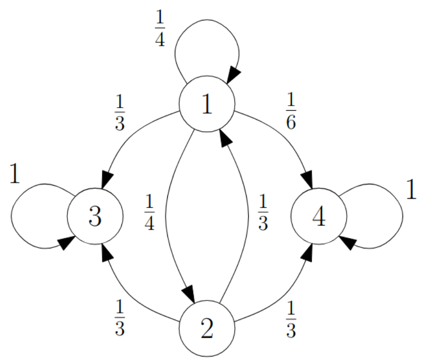
Problem 1. Consider a discrete-time Markov chain (DTMC) $\{X_n\}_{n\in\mathbb{N}}$ whose one-step transition probabilities are shown below.



1. *[2 points]* Classify the states of this DTMC.
2. *[5 points]* Suppose that the chain starts at $X_0 = 2$. Find the probability that $X_2 = 4$.
3. *[5 points]* What is the probability of eventually visiting state 4, given that the initial state is $X_0 = 1$?

Problem 2. Consider the DTMC over state space $\mathcal{S} = \{1, 2, 3\}$ with transition matrix

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 \end{pmatrix}$$

1. *[5 points]* Does this Markov chain converge? Justify your answer.
2. *[5 points]* Find the matrix which is the limit of $\mathbf{Q}^n$ as $n \to \infty$.
3. *[5 points]* Find the mean recurrence time to state 3, i.e., the expected number of transitions it takes to return to state 3, starting from state 3.

Problem 3. The inter-arrival times for cars passing a checkpoint are independent random variables with PDF

$$f_T(t) = \begin{cases} 2e^{-2t}, & t > 0 \\ 0, & \text{otherwise} \end{cases}$$

where the inter-arrival times are measured in minutes. The successive experimental values of the durations of these inter-arrival times are recorded on small computer cards. The recording operation occupies a negligible time period following each arrival. Each card has space for three entries. As soon as a card is filled, it is replaced by the next card.

1. *[5 points]* Given that no car has arrived in the last four minutes, determine the PMF for random variable K, the number of cars to arrive in the next six minutes.
2. *[5 points]* Determine the PDF and the expected value for the total time required to use up the first two computer cards.

Problem 4.

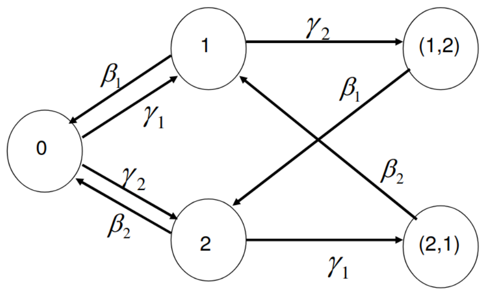
1. *[5 points]* Is the Metropolis-Hastings algorithm guaranteed to converge to the distribution of interest if we choose a **reducible** proposal matrix? Justify your answer.

Problem 5. Consider two copier machines that are maintained by a single repairman. Machine $i \in \{1, 2\}$ functions for an exponentially distributed amount of time with mean $1/\gamma_i$, with $\gamma_i > 0$, before it breaks down. The repair times for copier $i \in \{1, 2\}$ are exponential with mean $1/\beta_i$, with $\beta_i > 0$, but the repairman can only work on one machine at a time. Assume that the machines are repaired in the order in which they fail.

We wish to construct a continuous-time Markov chain (CTMC) model of this system, with the goal of finding the long-run proportions of time that each copier is working and the repairman is busy. We use the 5 following states :

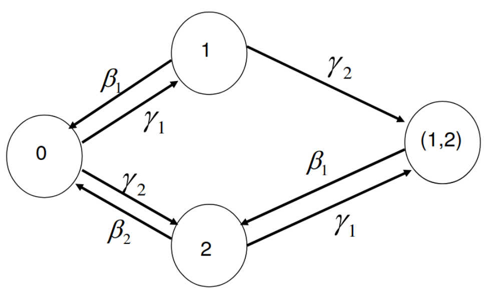
- 0 for no copiers failed,
- 1 for copier 1 is failed (and copier 2 is working),
- 2 for copier 2 is failed (and copier 1 is working),
- (1,2) for both copiers down (failed) with copier 1 having failed first and being repaired,
- (2,1) for both copiers down with copier 2 having failed first and being repaired.

The graph of this CTMC is as shown below



1. *[5 points]* Give the infinitesimal generator of this CTMC.
2. *[5 points]* Does this CTMC have a unique stationary distribution? Justify your answer.

Now let us consider an alternative repair strategy: Suppose that copier 1 is more important than copier 2, so that it is more important to have it working. Toward that end, suppose the repairman always work on copier 1 when both copiers are down. In particular, now suppose that the repairman stops working on copier 2 when it is down if copier 1 also subsequently fails, and immediately shifts his attention to copier 1, returning to work on copier 2 after copier 1 has been repaired. How do the long-run proportions change? With this alternative repair strategy, we can revise the state space. Now it does suffice to use 4 states, letting the state correspond to the set of failed copiers, because now we know what the repairman will do when both copiers are down; he will always work on copier 1. Thus it suffices to use the single state (1, 2) to indicate that both machines have failed. The graph that represents the alternative strategy is shown below.



4. *[6 points]* Suppose that the failure rates are $\gamma_1 = 1$ per month and $\gamma_2 = 3$ per month; and suppose the repair rates are $\beta_1 = 2$ per month and $\beta_2 = 4$ per month. Write down some equations(s) to retrieve a/the stationary distribution of this CTMC. Explain why it is solvable.
5. *[5 points]* Give an expression of the long-run proportion of time that copier 1 is working.

Re-sit exam : stochastic processes

June 5^{th}, 2025

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE ALLOWED TO DO SO

1. This is a closed book exam. One A4 page with notes in your own handwriting is allowed.
2. **DO NOT** write you name elsewhere other than at the bottom of this page.
3. Cheating is a **non-negotiable breach** that will lead to severe penalties.
4. **Show your work**. Even if your final answer is wrong or you can't finalize it, valid reasoning can get partial credit. Besides, correct answers with erroneous reasoning won't get full credit.
5. **Write legibly**. If it can't be read, it can't be graded.
6. If you get stuck on a question, move on to others. **Questions are not in order of difficulty**. You'll figure out what order they're in once you're done.
7. When asked to prove a given result, **be rigorous**. Approximate or **dishonest attempts to reach a result will be penalized**.
8. Questions are repeated in pages 3 to 13. Write your solutions in the space provided.
9. You have **two hours and 10 minutes** to complete the exam.

Attribute	Exercise	Problem	Total
Questions	3	13	16
Points	15	60	75
Score			

Full name : ....