Handout 3: The Bernoulli process

Due date : February 24^{th}

Brief recap

1. (Memorylessness) A distribution is said to be memoryless if a random variable X from that distribution satisfies $P(X > s + t \mid X > s) = P(X > t)$, for all $s, t \geq 0$.

2. We define the Bernoulli process as a sequence $X_1, X_2, \ldots$ of i.i.d. Bernoulli random variables X_i where for each i

$$P(X_i = 1) = P(\text{success at the } i\text{th trial}) = p,$$

$$P(X_i = 0) = P(\text{failure at the } i\text{th trial}) = 1-p,$$

3. Let S be the number of successes in n i.i.d. trials of a Bernoulli process. $\mathbf{S \sim Bin(n, p)}$. Its PMF, mean, and variance are:

$$P(S = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n,$$

$$E[S] = np, \quad \mathrm{V}[S] = np(1-p).$$

4. Let T be the number of trials up to (and including) the first success in a Bernoulli process. $\mathbf{T \sim Geom(p)}$. Its PMF, mean, and variance are:

$$P(T = t) = (1-p)^{t-1} p, \quad t = 1, 2, \ldots,$$

$$E[T] = \frac{1}{p}, \quad \mathrm{V}[T] = \frac{1-p}{p^2}.$$

5. The geometric distribution is memoryless.

6. The number $T - n$ of trials until the first success after time n has a geometric distribution with parameter p, and is independent of the past.

$$P(T - n = k) = (1-p)^k p, \quad k = 0, 1, 2, \ldots$$

7. For any given time n, the sequence of random variables $X_{n+1}, X_{n+2}, \ldots$ (the future of the process) is also a Bernoulli process, and is independent from $X_1, \ldots, X_n$ (the past of the process).

8. The PMF of arrival times Y_k in a Bernoulli process is given by:

$$P(Y_k = t) = \binom{t-1}{k-1} p^k (1-p)^{t-k}, \text{ for } t \geq k,$$

and 0 otherwise. This is known as the **Pascal PMF** of order k.

Exercise 1. X is a random variable with memoryless distribution with CDF F and PMF $p_i = P(X = i)$. Find an expression for $P(X \geq j + k)$ in terms of $F(j)$, $F(k)$, p_j, p_k.

Exercise 2. Suppose that, at the CC entrance, a taxi arrives every 5 minutes, and the probability that an arriving taxi is available (i.e. hasn't been specifically ordered by someone) is $\frac{1}{5}$.

1. Suppose you have just arrived at the entrance and are waiting for an available taxi. When you arrive, you just miss an available taxi. What is the probability that you will have to wait at least 15 minutes (3 taxi arrivals) for an available taxi?

2. Suppose you are at the entrance, and 10 taxis have arrived and left, and none of them was available. What is the probability that you will have to wait at least 15 minutes (3 taxi arrivals) for the next available taxi?

Exercise 3. You noticed that you haven't left campus for quite a long time. Aiming for some greenery and fresh air, you decided to go on a hike to Ourika. Yet, beautiful landscapes sometimes come with their own pitfalls. Each second, a mosquito lands on your neck with probability 0.5. If one lands, with probability 0.2 it bites you, and with probability 0.8 it never bothers you, independently of other mosquitoes.

1. What is the expected time between consecutive mosquito bites? What is the variance of the time between consecutive mosquito bites?

2. In addition, a bee lands on your neck with probability 0.05 (bees are generally peaceful). If one lands, with probability 0.7 it bites you, and with probability 0.3, it doesn't harm you, independently of any other insect bites (mosquitoes and other bees). Find the expected time between consecutive insect bites? Give the variance of the time between consecutive insect bites?

Exercise 4. We perform an experiment comprising a series of independent trials. On each trial, we simultaneously flip a set of three fair coins.

1. Given that we have just had a trial with 3 tails, what is the probability that both of the next two trials will also have this result?
2. Whenever all three coins land on the same side in any given trial, the trial is called a success.
   (a) Find the PMF for K, the number of trials up to, but not including, the second success.
   (b) Find the expectation and variance of M, the number of tails that occur before the first success.

Exercise 5. You ran out of exercises but you want to practice more for the midterm. Hence, you decide to make up one! You conduct an experiment like the one in exercise 4, except that you use 4 coins for the first trial, and then you obey the following rule: Whenever all of the coins land on the same side in a trial, you permanently remove one coin from the experiment and continue with the trials. You follow this rule until the third time you remove a coin, at which point the experiment ceases (you've other courses to study for). Find $E[N]$, where N is the number of trials in your experiment.

Exercise 6. Suppose there are n papers in a drawer. You draw a paper and sign it, and then, instead of filing it away, you place the paper back into the drawer. If any paper is equally likely to be drawn each time, independent of all other draws, what is the expected number of papers that you will draw before signing all n papers? You may leave your answer in the form of a summation.

Exercise 7. Let L17 be described by a Pascal PMF of order 17. Find the numerical values of a and b in the following equation. Explain your work.

$$\sum_{i=42}^{\infty} p_{L_{17}}(i) = \sum_{x=0}^{a} \binom{b}{x} p^x (1-p)^{b-x}$$

Exercise 8. Two students are independently performing independent Bernoulli trials. For concreteness, assume that student A is flipping a 1dH coin with probability p_1 of Heads and student B is flipping a 5dhs coin with probability p_2 of Heads. Let $X_1, X_2, \ldots$ be student A's results and $Y_1, Y_2, \ldots$ be student B's results, with $X_i \sim \text{Bern}(p_1)$ and $Y_j \sim \text{Bern}(p_2)$.

Find the distribution and expected value of the first time at which they are simultaneously successful, i.e., the smallest n such that $X_n = Y_n = 1$.

References and acknowledgments: Introduction to probability (J. Blitzstein and J. Huang) - Introduction to probability (D. Bertsekas and J. Tsitsiklis).