1. (7 points) Compute the following two improper integrals, or explain why they do not converge. **Simplify your answer as much as possible.**

(a)

$$\int_{-\infty}^{-1} \frac{e^{1/x}}{x^2}\,dx.$$

$$= \lim_{t\to-\infty} \int_t^{-1} \frac{e^{1/x}}{x^2}\,dx$$

$u = \frac{1}{x}$

$du = \frac{-1}{x^2}\,dx$

$$= \lim_{t\to-\infty} \left[-e^{1/x}\right]_t^{-1}$$

$$= \lim_{t\to-\infty} \left(-e^{-1} + e^{1/t}\right)$$

$$= -\frac{1}{e} + e^0$$

$$= -\frac{1}{e} + 1 = \frac{e-1}{e}$$

(b)

$$\int_0^1 \frac{e^{1/x}}{x^2}\,dx.$$

$$= \lim_{t\to0^+} \int_t^1 \frac{e^{1/x}}{x^2}\,dx$$

$$= \lim_{t\to0^+} \left[-e^{1/x}\right]_t^1$$

$$= \lim_{t\to0^+} \left(-e^1 + e^{1/t}\right)$$

as $t\to0^+$

$1/t \to \infty$

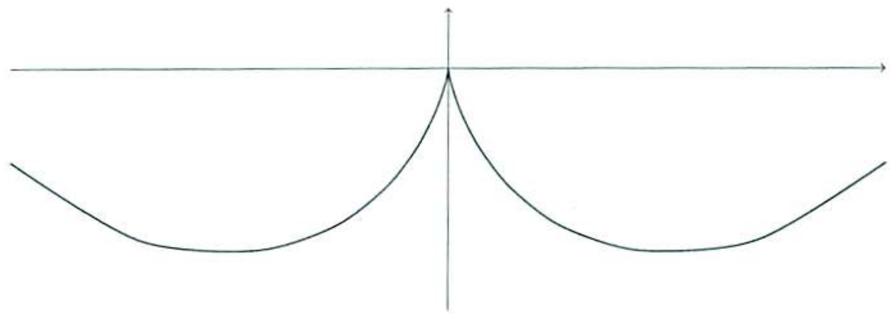
so $e^{1/t} \to \infty$

$\therefore$ this integral diverges.

2. (14 points) Let C be the parametrised curve with equation

$$x = \frac{6}{5}t^5, \quad y = \frac{t^6}{6} - \frac{9}{4}t^4,$$

as shown in the diagram below.



(a) Find the point(s) where C has a horizontal tangent. **Simplify your answer as much as possible.**

C has a horizontal tangent when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

$\frac{dy}{dt} = 0$ when $t^5 - 9t^3 = 0$

$t^3(t^2 - 9) = 0$

$t = 0$ or $t = 3$ or $t = -3$.

When $t=0$: $\frac{dx}{dt} = 6t^4 = 0$ $\quad$ $t=0$ corresponds to the point $(0,0)$, and from the picture we can see that there is no horizontal tangent here.

when $t=3$: $\frac{dx}{dt} = 6(3)^4 \neq 0$

when $t=-3$: $\frac{dx}{dt} = 6(-3)^4 \neq 0$

$\Big\}$ $\therefore$ these do give horizontal tangents.

$t=3 \Rightarrow x = \frac{6}{5} 3^5 = \frac{1458}{5}, \quad y = \frac{3^6}{6} - \frac{9(3)^4}{4} = 3^6\left(\frac{1}{6} - \frac{1}{4}\right) = -\frac{243}{4}$

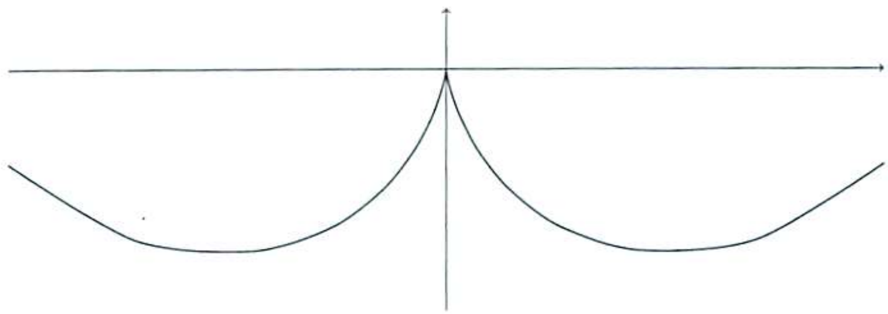
$t=-3 \Rightarrow x = \frac{6}{5}(-3)^5 = \frac{-1458}{5}, \quad y = \frac{(-3)^6}{6} - \frac{9(-3)^4}{4} = \frac{-243}{4}$

$\therefore$ C has horizontal tangents at $\left(\frac{1458}{5}, \frac{-243}{4}\right)$ and $\left(\frac{-1458}{5}, \frac{-243}{4}\right)$

(Question 2 continues on the next page.)

(b) For your convenience, here again is the information about the parametrised curve C:

$$x = \frac{6}{5}t^5, \quad y = \frac{t^6}{6} - \frac{9}{4}t^4.$$



Find the length of the part of C with $-2 \leq t \leq -1$. **Simplify your answer as much as possible.**

$$\text{length} = \int_{-2}^{-1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_{-2}^{-1} \sqrt{(6t^4)^2 + (t^5 - 9t^3)^2}\, dt$$

$$= \int_{-2}^{-1} \sqrt{36t^8 + t^{10} - 18t^8 + 81t^6}\, dt$$

$$= \int_{-2}^{-1} \sqrt{t^{10} + 18t^8 + 81t^6}\, dt$$

$$= \int_{-2}^{-1} \sqrt{t^6(t^2+9)^2}\, dt$$

$$= \int_{-2}^{-1} |t^3||t^2+9|\, dt \qquad \text{for } -2 \leq t \leq -1,\ t^2+9 > 0,\ t^3 < 0$$

$$= \int_{-2}^{-1} -t^3(t^2+9)\, dt$$

$$= \left[-\frac{t^6}{6} - \frac{9t^4}{4}\right]_{-2}^{-1}$$

$$= \left(-\frac{1}{6} - \frac{9}{4}\right) - \left(-\frac{64}{6} - \frac{9}{4}16\right) = \frac{177}{4}.$$