The Coq Proof Assistant Tutorial

Software Verification

Amy Pritchard
University at Buffalo, SUNY
amypritc@buffalo.edu

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CoqIDE v8.10.2 (Updated up to Section 7.4)

CoqIDE v8.7.2 (to be removed once fully updated)

Note: Screenshots and tutorial are based on the CoqIDE for MacOS - there may be differences between OS versions.

Contents

1 The COQ Proof Assistant Web Page

`https://coq.inria.fr`

Here you can find all sorts of info about the COQ Proof Assistant, including the following:

- About COQ (`https://coq.inria.fr/about-coq`)
- Getting COQ (`https://coq.inria.fr/download`)
- Documentation, i.e. books, tutorials, etc. (`https://coq.inria.fr/documentation`)
- Reference Manual (`https://coq.inria.fr/distrib/current/refman/`).
- Standard Library (`https://coq.inria.fr/distrib/current/stdlib/`)
- Community for discussion, contribution, events, etc. (`https://coq.inria.fr/community`)
- News about new releases (`https://coq.inria.fr/news/`)

2 Installation

For general installation options and instructions, go to `https://coq.inria.fr/download`.

To get an installer for the latest version of the CoqIDE for Windows or MacOS, go to `https://github.com/coq/coq/releases/latest` and scroll down to the bottom of the page.

To get the installer for Windows or MacOS for CoqIDE v8.10.2 (used for this tutorial), go to `https://github.com/coq/coq/releases/tag/V8.10.2` and scroll down to the bottom of the page.

To install Coq via OPAM on MacOS or Linux, go to `https://coq.inria.fr/opam-using.html` for step by step instructions.

3 The Coq IDE

3.1 CoqIDE version 8.10.2

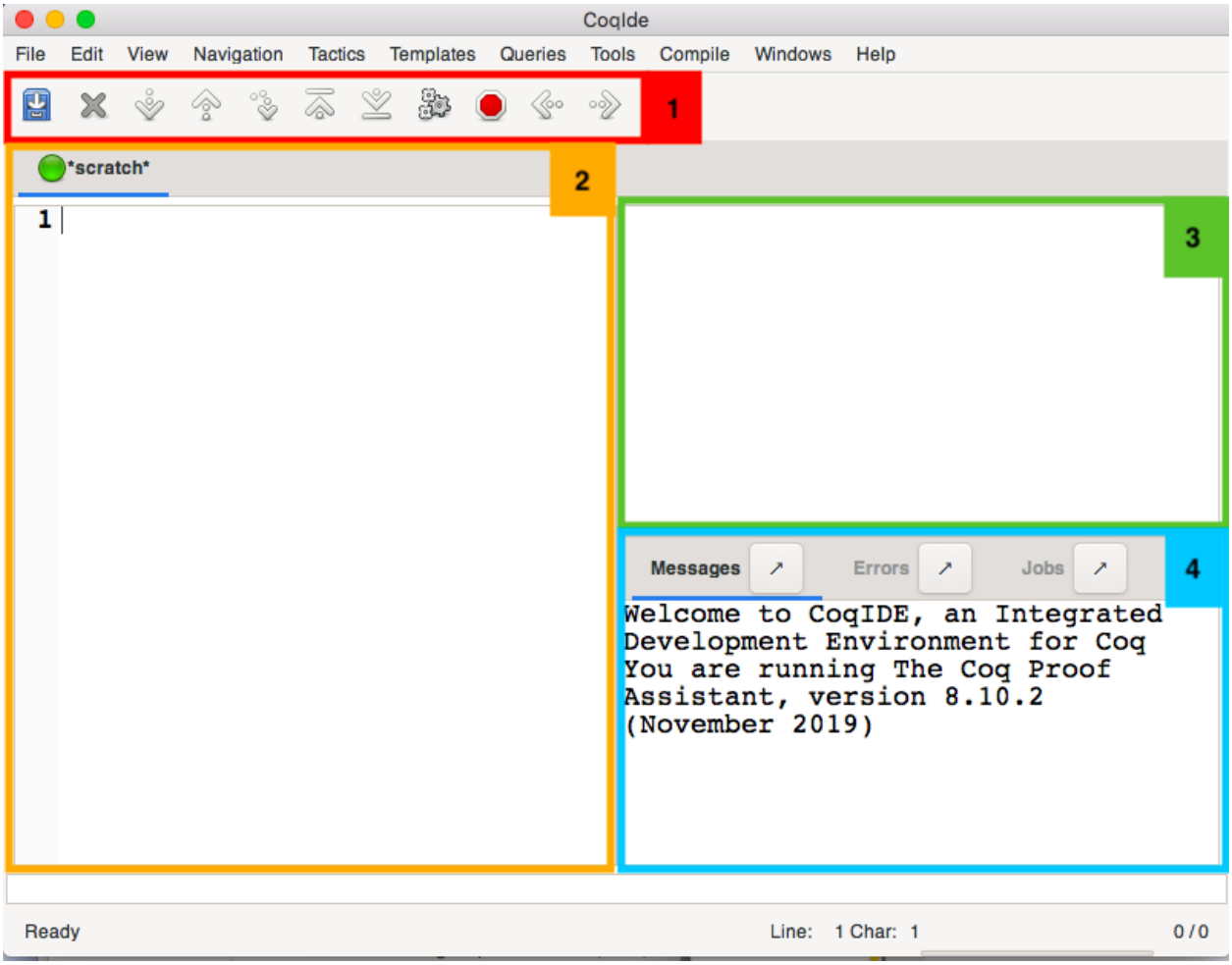


Figure 1: CoqIDE v8.10.2 (1) Toolbar. (2) Script Buffer. (3) Goal Window. (4) Message Window.

3.1.1 Toolbar for CoqIDE v8.10.2

Save current buffer. If it hasn't been previously saved, functions as save as; use the extension .v to save as a Coq file.

Close current buffer. Gives a warning if the file has unsaved changes.

Forward one command. Steps forward to evaluate the next command in the current file.

Backward one command. Steps backward one command in the file, returns the state to where it was before evaluating that command.

Go to cursor. Evaluate all commands in file up to where the cursor currently is.

Restart Coq. Returns to the top of the file, where no commands have been evaluated.

Go to end. Evaluate to the bottom of the file. Does not work as well with load commands and require import commands.

Fully check the document. Submits proof terms to the Coq kernel for type checking.

Interrupt computations. Stops computation at whatever point was reached before pressing the button.

Next Occurrence. Goes to the next occurrence of whatever the cursor is currently by. Works well for longer words.

Previous Occurrence. Goes to the previous occurrence of whatever the cursor is currently by. Works well for longer words.

3.2 CoqIDE version 8.7.2

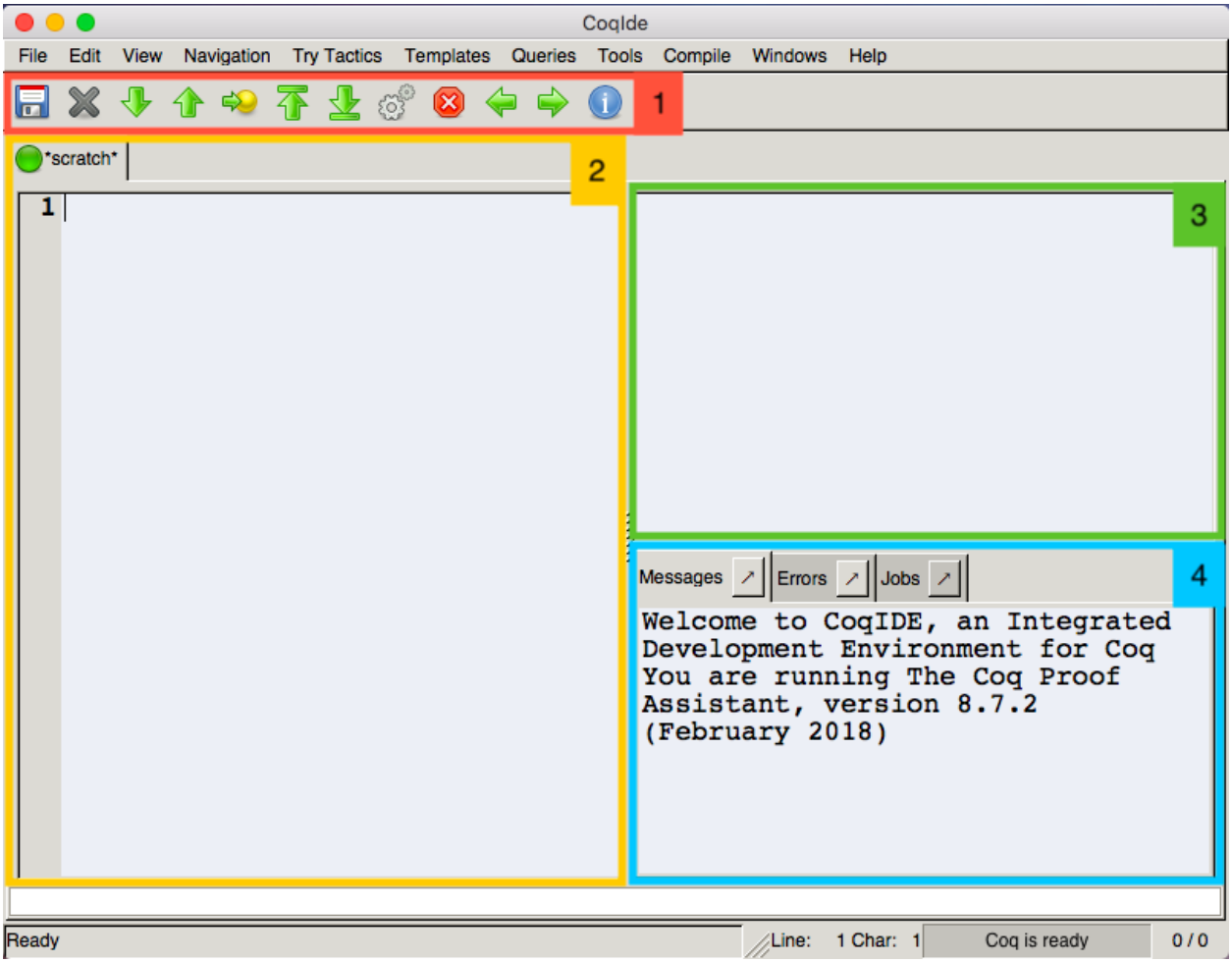


Figure 2: CoqIDE v8.7.2 (1) Toolbar. (2) Script Buffer. (3) Goal Window. (4) Message Window.

3.2.1 Toolbar for CoqIDE v8.7.2

Save current buffer. If it hasn't been previously saved, functions as save as; use the extension .v to save as a Coq file.

Close current buffer. Gives a warning if the file has unsaved changes.

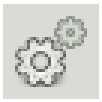
Forward one command. Steps forward to evaluate the next command in the current file.

Backward one command. Steps backward one command in the file, returns the state to where it was before evaluating that command.

Go to cursor. Evaluate all commands in file up to where the cursor currently is.

Restart Coq. Returns to the top of the file, where no commands have been evaluated.

Go to end. Evaluate to the bottom of the file. Does not work as well with load commands and require import commands.



Fully check the document. Submits proof terms to the Coq kernel for type checking.

Interrupt computations. Stops computation at whatever point was reached before pressing the button.

Next Occurrence. Goes to the next occurrence of whatever the cursor is currently by. Works well for longer words.

Previous Occurrence. Goes to the previous occurrence of whatever the cursor is currently by. Works well for longer words.

Proof Wizard. Tries to apply a given set of tactics in order. The tactics to be attempted can be customized by going to: Edit tab → Preferences, then select Tactics Wizard in the leftmost pane of the Customizations window.

3.3 Script Buffer

Here you can type out new definitions and proofs in a new buffer, or open a buffer from a saved file by going to the File tab → Open, then choosing the file. You can edit and save new and existing buffers here.

This block represents a script buffer in this tutorial.

3.4 Goal Window

Goals to be proven will be displayed here. This window will be empty unless you're in a proof environment; inside the proof environment, it will display what you goal are currently proving, what goals are left to be proved, or that there are no more subgoals and your proof is complete.

This block represents the goal window in this tutorial.

3.5 Message Window

Any messages resulting from an executed command will be displayed here. Results from queries are also printed out here. Clicking the arrow in the corner of the Messages, Errors, or Jobs tab here will create a separate window for that tab. When the separate window is closed, it will return to the main Coq IDE window.

This block represents the message window in this tutorial.

4 Coq Basics

The language of Coq is functional, and has similarities to OCaml. If you are not familiar with functional programming, this section will help introduce some basic syntax and concepts to get you started. See file "Basics.v"

4.1 Comments

Comments in Coq are surrounded by (* *).

```
(*  This is a comment in Coq.  *)
```

4.2 Commands

All commands in Coq must end with a period. The various available commands are discussed throughout this document; please see the respective sections for more information on any specific commands.

4.3 Pattern Matching

Pattern matching is a useful way to specify what should occur when you see a specific option of a type. For example, with booleans you have the options of true and false. In general, you have something like this for a single variable:

```
match var with
| opt1 => expr1
| opt2 => expr2
end
```

```
match b with
| true => false
| false => true
end
```

and something like this if you'd like to match over a combination of variables:

```
match (var1, var2) with
| (var1opt1, var2opt1) => expr1
| (var1opt2, var2opt2) => expr2
| ... => ...
| (_ , _ ) => exprX
end
```

The underscore in the last option allows you to specify only the options that you care about at the top, and then for all other cases, do exprX. The underscore is not necessary if you've listed all possible options or combinations of options as possible matches.

See example 10.2 for some simple boolean functions using pattern matching.

4.4 Lists

Coq has built in notation for lists that you can use; however, to use the notations, you must make sure to load the List library beforehand.

```
Load List.
```

Lists cannot contain elements of different types. If you want to have a list with different types, you must first create a new user-defined type (see defining Inductive objects, section **??**) or use a tuple (see **??**). From there, you can use the following notations (general case on the left, list of numbers on the right):

```
v1::v2:: ... ::[ ]
```

```
1::2::3::4::5::6::7::8::9::0::[ ]
```

```
[ v1; ...; vN ]
```

```
[ 1; 2; 3; 4; 5; 6; 7; 8; 9; 0 ]
```

The empty list is always denoted as [ ]; When using the double colon appended list, you must have the empty list as the right-most element.
You can also define your own lists like they are defined in Coq; for example, this defines a list of nat numbers:

```
Inductive natlist : Type := | nil | cons (n : nat) (l : natlist).
```

The following functions are defined in the List library; for examples of the use of most of these, please see Example 10.5.

length :	number of elements in list
head :	first element (with default)
tail :	all but first element
app :	concatenation
rev :	reverse
nth :	accessing n-th element (with default)
map :	applying a function
flat_map :	applying a function returning lists
fold_left :	iterator (from head to tail)
fold_right :	iterator (from tail to head)

4.5 Tuples

A tuple is given by two or more comma separated objects enclosed in parentheses. Tuples can contain items of the same or different types.

```
(v1, ..., vN)
```

```
(1, 2, 3)
```

4.6 Boolean Expressions for Branches

To use boolean expressions on numbers in an if then else, make sure you've imported Arith and loaded Bool:

```
Require Import Arith.
Load Bool.
```

Checking less than:

```
x <? y
```

Checking equal to:

```
x =? y
```

Checking less than or equal to:

```
x <=? y
```

To my knowledge, there is not pre-defined notation in Coq for greater than or not equal operations - however, you can get around this fairly easily by defining the functionality and notations yourself (see Example 10.3).

4.7 Option Types

When using some built-in functions in Coq, you may come across option types. Option types are particularly useful when you want to return an element if it is found, or be told explicitly that it does not exist. In other languages, you would either need to have two separate functions, one to check if it exists and one to obtain the value, or return some sort of 'neutral' value, like -1 or 0, to indicate an item wasn't found. Option types are defined as:

```
Inductive option (A : Type ) : Type :=
  | None : option A
  | Some : A -> option A
```

and an example of it being used:

```
Fixpoint at_n (n : nat) (l : Datatypes.list nat) : option nat :=
  match l with
    | [ ] => None
    | hd :: tl => if (n =? 0)
                  then Some hd
                  else at_n (n - 1) tl
  end.
```

In this function, we are recursively going through the list to find the item at position n in the list. To do this, at each iteration we are checking if we have found the empty list, if so we return None; otherwise we pull apart the head element of the list from the tail of the list, and if n is 0, then we return Some with that head element; if n is greater than 0, then we make the recursive call on n -1 and the tail of the list. The function will execute until we either find the empty list or the element at position n. We can then use pattern matching on the option type to obtain and use the value that was found or do something else having found that the value doesn't exist.

5 Using External Files/Libraries

5.1 Standard Libraries

Logic	Classical logic and dependent equality
Arith	Basic Peano arithmetic
PArith	Basic positive integer arithmetic
NArith	Basic binary natural number arithmetic
ZArith	Basic relative integer arithmetic
Numbers	Various approaches to natural, integer and cyclic numbers (currently axiomatically and on top of 2^{31} binary words)
Bool	Booleans (basic functions and results)
Lists	Monomorphic and polymorphic lists (basic functions and results), Streams (infinite sequences defined with co-inductive types)
Sets	Sets (classical, constructive, finite, infinite, power set, etc.)
FSets	Specification and implementations of finite sets and finite maps (by lists and by AVL trees)
Reals	Axiomatization of real numbers (classical, basic functions, integer part, fractional part, limit, derivative, Cauchy series, power series and results,...)
Relations	Relations (definitions and basic results)
Sorting	Sorted list (basic definitions and heapsort correctness)
Strings	8-bits characters and strings
Wellfounded	Well-founded relations (basic results)

5.2 Require

To use the standard libraries or other compiled files, you first need to tell the environment that it needs to load the compiled file. Require adds the specified module and all of its dependencies to the environment.

```
Require Logic.
```

Require Import loads the specified module and its dependences, then imports the contents of the specified module.

```
Require Import Bool.
```

Require Export acts like Require Import, but will ensure that any module B that uses Require Import on the module A that contained the Require Export command will import both module A and the one specified in the Require Export command.

```
Require Export Bool.
```

5.3 Load

This is used to load a file or library into the current environment. To load one of the standard libraries, you can simply use the Load command.

```
Load Arith.
```

However, to load any other existing file, you will likely need to specify where to look for the file; to do this, there is the Add LoadPath command. All the commands in the loaded file will be evaluated

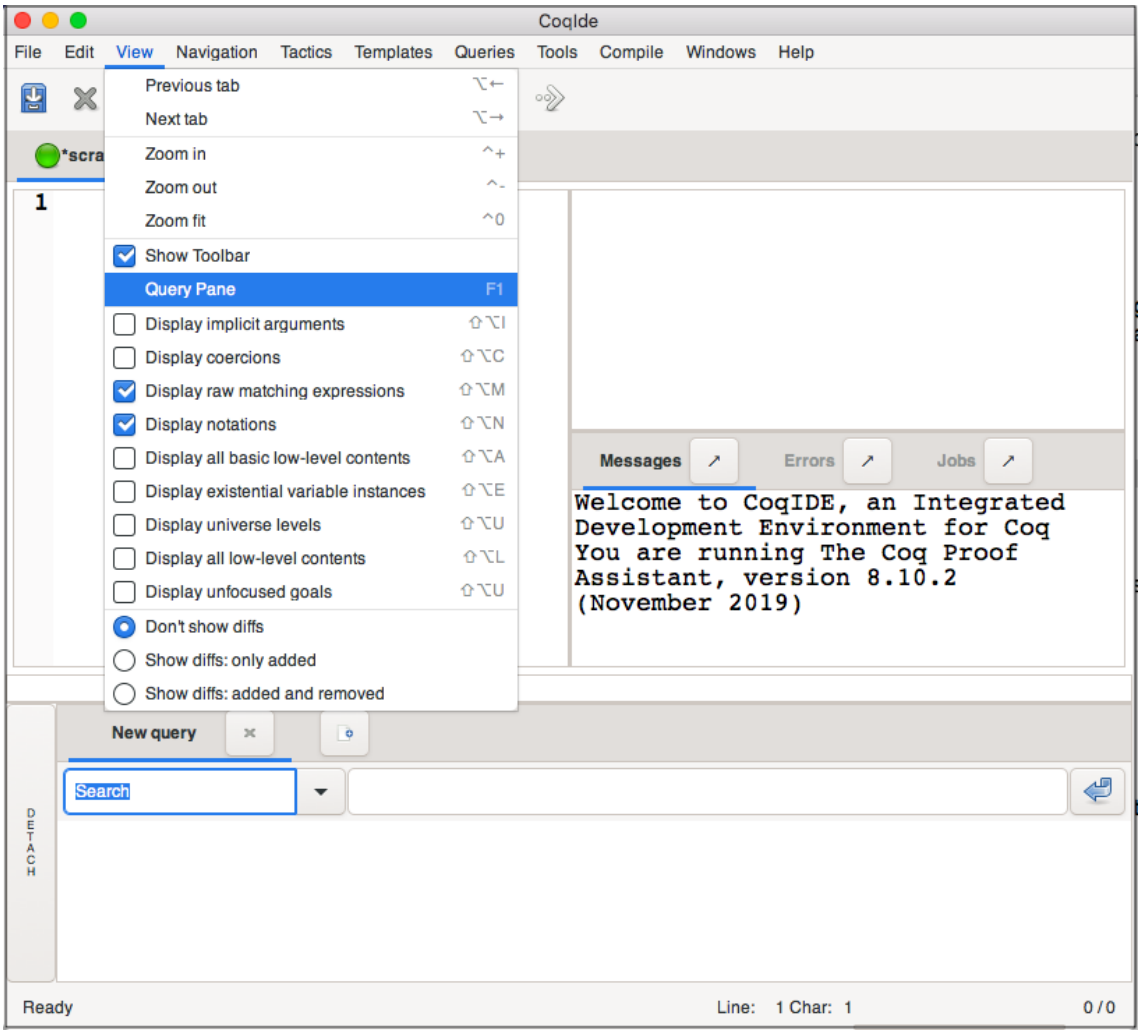
```
Add LoadPath "/myDirectory/path/".
Load myFile.
```

6 Queries

You can insert queries into your file (despite the Coq IDE giving a warning), or you can use the query pane. The results of a query will appear in the messages pane. There are many types of queries, including the ones described below. See file "Queries.v".

6.1 Query Pane

To open the query pane, either press F1 or go to the View tab, then down to the Query Pane option. The query pane is more convenient when you're in the middle of a proof but need to look up something; however, when doing multiple queries from the query pane, the results will show up in the messages tab without clearing the results of previous queries.



You can detach the query pane into its own window by pressing the Detach button along the lefthand side of the query pane.

6.2 Search

Searches the environment for the given name, then displays the name and type of all objects that contains the name. This is useful to find out information about libraries and pre-defined theorems.

```
Search plus.
```

```
plus_O_n: forall n : nat, 0 + n = n
plus_n_O: forall n : nat, n = n + 0
plus_n_Sm: forall n m : nat, S (n + m) = n + S m
plus_Sn_m: forall n m : nat, S n + m = S (n + m)
mult_n_Sm: forall n m : nat, n * m + n = n * S m
f_equal2_plus: forall x1 y1 x2 y2 : nat, x1 = y1 -> x2 = y2 -> x1 + x2 = y1 + y2
nat_rect_plus:
  forall (n m : nat) (A : Type) (f : A -> A) (x : A),
  nat_rect (fun _ : nat => A) x (fun _ : nat => f) (n + m) =
  nat_rect (fun _ : nat => A) (nat_rect (fun _ : nat => A) x (fun _ : nat => f) m)
    (fun _ : nat => f) n
```

Searching multiple names obtains more specific results.

```
Search plus 0.
```

```
plus_O_n: forall n : nat, 0 + n = n
plus_n_O: forall n : nat, n = n + 0
```

If you haven't loaded the library corresponding to what you are searching, you will get fewer results; you may need to load the library if you can't find what you need. For example,

```
Search plus Nat.odd.
```

yields no results when the Arith library hasn't been added, but once we have imported the library we receive five results.

```
Require Import Arith.
Search plus Nat.odd.
```

```
Nat.odd_add_even: forall n m : nat, Nat.Even m -> Nat.odd (n + m) = Nat.odd n
Nat.odd_add: forall n m : nat, Nat.odd (n + m) = xorb (Nat.odd n) (Nat.odd m)
Nat.odd_add_mul_even: forall n m p : nat, Nat.Even m -> Nat.odd (n + m * p) = Nat.odd n
Nat.odd_add_mul_2: forall n m : nat, Nat.odd (n + 2 * m) = Nat.odd n
Nat.div2_odd: forall a : nat, a = 2 * Nat.div2 a + Nat.b2n (Nat.odd a)
```

You can also search for patterns: enclose the pattern in parentheses, and use an underscore for arbitrary terms.

```
Search (~ _ <-> _ ).
```

```
neg_false: forall A : Prop, ~ A <-> (A <-> False)
not_iff_compat: forall A B : Prop, A <-> B -> ~ A <-> ~ B
```

6.3 SearchRewrite

Searches the environment for a statement with an equality that contains the given pattern on at least one side.

```
SearchRewrite (_ * _ / _ ).
```

```
Nat.div_mul: forall a b : nat, b <> 0 -> a * b / b = a
Nat.divide_div_mul_exact: forall a b c : nat, b <> 0 -> Nat.divide b a -> c * a / b = c * (a / b)
Nat.div_mul_cancel_l: forall a b c : nat, b <> 0 -> c <> 0 -> c * a / (c * b) = a / b
Nat.div_mul_cancel_r: forall a b c : nat, b <> 0 -> c <> 0 -> a * c / (b * c) = a / b
Nat.lcm_equiv1: forall a b : nat, Nat.gcd a b <> 0 -> a * (b / Nat.gcd a b) = a * b / Nat.gcd a b
Nat.lcm_equiv2: forall a b : nat, Nat.gcd a b <> 0 -> a / Nat.gcd a b * b = a * b / Nat.gcd a b
```

6.4 Check

Displays the type of the given term.

```
Check 0.
```

```
0 : nat
```

```
Check nat.
```

```
nat : Set
```

```
Check plus.
```

```
Nat.add : nat -> nat -> nat
```

6.5 About

Displays information about the object with the given name (i.e. its kind, type, arguments).

```
About plus.
```

```
Notation plus := Init.Nat.add
Expands to: Notation Coq.Init.Peano.plus
```

6.6 Locate

Displays the full name of the given term and what Coq module it is defined in.

```
Locate plus.
```

```
Notation Coq.Init.Peano.plus
```

```
Locate nat.
```

```
Inductive Coq.Init.Datatypes.nat
```

6.7 Print

Displays information about the given name.

```
Print plus.
```

```
Notation plus := Init.Nat.add
```

It is possible to see what is currently loaded and imported in the environment with the following:

```
Print Libraries.
```

```
Loaded and imported library files:
  Coq.Init.Notations
  Coq.Init.Logic
  Coq.Init.Logic_Type
  Coq.Init.Datatypes
  Coq.Init.Specif
  Coq.Init.Peano
  Coq.Init.Wf
  Coq.Init.Tactics
  Coq.Init.Tauto
  Coq.Init.Prelude
Loaded and not imported library files:
  Coq.Init.Nat
```

6.8 Print Assumptions

Displays all assumptions depended upon by the object of the given name.

```
Print Assumptions plus.
```

```
Closed under the global context
```

7 Defining in Coq

Please see file "Defining.v" to follow along with this code in the CoqIDE.

7.1 Sorts

There are three main Sorts for defining types: **Prop**, **Set**, and **Type**.

- **Prop**: This type is for logical propositions.
- **Set**: This type is for small sets, such as booleans (bool) and natural numbers (nat).
- **Type**: This type can be used for small sets as well as larger sets - it encompasses both **Prop** and **Set**.

These are used in defining Inductive types, as shown in the next section.

7.2 Inductive

The command Inductive is used to define simple inductive types and the constructors used in the type. The name is placed directly after the keyword Inductive, when a colon followed by the Sort (discussed in the previous section) of the inductive type you are defining. This is followed by := and the constructors (i.e. elements) included in that type. For example, boolean values are defined in Coq as follows:

```
Inductive bool : Set :=
  | true
  | false.
```

```
bool is defined
bool_rect is defined
bool_ind is defined
bool_rec is defined
```

and nat numbers are defined as:

```
Inductive nat : Set :=
  | O : nat
  | S : nat -> nat.
```

```
nat is defined
nat_rect is defined
nat_ind is defined
nat_rec is defined
```

You can see what all the definitions it created are by using Print:

```
Print nat.
```

```
Inductive nat : Set := O : nat | S : nat -> nat
```

```
Print nat_rect.
```

```
nat_rect =
fun (P : nat -> Type) (f : P O) (f0 : forall n : nat, P n -> P (S n)) =>
fix F (n : nat) : P n := match n as n0 return (P n0) with
            | O => f
            | S n0 => f0 n0 (F n0)
            end
     : forall P : nat -> Type, P O -> (forall n : nat, P n -> P (S n)) -> forall n : nat, P n

Argument scopes are [function_scope _ function_scope _ ]
```

```
Print nat_ind.
```

```
nat_rect =
fun (P : nat -> Prop) (f : P O) (f0 : forall n : nat, P n -> P (S n)) =>
fix F (n : nat) : P n := match n as n0 return (P n0) with
              | O => f
              | S n0 => f0 n0 (F n0)
              end
       : forall P : nat -> Prop, P O -> (forall n : nat, P n -> P (S n)) -> forall n : nat, P n

Argument scopes are [function_scope _ function_scope _ ]
```

```
Print nat_rec.
```

```
nat_rec =
fun P : nat -> Set => nat_rect P
       : forall P : nat -> Set, P O -> (forall n : nat, P n -> P (S n)) -> forall n : nat, P n

Argument scopes are [function_scope _ function_scope _ ]
```

You can use these properties of what you've defined in proofs.

Similarly, you can define days of the week:

```
Inductive day: Type :=
  | monday : day
  | tuesday : day
  | wednesday : day
  | thursday : day
  | friday : day
  | saturday : day
  | sunday : day.
```

```
day is defined
day_rect is defined
day_ind is defined
day_rec is defined
```

or lists of natural numbers:

```
Inductive natlist Type :=
    | nil
    | cons (n:Datatypes.nat) (l:natlist).
```

```
list is defined
list_rect is defined
list_ind is defined
list_rec is defined
```

It is also possible to define polymorphic lists:

```
Inductive list (A: Set) : Set :=
    | nil : list A
    | cons : A -> list A.
```

```
list is defined
list_rect is defined
list_ind is defined
list_rec is defined
```

(polymorphic lists will not be used in the remainder of the tutorial - it is just an example of a more complex construct that can be defined in COQ).

7.3 Definition

The command Definition is used to bind an name to some term. The name is always placed directly after the keyword Definition, and the term to bind to the name is given after :=. For example, we can give the name x a simple value of 4:

```
Definition x := 4.
```

```
x is defined
```

or we can use this to define functions, such as the following simple function (using the previous weekday inductive type definition), taking a weekday as input and giving back the weekday as output. Here, we are specifying the parameter d of type day must be given to the function. When giving a parameter, you give the $(paramName : type)$ as in $(d : day)$. The parameters are then followed by the return type, as in $: returnType$. This is shown in the following example.

```
Definition next_weekday (d:day) : day :=
  match d with
    | monday => tuesday
    | tuesday => wednesday
    | wednesday => thursday
    | thursday => friday
    | friday => monday
    | saturday => monday
    | sunday => monday
  end.
```

```
next_weekday is defined
```

Another simple example function definition (using the previous nat number inductive type definition) is taking a nat number and returning that result of adding 2 to that nat number:

```
Definition plus2 (n:nat) : nat :=
  match n with
    | O => S (S O)
    | _ => S (S n)
  end.
```

```
plus2 is defined
```

You can also give multiple parameters, as shown in the example below using 3 parameters:

```
Definition choose1 (b: bool) (n1: Datatypes.nat) (n2: Datatypes.nat) :
Datatypes.nat :=
  match b with
    | true => n1
    | false => n2
  end.
```

```
choose1 is defined
```

We have to specify that we would like to use Datatypes.nat as our type in order to use natural numbers (i.e. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9), as we have not specified how to interpret these numbers from our definition of nat (you can do this using the Notation command - using this command will be discussed in the following subsection 7.4). Alternately, we can define the same function without giving the types of the parameters (however, it is always best practice to declare the types of all parameters, to ensure they are interpreted as you expect them to be). When you do not specify the types of parameters, the parentheses around the parameter names are optional.

```
Definition choose1′ b n1 n2 : Datatypes.nat :=
  match b with
    | true => n1
    | false => n2
  end.
```

```
choose1′ is defined
```

Both *choose1* and *choose1′* have the same functionality.

7.4 Notation

The command Notation is used to define a short-hand way of writing a concept we have previously defined - for example, take our list definition from subsection 7.2. We can then make the shorthand notation where [ ] refers to *nil*, or the empty list, and $x :: xs$ refers to (*cons* x xs), or appending some number x to some list xs.

```
Notation "[ ]" := nil.
Notation "x :: xs" := (cons x xs).
```

```
Setting notation at level 0.
```

Now we are able to write lists more simply (shown compared to the identical list using *cons* and *nil*):

```
Compute 3::2::1::[ ].
Compute (cons 3 (cons 2 (cons 1 nil))).
```

```
= 3 :: 2 :: 1 :: [ ]
: natlist
```

This can be used to define any number of concepts that you may desire, including notations for the boolean expressions for checking where some number is greater than some other number (which are not natively defined, as discussed in subsection 4.6). These notations and definitions are shown in example 10.3.

7.5 Fixpoint

7.6 Compute

The command Compute evaluates the term that follows it.

8 Proof Environment

8.1 Assertions

To switch on the proof environment, you first need to use one of the keywords for an assertion. It does not matter which one you choose, all have the same behavior - it is more a matter of personal preference.
These keywords are:

Theorem, Lemma, Corollary, Proposition, Fact, Goal, Example, Remark

After using one of the keywords, you will give your assertion a name, followed by a semi-colon, then write out the body of your assertion, followed by a period.
The theorem template (Templates tab → Theorem) is shown below:

```
Theorem new_theorem : .
Proof.

Qed.
```

8.2 Proof Environment Commands

Proof.	Begins the proof.
Qed.	Ends and saves the proof; will fail if the assertion is not fully proven.
Admitted.	Gives up the current proof; declares the initial goal as an axiom that can be used in other proofs (but will need to be fully proven later).
Abort.	Cancels the current proof development.

9 Proof Tactics

This section contains brief descriptions of some commonly used proof tactics. For the full breakdown of all Coq tactics, please refer to the Tactic Index of the Coq documentation that can be found here: `https://coq.inria.fr/distrib/current/refman/coq-tacindex.html`.

9.1 simpl

Simplify both sides of an equation.
Examples using this tactic: 10.7, 10.8

9.2 reflexivity

Check if both sides of an equation are equal. Will do more simplifications that simpl; tries unfolding and expanding definitions.
Examples using this tactic: 10.8

9.3 trivial

A restriction of auto that is not recursive. Tries simple hints, like solving trivial equalities such as x=x.
Examples using this tactic: 10.7, 10.8

9.4 auto

First attempts to use assumption, then uses intros and generates any hypotheses to use as hints and attempt to apply.
Examples using this tactic:

9.5 ring

Very useful when proving over numbers. Solves equations upon polynomial expressions of a ring structure. Normalizes and compares results. Uses properties like associativity, commutativity, distributivity, and constant propagation.
Examples using this tactic: 10.8

9.6 discriminate

Proves any goal in an assumption stating two structurally different terms of an inductive set are equal. For example, S (S O) = S O.
Examples using this tactic:

9.7 assumption

Looks in the local context for the hypothesis. The type must be convertible to the goal.
Examples using this tactic:

9.8 intros

Introduces variables from the goal into the proof environment for use.
Examples using this tactic: 10.7, 10.8

9.9 contradiction

Attempts to find a hypothesis equivalent to:

- an empty inductive type (i.e. false)
- the negation of a single inductive type (i.e. true, x=x)
- two contradictory hypotheses

Examples using this tactic:

9.10 induction

The object must be of an inductive type to use induction. This generates subgoals and an induction hypothesis.
Examples using this tactic: 10.7

9.11 functional induction

Very useful for inductive/recursive functions. Performs case analysis and induction on the definition of a function. To use functional induction, you need to make sure to require and load FunInd, and then define the functional induction scheme.

```
Require FunInd.
Load FunInd.

Functional Scheme name_ind := Induction for name Sort Prop.
```

Examples using this tactic: 10.8

9.12 destruct

Creates subgoals from a more complex goal. Subgoals must be proven separately. Can be used with any inductively defined type. Can be used nested if a generated subgoal needs to be broken up further. Allows you to specify the names of variables to be used in proving the subgoals.
Examples using this tactic:

9.13 rewrite

A way to apply previously defined assumptions. Can use arrows $<-$, $->$ to give directionality for the application. Example 10.8 demonstrates the use of arrows to give directionality to rewrite.
Examples using this tactic: 10.7, 10.8

9.14 apply

Attempts to match the current goal against the conclusion of the given term.
Examples using this tactic:

10 Examples

10.1 Cards

A simple example defining suits and values for cards, and what a valid card is. In the function check_num, we check if the number given is less than 11 and greater than 1. In function is_valid_card, we pattern match against various types of cards. Here, the given card is represented by the variable x. The type card is defined to be c (s: suit) (v: val), and we know that an s of type suit can only be one of the valid suits we defined, so we can use some variable, say q, to represent allowing any suit type; the part we need to ensure is valid is the value given to the suit, since a valid card can only have a value of an ace, king, queen, jack, or number values 2 thru 10.

```
Load List.
Load Bool.

Inductive suit : Type := | heart | diamond | spade | club.
Inductive val : Type := | ace | king | queen | jack | num (n: nat).
Inductive card : Type := | c (s: suit) (v: val).

(* Card: 2 of Spades *)
Check (c spade (num 2)).

(* List of suits *)
Check (ace::king::[ ]).

(* List of Cards *)
Check [c heart ace; c diamond king; c spade queen].

Definition check_num (x: nat) : bool :=
  if (x <? 11)
  then if (1 <? x)
    then true
    else false
  else false.

Definition is_valid_card (x: card) : bool :=
  match x with
  | c q ace => true
  | c q king => true
  | c q queen => true
  | c q jack => true
  | c q (num n) =>
      if (check_num n)
      then true
      else false
  end.

(* Ace of Hearts is a valid card *)
Compute is_valid_card (c heart ace).

(* 11 of Diamonds is NOT a valid card *)
Compute is_valid_card (c diamond (num 11)).
```

10.2 Boolean Operations

Simple example of the definition of booleans and some boolean operations.

```
Inductive bool := | true | false.

Definition not (b : bool) : bool :=
  match b with
  | true => false
  | false => true
  end.

Definition and (b1 b2 : bool) : bool :=
  match (b1, b2) with
  | (true, true) => true
  | (_ , _ ) => false
  end.

Definition or (b1 b2 : bool) : bool :=
  match (b1, b2) with
  | (false, false) => false
  | (_ , _ ) => true
  end.

Definition xor (b1 b2 : bool) : bool :=
  match (b1, b2) with
  | (true, false) => true
  | (false, true) => true
  | (_ , _ ) => false
  end.
```

10.3 Boolean Expressions for Branches

Simple example defining less than, equal to, and less than or equal to operations over numbers.

```
Require Import Arith.
Load Bool.

Definition lt (n m : nat) : bool :=
    if (n <? m)
    then true
    else false.
Notation "n < m" := (lt n m).

Definition gt (n m : nat) : bool :=
    if (m <=? n)
    then true
    else false.
Notation "n > m" := (gt n m).

Definition eq (n m : nat) : bool :=
    if (n =? m)
    then true
    else false.
Notation "n = m" := (eq n m).

Definition neq (n m : nat) : bool :=
    if (n =? m)
    then false
    else true.

Definition lteq (n m : nat) : bool :=
    if (n <=? m)
    then true
    else false.
Notation "n <= m" := (lteq n m).

Definition gteq (n m : nat) : bool :=
    if (m <? n)
    then true
    else false.
Notation "n >= m" := (gteq n m).
```

10.4 Days of the Week

This example ...

First, we define what a day is using the following definition:

```
Inductive day: Type :=
  | monday : day
  | tuesday : day
  | wednesday : day
  | thursday : day
  | friday : day
  | saturday : day
  | sunday : day.
```

Then we define a simple function to compute what the next weekday is after a given day. This takes a day as input, and gives back what the next weekday would be.

```
Definition next_weekday (d:day): day :=
  match d with
    | monday => tuesday
    | tuesday => wednesday
    | wednesday => thursday
    | thursday => friday
    | friday => monday
    | saturday => monday
    | sunday => monday
  end.
```

We can perform some simple computations to check the correctness of our function:

```
Compute (next_weekday friday).
```

```
= monday : day
```

```
Compute next_weekday (next_weekday friday).
```

```
= tuesday : day
```

We can also do a simple proof to check that the results are as expected.

```
Example test_next_weekday:
  (next_weekday (next_weekday saturday))
    = tuesday.
Proof.
```

```
1 subgoal
----------------------------------(1/1)
next_weekday (next_weekday saturday) =
tuesday
```

10.5 List

A simple example to demonstrate some uses of the built-in lists and list functions, and define a search function for lists of nat numbers. To use the notations and functions, you'll want to load List and open the list_scope so everything works properly.

```
Require Import Arith.
Load List.
Open Scope list_scope.

Fixpoint search (l : Datatypes.list nat) (n: nat) : bool :=
  match l with
    | [ ] => false
    | hd :: tl =>
        if (n =? hd)
        then true
        else search tl n
  end.
```

Uses of the list functions and results:

```
Compute search [1;3;6;8] 5.
```
```
= false : bool
```

```
Compute length [1;2;3].
```
```
= 3 : nat
```

```
Compute head [2;4;6].
```
```
= Some 2 : option nat
```

```
Compute tail [3;6;9].
```
```
= [6; 9] : Datatypes.list nat
```

```
Compute app [1;2] [3;4].
```
```
= [1; 2; 3; 4] : Datatypes.list nat
```

```
Compute [1;2] ++ [3;4].
```
```
= [1; 2; 3; 4] : Datatypes.list nat
```

```
Compute 1::[2;3;5;7].
```
```
= [1; 2; 3; 5; 7] : Datatypes.list nat
```

```
Compute rev [9;7;5;3;1].
```
```
= [1; 3; 5; 7; 9] : Datatypes.list nat
```

```
Compute nth 0 [2;5;7;9].
```
```
= fun _ : nat => 2 : nat -> nat
```

```
Compute nth 4 [2;5;7;9].
(* Not found *)
```

```
= fun default : nat => default
: nat -> nat
```

```
Compute map (Nat.mul 3) [1;2;3].
```

```
= [3; 6; 9] : Datatypes.list nat
```

10.6 List Rev

10.7 Count to n

A simple example defining a function that takes a number n and counts to n. We then work through proving that count n = n.

```
Require Import Arith.

Fixpoint count (n : nat) : nat :=
  match n with
    | O => 0
    | S p => 1 + count p
  end.

Compute count 10.
```

Now, we can state our theorem, and begin our proof. We will see that we are initially given one subgoal as follows:

```
Theorem count_to_n :
  forall n:nat, count n = n.
Proof.
```

```
1 subgoal
______________________________________(1/1)
forall n : nat, count n = n
```

Then we need to introduce the variable n into the proof environment:

```
Proof.
intros n.
```

```
1 subgoal
n : nat
______________________________________(1/1)
count n = n
```

We will use induction to split this goal into two subgoals because nat numbers are inductively defined:

```
Proof.
intros n.
induction n as [| n' IHn'].
```

```
2 subgoals
______________________________________(1/2)
count 0 = 0
______________________________________(2/2)
count (S n') = S n'
```

Then we enter into a subgoal environment, and see that it focuses on the 1st subgoal:

```
Proof.
intros n.
induction n as [| n' IHn'].
{ -
```

```
1 subgoal
______________________________________(1/1)
count 0 = 0
```

This result is trivial to solve, and we can use the proof tactic trivial:

```
Proof.
intros n.
induction n as [| n' IHn'].
{ - trivial.
```

```
This subproof is complete, but there are some
unfocused goals:
______________________________________(1/1)
count (S n') = S n'
```

Now we close out this subgoal and move to the next involving the induction hypothesis:

```
Proof.
intros n.
induction n as [| n' IHn'].
{ - trivial. }
{ -
```

```
1 subgoal
n' : nat
IHn' : count n' = n'
----------------------------------------(1/1)
count (S n') = S n'
```

Our first goal here is to obtain what is on the left-hand side of the induction hypothesis equation within our subgoal. we can do this using the proof tactic simpl:

```
Proof.
intros n.
induction n as [| n' IHn'].
{ - trivial. }
{ - simpl.
```

```
1 subgoal
n' : nat
IHn' : count n' = n'
----------------------------------------(1/1)
S (count n') = S n'
```

Now we are able to use the rewrite proof tactic to rewrite our goal using the induction hypothesis:

```
Proof.
intros n.
induction n as [| n' IHn'].
{ - trivial. }
{ - simpl. rewrite IHn'.
```

```
1 subgoal
n' : nat
IHn' : count n' = n'
----------------------------------------(1/1)
S n' = S n'
```

As you can see, both sides of our goal are equal, and we can use trivial again to finish off this goal, and Coq will then tell us we have no more subgoals:

```
Proof.
intros n.
induction n as [| n' IHn'].
{ - trivial. }
{ - simpl. rewrite IHn'. trivial.
```

```
No more subgoals.
```

Then we can close out this proof and see that it is now defined in the environment:

```
Theorem count_to_n :
  forall n:nat, count n = n.
Proof.
intros n.
induction n as [| n' IHn'].
{ - trivial. }
{ - simpl. rewrite IHn'. trivial. }
Qed.
```

```
count_to_n is defined
```

10.8 Sum to n

This example defines a recursive function that takes a number n and returns the sum of the numbers from one to n. It then proves that for all nat numbers, 2 * sum n = (1 + n) * n. This assertion is equivalent to sum n = ((1 + n) * n) / 2; however, it is much easier to prove when we do not involve division. First, a longer proof using properties of addition and multiplication will be shown, then the shorter version that applies the ring tactic to take care of the application of many of these properties for us.

```
Load Arith.

Fixpoint sum (n : nat) : nat :=
  match n with
  | O => 0
  | S p => (S p) + (sum p)
  end.
```

In this example, we will use functional induction to help us successfully complete our proof over the recursive function sum. To do this, we must first make sure to require and load FunInd, and then define the functional induction scheme.

10.9 Sum of Squares

11 Troubleshooting

11.1 Empty IDE

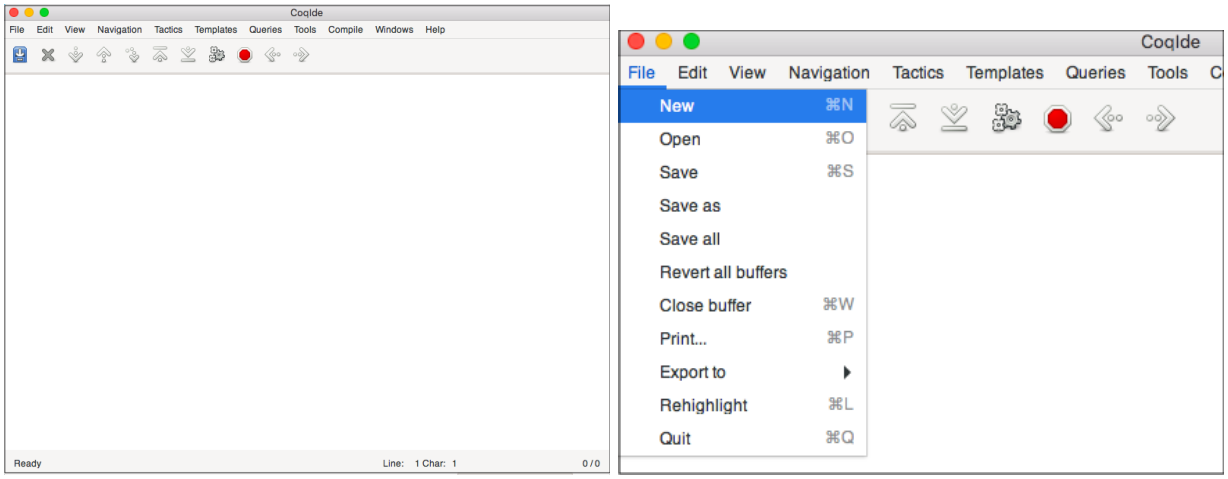


Figure 3: CoqIDE v8.10.2 (a) empty IDE screen (b) getting new scratch script buffer open

Go to "File" then "New" to get a new scratch script buffer open if you've accidentally closed out all of your script buffers. Alternately, you can go to "File" then "Open" if you'd like to open a saved file.

11.2 Can't close "Load File" Window

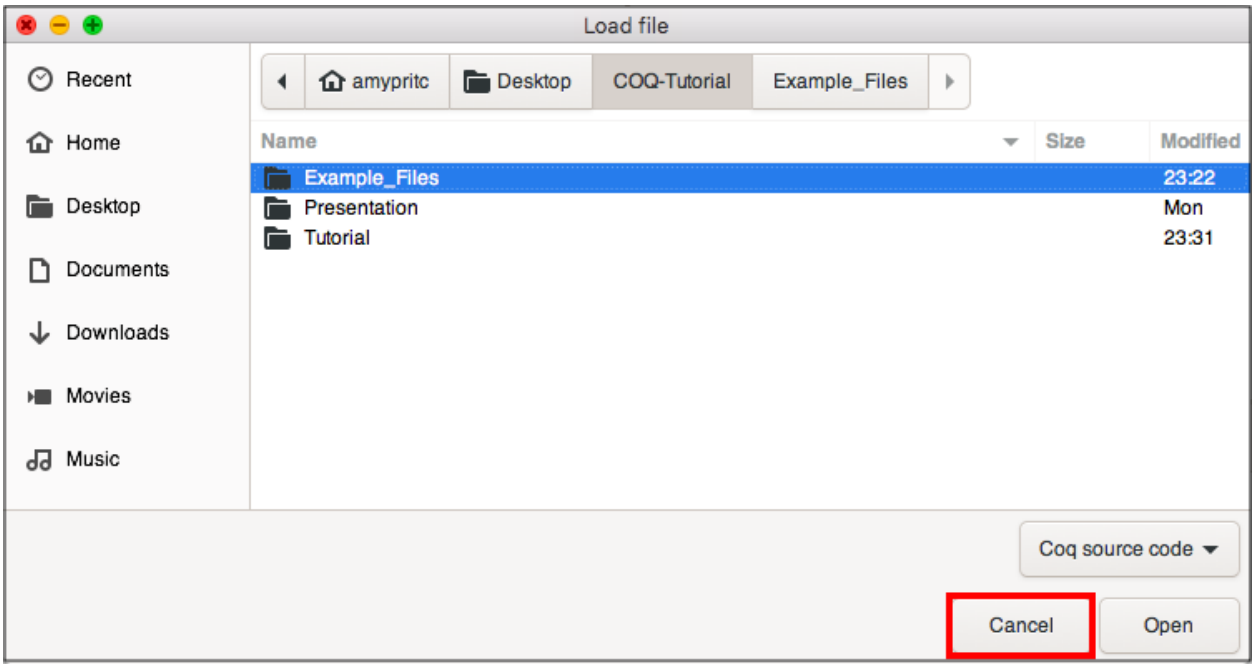


Figure 4: CoqIDE v8.10.2 Load File window

If the "Load File" window won't close using the red X in the upper left-hand corner, use the "Cancel" button in the lower right (outlined in red).

12 References

Coq Proof Assistant Webpage
https://coq.inria.fr

Coq Documentation
https://coq.inria.fr/distrib/current/refman/

Christine Paulin-Mohring's course notes on Coq
https://www.lri.fr/~paulin/LASER/course-notes.pdf

DeepSpec Summer School COQ Intensive July 2017
https://deepspec.org/event/dsss17/coq_intensive.html

Software Foundations Books
https://softwarefoundations.cis.upenn.edu

Induction in Coq
http://www.cs.cornell.edu/courses/cs3110/2018sp/l/22-coq-induction/notes.html

CompCert: Verified C Compiler
http://compcert.inria.fr/motivations.html

CertiCrypt: Computer-Aided Cryptographic Proofs in Coq
http://certicrypt.gforge.inria.fr