Last time: Riesz representation theorem:

if $\dim V < \infty$, then V and $\hat{V}$ are conjugate isomorphic via:

$$
\begin{array}{cccc}
 & V & & \hat{V} \\
\Phi: & \gamma & \longmapsto & \langle \gamma, - \rangle \\
 & ? & \longleftarrow & \phi \ : \Lambda \\
 & \alpha_i & \longleftarrow & \phi_i
\end{array}
$$

(conjugate linear)

($\{\phi_1, \dots, \phi_n\}$ is dual basis to an orthonormal basis $\{\alpha_1, \dots, \alpha_n\}$ of V)

A reason for this: to translate questions about $\hat{V}$ into questions about V.

Eg. Given $\sigma: U \longrightarrow V$, its dual $\hat{\sigma}: \hat{V} \longrightarrow \hat{U}$ is $\hat{\sigma}(\phi) = \phi \circ \sigma$

$$
\begin{tikzcd}
\hat{V} \arrow[r, "\hat{\sigma}"] \arrow[d, shift left, "\Lambda_V"] & \hat{U} \arrow[d, shift left, "\Lambda_U"] \\
V \arrow[r, "\sigma^*"] \arrow[u, shift left, "\Phi_V"] & U \arrow[u, shift left, "\Phi_U"]
\end{tikzcd}
$$

the adjoint of σ

$$\sigma^* = \Lambda_U \circ \hat{\sigma} \circ \Phi_V$$

i.e. $\Lambda_U^{-1} \circ \sigma^* = \hat{\sigma} \circ \Phi_V$

i.e. $\Phi_U \circ \sigma^* = \hat{\sigma} \circ \Phi_V$

i.e. $\Phi_U(\sigma^*(\alpha)) = \hat{\sigma}(\Phi_V(\alpha)) \quad \forall \alpha \in V$

i.e. $\langle \sigma^*(\alpha), - \rangle_U = \hat{\sigma}[\langle \alpha, - \rangle_V]$ [definition of $\hat{\sigma}$]

$= \langle \alpha, \sigma(-) \rangle_V$

i.e. $\forall \alpha \in V, \beta \in U$

$$\langle \sigma^*(\alpha), \beta \rangle_U = \langle \alpha, \sigma(\beta) \rangle_V \quad (*)$$

Alternatively, we can define the adjoint by (*) (Th. 10.2.4)

(see HW5 Q9)

Given $\sigma \in L(U,V)$:

For each $\alpha \in V$, define $\phi^{(\alpha)} : U \to \mathbb{F}$ by $\phi^{\alpha}(\beta) = \langle \alpha, \sigma(\beta) \rangle_V$

i.e. $\phi^{(\alpha)} = \phi_\alpha \circ \sigma$, a composition of linear transformations

$\therefore$ linear, i.e. $\phi^{(\alpha)} \in \hat{U}$.

$\therefore$ by Riesz representation, $\phi^{(\alpha)} = \langle \gamma, - \rangle_U = \phi_\gamma$ for some $\gamma \in U$.

There is such a γ for each α, so let $\gamma = \sigma^*(\alpha)$.

Then check σ^* is linear.

Warning 1: σ^* depends on the choice of $\langle \, , \rangle$. (see HW5 Q9)

(Warning 2: in some function spaces, adjoints are defined differently, because of domain/codomain problems — see HW5 Q9 fixed version)

Given $\sigma \in L(U,V)$ $(\dim U, \dim V < \infty)$

? What is ${}_{\mathcal{B}\leftarrow\mathcal{A}}[\sigma^*]$,

where $\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\}$ is an orthonormal basis of V
$\mathcal{B} = \{\beta_1, \cdots, \beta_m\}$ " " " " of U?

$${}_{\mathcal{B}\leftarrow\mathcal{A}}[\sigma^*] = \begin{pmatrix} | & & | \\ [\sigma^*(\alpha_1)]_{\mathcal{B}} & \cdots & [\sigma^*(\alpha_n)]_{\mathcal{B}} \\ | & & | \end{pmatrix}$$

$\downarrow$

if $[\sigma^*(\alpha_1)]_{\mathcal{B}} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

this means $\sigma^*(\alpha_1) = b_1\beta_1 + \cdots + b_m\beta_m$.

$\langle \beta_i, \sigma^*(\alpha_1)\rangle_U$
$= \langle \beta_i, b_1\beta_1 + \cdots + b_m\beta_m\rangle_U$
$= b_1\langle\beta_i,\beta_1\rangle + \cdots + b_m\langle\beta_i,\beta_m\rangle$
$= b_i\langle\beta_i,\beta_i\rangle = b_i$

by similar argument:
row i, column j of ${}_{\mathcal{B}\leftarrow\mathcal{A}}[\sigma^*]$ is $\langle\beta_i, \sigma^*(\alpha_j)\rangle_U$

Same argument using σ instead of σ^*:
row i, column j of ${}_{\mathcal{A}\leftarrow\mathcal{B}}[\sigma]$ is $\langle\alpha_i, \sigma(\beta_j)\rangle_V$

Now $\langle\beta_i, \sigma^*(\alpha_j)\rangle_U = \overline{\langle\sigma^*(\alpha_j), \beta_i\rangle_U} = \overline{\langle\alpha_j, \sigma(\beta_i)\rangle_V}$

$\therefore$ row i, column j of ${}_{\mathcal{B}\leftarrow\mathcal{A}}[\sigma^*]$ = row j, column i of $\overline{{}_{\mathcal{A}\leftarrow\mathcal{B}}[\sigma]}$

Th:
$${}_{\mathcal{B}\leftarrow\mathcal{A}}[\sigma^*] = \overline{{}_{\mathcal{A}\leftarrow\mathcal{B}}[\sigma]}^{T}$$

if $\mathcal{A}, \mathcal{B}$ are orthonormal bases

From 2207: $(\text{Row } A)^{\perp} = \text{Nul } A$ (over $\mathbb{R}$)

so $(\text{Row } A^T)^{\perp} = \text{Nul}(A^T)$

$((\text{Col } A)^{\perp} = \text{Nul}(A^T)$

Th 10.2.13: For $\sigma \in L(U,V)$, $(\text{range } \sigma)^{\perp} = \ker(\sigma^*)$

Proof: $(\text{range } \sigma)^{\perp} = \{\beta \mid \langle \alpha, \beta \rangle = 0 \ \forall \alpha \in \text{range } \sigma\}$

$= \{\beta \mid \langle \sigma(\gamma), \beta \rangle = 0 \ \forall \gamma \in U\}$

$= \{\beta \mid \langle \gamma, \sigma^*(\beta) \rangle = 0 \ \forall \gamma \in U\}$

$= \{\beta \mid \sigma^*(\beta) = \vec{0}\} = \ker(\sigma^*)$.

Def 10.2.16 $\sigma: V \to V$ is <u>self-adjoint</u> if $\sigma = \sigma^*$.

Ex: Let $V = C^0([a,b])$ with $\langle f,g \rangle = \int_a^b f(x)g(x)\,dx$.

Fix $h \in V$.

Then $\sigma(f) = hf$ i.e. $\sigma(f)(x) = h(x)f(x)$,

i.e. σ = multiplication by h, is self-adjoint:

$$\langle f, \sigma(g) \rangle = \langle f, hg \rangle = \int_a^b f(x)h(x)g(x)\,dx$$

$$= \langle fh, g \rangle \quad \forall f,g.$$

$$\therefore \sigma^*(f) = fh = hf = \sigma(f).$$

From theorem above: if $\sigma: V \to V$ is self-adjoint, and $\mathcal{A}$ is an <u>orthonormal</u> basis of V, then

$$[\sigma]_{\mathcal{A}} = [\sigma^*]_{\mathcal{A}} = \overline{[\sigma]_{\mathcal{A}}}^T$$

$\therefore$ if V is over $\mathbb{R}$, then $[\sigma]_{\mathcal{A}}$ is symmetric.

if V is over $\mathbb{C}$, then $[\sigma]_{\mathcal{A}}$ is <u>Hermitian</u>.