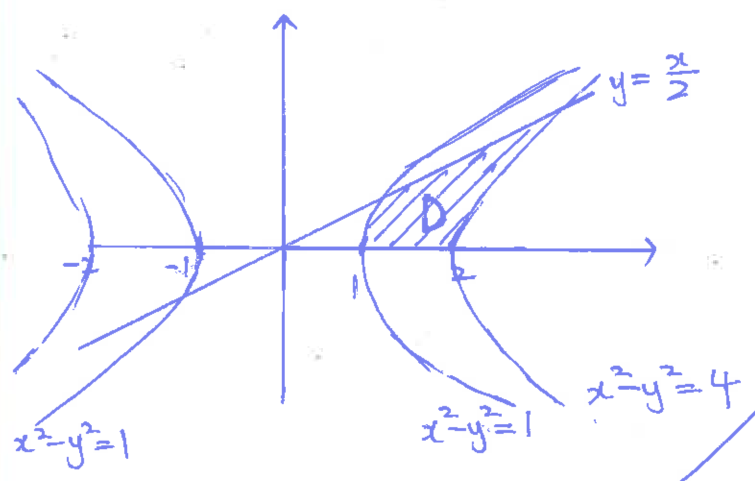
Example: Evaluate $\iint_D \frac{1}{x^2}\, dA$, where D is the region bounded by $x^2 - y^2 = 1$, $x^2 - y^2 = 4$, $y = 0$ and $y = \frac{x}{2}$, $\boxed{\text{with } x>0}$

hyperbola

i.e. $y - \frac{x}{2} = 0$ — but what to do with $y=0$?

better: $\frac{y}{x} = \frac{1}{2}$



Let $u = x^2 - y^2$, $v = \frac{y}{x}$

① So D corresponds to (S)

$1 \le u \le 4$, $0 \le v \le \frac{1}{2}$.

② $\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \det\begin{pmatrix} 2x & -2y \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix}$

$= 2 - \frac{2y^2}{x^2}$.

$= \frac{2}{x^2}(x^2 - y^2)$

Is this positive or negative?

and $x^2 - y^2 \ge 1 > 0$

and $\frac{2}{x^2} > 0$

so this is positive

$$
\begin{aligned}
\iint_D \frac{1}{x^2}\, dA &= \int_0^{\frac{1}{2}} \int_1^4 \frac{1}{x^2} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv \\
&= \int_0^{\frac{1}{2}} \int_1^4 \frac{1}{x^2} \frac{1}{\left| \frac{\partial(u,v)}{\partial(x,y)} \right|} du\, dv \\
&= \int_0^{\frac{1}{2}} \int_1^4 \frac{1}{x^2} \frac{1}{2 - \frac{2y^2}{x^2}} du\, dv \\
&= \int_0^{\frac{1}{2}} \int_1^4 \frac{1}{2x^2 - 2y^2} du\, dv \\
&= \int_0^{\frac{1}{2}} \int_1^4 \frac{1}{2} \frac{1}{u} du\, dv \\
&= \int_0^{\frac{1}{2}} \frac{1}{2} \ln u \Big|_{u=1}^{u=4} dv = \int_0^{\frac{1}{2}} \frac{1}{2} \ln 4\, dv = \frac{1}{2} \ln 4\, v \Big|_0^{\frac{1}{2}} = \frac{1}{4} \ln 4 .
\end{aligned}
$$

3