How to find $[\iota]_{\mathcal{A}\leftarrow\mathcal{B}}$?

1) From definition of matrix representation:

$$[\iota]_{\mathcal{A}\leftarrow\mathcal{B}} = \begin{pmatrix} | & & | \\ [\iota(\beta_1)]_{\mathcal{A}} & \cdots & [\iota(\beta_n)]_{\mathcal{A}} \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ [\beta_1]_{\mathcal{A}} & \cdots & [\beta_n]_{\mathcal{A}} \\ | & & | \end{pmatrix}$$

(i.e. "take $\underbrace{\text{new}}_{\mathcal{A}}$ coordinates of $\underbrace{\text{old}}_{\mathcal{B}}$ basis vectors")

2) when $[\beta_i]_{\mathcal{A}}$ is hard to find:

Notice $[\iota]_{\mathcal{A}\leftarrow\mathcal{B}}[\iota]_{\mathcal{B}\leftarrow\mathcal{A}} = [\iota\circ\iota]_{\mathcal{A}\leftarrow\mathcal{A}} = [\iota]_{\mathcal{A}\leftarrow\mathcal{A}} = I$ identity matrix, $\because$ same input and output basis.

also $[\iota]_{\mathcal{B}\leftarrow\mathcal{A}}[\iota]_{\mathcal{A}\leftarrow\mathcal{B}} = [\iota\circ\iota]_{\mathcal{B}\leftarrow\mathcal{B}} = [\iota]_{\mathcal{B}\leftarrow\mathcal{B}} = I$

$\therefore$ $[\iota]_{\mathcal{A}\leftarrow\mathcal{B}} = \left([\iota]_{\mathcal{B}\leftarrow\mathcal{A}}\right)^{-1}$ — useful when it is easy to find $[\alpha_i]_{\mathcal{B}}$.

Particular case of Th 7.3.1 : if $V=U$, and $\mathcal{A}=\mathcal{C}$, $\mathcal{B}=\mathcal{D}$

$$\underset{\mathcal{B}\leftarrow\mathcal{B}}{[\sigma]} = \underset{\mathcal{B}\leftarrow\mathcal{A}}{[\iota]}\,\underset{\mathcal{A}\leftarrow\mathcal{A}}{[\sigma]}\,\underset{\mathcal{A}\leftarrow\mathcal{B}}{[\iota]}$$

$$= \left(\underset{\mathcal{A}\leftarrow\mathcal{B}}{[\iota]}\right)^{-1}\underset{\mathcal{A}\leftarrow\mathcal{A}}{[\sigma]}\,\underset{\mathcal{A}\leftarrow\mathcal{B}}{[\iota]}$$

$$= \underset{\mathcal{B}\leftarrow\mathcal{A}}{[\iota]}\,\underset{\mathcal{A}\leftarrow\mathcal{A}}{[\sigma]}\left(\underset{\mathcal{B}\leftarrow\mathcal{A}}{[\iota]}\right)^{-1}$$

Ex: $\sigma : P_{<2}(\mathbb{R}) \longrightarrow P_{<4}(\mathbb{R})$

$[\sigma(f)](x) = f(x)(2+x^2)$

$\mathcal{A} = \{1, x\}$, $\mathcal{C} = \{1, x, x^2, x^3\}$

$\mathcal{B} = \{1, 1+x\}$, $\mathcal{D} = \{1, -1+x, 1+x^2, x^2+x^3\}$

Find $\underset{\mathcal{D}\leftarrow\mathcal{B}}{[\sigma]}$

Answer 1: use change of basis:

$$\underset{\mathcal{D}\leftarrow\mathcal{B}}{[\sigma]} = \underset{\mathcal{D}\leftarrow\mathcal{C}}{[\iota]}\,\underset{\mathcal{C}\leftarrow\mathcal{A}}{[\sigma]}\,\underset{\mathcal{A}\leftarrow\mathcal{B}}{[\iota]}$$

$$\underset{\mathcal{C}\leftarrow\mathcal{A}}{[\sigma]} = \left( [\sigma(1)]_{\mathcal{C}} \;\; [\sigma(x)]_{\mathcal{C}} \right) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$ (see §7.2)

$$\underset{\mathcal{A}\leftarrow\mathcal{B}}{[\iota]} = \left( [1]_{\mathcal{A}} \;\; [1+x]_{\mathcal{A}} \right)$$

$$= \begin{array}{c|cc} & 1 & 1+x \\ \hline 1 & 1 & 1 \\ x & 0 & 1 \end{array} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\underset{\mathcal{D}\leftarrow\mathcal{C}}{[\iota]} = \begin{array}{cccc|c} 1 & x & x^2 & x^3 & \\ \hline & & & & 1 \\ & & & & -1+x \\ & & & & 1+x^2 \\ & & & & x^2+x^3 \end{array}$$

is hard —
∵ need to solve
$x^{\wedge} = ?1 + ?(-1+x)$
$+?(1+x^2) + ?(x^2+x^3)$

So use $\underset{D \leftarrow C}{[\iota]} = \left(\underset{C \leftarrow D}{[\iota]}\right)^{-1}$ :

$$\underset{C \leftarrow D}{[\iota]} = \begin{array}{cc} \begin{array}{cccc} 1 & -1+x & 1+x^2 & x^2+x^3 \end{array} & \\ \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \end{array} \end{array}$$

Invert (e.g. by RREF): $\underset{D \leftarrow C}{[\iota]} = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\therefore \underset{D \leftarrow B}{[\sigma]} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

allowed if question says "you may give your answer as a product of matrices and/or inverses"

$$= \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 0 & 2 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Answer 2 : direct computation:

$${}_{\mathbb{D}}[\sigma]_{\mathcal{B}} = \begin{array}{c} \begin{array}{cc} 1 & 1+x \end{array} \\ \begin{pmatrix} 1 & 4 \\ 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} 1 \\ -1+x \\ 1+x^2 \\ x^2+x^3 \end{array}$$

$$\sigma(1) = 1(2+x^2) = 2+x^2$$

$$\sigma(1+x) = (1+x)(2+x^2) = 2+2x+x^2+x^3$$

this part can be hard, depending on $\mathbb{D}$.