§10.1 Inner product

Motivation: dot product on $\mathbb{R}^n$ is useful

e.g. if $\{\alpha_1, \cdots, \alpha_k\}$ is an orthogonal basis for $W \subseteq \mathbb{R}^n$

then the closest point in W to β / best approximation in W

to β is $\mathrm{Proj}_W(\beta) = \dfrac{\beta \cdot \alpha_1}{\alpha_1 \cdot \alpha_1}\alpha_1 + \cdots + \dfrac{\beta \cdot \alpha_k}{\alpha_k \cdot \alpha_k}\alpha_k$

Want to similarly approximate functions in other vector spaces,

i.e. want to define $\alpha \cdot \beta$ for $\alpha, \beta \in V$ — use a symmetric bilinear form that's positive definite.

Def 10.1.1 Let V be a vector space $\boxed{\text{over } \mathbb{R}}$.

$\langle \, , \rangle$ is an inner product on V if:

a. symmetric $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle$

b. bilinear $\langle \alpha, a\beta_1 + \beta_2 \rangle = a\langle \alpha, \beta_1 \rangle + \langle \alpha, \beta_2 \rangle$

(c. consequence of a and b:
$\langle a\alpha_1 + \alpha_2, \beta \rangle = a\langle \alpha_1, \beta \rangle + \langle \alpha_2, \beta \rangle$)

d. positive definite $\langle \alpha, \alpha \rangle > 0 \ \forall \alpha \neq \vec{0}$.

Ex:

Ex:. dot product on $\mathbb{R}^n$: $\langle \alpha, \beta \rangle = \alpha \cdot \beta$

- weighted dot product
  e.g. in $\mathbb{R}^3$ $\left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\rangle = x_1y_1 + 2x_2y_2 + 5x_3y_3$

- in $C^0([a,b])$: $\langle f, g \rangle = \int_a^b w(x) f(x) g(x)\, dx$

  where $w(x)$ is a positive weight function on (a,b)

  e.g. $\langle f, g \rangle = \int_{-1}^{1} (1+x) f(x) g(x)\, dx$
  $(w(x) = 1+x)$

- in $M_{n,n}(\mathbb{R})$:

  Def: the trace of $X \in M_{n,n}(\mathbb{R})$ is $\mathrm{Tr}(X) = X_{11} + X_{22} + \cdots + X_{nn}$

  e.g. $\mathrm{Tr}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a + e + i$

  $\langle A, B \rangle = \mathrm{Tr}(A^T B)$

  e.g On $M_{2,2}(\mathbb{R})$:

$$\left\langle \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \right\rangle = \mathrm{Tr}\left(\begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \right)$$

$$= \mathrm{Tr}\begin{pmatrix} a_1b_1 + a_3b_3 & a_1b_2 + a_3b_4 \\ a_2b_1 + a_4b_3 & a_2b_2 + a_4b_4 \end{pmatrix}$$

dot product on $\mathbb{R}^4$

$$= (a_1b_1 + a_3b_3) + (a_2b_2 + a_4b_4)$$

In general $\langle A, B \rangle = [A]_{\mathcal{A}} \cdot [B]_{\mathcal{A}}$ if $\mathcal{A} = \{E^{1,1}, E^{1,2}, \cdots, E^{n,n}\}$ is the standard basis of $M_{n,n}(\mathbb{R})$.

Over $\mathbb{C}$, we have to define inner products differently, because the condition $\langle \alpha, \alpha \rangle > 0 \ \forall \alpha \neq \vec{0}$ does not make sense if $\langle \alpha, \alpha \rangle \in \mathbb{C} \setminus \mathbb{R}$

Def 10.1.1 / 9.6.1 Let V be a vector space $\boxed{\text{over } \mathbb{C}}$.

$\langle \, , \, \rangle$ is an inner product on V if:

a. Hermitian $\overline{\langle \alpha, \beta \rangle} = \langle \beta, \alpha \rangle$ where $\overline{}$ means complex conjugation (e.g. $\overline{2+i} = 2-i$)

(then $\overline{\langle \alpha, \alpha \rangle} = \langle \alpha, \alpha \rangle$ so $\langle \alpha, \alpha \rangle \in \mathbb{R}$)

b. sesquilinear $\langle \alpha, a\beta_1 + \beta_2 \rangle = a\langle \alpha, \beta_1 \rangle + \langle \alpha, \beta_2 \rangle$

(c. consequence of a and b: $\langle a\alpha_1 + \alpha_2, \beta \rangle = \boxed{\overline{a}} \langle \alpha_1, \beta \rangle + \langle \alpha_2, \beta \rangle$)

d. positive definite $\langle \alpha, \alpha \rangle > 0 \ \forall \alpha \neq \vec{0}$.

(There is a theory for Hermitian sesquilinear forms, like the one for symmetric bilinear forms

e.g. polarisation identity,
diagonalisation theorem and algorithm,
classification by definiteness,
... – see §9.6.)

Ex: Dot product on $\mathbb{C}^n$

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \overline{x_1} y_1 + \cdots + \overline{x_n} y_n$$

e.g. on $\mathbb{C}^2$: $\begin{pmatrix} 1 \\ 1+i \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2i \end{pmatrix} = 1 \cdot 3 + (1-i)(-2i) = 3 - 2i - 2 = 1 - 2i$

Ex: $V = \{ f : \mathbb{R} \to \mathbb{C}$ continuous on $[a,b] \}$

$$\langle f, g \rangle = \int_a^b \overline{f(x)}\, g(x)\, dx$$

(computation not examinable)

No inner product over other fields
e.g. $\{0,1\} = \mathbb{F}_2$

Define the <u>length</u> or <u>norm</u> of $\alpha \in V$ to be $\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle}$

Then
$$\begin{aligned} \|a\alpha\|^2 &= \langle a\alpha, a\alpha \rangle \\ &= \overline{a} \langle \alpha, a\alpha \rangle \\ &= \overline{a} a \langle \alpha, \alpha \rangle \\ &= |a|^2 \langle \alpha, \alpha \rangle \end{aligned}$$
so $\|a\alpha\| = |a|\,\|\alpha\|$