

Th 9.4.13 Polarisation identity — to find the symmetric bilinear form f given q with $q(\alpha) = f(\alpha, \alpha)$.

$$2f(\alpha, \beta) = q(\alpha + \beta) - q(\alpha) - q(\beta)$$

\therefore If $1+1 \neq 0$ in F (i.e. can divide by 2), then quadratic forms are in bijection with symmetric bilinear forms.

Proof:
$$\begin{aligned} q(\alpha + \beta) &= f(\alpha + \beta, \alpha + \beta) \\ &= f(\alpha, \alpha + \beta) + f(\beta, \alpha + \beta) \\ &= f(\alpha, \alpha) + f(\alpha, \beta) + f(\beta, \alpha) + f(\beta, \beta) \\ &= q(\alpha) + 2f(\alpha, \beta) + q(\beta, \beta) \end{aligned}$$

Change of coordinates for bilinear forms

[?] How is $\{f\}_A$ and $\{f\}_B$ related?

$$q(\alpha) = [\alpha]_A^T \{f\}_A [\alpha]_A$$

$$= \left([\underset{A \leftarrow B}{C}] [\alpha]_B \right)^T \{f\}_A \left(\underset{A \leftarrow B}{[\underset{B}{C}]} [\alpha]_B \right)$$

$$= [\alpha]_B^T \left(\underset{A \leftarrow B}{[\underset{B}{C}]}^T \{f\}_A \underset{A \leftarrow B}{[\underset{B}{C}]} \right) [\alpha]_B$$

Compare $q(\alpha) = [\alpha]_B^T \{f\}_B [\alpha]_B$

Th:
$$\{f\}_B = \underset{A \leftarrow B}{[\underset{B}{C}]}^T \{f\}_A \underset{A \leftarrow B}{[\underset{B}{C}]}$$

Def 9.4.4: A and B are congruent if $B = P^T A P$
for some invertible P .

Goal: Given a symmetric matrix A ,
find a diagonal D and invertible P
such that $D = P^T A P$.

Diagonalising a quadratic form by row and column operations

Observe:

$$\begin{pmatrix} 1 & . & k \\ . & 1 & . \\ . & . & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a+kg & b+kh & c+ki \\ d & e & f \\ g & h & i \end{pmatrix} \quad \begin{matrix} R_1 + kR_3 \\ \\ \end{matrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ k & . & 1 \end{pmatrix} = \begin{pmatrix} a+kc & b & c \\ d+kf & e & f \\ g+ki & h & i \end{pmatrix} \quad \begin{matrix} \\ C_1 + kC_3 \\ \end{matrix}$$

In general: $i=1, j=3$ in example above

$$(I + kE_{ij})X = \text{do } (R_i \rightarrow R_i + kR_j) \text{ to } X$$

$$X(I + kE_{ji}) = \text{do } (C_i \rightarrow C_i + kC_j) \text{ to } X$$

$= (I + kE_{ij})^T$ $i \neq j$

\therefore if S is of the form $I + kE_{ij}$ (i.e. an elementary matrix)
then SXS^T is the result of doing the
corresponding row and column operations to X .

Goal: do row and column operations to make
 A diagonal, i.e. find S_1, S_2, \dots, S_r such

$$\text{that } S_r \dots S_2 S_1 A S_1^T S_2^T \dots S_r^T = D$$

$$\underbrace{(S_r \dots S_1)}_{PT} A \underbrace{(S_r \dots S_1)^T}_P = D$$

\therefore if we do the same row operations to I ,
then the result $S_r \dots S_1 I$, so
transposing then gives P .

- Start with $(A|I)$
— this is NOT a linear system.

- Do row operations to both "sides"
 - Do column operations to left side only
- } repeat
- Result is $(D|PT)$

Tip: use replacement operation only!

interchange and scaling have strange effects, \therefore we are doing both row and column operations

Ex 9.4.20: $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} \boxed{0} & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array} \right)$$

put pivot here

$C_1 + C_2$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array} \right) R_1 + R_2$$

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \end{array} \right)$$

check "left side" is symmetric

make 0s under pivot

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & 1 \end{array} \right) \begin{array}{l} R_2 - \frac{1}{2}R_1 \\ R_3 - \frac{3}{2}R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} \boxed{2} & 0 & 0 & 1 & 1 & 0 \\ 0 & \boxed{-\frac{1}{2}} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & 1 \end{array} \right) \begin{array}{l} C_2 - \frac{1}{2}C_1 \\ C_3 - \frac{3}{2}C_1 \end{array}$$

next pivot

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -4 & -1 & -2 & 1 \end{array} \right) \quad R_3 - R_2$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -4 & -1 & -2 & 1 \end{array} \right) \quad \underbrace{\hspace{1cm}}_D \quad \underbrace{\hspace{1cm}}_{P^T} \quad C_3 - C_2$$

$$\therefore D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -4 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 1 & \frac{1}{2} & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

(P, D not unique)

If the diagonal entries are never 0 in this process, then P is upper-triangular — this is related to Cholesky factorisation $A = LDL^T$ — see numerical linear algebra — twice as fast as row-reduction.