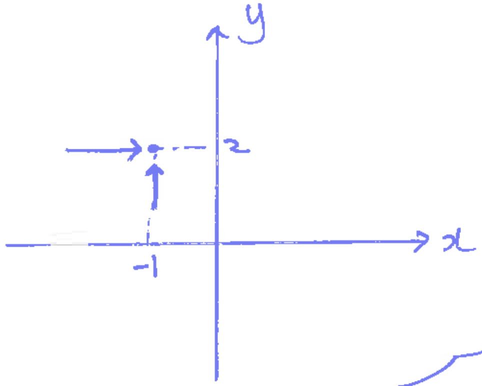
Example: Show that the limit $\displaystyle\lim_{(x,y)\to(-1,2)} \frac{x^2-1}{4x^2-y^2}$ does not exist.



along the path $y=2$: $\displaystyle\lim_{x\to-1} \frac{x^2-1}{4x^2-2^2} = \lim_{x\to-1} \frac{x^2-1}{4(x^2-1)} = \frac{1}{4}$

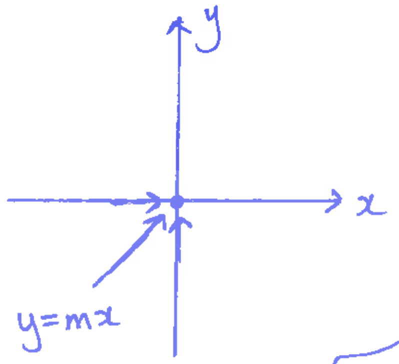
$x(t)=t,\ y(t)=2$

along the path $x=-1$: $\displaystyle\lim_{y\to 2} \frac{(-1)^2-1}{4(-1)^2-y^2} = \lim_{y\to2} \frac{0}{4-y^2} = 0 \neq \frac{1}{4}$

$x(t)=-1,\ y(t)=t$

So the function has different limits along two paths, so the 2D limit doesn't exist.

Example: Show that the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^3}$ does not exist.



along the path $y=0$: $\displaystyle\lim_{x\to 0} \frac{x^2\cdot 0}{x^3+0^3} = \lim_{x\to 0}\frac{0}{x^3} = 0$

along the path $x=0$: $\displaystyle\lim_{y\to 0} \frac{0^2 y}{0^3+y^3} = \lim_{y\to 0}\frac{0}{y^3} = 0$

same limit as for the path $y=0$

$\therefore$ not helpful

along a path of the form $y=mx$: $\displaystyle\lim_{x\to 0} \frac{x^2(mx)}{x^3+(mx)^3} = \lim_{x\to 0}\frac{mx^3}{(1+m^3)x^3} = \frac{m}{1+m^3}$

$\therefore$ choose m later so that the limit along this path is not 0.

e.g. when $m=1$, this limit is $\frac{1}{2} \neq 0$.

so the 2D limit doesn't exist.

Example: Show that the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4}{x^2y+y^2}$ does not exist.

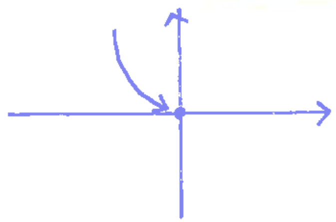
~~along the path $y=0$: $\displaystyle\lim_{x\to 0}\frac{x^4}{x^2\cdot 0+0^2}$~~

can't use the path $y=0$ because it is not in the domain (function is not defined when $y=0$)

along the path $x=0$: $\displaystyle\lim_{y\to 0}\frac{0^4}{0^2y+y^2} = \lim_{y\to 0}\frac{0}{y^2} = 0$

along the path $y=mx$: $\displaystyle\lim_{x\to 0}\frac{x^4}{x^2(mx)+(mx)^2} = \lim_{x\to 0}\frac{x^4}{mx^3+m^2x^2}$

$\displaystyle = \lim_{x\to 0}\frac{x^2}{mx+m^2} = 0$ if $m\neq 0$.



along the path $y=x^2$: $\displaystyle\lim_{x\to 0}\frac{x^4}{x^2(x^2)+(x^2)^2} = \lim_{x\to 0}\frac{x^4}{2x^4} = \frac{1}{2} \neq 0$

so the limit doesn't exist.

Now we give some strategies for showing that a 2-variable limit does exist. This will use the concept of continuity, which has the same definition as the 1D case.

Definition: An n-variable function $\mathbf{f} : \mathcal{D} \to \mathbb{R}^m$ is *continuous* at a point $(a_1, \ldots, a_n)$ in the domain $\mathcal{D}$ if

$$\lim_{(x_1,\ldots x_n)\to(a_1,\ldots,a_n)} \mathbf{f}(x_1, \ldots, x_n) = \mathbf{f}(a_1, \ldots, a_n).$$

As in the 1D case, elementary functions (i.e. sums, products and compositions of $x^n, e^x, \ln x, \sin x, \cos x$) are continuous. So the following example is easy:

Example: Evaluate the limit $\displaystyle\lim_{(x,y)\to(-1,2)} \frac{x^2-2}{y^2-1}$, or prove that it does not exist.

$\frac{x^2-2}{y^2-1}$ is a continuous function, and $(-1,2)$ is in the domain,

so the limit is $\frac{(-1)^2-2}{2^2-1} = \frac{-1}{3}$

Here's a simple 2-dimensional version of the standard 1D squeeze theorem example (see Homework 3 final question).

Example: Evaluate $\lim_{(x,y)\to(0,0)} xy^2 \sin\left(\frac{1}{y}\right)$, or prove that the limit does not exist.

$$\left|xy^2 \sin\left(\tfrac{1}{y}\right)\right| = |xy^2|\left|\sin\left(\tfrac{1}{y}\right)\right| \leq |xy^2|$$ because the image of sin is between -1 and 1

$|xy^2|$ is continuous so $\lim_{(x,y)\to(0,0)} |xy^2| = |0\cdot 0^2| = 0$.

By squeeze theorem, $\lim_{(x,y)\to(0,0)} xy^2 \sin\left(\frac{1}{y}\right) = 0$.

Most applications of the Squeeze Theorem in 2D don't involve trigonometric functions, and look more like this next example:

Example: Evaluate the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{yx^2}{x^2+y^2}$, or prove that it does not exist.

Since we don't know if this has a limit, try some paths first:

along $y=0$: $\displaystyle\lim_{x\to 0} \frac{0}{x^2} = 0$

along $x=0$: $\displaystyle\lim_{y\to 0} \frac{0}{y^2} = 0$

along $y=mx$: $\displaystyle\lim_{x\to 0} \frac{mx^3}{x^2+m^2x^2} = \lim_{x\to 0} \frac{mx}{1+m^2} = 0$

along $y=x^n$ $(n>1)$: $\displaystyle\lim_{x\to 0} \frac{x^n x^2}{x^2+x^{2n}} = \lim_{x\to 0} \frac{x^n}{1+x^{2n-2}} = 0$ (if $n>1$)

maybe the limit is 0

Try to use squeeze theorem to prove that the limit is 0: i.e. need $g(x,y)$ such that $\left|\frac{yx^2}{x^2+y^2}\right| \leq g(x,y)$ and $\displaystyle\lim_{(x,y)\to(0,0)} g(x,y) = 0$

i.e. $|yx^2| \leq g(x,y)(x^2+y^2) = g(x,y)x^2 + g(x,y)y^2$

so if $g(x,y)$ is $|y|$: $\quad |yx^2| + \text{extra}$

$|y|x^2 \leq |y|(x^2+y^2)$

$\left|\frac{yx^2}{x^2+y^2}\right| = \frac{|y|x^2}{x^2+y^2} \leq |y|$

and $|y|$ is continuous

so $\displaystyle\lim_{(x,y)\to(0,0)} |y| = |0| = 0$

so by squeeze theorem, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{yx^2}{x^2+y^2} = 0$.