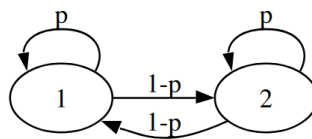


Handout 2: Markov chains

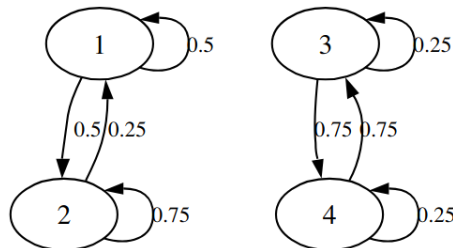
Due date :

Exercise 1. Consider the Markov chain shown below, where $0 < p < 1$.



1. Write down the transition matrix Q for this chain.
2. Find the stationary distribution of the chain.
3. What is the limit of Q_n as $n \rightarrow \infty$?

Exercise 2. Consider the Markov chain shown below, with state space $\{1, 2, 3, 4\}$.



1. Write down the transition matrix Q for this chain.
2. Which states (if any) are recurrent? Which states (if any) are transient?
3. Find two different stationary distributions for the chain.

Exercise 3. Consider a Markov chain with transition matrix

$$P = \begin{pmatrix} 0.4 & 0.6 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.2 & 0.8 \end{pmatrix}$$

1. States i and j communicate if i is accessible from j and j is accessible from i . Find all its communication classes.
2. Given

$$\lim_{n \rightarrow \infty} P^n = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 2/7 & 5/7 \\ 0 & 0 & 2/7 & 5/7 \end{pmatrix}$$

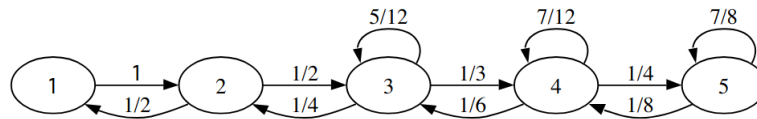
Does the Markov chain have a limiting distribution? Why?

3. Find all of its stationary distributions.

Exercise 4. A component of a computer has an active life, measured in discrete units, that is a random variable T where $P(T = k) = ak$ for $k = 1, 2, \dots$. Suppose one starts with a fresh component and each component is replaced by a new component upon failure. Let X_n be the age of the component in service at time n .

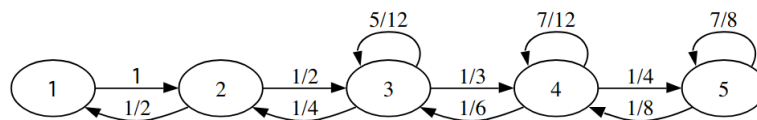
1. Let $p_i = P(X_{n+1} = i+1 | X_n = i)$ and $q_i = P(X_{n+1} = 0 | X_n = i)$ for all n . Write down the stochastic matrix of $\{X_n\}$.
2. Solve p_i, q_i in terms of a_i .

Exercise 5. Find the stationary distribution of the Markov chain shown below, without using matrices. The number above each arrow is the corresponding transition probability.



Exercise 6. Consider a Markov chain on the state space $\{1, 2, \dots, 7\}$, and transitions given by moving one step clockwise or counterclockwise with equal probabilities. For example, from state 6, the chain moves to state 7 or state 5 with probability 0.5 each; from state 7, the chain moves to state 1 or state 6 with probability 0.5 each. The chain starts at state 1.

1. Draw the Markov graph of this chain.
2. Find the stationary distribution of this chain.
3. Consider a new chain obtained by unfolding the circle. From state 1 the chain always goes to state 2, and from state 7 the chain always goes to state 6. Draw the Markov graph and find the new stationary distribution.



Exercise 7. (Transition matrix) A component of a computer has an active life, measured in discrete units, that is a random variable T where $P(T = k) = a_k$ for $k = 1, 2, \dots$. Suppose one starts with a fresh component and each component is replaced by a new component upon failure. Let X_n be the age of the component in service at time n .

1. Let $p_i = P(X_n + 1 = i + 1 | X_n = i)$ and $q_i = P(X_n + 1 = 0 | X_n = i)$ for all n . Write down the stochastic matrix of $\{X_n\}$.
2. Find p_i, q_i in terms of a_i .

Exercise 8. There are two urns with a total of $2N$ distinguishable balls. Initially, the first urn has N white balls and the second urn has N black balls. At each stage, we pick a ball at random from each urn and interchange them. Let X_n be the number of black balls in the first urn at time n . This is a Markov chain on the state space $\{0, 1, \dots, N\}$.

1. Give the transition probabilities of the chain.
2. Show that (s_0, s_1, \dots, s_N) where

$$s_i = \frac{\binom{N}{i} \binom{N}{N-i}}{\binom{2N}{N}}. \quad (1)$$

Exercise 9. A cat and a mouse move independently back and forth between two rooms. At each time step, the cat moves from the current room to the other room with probability 0.8. Starting from room 1, the mouse moves to Room 2 with probability 0.3 (and remains otherwise). Starting from room 2, the mouse moves to room 1 with probability 0.6 (and remains otherwise).

1. Find the stationary distributions of the cat chain and of the mouse chain.
2. Note that there are 4 possible (cat, mouse) states: both in room 1, cat in room 1 and mouse in room 2, cat in room 2 and mouse in room 1, and both in room 2. Number these cases 1, 2, 3, 4, respectively, and let Z_n be the number representing the (cat, mouse) state at time n . Is Z_0, Z_1, Z_2, \dots a Markov chain?
3. Now suppose that the cat will eat the mouse if they are in the same room. We wish to know the expected time (number of steps taken) until the cat eats the mouse for two initial configurations: when the cat starts in room 1 and the mouse starts in room 2, and vice versa. Set up a system of two linear equations in two unknowns whose solution is the desired values.

References and acknowledgments: Introduction to probability (Blitzstein and Huang) - Lisa Ruan - Christy Huo.