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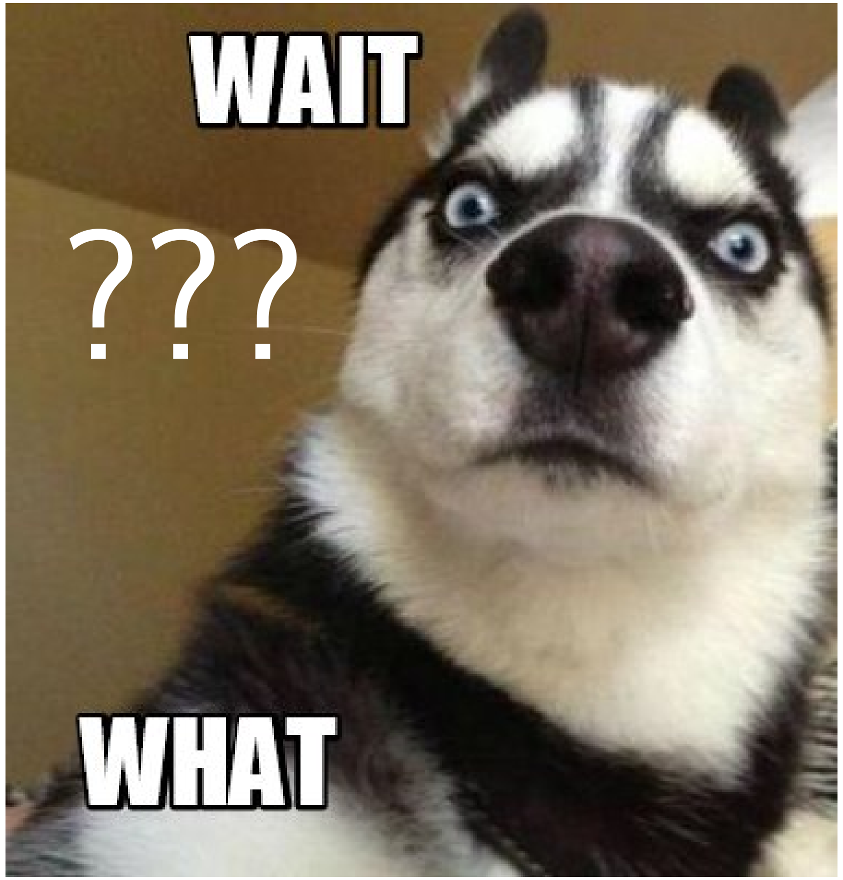

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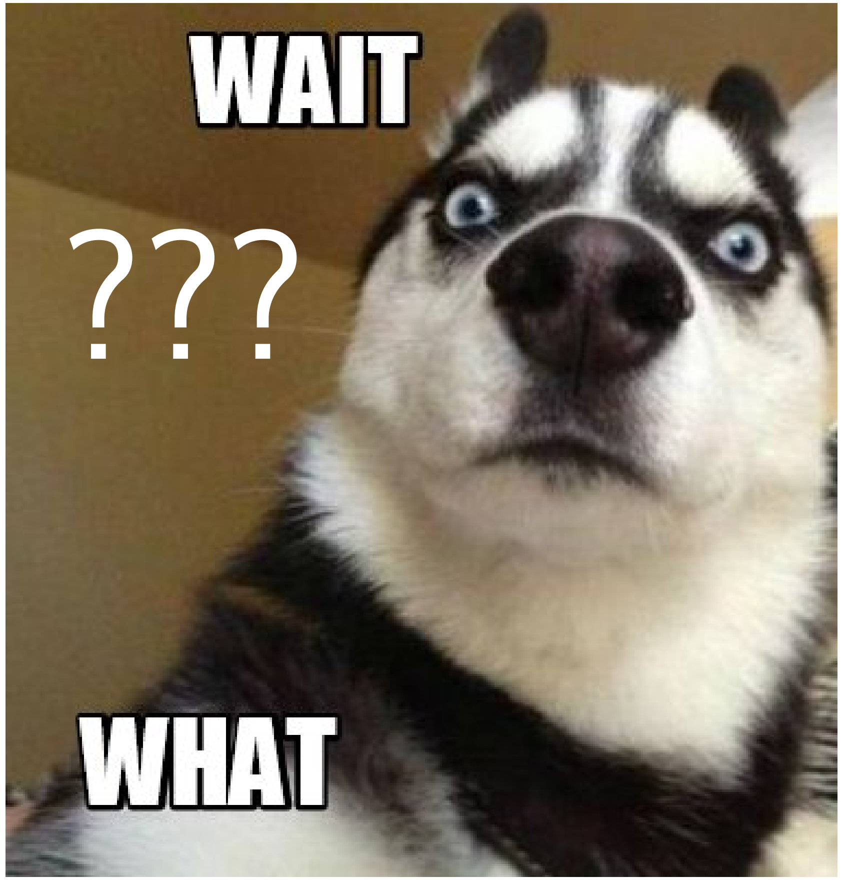

In mathematics, there are many situations where we need to "add things" (e.g. numbers, functions, shapes), and linear algebra is about the properties that are common to all these different "additions". This means we only need to study these properties once, not separately for each type of "addition" (better explanation in Week 7).

Because so many problems require "adding things", linear algebra is one of the best tools in mathematics.

(picture from mememaker.net)

The concepts in linear algebra are important for many branches of mathematics:

Major Requirements for Graduation:

Core Courses (3 units each):

MATH1005 Calculus I

MATH2225 Calculus II

MATH2205 Multivariate Calculus

MATH2206 Probability & Statistics

MATH2207 Linear Algebra

MATH2215 Mathematical Analysis

MATH2216 Statistical Methods and Theory

MATH3205 Linear Programming and Integer Programming

MATH3206 Numerical Methods I

MATH3405 Ordinary Differential Equations

MATH3805 Regression Analysis

MATH3806 Multivariate Statistical Methods

MATH4998 Mathematical Science Project I

All these classes list Linear Algebra as a prerequisite
(Info from math department website)

This class is about more than calculations. From the official syllabus:

Course Intended Learning Outcomes (CILOs):

Upon successful completion of this course, students should be able to:

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1	Explain the concept/theory in linear algebra, to develop dynamic and graphical views to the related issues of the chosen topics as outlined in "course content," and to formally prove theorems

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Linear algebra is used in future courses in entirely different ways. So it's not enough to know routine calculations; you need to understand the concepts and ideas, to solve problems you haven't seen before on the exam. This will require words and not just formulae.

For many people, this is different from their previous math classes, and will require a lot of study time.
(Week 1 is straightforward computation; the abstract theory starts in Week 2.)

§1.1: Systems of Linear Equations

Linear Algebra starts with linear equations.

Example: $y = 5x + 2$ is a linear equation. We can take all the variables to the left hand side and rewrite this as $(-5)x + (1)y = 2$.

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Example: $3(x_1 + 2x_2) + 1 = x_1 + 1 \longrightarrow (2)x_1 + (6)x_2 = 0$

Example: $x_2 = \sqrt{2}(\sqrt{6} - x_1) + x_3 \longrightarrow \sqrt{2}x_1 + (1)x_2 + (-1)x_3 = 2\sqrt{3}$

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The following two equations are not linear, why?

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The following two equations are not linear, why?

$x_2 = 2\sqrt{x_1}$ $\qquad$ $xy + x = e^5$

The problem is that the variables are not only multiplied by numbers.

In general, a linear equation is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

$x_1, x_2, \ldots x_n$ are the variables. $\qquad$ $a_1, a_2, \ldots a_n$ are the coefficients.

A linear equation has the form $a_1x_1 + a_2x_2 + \ldots a_nx_n = b$.

Definition: A *system of linear equations* (or a *linear system*) is a collection of linear equations involving the same set of variables.

Example:

$$\begin{array}{rrrcr} x & +y & & = & 3 \\ 3x & & +2z & = & -2 \end{array}$$

is a system of 2 equations in 3 variables, x, y, z. Notice that not every variable appears in every equation.

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Definition: A *solution* of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$ respectively.

Definition: The *solution set* of a linear system is the set of all possible solutions.

Example: One solution to the above system is $(x, y, z) = (2, 1, -4)$, because $2 + 1 = 3$ and $3(2) + 2(-4) = -2$.

Question: Is there another solution? How many solutions are there?

Definition: A linear system is *consistent* if it has a solution,
and *inconsistent* if it does not have a solution.

Fact: (which we will prove in the next class) A linear system has either

- exactly one solution — consistent
- infinitely many solutions — consistent
- no solutions — inconsistent

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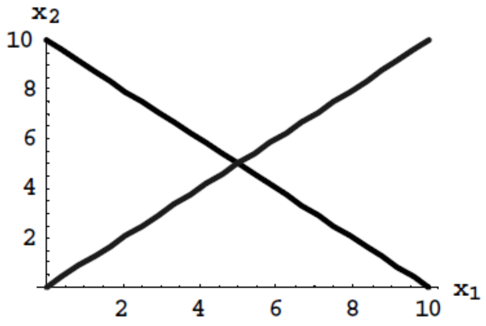
- no solutions — inconsistent

EXAMPLE Two equations in two variables:

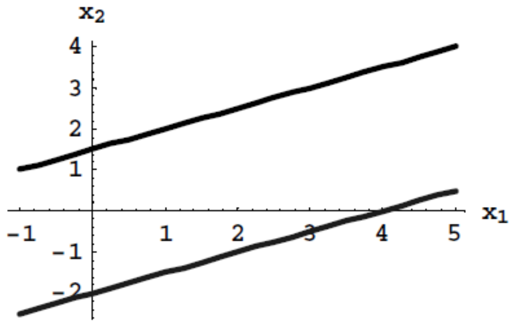
$$\begin{aligned} x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0 \end{aligned}$$



one unique solution

consistent

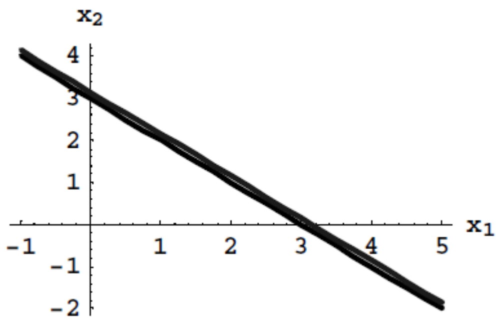
$$\begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8 \end{aligned}$$



no solution

inconsistent

$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$



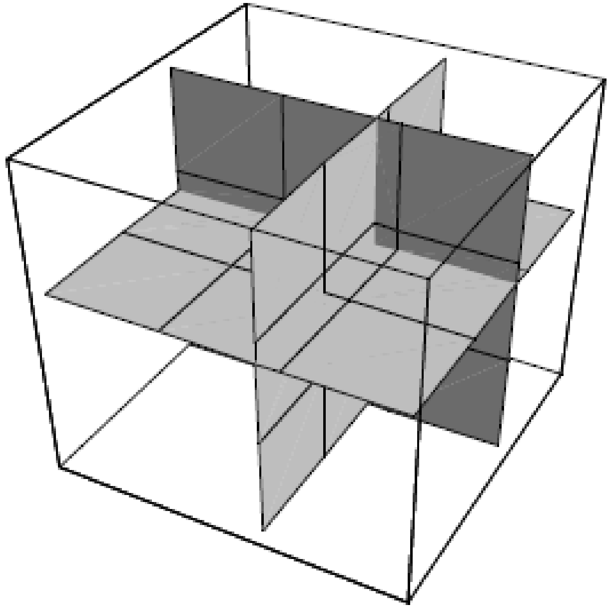
infinitely many solutions

consistent

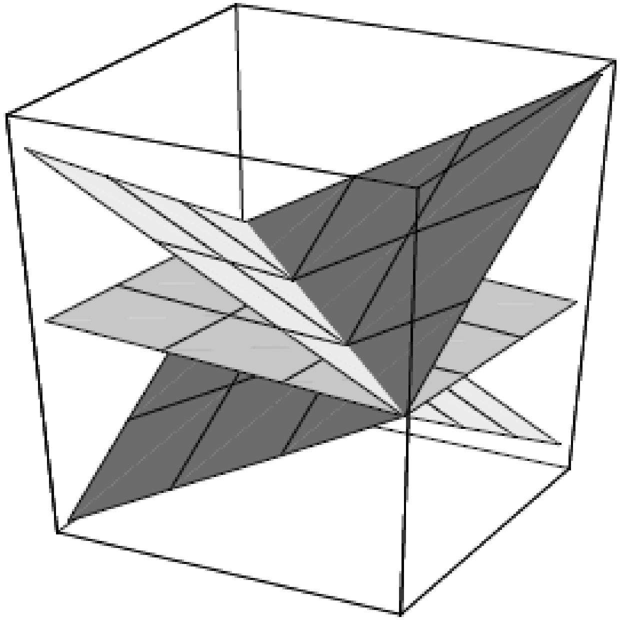
i.e. $ax + by + cz = d$

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

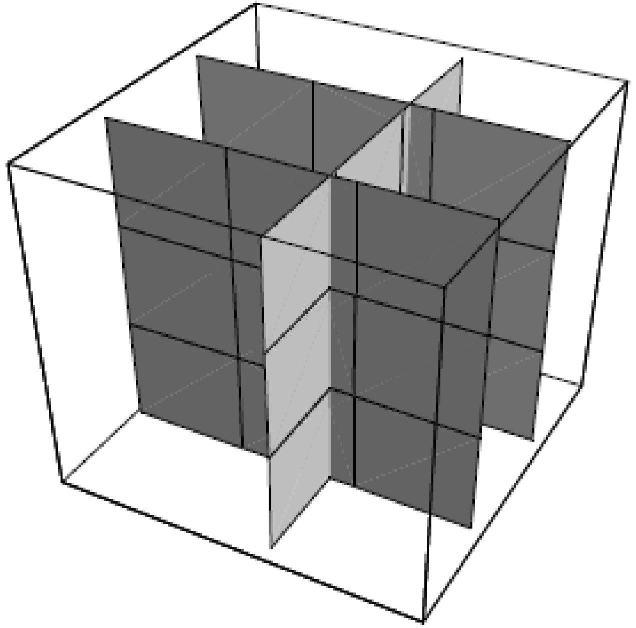
i) The planes intersect in one point. *(one solution)*



ii) The planes intersect in one line. *(infinitely many solutions)*



iii) There is no point in common to all three planes. *(no solution)*



Which of these cases are consistent?

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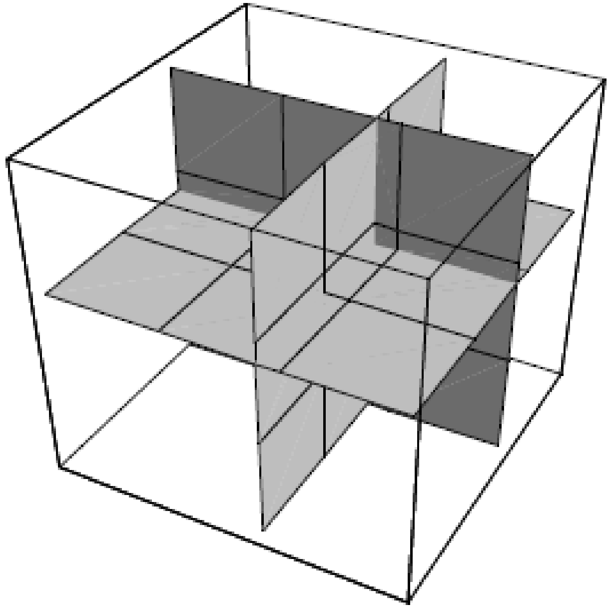

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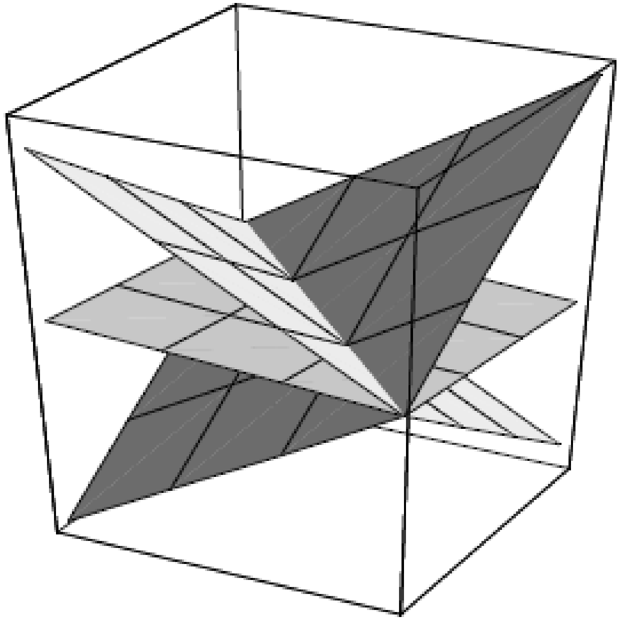

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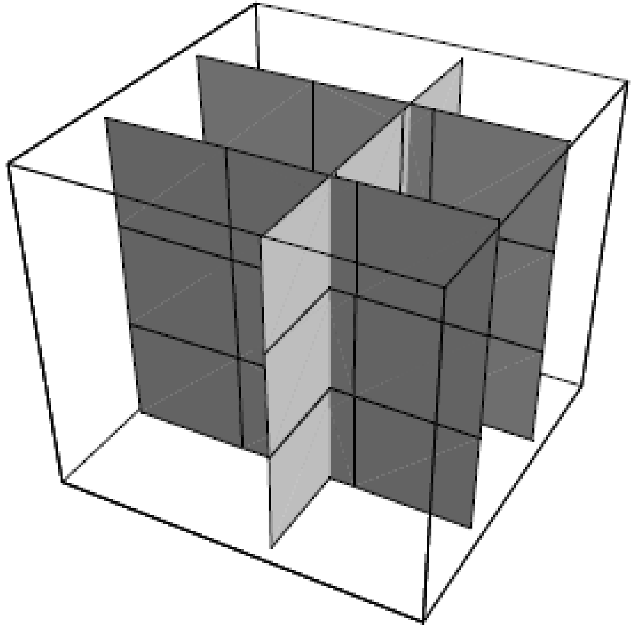

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Our goal for this week is to develop an efficient algorithm to solve a linear system.

Example:

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

$$x_2 = 2$$

add the two equations to eliminate x_1

$$x_1 = 3$$

substitute for x_2 in the first equation to find x_1

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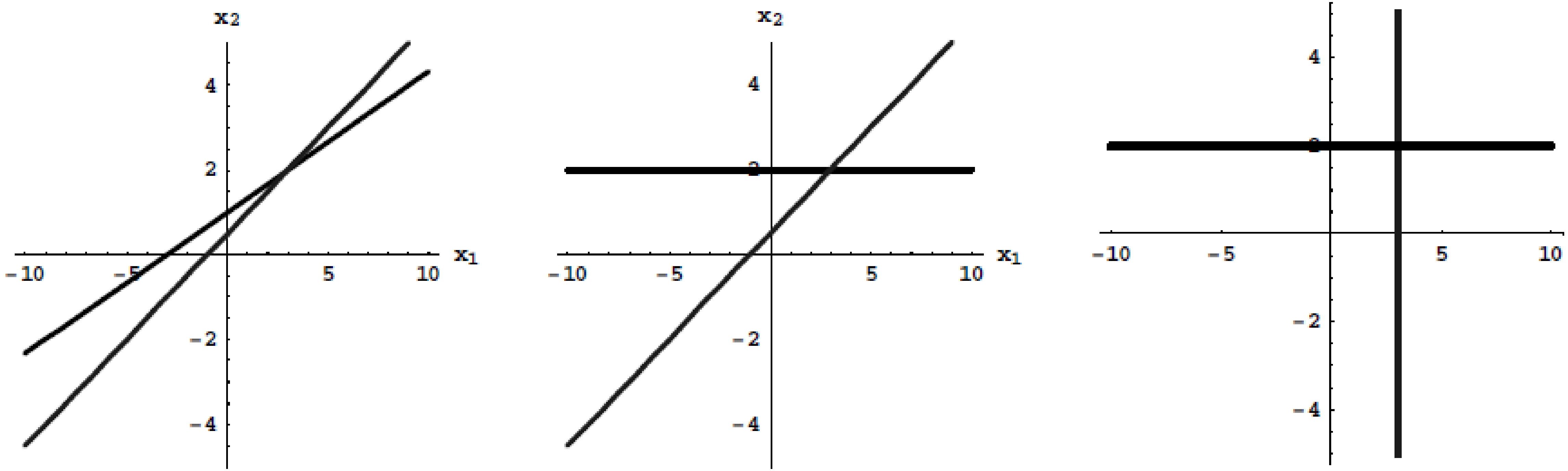

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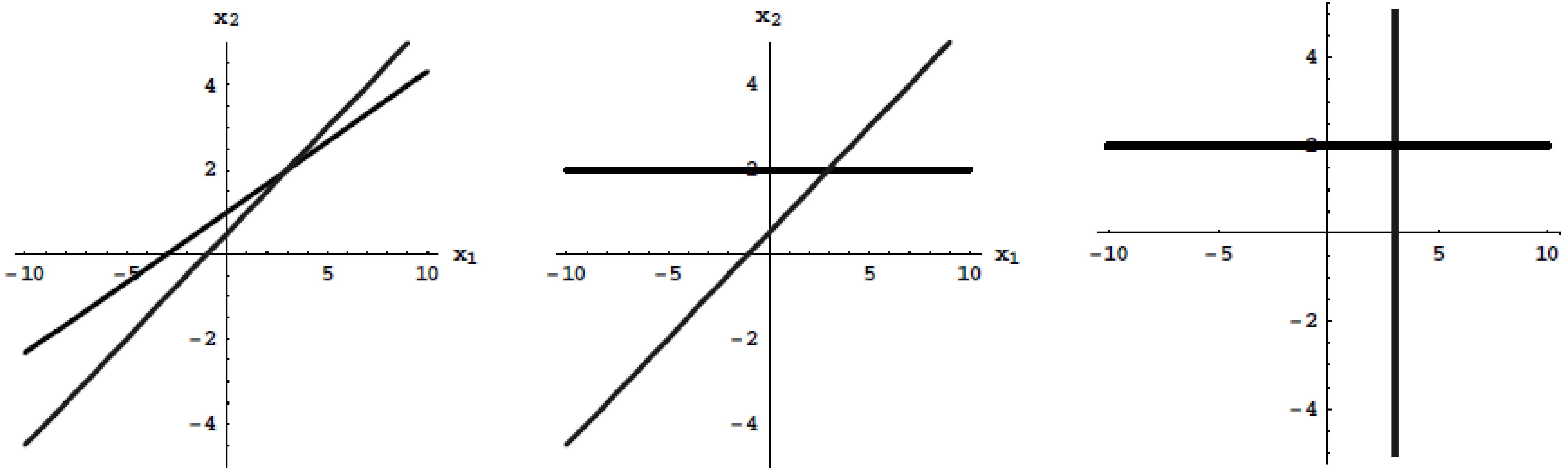

Definition: Two linear systems are *equivalent* if they have the same solution set.

So the three linear systems above are different but equivalent.

A general strategy for solving a linear system: replace one system with an equivalent system that is easier to solve.

We simplify the writing by using matrix notation, recording only the coefficients and not the variables.

$$
\begin{array}{ll} R_1 & x_1 - 2x_2 = -1 \\ R_2 & -x_1 + 3x_2 = 3 \end{array}
\quad \xrightarrow{R_2 + R_1} \quad
\begin{array}{l} x_1 - 2x_2 = -1 \\ x_2 = 2 \end{array}
\quad \xrightarrow{R_1 + 2R_2} \quad
\begin{array}{l} x_1 = 3 \\ x_2 = 2 \end{array}
$$

$$
\left[\begin{array}{rr|r} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]
\longrightarrow
\left[\begin{array}{rr|r} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right]
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coefficient of x_1 coefficient of x_2 right hand side

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coefficient of x_1 coefficient of x_2 right hand side

The augmented matrix of a linear system contains the right hand side: $\left[\begin{array}{rr|r} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$

The coefficient matrix of a linear system is the left hand side only: $\left[\begin{array}{rr} 1 & -2 \\ -1 & 3 \end{array}\right]$

(The textbook does not put a vertical line between the coefficient matrix and the right hand side, but I strongly recommend that you do to avoid confusion.)

$$\begin{array}{lrcrcr} R_1 & x_1 & - & 2x_2 & = & -1 \\ R_2 & -x_1 & + & 3x_2 & = & 3 \end{array} \xrightarrow{R_2 + R_1 \to} \begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array} \xrightarrow{R_1 + 2R_2 \to} \begin{array}{rcr} x_1 & = & 3 \\ x_2 & = & 2 \end{array}$$

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In this example, we solved the linear system by applying elementary row operations to the augmented matrix (we only used 1. above, the others will be useful later):

1. Replacement: add a multiple of one row to another row. $R_i \to R_i + cR_j$
2. Interchange: interchange two rows. $R_i \to R_j,\ R_j \to R_i$
3. Scaling: multiply all entries in a row by a nonzero constant. $R_i \to cR_i,\ c \neq 0$

$$\begin{array}{lrcrcr} R_1 & x_1 & - & 2x_2 & = & -1 \\ R_2 & -x_1 & + & 3x_2 & = & 3 \end{array} \quad \longrightarrow \quad \begin{array}{lrcrcr} & x_1 & - & 2x_2 & = & -1 \\ R_2 + R_1 \rightarrow & & & x_2 & = & 2 \end{array} \quad \begin{array}{lrcr} R_1 + 2R_2 \rightarrow & x_1 & = & 3 \\ & x_2 & = & 2 \end{array}$$

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Definition: Two matrices are *row equivalent* if one can be transformed into the other by a sequence of elementary row operations.

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Definition: Two matrices are *row equivalent* if one can be transformed into the other by a sequence of elementary row operations.

Fact: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set, i.e. they are equivalent linear systems.

Warning: Do not do multiple elementary row operations at the same time, except adding multiples of the same row to several rows.

These are NOT equivalent systems: in the system on the right, x_1 can take any value, which is not true for the system on the left.

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \qquad \begin{aligned} x_2 &= 2 \\ x_2 &= 2 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \begin{array}{l} \longleftarrow R_1 + R_2 \\ \longleftarrow R_2 + R_1 \end{array}$$

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} \longleftarrow R_1 - R_3 \\ \longleftarrow R_2 + 4R_3 \\ \end{array}$$

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Sometimes we are not interested in the exact value of the solutions, just the number of solutions. In other words:

1. Existence of solutions: is the system consistent?
2. Uniqueness of solutions: if a solution exists, is it the only one?

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1. Existence of solutions: is the system consistent?
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Answering this requires less work than finding the solution.

Example:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

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We can stop here: back-substitution shows that we can find a unique solution.

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$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \boxed{\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]}$$

echelon form

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -3 \\ & & x_2 & = & 16 \\ & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{rcr} x_1 & = & 29 \\ x_2 & = & 16 \\ x_3 & = & 3 \end{array} \qquad \boxed{\left[\begin{array}{rrr|r} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]}$$

reduced echelon form

Here is the example from p10. Notice that we use row operations to first put the matrix into echelon form, and then into reduced echelon form.

Can we always do this for any linear system?

Theorem: Any matrix A is row-equivalent to exactly one reduced echelon matrix, which is called its reduced echelon form and written $\text{rref}(A)$.

So our general strategy for solving a linear system is: apply row operations to its augmented matrix to obtain its rref.

And our general strategy for determining existence/uniqueness of solutions is: apply row operations to its augmented matrix to obtain an echelon form, i.e. a row-equivalent echelon matrix.

These processes of row operations (to get to echelon or reduced echelon form) are called row reduction.

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And our general strategy for determining existence/uniqueness of solutions is: apply row operations to its augmented matrix to obtain an echelon form, i.e. a row-equivalent echelon matrix.

Warning: an echelon form is not unique. Its entries depend on the row operations we used. But its pattern of $\blacksquare$ and $*$ is unique.

These processes of row operations (to get to echelon or reduced echelon form) are called row reduction.

Row reduction:

augmented matrix of linear system $\longrightarrow$ echelon form $\longrightarrow$ reduced echelon form

echelon form $\downarrow$ existence and uniqueness of solutions

reduced echelon form $\downarrow$ solution set

Row reduction:

$$\begin{tikzcd} \text{augmented matrix of linear system} \arrow[r] & \text{echelon form} \arrow[r] \arrow[d] & \text{reduced echelon form} \arrow[d] \\ & \text{existence and uniqueness of solutions} & \text{solution set} \end{tikzcd}$$

The rest of this section:

- The row reduction algorithm (p21-25);
- Getting the solution, existence/uniqueness from the (reduced) echelon form (p26-29).

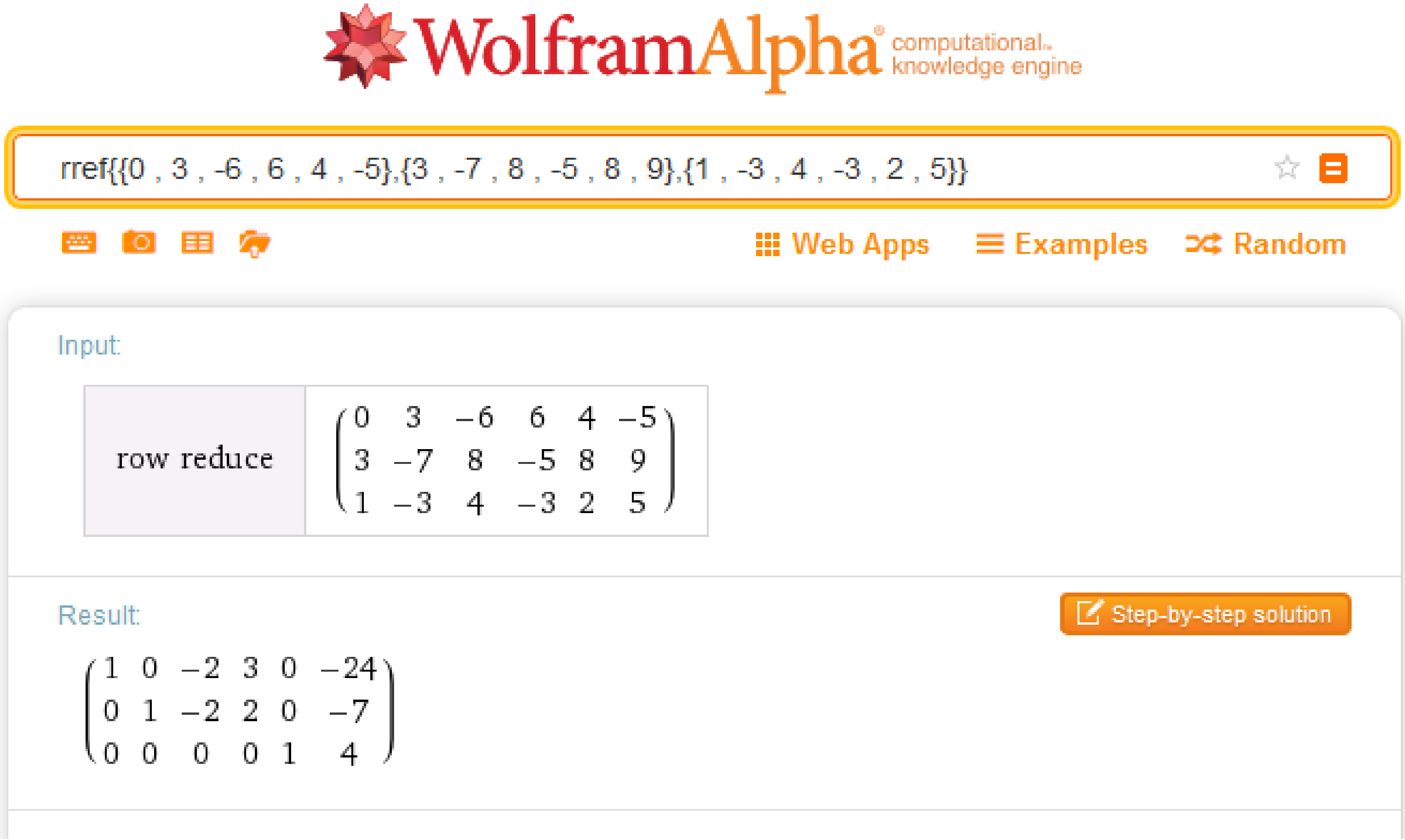
Important terms in the row reduction algorithm:

- pivot position: the position of a leading entry in a row-equivalent echelon matrix.
- pivot: a nonzero entry of the matrix that is used in a pivot position to create zeroes below it.
- pivot column: a column containing a pivot position.

The black squares are the pivot positions.

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$$

Check your answer: www.wolframalpha.com



Getting the solution set from the reduced echelon form:

A basic variable is a variable corresponding to a pivot column.
All other variables are free variables.

6. Write each row of the augmented matrix as a linear equation.

Example:

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$$\begin{aligned} x_1 \quad -2x_3 + 3x_4 \quad &= -24 \\ x_2 - 2x_3 + 2x_4 \quad &= -7 \\ x_5 &= 4 \end{aligned}$$

basic variables: x_1, x_2, x_5, free variables: x_3, x_4.

The free variables can take any value. These values then uniquely determine the basic variables.

7. Take the free variables in the equations to the right hand side, and add equations of the form "free variable = itself", so we have equations for each variable in terms of the free variables.

Example:
$$\begin{aligned} x_1 &= -24 + 2x_3 - 3x_4 \\ x_2 &= -7 + 2x_3 - 2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \\ x_5 &= 4 \end{aligned}$$

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Example:

$$\begin{aligned} x_1 &= -24 + 2x_3 - 3x_4 \\ x_2 &= -7 + 2x_3 - 2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \\ x_5 &= 4 \end{aligned}$$

So the solution set is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -24 + 2s - 3t \\ -7 + 2s - 2t \\ s \\ t \\ 4 \end{pmatrix}$$

where s and t can take any value.

What this means: for every choice of s and t, we get a different solution:

e.g. $s = 0, t = 1$: $(x_1, x_2, x_3, x_4, x_5) = (-27, -9, 0, 1, 4)$

$s = 1, t = -1$: $(x_1, x_2, x_3, x_4, x_5) = (-19, -3, 1, -1, 4)$

and infinitely many others. (Exercise: check these two are solutions.)

We will see a better way to write the solution set next week (Week 2 p29-31, §1.5).

Answering existence and uniqueness of solutions from the echelon form

Example: On p14 we found $\left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \\ 0 & 3 & -6 & 8 \end{array}\right] \xrightarrow{\text{row-reduction}} \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 5 \\ 0 & 0 & 0 & 3 \end{array}\right]$

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The last equation says $0x_1 + 0x_2 + 0x_3 = 3$, so this system is inconsistent.
Generalising this observation gives us "half" of the following theorem:

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The last equation says $0x_1 + 0x_2 + 0x_3 = 3$, so this system is inconsistent. Generalising this observation gives us "half" of the following theorem:

Theorem 2: Existence and Uniqueness:

A linear system is consistent if and only if an echelon form of its augmented matrix has no row of the form $[0 \dots 0|\blacksquare]$ with $\blacksquare \neq 0$.

Answering existence and uniqueness of solutions from the echelon form

Example: On p14 we found
$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \\ 0 & 3 & -6 & 8 \end{array}\right] \xrightarrow{\text{row-reduction}} \left[\begin{array}{ccc|c} \boxed{1} & -2 & 3 & -1 \\ 0 & \boxed{3} & -6 & 5 \\ 0 & 0 & 0 & \boxed{3} \end{array}\right]$$

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Theorem 2: Existence and Uniqueness:
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Be careful with the logic here: this theorem says "if and only if", which means it claims two different things:

- If a linear system is consistent, then an echelon form of its augmented matrix cannot contain $[0 \dots 0|\blacksquare]$ with $\blacksquare \neq 0$.
  This is the observation from the example above.

- If there is no row $[0 \dots 0|\blacksquare]$ with $\blacksquare \neq 0$ in an echelon form of the augmented matrix, then the system is consistent.

  This is because we can continue the row-reduction to the rref, and then the solution method of p26-27 will give us solutions.

- If there is no row $[0 \ldots 0|\blacksquare]$ with $\blacksquare \neq 0$ in an echelon form of the augmented matrix, then the system is consistent.
  This is because we can continue the row-reduction to the rref, and then the solution method of p26-27 will give us solutions.

As for the uniqueness of solutions:

Theorem 2: Existence and Uniqueness:
If a linear system is consistent, then:
- it has a unique solution if there are no free variables;
- it has infinitely many solutions if there are free variables.

In particular, this proves the fact we saw earlier, that a linear system has either a unique solution, infinitely many solutions, or no solutions.

Warning: In general, the existence of solutions is unrelated to the uniqueness of solutions. (We will meet an important exception in §2.3.)