Standard example: Evaluate $\int xe^x\,dx$.

$$\int U\,dV = UV - \int V\,dU$$

1. Choose U, dV

$U = x \qquad dV = e^x dx$

2. differentiate U, integrate dV

$dU = dx \qquad V = e^x$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

Standard example: Evaluate $\displaystyle\int x \sin x \, dx$.

$$\int U\,dV = UV - \int V\,dU$$

Integration by parts:

$U = x \qquad dV = \sin x \, dx$

$dU = dx \qquad V = -\cos x$

$$= x(-\cos x) - \int -\cos x \, dx$$

$$= -x\cos x - (-\sin x) + C$$

$$= -x\cos x + \sin x + C$$

Note U, V do not appear in the main calculation

It's easy to make sign errors: differentiate to check:

$$\frac{d}{dx}(-x\cos x + \sin x)$$

$$= \cancel{-\cos x} - x(-\sin x) + \cancel{\cos x}$$

$$= x \sin x \checkmark$$

Standard example: Evaluate $\int x \ln x \, dx$.

$$\int U\,dV = UV - \int V\,dU$$

$U = x$ $\quad dV = \ln x \; dx$

$dU = dx$ $\quad V = ?$

Try again

$U = \ln x$ $\quad dV = x \; dx$

$dU = \frac{1}{x}\,dx$ $\quad V = \frac{x^2}{2}$

$$= \ln x \, \frac{x^2}{2} - \int \frac{x^2}{2} \frac{1}{x} \, dx$$

$$= \ln x \, \frac{x^2}{2} - \int \frac{x}{2} \, dx$$

$$= \ln x \, \frac{x^2}{2} - \frac{x^2}{4} + C$$

Differentiate to check!

Sometimes, after integration by parts, our new integral again requires integration by parts:

$$\int U\,dV = UV - \int V\,dU$$

Example: Evaluate $\int_0^2 (xe^x)^2\,dx$.

$$= \int_0^2 x^2 e^{2x}\,dx$$

substitution $u = xe^x$?
$du = \underbrace{e^x + xe^x}_{\text{not in integrand}}\,dx$
$\therefore$ maybe not a good idea

$$= x^2\frac{e^{2x}}{2}\Big|_0^2 - \int_0^2 \frac{e^{2x}}{\cancel{2}}\cancel{2}x\,dx$$

integration by parts:
$U = x^2$ $\quad dV = e^{2x}\,dx$
$dU = 2x\,dx$ $\quad V = \frac{e^{2x}}{2}$

$$= \frac{4e^4}{2} - \left(x\frac{e^{2x}}{2}\Big|_0^2 - \int_0^2 \frac{e^{2x}}{2}\,dx \right)$$

integration by parts again:
$U = x$ $\quad dV = e^{2x}\,dx$
$dU = dx$ $\quad V = \frac{e^{2x}}{2}$

$$= \frac{4e^4}{2} - \left(\frac{2e^4}{2} - \frac{e^{2x}}{4}\Big|_0^2 \right)$$

$$= 2e^4 - \left(e^4 - \left(\frac{e^4}{4} - \frac{1}{4} \right) \right) = \frac{1}{4}(5e^4 - 1)$$

Some integrals are best calculated using a substitution and then integration by parts. (It can also happen that, after integration by parts, the new integral requires a substitution.)

$$\int U\,dV = UV - \int V\,dU$$

Example: Evaluate $\int x^3 e^{x^2}\,dx$.

substitution

$u = x^2$

$du = 2x\,dx$

$$= \int \frac{x^2}{2} e^{x^2}\, 2x\,dx$$

$$= \int \frac{u}{2} e^u\,du$$

integration by parts:

$U = \frac{u}{2}$ $\quad dV = e^u\,du$

$dU = \frac{1}{2}du$ $\quad V = e^u$

$$= \frac{u}{2}e^u - \int e^u \frac{1}{2}\,du$$

$$= \frac{u}{2}e^u - \frac{e^u}{2} + C$$

$$= \frac{x^2}{2}e^{x^2} - \frac{e^{x^2}}{2} + C$$

$U = x^3$ $\quad dV = e^{x^2}dx$

$dU = 3x^2\,dx$ $\quad V = \cancel{\frac{e^{x^2}}{2x}}$

$$\frac{d}{dx}\left(\frac{e^{x^2}}{2x}\right) = \frac{(2x)^2e^{x^2} - 2e^{x^2}}{(2x)^2}$$

! Always consider substitution first.

$e^{g(x)} \longrightarrow$ substitute for $g(x)$