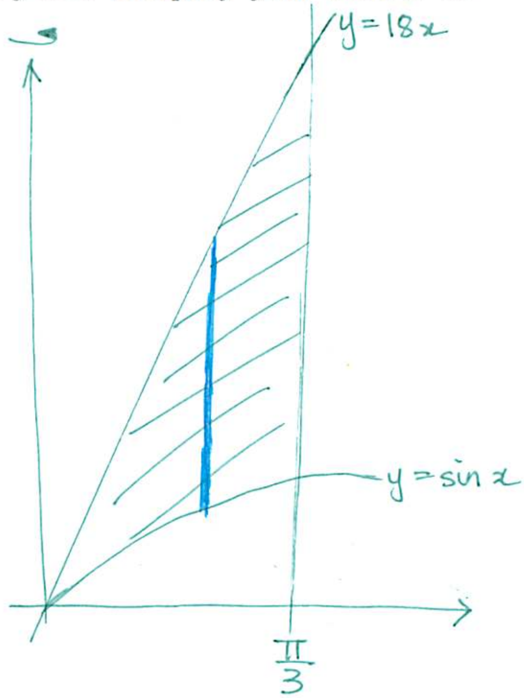
1. (7 points) Let R be the region bounded by the curves

$$y = \sin x, \quad y = 18x, \quad x = \frac{\pi}{3}.$$

Find the volume of the solid obtained by rotating R about the y-axis. **Simplify your answer as much as possible.**



Use cylindrical shells:

$$\text{Volume} = \int_0^{\frac{\pi}{3}} 2\pi x(18x - \sin x)\,dx$$

$$= \int_0^{\frac{\pi}{3}} 36\pi x^2 - 2\pi x \sin x \, dx$$

$$= \left[36\pi \frac{x^3}{3}\right]_0^{\frac{\pi}{3}} - 2\pi \int_0^{\frac{\pi}{3}} x \sin x \, dx$$

$$= 12\pi\left(\frac{\pi}{3}\right)^3 - 2\pi \int_0^{\frac{\pi}{3}} x \sin x \, dx$$

Integration by parts:

$$\int_0^{\frac{\pi}{3}} x \sin x \, dx = \Big[x(-\cos x)\Big]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} -\cos x \, dx \qquad \begin{array}{ll} u = x & dv = \sin x\,dx \\ du = dx & v = -\cos x \end{array}$$

$$= \frac{\pi}{3}\left(-\frac{1}{2}\right) - 0 - \Big[-\sin x\Big]_0^{\frac{\pi}{3}}$$

$$= -\frac{\pi}{6} - \left(-\frac{\sqrt{3}}{2} + 0\right) = -\frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

So $\text{Volume} = \dfrac{12\pi^4}{3^3} - 2\pi\left(-\dfrac{\pi}{6} + \dfrac{\sqrt{3}}{2}\right) = \dfrac{4\pi^4}{9} + \dfrac{\pi^2}{3} - \sqrt{3}\,\pi$

2. (7 points) Compute the following integral:

$$\int \frac{x+4}{x(x-1)^2}dx.$$

Partial fractions:

$$\frac{x+4}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$x+4 = A(x-1)^2 + Bx(x-1) + Cx$$

$x=1$: $\quad 5 = C \quad \Rightarrow C=5$

$x=0$: $\quad 4 = A(-1)^2 \quad \Rightarrow A=4$

coeff of x^2: $\quad 0 = A+B \quad \Rightarrow B=-4$

$$\therefore \int \frac{x+4}{x(x-1)^2}dx = \int \frac{4}{x} - \frac{4}{x-1} + \frac{5}{(x-1)^2}dx$$

$$= 4\ln|x| - 4\ln|x-1| + 5\frac{(x-1)^{-1}}{-1} + C$$

$$= 4\ln|x| - 4\ln|x-1| - \frac{5}{x-1} + C$$

3. (7 points) Compute the following integral:

$$\int (x^2+9)^{-\frac{5}{2}}\,dx.$$

substitution:

$x = 3\tan\theta$

$dx = 3\sec^2\theta$

$x^2+9 = 9\tan\theta + 9$

$= 9\sec^2\theta$

$$= \int (3\sec\theta)^{-5}(3\sec^2\theta)\,d\theta$$

$$= \int 3^{-4}(\sec\theta)^{-3}\,d\theta$$

$$= \int \frac{1}{81}\cos^3\theta\,d\theta$$

$$= \int \frac{1}{81}(1-\sin^2\theta)\cos\theta\,d\theta$$

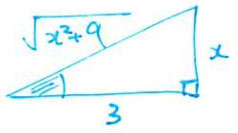
substitution

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= \frac{1}{81}\left(\sin\theta - \frac{\sin^3\theta}{3}\right) + C$$

$$= \frac{1}{81}\frac{x}{\sqrt{x^2+9}} - \frac{1}{243}\left(\frac{x}{\sqrt{x^2+9}}\right)^3 + C$$



$\frac{x}{3} = \tan\theta = \frac{\text{opp}}{\text{adj}}$

$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{x^2+9}}$