In this week's notes, we are interested in finding the *local maxima* / *local minima* of a multivariate function $f$, i.e. the points $(a_1, \dots, a_n)$ such that

$$\begin{matrix} f(a_1,\dots,a_n) \geq f(x_1,\dots,x_n) \\ f(a_1,\dots,a_n) \leq f(x_1,\dots,x_n) \end{matrix}$$

for all $(x_1, \dots, x_n)$ close to $f(a_1, \dots, a_n)$.

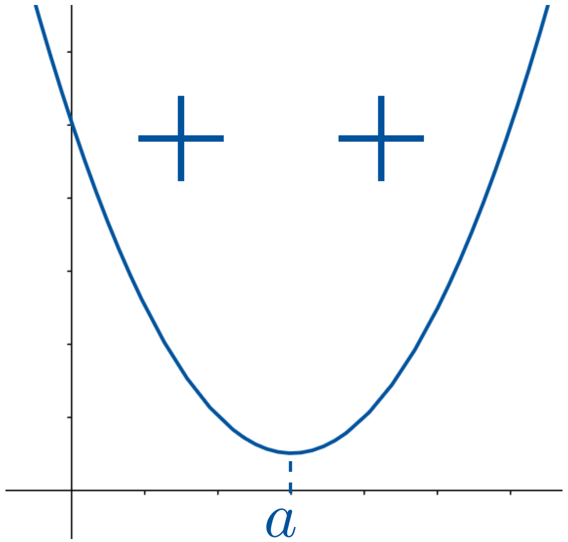
If $\nabla f(a_1, \dots, a_n) \neq \mathbf{0}$, then $f$ is increasing in the direction $\nabla f(a_1, \dots, a_n)$ and decreasing in the direction $-\nabla f(a_1, \dots, a_n)$, so $(a_1, \dots, a_n)$ cannot be a local maximum or minimum. So a local maximum or minimum must be a critical point.

**Definition**: A point $(a_1, \dots, a_n)$ is a *critical point* of $f : \mathbb{R}^n \to \mathbb{R}$ if $\nabla f(a_1, \dots, a_n) = \mathbf{0}$, i.e. if all its partial derivatives are 0.

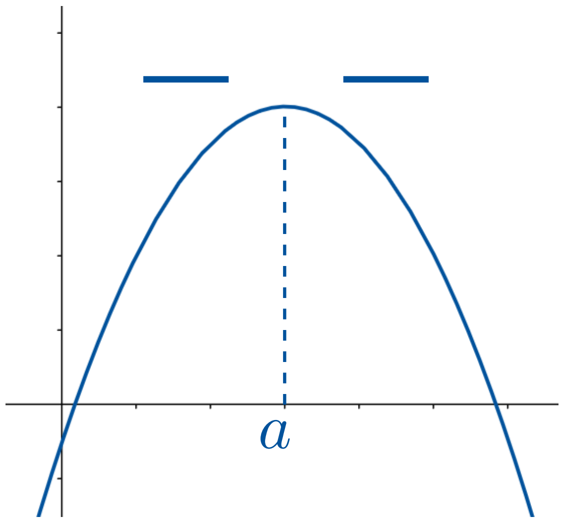
But not every critical point is a local maximum or minimum as we will see.

# §13.1: Classifying Critical Points

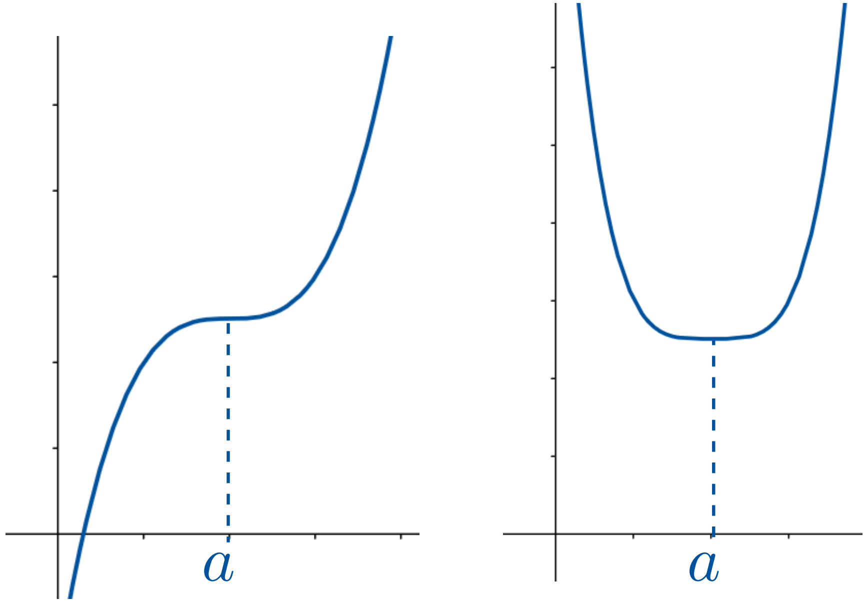
Recall that a critical point of a single-variable function $f$ is where the derivative $f'$ is zero. A standard way to determine whether it is a local maximum, a local minimum, or neither, is the second derivative test:




if $f''(a) > 0$ then $a$ is a local minimum

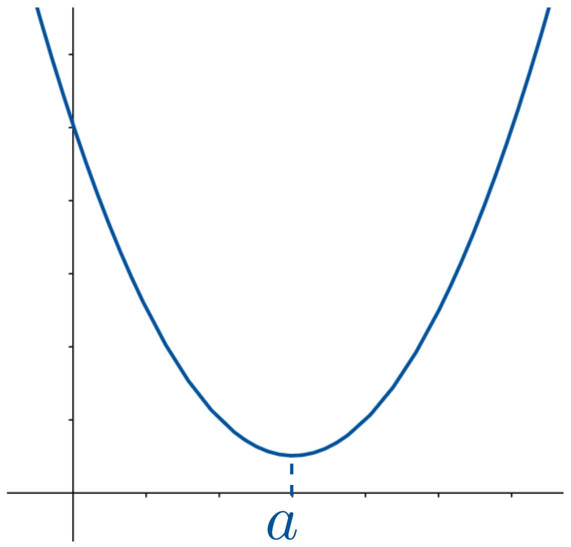



if $f''(a) < 0$ then $a$ is a local maximum

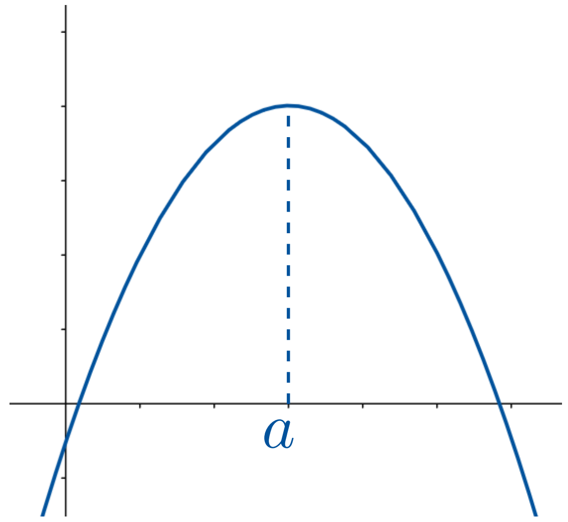



if $f''(a) = 0$ then we need to investigate further

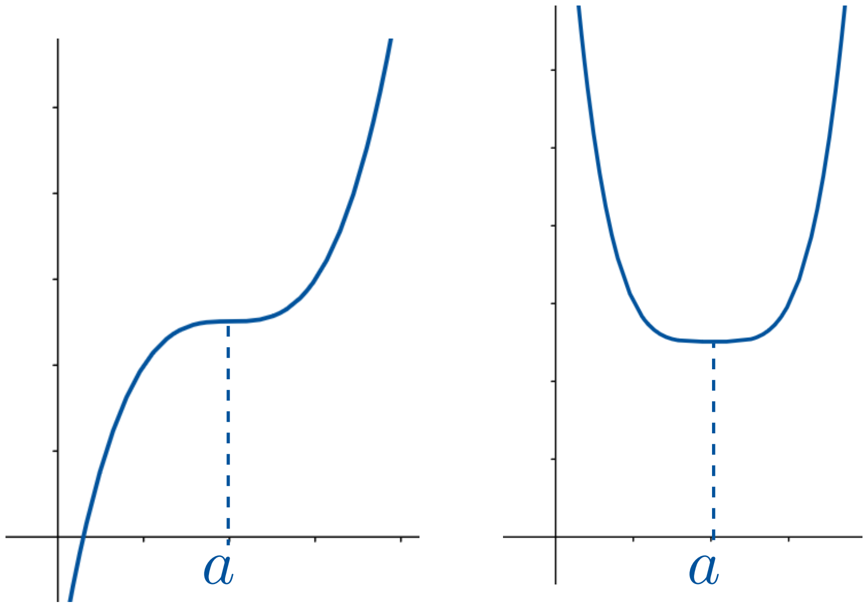
The reason is clear from considering the change in the slope of the graph, but because graphs of multivariate functions are hard to visualise, we give a different justification on the next page.




if $f''(a) > 0$ then $a$ is a local minimum




if $f''(a) < 0$ then $a$ is a local maximum




if $f''(a) = 0$ then we need to investigate further

The second-order Taylor polynomial of $f$ at $a$ is

is 0 if $a$ is a critical point

$$f(a+h) \approx f(a) + f'(a)h + \frac{f''(a)}{2!}h^2$$

is positive if $h \neq 0$, i.e. $x \neq a$

$$= f(a) + \frac{f''(a)}{2!}h^2 \quad \begin{cases} > f(a) & \text{if } f''(a) > 0 \text{ and } h \neq 0 \\ < f(a) & \text{if } f''(a) < 0 \text{ and } h \neq 0 \end{cases}$$

Here is a simplified example of how to use second order Taylor polynomials to classify critical points of multivariate functions.

**Example**: Find and classify the critical points of $f(x, y) = y^2 - x^3 + x$.

Now we develop a multivariate second derivative test by copying the previous example's argument in general.
The second-order Taylor polynomial of $f$ about $(a,b)$ is

$$f(a+h,b+k) \approx f(a,b) + f_x(a,b)h + f_y(a,b)k + \frac{\boxed{f_{xx}(a,b)h^2 + 2f_{xy}(a,b)hk + f_{yy}(a,b)k^2}}{2!},$$

is 0 if $(a,b)$ is a critical point

we need the "sign" of the numerator

**Definition**: A function $Q : \mathbb{R}^n \to \mathbb{R}$ is a *quadratic form* if it is homogeneous of degree two i.e. a linear combination of $x_i x_j$. A quadratic form $Q$ is:

- *positive definite* if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$; $\Longrightarrow$ maximum
- *negative definite* if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$; $\Longrightarrow$ minimum
- *indefinite* if $Q(\mathbf{x}) > 0$ for some $\mathbf{x} \neq \mathbf{0}$, and $Q(\mathbf{x}) < 0$ for some other $\mathbf{x} \neq \mathbf{0}$. $\Longrightarrow$ not maximum nor minimum

A quadratic form can be at most one of the three types. But it is possible to be none of the three types, e.g. $Q(h,k) = h^2$. (see later)

**Definition**: A quadratic form $Q$ is:

- *positive definite* if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$; $\implies$ maximum
- *negative definite* if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$; $\implies$ minimum
- *indefinite* if $Q(\mathbf{x}) > 0$ for some $\mathbf{x} \neq \mathbf{0}$, and $Q(\mathbf{x}) < 0$ for some other $\mathbf{x} \neq \mathbf{0}$. $\implies$ not maximum nor minimum

Let's start with the 2-variable case: any 2-variable quadratic form has the form $Ah^2 + 2Bhk + Ck^2$. (We are interested in $f_{xx}(a,b)h^2 + 2f_{xy}(a,b)hk + f_{yy}(a,b)k^2$.)

In the previous example where $B = 0$, we can quickly tell the definiteness from the signs of $A$ and $C$. In the general case, we will have to complete the square:

$$Ah^2 + 2Bhk + Ck^2 = A\left(h + \frac{B}{A}k\right)^2 + \frac{AC - B^2}{A}k^2$$

$$\det\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

So $Q(x)$ is positive definite

if $A$ and $\frac{AC-B^2}{A}$ are both positive, i.e. $A > 0$ and $\boxed{AC - B^2} > 0$;

$Q(x)$ is negative definite

if $A$ and $\frac{AC-B^2}{A}$ are both negative, i.e. $A < 0$ and $AC - B^2 > 0$;

$Q(x)$ is indefinite if $A$ and $\frac{AC-B^2}{A}$ have different signs, i.e. $A \neq 0$ and $AC - B^2 < 0$.