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In Search of Insight

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This paper describes the process of attaining the insight required to solve a particular problem—the Mutilated Checkerboard (MC) problem. It shows that attaining insight requires discovering an effective problem representation, and that performance on insight problems can be predicted from the availability of generators and constraints in the search for such a representation. To test these claims we varied the salience of features leading to the critical concept of parity in the MC problem. Using chronometric measures, verbal protocols, and computer simulations, we explored first why it is difficult to find a representation for the Checkerboard problem, and then tested four potential sources of search constraint for reducing the difficulty: cue salience manipulations, prior knowledge, hints, and heuristics. While subjects used each of these four sources of constraint, a particular heuristic—noticing properties of the situation that remained invariant during solution attempts (the Notice Invariants heuristic)—proved to be a particularly powerful means for focusing search. In conjunction with hints and independently, it played a major part in producing the insight that yielded an effective problem representation and solution.

Most of us have experienced insight while trying to solve a problem—an AHA! experience in which we suddenly felt that we knew the answer, even if not its details. Although numerous definitions of insight have been offered (Dominowski, 1981; Ellen, 1982; Ohlsson, 1984a, 1984b; Weisberg & Alba, 1981a, 1981b, 1982), many researchers consider the subjective AHA! feeling to be a critical component (Duncker, 1945; Kohler, 1956, 1969; Posner, 1973; Worthy, 1975). Moreover, the AHA! experience figures prominently in anecdotal accounts of insightful discoveries (Ghiselin, 1952; Hadamard, 1949; Haefele, 1962). For the purposes of this paper,

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we will use insight to refer to a subjective AHA! experience during problem solving.

While there are many potential causes of surprise in problem solving, we will focus on insight resulting from changes in representation. Such changes seem to typify many of the problems used in past experimental studies. For example, subjects in Duncker's functional fixedness experiments must re-represent the function of a critical object before they can reach a solution. Duncker clearly considered this "restructuring" to be the essential source of difficulty (as well as of the Aha! experience) when he wrote:

> It has often been pointed out that such restructurations play an important role in thinking, in problem solving. The decisive points in thought processes, the moments of sudden comprehension, of the "Aha!," of the new, are always at the same time moments in which such a sudden restructuring of the thought material takes place, in which something "tips over." (Duncker, 1945, p. 29)

A second example of insight co-occurring with representational change can be found in Kohler's experiments with chimpanzees. The way in which the ape Sultan insightfully joins two sticks to form a pole long enough to reach some bananas outside his cage surely involves representational change (Kohler, 1956). Before his insight, Sultan knew he could use a single stick as a pole, and in fact tried to reach the fruit this way but failed. After his insight, Sultan clearly acquired the concept of joining two sticks to make one, and was able to use this method on subsequent trials with little delay. This change in Sultan's representation of the potential uses for a stick is quite sudden, and seems to be accompanied by the chimpanzee equivalent of an AHA!:

> Sultan first of all squats indifferently on the box, which has been left standing a little back from the railings; then he gets up, picks up the two sticks, sits down again on the box and plays carelessly with them. While doing this, it happens that he finds himself holding one rod in either hand in such a way that they lie in a straight line; he pushes the thinner one a little way into the opening of the thicker, jumps up and is already on the run towards the railings, to which he has up till now half turned his back, and begins to draw a banana towards him with the double stick. (Kohler, 1956, p. 127)

Similar accounts of change of representation can be given for many other insight problems including the nine dots problem (Burnham & Davis, 1969; Lung & Dominowski, 1985; Weisberg & Alba, 1981a), the match stick problems (Katona, 1940), and the two-string problem (Maier, 1931). However none of the published accounts of these problems provides a process theory showing how change of representation occurs.

Since Duncker, Kohler, and other Gestaltists wrote on insight, much progress has been made to provide an information processing account of problem solving (e.g., Newell & Simon, 1972). However, the IP theory has had little to say about the AHA! phenomenon or insight generally. Newell and Simon (1972, pp. 90–91) observe explicitly that their theory does not deal with changes in problem representation. They report having observed only one subject (S8 on Cryptarithmetic problems) who paid specific attention to choosing a problem space. (Throughout this paper, we will use the terms *problem space* and *representation* as synonyms.) And Hayes and Simon, in their work on problem understanding (Hayes & Simon, 1974; Simon & Hayes, 1976) demonstrate that subjects do not initially choose deliberately among problem representations, but almost always adopt the representation suggested by the verbal problem statement.

The research reported here represents a major extension of the standard information processing theory of problem solving to handle insight problems that require a change in representation for their solution. More specifically, we show that the same processes that are ordinarily used to search *within* problem space can be used to search *for* a problem space (problem representation). But since previous evidence shows that subjects do not often switch their representations, we must also explain how the search for a new representation is initiated and under what conditions it has chances of success.

Diamonds in the Dark

Imagine that you are searching for a diamond in a huge, dark room. What do you do? One option is to grope blindly in the dark. If you have a specific idea of where the diamond might be, this may even be an effective strategy. But after groping blindly for several minutes, you might decide to abandon the search for the diamond, and to search instead for a light switch. If one could be found, and the light turned on, the location of the diamond could be evident almost at once.

Subjects asked to solve insight problems are faced with an analogous situation. The subjects believe at first that they know just what to do, only to discover that they are groping in the dark. Re-representing the problem is equivalent to turning on the light. But how is this change of representation brought about?

Our metaphor suggests that search is one way to achieve a new representation. Subjects trying to solve an insight problem may switch from search for the solution within a given representation to search for a new representation altogether. Viewing change of representation as a search process allows us to account for both insightful and routine problem solving in the same theoretical framework. Moreover, this extension suggests that we ask about insightful problem solving many of the same questions that have been fruitfully asked about more routine types of problem solving.

Effective search relies on search generators that are sufficiently selective and constrained to look mostly in the likely places. Wanting or needing a new problem representation does not automatically provide a generator adequate for finding it. Hence, we must discover how subjects constrain their search for a new representation when the initial one does not suffice, and how the new representation constrains their search for the solution. We must also discover what triggers subjects to shift from searching for a problem solution to searching for a better problem space in which to conduct the solution search.[1]

Heuristic Search—Adding Rigor to the Metaphor

Newell and Simon's (1972) conception of problem solving as heuristic search through a problem space, combined with Simon and Lea's (1974) subsequent notion of a dual problem space for instances and hypotheses provide foundations for a rigorous theory. The dark room maps nicely onto the search space of the task environment—a space of all possible actions that might be taken to solve a particular problem. Each representation corresponds to a problem space, and problem solvers apply operators associated with that space in an attempt to transform their current (physical or mental) state into a state that satisfies their goal.

Within a given problem space, the trick lies in searching for the right operator to apply next. But if no operators seem to yield progress, one must search for a new problem space to explore. Both search within a problem space and search in the meta-level space of possible problem spaces is often enormously difficult unless constraints for the search can be found.

The crux in problem solving is selectivity in search. Humans almost never search completely randomly. For example, no one would try to find their misplaced car keys in that way. Heuristics that serve to constrain search can range from general ("When something is lost, think of the last time you had it") to specific ("I often leave my car keys in the ignition when I go shopping because I need both hands to carry groceries"). Cues in the environment (e.g., a buzzer that sounds when you leave the car keys in the ignition) can also serve to constrain search. A wide range of problem solving behavior can be understood by examining the heuristics and other sources of constraint on search. We will undertake to understand insight by similarly identifying the sources of search constraint that lead to success with the Mutilated Checkerboard problem.

[1] Depending on the type of problem, a fair amount of problem solving search *making use of* the critical representation may still be needed to complete the problem solution (e.g., the nine-dots problem, Weisberg & Alba, 1981a).

The Mutilated Checkerboard (MC) Problem

The MC problem has somewhat of a reputation both as an insight/puzzle problem (Anderson, 1985; Wickelgren, 1974) and as a challenge to problem solving programs in AI (Korf, 1980; McCarthy, 1964; Newell, 1965). Its difficulty stems from the fact that the initial representation that problem solvers almost always form fails to solve the problem. Subjects need to change their representation in a nonobvious way. Therefore, this problem falls outside the limits of the standard information processing theory of problem solving, and providing an account of it requires an essential extension of that theory.

The classic MC problem (see Fig. 1) employs a standard 8 × 8 checkerboard, two of whose diagonally opposite corners have been removed. Subjects are told to imagine placing dominos on the board so that one domino covers two horizontally or vertically (but not diagonally) adjacent squares. The problem is either to show how 31 dominos would cover the 62 remaining squares, or to prove logically that a complete covering is impossible. (If you have never seen this problem before, you might want to try it now, before reading the solution.)

Since each of the 31 dominos covers two squares, a covering initially seems possible. To see why a complete covering is actually *impossible,* observe that a domino must always cover a black and a white square. But removing two squares of the same color (the diagonal corners) from the 8 × 8 board has left an imbalance between the number of black and white squares that remain. After covering 30 black-white pairs with 30 dominos,

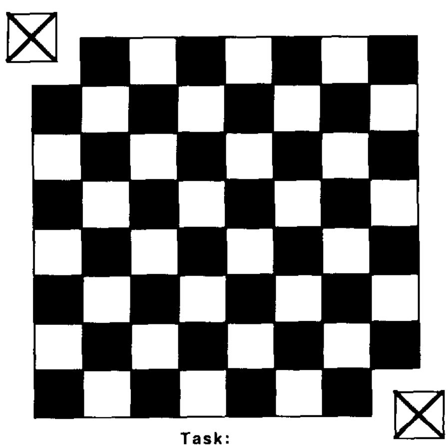


FIG. 1. The classic mutilated checkerboard (MC) problem.

the problem solver is always left in the impossible situation of having to cover two same-colored squares with the single remaining domino. The alternation of squares (black-white, odd-even, etc.) that is critical to this analysis is called *parity*.

Pilot Work on Problem Difficulty

To solve the MC problem insightfully, subjects must switch from an initial representation that considers only numbers of squares and dominos and their geometrical arrangement, to a new representation that takes the parity of the squares into account as well. One way of representing parity is to partition the squares into two equivalence classes: black squares and white squares.[2] Switching to such a representation allows subjects to reason about the numbers of squares of each type, and to make the crucial inferences needed to solve the problem (Korf, 1980). Our initial pilot work explored the assumption that this switch in representation is a major source of the problem's difficulty.

We first tried to get a sense of the magnitude of the problem's difficulty by estimating the size of the search space. An obvious way to prove the problem impossible is by trying coverings exhaustively. We constructed a computer program incorporating some simple covering heuristics (e.g., look ahead one move and try to constrain maximally the number of possible next placements). The program required 758,148 domino placements in order to prove the problem impossible by exhaustion, an unacceptable strategy for human subjects.

Of course, a clever subject might discover heuristics that would reduce the search space further, and might be patient enough to make the entire search, but such cleverness and persistence are rare. Allen Newell (personal communication) reports that he did succeed in solving the problem by exhaustion, using heuristics that reduced the search tree to a few thousand possibilities. Even with this reduction, few subjects would persevere to the end. To solve the problem, subjects must explore other problem spaces.

In a pilot experiment involving the MC conducted by Deepak Kulkarni (Kulkarni, personal communication), a graduate student in Chemical Engineering spent 18 hours and filled 61 pages of a lab notebook with notes, yet still did not solve the problem! While the notebook contains numerous drawings of boards and potential domino placements, the boards were never drawn with alternating squares shaded differently. As the student

[2] There is nothing special about black and white except that color is a readily available classification scheme. Other systems, such as labeling squares even and odd, would work equally well.

was given only a written description of the problem, and not an actual checkerboard, presumably the color of the squares was not available to him unless he shaded them himself. (The significance of this seemly minor detail will become apparent when we discuss our own BREAD & BUTTER experiment below.)

The graduate student, although quite persistent, eventually tried other methods besides exhaustive coverings. Many of his boards were labeled with x and y coordinates, and many pages were devoted to mathematical analyses of various sorts (e.g., equations involving the number of squares as a parameter, degrees of freedom tables, etc.). Some of his later attempts to prove the problem impossible, such as the "anti-puzzle" approach,[3] abandoned the space of exhaustive coverings and searched in a meta-level space of possible new approaches to the problem.

In other pilot research with the MC problem, none of our subjects was able to solve the problem within an hour without being given one or more hints. Again, subjects switched from searching in the initial covering problem space to a meta-level space of potential new representations. The pilot subjects had little difficulty in generating a rough proof of impossibility once they noticed that the alternation of the colors of the squares (more generally, their parity) was important. Most of their time was spent either fruitlessly trying various coverings or searching for new approaches to the problem. When subjects finally paid attention to the parity of the squares, many experienced a sudden insight leading to the problem's solution.

THEORY AND EXPERIMENT

We have argued that the MC problem is hard because solving it requires discovering a representation within which the solution can be found easily, a representation that substitutes the number of squares of each parity for the actual detailed geometry of the checkerboard. The initial problem space provided by the instructions for the problem, the space of all possible coverings, is too large to permit humans to demonstrate impossibility by searching for all legal coverings.

The solver must therefore undertake a search in another space: the meta-level space of possible problem spaces for the MC problem. The

[3] His idea was: "The given puzzle is equivalent to the following anti-puzzle: Given a $n \times n$ chessboard and dominos, arrange the dominos so that two opposite squares are covered and all others are empty. Solution: [There is] no possible way to cover the 2 opposite corner squares on any given chessboard *without* covering other squares. [Therefore the original checkerboard problem must also be impossible]." To show why this proof fails, one need only remove one square of each color from the edges of the board. His argument still holds, only now the problem *is* possible!

decision to search for a new representation is motivated by the "Try a Switch" meta-heuristic: If at first you do not succeed, search for a different problem space. But the space of possible problem spaces is exceedingly ill-defined, in fact, infinite. People don't have a generator for "all possible problem spaces," and if they did, it would not likely soon yield the right representation for the problem that happened to be at hand. To find an appropriate representation, the subject has to have or obtain strong constraints that guide search and make it highly selective. Therefore, we would expect the "Try a Switch" heuristic to incorporate two initiating conditions: (a) frustration from lack of progress in the current problem space, and (b) attention to cues that can guide the generation of alternative problem spaces.

Four Sources of Search Constraint

The remainder of this paper explains the process of changing representations in the MC problem using the concept of search constraint as the unifying framework. Based upon our pilot studies and other research along these lines (e.g., see Newell, Shaw, & Simon, 1962), we have identified four major sources of search constraint that seem relevant to the MC problem, and insight problems in general:

First, *features of the problem itself* provide cues that constrain search. Manipulating the salience of these cues might affect the solver's ability to attain insight. For example, Janet Davidson found that highlighting relevant information improved children's performance on a group of insight problems (Davidson, 1986). Similarly, if the critical feature of parity in the MC problem could somehow be made more salient, we would expect subjects to switch representations and discover the reason for the problem's impossibility more easily.

A second source of constraints, *hints from the experimenter,* tells the subject what features of the problem situation are relevant to solving it. Focusing on these features ought to reduce greatly the time spent exploring unproductive problem spaces. Experiments dating back at least to Maier's (1931) famous two string problem testify to the effectiveness of even seemingly unsubstantial hints (e.g., the mere "accidental" brushing of a string by the experimenter).

Both hints and cue salience are external sources of search constraint. Because they are to be found in the problem environment rather than in the problem solver's head, they can be easily manipulated, and the effects can be observed across subjects. Interacting with these external factors, however, are two internal sources of search constraint: domain specific prior knowledge and widely applicable heuristic knowledge (e.g., see Larkin, Reif, Carbonell, & Gugliotta, 1985).

Relevant domain knowledge can make solving a problem routine. However, the solver who unwittingly applies domain knowledge that is in fact irrelevant may spend a long time exploring unfruitful paths. For example, the Chemical Engineering graduate student mentioned earlier spent most of his 18 hours exploring fruitless mathematical approaches. Someone with less mathematical experience might have exhausted more quickly his stock of possibilities, while someone with slightly different knowledge (e.g., experience with parity problems) might have been able to constrain search in a productive manner.

The use of *heuristics* promises to be the most interesting source of search constraint for it offers the clearest opportunity for exploring individual differences between problem solvers. A heuristic that we will show to be especially important for insight problems is the rule of attending to features of a problem situation that remain invariant. Scientific laws express invariant relations among variables, and recent research in the domain of scientific discovery (e.g., see Langley, Simon, Bradshaw, & Zytkow, 1987) shows that features of a situation may be noticed because they remain invariant or recur repeatedly. For example in the MC problem, subjects may notice that both of the deleted squares are always of the same color, or that the squares they fail to cover are always of the color opposite to that of the deleted squares. The heuristic of attending to invariants appears not to have been discussed previously as a factor in the solution of insight problems.

All of these sources of constraint direct search in the meta-level problem space of possible representations. Cues, hints, prior experience, and heuristics each guide the solver to a particular representation which can then be explored for a solution. Some of these sources of constraint also operate at the level of search within a particular representation. However, since insight in the MC problem appears to occur when subjects discover the appropriate representation and before they find the exact formulation of the proof, we will focus on search constraints in the meta-level problem space.

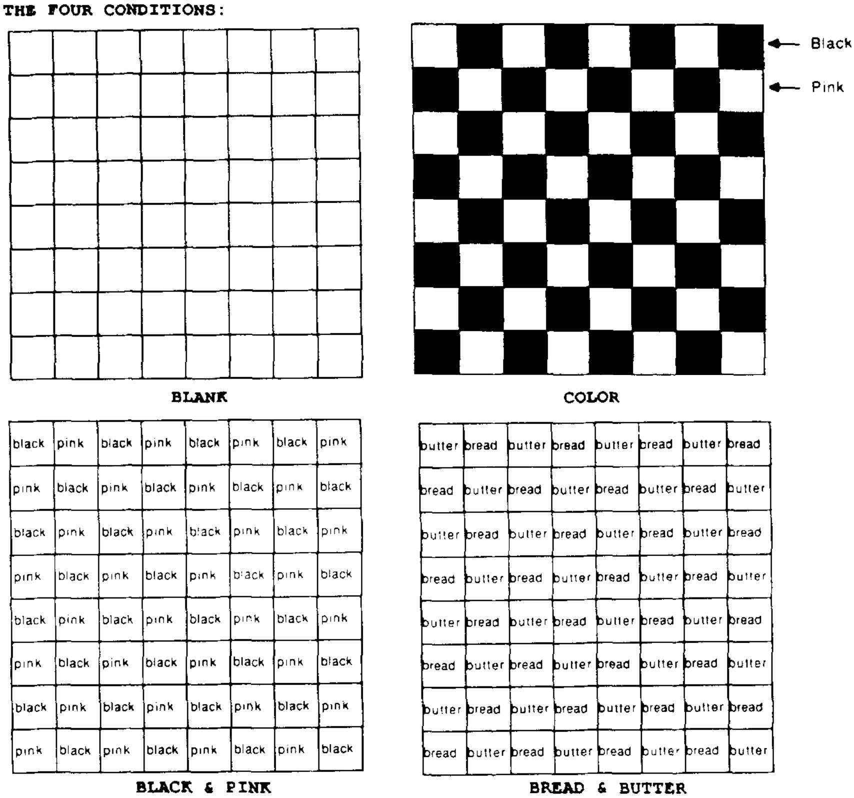
Description of the BREAD & BUTTER Experiment

To test our theory, we must first verify that the problem difficulty stems from the initial absence of constraints that could guide the search for a new representation, and second examine potential sources of such constraints. Through manipulating the salience of the parity cue by providing hints selectively, we could assess the effect of external sources of search constraint. A detailed analysis of thinking aloud protocols allowed us to determine the effects of internal sources of search constraint (i.e., heuristics and domain specific knowledge).

As shown in Fig. 2, subjects in the BREAD & BUTTER experiment received one of four types of checkerboards which varied with respect to

- 25 SUBJECTS (23 Naive, 2 Excluded -- suspect prior knowledge)
- 4 CONDITIONS, 5-7 Subjects Per Condition
- HINTS GIVEN AT REGULAR INTERVALS

THE FOUR CONDITIONS:



(Note: Boards not drawn to actual size)

PREDICTION:

BLANK > COLOR > BLACK & PINK > BREAD & BUTTER
Most Difficult..Easiest

FIG. 2. Experiment 2 at a glance.

the salience of the critical cue, parity. Hints were provided systematically (if needed) to ensure that all subjects solved the problem within an hour. We predicted that subjects in the higher cue salience conditions would take less time and require fewer hints to solve the problem than subjects in the lower cue salience conditions. We further expected to find evidence of the use of heuristics in subjects' verbal protocols. Our specific predictions follow the description of our methodology.

Method

Materials. Instructions, identical for all subjects were typed on a standard (8.5 × 11 in.) sheet of paper. The instructions referred to the "board" but did not mention color or "checkerboard."

The boards were all 8 × 8 matrices of ¾ in. squares. These squares were either left blank, filled in with black and pink color in the fashion of a checkerboard, filled with the words "black" and "pink" in checkerboard fashion, or filled with the words "bread" and "butter" in checkerboard fashion. Blue Xs made of sticky paper were placed on the boards to indicate the squares removed.

No real dominos were used, but subjects were allowed to write on the boards with either a pen or pencil. A cassette tape recorder with a condenser microphone was used to record thinking aloud protocols.

Procedure. Each subject was run individually. The subject was asked a series of questions regarding class level, major, and previous problem-solving experience. Next, subjects were given practice thinking aloud on an unrelated task (e.g., mental multiplication of 2 three-digit numbers). Questions and practice took about 10 min.

During the remaining 50 min (the session was limited to approximately 1 h), subjects were presented with the instructions and asked to solve the problem. After the subjects had read the instructions, the experimenter carefully placed the blue Xs on the upper left and lower right hand corners of the board while the subject watched. The experimenter then sat behind the subject where he could view the subject's behavior but where the subject could not see him without turning around.

Because we wished all subjects to solve the problem, hints were given periodically to subjects if and when they failed to make good progress towards the solution. There were four types of hints given in all: IMPOSSIBLE, INSIGHT, PARITY, and COUNT.

The IMPOSSIBLE hint informed subjects that covering the squares was indeed impossible and that they should therefore direct their efforts toward finding a logical proof. The INSIGHT hint suggested that there was a trick way of looking at the problem that did not involve exhaustive covering. Generally the IMPOSSIBLE hint and the INSIGHT hint were given together after approximately 15 min if there was any indication that the subject might still believe covering the board possible, and if the subject was still trying covering solutions.

Approximately 15 min after the INSIGHT hint was given, if the subjects still had not solved the problem, subjects were told that the color (or words) on the squares might help them solve the problem—the PARITY hint.

The PARITY hint differed slightly with condition. For the BLANK condition, subjects were told to take their pencils and color in every other square in the fashion of a checkerboard. They were then to look at the resulting pattern and see if that might help them solve the problem. Subjects in the COLOR condition were told to look at the colors of the squares in the checkerboard to see if that might lend them some insight. Subjects in the BLACK & PINK and BREAD & BUTTER conditions were told to pay attention to the words that were inside the squares in order to gain an insight.

Subjects who did not solve the problem after receiving the PARITY hint, and who had been working on the problem for more than 40 min were told to count the numbers of different types of squares (e.g., "count the number of blacks vs. pinks")—the COUNT hint. In the rare event that a subject failed to solve the problem after being told to count, increasingly directive hints were given until the problem was solved. The problem was considered solved, when subjects were able to generate a rough proof of the impossibility of covering the board.

A Rough Proof consisted of a statement by the subject indicating how to solve the probem using the parity representation. The statement needed both to recognize the importance of parity (e.g., describing the color of the squares or words on them as being crucial to the solution) and to express some conviction that the problem had been solved. Time to Rough Proof was used as the dependent measure, for subjects varied considerably in their knowledge as to what constituted a formal proof.

Dependent measures. Times to Rough Proof and to the first mention of parity (e.g., the color or words on the squares) were derived from the tape recordings of all 23 subjects. Notations were made about each of the 23 tapes, and 8 were selected (to include two subjects from each group—one above and one below the median solution time for that group) for complete transcription and detailed analysis. The coding system used for these 8 protocols is detailed in Appendix A.

Subjects. Twenty-five CMU undergraduates fulfilling a course requirement in Introductory Psychology served as subjects. The subjects were naive, assuming that they were honest in denying any previous experience with problems involving checkerboards and dominos. Because two subjects were suspected of having seen the problem before, their data were excluded from the analyses.

Subjects were arbitrarily assigned to conditions (before they were seen), with 5 to 7 subjects per condition.

Predictions

First, we predicted that parity[4] would be a critical cue: that once subjects considered the possible importance of the parity of the squares, they would change their representations and soon thereafter have an AHA! experience. Similarly we predicted that most of the solution time would be spent in ineffective attempts to cover the board or search for a better representation.

Reasoning from the perceptual characteristics of the checkerboard in each condition to the salience of the parity cue, we inferred that the four problem conditions would be ordered, from hardest to easiest, as follows:

Because the BLANK condition offered no visual pattern for subjects to use for noticing parity and its relevance, we predicted that this condition would be most difficult. The COLOR condition was predicted to be second in difficulty, because actual colors are so much a part of one's normal conception of a checkerboard that their possible relevance to solution of the problem would tend to be overlooked. On the other hand, because words printed on the squares should seem unusual and attract attention, we predicted that the conditions with words would evoke parity sooner than the other two. In addition, as the words "Bread" and "Butter" seem to "go together," we thought this particular choice of words might emphasize the concept of a pair. Because realizing that a domino must cover a *pair* of differently labeled squares is crucial to the insight, we predicted that the BREAD & BUTTER condition would be the easiest of the four conditions.

In short, we predicted that the salience, defined as the strikingness or unusualness of the boards together with whether or not they engendered the idea of a "pair," would be responsible for speed of solution. To confirm that our manipulation really did influence sa-

[4] In the context of the BREAD & BUTTER experiment, we use the term parity to refer to the actual featues of the squares that allow them to be divided into two equivalence classes. These features vary of course depending upon the experimental condition. In the COLOR condition, the features would be actual colors of the squares, while in the BLACK & PINK and BREAD & BUTTER conditions "parity" refers to the words labelling the squares.

lience, we conducted a small rating study. Twenty subjects were asked to rate and rank the four boards according to strikingness, rarity (unusualness), remarkableness, and suggestiveness of "pairs." The actual questions used can be found in Appendix A. When the raw scores are normalized and converted to a 4-point scale where 4 is least salient, and 1 is the most salient, the mean results are as follows: BLANK: 3.63, COLORED: 2.44, BLACK&PINK: 2.45, BREAD&BUTTER: 1.97. These ratings correspond to our predictions, with the exception that the COLORED and BLACK&PINK boards seem about equal in overall rank. In retrospect it seems reasonable that the high salience of real colors in general compensated for the relative commonness of the checkerboard. As we shall see, the solution times correspond best to the salience ordering empirically determined by the rating study.

We also predicted that the IMPOSSIBLE and INSIGHT hints would encourage subjects to switch from searching in the original problem space to searching for a new representation, while the PARITY and COUNT hints would help subjects constrain the latter search. Hints mentioning the parity pattern were expected to be especially effective because parity is central to the representation needed to solve the problem insightfully. We also hoped to find evidence of problem space search and search for a representation (and of the heuristics guiding these searches) in the verbal protocols.

Results

We have claimed first that the difficulty of the MC problem stems from search, and second that performance on the problem can be predicted by the sources of constraint for this search. The evidence marshaled to test these claims is of two types. First there are solution times as well as simple statistical counts that employ data from all 23 subjects. Second, detailed protocol analyses have been performed on a more manageable subset of eight protocols. These eight protocols, consisting of one fast and one slow subject (based on median splits of solution time) selected arbitrarily from each of the four conditions, also serve as the basis for our claims regarding individual differences.

Search as the Source of Problem Difficulty

The behavior of pilot subjects could be divided into segments, alternating between search for a solution in some problem space and search for a new problem space. If and when a problem space based on parity is found, there is a final segment of search within that space for a reasoned argument for the impossibility of a covering. In this section we argue that the principal difficulty stems from the search prior to attending to parity and finding an appropriate representation using it, and not from great difficulty in discovering the necessary logical inferences once the concept of parity has been attended to and represented.

Describing Search

Subjects search at two levels. When they have a particular representation that they believe will allow them to solve the problem, subjects search within the corresponding problem space. For example, the task instructions suggest that a simple covering of the board might be possible, leading subjects to search initially within a COVERINGS problem space.

As covering attempts fail, subjects are forced to search in the meta-level space of potential representations to find their next approach. Once found, this new approach constitutes a new problem space which can then be searched. Table 1 lists a sample of the problem spaces actually used by subjects in the BREAD & BUTTER experiment.

The first five approaches listed, corresponding to small shifts in representation around the theme of covering, are typical of the problem spaces explored early in problem solving. The remaining approaches correspond to the more radical changes in representation that typify later problem solving efforts.

Later, we will elaborate on search in the meta-level space. At this point, we wish only to distinguish search at the two levels, and to emphasize that the brief period (typically less than a minute) of rapid reasoning immediately preceding subjects' solutions is highly constrained search within the Parity problem space.

Simulating Rapid Inferential Search

Figure 3 presents excerpts from three typical subjects in the BREAD & BUTTER experiment which capture their behavior just before, during, and just after the AHA! experience. Each episode lasts less than a minute and a half, with the actual insight being much more rapid. Each contains

TABLE 1
Some Problem Spaces Used by Subjects

Covering spaces
Try placing all dominos horizontally.
Try placing all dominos vertically.
Try placing dominos in a spiral pattern.
Try placing dominos in a zig-zag pattern.
Try decomposing board into smaller areas, and cover each area.
Mathematical spaces
Consider whether a path between the Mutilated squares contains an even or odd number of squares.
Create mathematical expressions describing the quantities of dominos or squares.
Analogy/comparison spaces
Consider if legal moves in the game of checkers are related to the problem.
Try to draw an analogy between the Checkerboard problem and the 8-puzzle problem.
Try solving the problem if a different pair of squares was mutilated.
Physical manipulation spaces
Try rotating the board to see if that changes one's perspective.
Draw on the board.
Parity space
Consider how color might help solve the problem.
Explore why words might be on the squares.

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SUBJECT 1 (a BREAD & BUTTER subject):                  EXCERPT LASTS:  70 Secs.

1:  Just by trial and error I can only find 31 places ... I dunno,
        maybe someone else would have counted the spaces and just said that
        you could fit 31, but if you try it out on the paper, you can only
        fit 30.                  (pause & distracted chattering)

E:  Keep trying.

1:  Maybe it has to do with the words on the page?  I haven't tried anything
        with that.
    Maybe that's it.  Ok, dominos, umm, the dominos can only fit ... alright,
        the dominos can fit over two squares, and no matter which way you put
        it because it cannot go diagonally, it has to fit over a butter and
        a bread.  And because you crossed out two breads, it has to leave two
        butters left over so it doesn't ... only 30, it won't fit.  Is that
        the answer?
--------------------------------------------------------------------------------
SUBJECT 2  (a COLOR subject):                          EXCERPT LASTS:  48 Secs.

2:  There's an even number of squares, so it's possible depending on the
        placement ... so it has to be the placement.     (pause)

2:  How about a different placement?  We could try that.
        Well, if we place the Xs in different corners, then it'd be really
        simple ... other than opposite ... ummmmm ....How about a black and
        a pink .....  Oh, we always have to cover a black and a pink square...
        at the same time time .... Uh, there's no way to avoid that ... ummm.

    Oh!, There's two black squares covered up and ... since you always
        have to cover up a black and a pink square, there's no way you can
        do it.
--------------------------------------------------------------------------------
SUBJECT 7 (a COLOR subject requiring a hint):          EXCERPT LASTS:  36 Secs.

E:  What about the color?  Can you use color to help you out?

7:  There's two pinks next to each other .... Oh God!! You're taking two black
        out?  And you would need to take a black and a white out ... a black
        and a pink out.                                 (pause)

7:  So you're leaving ... OH!!! Jeez!  So you're leaving .... it's short --
        how many, you're leaving uhhhh .... there's more pinks than black,
        and in order to complete it you'd have to connect two pinks but you
        can't because they are diagonally ... is that getting close? ...
        since they are diagonally connected ... and so you're always gonna
        end up with two extra pinks ... because their mates were taken out.
```

FIG. 3. The AHA! experience (three protocol excerpts).

sufficient information to constitute a Rough Proof of the problem's impossibility. While there are individual differences in the routes taken to the solution, the subjects all seem to make a series of rapid inferences directly following their insight. After having focused attention on parity, the search for a Rough Proof seems not to be difficult.

One way to specify rigorously the work required to generate a Rough Proof once parity has been attended to is to simulate the actual switch of representation and subsequent processes in a computer program. To this end, we built a computer simulation, SWITCH (Kaplan, 1988), using the

Soar architecture (Laird, Newell, & Rosenbloom, 1987). The SOAR architecture is especially appropriate for this task since it carries on all its search within a problem space, and shifts from one problem space to another whenever an impasse is encountered.

Because we intend to describe SWITCH in detail in another paper, we will comment here only on what it taught us about the representation shift process and the subsequent search for a proof of impossibility. SWITCH was provided with a representation of the actual (external) checkerboard, and a model of the typical subject's internal representation of the board and knowledge prior to receiving the PARITY hint. This internal representation ignored the parity of the squares. SWITCH was also assumed to know some inference rules for deriving new knowledge in a representation. Finally, it was given some general search procedures, including procedures for selecting inference rules, and heuristics for shifting to a search for a new representation.

When a solution was not found in the initial problem space, and SWITCH was given the PARITY hint, it used the information provided by the hint to add color to its internal representation of the squares. In the search in the new representation, the inference rules could be used to add still more information. In the course of this search, SWITCH inferred that a domino must cover a square of each color, and that there were more squares of one color than of the other. A new inference, combining these two conclusions, found a contradiction—which provided the proof of impossibility. By examining the search path that led to the proof, SWITCH was able to provide a coherent statement of it. Hence the same inference rules that were unable to find the proof without the hint found it readily with the hint.

The best way to get a feel for how the simulation works is to examine a production and see what it does. Figure 4 presents (an English translation of) the production that implements the switch from representing "generic" squares to representing squares with color. While the production may be instantiated with the concepts of "squares" and "color" (see the parenthesized comments), it corresponds to general knowledge that subjects might have about making analogical mappings of any kind.

Using information from the real world, the simulation is able to shift from an initial representation of "square," to a representation of "black square" or "white square." A similar production allows the simulation to elaborate old propositions using the new concepts of colored squares. Thus the proposition "A domino covers a square and a square" becomes "A domino covers a black square and a white square."

SWITCH behaves in a psychologically plausible manner. First, the order in which it notices facts corresponds roughly to the order found in the protocols of human subjects. For example, SWITCH infers that a

```
Production: elaborate-concept-by-analogy

IF:   The goal is to prove the problem impossible, AND
      The operator is to elaborate a representation, AND
      A hint exists saying pay attention to some attribute (e.g. COLOR), AND
      Some representational concepts (e.g. the concept of squares) exist
      that have no value for the attribute in question (e.g. COLOR), AND
      There are some real world referents for the representational concepts
      that can be referred to (e.g. the squares which really exist
      on the board)

THEN: Map the value of the hinted-at attribute (e.g. COLOR) from the real
      world objects (e.g. real squares) to the representational concept of the
      objects (e.g. representation of squares).
```

FIG. 4. A sample production from switch.

domino must cover a black and a white square AFTER receiving the PARITY hint. Similarly, every human subject (11 in all) requiring the PARITY hint mentioned this property of dominos AFTER receiving the PARITY hint. Of the 12 subjects who noticed parity on their own, only two mentioned that a domino covers one square of each type as their first remark including parity. The others, like SWITCH, noticed parity first, and later made an inference that dominos must cover one square of each type.

Without making strong claims for detailed psychological validity at the level of individual productions, we believe that the overall qualitative behavior of SWITCH is quite similar to that of human subjects who have recognized the significance of parity. SWITCH provides a sufficient set of processes to explain this behavior. The straightforward way in which SWITCH changes representation and then generates a Rough Proof reinforces our view that the problem difficulty has relatively little to do with discovering a proof once parity has been attended to. Rather, the difficulty seems to lie in the less selective search that precedes this rapid (because it is well-constrained) reasoning.

Human Data on Problem Difficulty

Converging evidence for the proposition that the MC problem's difficulty stems from search comes from analysis of the time spent by subjects at various points along the solution path. Figure 5 illustrates the prototypical solution path along with the mean time spent by subjects at various stages.

After reading the instructions, subjects invariably tried different methods of placing dominos on the board to see if the problem might be solved easily. The length of this covering stage varied widely, from about 2 to

```
APPROXIMATE TIME LINE:                            SUBJECTS' BEHAVIOR:

ELAPSED
TIME                                             ======================
                                                 |  READ INSTRUCTIONS   |
                                                 |                      |
0  Minutes   START OF problem.........>|---------------------|
                                                 |                      |
                                                 |                      |
                                                 |                      |
5  Minutes                                       |                      |
                                                 |        MAINLY        |
                                                 |  ATTEMPT COVERINGS   |
                                                 |                      |
                                                 |                      |
10 Minutes                                       |                      |
                                                 |                      |
                                                 |                      |
               IMPOSSIBLE                        |                      |
                 and/or                          |                      |
15 Minutes       INSIGHT     -->                 |----------------------|
               Hints Given                       |                      |
                if needed                        |  SEARCH FOR, & TRY   |
                                                 | NEW REPRESENTATIONS  |
                                                 |                      |
20 Minutes     Mean Time to  -->                 |                      |
               1st Mention Parity                |                      |
                                                 |                      |
                                                 |                      |
                                                 |  SETTLE UPON PARITY  |
25 Minutes     Mean Time to                      |    PROBLEM SPACE     |
               Solve Problem -->                 |*GENERATE ROUGH PROOF*|
                                                 |----------------------|
                                                 |                      |
                                                 |      ADDITIONAL      |
30 Minutes     PARITY Hint                       |      SEARCH TIME     |
               given if needed -->               |        SPENT         |
                                                 |       BY SLOWER      |
                                                 |        SUBJECTS      |
                                                 |                      |
35 Minutes                                       |                      |
                                                 |                      |
                                                 |                      |
                                                 |                      |
                                                 |                      |
40 Minutes     COUNT Hint                        |                      |
               given if needed -->               |                      |
                                                 |                      |
                                                 |                      |
45 Minutes                                       |                      |
               Slowest Subject                   |                      |
               Solves Problem  -->               |                      |
                                                 ======================
```

FIG. 5. A prototypical solution path.

almost 20 min. It ended when the subject either realized or was told explicitly (via the IMPOSSIBLE and/or INSIGHT hints) that the problem was impossible and there must be a better approach than trying all possible coverings.

At this point, subjects entered the "search for a new representation" stage, characterized by fewer covering attempts and more search for new approaches to the problem. Some of the common approaches explored in this phase included (Table 1): symmetry, moving the position of the blue Xs, decomposition of the board into smaller boards, various mathematical approaches, counting the number of squares in rows or columns, and finally using the parity of squares (e.g., color). Once the subjects focused attention on the parity of squares they were usually able to generate a Rough Proof rapidly.

As Figure 5 indicates, the time subjects spent just *generating* a Rough Proof[5] is quite small (3%) in comparison with the total time. Most of this proof generation time seemed to be spent finding the right words to communicate the series of rapid inferences that typically followed noticing parity.

The vast majority (77%) of the time was spent searching before there is any mention of parity in the protocols. Taken together, the vary rapid generation of a proof and the large amount of search before any mention of parity constitute strong evidence that the difficulty of the MC problem stems from search for the correct representation, not search once this representation has been found (as might be the case in other problem-solving tasks, e.g., see Wason & Johnson-Laird, 1972).

However, it may still seem odd that the gap between first mention of parity and until just before the generation of the Rough Proof (approximately 20% of the total time) is as large as it is. Twelve of the 23 subjects did, in fact, generate a Rough Proof almost immediately after their first mention of parity; but the remaining 11 subjects mentioned parity some time before generating a proof, with an average gap of about 12 min. If parity really triggers the new representation as we suggest, why don't all subjects generate a Rough Proof immediately after mentioning parity?

Earlier, we remarked that a switch of representation depends on *two* conditions: (a) dissatisfaction with the current representation, and (b) appropriate cues to guide the search for the new representation. Parity may be noticed while the subject is still actively pursuing the search in some problem space other than the PARITY space. In these cases, subjects may mention parity in passing (e.g., "we can cover that . . . except for the rightmost *pink* one in the second top row"). In more interesting (and more common) cases, parity competes with other cues in the search for a new representation. Subjects are blocked from exploring parity by their preoccupation with other cues and representations that we (but not

[5] Time for Rough Proof generation was measured from the moment either just before parity was noticed for the first time or when parity was last focused on (in the case of subjects who mentioned parity one or more times but then persisted in exploring other approaches) through the subject's statement of a Rough Proof.

they) know to be irrelevant. Attention to parity is a (nearly) necessary, but not sufficient, condition for constructing an effective representation.

The empirical data supporting these arguments come from eight subjects whose protocols had been fully transcribed. Of these eight, three generated a rough proof immediately after first mentioning parity. The remaining five are among the 11 subjects in total who showed a delay between first mention of parity and generation of a rough proof. Table 2 shows four categories that account for the time spent between first mention of parity and generation of a rough proof. (Appendix A contains the criteria used to code the protocols into the four categories.)

The entire delay period averages 20% of subjects' problem solving time (about 10 min per subject). About 2% (1 min) can be immediately disregarded as time used by the experimenter to give hints, make comments, and prompt the subject to keep talking. Roughly 6% (3 min) was spent trying coverings. During this time, subjects had no reason to focus on parity because they were still exploring covering approaches. Similarly, 7.24% (3½ min) of the time was spent exploring other problem spaces besides COVERINGS or PARITY (e.g. MATH, ANALOGY, others listed in Table 1). While we know these explorations to be fruitless, they effectively competed with PARITY for the subjects' attention. Finally, on average, something less than 5% (2½ min) of subjects' total time was spent actually searching in the PARITY problem space.

Given that our fine-grained analysis lends additional support to the claim that problem difficulty stems mainly from search in the wrong problem spaces (even after first mention of parity), it is worth considering in some detail the external and internal sources of search constraint, including irrelevant knowledge and cues that caused subjects to delay their responses to the parity cue.

External Sources of Search Constraint: Cue Salience and Hints

If search is the primary source of difficulty, examining the potential sources of search constraint will help us understand how subjects manage

TABLE 2
Breakdown of Time Spent after First Mention of Parity, before Rough Proof

Time spent in:	Mean % time spent
Experimenter hints/comments	1.96
COVERING problem spaces	5.92
Other nonparity spaces	7.24
Parity problem space	4.88
Total	20.00%

to change representations and experience insight. Increasing the salience of parity and providing hints are ways of providing external sources of search constraint.

Specifically, we predict that solution times will be ordered by the expected salience of parity in the four conditions. According to our salience rating study, the BLANK condition should be most difficult, followed by either the COLOR or BLACK & PINK condition, followed by the BREAD & BUTTER condition which should be easiest. Table 3 presents the relevant results. The first column, showing the mean times required by subjects of different groups to first mention parity, serves as a check on our salience manipulation. As we originally predicted, the BREAD & BUTTER board was the most salient (i.e., caused subjects to mention parity earliest), followed by the BLACK & PINK board, the COLOR board, and lastly the BLANK board. Note however, that the difference between the COLOR and BLACK & PINK conditions is the smallest of all the intergroup differences. A one-way analysis of variance indicates that the overall difference between groups is highly statistically significant ($F[3,19] = 12.05, p < .0001$).

The second column of Table 3 gives the mean times required by subjects to decide that covering is impossible. Again the times are ordered as we originally predicted with the smallest intergroup difference occurring between the COLOR and BLACK & PINK conditions. However, a one-way analysis of variance shows that the overall difference between groups is not significant ($F[3,19] = .83, p > .48$) suggesting that time to declare the problem impossible may be relatively constant.

The third column confirms that time to Rough Proof is ordered according to salience (as measured in the empirical salience-rating study). Here, a one-way analysis of variance reveals a statistically significant overall difference ($F[3,19] = 4.08, p < .025$). Note that the COLOR and PINK & BLACK groups differ in mean solution time by only 3 s.

The fourth column, showing the number of approaches tried in each

TABLE 3
Mean Times to Points on Solution Path and Mean Number of Approaches Tried

Condition	Time to first mention parity	Time to declare impossible	Time to rough proof	Number of approaches
BLANK	1980 sec.	828 sec.	2242 sec.	9.14
COLOR	1265 sec.	672 sec.	1375 sec.	5.80
BLACK & PINK	905 sec.	639 sec.	1378 sec.	5.16
BREAD & BUTTER	342 sec.	483 sec.	995 sec.	4.00
Mean total	1188 sec.	670 sec.	1557 sec.	6.26

experimental condition, supports the originally predicted rank ordering. Some examples of typical approaches were mentioned earlier (see Table 1). A one-way analysis of variance reveals a highly statistically significant overall difference between conditions ($F[3,19] = 11.09$, $p < .002$). Note again however, that the difference between the COLOR and PINK & BLACK groups is quite small.

Table 4 indicates which of the originally predicted differences shown in Table 3 are statistically significant. The BLANK group was statistically different from the other three groups on all three measures. While the BREAD & BUTTER group was only significantly faster than the other groups to first mention parity, it nevertheless fit the predicted pattern in each of the four columns of Table 3. Assuming no difference between the COLOR and BLACK & PINK groups, we are left with data supporting the ordering of BLANK > COLOR/PINK & BLACK > BREAD & BUTTER in all four columns of Table 3. And of course, the groups with the longest solution times also received the most hints (see Table 7) thereby cutting their solution times well below what they would have been without hints. Hence the differences in times seriously underestimate the differences in performance.

Problems that require the *invention* of new cues (eg., the BLANK group) are much more difficult than those that require only *noticing* cues already present (e.g., the remaining three groups). For the former, without some source of constraints the space of possible inventions is huge, while when perceptual cues are present, typically only a limited number of salient features are noticed. In a problem containing relevant cues, subjects need only notice them to constrain their search effectively. We

TABLE 4
Pairwise Comparisons between Groups of Subjects ($df = 19$)

	COLOR	BLACK & PINK	BREAD & BUTTER
Comparisons of mean time to first mention parity			
BLANK	$t = 2.54, p < .01$	$t = 3.99, p < .001$	$t = 5.78, p < .001$
COLOR	—	ns	$t = 3.01, p < .005$
BLACK & PINK	—	—	$t = 1.92, p < .05$
Comparisons of mean time to rough proof			
	COLOR	BLACK & PINK	BREAD & BUTTER
BLANK	$t = 2.27, p < .025$	$t = 2.38, p < .025$	$t = 3.26, p < .005$
COLOR	—	ns	ns
BLACK & PINK	—	—	ns
Comparisons of mean number of approaches tried			
	COLOR	BLACK & PINK	BREAD & BUTTER
BLANK	$t = 3.43, p < .003$	$t = 4.30, p < .001$	$t = 5.28, p < .001$
COLOR	—	ns	ns
BLACK & PINK	—	—	ns

shall discuss later the hypothesis that one of the distinguishing characteristics of insightful problem solvers is that they are good noticers.

Before we leave Table 3, we should note that the mean times from first mention of parity to times of rough proof are longer for the BREAD & BUTTER and PINK & BLACK groups (653 and 473 s, respectively) than for the COLOR and BLANK groups (110 and 262 s, respectively). It may seem surprising that some of the time that subjects in the first two groups gain by noticing parity quite early they lose again by being slower than the others in exploiting this critical information. This is an important datum that we will have to account for later in our detailed analysis of the solution process.

Hints as Search Constraint

Hints required by subjects provide a final source of converging evidence for cue salience as a search constraint. We would expect subjects whose search is less constrained by elements in the problem (i.e, the low salience subjects) to require more explicit hints. However, to justify this prediction, we must show that hints are effective in constraining search; and we must clarify the relation between number of hints and solution time.

As a first step, we consider again the data in Table 3, and in particular, the interval from the time subjects decide or are told that covering is impossible to the time of their rough proofs. The mean times, beginning with the BREAD & BUTTER group and ending with the BLANK group are 512, 739, 703, and 1414 s, respectively. It appears that the decision about impossibility, except for the BLANK group, is a "leveller," substantially reducing the intergroup time differences. If the information for parity is perceptually available, as it is for all except the BLANK group, the decision to shift from a covering problem space to search for a new representation cuts the solution time nearly in half. We saw in the last section that the IMPOSSIBLE hint greatly flattened the differences between groups in the times they required to make this shift.

To test whether hints actually constrain search, we counted the number of parity statements relevant[6] to the problem's solution occurring before and after a hint. The hypothesis predicts that more of these relevant statements will occur after hints. As the first row of Table 5 shows, subjects did generate more relevant parity statements after hints than

[6] A relevant parity statement is a factual statement mentioning parity that is on a direct path to the insightful solution. (More formally, a relevant parity statement is defined by the criteria for relevant parity invariants listed in Appendix A.) Number of relevant parity statements was chosen as the dependent measure since these statements represent the ideal direction in which search should be constrained.

TABLE 5
Mean Number of Relevant Parity Statements before/after Hints

Type of hint considered	No. of hints of this type	Mean no. statements before hint	Mean no. statements after hint
Any hint	13	.46	1.62
PARITY hint	4	0	3.0

Note. n = 8 Subjects.

before them. Row 2 shows an even greater difference in the number of relevant statements generated before and after the PARITY hint. Both differences are statistically significant ($p < .05$, one-tailed t test). To check whether these effects might simply result from subjects just talking more after hints, we compared the number of *total* (i.e., both relevant and irrelevant) statements[7] before and after hints and found the difference (2.08 versus 2.69) not to be statistically significant. From these results it appears that hints help by steering subjects to the appropriate part of the meta-level search space.

Some hints might constrain search more effectively than others, and our model predicts that the PARITY hint will be particularly effective. Table 6 shows that the PARITY hint was more effective than all the other hints combined in terms of the number of subjects who reached a solution soon after a hint was given.

The numbers in parentheses in Table 6 show how many subjects in each group received a given hint. For example, in the BREAD & BUTTER group, two out of the five subjects received the IMPOSSIBLE hint and the INSIGHT hint. Both subjects noticed parity on their own and thus did not require the PARITY hint. One subject was able to solve the problem with no further hints. The other was unable to see immediately the connection between parity and the problem's solution (for reasons that will be discussed in the next section) and eventually had to be told to count the number of different types of squares (the COUNT hint).

Because subjects can discover the content of a hint on their own, the number of hints required is a separate dependent measure from solution time. A subject could take a long time to solve the problem, yet still require relatively few hints. However, if hints act to constrain search, and if subjects in the low salience conditions suffer from a relative lack of external sources of search constraint, we would expect these subjects to require more hints. The differences in the mean number of hints required

[7] Again Appendix A specifies the coding criteria used here. See relevant and irrelevant invariants.

TABLE 6
Number of Subjects Solving after Various Hints (No. of Subjects in Each Group Who Received Each Hint; $n = 23$ Subjects)

Group	No hint	Impossible hint	Insight hint	Parity hint	Other hints[a]
BREAD & BUTTER	3 (5)	0 (2)	1 (2)	0 (0)	1 (1)
BLACK & PINK	2 (6)	1 (3)	1 (3)	1 (2)	1 (1)
COLOR	1 (5)	0 (4)	2 (4)	2 (2)	0 (0)
BLANK	0 (7)	0 (7)	0 (7)	6 (7)	1 (1)
All groups	6 (23)	1 (16)	4 (16)	9 (11)	3 (3)

[a] Other hints included the COUNT hint, and other very specific hints which were given only if the PARITY hint proved ineffective.

($F[3,19] = 3.49$, $p < .05$), shown in Table 7, exhibit the same rank ordering as time to first mention of parity, solution time, the number of approaches tried, and the percentage of subjects requiring the IMPOSSIBLE, INSIGHT, and PARITY hints (see Table 3). All of these dependent measures converge on cue salience and hints as major determinants of subjects' performance on the MC problem.

Internal Sources of Search Constraint

While aggregate data like solution time or number of hints provide a good means for understanding the effects of external sources of search constraint, such data say little about factors more internal to the problem solver—factors that may account for individual differences in performance. To explore these, we turn to a much richer and denser record of problem solving activity—protocols (Ericsson & Simon, 1984). Specifically, we will use protocol analysis to show how domain knowledge and knowledge of certain general heuristics constrain search.

Past research on expertise (e.g., see Chase & Simon, 1973; Chi, Feltovich, & Glaser, 1981; de Groot, 1965) has typically emphasized the

TABLE 7
Mean Number of Hints Required

Group	Mean number of hints
BLANK	3.14
COLOR	2.00*
BLACK & PINK	1.50**
BREAD & BUTTER	1.00**
All groups	2.00

* $p < .10$, ** $p < .01$ (one tailed t test). Differences compared with BLANK group.

power of relevant domain-specific prior knowledge, but has not stressed the potential adverse effects of bringing inappropriate knowledge to bear. The first part of our discussion of external sources of search constraint attempts to answer some questions raised earlier (e.g., why some subjects solve the problem immediately upon noticing parity while others do not) by examining the adverse effects of obtaining knowledge prematurely, and of misapplying domain specific knowledge.

The second (and major) part of our discussion will focus on heuristic knowledge that may be applicable to a wide variety of problems. After sketching a sample of the heuristics used by subjects, we will focus on the Notice Invariants heuristic as a source of individual differences. We will see that this heuristic separates cleanly the good from the poor problem solvers, regardless of experimental condition, suggesting that generality of a heuristic need not necessarily lessens its power.

Prior Knowledge as Search Constraint

Knowledge is a two-edged sword. For most problems, knowledge allows one to hack away irrelevant details and focus on the problem elements that are likely to be critical for a solution. But in insight problems, where the answer often lies in a very obscure place, inappropriate or irrelevant knowledge may guide search to an unproductive region of the problem space, as we saw earlier in the case of the unfortunate Chemical Engineering graduate student.[8]

We consider first the consequence of obtaining knowledge when its relevance cannot be recognized. Four subjects, three in the BREAD & BUTTER group and one in the PINK & BLACK group, mentioned parity before they had decided that covering was impossible. These subjects required an average of 1142.3 s from first mention of parity to achievement of a rough proof, while the remaining seven subjects in these two groups took only an average of 218.6 s. No member of the first group bettered the average of the second, and no members of the second were slower than the average of the first. Noticing parity usually leads rapidly to a solution when the goal is to prove a covering impossible, but it seems to contribute not at all when the goal is to *find* a covering.

This analysis clarifies the apparently anomalous finding mentioned in the previous section: that subjects in the two conditions with the more salient cues took longer from first mention of parity to solution than subjects in the other two conditions (the times were 653, 473, 110, and 262 s, respectively). We have just seen that the first two times drop to an

[8] Other studies have found similar results in the other domains, for example the domain of algebra word problems (Paige & Simon, 1966).

average of 219 s when we remove those subjects who mentioned parity while they were still searching for a covering, quite comparable to the times for the subjects in the third and fourth groups.

Another way to look at these data is to replace time when parity was first mentioned by time when covering was judged impossible for those subjects who noticed parity "prematurely." Now the times for the first two groups are 421 and 431 s, respectively, while the times for the other two groups remain at 110 and 262 s, respectively. This new estimate for the first two groups remains higher than the times for the third and fourth groups; however it is a conservative estimate. It does not measure when the subjects who had noticed parity prematurely again brought it back to notice as they were striving to prove impossibility; it seriously overestimates, for these subjects, the interval of time from noticing parity after it became relevant to time of solution.

Perhaps the fairest measure is to calculate the mean intervals between first mention of parity *after* the problem is known to be impossible and rough proof generation. This calculation results in means that are remarkably similar across groups (183, 251, 110, and 262 s, respectively).

We take up next how performance is harmed when a subject attends to information he has that is irrelevant (by hindsight) to solving the problem. (We do not explain here the *cause* for this behavior, but its consequence.) Consider S9, a subject in the BREAD & BUTTER condition, who took a long time to find a solution although he was in the easiest condition and first noticed parity (in 230 s) without the PARITY hint.

S9 began by reading the instructions, calculating whether the *number* of dominos was sufficient to cover the number of remaining squares, and trying a number of coverings:

> So, I have 31 dominos, they cover two squares apiece, that's 62 squares . . .
> . . . logically, it should cover it . . .
> . . . maybe we can cut out this section of the board right here. This middle section, that gives 6 × 6 that's . . . 36 . . . 36 squares . . . and that can be . . . covered.

In these beginning statements, S9's calculations suggest a COVERINGS problem space, but not an ordinary one, for he considers cutting out sections of the board rather than placing individual dominos.

As his covering attempts fail, S9 searches for invariants that might serve as explanations, also to constrain and direct his search.

> . . . we seem to be missing one [domino placement] always . . .
> . . . we still seem to be missing one . . .
> . . . Argh [pause] missed it by one . . .
> . . . this an eight by eight checkerboard [pause] and it should be able to cover [pause] but since both of these numbers are [pause] even, and 31 is odd [pause] but it does make 64. I'm gonna say that [pause] it can't be done.

Although unconvinced of the reason, S9 has detected an invariant—namely, that all his attempts seem to fail. This argues that the problem is impossible, although his early calculations indicate that a covering should exist. Lacking a clear way to resolve the conflict, S9 persists both in trying coverings and in stating periodically that the problem is impossible. After 15 min, the experimenter confirms that the problem is indeed impossible, and then S9 generates a series of potential explanations for the impossibility:

> . . . the two Xs have to be side by side in order for it to be done . . .
> . . . Maybe 'cause it's [pause] it's 31 which is an odd number, maybe? uhhh [pause] even though it is multiplied by two [pause] it [pause] doesn't go into 64? . . .
> . . . I think it's because [pause] ummm if you were able to cover it with 32 dominos, and have no Xs, then 32 is like 2 to the 5th [pause] and ummm [pause] 64 is 2 to the 6th. If [pause] it's only off by a factor of 2 and it's multiplied by a factor of 2 [pause] . . .

S9 is noticing invariant properties of the board, but mathematical invariants rather than perceptual invariants. It is not that S9 simply fails to notice BREAD & BUTTER, for early during problem solving, he mentions:

> This is tough. Come on [pause] bread and butter [pause] 64, two squares [pause] it should cover [pause] uhhhh could fit in a logical pattern.

And later:

> . . . could it be bread will not fit on butter? [long pause] can't find [pause] a logical way . . .

S9 has two competing sources of search constraint, the highly salient cue, BREAD & BUTTER, and mathematical knowledge that seems relevant. Thus, although S9 returns to BREAD & BUTTER repeatedly as his mathematical approaches fail, only the continued failure of his mathematical ideas drives him to focus on BREAD & BUTTER intensely. At this point, S9 notices that if the board were coverable then one X would be on a bread, the other on a butter, and the number of spaces between them would be even.

For S9, the only critical fact that is missing is the notion that a domino must cover one square of each type, but he remains unwilling to explore his new discovery without a clear idea of its logical relevance. Instead he persists in trying to relate the parity invariants to his previous mathematical ideas:

> . . . if you designate butter odd [pause] as odd [pause] and bread as even [pause] then in many—any—situation where we're trying to figure out [pause] a square root of a number minus [pause] two [pause] we need odd even . . .
> . . . But logically this doesn't make sense [pause] I can't explain this logically. OK, but it works logically, [pause] this has no cause and effect . . .

Only after the hint to count the numbers of bread and butters on the board is S9 able to solve the problem.

The behavior of S9 is in direct contrast to the behavior of another BREAD & BUTTER subject who solved the problem quite rapidly, S1. Like S9, S1 noticed the words BREAD & BUTTER early in problem solving, and startled the experimenter by immediately asking:

> The . . . words don't matter do they? . . . in this problem?

The experimenter answered, "No," believing that any other response would give away the answer before the problem was even begun. Since she was put on the wrong track at the start of the problem, one might expect that S1 would require a long time to reach a solution. Not so. Instead, S1 suggests one "proof" after another—most of which amount to statements that particular coverings will not work. When these proofs are rejected, she returns to BREAD & BUTTER (see Fig. 3 for an excerpt from her protocol), and solves the problem rapidly. Inexperience with the nature of formal proofs (as evidenced by her attempts to call failed coverings "proofs") together with her willingness to try new approaches worked to S1's advantage.

Having seen that availability of irrelevant domain-specific knowledge explains in part why some subjects solve the problem immediately after noticing parity, but others do not, we turn to search heuristics as another reason why some subjects are faster solvers than others.

Heuristics

Cues, hints, and domain-specific prior knowledge are all domain-specific constraints that help determine how subjects will perform the task. We can influence performance by modifying the task to change cue salience, by providing hints, or by selecting subjects possessing certain kinds of knowledge. But what might we do to promote insights over more than one domain? What leads one person to insight, and not another?

We believe that the answer to these questions lies partly in the heuristics people use. The first step in exploring this hypothesis is to get some hard evidence that people use some general heuristics by counting the frequency of their use.

General use of heuristics. We hypothesized that subjects might use at least three very general heuristics: Noticing Invariants, Forming Hypotheses, and Comparing Alternative Board Situations. The notes made from the 23 verbal protocols reveal that each subject used each of these heuristics at least once. A more detailed analysis of the eight verbatim transcripts provided the data shown in Table 8.

The clearest result is that subjects used the Notice Invariant heuristic more often than Hypothesize, which in turn was used more often than

TABLE 8
Overall Frequency of Use (and Rate of Use) of Heuristics (n = 8 Subjects)

Subject	Notice-invariants			Hypothesize		Compare	
1 BREAD & BUTTER	(Fast)	26	(2.7/min)	7	(.73/min)	3	(.31/min)
14 BLACK & PINK	(Fast)	9	(1.7/min)	6	(1.1/min)	1	(.19/min)
2 Color	(Fast)	27	(2.6/min)	3	(.29/min)	2	(.20/min)
17 BLANK	(Fast)	33	(1.0/min)	20	(.61/min)	5	(.15/min)
9 BREAD & BUTTER	(Slow)	47	(1.1/min)	29	(.70/min)	8	(.19/min)
13 BLACK & PINK	(Slow)	43	(1.1/min)	32	(.85/min)	6	(.16/min)
16 Color	(Slow)	31	(1.4/min)	20	(.91/min)	5	(.22/min)
4 BLANK	(Slow)	107	(2.3/min)	51	(1.1/min)	12	(.25/min)
Total		323		168		42	
Mean		40.4		21.0		5.3	

Compare. This ordering is intuitively plausible since we would expect subjects to notice more facts than they actually hypothesize about, and to form more hypotheses than they actually test.

Noticing Invariants. In describing the processes underlying insight, we have already specified a means for actually switching representations (i.e., the SWITCH simulation discussed above). The simulation demonstrates that once the decision has been reached to search for a new representation, attention to the parity cue is sufficient to construct the new representation by modifying the initial one and to find the solution by means of a reasoning process. What leads subjects to consider the critical cues that are prerequisite to such a switch?

The Notice Invariants heuristic is of great importance in permitting subjects, particularly those not given strong perceptual or verbal hints, to narrow their search for new representations. Subjects are not equipped with generators for searching the space of "all possible representations." If they become discouraged with the progress of their search through the current problem space, their discouragement does not automatically guide them to a new space.

It has been observed (Ericsson & Simon, 1984; Newell & Simon, 1972) that under these circumstances, subjects giving thinking aloud protocols often fall silent. When asked what they are thinking about, they may say "I'm not thinking about anything," an indication that they have no available strategy for the meta-level search for the new representation. Such behavior has frequently been seen in novice chess players choosing a move, and novice subjects solving puzzles or mathematics or physics problems.

In the present study (Table 6) none of the seven subjects in the BLANK condition solved the problem without receiving the explicit parity hint, but 11 of the 16 subjects who had a perceptual cue available (color or paired words) solved without this hint. Of these 11, however, 5 required the hint that a covering was impossible and/or the insight hint in order to generate the required new representation. It appears that the sheer need to find a new representation is insufficient, unless some guidance is provided to the search either by a parity hint or an appropriate perceptual cue. Conversely, noticing the parity cue was efficacious only half the time without an impossibility and/or insight hint.

Let us examine further the conditions under which subjects acquire constraints that enable them to generate promising new representations. Specifically, how do subjects in the MC problem, after a time, generate representations that incorporate the relevant feature, parity?

We hypothesize that, while solving a difficult problem, people are attentive, at least intermittently, *to features of the problem display.* In particular, *features that are invariant*—do not change as the situation changes—will sooner or later attract attention and be remembered. For example, in the MC problem, the two squares that cannot be covered are always of the same color, opposite to the color of the two squares that have been removed. Another invariant (this one not directly relevant to the solution) is that the problem is solvable if the two squares removed are an even number of squares apart, insoluble if they are an odd number of squares apart.[9]

Table 9 lists some invariants frequently noticed by the eight subjects. All the invariants relevant[10] to the insightful solution are listed as well as the more common irrelevant ones (i.e., those that are *not* on the direct solution path). The range of invariants noticed demonstrates the generality of the heuristic.

Precisely because noticing invariants is a widely applicable rule of thumb for searching in ill-defined domains, there can be no guarantee that those noticed will be the critical ones for the particular problem. Nevertheless, the constraints offered by the Notice Invariant heuristic are a vast improvement over blind trial and error search. Specifically, noticing invariants provides a generator for possible problem spaces, many of which incorporate relevant invariants.

[9] The reader can verify that even/odd spacing is a true invariant in the following manner: Turn to the checkerboard illustrated in Fig. 1. Starting with any square next to one of the Xs, find a path to any square next to the diagonally opposite X, using only horizontal or vertical movements. No matter what the shape of the path, the number of squares it contains will prove to be odd. However, if one of the Xs is moved so that a covering becomes possible (i.e., the Xs are on different colors), and a new path is found by the same method, this new path will contain an even number of squares.

[10] See Appendix A for a precise definition of "relevance" and "invariant."

TABLE 9
Invariant Properties Mentioned Repeatedly by Subjects (n = 8 Subjects)

Description of invariant property	% Subjects (n = 8) who noticed it	Relevance
Problem is impossible /coverings fail	88%	Relevant
Domino covers two of different parity	75%	Relevant
Restatement of the givens (e.g., 31 dominos 62 squares left) or the goal	63%	Relevant
Squares of the same parity are removed/two removed of the same parity → impossible	50%	Relevant
Adjacent squares are of different parity/diagonal squares are same parity	38%	Relevant
A domino covers two adjacent squares/domino cannot cover diagonally	25%	Relevant
An imbalance exists between the number of squares of different parity	13%	Relevant
(Two diagonal) squares are always left over when a covering attempt fails	63%	Irrelevant
Various mathematical properties related to the (number of) squares or dominos	63%	Irrelevant
Various patterns of covering/symmetries of the board	50%	Irrelevant
The problem's possibility depends upon the position (not parity) of the Xs	50%	Irrelevant
Odd number of spaces between Xs indicates problem impossible, even number → possible	38%	Irrelevant

Note. Listed first are all the invariants on the direct path to the insightful parity solution (i.e. the relevant invariants). Listed second are some common invariants that are not on the direct path to the insightful parity solution (i.e. the irrelevant invariants).

The objection may be raised that an infinite number of things might be noticed in any actual physical situation, and that noticing invariants therefore does not provide any real selectivity. However, we are not concerned with noticing in the abstract, but with the features of a situation that are actually salient for human beings. While subjects are searching in a COVERINGS problem space, their attention is directed to the checkerboard and the imagined dominoes. The hypothesis asserts that the perceptual features of this display will be noticed more often than other features of the experimental situation (of which there are surely an infinity).

But Notice Invariants is even more restrictive. It directs attention especially at features of the display that remain the same from one covering attempt to the next. As the subject checks to see how each attempted covering fails, the situation is optimal for encoding information about the obstacle that blocks success. Attention would focus upon the squares not covered, and it could be noticed that they were always of the same color, or labeled with the same labels.

A referee has called to our attention a special variant of the Notice Invariants heuristic. Some subjects moved the locations of the Xs to see which mutilated boards could be covered and which could be not. By noticing invariant properties of the "possible" boards that were *different* from the invariant properties of the "impossible" boards, they could reach the conclusion that if the squares removed are of the same color, covering is impossible; if they are of opposite color, covering is possible; also that if there are an odd number of spaces between the Xs covering is impossible; if there are an even number, possible.

These subjects were essentially running a controlled experiment with the location of the Xs as the independent variable, and using the Notice Invariants heuristic to generalize the results. In fact, 50% of all subjects mentioned the first (and relevant) invariant, and 38% mentioned the (irrelevant) second one. Of course they may have obtained this knowledge by other routes than the one just described—for example, *after* they had noticed the parity property or received the parity hint.

Converging evidence about what subjects are likely to notice in the experimental situation is provided by the small rating study we conducted. The final question on the salience rating questionnaire (see Appendix A) asked subjects to write down everything they noticed about two diagonally adjacent squares in the lower righthand corner of each board. Table 10 lists the total frequency of different categories of responses for each of the four boards. Color or type of square, the fact that the squares are diagonally adjacent and are of the same type, and the

TABLE 10
Frequency of Remarks about Squares (n = 20 Subjects)—Salience Rating Study

Remark category	BLANK	COLORED	BLACK & PINK	BREAD & BUTTER	Total
Color/type of square	14	29	16	18	77
Same type	11	16	19	20	66
Position/location of square	13	9	9	12	43
Diagonal/adjacent/touching	10	11	8	12	41
Semantic associations	10	8	7	10	35
Other misc.	7	7	6	7	27
Appearance/position of words	0	0	6	4	10
Appearance of lines/shading	4	6	0	1	11
Size of squares	2	3	1	1	7
Total	71	89	72	85	317

location of the squares and semantic associations to the appearance or words on the squares (including the concept of "pairing") are the features mentioned most often. Color or type of square, the two responses relevant to parity, account for nearly one-half of all the responses of subjects not in the BLANK condition.

There is a good deal of independent evidence that invariant information of this kind is likely to be noticed. Research on detection of serial patterns (Kotovsky & Simon, 1973; Simon & Kotovsky, 1963) has shown that the relation of "same" between two stimuli or portions of stimuli has high perceptual salience, even taking precedence over the "next" relation that holds between two successive integers or two successive letters of the alphabet or some other familiar sequence. Further, the "same" relation, or repetition, is a central feature of almost all decorative design. Hence, there is ample psychological basis for the Noticing Invariants heuristic.

The protocols tell us some of the things our subjects actually noticed. As our hypothesis predicts, the features noticed virtually all refer to the displayed checkerboard to which the subjects are attending; other features of the experimental situation are rarely mentioned. Among the features frequently mentioned was the PARITY cue, along with other, irrelevant, cues. The protocols show clearly that the subjects do attend to the properties of the checkerboard when they are trying to cover it, and that, in fact, they frequently notice and mention features that are present after every attempted covering—the invariant features.

Individual Differences

At least two potential sources of individual differences are suggested by the Notice Invariant heuristic. First, there may be differences in the number of things noticed, as measured by the total number of invariants generated. Second, there may be differences in the types of things noticed. To measure these we would divide the invariants according to categories, as described in Appendix A (e.g., relevant, irrelevant, etc.).

Quantity of Invariants

In Table 8 we found no significant difference in the *overall* rate of generating invariants between fast and slow subjects. However we must also examine when the invariants are generated. When the eight protocols are divided into time slices (each roughly 100 s long), we see (Fig. 6a) that the fast subjects generate significantly more invariants than the slow subjects in the first 5 min ($t = 3.25$, $p < .025$). Fast subjects notice more things, earlier.

Since the fast subjects drop out as they solve the problem, the means for the fast subjects are based on progressively fewer data points, ex-

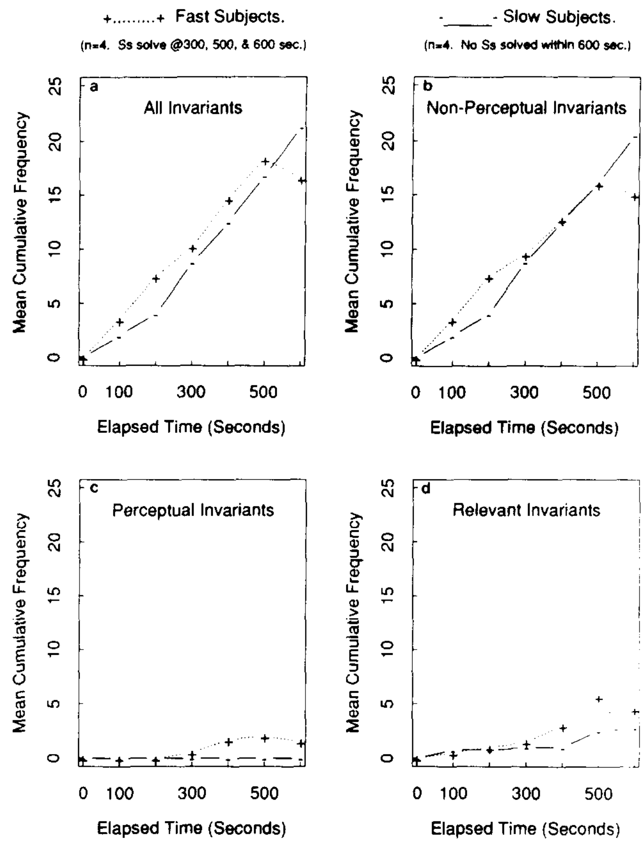

FIG. 6. Frequencies of different types of invariants (first 10 min).

plaining the drop in mean cumulative frequency from time slice 5 to 6. However, the fast subjects take a quick lead in the number of invariants generated. The slow subjects eventually "catch up," but not until most of the fast subjects (except in the BLANK condition where the relevant cue is unavailable) have already noticed important facts about the problem and solved it!

Perceptual Invariants

Will noticing any of the facts at all help, or does one type of invariant rather than another play a key role?

Both the cue salience results and interference from prior knowledge suggest that paying attention to the unique properties of the MC problem—the perceptual properties of the board—is likely to facilitate success. Since the problem instructions are quite simple, subjects who tend to look to the problem for constraints on search end up literally staring at the board, and noticing. Could this be what the fast subjects are doing?

To test this hypothesis, we classified invariants as perceptual (e.g., related to color, the position of Xs, or other visual aspects of the problem) or nonperceptual (e.g., related to strategies such as decomposition, mathematical properties, or other conceptual rather than visual properties of the problem).

Figures 6b and 6c show the cumulative frequencies of nonperceptual and perceptual invariants, respectively, generated by fast and slow subjects during the first 10 min of problem solving. The pattern for nonperceptual invariants is quite similar to the pattern for invariants in general (Fig. 6a). Fast subjects generate significantly more ($t = 2.34$, $p < .05$) during the first 5 min, then the slow subjects catch up.

The results for perceptual invariants are much more striking. Here we find a clear and statistically significant separation ($t = 2.61$, $p < .025$) between fast and slow subjects that extends over the first 10 min. Every single fast subject generated one or more perceptual invariants within the first 10 min, while none of the slow subjects did so! In fact, the slow subjects *never* catch up on this measure until after all the fast subjects have already solved the problem.

Eliminating an uninteresting explanation. One uninteresting explanation of these results would be that fast subjects (almost by definition) notice more relevant things about the problem, and not specifically more perceptual cues. However, not all of the perceptual invariants are relevant to the problem solution (e.g., noticing that the Xs are diagonal from each other, while a perceptual property, is not on a direct path to the solution). Conversely, nonperceptual properties (e.g., the knowledge that a domino is only allowed to cover adjacent squares) can also be relevant. A more convincing argument, however is made by the data on *relevant* invariants, perceptual and nonperceptual combined, plotted in Fig. 6d. There we find no significant difference between fast and slow subjects during the first 5 min of problem solving ($t = .76$, $p > .25$).

The pattern is quite different from Fig. 6c. Slow subjects generate *relevant* invariants from the very start of the problem—in contrast to their striking failure (shown in Fig. 6c) to generate *perceptual* invariants—and even start off slightly ahead of the fast subjects. Therefore, being "fast" does not translate into simply "generating more relevant invariants."

We would expect an increase in the number of relevant invariants generated over time (for both groups) as subjects home in on the solution. The increase is sharper for the fast subjects because they approach the

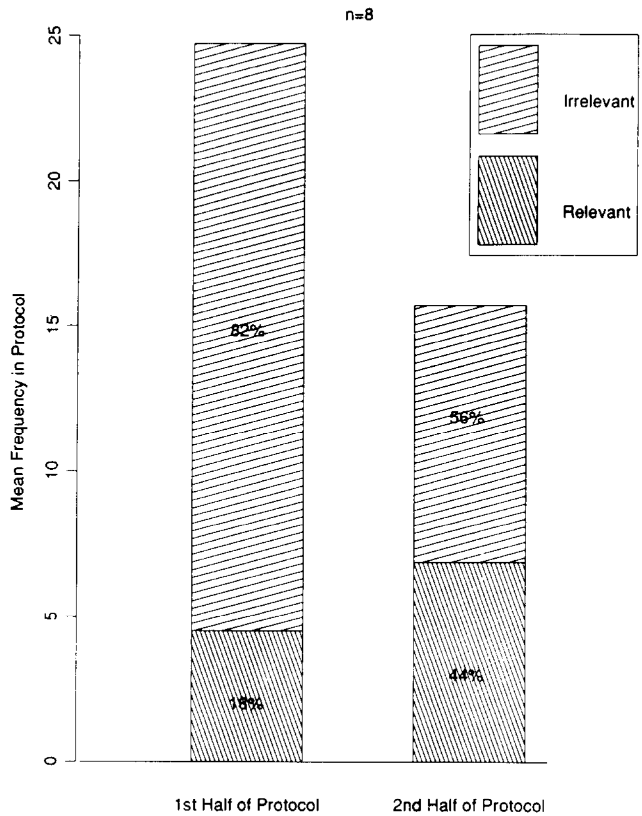

FIG. 7. Number and type of invariants noticed in first and second halves of protocol.

solution sooner. As converging evidence, Fig. 7 shows that subjects notice more total invariants in the first half of their protocols than in the second half. However, as the "homing in" hypothesis predicts, subjects notice significantly more *relevant* invariants ($p < .05$, one-tailed) and significantly less irrelevant invariants ($p < .05$) in the second half of their protocols. We are left with the interpretation that fast subjects possess a heuristic that the slow subjects do not—of paying attention to the perceptual features of the problem.

As one further test that this contributes to their greater success, we compare (in Table 11) the order in which subjects mentioned their first perceptual invariant (relevant or irrelevant) with the order in which they solved the problem. The rank order correlation between first mention and solution is .98 for the fast subjects, and .86 overall. At least for fast subjects, noticing perceptual invariants is almost a perfect predictor of

TABLE 11
Rank Orders of Times to First Mention Perceptual Invariant and to solution
(n = 8 Subjects)

Subject & condition	Rank order time to first percept. invar.	Rank order time to problem solved
s14 P&B fast	1	1
S1 B&B fast	2	2
S2 Clr fast	3	3
S17 Blnk fast	4	5
S13 P&B slow	5	6
S9 B&B slow	6	7
S16 Clr slow	7	4
S4 Blnk slow	8	8

speed in solving the MC problem. The difference in times between fast and slow subjects (in the same experimental conditions) to first mention of perceptual invariants is highly significant ($p < .01$, one-tailed t test).

Both the differences in number of perceptual invariants mentioned, and the correlation data strongly suggest that fast subjects are using qualitatively different heuristics from slow subjects. The exact nature of these heuristics is unclear. Are fast subjects simply perceptually driven, and would they be able to apply the more general heuristic of attending to problem features in nonperceptual domains? This is, of course, an empirical question. But the fact that fast subjects generate more invariants overall suggests that their advantage is not purely perceptual. From their own experience, most researchers can certainly attest to the benefit of "listening to what the data are saying"—which does not always involve visual features.

Flexibility in noticing

As a final exploration of individual differences, we tested the relation between the advice to "Think flexibly!" and the promising heuristic to "Pay attention to invariants!". One measure of flexibility would be the number of different types of invariants (where type is defined by the coding categories listed in Appendix A) generated by fast and slow subjects. Since the fastest subject solved the MC problem in 5 min, we can only present a full comparison for that interval. Figure 8 shows the mean number of coding categories (for invariants) covered by fast and slow subjects during this time interval.

The difference between the number of categories covered by fast and slow subjects is statistically significant ($p < .02$, one-tailed). We already know that fast subjects generate more invariants overall. However, since fast subjects notice significantly more nonperceptual invariants than slow

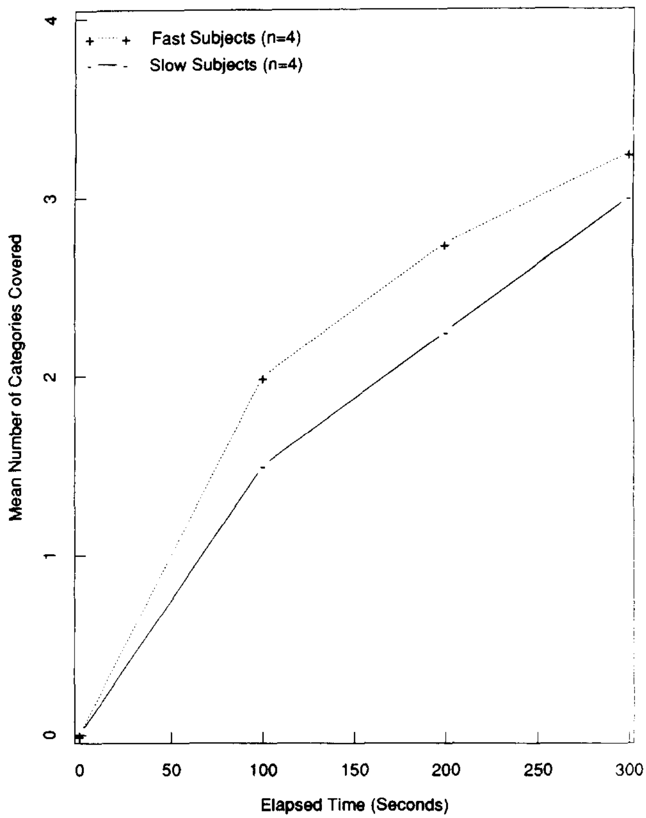

FIG. 8. Categories covered (first 5 min).

subjects during this same time period (Fig. 6b), the difference in category coverage cannot be due primarily to the difference in the number of perceptual invariants noticed. Rather, fast subjects are noticing not only more things, but a wider variety of things.

CONCLUSIONS

In this paper, we have proposed an extension of the theory of problem solving, viewed as selective search, to problems that require insight. The extension was achieved by proposing that insight problems are solved by shifting the problem representation, and that this shift is also accomplished by search—in this case search in a space of possible representations. Since this search is unfeasible without appropriate search constraints, we also explored the nature of these constraints, discovering some significant new mechanisms that have not previously been described. Our account is supported by a substantial body of empirical data.

We began our story with a metaphor comparing solving insight problems to searching for a diamond in a dark room. We argued that the task would be hopeless without some source of search constraint. In the domain of the MC problem, we identified four potential sources of constraint: perceptual cues in the problem, hints provided by the experimenter, prior knowledge that the subject might bring to the problem, and heuristics—in particular the Notice Invariants heuristic.

After establishing that the difficulty of the MC problem stems from search, we examined each potential source of search constraint in turn. The fact that subjects solving more salient versions of the MC problem attained insight sooner attests to the power of cue salience as a source of constraint. Hints, especially the PARITY hint, were shown to be quite effective sources of constraint. Subjects generated more statements relevant to the solution path after a hint than before it. Protocol evidence showed that prior domain knowledge, while constraining search, can actually be counterproductive if it leads to search in the wrong part of the space (as it is likely to do with insight problems).

Perhaps our most interesting results concerned use of the Notice Invariant heuristic. Focusing attention on invariant features of the problem situation allows subjects to convert a search in an enormous and unmanageable space (in which they have no relevant generators) to a search in a smaller space (with generators available).

We found that all subjects used this heuristic commonly, and that fast subjects used it more often than slow subjects early in problem solving. Furthermore, fast subjects differed from slow subjects in the types of invariants they noticed. In particular, of the eight subjects whose behavior was examined in detail, all fast subjects noticed perceptual invariants within the first 10 min of problem solving, whereas none of the slow subjects did so. This result cuts across experimental conditions. Fast subjects also tended to notice a wider variety of invariants than slow subjects during the initial minutes of problem solving, suggesting that flexibility, or the willingness to try a variety of things, may facilitate insight.

The "Notice Invariant" results constitute a significant step toward identifying a heuristic that can facilitate insight across a wide variety of domains. Although hints, cue salience, and prior knowledge all constrain search, it is difficult to specify how they will have their effect without knowing the nature of the problem (and its solution) beforehand. The essence of discovery however, is that you do *not* know beforehand where the solution may lie. If noticing invariants, and in particular perceptual invariants, provided even a little search constraint for the ill-defined task of discovery, then we have a cause for celebration. As this paper has tried to show, a little search constraint goes a long way.

APPENDIX A

Rating Questionnaire & Coding Systems

Rating Questionnaire

I.D. # ____________

Attached are four boards. Please answer the following questions about the boards by circling numbers on a 1 . . . 5 rating scale:

(1) How salient (striking or conspicuous) are the markings that differentiate the two types of squares on each board?

	NOT AT ALL SALIENT				VERY SALIENT
blank	1	2	3	4	5
pink and black words	1	2	3	4	5
black and white	1	2	3	4	5
bread and butter words	1	2	3	4	5

(2) How common are the markings on each board:

	VERY RARE				VERY COMMON
bread & butter words	1	2	3	4	5
pink & black words	1	2	3	4	5
black and white	1	2	3	4	5
blank	1	2	3	4	5

(3) Which marking do you think would be most likely to attract your attention and cause you to remark that the squares come in two different types? ____________
Which marking would be second most likely? ____________
Which marking would be third most likely? ____________
Which marking would be least likely? ____________

(4) Which marking best suggests the idea that squares might be related in pairs? ______

(5) Two diagonally adjacent squares in the lower righthand corner of each board were circled. The following question was asked verbally:

"You will notice that two of the squares have been circled on each board. In the space at the top of each board, please write down everything you notice about the two circled squares."

Coding System for Time Spent in Various Problem Spaces

1) Clean copies of verbatim transcripts for eight subjects were printed. Of these eight, five subjects had a delay between first mentioning parity and generating a rough ptoof. These five transcripts were subjected to further analysis.

2) Phrases, clauses, or sentences were matched to one of several problem spaces listed below. The match was based on a subjective estimate of the similarity between the content of the phrase, clause or sentence, and the content criteria for each problem space (described below). Each phrase, clause, or sentence could be assigned to only one problem space. In

case of a conflict where a phrase could be assigned to either the PARITY space or another space, the conservative policy was adopted of always assigning that phrase to the PARITY space.

3) Pauses were assumed to occur in the same problem space as the previous phrase, clause, or sentence.

4) It was assumed that for each subject the verbalization rate was approximately constant during the time period of interest. Based on this assumption, the time spent in each problem space was calculated by counting the number of words associated with each problem space and dividing by the verbalization rate. The resulting times (plus any pause times) spent in each problem space were then computed for each for the five subjects. The mean percentage of time spent in each space was then calculated. For purposes of Table 2, the time spent in the MATH, EVEN/ODD, ANALOGY, REARRANGE, PHYSCAL MANIPULATION, and META COMMENTS, problem spaces were collapsed into a single OTHER category.

5) The problems spaces and their defining criteria appear below:

Problem space	Defining content
Parity approaches	Any mention of the markings on the squares (parity) or mention of the fact that a domino covers two squares.
Covering spaces	Any mention of placing dominos in regular patterns. Any counting of domino placements. Any attempts to cover the board. Decomposition of board into smaller boards.
Math (general)	Any numerical analysis of the number of squares on the board or the number remaining to be covered.
Even/odd path	Attempts which focus on the fact that the path between removed squares may have even or odd numbers of squares.
Analogy to other puzzles	Attempts to recall or draw analogies between the MC problem and other puzzles/problems.
Rearrange board	Attempts to remove other squares from the board in order to determine whether the problem remains impossible. Comparison of current board to other possible board states.
Physical manipulation	Attempts to change perspective on the problem by rotating the board, drawing on it, or performing some other physical action.
Meta comments	Comments regarding progress made so far or high level evaluations of attempts made. Remarks directed at experimenter. Expressions of frustration. Re-reading of the instructions.
Experimenter comments	Any words or time used by the experimenter to make comments or give hints.

Coding System for Invariants, Comparisons, and Hypotheses

Each of the eight verbatim transcripts were coded for occurrences of INVARIANTS, COMPARISONS, and HYPOTHESES in the second phase.

These categories were defined as follows:

INVARIANTS:
An invariant is a fact that is mentioned repeatedly, and/or is qualified with one of the words "always", "any", "every", or "never."[11]

COMPARISON:
A comparison is the mention of two actual or hypothetical board situations in a single sentence, or in two consecutive sentences.

HYPOTHESIS:
A hypothesis is defined as an "if" statement that proposes, or refers, to a situation that could exist or an action that could be taken.

These categories are not mutually exclusive. For example the phrase "if the covering always fails . . ." would be coded both as an invariant and a hypothesis. Invariants, hypotheses, and comparisons were coded on separate passes through the transcripts using fresh copies of the transcript for each pass. The phrase or group of words matching the category was marked with a highlighter pen.

The third phase of coding involved categorizing the content of each INVARIANT as one of 19 mutually exclusive types. The content of each INVARIANT was matched against the defining content (information in parentheses below) of each type in the following order:[12]

1) Color-Imbalance (inc.: unequal numbers of two colors, 32 black & 30 pink, or vice versa, 2 pinks or blacks MUST be left)
2) Color-Covered (incl.: domino covers a pink & black, domino covers 2 of different colors)
3) Color-Position (incl.: pink & black adjacent, squares of same color are diagonal, pink & black go together, pink are left)
4) Color-Removed (incl.: same color squares Xed, Xs are both black, Xs on pink & black → possible, Xs on same color → impossible,)
5) Infer-Possibility (incl.: the problem is impossible)
6) Infer-Coverability (incl.: domino covers two adjacent squares, 2 remaining squares are not coverable, one domino remains after covering 30 squares)
7) Infer-Position (inc.: horizontal or vertical squares are adjacent)
8) Given-Covering-Properties (incl.: domino covers horizontally or vertically, domino cannot cover diagonally)
9) Given-Resources (incl. 8 × 8 board, 64 squares, 62 squares remaining, 2 squares removed, 31 dominos available)
10) Given-Goal (incl.: prove logically impossible, find covering)
11) Other-Color (incl.: any statements mentioning the COLOR of squares, excepting statements specifically covered above)
12) Xposition & Possibility (incl.: move Xs, Xs adjacent → possible, Xs moved → possible, even/odd spacing & possibility, Xs not diagonal → possible,

[11] These criteria were used to filter out facts that a subject might state in passing. We wished to identify the facts that *subjects believed* to be important and invariant. In fact, subjects' notion of what facts were invariant coincide quite well with objective analysis.

[12] Each INVARIANT was categorized on the basis of the *first* successful match with a type.

position of Xs is responsible for impossibility, Xs make problem impossible)
13) Xposition-General (incl.: Xs are diagonal, Xs exist)
14) Type-Left (incl.: two squares always left, squares left are diagonal)
15) Covering-Failures (only 30, not 31, can't do it, doesn't work, etc. EXCLUDES DIRECT STATEMENTS OF PROBLEM'S IMPOSSIBILITY)
16) Math (incl.: even/odd numbers—NOT SPACING—, algebra, facts about the numbers involved, counting features of board)
17) Decomposition (incl.: CONSIDERING $n \times n$ boards, mentioning decomposition, trying different shaped boards)
18) Covering (incl. ALL STATEMENTS ABOUT SPECIFIC DOMINO PLACEMENTS AND STRATEGIES OR SYSTEMATIC DOMINO PLACEMENT)
19) Other (incl.: Off-track ideas of rare frequency)

Once all INVARIANTS had been coded according to content, it was possible to define subsets of the 19 content types that define other meaningful categories. Specifically:

Types 1—10 = RELEVANT INVARIANTS: i.e., invariants that we would expect to be generated by a subject following a direct path to the insightful solution.
Types 11–19 = IRRELEVANT INVARIANTS: i.e., invariants that do not directly lead to the insightful solution.
Types 2–4, 11, 13, 14 = PERCEPTUAL INVARIANTS: i.e., features/properties of the board that are evident at a glance.
Types 1, 5–10, 12, 15–19 = NONPERCEPTUAL INVARIANTS.
Types 1–4 = RELEVANT PARITY INVARIANTS.
Type 18 = COVERING INVARIANTS.

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