

Remember that there are two rules for differentiating complicated functions: the chain rule and the product rule. (The quotient rule is a combination of these two rules, since $\frac{u}{v} = uv^{-1}$.)

Since FTC says that integration is antidifferentiation, we can derive from these differentiation rules two techniques of integration:

chain rule \longrightarrow method of substitution (p2-15, §5.6)
 product rule \longrightarrow integration by parts (p16-22, §6.1)

These techniques are **not** rules. They do not give us the answer; they only **change our integral to a new integral**, which we hope will be easier to evaluate. There are no rules in integration: there is no guaranteed algorithm to integrate a function. Using the techniques require some creativity, and there are often multiple efficient ways to calculate the same integral.

§5.6: The Method of Substitution

(The letters used here are different from in the textbook.)

Recall the chain rule for differentiation: $\frac{d}{dx}F(g(x)) = F'(g(x))g'(x)$

Take the antiderivative of both sides: $F(g(x)) + C = \int F'(g(x))g'(x) dx$

Write u for $g(x)$: $F(u) + C = \int F'(u)\frac{du}{dx} dx$

Write f for F' : $\int f(u) du = \int f(u)\frac{du}{dx} dx$.

Hence, if we can identify a function $u(x)$ such that our integrand is a product, of the composition $f(u(x))$ and the derivative $\frac{du}{dx}$ then we can rewrite our integral as $\int f(u) du$.

$$\int f(u)\frac{du}{dx} dx = \int f(u) du. \quad (\text{i.e. we can treat } \frac{du}{dx} \text{ formally like a fraction})$$

Example: Evaluate $\int \cos(x^3) 3x^2 dx$.

$$\int f(u)\frac{du}{dx} dx = \int f(u) du.$$

Example: Evaluate $\int e^{3x} dx$.

$$\int f(u)\frac{du}{dx} dx = \int f(u) du.$$

Example: Evaluate $\int x\sqrt{1+x^2} dx$.

$$\int f(u) \frac{du}{dx} dx = \int f(u) du.$$

There are two ways to calculate a definite integral by substitution:

1. Find the indefinite integral and then substitute in the limits for x ;
2. (Usually faster) Change the limits into limits for u .

Example: Evaluate $\int_0^1 x\sqrt{1+x^2} dx$.

Two other correct ways to use method 1:

$$\begin{aligned} & \int x\sqrt{1+x^2} dx \\ &= \int \frac{1}{2}\sqrt{u} du \\ &= \frac{u^{3/2}}{2(3/2)} + C \\ &= \frac{1}{3}\sqrt{1+x^2}^3 + C, \end{aligned}$$

so

$$\begin{aligned} & \int_0^1 x\sqrt{1+x^2} dx \\ &= \frac{1}{3}\sqrt{1+x^2}^3 \Big|_1^0 = \frac{1}{3}(\sqrt{2}^3 - 1). \end{aligned}$$

$$\begin{aligned} & \int_0^1 x\sqrt{1+x^2} dx \\ &= \int_{x=0}^{x=1} \frac{1}{2}\sqrt{u} du \\ &= \frac{u^{3/2}}{2(3/2)} \Big|_{x=0}^{x=1} \\ &= \frac{1}{3}\sqrt{1+x^2}^3 \Big|_1^0 = \frac{1}{3}(\sqrt{2}^3 - 1). \end{aligned}$$

Do **not** write $\int_0^1 \frac{1}{2}\sqrt{u} du$ - that would mean you want to evaluate at $u = 0, 1$.

Note that the final two steps in method 1 are to change the indefinite integral from u to x , then substitute the limits of x . In method 2 below, we combine these two steps – simply substitute the corresponding limits for u .

Example: Evaluate $\int_0^1 x\sqrt{1+x^2} dx$.

$$\int f(u) \frac{du}{dx} dx = \int f(u) du.$$

Tips for choosing a good u :

- If the integrand contains a composite function e.g. $e^{g(x)}$, $\cos(g(x))$, $\sin(g(x))$, $\sqrt{g(x)}$, $\frac{1}{g(x)}$, try $u = g(x)$.
- Choose a u for which $\frac{du}{dx}$ appears in the integrand.

The best way to get better at choosing u is to do lots of problems, and **think about** why your chosen u was effective.

Very important: make sure your integrand is **entirely in terms of u** (no x s) before you start integrating.

Harder example: Evaluate $\int_0^1 x^3 \sqrt{1-x^2} dx$.

Harder example: Evaluate $\int_0^1 \frac{x^2}{1+x^6} dx$.

Using various trigonometric identities and the method of substitution, we can obtain the integrals of many trigonometric functions - these will be given to you on the exams.

Examples:

$$\begin{aligned} \int \cos^2 x dx &= \int \frac{1}{2}(1 + \cos(2x)) dx && \text{by the identity } \cos(2x) = 2\cos^2 x - 1 \\ &= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C && \text{substitution } u = 2x \text{ in the second term} \\ &= \frac{1}{2}x + \frac{1}{2}\sin x \cos x + C. && \text{by the identity } \sin(2x) = 2\sin x \cos x \end{aligned}$$

$$\begin{aligned} \int \cos^3 x dx &= \int \cos x(1 - \sin^2 x) dx && \text{by the identity } \cos^2 x + \sin^2 x = 1 \\ &= \int \cos x - \cos x \sin^2 x dx \\ &= \sin x + \frac{1}{3}\sin^3 x + C. && \text{substitution } u = \sin x \text{ in the second term} \end{aligned}$$

The full list of trigonometric-power integrals you will be given in exams:

$$\int \sin^2 x \, dx = \frac{1}{2}(x - \sin x \cos x) + C,$$

$$\int \sin^3 x \, dx = -\cos x + \frac{1}{3} \cos^3 x + C.$$

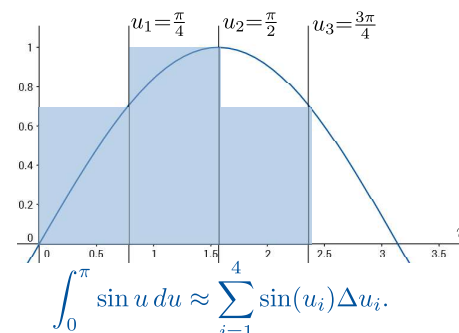
$$\int \sin^4 x \, dx = \frac{1}{8}(3x - 3 \sin x \cos x - 2 \sin^3 x \cos x) + C,$$

$$\int \cos^2 x \, dx = \frac{1}{2}(x + \sin x \cos x) + C,$$

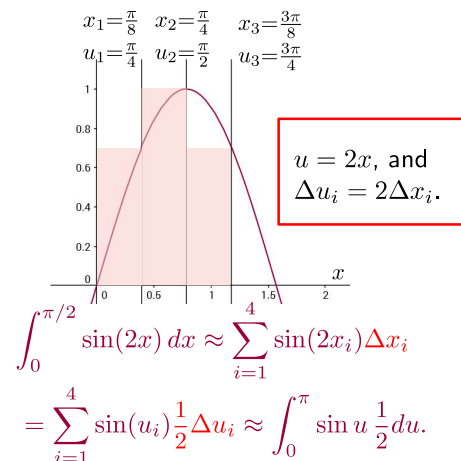
$$\int \cos^3 x \, dx = \sin x - \frac{1}{3} \sin^3 x + C.$$

$$\int \cos^4 x \, dx = \frac{1}{8}(3x + 3 \sin x \cos x + 2 \cos^3 x \sin x) + C.$$

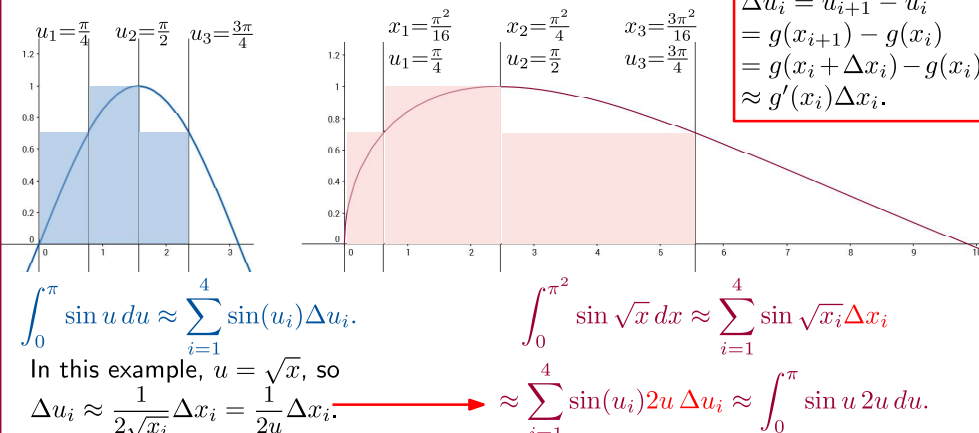
To prepare for a multivariate version of substitution (see the final week), we need to understand single-variable substitution geometrically, i.e. in terms of approximating the area under a curve with Riemann sums:



The heights of the two sets of approximating rectangles are the same, but on the right the rectangles are half as wide.



When u is not a linear function of x , the widths of the rectangles stretch by different amounts.



§6.1: Integration by Parts

Recall the product rule for differentiation:

$$\frac{d}{dx}(U(x)V(x)) = U(x) \frac{dV}{dx} + V \frac{dU}{dx}$$

Take the antiderivative of both sides:

$$U(x)V(x) = \int U(x) \frac{dV}{dx} \, dx + \int V \frac{dU}{dx} \, dx$$

Rearranging:

$$\int U(x) \frac{dV}{dx} \, dx = U(x)V(x) - \int V \frac{dU}{dx} \, dx$$

A shorthand that is easy to remember:

$$\int U \, dV = UV - \int V \, dU$$

Standard example: Evaluate $\int x e^x dx$.

$$\int U dV = UV - \int V dU$$

Standard example: Evaluate $\int x \sin x dx$.

$$\int U dV = UV - \int V dU$$

Standard example: Evaluate $\int x \ln x dx$.

$$\int U dV = UV - \int V dU$$

The technique of integration by parts relies on separating your integrand into two parts, a U and a $\frac{dV}{dx}$. Because we need to calculate $\int V dU$, we want U to be easy to differentiate and V to be easy to integrate. One good strategy to choose these parts is the DETAIL rule:

dV should be the part of the integrand that appears highest in this list:

Exponential: e^x	} nice to integrate
Trigonometric: $\sin x, \cos x$	
Algebraic: x^n	} hard to integrate
Inverse trigonometric: $\sin^{-1} x, \tan^{-1} x$	
Logarithmic: $\ln x$	

In our previous examples:

$x e^x$ (p17) is a product of an algebraic and an exponential function, and exponential is higher on the list, so $dV = e^x dx$ and $U = x$.

$x \ln x$ (p19) is a product of an algebraic and a logarithmic function, and algebraic is higher on the list, so $dV = x dx$ and $U = \ln x$.

Sometimes, after integration by parts, our new integral again requires integration by parts:

Example: Evaluate $\int_0^2 (xe^x)^2 dx$.

$$\int U dV = UV - \int V dU$$

Some integrals are best calculated using a substitution and then integration by parts. (It can also happen that, after integration by parts, the new integral requires a substitution.)

Example: Evaluate $\int x^3 e^{x^2} dx$.

$$\int U dV = UV - \int V dU$$

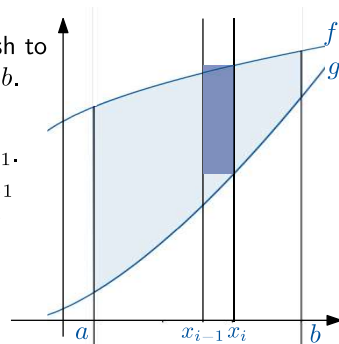
§5.7: Areas of Plane Regions

This is a simple, first example where we calculate a geometric quantity by writing it as a limit of a Riemann sum, identifying it as an integral, and then using FTC2. (We will do more of this in higher dimensions).

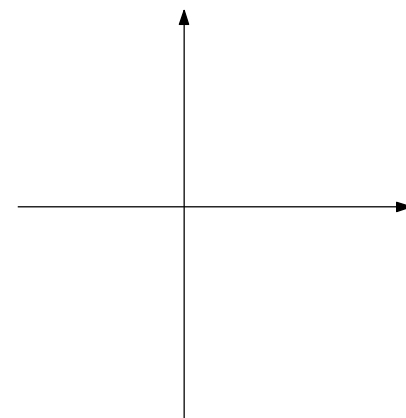
Given functions $f, g : [a, b] \rightarrow \mathbb{R}$ with $f(x) \geq g(x)$ we wish to find the area bounded by $y = f(x), y = g(x), x = a, x = b$.

1. Divide $[a, b]$ into n subintervals by choosing x_i with $a = x_0 < x_1 < \cdots < x_n = b$, and let $\Delta x_i = x_i - x_{i-1}$.
2. Approximate the part of the desired area between x_{i-1} and x_i by a rectangle, whose width is Δx_i and whose height is $f(x_i^*) - g(x_i^*)$, for some $x_i^* \in [x_{i-1}, x_i]$.
3. So the area is

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (f(x_i^*) - g(x_i^*)) \Delta x_i = \int_a^b f(x) - g(x) dx.$$



Example: Find the area of the region bounded by $y = x^2 - 4$ and $y = -x^2 + 2x$.



Example: Find the area of the region bounded by $y = 2\sqrt{x}$, $y = 3 - x$ and $y = 0$.

