MATH 2207: Linear Algebra
Homework 3, due 15:45 Monday 5 March 2018

You must justify your answers to receive full credit.

1. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation satisfying

$$F\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, \quad F\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -7 \\ 8 \end{bmatrix}.$$

   a) Find the standard matrix of F.

   b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection through the x_2-axis. Find the standard matrix of the composition $F \circ T$.

2. Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

   Calculate the following quantities, or explain why they are not defined.

   a) AB b) BAB^T c) $B + 2I_3$ d) $\mathbf{x}^T B$
   e) $\det B$ f) A^3 g) A^{301} h) B^3

3. Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & -2 & 3 & 5 \\ 1 & p & 1 & 3 \end{bmatrix}.$$

   a) Find all values of p such that A is invertible.

   b) Find the inverse of A when $p = 0$.

4. Suppose

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

   Find the following determinants, and explain your answers:

   a) $\begin{vmatrix} a & b & c \\ 5d & 5e & 5f \\ g & h & i \end{vmatrix}$ b) $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$

   c) $\begin{vmatrix} a & b & c \\ d & e & f \\ g+3a & h+3b & i+3c \end{vmatrix}$ d) $\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}$ e) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$

5. Let D be the determinant

$$D = \begin{vmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \\ 1 & x & x^2 & x^3 \end{vmatrix}$$

(so D is a function of x).

a) Without computation, find three solutions to $D(x) = 0$. Explain your answer.

b) **Optional** (outside syllabus): Explain why these three are the only solutions to $D(x) = 0$. (Hint: $D(x)$ is a polynomial in x, of what degree?)
This is an example of a "Vandermonde determinant".

6. State whether each of the following statements is (always) true or (sometimes) false. If it is true, give a brief justification; if it is false, give a counterexample with an explanation.

a) If A, B, C, X are non-zero matrices that satisfy $AB = C$ and $AX = C$, then $B = X$.

b) Let A be a 4×4 matrix and $\mathbf{b}$ be a vector in $\mathbb{R}^4$ such that $A\mathbf{x} = \mathbf{b}$ has more than one solution. Then the columns of A do not span $\mathbb{R}^4$.

c) Let A be a 4×5 matrix and $\mathbf{b}$ be a vector in $\mathbb{R}^4$ such that $A\mathbf{x} = \mathbf{b}$ has more than one solution. Then the columns of A do not span $\mathbb{R}^4$.

d) If $S : \mathbb{R}^3 \to \mathbb{R}^3$ is onto and $T : \mathbb{R}^3 \to \mathbb{R}^3$ is one-to-one, then $S \circ T$ is one-to-one.

e) If A is a square matrix, then $\det(-A) = -\det(A)$.

f) The set $Q = \{a + bt + ct^2 \mid a + b + c = 0\}$ is a subspace of $\mathbb{P}_2$.

7. **Optional Problem** (outside syllabus): Consider a group of 5 students. Student 1 is friends on Facebook with each of the other four students. Also, Student 4 is Facebook friends with Student 3 and Student 5. There are no other Facebook friendships among the 5 students.

Let A be a 5×5 matrix, where the entry in row i and column j is 1 if Student i and Student j are Facebook friends, and 0 if Student i and Student j are not Facebook friends. So A is a symmetric matrix. (We assume Facebook does not allow a student to be friends with himself or herself, so all diagonal entries of A are zero.)

a) Write down the matrix A.

b) Let u be the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Calculate $A\mathbf{u}$. What is the meaning of the ith entry of $A\mathbf{u}$?

c) Calculate A^2. What is the meaning of the (i, j)-entry of A^2, when $i \neq j$?
This is the beginnings of the subject of "algebraic graph theory".

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