Handout 3: Bernoulli and Poisson processes

Due date : February 24^{th}

Brief recap

1. (Memorylessness) A distribution is said to be memoryless if a random variable X from that distribution satisfies $P(X \geq s+t \mid X \geq s) = P(X \geq t)$, for all $s, t > 0$.

2. The PMF of interarrival times Y_k in a Bernoulli process is given by:

$$p_{Y_k}(t) = \binom{t-1}{k-1} p^k (1-p)^{t-k}, \quad t = k, k+1, \ldots,$$

   This is known as the Pascal PMF of order k.

3. A sequence of arrivals in continuous time is called a Poisson process with rate λ if

   (a) The number of arrivals in disjoint time intervals are independent.

   (b) The number of arrivals in an interval of length τ is given by the Poisson distribution $P(k, \tau) = \frac{(\lambda\tau)^k}{k!} e^{-\lambda\tau}$.

4. The number of arrivals N_τ over an interval of length τ in a Poisson process satisfies $E[N_\tau] = \lambda\tau$ and $V[N_\tau] = \lambda\tau$.

5. (Merging) Let $\{N_1(t)\}_{t>0}$ and $\{N_2(t)\}_{t>0}$ be independent Poisson processes with rates λ_1 and λ_2, respectively. The merged process $\{N(t) = N_1(t) + N_2(t)\}_{t>0}$ is a Poisson process with rate $\lambda_1 + \lambda_2$.

6. (Splitting) Let $\{N_1(t)\}_{t>0}$ be a Poisson process with rate λ, and classify each arrival in the process as a type-1 event with probability p and a type-2 event with probability $1-p$, independently. Then the type-1 events form a Poisson process with rate λp, the type-2 events form a Poisson process with rate $\lambda(1-p)$, and these two processes are independent.

Exercise 1. X is a random variable with memoryless distribution with CDF F and PMF $p_i = P(X = i)$. Find an expression for $P(X \geq j+k)$ in terms of $F(j)$, $F(k)$, p_j, p_k.

Exercise 2. Let N_t be the number of arrivals up until time t in a Poisson process of rate λ, and let T_n be the time of the n-th arrival. Consider statements of the form

$$P(N_t \gtrless_1 n) = P(T_n \gtrless_2 t),$$

where $\gtrless_1$ and $\gtrless_2$ are replaced by symbols from the list $<, \leq, \geq, >$. Which of these statements are true?

Exercise 3. Passengers arrive at a bus stop according to a Poisson process with rate λ. The arrivals of buses are exactly t minutes apart.

1. Show that on average, the sum of the waiting times of the riders on one of the buses is $\frac{1}{2}\lambda t^2$.

Exercise 4. Suppose we have a couple of Poisson processes with rates λ_1, λ_2 respectively. For an interval of length δ small enough, give an approximation of $P((k_1, k_2); \delta)$ for $k_1, k_2 \in \{1, 2\}$. A student receives phone calls according to a Poisson process with rate λ. Unfortunately she has lost her cell phone charger. The battery's remaining life is a random variable T with mean μ and variance σ^2. Let $N(T)$ be the number of phone calls she receives before the battery dies.

1. Find $\mathrm{E}[N(T)]$ and $\mathrm{V}[N(T)]$.

Exercise 5. Emails arrive in your inbox according to a Poisson process with rate λ, measured in emails per hour. Each email is studies-related with probability p and personal with probability $1-p$. The amount of time it takes to answer a studies-related email is a random variable with mean μ_W

and variance σ_W^2. The amount of time it takes to answer a personal email has mean μ_P and variance σ_P^2. The response times for different emails are independent.

What is the average amount of time you have to spend answering all the emails that arrive in a t-hour interval? What about the variance?

Exercise 6. On a whatsApp question-and-answer group, $N \sim \text{Pois}(\lambda_1)$ questions will be posted tomorrow, with λ_1 measured in questions/day. Given N , the post times are i.i.d. and uniformly distributed over the day (a day begins and ends at midnight). When a question is posted, it takes an $\text{Exp}(\lambda_2)$ amount of time (in days) for an answer to be posted, independently of what happens with other questions.

1. Find the probability that a question posted at a uniformly random time tomorrow will not yet have been answered by the end of that day.

Exercise 7. In an endless football match, goals are scored according to a Poisson process with rate λ. Each goal is made by team A with probability p and team B with probability $1-p$. For $j > 1$, we say that the jth goal is a turnaround if it is made by a different team than the $(j-1)$st goal. For example, in the sequence AABBA, the 3rd and 5th goals are turnarounds.

1. In n goals, what is the expected number of turnarounds?

2. What is the expected time between turnarounds, in continuous time?

Exercise 8. Bees in a forest are distributed according to a 3D Poisson process with rate λ. What is the distribution of the distance from a hiker to the nearest bee?

Exercise 9. Suppose cars enter a one-way highway from a common entrance, following a Poisson process with rate λ. The i-th car has velocity V_i and travels at this velocity forever; no time is lost when one car overtakes another car. Assume the V_i are i.i.d. discrete random variables whose support is a finite set of positive values. The process starts at time 0, and we'll consider the highway entrance to be at location 0.

For fixed locations a and b on the highway with $0 < a < b$, let Z_t be the number of cars located in the interval $[a, b]$ at time t. (For instance, on an interstate highway running west to east through the midwestern United States, a could be Kansas City and b could be St. Louis; then Z_t would be the number of cars on the highway that are in the state of Missouri at time t.) Figure 13.6 illustrates the setup of the problem and the definition of Z_t.

Assume t is large enough that $t > \frac{b}{V_i}$ for all possible values of V_i. Show that Z_t has a Poisson distribution with mean $\lambda(b-a)\mathbb{E}(V_i^{-1})$.

Exercise 10. Two students are independently performing independent Bernoulli trials. For concreteness, assume that student A is flipping a 1dH coin with probability p_1 of Heads and student B is flipping a 5dhs coin with probability p_2 of Heads. Let $X_1, X_2, \ldots$ be student A's results and $Y_1, Y_2, \ldots$ be student B's results, with $X_i \sim \text{Bern}(p_1)$ and $Y_j \sim \text{Bern}(p_2)$.

Find the distribution and expected value of the first time at which they are simultaneously successful, i.e., the smallest n such that $X_n = Y_n = 1$.

References and acknowledgments: Introduction to probability (J. Blitzstein and J. Huang) - Introduction to probability (D. Bertsekas and J. Tsitsiklis).