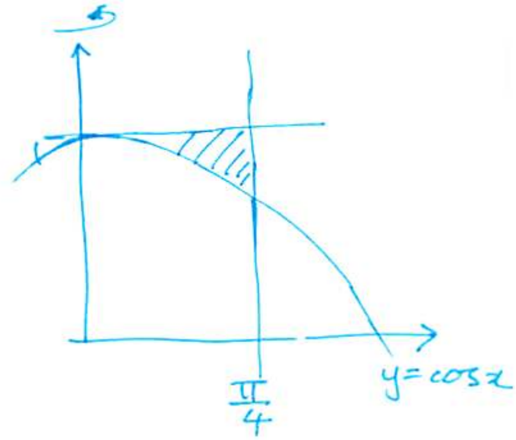
1. (7 points) Let R be the region bounded by the curves

$$y = \cos x, \quad y = 1, \quad x = \frac{\pi}{4}, \quad \text{with } x \le \frac{\pi}{4}$$

Find the volume of the solid obtained by rotating R about the y-axis. **Simplify your answer as much as possible.**

Use cylindrical shells



$$\text{volume} = \int_0^{\frac{\pi}{4}} 2\pi x(1-\cos x)\,dx$$

$$= \int_0^{\frac{\pi}{4}} 2\pi x - 2\pi x\cos x\,dx$$

$$= \left[2\pi \frac{x^2}{2}\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2\pi x\cos x\,dx$$

$$= \pi\left(\frac{\pi}{4}\right)^2 - 2\pi\int_0^{\frac{\pi}{4}} x\cos x\,dx$$

Integration by parts:

$$\int_0^{\frac{\pi}{4}} x\cos x\,dx = \Big[x\sin x\Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sin x\,dx$$

$u = x$ $\quad dv = \cos x\,dx$

$du = dx$ $\quad v = \sin x$

$$= \left(\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right) - 0\right) - \Big[-\cos x\Big]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4\sqrt{2}} - \left(-\frac{1}{\sqrt{2}} + 1\right) = \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 1$$

$$\text{So volume} = \frac{\pi^3}{16} - 2\pi\left(\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right)$$

$$= \frac{\pi^3}{16} - \frac{\pi^2}{2\sqrt{2}} - \sqrt{2}\pi + 2\pi$$

2. (7 points) Compute the following integral:

$$\int (9-x^2)^{-\frac{5}{2}}\,dx.$$

substitution

$x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

so $9-x^2 = 9-9\sin^2\theta$

$= 9\cos^2\theta$

$$= \int (3\cos\theta)^{-5}(3\cos\theta\, d\theta)$$

$$= \int 3^{-4}(\cos\theta)^{-4}\, d\theta$$

$$= \int 3^{-4}\sec^4\theta\, d\theta$$

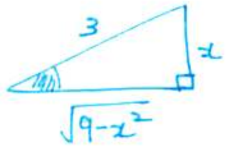
$$= \int 3^{-4}(1+\tan^2\theta)\sec^2\theta\, d\theta$$

substitution

$u = \tan\theta$

$$= 3^{-4}\left(\tan\theta + \frac{\tan^3\theta}{3}\right) + C$$

$$= \frac{1}{81}\ \frac{x}{\sqrt{9-x^2}} + \frac{1}{27}\left(\frac{x}{\sqrt{9-x^2}}\right)^3 + C$$



$\sin\theta = \frac{\text{opp.}}{\text{hyp}} = \frac{x}{3}$

$\tan\theta = \frac{\text{opp.}}{\text{adj}} = \frac{x}{\sqrt{9-x^2}}$

3. (7 points) Compute the following integral:

$$\int \frac{1-3x}{x(x^2+1)}dx.$$

Partial fractions:

$$\frac{1-3x}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1-3x = A(x^2+1) + (Bx+C)x$$

$x=0$: $1 = A$

coeff of x^2: $0 = A+B \Rightarrow B=-1$

coeff of x: $-3 = C$

So $$\int \frac{1-3x}{x(x^2+1)}dx = \int \frac{1}{x} - \frac{x}{x^2+1} - \frac{3}{x^2+1}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| - 3\arctan x + C$$