Last time:

(6.3.6) span(S) is the intersection of all subspaces containing S.

(6.3.8) i.e. if any subspace $W \supseteq S$, then $W \supseteq \text{span}(S)$.

(6.3.9) Equivalently.

$$\text{span}(S) = \left\{ \underbrace{\sum_{i=1}^{n} a_i \alpha_i}_{\cup} \;\middle|\; \begin{array}{l} \alpha_i \in S \\ a_i \in F \\ \text{some } n \in \mathbb{N} \end{array} \right\}$$

Proof of 6.3.9:

show $\operatorname{span}(S) \subseteq U$: use 6.3.8

$U \supseteq S$ (in the sum, take $n=1$ and $a_1=1$)

and U is a subspace

$\because \vec{0} \in U$ (when $n=0$ – empty linear combination)

$$c\left(\sum_{i=1}^{n} a_i \alpha_i\right) + \sum_{i=1}^{m} b_i \beta_i$$

$$= \sum_{i=1}^{n} (c a_i) \alpha_i + \sum_{i=1}^{m} b_i \beta_i$$

a linear combination of vectors in S.

show $U \subseteq \operatorname{span}(S)$

= intersection of all subspaces containing S

$\therefore$ enough to show that $U \subseteq$ any subspace containing S, call this subspace W.

Take $\alpha \in U$

i.e. $\alpha = a_1\alpha_1 + \dots + a_n\alpha_n$, $\alpha_i \in S$.

$W \supseteq S \ni \alpha_i$ and W is a subspace

$\therefore$ closed under linear combinations

$\therefore \alpha \in W$.

§6.4 Bases

Def 6.4.1 $A \subseteq V$ is a basis of V if:

- A is linearly independent
- $\operatorname{span}(A) = V$

Ex: standard basis of $\mathbb{F}^n$

$$= \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$$

standard basis of $P_{\leq n}(\mathbb{F}) = \{1, x, x^2, \dots, x^{n-1}\}$

standard basis of $\mathbb{F}[x] = \{1, x, x^2, \dots\}$

standard basis of $M_{2,2}(\mathbb{F})$:

$$\left(\begin{array}{cccc} E^{1,1}, & E^{1,2}, & E^{2,1}, & E^{2,2} \\ \| & \| & \| & \| \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \right)$$

and similarly for $M_{m,n}(\mathbb{F})$.

A basis of $W = \left\{ \begin{pmatrix} a \\ 2a \\ 0 \\ b \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}$

$$\begin{array}{c} \| \\ a\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

More methods to find bases of subspaces later.

The point of bases is unique representation:

<u>Prop 6.4.5</u> Let $\mathcal{A}$ be a linearly independent set. If

$\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n$ and

$\alpha = b_1\beta_1 + \cdots + b_m\beta_m$ with

$\alpha_i, \beta_j \in \mathcal{A}$, α_i distinct, β_j distinct

$a_i, b_j \neq 0$, then, after reordering

$m = n$, $\alpha_i = \beta_i$, $a_i = b_i$.

(And, if $\mathcal{A}$ is a basis of V, i.e. also spans V, then every $\alpha \in V$ can be written as a linear combination of vectors in $\mathcal{A}$.)

see also 2207 week 8 p11)

<u>Proof</u>: reorder the α_i, β_j so that

$\alpha_1 = \beta_1, \alpha_2 = \beta_2, \cdots, \alpha_k = \beta_k$,

$\alpha_{k+1}, \cdots, \alpha_n, \beta_{k+1}, \cdots, \beta_m$ are all different. (k can be 0)

Then $a_1\alpha_1 + \cdots + a_n\alpha_n = b_1\beta_1 + \cdots + b_m\beta_m$.

$(a_1 - b_1)\alpha_1 + \cdots + (a_k - b_k)\alpha_k + a_{k+1}\alpha_{k+1} + \cdots + a_n\alpha_n - b_{k+1}\beta_{k+1} - \cdots - b_m\beta_m = \vec{0}$.

$\mathcal{A}$ is linearly independent $\therefore$ it has no linear dependence relations

$\therefore a_1 - b_1 = 0, \cdots, a_k - b_k = 0, a_{k+1} = \cdots = a_n = 0$
$b_{k+1} = \cdots = b_m = 0$.

But we assumed $a_i, b_j \neq 0$ so $k = n = m$ $\therefore a_i = b_i$
$\alpha_i = \beta_i \ \forall i$

<u>How to find a basis 1:</u> by taking a subset of a spanning set.

The theory: Th 6.3.11 :

If α is a linear combination of other vectors in S, then $\mathrm{span}(S) = \mathrm{span}(S \setminus \{\alpha\})$.

$\therefore$ given a spanning set $\{\alpha_1, \cdots, \alpha_n\}$, remove one-by-one any α_i that is a linear combination of other α_j's.

(see 2207 week 8 p10 "spanning set theorem")

In practice: (for subspaces of $\mathbb{F}^n$; for other spaces, use coordinates)

remove ALL unnecessary α_i at the same time, using <u>casting out algorithm</u>: row reduce $\begin{pmatrix} | & & | \\ \alpha_1 & \cdots & \alpha_n \\ | & & | \end{pmatrix}$

take the α_i whose columns have pivots:

Ex 3.6.4. $W = \mathrm{Span}\left\{ \underset{\alpha_1}{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}, \underset{\alpha_2}{\begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix}}, \underset{\alpha_3}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}}, \underset{\alpha_4}{\begin{pmatrix} -1 \\ 1 \\ 1 \\ 2 \end{pmatrix}}, \underset{\alpha_5}{\begin{pmatrix} -2 \\ 3 \\ 2 \\ 7 \end{pmatrix}} \right\}$

$$\begin{pmatrix} 1 & 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & 1 & 3 \\ -1 & 2 & 1 & 1 & 2 \\ 1 & -1 & 0 & 2 & 7 \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} \boxed{1} & 0 & 1 & -1 & -2 \\ 0 & \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{echelon form}}$$

$\therefore \{\alpha_1, \alpha_2, \alpha_4\}$ is a basis for W.

Proof of 6.3.11

$\mathrm{span}(S \setminus \{\alpha\}) \subseteq \mathrm{span}(S)$

(use 6.3.8) $\mathrm{span}(S)$ is a subspace.

$\mathrm{span}(S) \supseteq S \supseteq S \setminus \{\alpha\}$.

$\mathrm{span}\, S \subseteq \mathrm{span}(S \setminus \{\alpha\})$. (use 6.3.8)

$\mathrm{span}(S \setminus \{\alpha\})$ is a subspace and

$\mathrm{span}(S \setminus \{\alpha\}) \supseteq S \setminus \{\alpha\}$. and

$\mathrm{span}(S \setminus \{\alpha\}) \ni \alpha \quad \because \alpha = a_1\alpha_1 + \cdots + a_n\alpha_n$

with $\alpha_i \in S \setminus \{\alpha\}$.

$\subseteq \mathrm{span}(S \setminus \{\alpha\})$

and $\mathrm{span}(S \setminus \{\alpha\})$ is closed under linear combinations.

(Different proof using Th. 6.3.9

— substitute for α in the linear combination)