Problem 1. Let X_i ($i = 1, 2, \dots$) be i.i.d. random variables with mean 0 and variance 2; and let Y_j ($j = 1, 2, \dots$) be i.i.d. random variables with mean 2. Assume that all variables X_i, Y_j are independent. Consider the following statements :

(A) *[2 points]* $\frac{X_1+\cdots+X_n}{n}$ converges to 0 in probability as $n \to \infty$.

(B) *[2 points]* $\frac{X_1^2+\cdots+X_n^2}{n}$ converges to 2 in probability as $n \to \infty$.

(C) *[2 points]* $\frac{X_1Y_1+\cdots+X_nY_n}{n}$ converges to 0 in probability as $n \to \infty$.

Which of these statements are always true? Write `True` or `False` in each of the boxes below.

A.	**B.**	**C.**

1. *[3 points]* Whether `True` or `False`, justify your answer to question (B).

Problem 2. In this problem set, we study the case of a CC-CI1 student (student A) who hadn't started working for this exam early enough to be able to go through all the exercises in the handouts (hopefully not your case!). Student A comes up with a strategy to study for the mid-term. He takes the 4 handouts in their chronological order and flips a coin for each exercise. If the coin lands heads, student A studies the exercise. If it lands tails, he doesn't study the exercise and moves on, repeating the strategy for each new exercise. The coin has probability $p > 0$ of landing heads.

1. *[5 points]* Student A gets into classroom 105 early for the mid-term. He tells his classmate that he studied m exercises. Let X be the number of coin flips it took student A to study the m exercises. Give the probability distribution of X.

2. *[6 points]* Let m_1 and m_2 be two integers such that $0 < m_1 \leq m_2$ and let N_1 and N_2 be the number of exercises done by rounds m_1 and m_2 respectively. Prove that the conditional probability distribution of $N_1 \mid N_2$ is given by

$$P(N_1 = k_1 \mid N_2 = k_2) = \frac{\binom{m_1}{k_1}\binom{m_2-m_1}{k_2-k_1}}{\binom{m_2}{k_2}}. \tag{1}$$

Two other CC-CI1 students (student B and student C) heard of student A's strategy and decided to improve it through cooperation with a 2-step strategy.

- Step 1: For a given time, student B works on the exercises of handouts 1 and 3, and student C works of the exercises of handouts 2 and 4;
- Step 2: They explain to each other the exercises they worked on, thereby saving some time.

During step 1, the two students use two independent coins that both have probability $p > 0$ of landing heads. Each student takes a coin and runs the same strategy as student A above.

At the end of step 1, student B has flipped the coin n_1 times and student C has flipped the coin n_2 times.

3. *[4 points]* Let Y_B be the number of exercises student B worked on. Give the probability distribution of Y_B.

4. *[5 points]* Let Z be the total number (i.e. the sum) of exercises both students B and C worked on. Give the probability distribution of Z.

Suppose all students in the class applied student A's strategy. The number of exercises W that each student ends up studying has mean $\mu > 0$ and standard deviation $\sigma > 0$.

5. *[5 points]* Give an expression of a nontrivial upper bound on the probability that the number of exercises you studied is 2 standard deviations away from the mean. Justify your answer.

Problem 3. Imagine that this exam was entirely a multiple-choice exam as in questions A, B, C of Problem 1. You may assume that the number of questions is infinite. *Simultaneously, but independently*, your conscious and subconscious faculties are generating answers for you, each in a Poisson manner. (Your conscious and subconscious are always working on different questions.) Conscious responses are generated at the rate $\lambda_c > 0$ responses per minute. Subconscious responses are generated at the rate $\lambda_s > 0$ responses per minute. Assume $\lambda_c \neq \lambda_s$. Each conscious response is an independent Bernoulli trial with probability $p_c > 0$ of being correct. Similarly, each subconscious response is an independent Bernoulli trial with probability $p_s > 0$ of being correct. You respond only once to each question, and you can assume that your time for recording these conscious and subconscious responses is negligible.

1. *[5 points]* Determine the probability distribution of the number of conscious responses you make in an interval of T minutes.

2. *[5 points]* If we pick any question to which you have responded, what is the probability that your answer to that question represents a **subconscious** correct response?

3. *[5 points]* Knowing that you have answered 10 questions from the beginning of the exam to time $t_1 > 0$. What is the probability that, in the interval $]t_1, t_1 + T]$, you will make exactly r conscious responses and s subconscious responses?

4. *[5 points]* Determine the probability distribution of random variable X, where X is the time from the start of the exam until you make 3 responses (conscious or subconscious).

Problem 4. You'd like to play a game after this exam. In preparation, you first want to run a probabilistic analysis of the game. After the professor collects the exam booklets, you are allowed to leave classroom 105. You decide the following :

- To move one step towards either the front door or the back door with equal probability.
- Your movements are on a vertical straight line. If you reach the same horizontal level as the front door, you leave from the front door, and if you reach the same horizontal level as the back door, you leave from the back door.

The back door is at $x = 0$ and the front door is at $x = T$, $T \in \mathbb{N}^*$. Your desk is at $x = n$, with $n \in \{0, 1, \ldots, T\}$. Let p_n be the probability that you leave from the front door given that your desk is initially at $x = n$.

1. *[5 points]* Write a *solvable* recurrence on p_n.

2. *[5 points]* What is the probability of leaving from the front door? A proof is required.

3. *[3 points]* What is the probability of leaving from the back door? A proof is required.

4. *[3 points]* What is the probability of remaining in classroom 105 forever? A proof is required.

If your chosen desk is at position n, let s_n be *the number of steps you make* until this game ends (because you reach either the front door or the back door).

5. *[5 points]* Give constants a, b, c such that

$$s_n = as_{n-1} + bs_{n-2} + c \tag{2}$$

for $1 < n < T$.

Mid-term exam : stochastic processes

March 26^{th}, 2025

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE ALLOWED TO DO SO

1. This is a closed book exam. **One A4 page with notes in your own handwriting** is allowed.
2. **DO NOT** write you name elsewhere other than at the bottom of this page.
3. Cheating is a **non-negotiable breach** that will lead to severe penalties.
4. Write your solutions in the space provided. If you need more space, write on the last page of the problem. Please keep your entire answer to a problem on that problem's pages.
5. **Show your work**. Even if your final answer is wrong, you could earn partial credit if the reasoning is correct. Besides, correct final answers with partially erroneous reasoning won't get full credit.
6. Be neat and write legibly. You may be graded not only on the correctness of your answers, but also on the clarity and the correctness of your reasoning.
7. If you get stuck on a problem, move on to others. Problems are not in order of difficulty. You'll figure out what order they're in once you're done.
8. When asked to give a probability distribution, **make sure to specify the range over which the formula holds**.
9. You have **two hours** to complete the exam.

Attribute	**Problem 1**	**Problem 2**	**Problem 3**	**Problem 4**	**Total**
Questions	4	5	4	5	18
Points	9	25	20	21	75
Score					

Full name : ....