Part_1-4

Ruiqi Yan (ry2417)

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Hurrican Data

hurricane703.csv collected the track data of 703 hurricanes in the North Atlantic area since 1950. For all the storms, their location (longitude & latitude) and maximum wind speed were recorded every 6 hours. The data includes the following variables

1. **ID**: ID of the hurricanes
2. **Season**: In which **year** the hurricane occurred
3. **Month**: In which **month** the hurricane occurred
4. **Nature**: Nature of the hurricane

- ET: Extra Tropical
- DS: Disturbance
- NR: Not Rated
- SS: Sub Tropical
- TS: Tropical Storm

5. **time**: dates and time of the record

6. **Latitude** and **Longitude**: The location of a hurricane check point
7. **Wind.kt** Maximum wind speed (in Knot) at each check point

load and clean data

Part I: Likelihood, Prior, Posterior and Conditional Posterior

Likelihood

$$L(\boldsymbol{Y} \mid \boldsymbol{\theta}) \propto \prod_{i=1}^{m} \left(\sigma^2\right)^{-\frac{n_i}{2}} \exp\left\{-\frac{1}{2\sigma^2}\left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i\right)^T\left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i\right)\right\}$$

where m is the number of hurricane and n_i is the number of observations for i_{th} hurricane

Prior

$$\pi\left(\boldsymbol{\theta} = (\boldsymbol{B}, \boldsymbol{\mu}, \Sigma^{-1}, \sigma^2)\right) \propto \left(\sigma^2\right)^{-1}\left|\Sigma^{-1}\right|^{d+1} \exp\left\{-\frac{1}{2}\operatorname{tr}\left(\Sigma^{-1}\right)\right\} \prod_{i=1}^{m}\left|\Sigma^{-1}\right|^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right)^T \Sigma^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu})\right\}$$

where d is the dimension of $\boldsymbol{\mu}$

Posterior

$$\pi(\boldsymbol{\theta} \mid \boldsymbol{Y}) \propto \left(\sigma^2\right)^{-1} \left|\Sigma^{-1}\right|^{d+1} \exp\left\{-\frac{1}{2} + \operatorname{tr}\left(\Sigma^{-1}\right)\right\}$$

$$\times \prod_{i=1}^{m} \left(\sigma^2\right)^{-\frac{n_i}{2}} \left|\Sigma^{-1}\right|^{\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2} \left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i\right)^T \left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i\right)\right\} \exp\left\{-\frac{1}{2} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right)^T \Sigma^{-1} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right)\right\}$$

$$= \left(\sigma^2\right)^{-\left(1+\frac{\sum_{i=1}^m n_i}{2}\right)} \left|\Sigma^{-1}\right|^{d+1+\frac{m}{2}} \exp\left\{-\frac{1}{2} \operatorname{tr}\left(\Sigma^{-1}\right)\right\} \exp\left\{-\frac{1}{2} \sum_{i=1}^{m} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right)^T \Sigma^{-1} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right)\right\}$$

$$\times \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{m} \left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i\right)^T \left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i\right)\right\}$$

Conditional Posterior

For $\boldsymbol{\beta}_i$:

$$\pi(\boldsymbol{\beta}_i \mid \boldsymbol{\theta}_{(-\boldsymbol{\beta}_i)}\boldsymbol{Y}) \propto \exp\left\{-\frac{1}{2} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right)^T \Sigma^{-1} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right) - \frac{1}{2\sigma^2} \left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i\right)^T \left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i\right)\right\}$$

For $\boldsymbol{\mu}$:

$$\begin{aligned}
\pi\left(\boldsymbol{\mu} \mid \boldsymbol{\theta}_{(-\boldsymbol{\mu})}, \boldsymbol{Y}\right) &\propto \exp\left\{-\frac{1}{2} \sum_{i=1}^{m} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right)^T \Sigma^{-1} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right)\right\} \\
(\boldsymbol{\beta}_i - \boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{\beta}_i - \boldsymbol{\mu}) &= \operatorname{tr}\left(\left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right)^T \Sigma^{-1} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right)\right) \\
&= \operatorname{tr}\left(\Sigma^{-1} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right) \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right)^T\right) \\
\pi\left(\boldsymbol{\mu} \mid \boldsymbol{\theta}_{(-\boldsymbol{\mu})}, Y\right) &\propto \exp\left\{-\frac{1}{2} \operatorname{tr}\left(\Sigma^{-1} \sum_{i=1}^{m} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right) \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right)^T\right)\right\} \\
&= \exp\left\{-\frac{1}{2} \operatorname{tr}\left(\Sigma^{-1} m \left(\boldsymbol{\mu} - \bar{\boldsymbol{\beta}}\right) \left(\boldsymbol{\mu} - \bar{\boldsymbol{\beta}}\right)^T\right)\right\} \\
&= \exp\left\{-\frac{1}{2} (\boldsymbol{\mu} - \bar{\boldsymbol{\beta}})^T \Sigma^{-1} m (\boldsymbol{\mu} - \bar{\boldsymbol{\beta}})\right\} \\
&\Rightarrow \boldsymbol{\mu} \mid \boldsymbol{\theta}_{(-\boldsymbol{\mu})}, \boldsymbol{Y} \sim N(\bar{\boldsymbol{\beta}}, \Sigma/m)
\end{aligned}$$

For σ^2:

$$\pi\left(\sigma^2 \mid \boldsymbol{\theta}_{(-\sigma^2)}, \boldsymbol{Y}\right) \propto \left(\sigma^2\right)^{-\left(1+\frac{\sum_{i=1}^m n_i}{2}\right)} \times \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{m} \left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i\right)^T \left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i\right)\right\}$$

$$\begin{aligned}
\pi\left(\Sigma^{-1} \mid \boldsymbol{\theta}_{(-\Sigma^{-1})}, \boldsymbol{Y}\right) &\propto \left|\Sigma^{-1}\right|^{d+1+\frac{m}{2}} \exp\left\{-\frac{1}{2} \operatorname{tr}\left(\Sigma^{-1}\right)\right\} \exp\left\{-\frac{1}{2} \sum_{i=1}^{m} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right)^T \Sigma^{-1} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right)\right\} \\
&= \left|\Sigma^{-1}\right|^{d+1+\frac{m}{2}} \exp\left\{-\frac{1}{2} \operatorname{tr}\left(\Sigma^{-1}\right) - \frac{1}{2} \operatorname{tr}\left(\Sigma^{-1} \sum_{i=1}^{m} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right) \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right)^T\right)\right\} \\
&= \left|\Sigma^{-1}\right|^{d+1+\frac{m}{2}} \exp\left\{-\frac{1}{2} \operatorname{tr}\left(\Sigma^{-1} \left(I + \sum_{i=1}^{m} \left(\beta_i - \mu\right) \left(\beta_1 - \mu\right)^T\right)\right)\right\} \\
&\Rightarrow \Sigma^{-1} \mid \boldsymbol{\theta}_{(-\Sigma^{-1})}, \boldsymbol{Y} \sim \text{Wishart}\left(3d + 3 + m, \left(\boldsymbol{I} + \sum_{i=1}^{m} \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right) \left(\boldsymbol{\beta}_i - \boldsymbol{\mu}\right)^T\right)^{-1}\right)
\end{aligned}$$

Part II: MCMC Algorithm

We could generate full conditional posterior distribution for partial parameters, so we apply hybrid algorithm consisting with Metropolis-Hastings steps and Gibbs steps.

MH steps for β_{ij}:

Sampling proposed β'_{ij} $j = 0, 1...4$ for i_{th} hurricane from proposal distribution $U\left(\beta_{ij}^{(t)} - a_{ij}, \beta_{ij}^{(t)} + a_{ij}\right)$, where a_{ij} is the search window for β_{ij}. Since the proposals are symmetry, the accepting or rejecting the proposed β'_{ij} depends on the ratio of posterior distribution. Some of the parameters in $\boldsymbol{\theta}$ could be cancelled out, so the ratio simplified to be:

$$\frac{\pi\left(\boldsymbol{\beta}'_i, \boldsymbol{\theta}^{(t)}_{(-\boldsymbol{\beta}_i)} \mid \boldsymbol{Y}\right)}{\pi\left(\boldsymbol{\beta}^{(t)}_i, \boldsymbol{\theta}^{(t)}_{(-\boldsymbol{\beta}_i)} \mid \boldsymbol{Y}\right)} = \frac{\exp\left\{-\frac{1}{2\sigma^{2(t)}}\left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}'_i\right)^T\left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}'_i\right) - \frac{1}{2}\left(\boldsymbol{\beta}'_i - \boldsymbol{\mu}^{(t)}\right)^T \Sigma^{-1^{(t)}}\left(\boldsymbol{\beta}'_i - \boldsymbol{\mu}^{(t)}\right)\right\}}{\exp\left\{-\frac{1}{2\sigma^{2(t)}}\left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}^{(t)}_i\right)^T\left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}^{(t)}_i\right) - \frac{1}{2}\left(\boldsymbol{\beta}^{(t)}_i - \boldsymbol{\mu}^{(t)}\right)^T \Sigma^{-1(t)}\left(\boldsymbol{\beta}^{(t)}_i - \boldsymbol{\mu}^{(t)}\right)\right\}}$$

where $\boldsymbol{\beta}'_i$ consisting with β'_{ij}, $\beta^{(t)}_{ik}$ for $k > j$ and $\beta^{(t+1)}_{ik}$ for $k < j$.

The log of the ratio:

$$\log\frac{\pi\left(\boldsymbol{\beta}'_i, \boldsymbol{\theta}^{(t)}_{(-\boldsymbol{\beta}_i)} \mid \boldsymbol{Y}\right)}{\pi\left(\boldsymbol{\beta}^{(t)}_i, \boldsymbol{\theta}^{(t)}_{(-\boldsymbol{\beta}_i)} \mid \boldsymbol{Y}\right)} = -\frac{1}{2}\left(\frac{\left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}'_i\right)^T\left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}'_i\right)}{\sigma^{2(t)}} + \left(\boldsymbol{\beta}'_i - \boldsymbol{\mu}^{(t)}\right)^T \Sigma^{-1^{(t)}}\left(\boldsymbol{\beta}'_i - \boldsymbol{\mu}^{(t)}\right)\right)$$
$$+\frac{1}{2}\left(\frac{\left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}^{(t)}_i\right)^T\left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}^{(t)}_i\right)}{\sigma^{2(t)}} + \left(\boldsymbol{\beta}^{(t)}_i - \boldsymbol{\mu}^{(t)}\right)^T \Sigma^{-1^{(t)}}\left(\boldsymbol{\beta}^{(t)}_i - \boldsymbol{\mu}^{(t)}\right)\right)$$

Then we randomly sample u from $U(0, 1)$ and compare $\log(u)$ with the log ratio, if the $\log(u)$ is smaller, we accept $\beta'_{ij} = \beta^{(t+1)}_{ij}$, otherwise we reject β'_{ij} and $\beta^{(t)}_{ij} = \beta^{(t+1)}_{ij}$.

Then, Gibb step for $\boldsymbol{\mu}$: Sample $\boldsymbol{\mu}^{(t+1)}$ from $N\left(\bar{\boldsymbol{\beta}}^{(t+1)}, \Sigma^{(t)}/m\right)$.

Next, MH step to generate $\sigma^{2'}$. The log posterior ratio:

$$\log\frac{\pi\left(\sigma^{2'}, \boldsymbol{\theta}^{(t)}_{(-\sigma^2)} \mid \boldsymbol{Y}\right)}{\pi\left(\sigma^{2^{(t)}}, \boldsymbol{\theta}^{(t)}_{(-\sigma^2)} \mid \boldsymbol{Y}\right)} = -\left(1 + \frac{M}{2}\right)\log(\sigma^{2'}) - \frac{1}{2\sigma^{2'}}\sum_{i=1}^{m}\left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}^{(t+1)}_i\right)^T\left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}^{(t+1)}_i\right)$$
$$+\left(1 + \frac{M}{2}\right)\log(\sigma^{2^{(t)}}) + \frac{1}{2\sigma^{2^{(t)}}}\sum_{i=1}^{m}\left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}^{(t+1)}_i\right)^T\left(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}^{(t+1)}_i\right)$$

where M is total number of observation for all hurricanes.

Check whether $\sigma^{2'}$ is positive, if not, we reject $\sigma^{2'}$. Then, we randomly sample u from $U(0, 1)$ and compare $\log(u)$ with the log ratio. If the $\log(u)$ is smaller, we accept $\sigma^{2'} = \sigma^{2^{(t+1)}}$.

Finally, we sample $\Sigma^{-1^{(t+1)}}$ from Wishart $\left(3d + 3 + m, \left(\boldsymbol{I} + \sum_{i=1}^{m}\left(\boldsymbol{\beta}^{(t+1)}_i - \boldsymbol{\mu}^{(t+1)}\right)\left(\boldsymbol{\beta}^{(t+1)}_i - \boldsymbol{\mu}^{(t+1)}\right)^T\right)^{-1}\right)$

Part III: Implementation and Estimations

Starting Value and Search Window

Table 1: Range of Search Window and Acceptance Rate for each paraemter in MH Algorithm

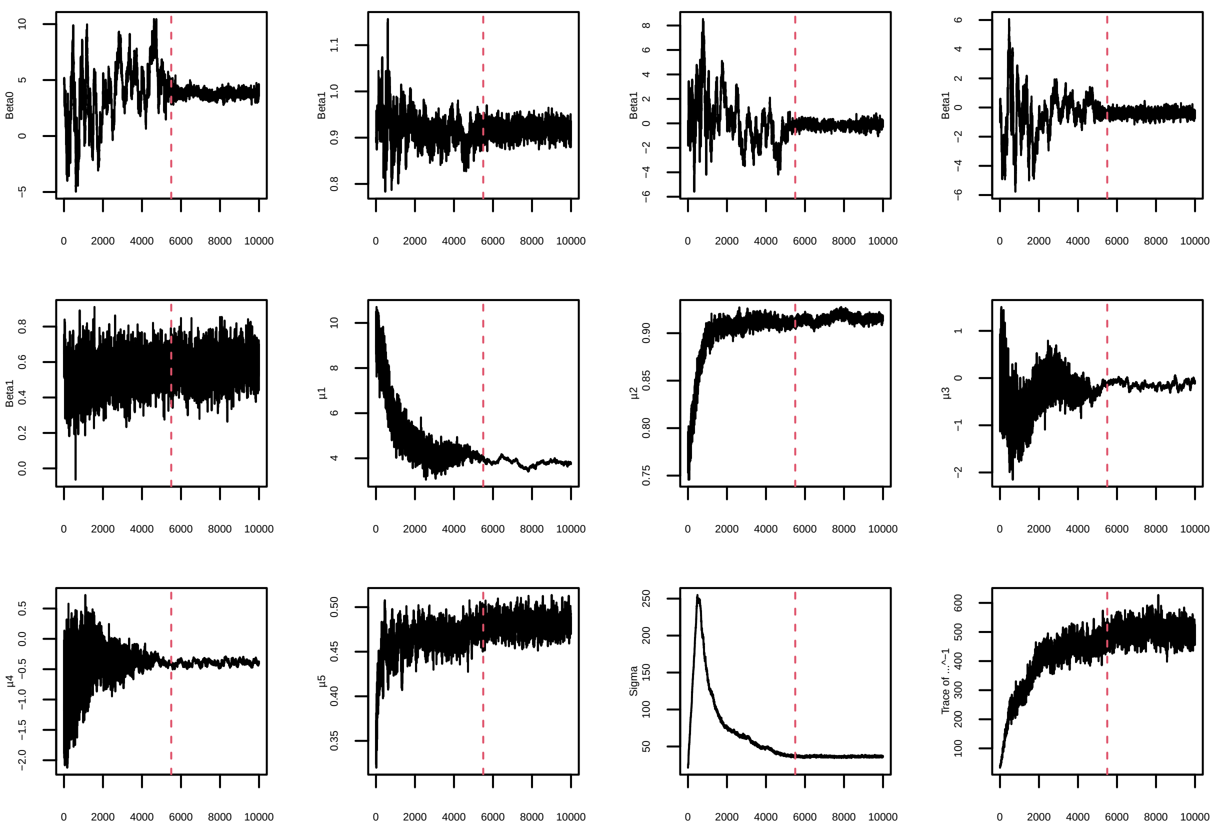
	search window	acceptance rate (%)
β_0	1.1	45.87-51.36
β_1	0.04-0.1	31.67-63.68
β_2	0.8-1	38.6-45.6
β_3	0.5-0.6	33.2-61.32
β_4	0.4-0.5	34.95-60.45
σ^2	2	44.93

Implement 10000 MCMC iterations

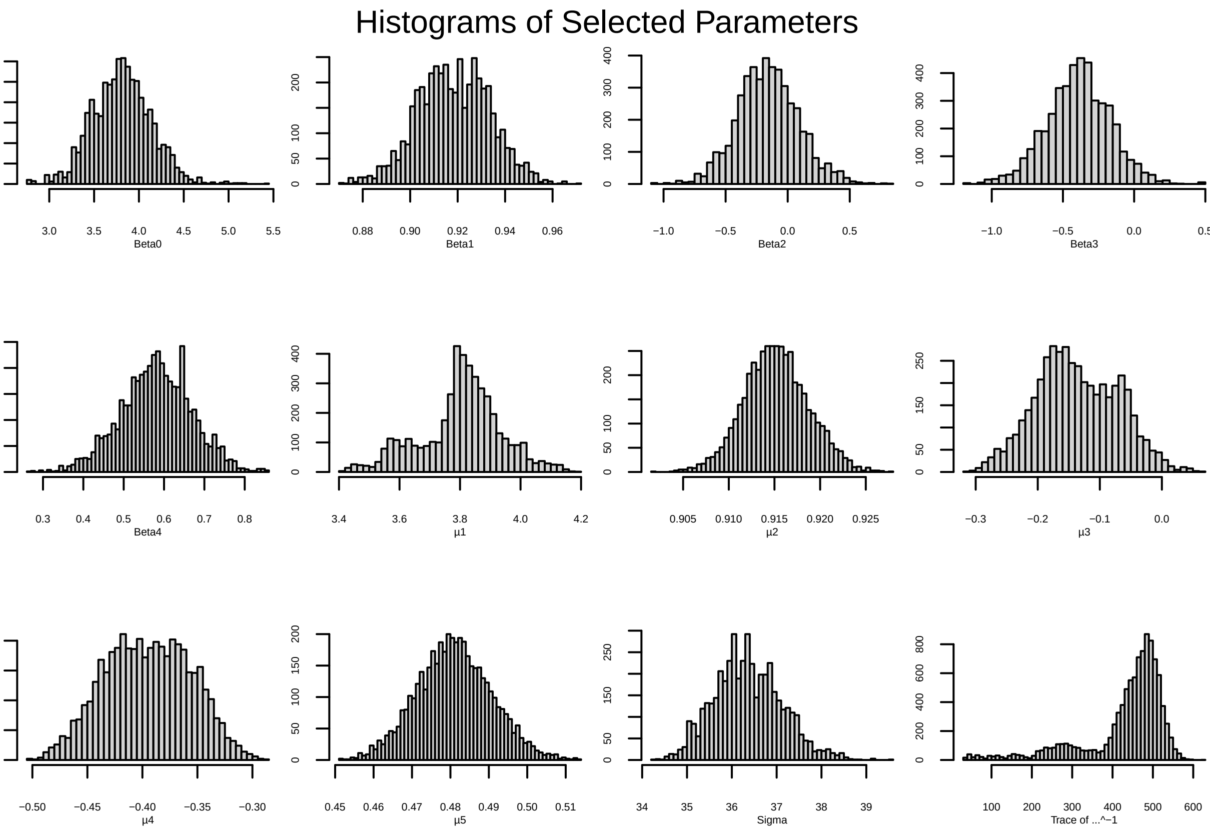
Acceptance Rate

Convergence trace plots

Covergence Plots of Selected Parameters



Histograms



Estimates (burn-in first 5500 runs)

Table 2: Bayesian Estimates for $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}^2$

	$\boldsymbol{\mu}_0$	$\boldsymbol{\mu}_1$	$\boldsymbol{\mu}_2$	$\boldsymbol{\mu}_3$	$\boldsymbol{\mu}_4$	$\boldsymbol{\sigma}^2$
Estimates	3.8	0.92	-0.14	-0.39	0.48	36.36

Table 3: Bayesian Estimate for Σ^{-1}

	$\boldsymbol{\beta}_0$	$\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_2$	$\boldsymbol{\beta}_3$	$\boldsymbol{\beta}_4$
$\boldsymbol{\beta}_0$	16.07	7.63	-1.78	5.39	1.48
$\boldsymbol{\beta}_1$	7.63	0.34	0.47	3.96	1.14
$\boldsymbol{\beta}_2$	-1.78	0.47	381.32	-6.32	24.35
$\boldsymbol{\beta}_3$	5.39	3.96	-6.32	21.43	-3.30
$\boldsymbol{\beta}_4$	1.48	1.14	24.35	-3.30	55.12

Beta estimate histograms

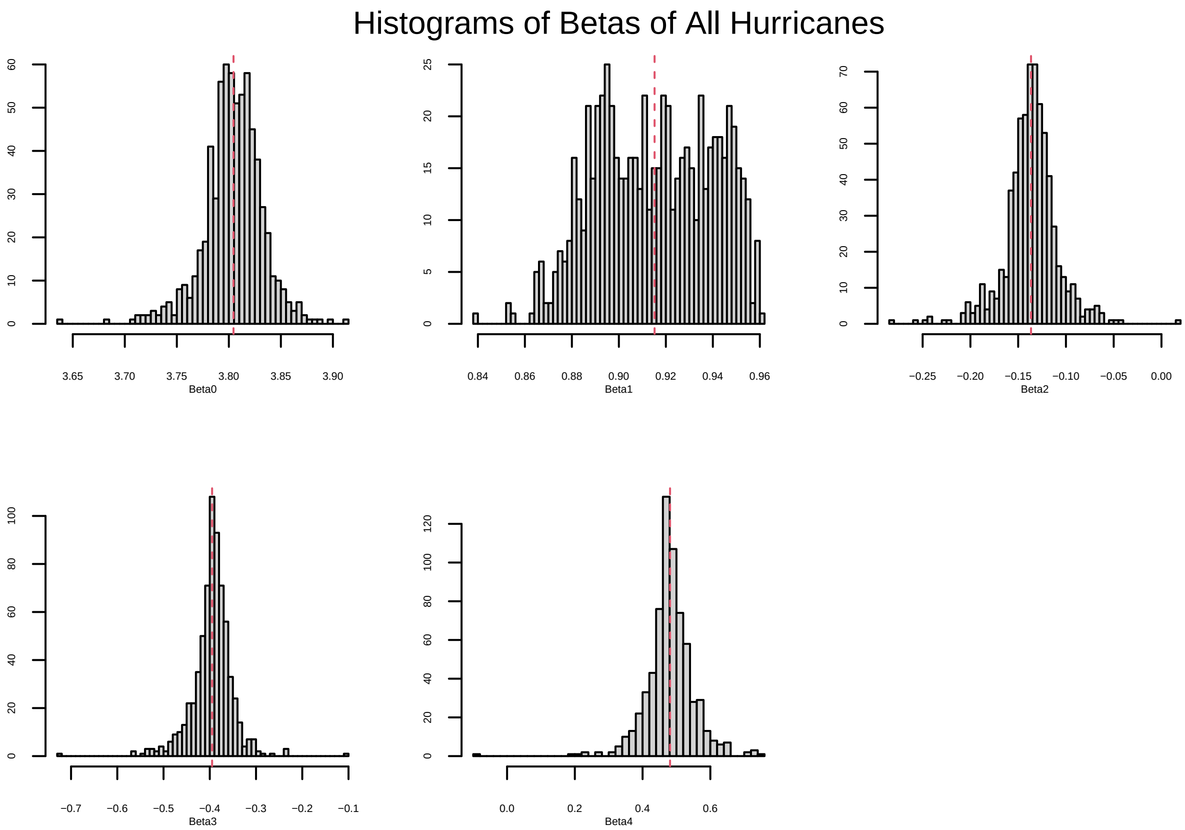


Chart of the estimates of mu, sigma and inverseCov

Table 4: R^2_{adj} for each hurricane

R^2_{adj}	Count of Hurricanes	Percentage(%)
0.7-1	472	69.3
0.3-0.7	117	17.2
0-0.3	31	4.6
< 0	61	9.0

Part IV: Model Performance