Homework 5, due 20:00 Thursday 2 May 2019 to Dr. Pang's mailbox

You must justify your answers to receive full credit.

1. Consider $\mathbb{C}^2$ with the standard dot product. Let

$$\alpha = \begin{pmatrix} 2+3i \\ -1-i \end{pmatrix}, \beta = \begin{pmatrix} -2i \\ 2 \end{pmatrix}, \gamma = \begin{pmatrix} -1+4i \\ 2-i \end{pmatrix}.$$

Calculate the following quantities:

a) $\alpha \cdot \beta$

b) $\beta \cdot \alpha$

c) the length of α

d) the distance between β and γ.

2. Prove the following "parallelogram law":

$$\|\alpha+\beta\|^2 + \|\alpha-\beta\|^2 = 2\|\alpha\|^2 + 2\|\beta\|^2.$$

3. Consider $P_{<3}(\mathbb{R})$, the vector space of polynomials over $\mathbb{R}$ of degree less than 3, with inner product

$$\langle f, g\rangle = \int_{-1}^{1} (1+3x) f(x) g(x)\, dx.$$

Construct an orthogonal basis for $P_{<3}(\mathbb{R})$. You may use the following:

$$\int_{-1}^{1} x^n\, dx = \begin{cases} 0 & \text{if } n \text{ odd} \\ \dfrac{2}{n+1} & \text{if } n \text{ even.} \end{cases}$$

(Click here for a hint)

4. Consider $M_{2,2}(\mathbb{R})$ with the inner product

$$\langle A, B\rangle = \mathrm{Tr}(A^T B).$$

Consider the subspace W with orthogonal basis $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\}$.

a) Calculate the orthogonal projection of $\begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}$ onto W.

b) Find the closest point in W to $\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$.

5. Recall that, for a square matrix X, its trace $\mathrm{Tr}(X)$ is the sum of the diagonal entries of X.

   a) Show that $\mathrm{Tr}(BC) = \mathrm{Tr}(CB)$ for all $B, C \in M_{2,2}(\mathbb{F})$.

   b) It is true that $\mathrm{Tr}(BC) = \mathrm{Tr}(CB)$ for all $B, C \in M_{n,n}(\mathbb{F})$ (you do not need to prove this). Show that, if $A, J \in M_{n,n}(\mathbb{F})$ are similar matrices, then $\mathrm{Tr}\, A = \mathrm{Tr}\, J$.

   c) Now suppose $F = \mathbb{C}$. Using part b or otherwise, explain why $\mathrm{Tr}\, A$ is the sum (with multiplicity) of the eigenvalues of A.

6. Prove the (simplified) Riesz Representation Theorem: let $\Phi : V \to \hat{V}$ be given by $\Phi(\gamma) = \langle \gamma, - \rangle$. Show that:

   a) Φ is conjugate-linear, i.e. $\Phi(a\gamma + \gamma') = \bar{a}\Phi(\gamma) + \Phi(\gamma')$.

   b) Φ is injective, i.e. $\Phi(\gamma) = \Phi(\gamma')$ means $\gamma = \gamma'$. (Please give full details.)

7. Let $V = P_{<2}(\mathbb{R})$, the vector space of polynomials over $\mathbb{R}$ of degree less than 2, with inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx.$$

   Define $\phi \in \hat{V}$ by $\phi(g) = g(-1)$.

   a) By direct calculation, find $f \in V$ such that $\langle f, g \rangle = \phi(g)$.

   b) to be released later

8. to be released later

Optional questions If you attempted seriously all the above questions, then your scores for the following questions may replace any lower scores for two of the above questions.

9. (Sturm-Liouville theory for PDEs) Let V be the subspace of continuous functions on $[0,1]$ defined by $V = \{f \in C^0([0,1]) \mid f(0) = f(1) = 0\}$. Let $\sigma : V \to V$ be the double-derivative operator, i.e. $\sigma(f) = f''$.

   a) Let V have inner product given by
   $$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$
   By integration by parts twice, or otherwise, show that σ is self-adjoint, i.e. $\langle f, \sigma(g)\rangle = \langle \sigma(f), g\rangle$.

   b) Now let V have inner product given by
   $$\langle f, g\rangle = \int_0^1 e^x f(x)g(x)\,dx.$$
   By integration by parts twice, or otherwise, find the function $\tau : V \to V$ such that $\langle f, \sigma(g)\rangle = \langle \tau(f), g\rangle$.

10. Let V be an inner product space, and let $f : V \to V$ be a self-adjoint function, i.e. for all $\alpha, \beta \in V$,
   $$\langle f(\alpha), \beta\rangle = \langle \alpha, f(\beta)\rangle.$$
   Show that f is linear.

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