§7.1: Diagonalisation of Symmetric Matrices

Symmetric matrices ($A = A^T$) arise naturally in many contexts, when a_{ij} depends on i and j but not on their order (e.g. the friendship matrix from Homework 3 Q7, the Hessian matrix of second partial derivatives from Multivariate Calculus). The goal of this section is to observe some very nice properties about the eigenvectors of a symmetric matrix.

Example: $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$ is a symmetric matrix.

$\begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, so $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is a -1-eigenvector.

$\begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, so $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a 4-eigenvector.

Notice that the eigenvectors are orthogonal: $\begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$. This is not a coincidence...

Theorem 1: Eigenvectors of Symmetric Matrices: If A is a symmetric matrix, then eigenvectors corresponding to distinct eigenvalues are orthogonal.
Compare: for an arbitrary matrix, eigenvectors corresponding to distinct eigenvalues are linearly independent (week 10 p22).

Proof: Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2. Then

$$(A\mathbf{v}_1) \cdot \mathbf{v}_2 = (\lambda_1 \mathbf{v}_1) \cdot \mathbf{v}_2 = \lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2),$$

and

$$\mathbf{v}_1 \cdot (A\mathbf{v}_2) = \mathbf{v}_1 \cdot (\lambda_2 \mathbf{v}_2) = \lambda_2(\mathbf{v}_1 \cdot \mathbf{v}_2).$$

But the two left hand sides above are equal, because (see also week 12 p25)

$$(A\mathbf{v}_1) \cdot \mathbf{v}_2 = (A\mathbf{v}_1)^T \mathbf{v}_2 = \mathbf{v}_1^T A^T \mathbf{v}_2 = \mathbf{v}_1^T A \mathbf{v}_2 = \mathbf{v}_1 \cdot (A\mathbf{v}_2).$$

A is symmetric

So the two right hand sides are equal: $\lambda_1(\mathbf{v}_1 \cdot \mathbf{v}_2) = \lambda_2(\mathbf{v}_1 \cdot \mathbf{v}_2)$. Since $\lambda_1 \neq \lambda_2$, it must be that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.

Remember from week 10 §5:
Definition: A square matrix A is *diagonalisable* if there is an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.
Diagonalisation Theorem: An $n \times n$ matrix A is diagonalisable if and only if A has n linearly independent eigenvectors. Those eigenvectors are the columns of P.

Given our previous observation, we are interested in when a matrix has n orthogonal eigenvectors. Because any scalar multiple of an eigenvector is also an eigenvector, this is the same as asking, when does a matrix have n orthonormal eigenvectors, i.e. when is the matrix P in the Diagonalisation Theorem an orthogonal matrix?
Definition: A square matrix A is *orthogonally diagonalisable* if there is an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1}$, or equivalently, $A = PDP^T$.

We can extend the previous theorem (being careful about eigenvectors with the same eigenvalue) to show that any diagonalisable symmetric matrix is orthogonally diagonalisable, see the example on the next page.

Example: Orthogonally diagonalise $B = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$, i.e. find an orthogonal P and diagonal D with $B = PDP^{-1}$:

Answer:

Step 1 Solve the characteristic equation $\det(B - \lambda I) = 0$ to find the eigenvalues.
Eigenvalues are 2 and 5.

Step 2 For each eigenvalue λ, solve $(B - \lambda I)\mathbf{x} = \mathbf{0}$ to find a basis for the λ-eigenspace.
This gives $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ as a basis for the 2-eigenspace, and $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ as a basis for the 5-eigenspace. Notice the 2-eigenvector is orthogonal to both the 5-eigenvectors, but the two 5-eigenvectors are not orthogonal.

Step 2A For each eigenspace of dimension > 1, find an orthogonal basis (e.g. by Gram-Schmidt) Applying Gram-Schmidt to the above basis for the 5-eigenspace gives $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$. To avoid fractions, let's use $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$, which is still an orthogonal set.

Step 2B Normalise all the eigenvectors
$\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$ is an orthonormal basis for the 2-eigenspace, and $\left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}$ is an orthonormal basis for the 5-eigenspace.

Step 3 Put the normalised eigenvectors from Step 2B as the columns of P.

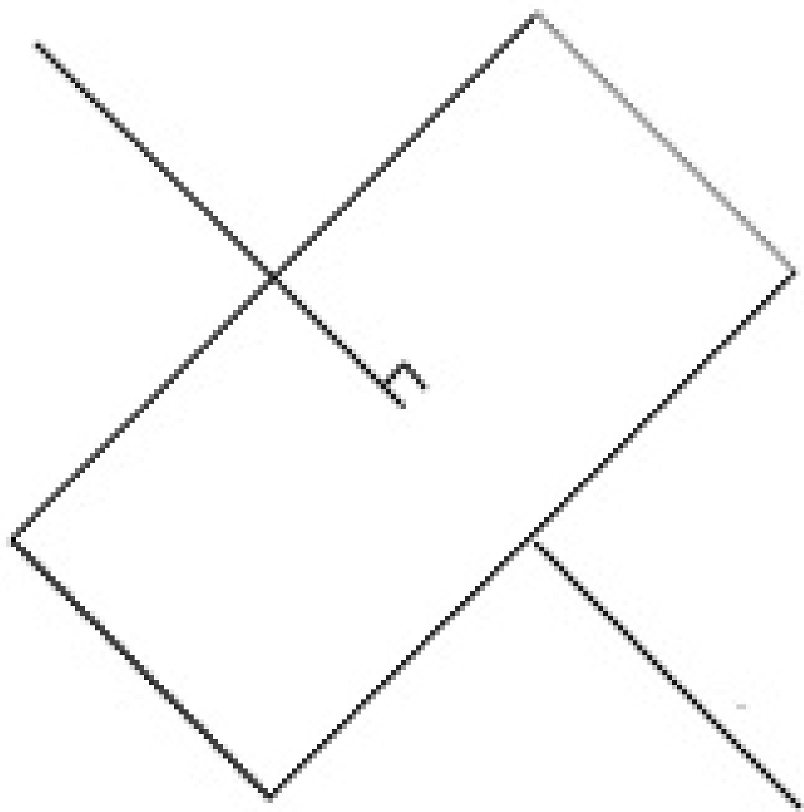
Step 4 Put the corresponding eigenvalues as the diagonal entries of D.

$$P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Check our answer:

$$PDP^T = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}.$$

A geometric illustration of "orthonormalising" the eigenvectors:

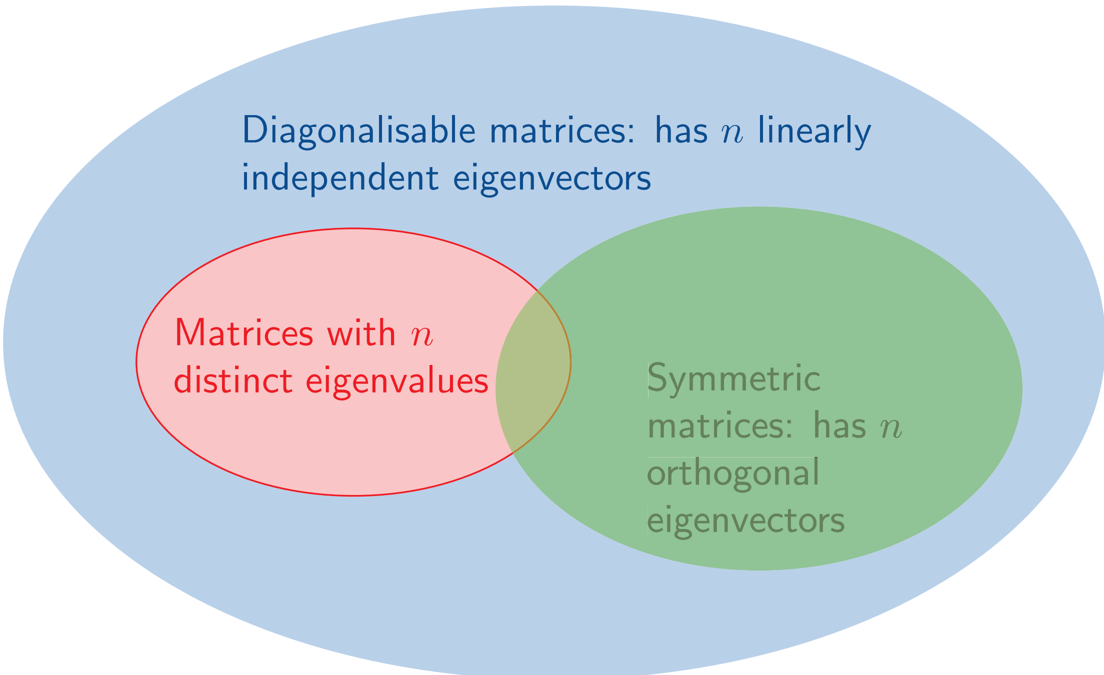


This algorithm shows that any diagonalisable symmetric matrix is orthogonally diagonalisable.
Amazingly, every symmetric matrix is diagonalisable:
Theorem 3: Spectral Theorem for Symmetric Matrices: A symmetric matrix is orthogonally diagonalisable, i.e. it has a orthonormal basis of eigenvectors.
(The name of the theorem is because the **set** of eigenvalues and multiplicities of a matrix is called its spectrum. There are spectral theorems for many types of linear transformations.)

The reverse direction is also true, and easy:
Theorem 2: Orthogonally diagonalisable matrices are symmetric: If $A = PDP^{-1}$ and P is orthogonal and D is diagonal, then A is symmetric.
Proof:

$$A^T = (PDP^{-1})^T \underset{P \text{ is orthogonal}}{=} (PDP^T)^T = (P^T)^T D^T P^T \underset{D \text{ is diagonal}}{=} PDP^T = A.$$

A diagram to summarise what we know about diagonalisation:



Non-examinable: ideas behind the proof of the spectral theorem

Because we need to work on subspaces of $\mathbb{R}^n$ in the proof, we consider self-adjoint linear transformations ($(T\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (T\mathbf{v})$) instead of symmetric matrices. So we want to show: a self-adjoint linear transformation has an orthogonal basis of eigenvectors.

The key ideas are:

1. Every linear transformation (on any vector space) has a complex eigenvector.
   Proof: Every polynomial has a solution if we allow complex numbers. Apply this to the characteristic polynomial.
2. Any complex eigenvector of a (real) self-adjoint linear transformation is a real eigenvector corresponding to a real eigenvalue. (We won't comment on the proof.)
3. Let $\mathbf{v}$ be an eigenvector of a self-adjoint linear transformation T, and $\mathbf{w}$ be any vector orthogonal to $\mathbf{v}$. Then $T(\mathbf{w})$ is still orthogonal to $\mathbf{v}$.
   Proof: $\mathbf{v} \cdot (T(\mathbf{w})) = (T(\mathbf{v})) \cdot \mathbf{w} = \lambda \mathbf{v} \cdot \mathbf{w} = \lambda 0 = 0$.

Putting these together: if $T : \mathbb{R}^n \to \mathbb{R}^n$ is self-adjoint, then by 1 and 2 it has a real eigenvector $\mathbf{v}$. Let $W = (\text{Span}\,\{\mathbf{v}\})^\perp$, the subspace of vectors orthogonal to $\mathbf{v}$. By 3, any vector in W stays in W after applying T (i.e. W is an invariant subspace under T), so we can consider the restriction $T : W \to W$, which is self-adjoint. So repeat this argument on W (i.e. use induction on the dimension of the domain of T).