Basic Concepts of Set Theory

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Contents

1 Introduction

Set theory, as the study of collections of objects and their relationships, is fundamental to mathematics, providing essential tools for organizing and manipulating data. In the realm of set theory, we encounter concepts such as sets, elements, subsets, unions, intersections, and complements.

Moreover, set theory serves as the foundation for understanding probability theory. By leveraging the principles of set theory, we gain deeper insights into probability concepts such as sample spaces, events, and probabilities themselves. Through concrete examples and intuitive explanations, we observe the interconnectedness of these two domains, enriching our understanding of both subjects. This synergy between set theory and probability theory enhances our ability to analyze and interpret data, making them invaluable tools in various fields of study and application.

2 Set Definitions

Definition 2.1. A set is a collection of objects called elements of the set.

Example. Set of computer science students, set of numbers.

Definition 2.2.
- **A set of sets** sometimes called class of sets.
- A set is usually denoted by a **capital letter** while an element is represented by a **lower-case letter**.
- A set is usually denoted by a capital letter while an element is represented by a lower-case letter.
  - If a is an element of set A, then we write $a \in A$
  - If a is not an element of set A, then we write $a \notin A$
- A set is specified by the content of two braces: $\{.\}$
- Two methods exist for specifying content:
  - The tabular method.

- The rule method.

Example. The set of all integers between 5 and 10 would be:

1. In tabular method such as:
   (a) $A = \{6, 7, 8, 9\}$.
   (b) $S = \{book, \; cellphone, \; mp3, \; paper, \; laptop\}$
1. In rule method such as:
   (a) A $= \{integers\,between\,5\,and\,10\,\}$

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1. A set with a large or infinite number of elements are best described by a statement or rule method. for example:
   (a) $\{Integers\,from\,5\,to\,10000\,inclusive\,\}$
   (b) $S = \{x|x\,is\,a\,city\,with\,a\,population\,over\,1\,million\}$

Definition 2.3. A set is called countable if its elements can be put in one-toone correspondence with natural numbers.

Example.
1. $A = \{1, 3, 5, 7\}$
2. $S = \{H, T\}$
3. $B = \{1, 2, 3, ...\}$

Definition 2.4. A set is called uncountable, if its elements not countable

Example.
1. $A = \{0.5 < x < 8.5\}$
2. $S = \{-0.5 \leq y \leq 12\}$
3. $B = \{t/t \geq 0\}$

Definition 2.5 (Subset)**.** The set A is called a subset of B if every element in A is also an element in B (A contained in B), we write

$$A \subseteq B \tag{1}$$

If at least one element exists in B which is not in A, then A is a proper subset of B, denoted by

$$A \subset B \tag{2}$$

- The null set is clearly a subset of all other sets.

Definition 2.6 (Disjoint Set)**.** Two sets A and B is called disjoint or mutually exclusive if they have no common elements :

$$A \cap B = AB = \emptyset \tag{3}$$

Example.
- $A = \{1, 3, 5, 7\}$
- $B = \{1, 2, 3; ...\}$
- $C = \{0.5 < x \leq 8.5\}$
- $D = \{0.0\}$
- $E = \{2, 4, 6, 8, 10, 12, 14\}$
- $F = \{-0.5 < x \leq 12.5\}$

1. A: tabular-specified countable and finite.
2. B: is also tabular-specified and countable but infinite.
3. C: rule- specified, uncountable and infinite
4. D and E are mutually exclusive.
5. F is uncountable and infinite
6. Set A is contained in set B, C and F.
7. $D \subset F$, $C \subset F$, $E \subset B$
8. Sets A, D and E are mutually exclusive.

- The largest set of objects under discussion in a given situation is called the universal set, denoted S.

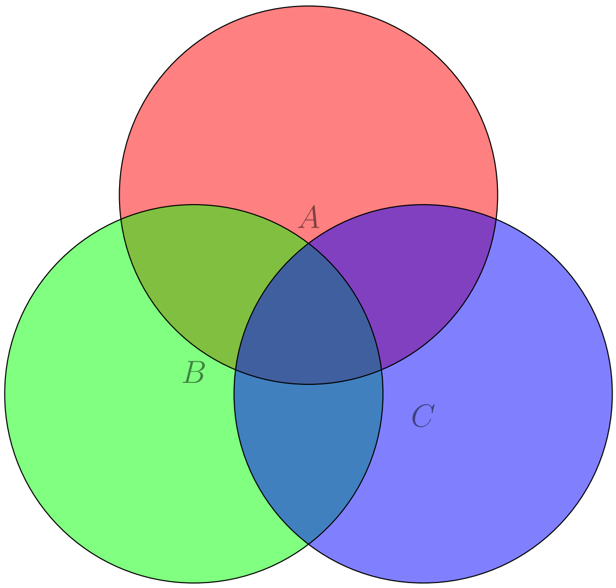
Example. In the problem of rolling a die, we are interested in the numbers that show on the upper face. The universal set is:

$$S = \{1, 2, 3, 4, 5, 6\} \tag{4}$$

- For any universal set with N elements, there are 2^N possible subsets of S.

3 Set Operations

- Geometrical representation of the sets using Venn diagram. The relationship between subsets and the universal set can be illustrated graphically using Venn diagram.
  - Sets are represented by closed-plane figures.



Definition 3.1 (Equality). Two sets A and B are equal if and only if they have the same elements. We write $A = B$

$$A \subseteq B \, and \, B \subseteq A \tag{5}$$

Definition 3.2 (Difference). The difference of two sets A and B, is the set containing all elements of A that are not present in B. We write

$$A \setminus B \tag{6}$$

Example. If $A = \{0.6 \leq x \leq 1.6\}$ and $B = \{1.0 \leq y \leq 2.5\}$
Then $A \setminus B = \{0.6 \leq t \leq 1\}$ and $B \setminus A = \{1.6 \leq t \leq 2.5\}$

✍ Note that $A \setminus B \neq B \setminus A$

Definition 3.3 (Union). The union of two sets A and B, written as $C = A \cup B$, is the set containing all elements of both A and B or both, Union sometimes called the sum of two sets

Definition 3.4 (Intersection). The intersection of two sets A and B, written as $C = A \cap B$, is the set of all elements common to both A and B or both, Intersection sometimes called the product of two sets

- ✍ For mutually exclusive sets A and B, $A \cap B = \emptyset$
- ✍ In general case, the union and intersection of of N sets $A_n, n = 1, 2, 3, ?., N$, become:

$$A_1 \cup A_2 \cup ... \cup A_N = \cup_{n=1}^{N} A_n \tag{7}$$

$$A_1 \cap A_2 \cap ... \cap A_N = \cap_{n=1}^{N} A_n \tag{8}$$

Definition 3.5 (Complement). The complement of set A, denoted by $\bar{A}$, is the set of all elements not in A.

- ✍ Note that:

$$\bar{S} = \emptyset,\ \emptyset = \bar{S},\ A \cup \bar{A} = S,\ and\ A \cap \bar{A} = \emptyset \tag{9}$$

Example. Given the four sets:

$$\begin{aligned} S &= \{1 \leq integer \leq 12\} \\ A &= \{1, 3, 5, 12\} \\ B &= \{2, 6, 7, 8, 9, 10, 11\} \\ C &= \{1, 3, 4, 6, 7, 8\} \end{aligned}$$

Then

$$\begin{aligned} &A \cup B = \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12\} \\ &A \cup C = \{1, 3, 4, 5, 6, 7, 8, 12\} \\ &B \cup C = \{1, 2, 3, 4, 6, 7, 8, 9, 10, 11\} \\ &A \cap B = \emptyset, A \cap C = \{1, 3\}, B \cap C = \{6, 7, 8\} \\ &\bar{A} = \{2, 4, 6, 7, 8, 9, 10, 11\} \\ &\bar{B} = \{1, 3, 4, 5, 12\} \\ &\bar{C} = \{2, 5, 9, 10, 11, 12\} \end{aligned}$$

4 Algebra of sets and duality

Property 1. Commutative Law

$$\begin{aligned} A \cup B = B \cup A \\ A \cap B = B \cap A \end{aligned}$$

Property 2. Distributive Law

$$\begin{aligned} A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \end{aligned}$$

Property 3. Associative Law

$$\begin{aligned} A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C \\ A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C \end{aligned}$$

Property 4. De Morgan's Law

$$\overline{A \cup B} = \bar{B} \cap \bar{A}$$
$$\overline{A \cap B} = \bar{B} \cup \bar{A}$$

- Replace unions by intersections, intersections by unions, by use of a venn

Property 5. Duality Principle In any an identity we replace unions by intersections, intersections by unions, S by $\emptyset$, and $\emptyset$ by S, then the identity is preserved.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$