Total Probability, Bayes' Theorem and Independent Events

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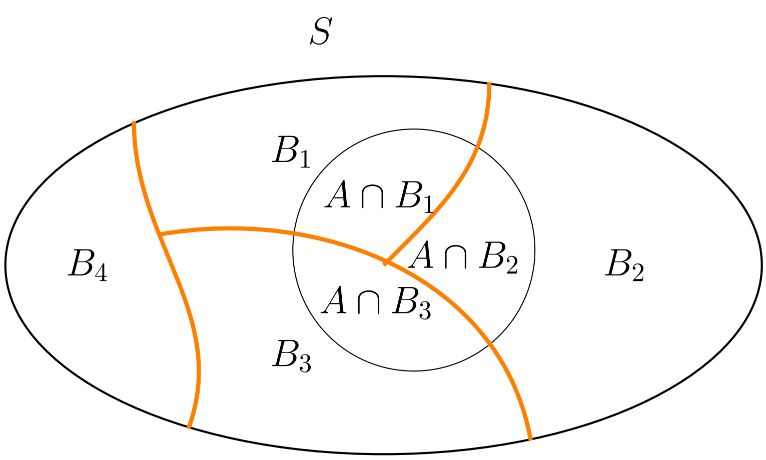
1 Total Probability and Bayes'Theorem

- If the events $B_1, B_2, \ldots B_N$ constitute a partition of the sample space S such that
  - $\cup_{n=1}^{N} B_n = S$ **(Collectively Exhaustive)**
  - $B_i \cap Bj$ for all i, j and $i \neq j$ **(mutually exclusive)**
- The probability $p(A)$ of any event defined on a sample space S can be expressed in terms of conditional probabilities:

$$p(A) = \sum_{n=1}^{N} p(A/B_n) \times B_n$$

Which is known as the total probability of event A



- Prove Since

$$A = A \cap S \Rightarrow A = A \cap \left(\cup_{n=1}^{N} B_n \right) = \cup_{n=1}^{N} (A \cap B_n)$$

The events $(A \cap B_n)$ are mutually exclusive, then

$$p(A) = p(A \cap S) = p\left(\cup_{n=1}^{N} (A \cap B_n) \right) = \sum_{n=1}^{N} p(A \cap B_n)$$

1.1 Bayes' Theorem

- Bayes' rule is one of the most important rules in probability theory.
- Bayes' theorem is often referred to as a theorem on the probability of causes.
- From the conditional probability (If $p(A) \neq 0$ and $p(B) \neq 0$):

$$p(A/B) = \frac{p(A \cap B)}{p(B)}$$

 and

$$p(A/B) = \frac{p(B \cap A)}{p(B)}$$

- Bayes' theorem is obtained by equating these two expressions:

$$p(B \cap A) = p(B) \times p(A/B) = p(A) \times p(B/A) \qquad (i)$$

$$p(A/B) = \frac{p(A) \times p(B/A)}{p(B)} \qquad (ii)\ \textbf{Bayes' theorem}$$

- Sample space and intersection of events: **Generalized Bayes' theorem**
  1. In the case of three events A_1, A_2, A_3 are mutually exclusive and collectively exhaustive (probabilities of all events = 1 and construct the sample space) ?
  2. B has some common in A_1, A_2 and A_3

$$B = (B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3)$$

  From equation (i)

$$p(B) = p(A_1) \times p(B/A_1) + p(A_2) \times p(B/A_2) + p(A_3) \times p(B/A_3)$$

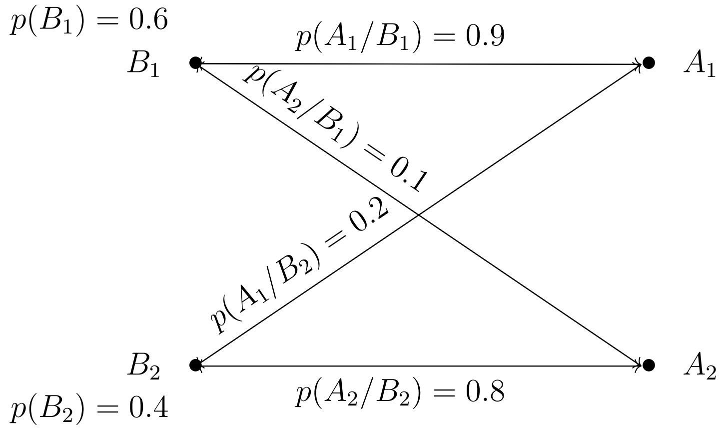
$$p(B) = \sum_{i=1}^{n} p(A_i) \times p(B/A_i) \qquad (iii)$$

- Putting all together (substitute (iii) in (ii)) we get;

$$p(A/B) = \frac{p(A) \times p(B/A)}{\sum_{i=1}^{n} p(A_i) \times p(B/A_i)} \qquad (iii)$$

2 Generalized form of the Bayes' Theorem

Example An elementary binary communication system consists of a transmitter that sends one of two possible symbols (a 1 or a 0) over a channel to a receiver. The channel occasionally causes errors to occur so that a 1 shows up at the receiver as a 0, and vice versa.



- noident Denote by B_1 qnd $B_@$ as the events the symbol before the channel,A_1 and A_2 as the events the symbol after the channel

✍ The probabilities of receiving symbol A_1 and A_2 are:

$$p(A_1) = P(A_1/B_1)p(B_1) + P(A_1/B_2)p(B_2) = 0.9 * 0,6 + 0.2 * 0.4 = 0.62$$

$$p(A_2) = P(A_2/B_1)p(B_1) + P(A_2/B_2)p(B_2) = 0.1 * 0,6 + 0.8 * 0.4 = 0.38$$

✍ The probability that B_1 is sent if A_1 is received (using Bayes' theorem):

$$p(B_1/A_1) = \frac{P(A_1/B_1)p(B_1)}{p(A_1)} = \frac{0.9 * 0.6}{0.62} = 0.87$$

✍ Similarly for:

$$p(B_2/A_1) = \frac{P(A_1/B_2)p(B_2)}{p(A_1)} = 0.13$$
$$p(A_1/B_1) = \frac{P(B_1/A_!)p(A_!)}{p(B_1)} = 0.84$$
$$p(A_2/B_2) = \frac{P(B_2/A_2)p(A_2)}{p(B_2)} = 0.16$$

✍ The total probability of error for the system:

$$p_e = p(A_1/B_2)p(B_2) + p(A_2/B_1)p(B_1) = 0.14$$

✍ Example In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25% respectively of the products. It is known from past experience that 2%, 3% and 2% of the products made by each machine, respectively, are defective. Now suppose that a finished product is randomly selected.

- What is the probability that it is defective?

✍ Solution:
Events:

1. A the product is defective
2. B_1 the product is made by machine B_1
3. B_2 the product is made by machine B_2
4. B_3 the product is made by machine B_3

Using total probability theorem

$$\begin{aligned} p(A) &= p(B_1) \times p(A/B_1) + p(B_2) \times p(A/B_2) + p(B_3) \times p(A/B_3) \\ &= 0.3 * 0,02 + 0.45 * 0.03??.03 + 0.25 * 0.02 \\ &= 0.006 + 0.0135 + 0,005 = 0,0245 \end{aligned}$$

- If a product was chosen randomly and found to be defective, what is the probability that it was made by machine B_3?

✍ Solution:Using Bayes' rule

$$\begin{aligned} p(B_3/A) &= \frac{p(B_3) \times p(A/B_3)}{p(B_1) \times p(A/B_1) + p(B_2) \times p(A/B_2) + p(B_3) \times p(A/B_3} \\ &= \frac{0.005}{0.006 + o.0135 + 0.005} = \frac{0.005}{0.245} = \frac{10}{49} \\ &= 0.006 + 0.0135 + 0,005 = 0,0245 \end{aligned}$$

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✍ Example A box contains 6 green balls, 4 black balls, and 10 yellow balls. All balls are equally likely (probable) to be drawn. What is the probability of drawing two green balls from the box if the first drawn ball is not replaced?

$$p(G \cap G) = p(G) \times p(G/G) = \frac{6}{20} * \frac{5}{19} = 0.0789$$

✍ ?In Bayes' theorem $p(B_n)$ are usually referred to as a Priori probabilities.

✍ ?The $p(A/B_n)$ are numbers typically known prior to conducting the experiment. The $p(B_n/A)$ are called a Posteriori probabilities

3 Independent Events

✍ The two nonzero probabilities events A and B are called statistically independent if the probability of occurrence of one event is not affected by the occurrence of the other event.

✍ ?If events A and B are statistically independent then they can both occur.

✍ Mathematically:

- First approach for testing the Independent Events

$$p(A/B) = p(A) \text{ and } p(B/A) = p(B)$$

Conditions B doesn't affect in A.

✍ Example From HBO example from previous lecture

$$\begin{aligned} p(WW/F) &= \frac{0.05}{0.54} = 0.093 \\ p(WW) &= 0.25 \end{aligned}$$

Therefore the events are not independent as $0.093 \neq 0.25$

✍ Example From HBO example from previous lecture

$$p(WW/F) = \frac{0.05}{0.54} = 0.093$$
$$p(WW) = 0.25$$

Therefore the events are not independent as $0.093 \neq 0.25$

- Second approach for testing the Independent Events:

$$p(A \cap B) = p(A).p(B)$$

✍ Example From HBO example from previous lecture

$$p(WW \cap F) = 0.05$$
$$p(WW) * p(F) = 0.25 * 0.54 = 0.14$$

Therefore the events are not independent as $0.05 \neq 0.14$

✍ Examplerolling a Dice and flipping a coin are independent events, so the probability of getting 2 and a Heads is

$$p = \frac{1}{6} * \frac{1}{2}$$

✍ Multiple Events In the case of three events A_1, A_2, A_3 are independent if they are pairwise independent:

$$p(A_j \cap A_k) = p(A_j).p(A_k), \ j = 1, 2, 3, \ k = 1, 2, 3, \ j \neq k$$
$$\text{And also independent as a triple}$$
$$p(A_1 \cap A_2 \cap A_3) = p(A_1).p(A_2).p(A_3)$$