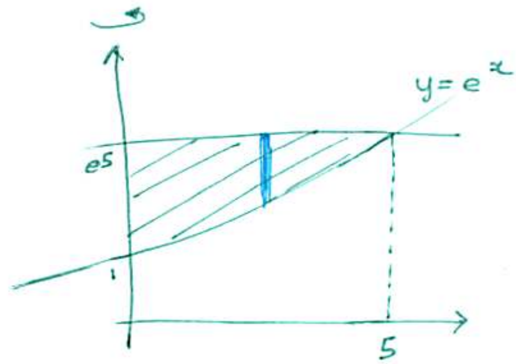
1. (7 points) Let R be the region bounded by the curves

$$y = e^x, \quad x = 0, \quad y = e^5.$$

Find the volume of the solid obtained by rotating R about the y-axis. **Simplify your answer as much as possible.**



Use cylindrical shells:

$$\text{volume} = \int_0^5 2\pi x(e^5 - e^x)\,dx$$

$$= \int_0^5 2\pi x e^5 - \int_0^5 2\pi x e^x\,dx$$

$$= \left[2\pi e^5 \frac{x^2}{2}\right]_0^5 - 2\pi \int_0^5 x e^x\,dx$$

$$= 25\pi e^5 - 2\pi \int_0^5 x e^x\,dx$$

Integration by parts:

$u = x \qquad dv = e^x\,dx$

$du = dx \qquad v = e^x$

$$\int_0^5 x e^x\,dx = \left[x e^x\right]_0^5 - \int_0^5 e^x\,dx$$

$$= (5e^5 - 0) - \left[e^x\right]_0^5$$

$$= 5e^5 - (e^5 - 1) = 4e^5 + 1$$

$$\text{So volume} = 25\pi e^5 - 2\pi(4e^5 + 1)$$

$$= 17\pi e^5 - 2\pi$$

2. (7 points) Compute the following integral:

$$\int \frac{x+12}{x(x-2)^2}dx.$$

Partial fractions:

$$\frac{x+12}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$x+12 = A(x-2)^2 + Bx(x-2) + Cx$$

$x=0$: $\quad 12 = A(-2)^2 \quad \Rightarrow A=3$

$x=2$: $\quad 2+12 = C(2) \quad \Rightarrow C=7$

coeff of x^2: $\quad 0 = A+B \quad \Rightarrow B=-3$

So
$$\int \frac{x+12}{x(x-2)^2}\,dx = \int \frac{3}{x} - \frac{3}{x-2} + \frac{7}{(x-2)^2}\,dx$$

$$= 3\ln|x| - 3\ln|x-2| + 7\frac{(x-2)^{-1}}{-1} + C$$

$$= 3\ln|x| - 3\ln|x-2| - \frac{7}{x-2} + C$$

3. (7 points) Compute the following integral:

$$\int (25 - x^2)^{-\frac{7}{2}}\, dx.$$

substitution:

$x = 5\sin\theta$

$dx = 5\cos\theta\, d\theta$

$25 - x^2 = 25 - 25\sin^2\theta$

$= 25\cos^2\theta$

$$= \int (5\cos\theta)^{-7}(5\cos\theta\, d\theta)$$

$$= \int 5^{-6}(\cos\theta)^{-6}\, d\theta$$

$$= \int 5^{-6}\sec^6\theta\, d\theta$$

$$= \int 5^{-6}(\tan^2\theta + 1)^2 \sec^2\theta\, d\theta$$

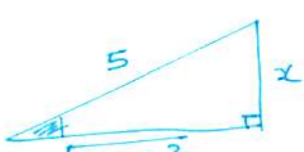
$$= \int 5^{-6}(\tan^4\theta + 2\tan^2\theta + 1)\sec^2\theta\, d\theta$$

substitution

$u = \tan\theta$

$$= 5^{-6}\left(\frac{\tan^5\theta}{5} + \frac{2\tan^3\theta}{3} + \tan\theta\right) + C$$

$$= 5^{-6}\left(\frac{1}{5}\left(\frac{x}{\sqrt{25-x^2}}\right)^5 + \frac{2}{3}\left(\frac{x}{\sqrt{25-x^2}}\right)^3 + \frac{x}{\sqrt{25-x^2}}\right) + C$$



$\frac{x}{5} = \sin\theta = \frac{\text{opp.}}{\text{hyp}}$

$\tan\theta = \frac{\text{opp.}}{\text{adj}} = \frac{x}{\sqrt{25-x^2}}$