Last time: Given $\sigma \in L(U,V)$, its dual $\hat{\sigma} \in L(\hat{V}, \hat{U})$ is given by $\hat{\sigma}(\phi) = \phi \circ \sigma$.

If $\mathcal{A}, \mathcal{B}$ are bases of U, V, then

Th. 9.3.3 $$[\hat{\sigma}]_{\hat{\mathcal{A}} \leftarrow \hat{\mathcal{B}}} = \left([\sigma]_{\mathcal{B} \leftarrow \mathcal{A}} \right)^T.$$

§ 9.2 Change of coordinates for linear forms

? Given $\mathcal{A}, \mathcal{B}$ bases of V, what is the matrix relating $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$, for $\hat{V}$?

i.e. what is $[\iota_{\hat{V}}]_{\hat{\mathcal{A}} \leftarrow \hat{\mathcal{B}}}$?

Note: $\iota_{\hat{V}} = \widehat{\iota_V}$ $\because$ $\widehat{\iota_V}(\phi) = \phi \circ \iota_V = \phi$

$\therefore$ From Th 9.3.3:

$$[\iota_{\hat{V}}]_{\hat{\mathcal{A}} \leftarrow \hat{\mathcal{B}}} = [\widehat{\iota_V}]_{\hat{\mathcal{A}} \leftarrow \hat{\mathcal{B}}} = \left([\iota_V]_{\mathcal{B} \leftarrow \mathcal{A}} \right)^T$$

equivalently

$$[\iota_{\hat{V}}]_{\hat{\mathcal{A}} \leftarrow \hat{\mathcal{B}}} = \left([\iota_V]_{\mathcal{A} \leftarrow \mathcal{B}} \right)^{T^{-1}}$$

A way to understand this when

$\mathcal{A}$ = standard basis of $\mathbb{F}^n$,

$\mathcal{B} = \{\beta_1, \cdots, \beta_n\}$

Let $\widehat{\mathcal{B}} = \{\psi_1, \cdots, \psi_n\} \subseteq \hat{V}$,

so $\psi_i : V \to \mathbb{F}$

Consider

$$\begin{pmatrix} — [\psi_1] — \\ \vdots \\ — [\psi_n] — \end{pmatrix} \begin{pmatrix} | & & | \\ \beta_1 & \cdots & \beta_n \\ | & & | \end{pmatrix} = \begin{pmatrix} \psi_1(\beta_1) & \psi_1(\beta_2) & \cdots & \psi_1(\beta_n) \\ \vdots & & & \vdots \\ \psi_n(\beta_1) & \cdots & \cdots & \psi_n(\beta_n) \end{pmatrix}$$

standard matrix of ψ_i (1×n matrix)

i,j entry of ↑ is $\psi_i(\beta_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

$\therefore$ matrix on RHS is I.

So $\begin{pmatrix} — [\psi_1] — \\ \vdots \\ — [\psi_n] — \end{pmatrix} = \begin{pmatrix} | & & | \\ \beta_1 & \cdots & \beta_n \\ | & & | \end{pmatrix}^{-1} = \left[\iota_V \right]_{\mathcal{A} \leftarrow \mathbb{B}}^{-1}$

$\| $

$\left[\iota_{\hat{V}} \right]_{\hat{\mathcal{A}} \leftarrow \hat{\mathbb{B}}}^{T}$ $\therefore$ the ith column of $\begin{pmatrix} — [\psi_1] — \\ \vdots \\ — [\psi_n] — \end{pmatrix}^{T}$ is $[\psi_i]^T = [\psi_i]_{\hat{\mathcal{A}}}$

Ex. 9.2.3 If $\mathbb{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^3 = V$

find $\hat{\mathbb{B}}$.

Method 1 : use change of coordinates matrix:

let $\mathcal{A} = \{e_1, e_2, e_3\}$, and let $\hat{\mathbb{B}} = \{\psi_1, \psi_2, \psi_3\}$.

The columns of $\underset{\hat{A}\leftarrow\hat{B}}{[\iota_{\hat{V}}]}$ are $[\psi_1]_{\hat{A}}, [\psi_2]_{\hat{A}}, [\psi_3]_{\hat{A}}$.

$$\underset{\hat{A}\leftarrow\hat{B}}{[\iota_{\hat{V}}]} = \underset{B\leftarrow A}{[\iota_V]}^T$$

columns of this are $[e_i]_B$.

$$= \left(\underset{A\leftarrow B}{[\iota_V]}^{-1}\right)^T$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}^{-1\,T}$$

row reduction to find inverse

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}^T$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

read the columns:

$[\psi_1]_{\hat{A}} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. i.e. $\psi_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2}x + \frac{1}{2}y - \frac{1}{2}z$

$\psi_1 = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2 - \frac{1}{2}\phi_3$ if $\hat{A} = \{\phi_1, \phi_2, \phi_3\}$

$$\psi_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{2}x + \frac{1}{2}y - \frac{1}{2}z$$

$$\psi_3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z.$$

Check: $\psi_i(\beta_i) = 1$ and $\psi_i(\beta_j) = 0$ if $i \neq j$.

e.g. $\psi_1(\beta_1) = \psi_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{2}1 + \frac{1}{2}0 - \frac{1}{2}(-1) = 1$ ✓

$\psi_2(\beta_3) = \psi_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = -\frac{1}{2}0 + \frac{1}{2}1 - \frac{1}{2}1 = 0$ ✓

Method 2: use definition of $\hat{B}$.

let $\psi_1\begin{pmatrix}x\\y\\z\end{pmatrix} = ax + by + cz$.

We need $\psi_1(\beta_1) = 1$, $\psi_1(\beta_2) = 0$, $\psi_1(\beta_3) = 0$.

$$\psi_1(\beta_1) = \psi_1\begin{pmatrix}1\\0\\-1\end{pmatrix} = a\cdot 1 + b\cdot 0 + c(-1) = 1$$

$$\underline{a - c = 1}$$

$$\psi_1(\beta_2) = \psi_1\begin{pmatrix}-1\\1\\0\end{pmatrix} = -a + b = 0$$

$$\psi_1(\beta_3) = \psi_1\begin{pmatrix}0\\1\\1\end{pmatrix} = b + c = 0$$

$$a = b = -c = \frac{1}{2}.$$

so $\psi_1\begin{pmatrix}x\\y\\z\end{pmatrix} = \frac{1}{2}x + \frac{1}{2}y - \frac{1}{2}z$

similarly solve for ψ_2, ψ_3.

The double dual:

Recall $\hat{V} = L(V, \mathbb{F})$, so $\hat{\hat{V}} = L(\hat{V}, \mathbb{F})$

i.e. each $f \in \hat{\hat{V}}$ is a function $\phi \longmapsto$ number,

where the input ϕ is a function on V.

One example: f = evaluation at some fixed $\alpha \in V$.

i.e. f is $\phi \longmapsto \phi(\alpha)$ i.e. $f(\phi) = \phi(\alpha)$.

Exercise: check this f is in $\hat{\hat{V}}$, i.e. f is linear (in its input in $\hat{V}$).

So, to every $\alpha \in V$, we can associate such an $f \in \hat{\hat{V}}$.

Th. 9.2.1: The function $J : V \longrightarrow \hat{\hat{V}}$ given by $J(\alpha)$ = evaluation at α, i.e.

$[J(\alpha)](\phi) = \phi(\alpha)$ is an injective linear transformation.