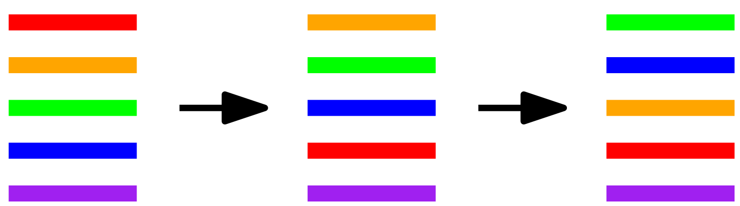
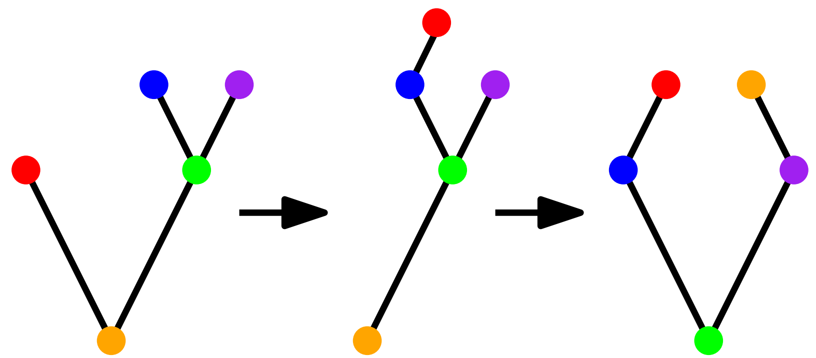
# A Uniform Analysis of Combinatorial Markov Chains via Hopf Algebras





C.Y. Amy Pang

LaCIM, Université du Québec à Montréal

presented at Hong Kong Baptist University, 19 Jan 2016

slides available at amypang.github.io/hkbu2016.pdf

# Motivation: a Dynamic Storage Allocation Problem

- You have $n$ files, arranged in a list.
- You request files one-by-one independently, removing one from the list and returning it in a possibly different position.
- You request file $i$ with a fixed, unknown, probability $p_i$.
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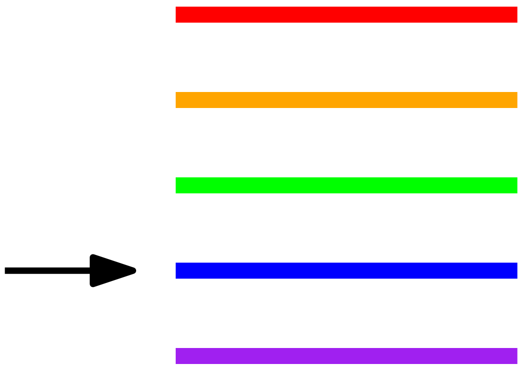
Question: where should you return a file to minimise the average search time?

# Two Answers by McCabe (1965)

Starting list



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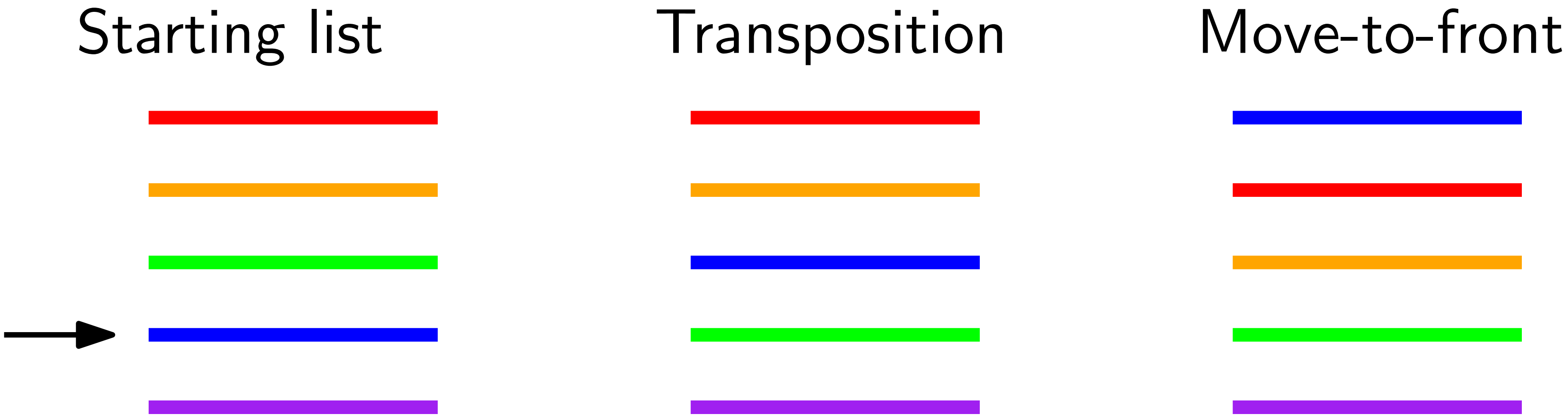

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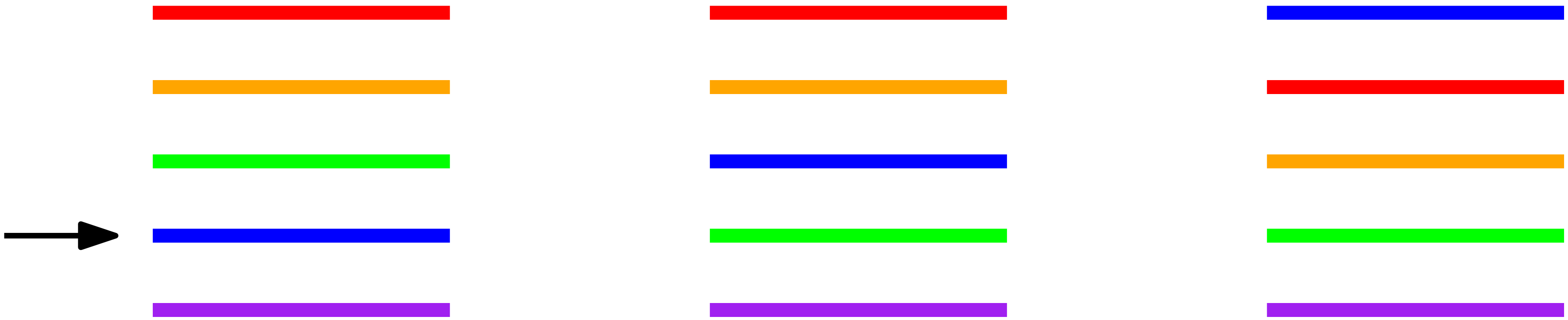

Stationary distribution: limiting probability of being in rainbow order

$${p_1}^4 {p_2}^3 {p_3}^2 p_4$$

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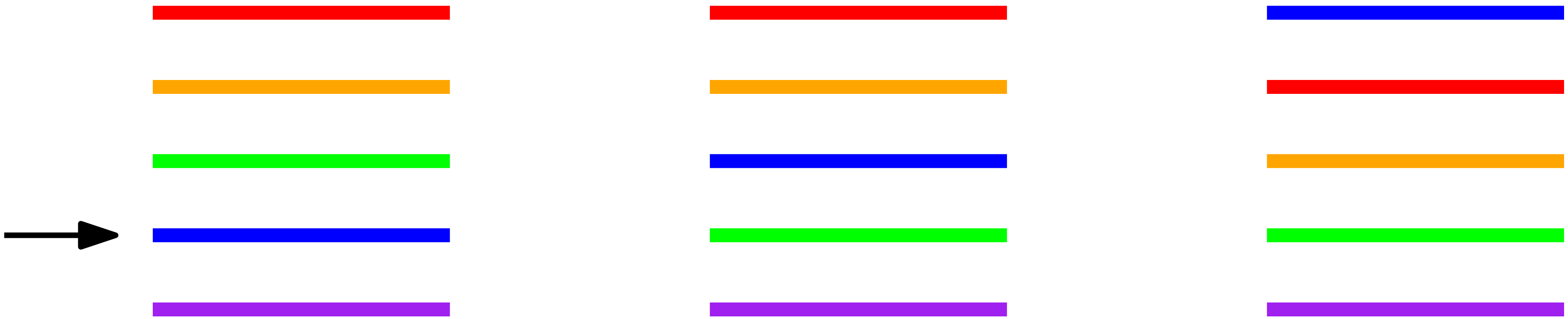

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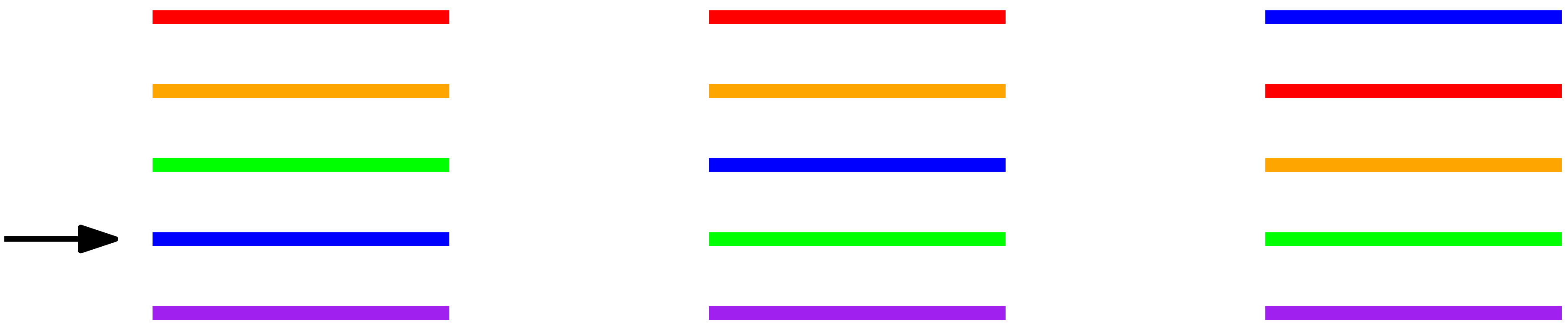

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Rivest (1976): lower average search time when in stationary distribution

Bitner (1979): reaches stationary distribution earlier

# Markov Chains

- $\mathcal{X}$ a (finite) state space. all possible orders of $n$ files
- $X_t$ a random variable taking values in $\mathcal{X}$, for each $t \in \mathbb{N}$.

  the order of the files after $t$ requests
- The process $\{X_t\}$ is memoryless, in that $\text{Prob}(X_{t+1} = y | X_t = x)$ is a number $K(x, y)$ independent of $X_1, X_2, ... X_{t-1}$ and of $t$.

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Important questions:

- Stationary distribution: $\sum_{x \in \mathcal{X}} \pi_x K(x, y) = \pi_y$.

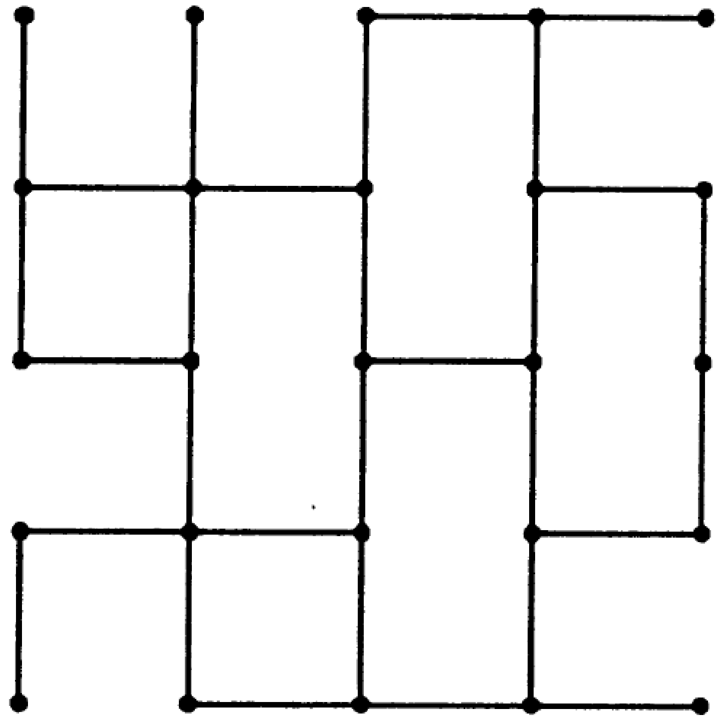
  eigenvector of eigenvalue 1
- Convergence rate: $||X_t - \pi|| \leq \epsilon$.

  subdominant eigenvalue (spectral gap)

# More applications of Markov Chains

To model a process:

- Exclusion process (Quastel 1992, Diaconis, Saloff-Coste 1993)
- DNA sequences (Ching, Fung, Ng 2004)



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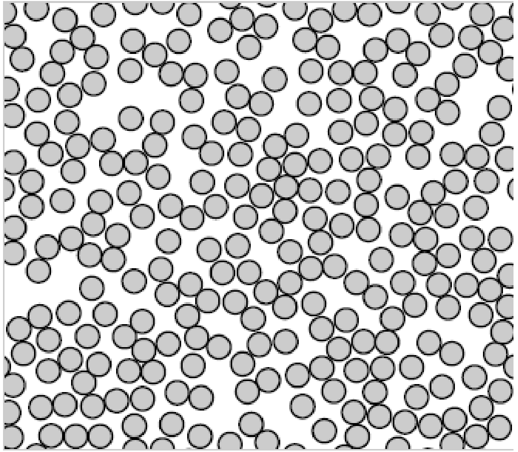
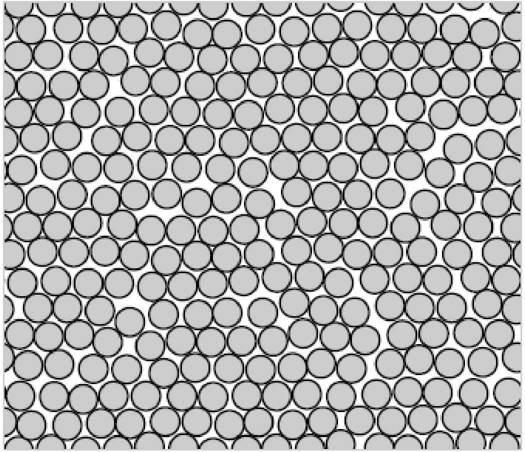


$\eta = 0.48$

(a) Low density



$\eta = 0.72$

(b) High density

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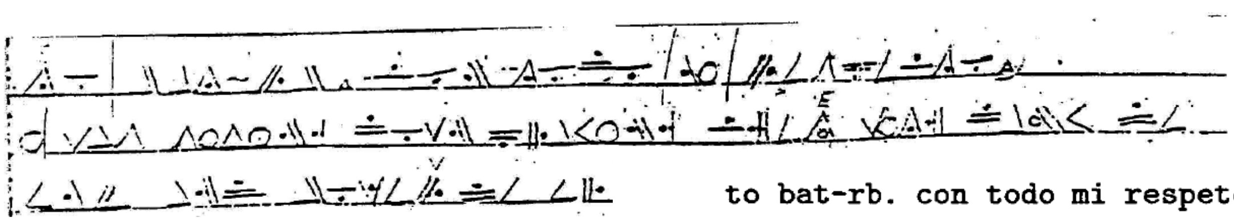
To obtain good approximations to optimisation problems:

- Data augmentation (Tanner, Wong 1987)
- Decoding prisoner communication (Connor 2003)



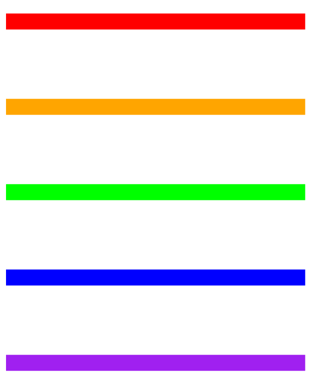
to bat-rb. con todo mi respeto. i was sitting down playing chess with danny de emf and boxer de el centro was sitting next to us. boxer was making loud and loud voices so i tell him por favor can you kick back homie cause im playing chess a minute later the vato starts back up again so this time i tell him con respecto homie can you kick back. the vato

# The Top-to-Random Shuffle

(time-reversal of move-to-front with equal request probabilities)

- Remove top card
- Reinsert this card at a uniformly chosen position

For example:



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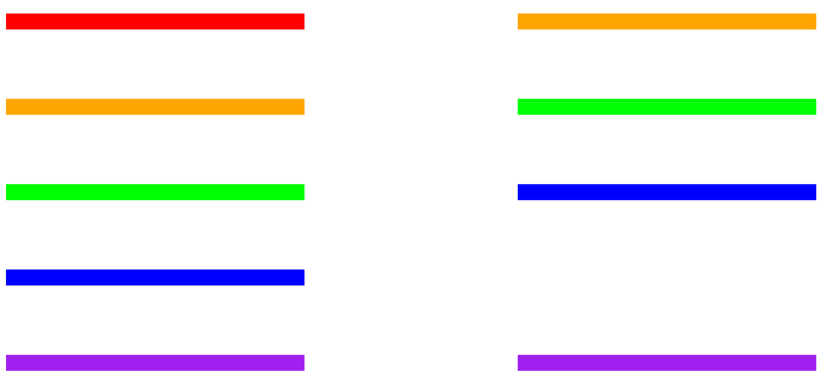

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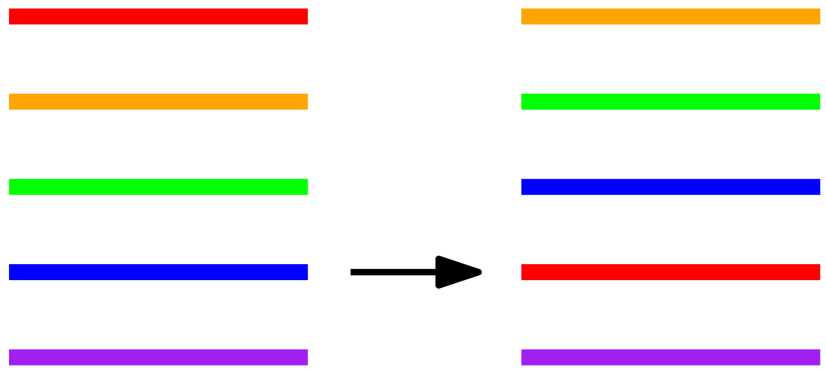

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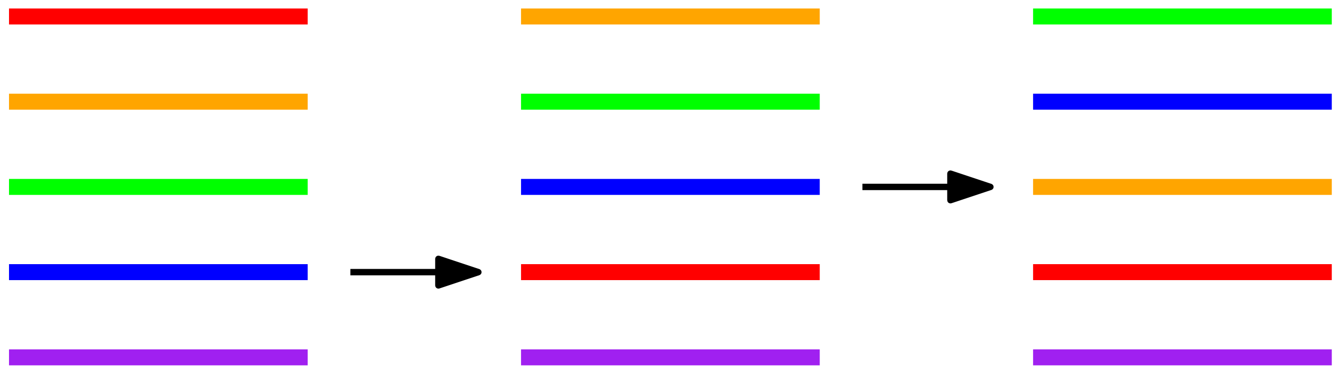

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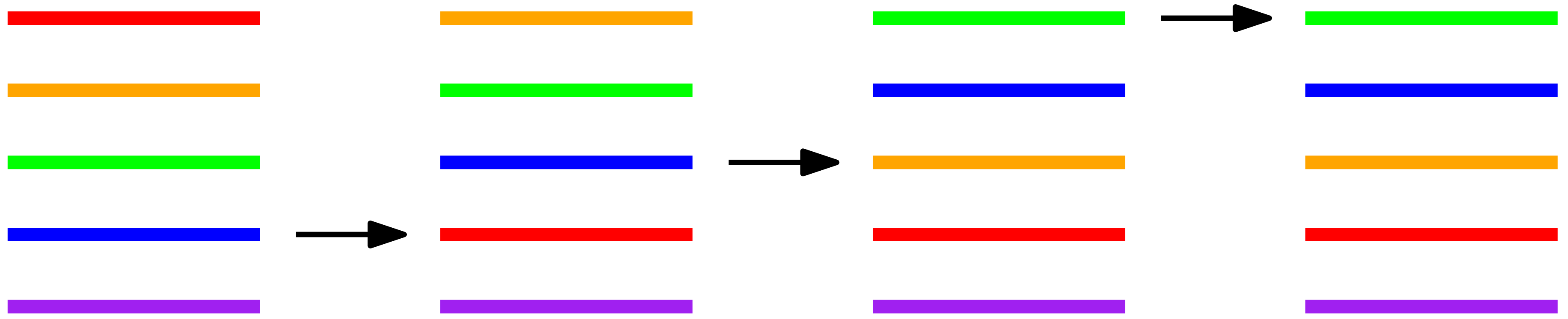

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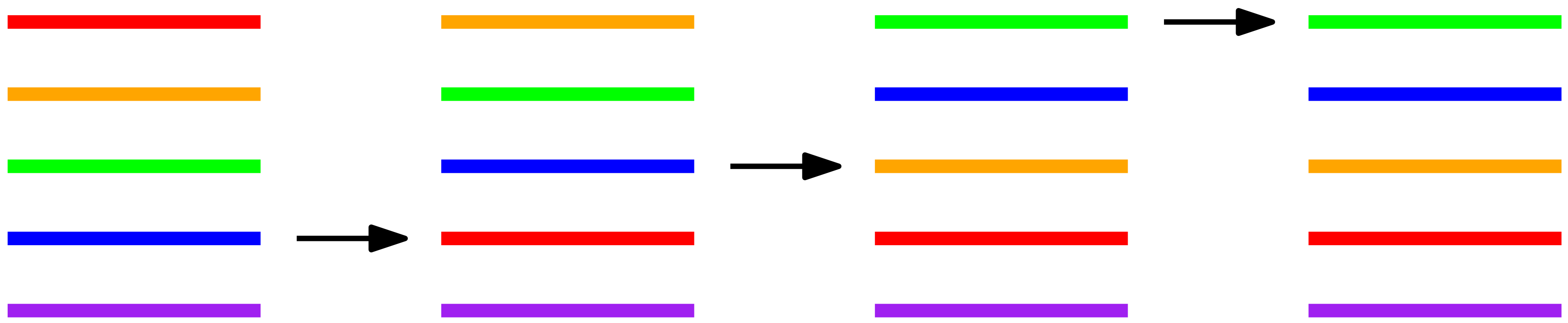
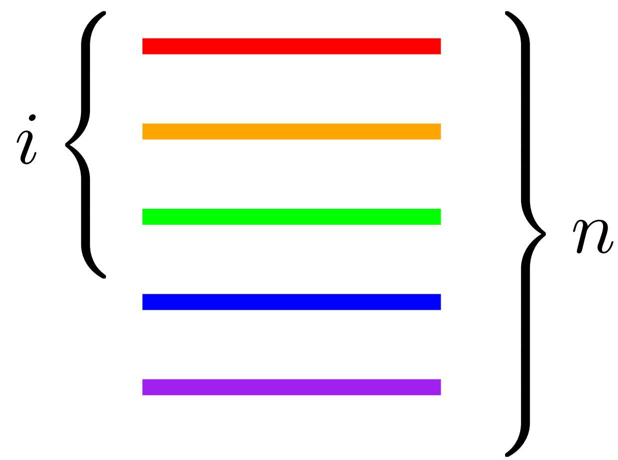


Aldous-Diaconis (1986): convergence rate $\sim n \log n$.

(asymptotically in $n$; 205 when $n = 52$).

# The Riffle Shuffle

- Cut the deck with symmetric binomial distribution;




$$\text{Prob} = 2^{-n}\binom{n}{i}$$

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Prob $= \frac{3}{5}$

Prob $= \frac{2}{5}$

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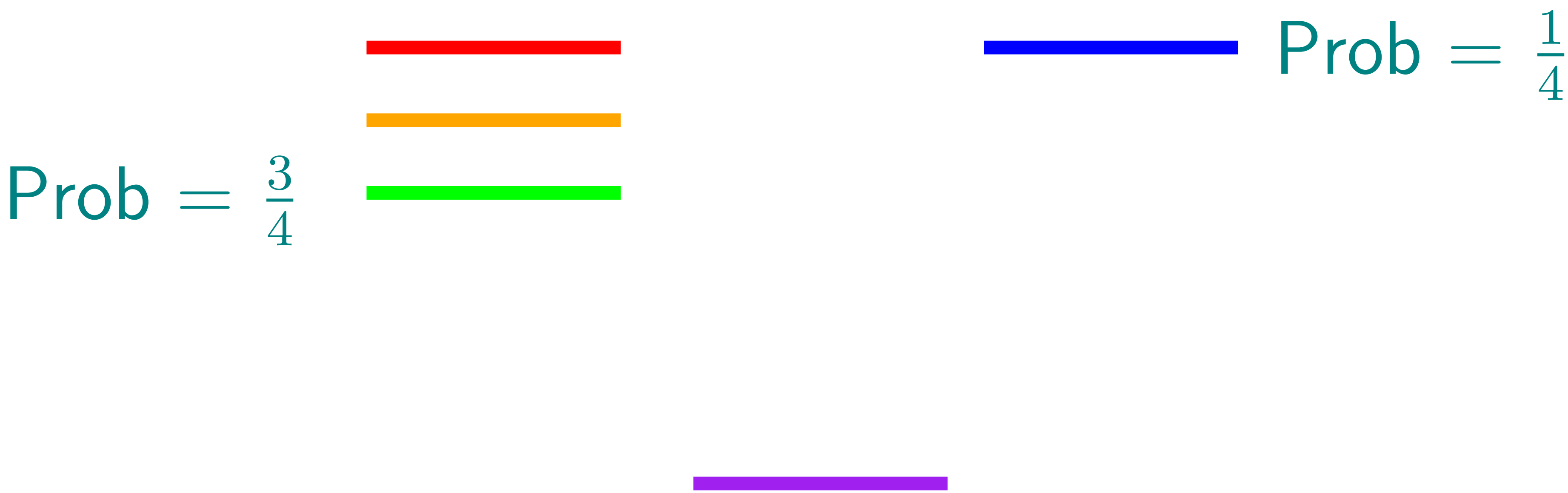

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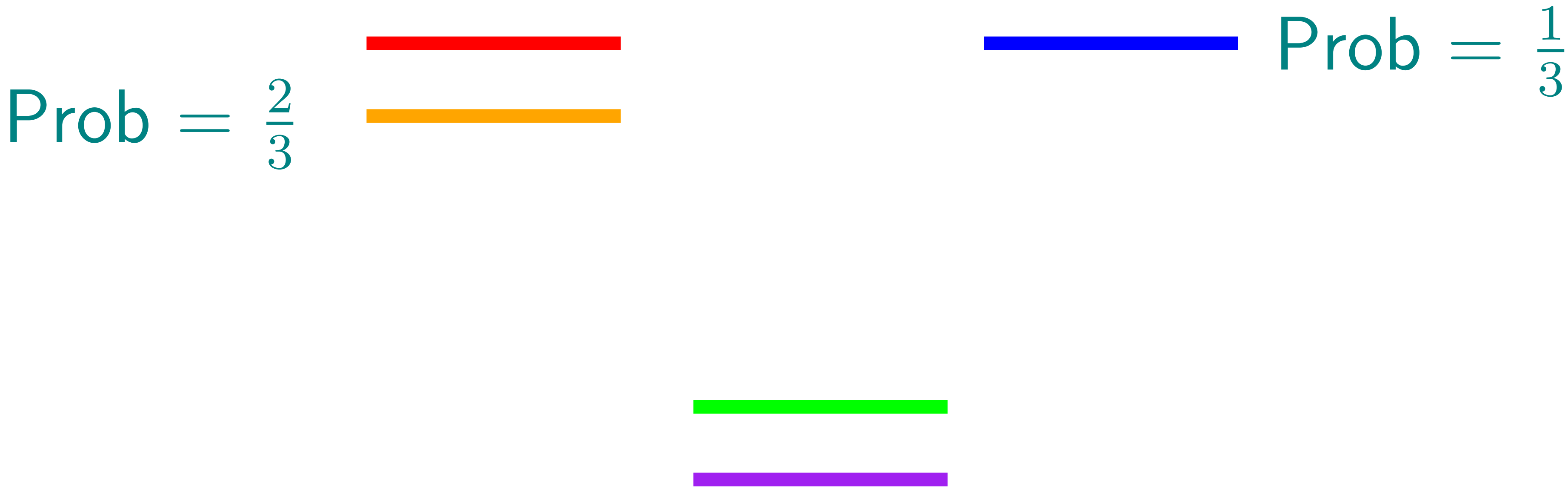

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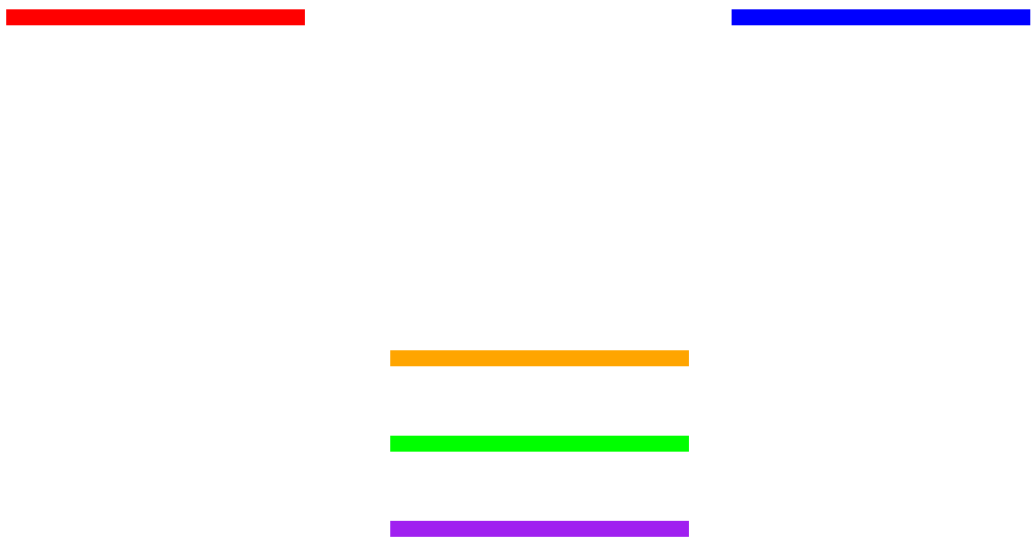
$\text{Prob} = \frac{1}{2}$



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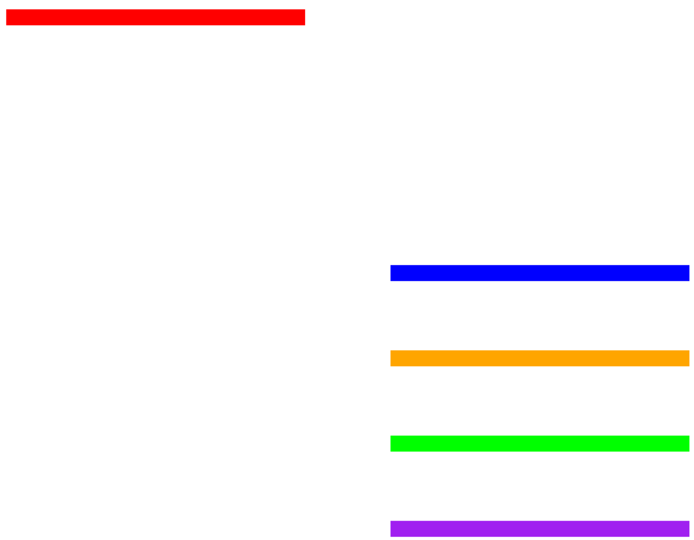
Prob $= \frac{1}{1}$



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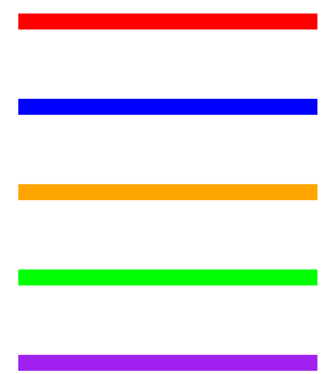

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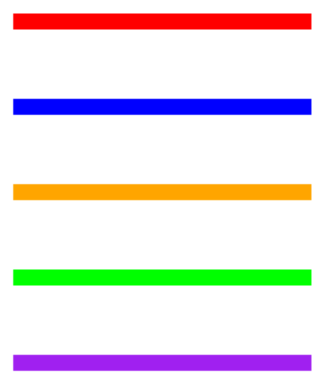

Bayer-Diaconis (1992): convergence rate $\sim \frac{3}{2}\log_2 n$.

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# Analysing Shuffles using Maps on Hopf Algebras

| | Top-to-random | Riffle |
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| **Theorem** : | (2015) | (+Diaconis, Ram 2014) |
| extensions of results by: | Diaconis-Fill-Pitman (1992) | Bayer-Diaconis (1992), Hanlon (1990) |

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| Multiplicities of eigenvalues, for all cards distinct | number of permutations of $n$ cards with $j$ fixed points. | number of permutations of $n$ cards with $j$ cycles. |

# Analysing Shuffles using Maps on Hopf Algebras

- An algorithm to compute an eigenbasis.

**Corollary** (2015): Start with $n$ distinct cards in ascending order. After $t$ top-to-random shuffles:

$$\text{Prob (descent at the bottom)} = \left(1 - \left(\frac{n-2}{n}\right)^t\right)\frac{1}{2}.$$

big card on small card

$$\text{Prob (peak at the bottom)} = \left(1 - \left(\frac{n-3}{n}\right)^t\right)\frac{1}{3}.$$

triple of cards with biggest in middle

## Analysing Shuffles using Maps on Hopf Algebras

- An algorithm to compute an eigenbasis.

**Corollary** (+Diaconis, Ram, 2014): Start with $n$ distinct cards in ascending order. After $t$ riffle shuffles:

Expect (number of descents) $= \left(1 - \left(\frac{1}{2}\right)^t\right)\frac{n-1}{2}$.

Expect (number of peaks) $= \left(1 - \left(\frac{1}{4}\right)^t\right)\frac{n-2}{3}$.

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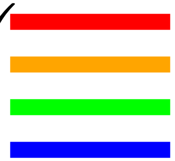
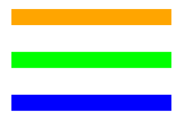
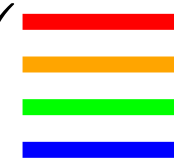
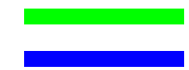
$\Delta_{1,3}\Bigg($  $\Bigg) =$  $\otimes$  ; $\Delta_{2,2}\Bigg($  $\Bigg) =$  $\otimes$ 

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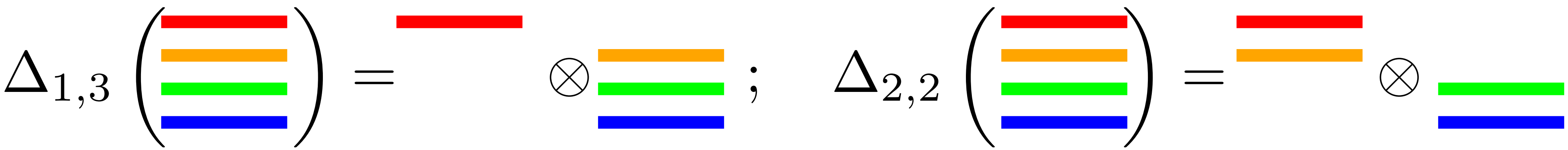

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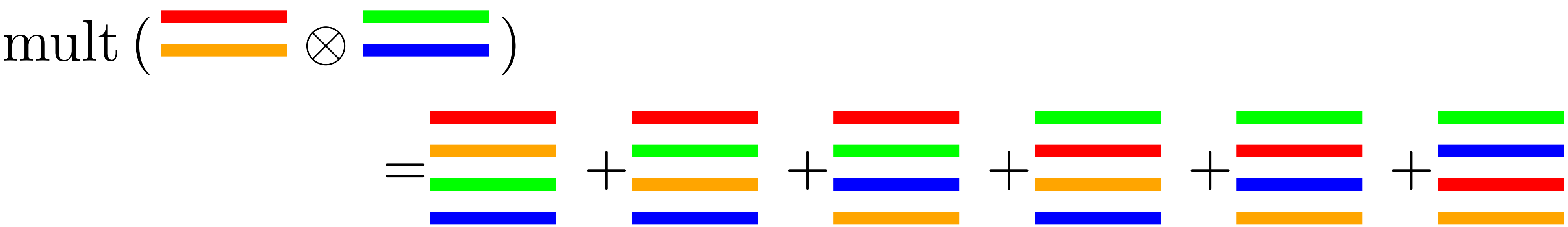

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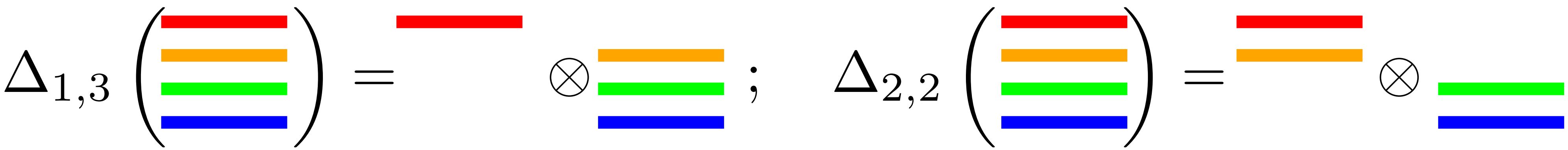

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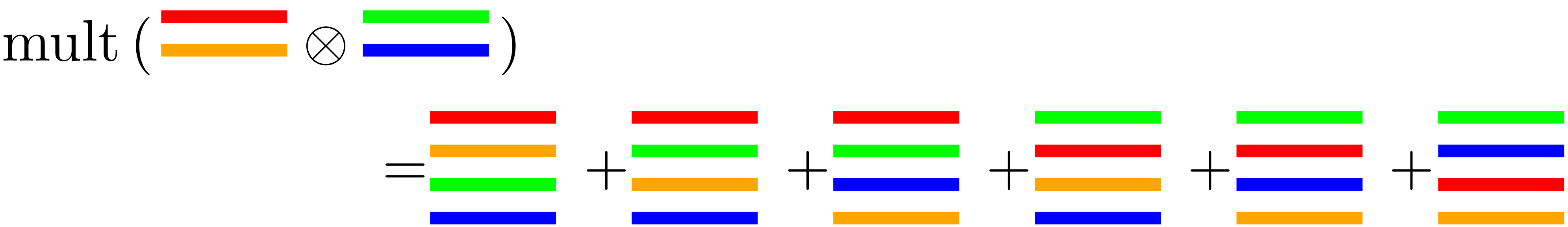

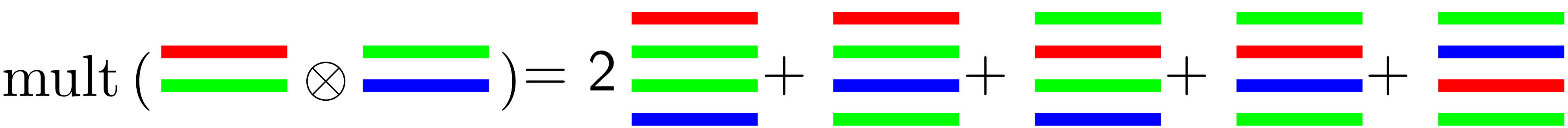

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**Top-to-random:**

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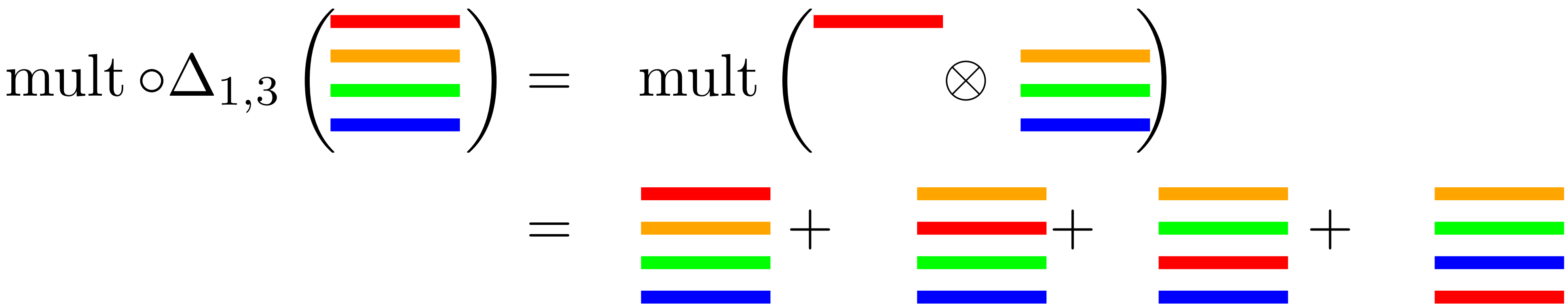

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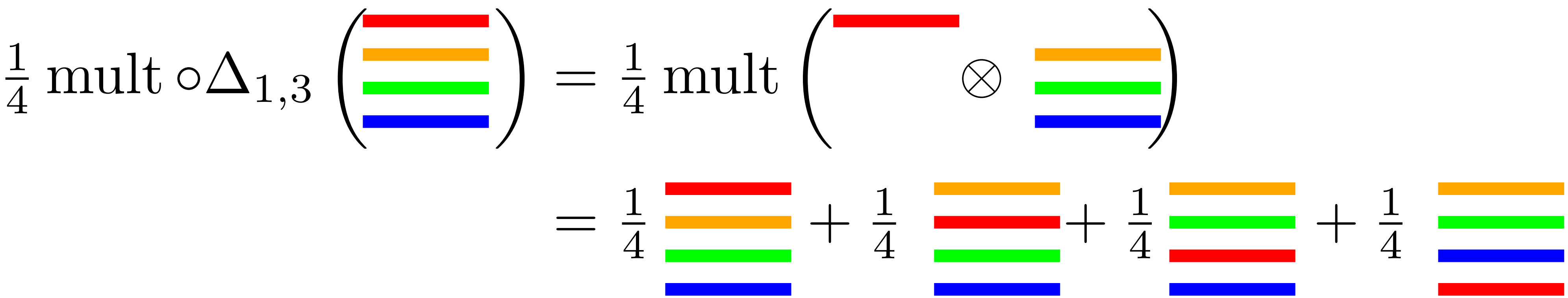

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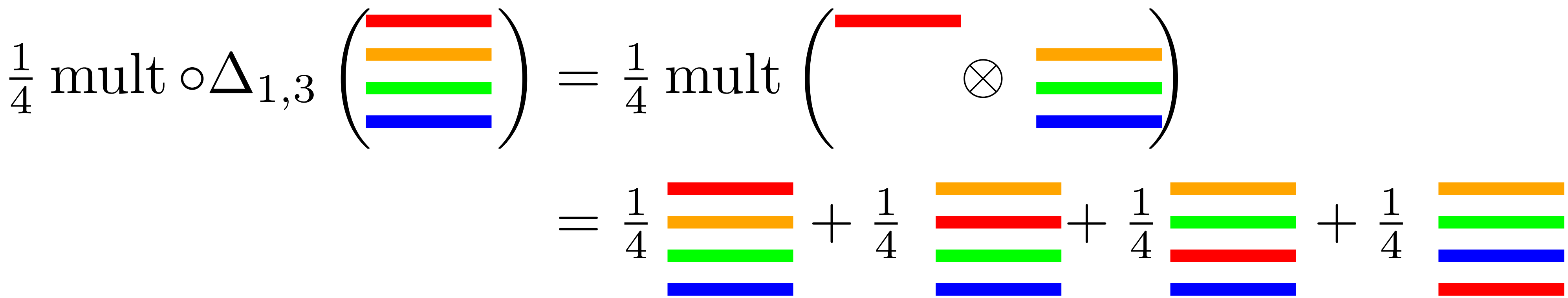


Riffle:

Prob$(x \to y)$= coefficient of $y$ in $\dfrac{1}{2^n}\,\mathrm{mult} \circ \displaystyle\sum_{i=0}^{n} \Delta_{i,n-i}(x)$.

# Chains on Other Combinatorial Objects

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| Markov chain | Hopf algebra / Hopf | free? | cofree? | basis | free-comm | stationary distribution |
|---|---|---|---|---|---|---|
| shuffling | shuffle algebra $\mathscr{S}$ | | x | words / decks of cards | | uniform |
| inverse-shuffling | free associative algeb | | | words / decks of cards | | uniform |
| edge-removal | $\mathscr{G}$ | | | unlabelled graphs | x | absorbing at empty graph |
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| restriction-then-induction | representations of sy | | x | irreducible representations | | plancherel |
| rock-breaking | symmetric functions | | x | elementary or complete | x | absorbing at $(1,1,\dots,1)$ |
| tree-pruning | Connes-Kreimer | | | rooted forests | x | absorbing at disconnected forest |
| descent-set-under-shuffling | quasisymmetric funct | | x | fundamental (compositions) | | proportion of permutations with this de |
| jeu-de-taquin | Poirier-Reutenauer F | | | standard Young tableaux | | proportion of standard tableaux with th |
| shuffle with standardisation | Malvenuto-Reutenau | | x | fundamental (permutations) | | uniform |

# The Top-to-Random Chain on Trees

Hopf algebra of trees: Butler (1972), Connes-Kreimer (1998)

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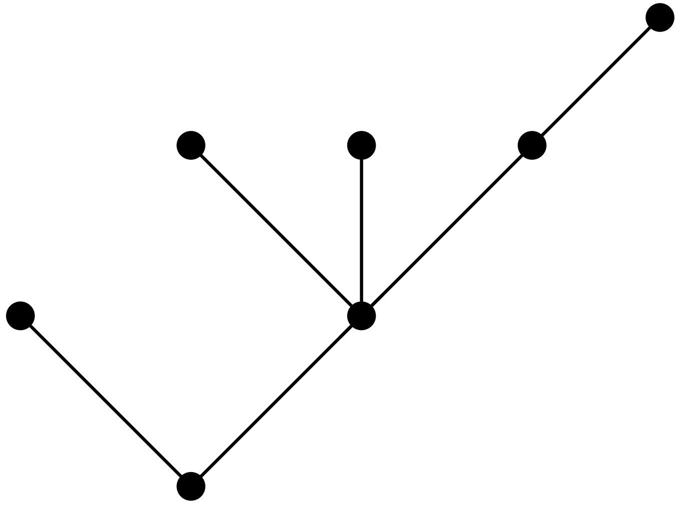

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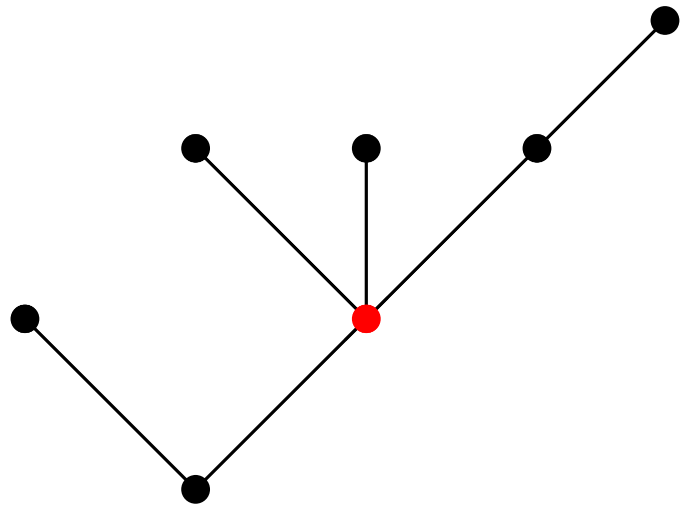

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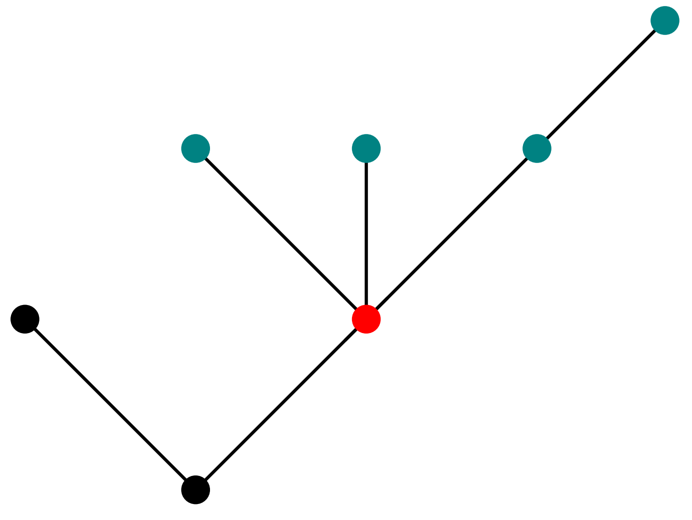

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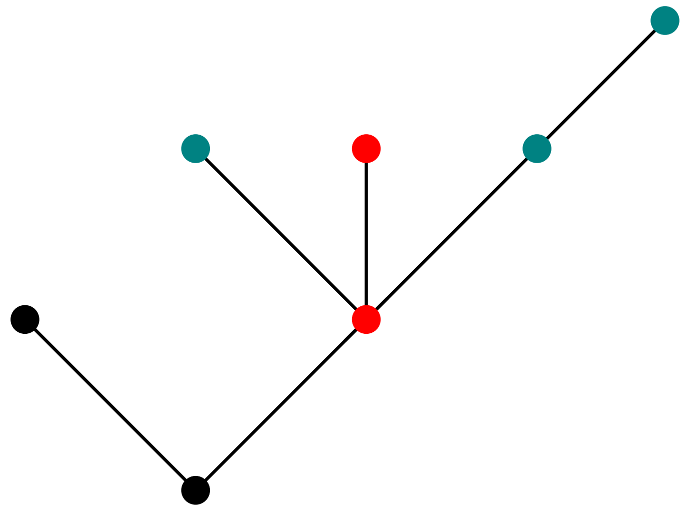

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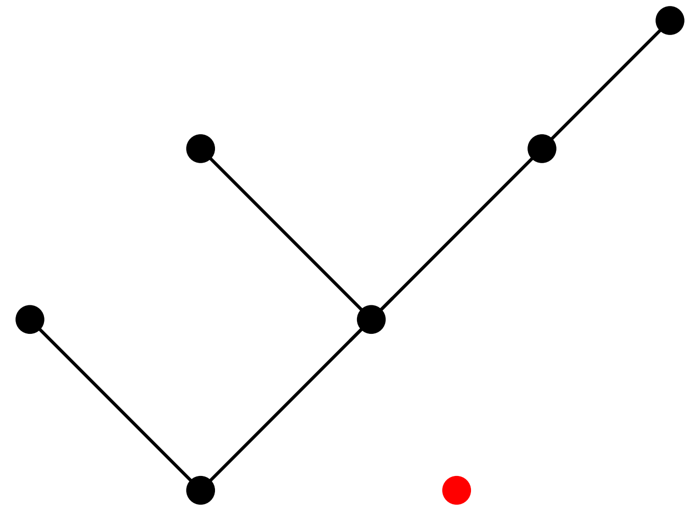

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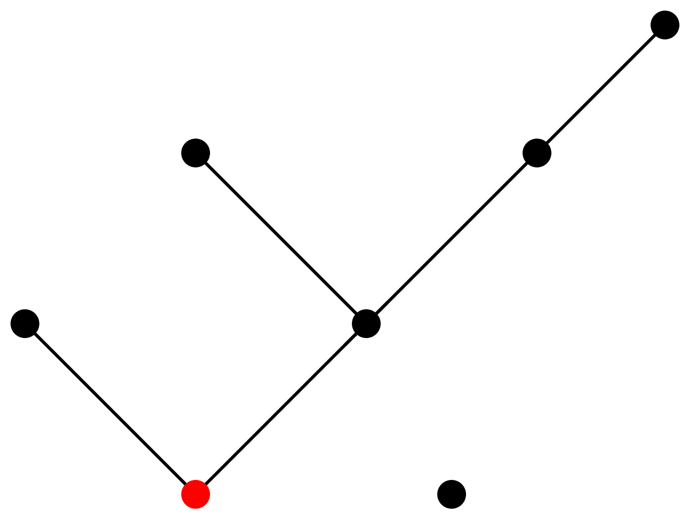

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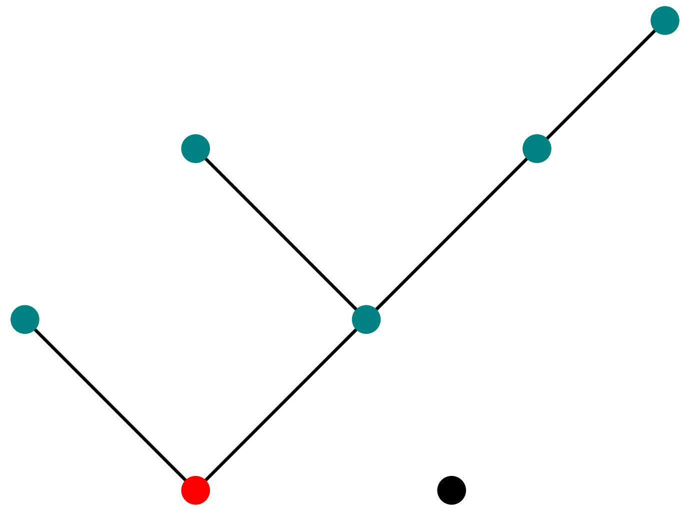

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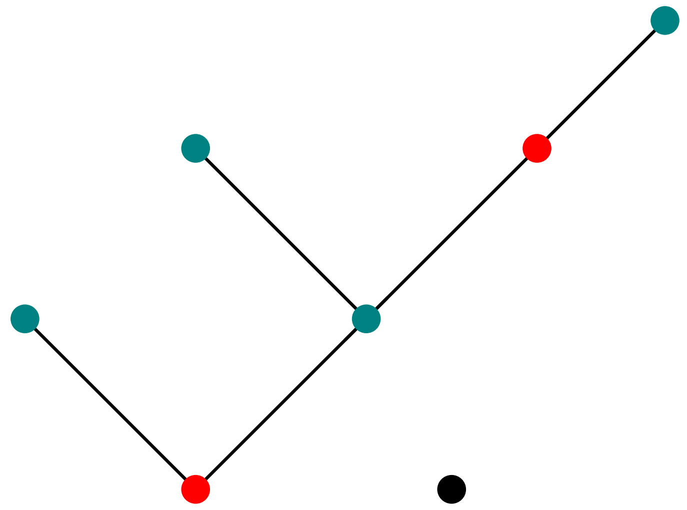

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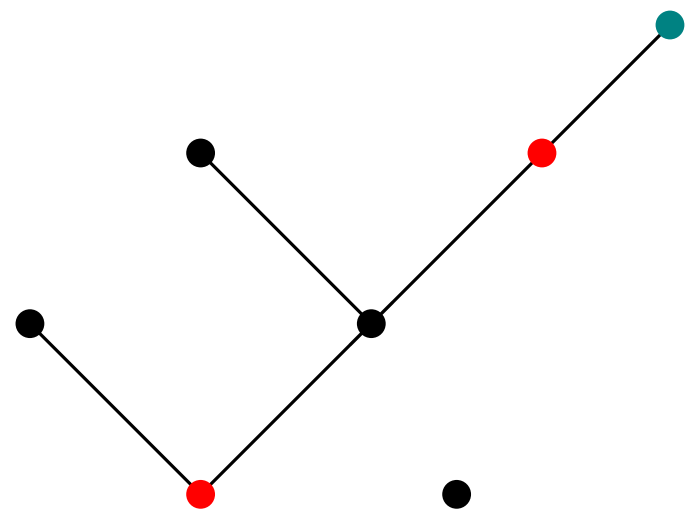

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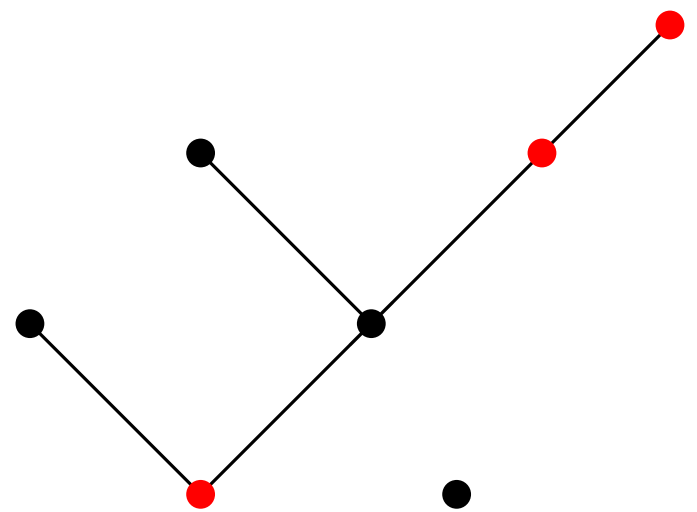

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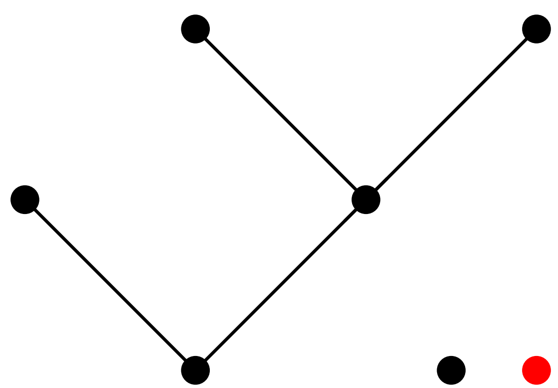

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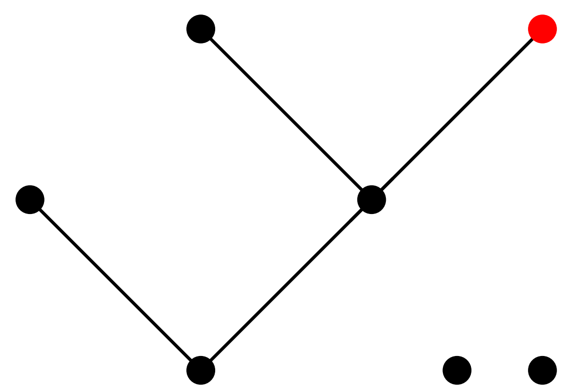

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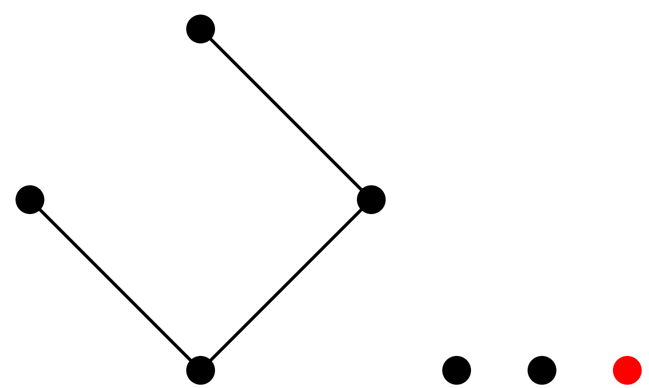

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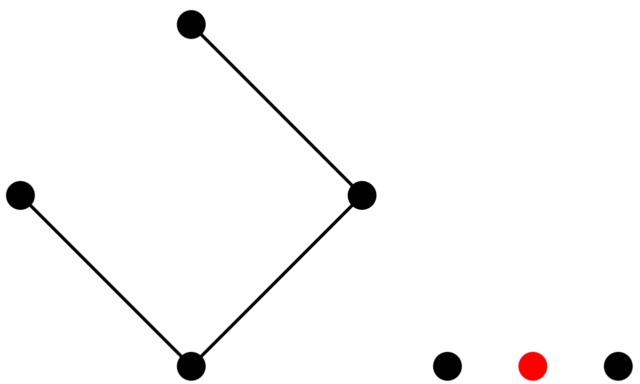

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**Theorem** (2015, 2016+): The eigenvalues are $0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-2}{n}, 1$.

Let $f_j$ be the number of $j$-tuples of vertices on different "branches". Then

$$\text{Expect}(f_j(X_t)) = \left(\frac{n-j}{n}\right)^t f_j(X_0).$$

# The Future

- More combinatorial objects (e.g. phylogenetic trees)
- More linear maps (e.g. move-to-front with arbitrary request probabilities)
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# Thank you!

These slides: amypang.github.io/hkbu2016.pdf


Reader-friendly summary: *Card-Shuffling via Convolutions of Projections on Combinatorial Hopf Algebras*, Discrete Math.Theor.Comput.Sci.Proc., 2015

Initial theory: (+Diaconis, Ram) *Hopf Algebras and Markov Chains: Two Examples and a Theory*,J. Algebraic Combin., 2014

Beyond eigen-information: *A Hopf-Algebraic Lift of the Down-Up Markov Chain on Partitions to Permutations* arXiv:1508.01570