Orthogonality

Def 10.1.7: Let V be an inner product space.

α, β are <u>orthogonal</u> if $\langle \alpha, \beta \rangle = 0$

(This is the same condition as $\langle \beta, \alpha \rangle = 0$)

Given $W \leq V$ a subspace, $\alpha \in V$ is <u>orthogonal to W</u> if $\langle \alpha, \beta \rangle = 0 \ \forall \beta \in W$

Ex: $V = C^0([-\pi, \pi])$, with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx$

Then $\sin x, \cos x$ are orthogonal: $\langle \sin x, \cos x \rangle = \int_{-\pi}^{\pi} \sin x \cos x \, dx = \int_{-\pi}^{\pi} \frac{\sin 2x}{2} dx = \left. \frac{\cos 2x}{4} \right|_{-\pi}^{\pi}$

$= \frac{1}{4} - \frac{1}{4} = 0$

Def 10.1.7: $S \subseteq V$ is an orthogonal set if $\forall \alpha, \beta \in S$ with $\alpha \neq \beta$, we have $\langle \alpha, \beta \rangle = 0$

S is an orthogonal basis for V if it is an orthogonal set and also a basis for V.

Th. 10.1.8: An orthogonal set of non-zero vectors is linearly independent.

Proof: see 2207 week 12 p3 — that argument still works for infinite sets because each linear dependence relation only involves finitely-many vectors.

Th. 10.1.17: Let $\{\xi_1, \cdots, \xi_k\}$ be an $\boxed{\text{orthogonal}}$ basis for $W \subseteq V$.

Then, for $\alpha \in V$

order is important

$$\mathrm{Proj}_W(\alpha) = \frac{\langle \xi_1, \alpha \rangle}{\langle \xi_1, \xi_1 \rangle} \xi_1 + \cdots + \frac{\langle \xi_k, \alpha \rangle}{\langle \xi_k, \xi_k \rangle} \xi_k$$

is the closest point in W to α.

i.e. $\|\alpha - \mathrm{Proj}_W(\alpha)\| < \|\alpha - \beta\| \quad \forall \beta \in W, \beta \neq \mathrm{Proj}_W(\alpha)$

and $\alpha - \mathrm{Proj}_W(\alpha)$ is orthogonal to W.

If W is infinite-dimensional, then $\mathrm{Proj}_W(\alpha)$ is the vector in W such that $\alpha - \mathrm{Proj}_W(\alpha)$ is orthogonal to W, <u>if</u> this vector exists.

Ex: (Fourier Series) $V = C^0([-\pi,\pi])$ $\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$

It can be shown that (using trig identities):

$$\int_{-\pi}^{\pi} \cos mx \cos kx \, dx = 0 \quad \text{if } m \neq k$$

$$\int_{-\pi}^{\pi} \sin mx \sin kx \, dx = 0 \quad \text{if } m \neq k$$

$$\int_{-\pi}^{\pi} \cos mx \sin kx \, dx = 0$$

$$\int_{-\pi}^{\pi} \cos mx \, dx = 0, \quad \int_{-\pi}^{\pi} \sin mx \, dx = 0.$$

$\therefore \{1, \cos x, \cos 2x, \cdots \cos nx, \sin x, \sin 2x, \cdots, \sin nx\}$ is an orthogonal set.

Given $f \in C^0([-\pi,\pi])$, its best approximation in $\mathrm{Span}\{1, \cos x, \cdots, \cos nx, \sin x, \cdots \sin nx\}$ is

$$\frac{\langle 1, f\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle \cos x, f\rangle}{\langle \cos x, \cos x\rangle} \cos x + \cdots$$

$$+ \cdots \frac{\langle \cos nx, f\rangle}{\langle \cos nx, \cos nx\rangle} \cos nx + \cdots$$

$\therefore$ coefficient of $\cos kx$ is

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos kx \, f(x)\, dx$$

$$\pi = \int_{-\pi}^{\pi} (\cos kx)^2 \, dx$$

Th. 10.1.10: Gram-Schmidt algorithm for producing orthogonal bases

Let $\{\alpha_1, \dots, \alpha_s\}$ be a basis for W.

Set $\xi_1 = \alpha_1 \qquad W_1 = \mathrm{Span}\{\xi_1\}$

$\xi_2 = \alpha_2 - \mathrm{Proj}_{W_1}(\alpha_2) \qquad W_2 = \mathrm{Span}\{\xi_1, \xi_2\}$

$\xi_3 = \alpha_3 - \mathrm{Proj}_{W_2}(\alpha_3)$

$\vdots$

Then $\{\xi_1, \dots, \xi_s\}$ is an orthogonal basis for W.

Ex: $V = C^0([-1,1])$ $\quad \langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$

We apply Gram-Schmidt to $1, x^2, x^4$ to find
the (even) Legendre polynomials
(one example of a family of orthogonal polynomials)

$\Xi_1 = 1 \quad W_1 = \mathrm{Span}\{\Xi_1\}$

$\Xi_2 = x^2 - \mathrm{Proj}_{W_1}(x^2)$

$$= x^2 - \frac{\langle 1, x^2\rangle}{\langle 1,1\rangle}\,1 = x^2 - \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} 1\,dx}\,1 = x^2 - \frac{\frac{x^3}{3}\Big|_{-1}^{1}}{2}\,1$$

$$= x^2 - \frac{1}{3}$$

Check: $\langle x^2 - \frac{1}{3}, 1\rangle = \int_{-1}^{1} x^2 - \frac{1}{3}\,dx = \frac{x^3}{3} - \frac{x}{3}\Big|_{-1}^{1} = 0$

$W_2 = \mathrm{Span}\{\Xi_1, \Xi_2\}$

$\Xi_3 = x^4 - \mathrm{Proj}_{W_2}(x^4)$

$$= x^4 - \left(\frac{\langle 1, x^4\rangle}{\langle 1,1\rangle}\,1 + \frac{\langle x^2 - \frac{1}{3}, x^4\rangle}{\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\rangle}\left(x^2 - \frac{1}{3}\right)\right)$$

$$= x^4 - \left(\frac{\int_{-1}^{1} x^4\,dx}{\int_{-1}^{1} 1\,dx}\,1 + \frac{\int_{-1}^{1} (x^2 - \frac{1}{3})x^4\,dx}{\int_{-1}^{1} (x^2 - \frac{1}{3})^2\,dx}\left(x^2 - \frac{1}{3}\right)\right)$$

$$= \cdots$$

$$= x^4 - \left(\frac{1}{5} + \frac{6}{7}\left(x^2 - \frac{1}{3}\right)\right)$$

$$= x^4 - \frac{6}{7}x^2 + \frac{3}{35}$$

diagonalise a quadratic form

$$P^{-1}AP \neq D = P^{T}AP$$

diagonal entries of $D \neq$ eigenvalues

if P is an orthogonal matrix,
then $P^{-1} = P^{T}$.