Handout 6: Discrete-time Markov chains (2)

Due date : April $17^{th}, 2025$

Brief recap :

In all that follows, we consider homogeneous discrete-time discrete space Markov chains with state space $\{1, 2, \ldots, M\}$.

1. A Markov chain is **ergodic** if it is both irreducible and aperiodic.

2. A row vector $\mathbf{s} = (s_1, s_2, \ldots, s_M)$ with $s_i \geq 0$ for all $i \in \{1, 2, \ldots, M\}$ and $\sum_{i=1}^{M} s_i = 1$ is said to be a **stationary distribution** for a Markov chain with transition matrix $\mathbf{Q}$ if $\mathbf{sQ} = \mathbf{s}$.

3. A Markov chain whose initial distribution is a stationary distribution will stay in the stationary distribution forever.

4. Stationary distribution is marginal, not conditional. When a Markov chain is at the stationary distribution, the PMF of X_n equals $\mathbf{s}$ for all $n \in \mathbb{N}^*$, but the conditional PMF of $X_n \mid X_{n-1}$ is still encoded by the the transition matrix $\mathbf{Q}$.

5. **(Existence and uniqueness theorem)**. If a Markov chain is irreducible, then it has a unique stationary distribution with strictly positive probability for every state (consequence of the Perron-Frobenius theorem).

6. Let $\mathbf{Q}$ be the transition matrix of a Markov chain. The chain is irreducible and aperiodic if and only if some power $\mathbf{Q}^m$, with $m \in \mathbb{N}^*$ is strictly positive in all entries.

7. **(Convergence theorem)**. If $X_0, X_1, \ldots$ is an *irreducible and aperiodic (i.e. ergodic)* Markov chain with stationary distribution $\mathbf{s}$ and transition matrix $\mathbf{Q}$, then $\lim\limits_{n\to\infty} P(X_n = i) = s_i$. In terms of the transition matrix, $\mathbf{Q}^n$ converges to a matrix in which each row is $\mathbf{s}$.

8. After a large number of steps, the probability that the chain is in state $i \in \{1, 2, \ldots, M\}$ is close to the stationary probability s_i, regardless of the chain's initial conditions.

9. Intuitively, aperiodicity is needed in order to rule out chains that go around in circles.

10. Let $X_0, X_1, \ldots$ be an irreducible Markov chain with stationary distribution $\mathbf{s}$. For $i \in \{1, 2, \ldots, M\}$, let r_i be the **expected return time** (number of rounds) to state i, given that it starts at i. Then
$$s_i = \frac{1}{r_i}.$$

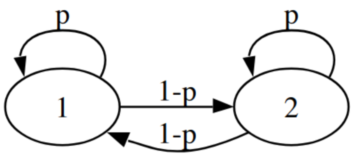
11. Let $\mathbf{Q} = (q_{ij})_{1\leq i,j\leq M}$ be the transition matrix of a Markov chain. Suppose the existence of $\mathbf{s} = (s_1, s_2, \ldots, s_M)$ with $s_i \geq 0$ for all $i \in \{1, 2, \ldots, M\}$ and $\sum_{i=1}^{M} s_i = 1$, such that
$$s_i q_{ij} = s_j q_{ji}, \quad \forall\, i, j \in \{1, 2, \ldots, M\}. \tag{1}$$
Equation (1) is called the reversibility or detailed balance condition The Markov chain is **reversible** with respect to $\mathbf{s}$ if it satisfies equation (1) .

12. Given a transition matrix $\mathbf{Q}$, if we can find a non-negative vector $\mathbf{s}$ whose components sum to 1 and which satisfies the reversibility condition, then $\mathbf{s}$ is a stationary distribution.

13. **(Reversible implies stationary)**. Suppose that $\mathbf{Q} = (q_{ij})_{1\leq i,j\leq M}$ is a transition matrix of a Markov chain that is reversible with respect to a nonnegative vector $\mathbf{s} = (s_1, s_2 \ldots, s_M)$ whose components sum to 1. Then s is a stationary distribution of the chain.

14. **(Birth-death chain).** A birth-death chain on states $\{1, 2, \ldots, M\}$ is a Markov chain with transition matrix $\mathbf{Q} = (q_{ij})_{1\leq i,j\leq M}$ such that $q_{ij} > 0$ if $|i - j| = 1$ and $q_{ij} = 0$ if $|i - j| \geq 2$. In other words, wa can go one step to the left or one step to the right (except at boundaries) but it's impossible to jump more than one step.

Exercise 2. It is reasonable to consider that your mood swings between cheerful (state 1) and sad (state 2) following the Markov chain shown below, where $0 < p < 1$.



1. Gie your mood swings' transition matrix $\mathbf{Q}$.
2. Find the stationary distribution of $\mathbf{Q}$.
3. What is the limit of $\mathbf{Q}^n$ as $n \to \infty$?

Exercise 3. Consider a Markov chain with transition matrix

$$\mathbf{Q} = \begin{pmatrix} 0.4 & 0.6 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.2 & 0.8 \end{pmatrix}$$

1. Find all its communicating classes.
2. Given

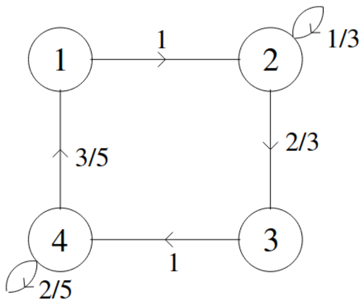
$$\lim_{n\to\infty} \mathbf{Q}^n = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 2/7 & 5/7 \\ 0 & 0 & 2/7 & 5/7 \end{pmatrix}$$

   Does the Markov chain have a limiting distribution? Why?
3. Find all of its stationary distributions.

Exercise 4. On any given week while taking MCS123 (a.k.a. Stochastic processes), a student can be either up-to-date on learning the material, or may have fallen behind. If the student is up-to-date in a given week, the probability that they will be up-to-date (or behind) in the next week is 0.8 (or 0.2, respectively). If the student is behind in the given week, the probability that they will be up-to-date (or behind) in the next week is 0.6 (or 0.4, respectively). We assume that these probabilities do not depend on whether the student was up-to-date or behind in previous weeks, so we can model the situation as a 2-state Markov chain where state U is the case when the student is up-to-date and state B is the case when the student is behind.

1. Calculate the average return time to state U.

Exercise 5. Consider the following Markov chain:



The steady-state probabilities for this process are:

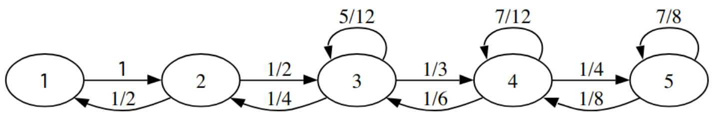
$$\pi_1 = \frac{6}{31}, \quad \pi_2 = \frac{9}{31}, \quad \pi_3 = \frac{6}{31}, \quad \pi_4 = \frac{10}{31}.$$

Assume the process is in state 1 just before the first transition.

1. Let K be the number of transitions up to and including the next transition on which the process returns to state 1. What are the expected value and variance of K?
2. What is the probability that the state of the system resulting from transition 1000 is neither the same as the state resulting from transition 999 nor the same as the state resulting from transition 1001?

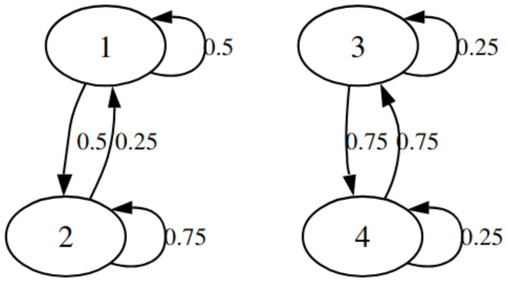
Exercise 6. Show that if each column of the transition matrix $\mathbf{Q}$ of a Markov chain sums to 1, then the uniform distribution over all states is a stationary distribution. (A non-negative matrix such that the row sums and the column sums are all equal to 1 is called a doubly stochastic matrix.)

Exercise 7. Find the stationary distribution of the Markov chain shown below, without using matrices.



Exercise 8. Consider the Markov chain shown below, with state space $\{1, 2, 3, 4\}$.

1. Write down the transition matrix Q for this chain.
2. Which states (if any) are recurrent? Which states (if any) are transient?
3. Find two different stationary distributions for the chain.



Exercise 9. You are currently a newRest customer. On each day that you are a newRest customer, you have probability $\frac{1}{2}$ of being a newRest customer the next day. Otherwise, you have an equally likely chance of becoming a CCI customer, a Food Truck customer, a BG corner customer, or a Rihla customer the next day. On any day you are a Food Truck customer, you have probability $\frac{1}{4}$ of switching to BG corner, probability $\frac{3}{8}$ of switching to newRest, and probability $\frac{3}{8}$ of switching to CCI the next day. On any day you are a CCI customer, you have probability $\frac{1}{2}$ of switching to Rihla, probability $\frac{3}{8}$ of switching to newRest, and probability $\frac{1}{8}$ of switching to a Food Truck the next day.

In answering the questions below, assume you will be a campus resident forever (God forbid!). Also assume, for questions (1) to (6) that once you test BG corner or Rihla's tasty food, you will not change restaurants again.

1. What is the probability that you will eventually leave the newRest?
2. What is the probability that you will eventually be in Rihla?
3. What is the expected number of days until you leave the newRest?
4. Every time you switch into newRest from 6-2 or 6-3, you buy yourself a mesemen. You don't have time to regularly wait inline for as long as it takes (you need to study!), so after you've eaten 2 msemens, you stop buying yourself msemen. What is the expected number of ice msemens you buy yourself before you leave the newRest?
5. You classmate started out just like you. He is now in Rihla. You don't know how long it took him to switch. What is the expected number of days it took him to switch to Rihla?
6. You decide that Rihla is not in your future (it's overpriced!). Accordingly, when you are a newRest customer, you stay at newRest for another day with probability $\frac{1}{2}$, and otherwise you have an equally likely chance of becoming any of the other options. When you are a CCI customer, your probability of entering the newRest or the Food Truck are in the same proportion as before. What is the expected number of days until you go to BG corner?
7. For this part only, assume that when you are in BG corner you are equally likely to stay in BG corner or switch to Rihla. Similarly, if you are in Rihla, you are equally likely to stay in Rihla or switch to BG corner. Calculate the probability of you going to each restaurant on any given day far into the future.
8. Suppose that if you are in BG corner or Rihla, you have probability $\frac{1}{8}$ of returning to the newRest, and otherwise you remain in your current restaurant. What is the expected number of days until you go to the newRest again? (Notice that we know you are at the newRest today, so if tomorrow you still go to the newRest, then the number of days until you go to the newRest again is 1).

Exercise 10. Consider a Markov chain on the state space $\{1, 2, \ldots, 7\}$, and transitions given by moving one step clockwise or counterclockwise with equal probabilities. For example, from state 6, the chain moves to state 7 or state 5 with probability 0.5 each; from state 7, the chain moves to state 1 or state 6 with probability 0.5 each. The chain starts at state 1.

1. Draw the Markov graph of this chain.
2. Find the stationary distribution of this chain.
3. Consider a new chain obtained by unfolding the circle. From state 1 the chain always goes to state 2, and from state 7 the chain always goes to state 6. Draw the Markov graph and find the new stationary distribution.

Exercise 11. There are two urns with a total of 2N distinguishable balls. Initially, the first urn has N white balls and the second urn has N black balls. At each stage, we pick a ball at random from each urn and interchange them. Let X_n be the number of black balls in the first urn at time n. This is a Markov chain on the state space $\{0, 1, \ldots, N\}$.

1. Give the transition probabilities of the chain.
2. Show that $(s_0, s_1, \ldots, s_N)$ where

$$s_i = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}}. \qquad (2)$$

Exercise 12. A cat and a mouse move independently back and forth between two rooms. At each time step, the cat moves from the current room to the other room with probability 0.8. Starting from room 1, the mouse moves to Room 2 with probability 0.3 (and remains otherwise). Starting from room 2, the mouse moves to room 1 with probability 0.6 (and remains otherwise).

1. Find the stationary distributions of the cat chain and of the mouse chain.
2. Note that there are 4 possible (cat, mouse) states: both in room 1, cat in room 1 and mouse in room 2, cat in room 2 and mouse in room 1, and both in room 2. Number these cases $1, 2, 3, 4$, respectively, and let Z_n be the number representing the (cat, mouse) state at time n. Is $Z_0, Z_1, Z_2, \ldots$ a Markov chain?
3. Now suppose that the cat will eat the mouse if they are in the same room. We wish to know the expected time (number of steps taken) until the cat eats the mouse for two initial configurations: when the cat starts in room 1 and the mouse starts in room 2, and vice versa. Set up a system of two linear equations in two unknowns whose solution is the desired values.

Exercise 13. An absent-minded professor has two umbrellas, used when commuting from home to work and back. If it rains and an umbrella is available, the professor takes it. If an umbrella is not available, the professor gets wet. If it does not rain, the professor does not take the umbrella. It rains on a given commute with probability p, independently for all days. We are interested in determining the steady-state probability that the professor will get wet on a given day. We model the process as a Markov chain with states $j = 0, 1, 2$. The state j represents the number of umbrellas the professor currently has.

1. Give the corresponding transition probabilities.

$p_{0,2} = 1, p_{2,1} = p, p_{1,2} = p, p_{1,1} = 1-p, p_{2,0} = 1-p$. The steady-state probabilities π_j (where $j = 0, 1, 2$) satisfy the following system of equations:

$$\pi_0 = \pi_2(1-p), \quad \pi_1 = (1-p)\pi_1 + p\pi_2, \quad \pi_2 = \pi_0 + p\pi_1.$$

From the second equation, we obtain $\pi_1 = \pi_2$. Substituting this into the first equation, and using the normalization condition $\pi_0 + \pi_1 + \pi_2 = 1$, we get:

$$\pi_1 = \pi_2 = \frac{1}{3-p}, \quad \pi_0 = \frac{1-p}{3-p}.$$

The steady-state probability that the professor gets wet is the probability of being in state zero (no umbrella) times the probability that it rains on a given day. Hence, the probability that the professor gets wet is :

$$P(\text{wet}) = \frac{p(1-p)}{3-p}.$$

References and acknowledgments: Introduction to probability (J. Blitzstein and J. Huang), Introduction to probability (D. Bertsekas and J. Tsitsiklis), Probability and random processes (Grimmett and Stirzaker), J. Blitzstein, C. Lee, J. Tsitsiklis, MIT OCW, STAT 171 (H. Lyu), UNI-ULM.