Example of a strange vector space over $\mathbb{R}$:

$\mathbb{R}^2$ with strange operations:

$$\begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + x' + 1 \\ y + y' \end{pmatrix}$$

$$c \boxed{\cdot} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx + c - 1 \\ cy \end{pmatrix}$$

Check some axioms

$$(\alpha + \beta) + \gamma$$

V2 LHS: $\left[ \begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \begin{pmatrix} x' \\ y' \end{pmatrix} \right] \boxed{+} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

$$= \begin{pmatrix} x + x' + 1 \\ y + y' \end{pmatrix} \boxed{+} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$= \begin{pmatrix} (x + x' + 1) + x_1 + 1 \\ (y + y') + y_1 \end{pmatrix}$$

$$= \begin{pmatrix} x + x' + x_1 + 2 \\ y + y' + y_1 \end{pmatrix}$$

RHS: $\alpha + (\beta + \gamma)$

$$\begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \left[ \begin{pmatrix} x' \\ y' \end{pmatrix} \boxed{+} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \begin{pmatrix} x' + x_1 + 1 \\ y' + y_1 \end{pmatrix}$$

$$= \begin{pmatrix} x + (x' + x_1 + 1) + 1 \\ y + (y' + y_1) \end{pmatrix} = \text{LHS} \checkmark$$

V4 $\vec{0}$ for this vector space is NOT $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$:

$$\begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x+0+1 \\ y+0 \end{pmatrix} \neq \begin{pmatrix} x \\ y \end{pmatrix}.$$

To find $\vec{0}$: solve $\begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \begin{pmatrix} ?_1 \\ ?_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \forall x, y.$

$$\begin{pmatrix} x + ?_1 + 1 \\ y + ?_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \forall x, y.$$

$$\Rightarrow \vec{0} = \begin{pmatrix} ?_1 \\ ?_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \checkmark$$

V5: $\forall \alpha$, $\exists (-\alpha)$ such that $\alpha + (-\alpha) = \vec{0}$.

Taking $-\alpha$ to be $(-1)\alpha$ always works.

$$\begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \left( -1 \boxed{\cdot} \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \begin{pmatrix} (-1)x + (-1) - 1 \\ -1y \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \begin{pmatrix} -x-2 \\ -y \end{pmatrix}$$

$$= \begin{pmatrix} x + (-x-2) + 1 \\ y + (-y) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \vec{0} \quad \checkmark$$

Exercise: check V7 – V10.

Non-example: strange operations that don't make a vector space:

$$\begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \max\{x, x'\} \\ \max\{y, y'\} \end{pmatrix}$$

$$\text{e.g. } \begin{pmatrix} 2 \\ 3 \end{pmatrix} \boxed{+} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

$$c \boxed{\cdot} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ cy \end{pmatrix}$$

V2, V3 are true (check yourself)

V4 does not hold:

$\vec{0} = \begin{pmatrix} a \\ b \end{pmatrix}$ must satisfy

$$\begin{pmatrix} x \\ y \end{pmatrix} \boxed{+} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \forall x, y \in \mathbb{R}.$$

i.e. $$\begin{pmatrix} \max\{x, a\} \\ \max\{y, b\} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

i.e. $a \leq x \quad \forall x \in \mathbb{R}$ — no $a, b$ satisfies this
$b \leq y \quad \forall y \in \mathbb{R}$

$\therefore$ this set does not contain a $\vec{0}$.

---

**Lemma 6.1.2 : Cancellation law**
small theorem

if $\alpha + \beta = \alpha + \gamma$, then $\beta = \gamma$.

Proof:

$$\alpha + \beta = \alpha + \gamma$$

Add $-\alpha$: $\quad -\alpha + (\alpha + \beta) = -\alpha + (\alpha + \gamma)$

V2: $\quad (-\alpha + \alpha) + \beta = (-\alpha + \alpha) + \gamma$

V3: $(\alpha+(-\alpha))+\beta=(\alpha+(-\alpha))+\gamma$

V5: $\vec{0}+\beta=\vec{0}+\gamma$

V4: $\beta=\gamma$

Corollary 6.1.3: if $\alpha+\beta=\alpha$, then $\beta=\vec{0}$.

follows from other results

(A way to prove something is the zero vector)

Proof: we know $\alpha+\beta=\alpha\underset{V4}{=}\alpha+\vec{0}$

Now use cancellation law with $\gamma=\vec{0}$.

Lemma 6.1.4 a,c multiply-by-zero law:

a. $\forall\alpha\in V$, $0\alpha=\vec{0}$

c. $\forall a\in\mathbb{F}$, $a\vec{0}=\vec{0}$

Proof: by 6.1.3, it's enough to show

$$\gamma + 0\alpha = \gamma \quad \text{and} \quad \gamma + a\vec{0} = \gamma \quad \forall \alpha \in V,\ a \in F,$$

and some $\gamma \in V$.

a. $(\gamma = 0\alpha)$

$$\begin{aligned} 0\alpha + 0\alpha &= (0+0)\alpha & V9 \\ &= 0\alpha \end{aligned}$$

c. $(\gamma = a\vec{0})$

$$\begin{aligned} a\vec{0} + a\vec{0} &= a(\vec{0}+\vec{0}) & V8 \\ &= a\vec{0} & V4 \end{aligned}$$