§7.3 Change of basis

? If A, B are bases of U
C, D are bases of V
$\sigma \in L(U,V)$
then how is $[\sigma]_{C \leftarrow A}$ related to $[\sigma]_{D \leftarrow B}$?

Answer: $\sigma = I_V \circ \sigma \circ I_U$ (I_U = identity transformation on U)

Th. 7.31 $[\sigma]_{D \leftarrow B} = [I_V]_{D \leftarrow C} [\sigma]_{C \leftarrow A} [I_U]_{A \leftarrow B}$

what is $[\,]_{A \leftarrow B}$?

$P_{A \leftarrow B}$ from 2207 ↓

$\therefore$ multiplying by $[I]_{A \leftarrow B}$ changes from B-coordinates to A-coordinates

By 7.2.11: $[\,]_{A \leftarrow B} [\alpha]_B = [L(\alpha)]_A = [\alpha]_A$

How to find $\underset{A \leftarrow}{[\,L\,]}_B$?

1) From definition of matrix representation:

$$\underset{A \leftarrow}{[\,L\,]}_B = \begin{pmatrix} | & & | \\ [L(\beta_1)]_A & \cdots & [L(\beta_n)]_A \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ [\beta_1]_A & \cdots & [\beta_n]_A \\ | & & | \end{pmatrix}$$

i.e. "take $\underset{A}{\text{new}}$ coordinates of $\underset{B}{\text{old}}$ basis vectors")

2) when $[\beta_i]_A$ is hard to find:

Notice $\underset{A \leftarrow}{[\,L\,]}_B \underset{B \leftarrow}{[\,L\,]}_A = \underset{A \leftarrow}{[\,L \circ L\,]}_A = \underset{A \leftarrow}{[\,L\,]}_A = I$ identity matrix, ∵ same input and output basis.

also $\underset{B \leftarrow}{[\,L\,]}_A \underset{A \leftarrow}{[\,L\,]}_B = \underset{B \leftarrow}{[\,L \circ L\,]}_B = \underset{B \leftarrow}{[\,L\,]}_B = I$

$$\therefore \underset{A \leftarrow}{[\,L\,]}_B = \left(\underset{B \leftarrow}{[\,L\,]}_A \right)^{-1}$$ — useful when it is easy to find $[\alpha_i]_B$.

Particular case of Th 7.3.1 : if $V=U$, and $\mathcal{A}=C$, $\mathcal{B}=D$

$$\underset{\mathcal{B}\leftarrow\mathcal{B}}{[\sigma]} = \underset{\mathcal{B}\leftarrow\mathcal{A}}{[\iota]}\,\underset{\mathcal{A}\leftarrow\mathcal{A}}{[\sigma]}\,\underset{\mathcal{A}\leftarrow\mathcal{B}}{[\iota]}$$

$$= \left(\underset{\mathcal{A}\leftarrow\mathcal{B}}{[\iota]}\right)^{-1}\,\underset{\mathcal{A}\leftarrow\mathcal{A}}{[\sigma]}\,\underset{\mathcal{A}\leftarrow\mathcal{B}}{[\iota]}$$

$$= \underset{\mathcal{B}\leftarrow\mathcal{A}}{[\iota]}\,\underset{\mathcal{A}\leftarrow\mathcal{A}}{[\sigma]}\left(\underset{\mathcal{B}\leftarrow\mathcal{A}}{[\iota]}\right)^{-1}$$

Ex: $\sigma: P_{\le 2}(\mathbb{R}) \to P_{\le 4}(\mathbb{R})$

$[\sigma(f)](x) = f(1)(2+x^2)$

$\mathcal{A} = \{1, x\}$, $C = \{1, x, x^2, x^3\}$

$\mathcal{B} = \{1, 1+x\}$, $D = \{1, -1+x, 1+x^2, x^2+x^3\}$

Find $\underset{D\leftarrow\mathcal{B}}{[\sigma]}$

Answer 1: use change of basis:

$$\underset{D\leftarrow\mathcal{B}}{[\sigma]} = \underset{D\leftarrow C}{[\iota]}\,\underset{C\leftarrow\mathcal{A}}{[\sigma]}\,\underset{\mathcal{A}\leftarrow\mathcal{B}}{[\iota]}$$

$$\underset{C\leftarrow\mathcal{A}}{[\sigma]} = \left([\sigma(1)]_C\ [\sigma(x)]_C\right) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(see §7.2)}$$

$$\underset{\mathcal{A}\leftarrow\mathcal{B}}{[\iota]} = \left([1]_{\mathcal{A}}\ [1+x]_{\mathcal{A}}\right) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\underset{D\leftarrow C}{[\iota]} = \begin{pmatrix} 1 & x & x^2 & x^3 \\ & & & \\ & & & \\ & & & \end{pmatrix} \begin{matrix} 1 \\ -1+x \\ 1+x^2 \\ x^2+x^3 \end{matrix}$$

is hard — ∵ need to solve

$x^? = ?1 + ?(-1+x) + ?(1+x^2) + ?(x^2+x^3)$

So use $\underset{D\leftarrow C}{[\iota]} = \left(\underset{C\leftarrow D}{[\iota]}\right)^{-1}$:

$$\underset{C\leftarrow D}{[\iota]} = \begin{pmatrix} 1 & -1+x & 1+x^2 & x^2+x^3 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} 1 \\ x \\ x^2 \\ x^3 \end{matrix}$$

Invert (e.g. by RREF): $\underset{D\leftarrow C}{[\iota]} = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\therefore \underset{D\leftarrow B}{[\sigma]} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \leftarrow$$

allowed if question says "you may give your answer as a product of matrices and/or inverses."

$$= \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 0 & 2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Answer 2 : direct computation:

$$D[\sigma]_B = \begin{matrix} 1 \\ \\ \\ \\ \\ 0 \end{matrix} \begin{pmatrix} 1 & 4 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{matrix} 1+x \\ \\ 1 \\ \\ -1+x \\ \\ 1+x^2 \\ \\ x^2+x^3 \end{matrix}$$

$\sigma(1) = 1(2+x^2) = 2+x^2$

$\sigma(1+x) = (1+x)(2+x^2) = 2+2x+x^2+x^3$

this part can be hard, depending on D.