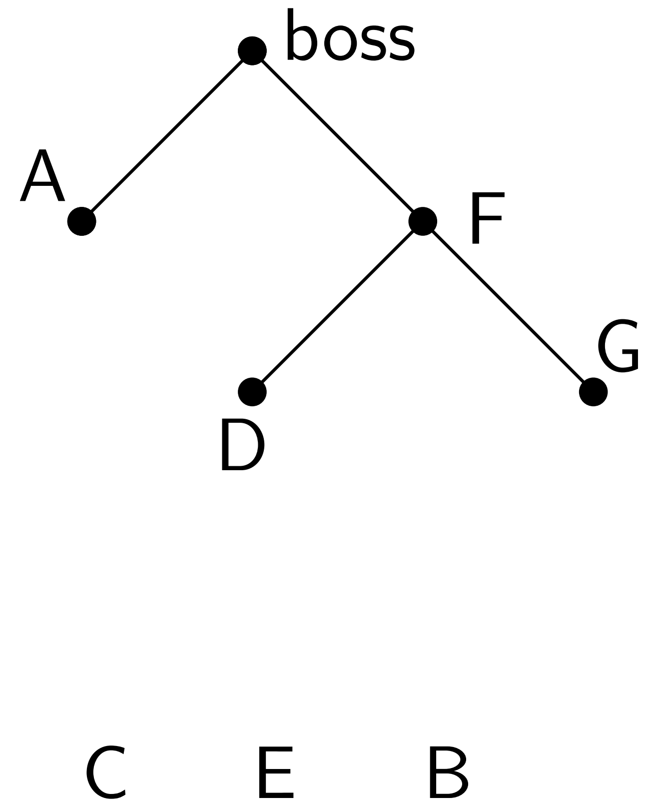

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- Fire a random employee with probability n_t/n, else no change
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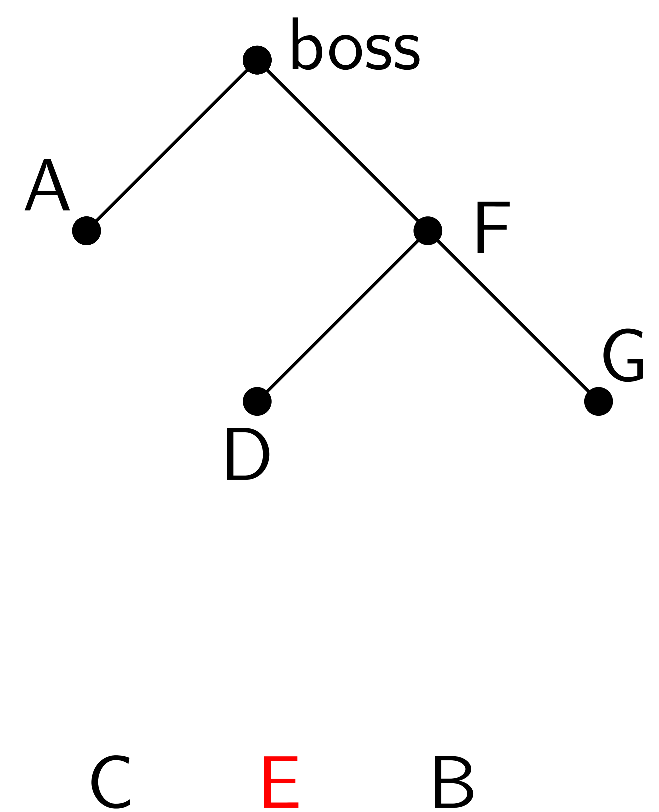

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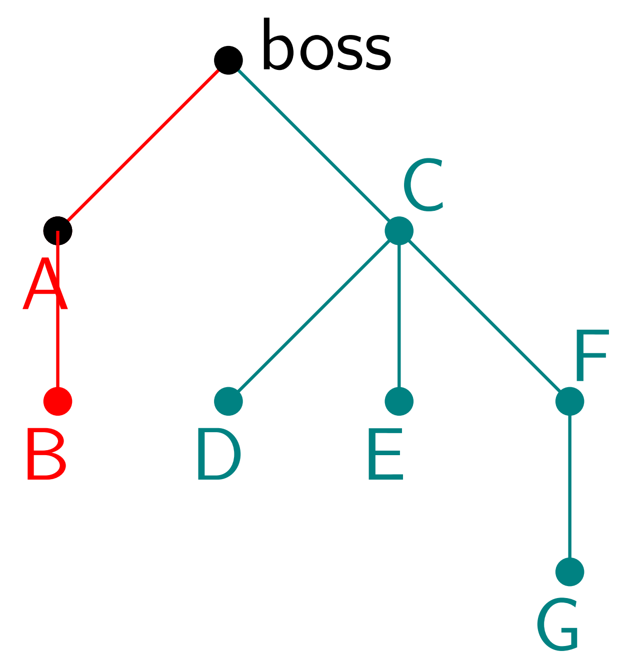

Theorem (2016): The eigenvalues are $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-2}{n}, 1.$

Expected number of teams of s_i employees from department i falls roughly by a factor of $\frac{n-1-\sum s_i}{n}$ per unit time.

The Top-to-Random Chain on Trees

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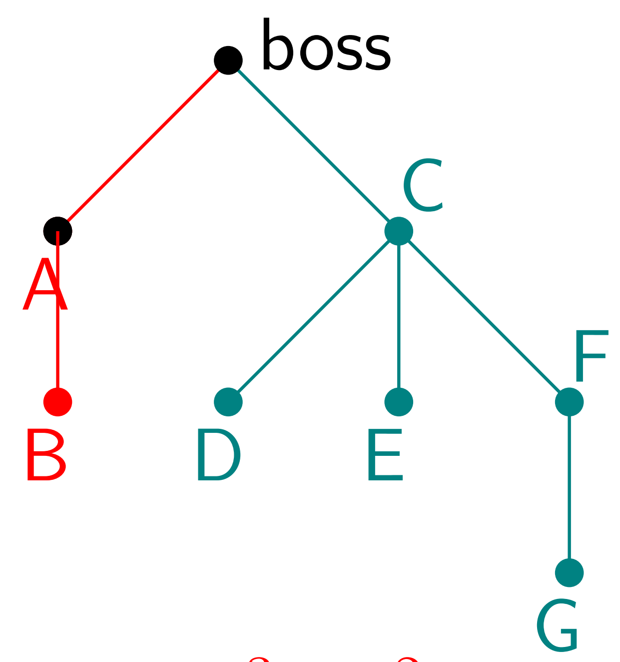
Transition probabilities from $\frac{1}{n}\,\mathrm{mult}\circ\Delta_{1,n-1}$

- Fire all employees with performance below q
- Promote a randomly chosen subordinate to replace him.
- Repeat the promotions until the vacated position has no subordinates.



Theorem (2016): The eigenvalues are $q^{-n}, \ldots, q^{-3}, q^{-2}, 1$.

Expected number of teams of s_i employees from department i falls roughly by a factor of $q^{1+\sum s_i}$ per unit time.

The Future

- More combinatorial objects (e.g. phylogenetic trees)
- More linear maps (e.g. move-to-front with arbitrary request probabilities, a version with involutions)
- Maps between Hopf algebras give relationship between chains (2015)
- Use probability to understand Hopf algebras (+Josuat-Vergès, 2016)
- How to characterise chains which come from Hopf algebras in this way??

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Thank you!

amypang.github.io/research.html amypang@hkbu.edu.hk