Ex. 6.5.8: Seeing the first part of the proof in an example:

In $\mathbb{R}^3$: $W_1 = \operatorname{Span}\left\{\begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}\right\}$, $W_2 = \operatorname{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$

To find α-basis of $W_1 \cap W_2$:

if $\alpha \in W_1 \cap W_2$ $\quad \alpha = a\begin{pmatrix}1\\0\\2\end{pmatrix} + b\begin{pmatrix}1\\2\\2\end{pmatrix} = c\begin{pmatrix}1\\1\\0\end{pmatrix} + d\begin{pmatrix}0\\1\\1\end{pmatrix}$. $\quad (*)$

solving e.g. $\begin{pmatrix}1 & 1 & -1 & 0\\ 0 & 2 & -1 & -1\\ 2 & 2 & 0 & -1\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \vec{0}$

row reduction

$$\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}-1/4\\3/4\\1/2\\1\end{pmatrix} t, \ t \in \mathbb{R}$$

Substitute into $*$: $\quad \alpha = \begin{pmatrix}1/2\\3/2\\1\end{pmatrix} t, \ t \in \mathbb{R}$ OR $\alpha = \begin{pmatrix}1\\3\\2\end{pmatrix} t' \ t' \in \mathbb{R}$.

$\therefore W_1 \cap W_2 = \text{Span}\left\{\begin{pmatrix}1\\3\\2\end{pmatrix}\right\}$

$\therefore$ basis of $W_1 \cap W_2$ is $\left\{\alpha = \begin{pmatrix}1\\3\\2\end{pmatrix}\right\}$

(i.e. dimensions in proof are $r = 1$, $s = 1$, $t = 1$.)

Extend to α, β- basis of W_1

e.g. method 2 : casting out algorithm : $\begin{array}{c} \begin{pmatrix}1 & 1 & 1\\3 & 0 & 2\\2 & 2 & 2\end{pmatrix} \\ \uparrow \alpha_1 \quad \underbrace{\qquad\quad}_{\text{spanning set for } W_1} \end{array} \longrightarrow$ row reduction shows first two columns have pivots, $\therefore$ basis of W_1 $= \left\{\alpha_1 = \begin{pmatrix}1\\3\\2\end{pmatrix}, \beta_1 = \begin{pmatrix}1\\0\\2\end{pmatrix}\right\}$

<u>OR</u> method 3: we are given $W_1 = \text{Span}\left\{\begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}\right\}$, and these vectors are linearly independent

$\therefore \dim W_1 = 2$.

And $\left\{\begin{pmatrix}1\\3\\2\end{pmatrix}, \begin{pmatrix}1\\0\\2\end{pmatrix}\right\}$ is linearly independent and contains 2 vectors in W_1,

$\therefore$ is a basis for W_1.

Similarly: extend to α-γ basis of W_2:

$$\left\{ \alpha_1 = \begin{pmatrix}1\\3\\2\end{pmatrix}, \gamma_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} \right\}$$

α-β-γ basis of $W_1 + W_2$:

$$\left\{ \alpha_1 = \begin{pmatrix}1\\3\\2\end{pmatrix}, \beta_1 = \begin{pmatrix}1\\0\\2\end{pmatrix}, \gamma_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} \right\}$$

A sum with "no overlap" — i.e. "overlap at $\vec{0}$ only" like a disjoint union.

Def 6.5.9: if $W_1 \cap W_2 = \{\vec{0}\}$, then $W_1 + W_2$ is a direct sum and is written $W_1 \oplus W_2$.

From 6.5.6: $\dim(W_1 \oplus W_2) = \dim W_1 + \dim W_2$

and from proof: (basis of W_1) $\cup$ (basis of W_2) $=$ basis of $W_1 \oplus W_2$.

Ex: ① in $\mathbb{R}^3$: $\operatorname{Span}\{e_1\} \oplus \operatorname{Span}\{e_2\} = \operatorname{Span}\{e_1, e_2\}$.

$\operatorname{Span}\{e_1, e_2\} + \operatorname{Span}\{e_2, e_3\}$ is not direct.

② $\mathbb{F}[x] = \mathbb{F}[x]_{odd} \oplus \mathbb{F}[x]_{even}$:

- $a_0 + a_1 x + a_2 x^2 + \cdots$
  $= \underbrace{(a_1 x + a_3 x^3 + \cdots)}_{\in\, \mathbb{F}[x]_{odd}} + \underbrace{(a_0 + a_2 x^2 + \cdots)}_{\in\, \mathbb{F}[x]_{even}}$

  shows $\boxed{\mathbb{F}[x] = \mathbb{F}[x]_{odd} + \mathbb{F}[x]_{even}}$
- and $\boxed{\mathbb{F}[x]_{odd} \cap \mathbb{F}[x]_{even} = \{\vec{0}\}}$:

  if $a_1 x + a_3 x^3 + \cdots = a_0 + a_2 x^2 + a_4 x^4 + \cdots$

  then $a_0 = a_1 = \cdots = 0$

OR if p is odd and even, then

$p(-x) = -p(x)$ $\quad$ $p(-x) = p(x)$ $\forall x$

so $-p(x) = p(x)$ $\therefore p(x) = 0$. $\forall x$

Direct sums have a unique representation property:

<u>Prop. 6.5.10</u>: W_1+W_2 is a direct sum $\Leftrightarrow$ $\forall \alpha \in W_1+W_2$, α has unique representation as $\alpha = \alpha_1+\alpha_2$ with $\alpha_i \in W_i$.

<u>Proof</u>: $\Leftarrow$ (by contradiction):

if W_1+W_2 is not direct, then $\exists \alpha \neq \vec{0} \in W_1 \cap W_2$.

α can be written as $\alpha = \alpha + \vec{0}$ and $\alpha = \vec{0}+\alpha$.

and these are different $\because \alpha \neq \vec{0}$.

$\Rightarrow$: Assume $W_1 \cap W_2 = \{\vec{0}\}$ and suppose $\alpha = \underset{W_1}{\underset{\cap}{\alpha_1}} + \underset{W_2}{\underset{\cap}{\alpha_2}} = \underset{W_1}{\underset{\cap}{\beta_1}} + \underset{W_2}{\underset{\cap}{\beta_2}}$

$$\overset{x}{\overset{\shortparallel}{\underset{\underset{W_1}{\cap}}{\underbrace{\alpha_1-\beta_1}}}} = \underset{\underset{W_2}{\cap}}{\underbrace{\beta_2-\alpha_2}}$$

LHS: $x \in W_1$

RHS: $x \in W_2$

$\therefore$ this vector is in $W_1 \cap W_2 = \{\vec{0}\}$.

$\therefore \alpha_1 - \beta_1 = \vec{0}$,

$\beta_2 - \alpha_2 = \vec{0}$