Reduced basis for variational inequalities
Application to contact mechanics

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Motivations for contact

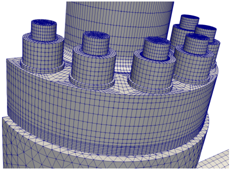
- Long-standing question in many contexts...
- Present industrial context
  - Behavior of valve components
  - Costly simulations ($\approx$ 12h using ‘`code_aster`’)
- Nonlinearities, constraints

Challenge

- Expensive for parameterized studies

Goal

- Nonlinearly constrained model reduction



Variational inequalities with linear constraints

Reduced basis procedures

- *Haasdonk, Salomon, Wohlmuth.* A reduced basis method for parametrized variational inequalities. 2012.
- *Balajewicz, Amsallem, Farhat.* Projection-based model reduction for contact problems. 2016.
- *Fauque, Ramiere, Ryckelynck.* Hybrid hyper-reduced modeling for contact mechanics problems. 2018.

Related work

- *Burkovska, Haasdonk, Salomon, Wohlmuth.* Reduced basis methods for pricing options with the Black–Scholes and Heston models. 2014.
- *Glas, Urban.* Numerical investigations of an error bound for reduced basis approximations of non-coercive variational inequalities. 2015.
- *Bader, Zhang, Veroy.* An empirical interpolation approach to reduced basis approximations for variational inequalities, 2016.
- *Burkovska.* PhD thesis. 2016.

Outline

Elastic contact problem

Strain tensor

$$\varepsilon(v) := \frac{1}{2}(\nabla v + \nabla v^T)$$

Stress tensor

$$\sigma(v) = \frac{E\nu}{(1+\nu)(1+2\nu)}\mathrm{tr}(\varepsilon(v))\mathcal{I} + \frac{E}{(1+\nu)}\varepsilon(v)$$

E: Young modulus ν: Poisson coefficient

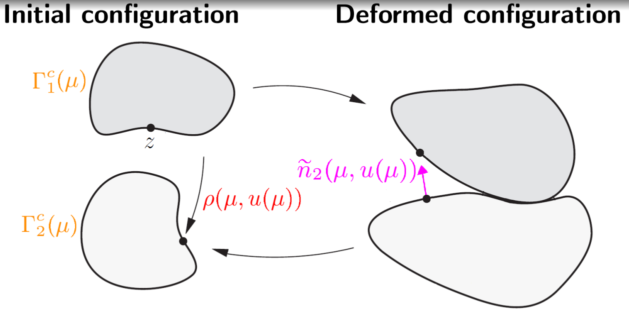
Parametric equilibrium condition

$$\nabla \cdot \sigma(u(\mu)) \;=\; \ell(\mu) \quad \text{in } \Omega(\mu)$$

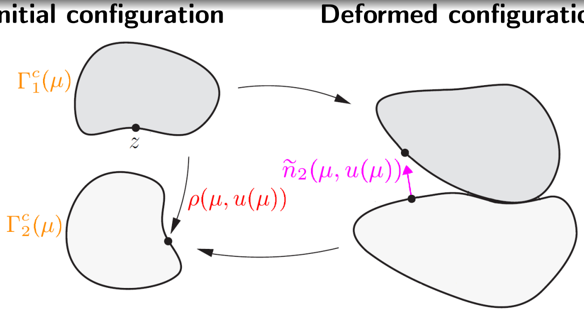
$$a(\mu; v, w) = \int_{\Omega(\mu)} \sigma(v) : \varepsilon(w) \quad \text{and} \quad f(\mu; w) = \int_{\Omega(\mu)} \ell(\mu) w$$

Other nonlinearities can be handled

Non-interpenetration condition



Non-interpenetration condition



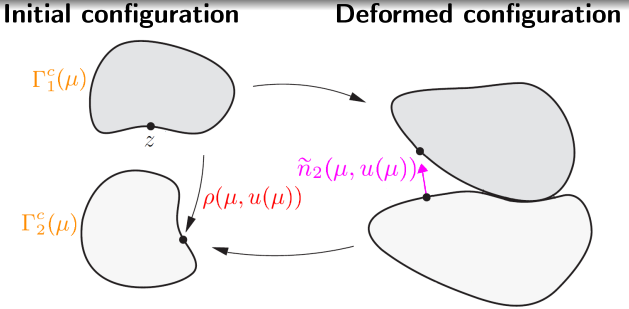
Consider $\mathcal{V}(\mu) = H^1(\Omega_1(\mu)) \times H^1(\Omega_2(\mu))$
Admissible solutions are denoted $u(\mu) = (u_1(\mu), u_2(\mu)) \in \mathcal{V}(\mu)$

For all $z \in \Gamma_1^c(\mu)$

$$\underbrace{\left(u_1(\mu)(z) - \left(u_2(\mu) \circ \rho(\mu, u(\mu))\right)(z)\right)}_{\text{displacement difference}} \cdot \tilde{n}_2(\mu, u(\mu))(z) \geqslant \underbrace{\left(\rho(\mu, u(\mu))(z) - z\right)}_{\text{initial gap}} \cdot \tilde{n}_2(\mu, u(\mu))(z)$$

$\Longrightarrow$ Proof : Benaceur, PhD thesis

Non-interpenetration condition



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$$\underbrace{\left(u_1(\mu)(z) - \left(u_2(\mu) \circ \rho(\mu, u(\mu))\right)(z)\right) \cdot \tilde{n}_2(\mu, u(\mu))(z)}_{-k(\mu, u(\mu); u(\mu))} \geqslant \underbrace{\left(\rho(\mu, u(\mu))(z) - z\right) \cdot \tilde{n}_2(\mu, u(\mu))(z)}_{-g(\mu, u(\mu))}$$

$\Longrightarrow$ Proof: Benaceur, PhD thesis

Model problem

$\Omega(\mu)$: bounded domain in $\mathbb{R}^d$ with a contact boundary $\Gamma_1^c(\mu) \subset \partial\Omega(\mu)$
$\mathcal{V}(\mu)$: Hilbert space on $\Omega(\mu)$
$\mathcal{P}$: parameter set

For many values $\mu \in \mathcal{P}$: Find $u(\mu) \in \mathcal{V}$ such that

$$u(\mu) = \underset{v \in \mathcal{V}(\mu)}{\operatorname{argmin}} \frac{1}{2} a(\mu; v, v) - f(\mu; v)$$

$$k(\mu, u(\mu); u(\mu)) \leqslant g(\mu, u(\mu)) \quad \text{a.e. on } \Gamma_1^c(\mu)$$

$k(\mu, \cdot; \cdot)$ is semi-linear : natural for contact problems
handy for iterative solution methods

Lagrangian formulation

We consider

- the convex cone $\mathcal{W}(\mu) := L^2(\Gamma_1^c(\mu), \mathbb{R}_+)$
- the Lagrangian $\mathcal{L}(\mu) : \mathcal{V}(\mu) \times \mathcal{W}(\mu) \to \mathbb{R}$ defined as

$$\boxed{\mathcal{L}(\mu)(v,\eta) := \frac{1}{2}a(\mu; v, v) - f(\mu; v) + \left(\int_{\Gamma_1^c(\mu)} k(\mu, v; v)\eta - \int_{\Gamma_1^c(\mu)} g(\mu, v)\eta\right)}$$

Find $(u(\mu), \lambda(\mu)) \in \mathcal{V}(\mu) \times \mathcal{W}(\mu)$ such that

$$(u(\mu), \lambda(\mu)) = \arg \underset{v \in \mathcal{V}(\mu), \eta \in \mathcal{W}(\mu)}{\text{minmax}} \mathcal{L}(\mu)(v,\eta)$$

$u(\mu)$ is the primal solution, $\lambda(\mu)$ is the dual solution

Discrete FEM formulation

1. FEM space ($\mathcal{N} \gg 1$)

$$V_{\mathcal{N}}(\mu) := \mathrm{span}\{\phi_1(\mu), \ldots, \phi_{\mathcal{N}}(\mu)\} \subset \mathcal{V}(\mu)$$

2. FEM convex cone ($\mathcal{R} \gg 1$)

$$W_{\mathcal{R}}(\mu) := \mathrm{span}_+\{\psi_1(\mu), \ldots, \psi_{\mathcal{R}}(\mu)\} \subset \mathcal{W}(\mu)$$

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Algebraic FEM formulation

$$(\mathbf{u}(\mu), \boldsymbol{\lambda}(\mu)) = \arg \min_{\mathbf{v}\in\mathbb{R}^{\mathcal{N}}} \max_{\boldsymbol{\eta}\in\mathbb{R}^{\mathcal{R}}_+} \Big\{ \frac{1}{2}\mathbf{v}^T \mathbf{A}(\mu)\mathbf{v} - \mathbf{v}^T \mathbf{f}(\mu) + \boldsymbol{\eta}^T \big(\mathbf{K}(\mu,\mathbf{v})\mathbf{v} - \mathbf{g}(\mu,\mathbf{v})\big) \Big\}$$

$$\mathbf{A}(\mu)_{ij} = a(\mu; \phi_j(\mu), \phi_i(\mu)) \qquad \mathbf{K}(\mu, w)_{ij} = \int_{\Gamma_1^c(\mu)} k(\mu, w; \phi_j(\mu))\psi_i(\mu)$$

Additional nonlinearity caused by spatial dicretization

The Kačanov method

- Iterative procedure
- Consists in solving the following problems : For all $k \geqslant 1$

$$\boxed{\begin{aligned}(\mathbf{u}^k(\mu), \boldsymbol{\lambda}^k(\mu)) = \arg \min_{\mathbf{v}\in\mathbb{R}^{\mathcal{N}}, \eta\in\mathbb{R}_+^{\mathcal{R}}} \max \ & \frac{1}{2}\mathbf{v}^T \mathbf{A}(\mu)\mathbf{v} - \mathbf{v}^T \mathbf{f}(\mu) \\ & + \boldsymbol{\eta}^T \left(\mathbf{K}(\mu, \mathbf{u}^{k-1}(\mu))\mathbf{v} - \mathbf{g}(\mu, \boldsymbol{\lambda}^{k-1}(\mu))\right)\end{aligned}}$$

- Stopping criteria

$$\frac{\|\mathbf{u}^k(\mu) - \mathbf{u}^{k-1}(\mu)\|_{\mathbb{R}^{\mathcal{N}}}}{\|\mathbf{u}^{k-1}(\mu)\|_{\mathbb{R}^{\mathcal{N}}}} \leqslant \epsilon_{\mathrm{ka}}^{\mathrm{pr}} \qquad \frac{\|\boldsymbol{\lambda}^k(\mu) - \boldsymbol{\lambda}^{k-1}(\mu)\|_{\mathbb{R}^{\mathcal{N}}}}{\|\boldsymbol{\lambda}^{k-1}(\mu)\|_{\mathbb{R}^{\mathcal{N}}}} \leqslant \epsilon_{\mathrm{ka}}^{\mathrm{du}}$$

The reduced-basis model: Reference configuration

Geometric mapping $h(\mu)$ defined on a reference domain $\breve{\Omega}$ (with $I = 2$)

$$\begin{aligned} h(\mu) : \breve{\Omega} &\to \Omega(\mu) \\ x &\mapsto \sum_{i=1}^{I} h_i(\mu, x)\mathbf{1}_{\breve{\Omega}_i}(x) \end{aligned}$$

The reduced-basis model: Reference configuration

Geometric mapping $h(\mu)$ defined on a reference domain $\check{\Omega}$ (with $I = 2$)

$$\boxed{\begin{aligned} h(\mu) : \check{\Omega} &\rightarrow \Omega(\mu) \\ x &\mapsto \sum_{i=1}^{I} h_i(\mu, x) \mathbf{1}_{\check{\Omega}_i}(x) \end{aligned}}$$

$$\Omega_i(\mu) = h(\mu)(\check{\Omega}_i) \quad \forall i \in \{1, 2\}$$

$$\Gamma_i^c(\mu) = h(\mu)(\check{\Gamma}_i^c) \quad \forall i \in \{1, 2\}$$

Reference Hilbert space

$$\check{\mathcal{V}} := H^1(\check{\Omega}; \mathbb{R}^d)$$

Parametric Hilbert space

$$\mathcal{V}(\mu) = \check{\mathcal{V}} \circ h(\mu)^{-1}$$

Reference convex cone

$$\check{\mathcal{W}} := L^2(\check{\Gamma}_1^c; \mathbb{R}_+)$$

Parametric convex cone

$$\mathcal{W}(\mu) = \check{\mathcal{W}} \circ h(\mu)^{-1}_{|\Gamma_1^c(\mu)}$$

Reduced basis spaces

Primal RB subspace

$$\check{V}_N \subset \check{V}_{\mathcal{N}} \subset \check{\mathcal{V}}$$

$$\check{V}_N = \mathrm{span}\{\check{\theta}_1, \dots, \check{\theta}_N\}$$

Dual RB subcone

$$\widecheck{W}_R \subset \widecheck{W}_{\mathcal{R}} \subset \widecheck{\mathcal{W}}$$

$$\widecheck{W}_R = \mathrm{span}_+\{\check{\xi}_1, \dots, \check{\xi}_R\}$$

Reduced basis spaces

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RB approximations

$$\widehat{u}(\mu) = \sum_{n=1}^{N} \widehat{u}_n(\mu)\check{\theta}_n \circ h(\mu)^{-1} \qquad \widehat{\lambda}(\mu) = \sum_{n=1}^{R} \widehat{\lambda}_n(\mu)\check{\xi}_n \circ h(\mu)^{-1}$$

Reduced problem

$$(\hat{\mathbf{u}}(\mu), \hat{\boldsymbol{\lambda}}(\mu)) = \arg \min_{\hat{\mathbf{v}} \in \mathbb{R}^N, \hat{\boldsymbol{\eta}} \in \mathbb{R}^R_+} \max \left\{ \frac{1}{2} \hat{\mathbf{v}}^T \widehat{\mathbf{A}}(\mu) \hat{\mathbf{v}} - \hat{\mathbf{v}}^T \widehat{\mathbf{f}}(\mu) + \hat{\boldsymbol{\eta}}^T \left(\widehat{\mathbf{K}}(\mu, \hat{\mathbf{v}}) \hat{\mathbf{v}} - \hat{\mathbf{g}}(\mu, \hat{\mathbf{v}}) \right) \right\}$$

$$\widehat{\mathbf{A}}(\mu) \in \mathbb{R}^{N \times N} \quad \widehat{\mathbf{f}}(\mu) \in \mathbb{R}^N \quad \widehat{\mathbf{K}}(\mu, \hat{\mathbf{v}}) \in \mathbb{R}^{R \times N} \quad \hat{\mathbf{g}}(\mu, \hat{\mathbf{v}}) \in \mathbb{R}^R$$

Reduced problem

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$$\widehat{\mathbf{A}}(\mu) \in \mathbb{R}^{N \times N} \quad \widehat{\mathbf{f}}(\mu) \in \mathbb{R}^N \quad \widehat{\mathbf{K}}(\mu, \widehat{\mathbf{v}}) \in \mathbb{R}^{R \times N} \quad \widehat{\mathbf{g}}(\mu, \widehat{\mathbf{v}}) \in \mathbb{R}^R$$

$$\widehat{\mathbf{A}}(\mu)_{pn} = a(\mu; \check{\theta}_n \circ h(\mu)^{-1}, \check{\theta}_p \circ h(\mu)^{-1})$$

$$\widehat{\mathbf{K}}(\mu, \widehat{\mathbf{v}})_{pn} = \int_{\Gamma_1^c(\mu)} k\left(\mu, \sum_{i=1}^N \widehat{v}_i \check{\theta}_i \circ h(\mu)^{-1}; \check{\theta}_n \circ h(\mu)^{-1}\right) \check{\xi}_p \circ h(\mu)^{-1}$$

Small (dense) matrices but require FEM reconstructions $\theta_n \circ h(\mu)^{-1}$...etc

$\Longrightarrow$ Need to separate (n, p)- and μ-dependencies

Functional separation

For the stiffness matrix, we need a decomposition of the form

$$\left(\widehat{\mathbf{A}}(\mu)\right)_{np} := \widehat{\mathbf{A}}(\mu, n, p) = \sum_{j=1}^{M^a} \alpha_j(\mu) \widehat{\mathbf{A}}_{j,np}$$

- matrices $\{\widehat{\mathbf{A}}_j\}_{j=1}^{M^a}$ are precomputed offline
- coefficients $\{\alpha_j(\mu)\}_{j=1}^{M^a}$ are evaluated online
- straightforward for affine transformations

For the matrix $\widehat{\mathbf{K}}(\mu, \hat{\mathbf{v}})$, search for a separated approximation

$$k(\mu, u(\mu); \phi_n)(h(\mu)(\check{x})) =: \kappa(\mu, n, \check{x}) \approx \sum_{j=1}^{M^k} \varphi_j(\mu) q_j(n, \check{x})$$

The functions $\{\varphi_j, q_j\}_{j=1}^{M^k}$ are built offline using the EIM

Barrault, Maday, Nguyen, Patera ('04)

Offline/Online efficient RB problem

Offline : using $\{q_j\}_{j=1}^{M^k}$, we build

- a matrix $\mathbf{B} \in \mathbb{R}^{M^k \times M^k}$
- a family of matrices $\{\mathbf{C}_j\}_{j=1}^{M^k}$ all in $\mathbb{R}^{R \times N}$

Online : For each new parameter $\mu \in \mathcal{P}$ and a vector $\hat{\mathbf{v}} \in \mathbb{R}^N$

- compute a family of functions $\{\widehat{\phi}_j(\mu, \hat{\mathbf{v}})\}_{j=1}^{M^k}$
- assemble

$$\widehat{\mathbf{K}}(\mu, \hat{\mathbf{v}}) \approx \mathbf{D}^{\kappa}(\mu, \hat{\mathbf{v}}) := \sum_{i,j=1}^{M^k} \mathbf{C}_j \mathbf{B}_{ji} \widehat{\kappa}_i(\mu, \hat{\mathbf{v}}) \in \mathbb{R}^{R \times N}$$

Similarly, we build an efficient approximation $\widehat{\gamma}(\mu, \hat{\mathbf{v}})$ for the gap function

Offline/Online efficient RB problem

Solve

$$(\widehat{\mathbf{u}}(\mu), \widehat{\boldsymbol{\lambda}}(\mu)) = \arg \min_{\widehat{\mathbf{v}} \in \mathbb{R}^N, \widehat{\boldsymbol{\eta}} \in \mathbb{R}^R_+} \max \left\{ \frac{1}{2} \widehat{\mathbf{v}}^T \widehat{\mathbf{A}}(\mu) \widehat{\mathbf{v}} - \widehat{\mathbf{v}}^T \widehat{\mathbf{f}}(\mu) + \widehat{\boldsymbol{\eta}}^T \left(\mathbf{D}^\kappa(\mu, \widehat{\mathbf{v}}) \widehat{\mathbf{v}} - \mathbf{D}^\gamma \widehat{\gamma}(\mu, \widehat{\mathbf{v}}) \right) \right\}$$

$$\mathbf{D}^\kappa(\mu, \widehat{\mathbf{v}}) \in \mathbb{R}^{R \times N} \quad \mathbf{D}^\gamma \in \mathbb{R}^{R \times M^\gamma} \quad \widehat{\gamma}(\mu, \widehat{\mathbf{v}}) \in \mathbb{R}^{M^\gamma}$$

- $\mathbf{D}^\kappa(\mu, \widehat{\mathbf{v}})$ results from the EIM on κ
- $\mathbf{D}^\gamma$ results from the EIM on γ

Online stage

Algorithm Online stage

Input : μ

$\{\widehat{\mathbf{f}}_j\}_{1\leqslant j\leqslant J^f}$, $\{\widehat{\mathbf{A}}_j\}_{1\leqslant j\leqslant J^a}$

$\{(n_i^\kappa, x_i^\kappa)\}_{1\leqslant i\leqslant M^k}$, $\{q_j^\kappa\}_{1\leqslant j\leqslant M^k}$

$\{x_i^\gamma\}_{1\leqslant i\leqslant M^g}$, $\{q_j^\gamma\}_{1\leqslant j\leqslant M^g}$, $\mathbf{B}^\kappa$, $\{\mathbf{C}_j^\kappa\}_{1\leqslant j\leqslant M^k}$ and $\mathbf{D}^\gamma$

1: Assemble the vector $\widehat{\mathbf{f}}(\mu)$ and the matrix $\widehat{\mathbf{A}}(\mu)$

2: Compute $\widehat{\boldsymbol{\kappa}}(\mu, \widehat{\mathbf{v}})$ and $\widehat{\boldsymbol{\gamma}}(\mu, \widehat{\mathbf{v}})$ EIM on the contact map

3: Compute $\mathbf{D}^\kappa(\mu)$ using $\widehat{\boldsymbol{\kappa}}(\mu, \widehat{\mathbf{v}})$ EIM on the gap map

4: Solve the reduced saddle-point problem to obtain $\widehat{\mathbf{u}}(\mu)$ and $\widehat{\boldsymbol{\lambda}}(\mu)$

Output : $\widehat{\mathbf{u}}(\mu)$ and $\widehat{\boldsymbol{\lambda}}(\mu)$

Basis constructions

Two goals

1. Build $\check{V}_N \subset \check{V}_{\mathcal{N}}$ of dimension $N \ll \mathcal{N}$
   $\Rightarrow$ POD✓
2. Build $\breve{W}_R \subset \breve{W}_{\mathcal{R}}$ of dimension $R \ll \mathcal{R}$
   Requirement: Positive basis vectors

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2. Build $\breve{W}_R \subset \breve{W}_{\mathcal{R}}$ of dimension $R \ll \mathcal{R}$
   Requirement: Positive basis vectors
   - POD✗
   - NMF✓ : considered in *Balajewicz, Amsallem, Farhat ('16)*
   - Angle-greedy algorithm✓ : introduced in *Haasdonk, Salomon, Wohlmuth ('12)*
   - Cone-projected greedy algorithm✓✓ : devised in this work (19')

Non-negative Matrix Factorization

We search for $\mathbf{W}$

$$\mathbf{W} = \mathrm{NMF}(\mathbf{T}, R)$$

- Input: integer R
- Output: R positive vectors

  Lee, Seung('01)

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NMF optimization problem

$$(\mathbf{W}, \mathbf{H}) = \underset{\substack{\tilde{\mathbf{W}} \in \mathbb{R}_+^{\mathcal{R}\times R} \\ \tilde{\mathbf{H}} \in \mathbb{R}_+^{R\times P}}}{\operatorname{argmin}} \|\mathbf{T} - \tilde{\mathbf{W}}\tilde{\mathbf{H}}\|^2$$

Functional $\|\mathbf{T} - \tilde{\mathbf{W}}\tilde{\mathbf{H}}\|$ is not convex in both variables $\tilde{\mathbf{W}}$ and $\tilde{\mathbf{H}}$ together $\implies$ only local minima

- Non-unique solution

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Functional $\|\mathbf{T} - \tilde{\mathbf{W}}\tilde{\mathbf{H}}\|$ is not convex in both variables $\tilde{\mathbf{W}}$ and $\tilde{\mathbf{H}}$ together $\Longrightarrow$ only local minima

- Non-unique solution ☹

Angle-greedy algorithm

Selection criterion

$$\mu_n \in \operatorname{argmax}_{\mu \in \mathcal{P}^{\mathrm{tr}}} \|\measuredangle \left(\check{\lambda}(\mu;\cdot), \breve{W}_{n-1}\right)\|_{\ell^\infty(\check{\Gamma}_1^{c,\mathrm{tr}})}$$

Dual set at iteration n

$$\check{K}_n := \operatorname{span}_+\{\check{\lambda}(\mu_1;\cdot), \ldots, \check{\lambda}(\mu_n;\cdot)\}$$

Stopping criterion

$$r_n < \epsilon_{\mathrm{du}}$$

- Input: Tolerance ϵ_{du} to reach
- Output: R positive vectors
- $\breve{W}_n$ is a linear space

Angle-greedy algorithm

Selection criterion

$$\mu_n \in \operatorname{argmax}_{\mu \in \mathcal{P}^{\mathrm{tr}}} \|\check{\lambda}(\mu;\cdot) - \Pi_{\breve{W}_{n-1}}(\check{\lambda}(\mu;\cdot))\|_{\ell^\infty(\check{\Gamma}_1^{c,\mathrm{tr}})}$$

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- Output: R positive vectors
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Cone-projected greedy algorithm

Selection criterion

$$\mu_n \in \operatorname{argmax}_{\mu \in \mathcal{P}^{\mathrm{tr}}} \|\check{\lambda}(\mu;\cdot) - \Pi_{\check{K}_{n-1}}(\check{\lambda}(\mu;\cdot))\|_{\ell^\infty(\check{\Gamma}_1^{c,\mathrm{tr}})}$$

Dual set at iteration n

$$\check{K}_n := \mathrm{span}_+\{\check{K}_{n-1}, \check{\lambda}(\mu_n;\cdot)\}$$

Stopping criterion

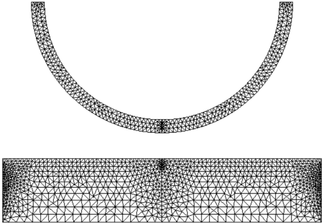
$$r_n < \epsilon_{\mathrm{du}}$$

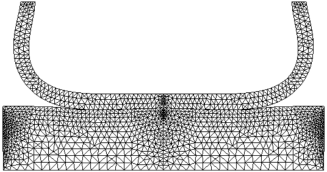
→ Python cvxopt library for positive projections

- Input: Tolerance ϵ_{du} to reach 🙂
- Output: R positive vectors

Ring on block

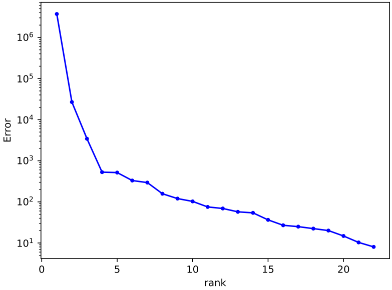
- Imposed displacement at the ring's top ends
- Parametric ring's radius $\mu \in \mathcal{P} := [0.95, 1.15]$, $\text{card}(\mathcal{P}^{\text{tr}}) = 21$
- Non-matching meshes, $\mathcal{N} = 2{\times}1590$ dofs, $\mathcal{R} = 50$ dofs
- Reference (left) and deformed (right) configuration



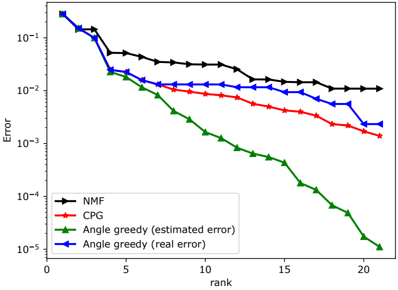


Ring on block

Basis construction



Primal basis



Dual basis

Cone-projected greedy algorithm

Algorithm Cone-projected weak greedy algorithm

Input : $\mathcal{P}^{\mathrm{tr}}$, $\breve{\Gamma}_1^c$ and $\epsilon_{\mathrm{du}} > 0$

1: Compute $\mathcal{S}_{\mathrm{du}} = \{\breve{\lambda}(\mu;\cdot)\}_{\mu\in\mathcal{P}^{\mathrm{tr}}}$

2: Set $\breve{K}_0 = \{0\}$, $n = 1$ and $r_1 = \max_{\mu\in\mathcal{P}^{\mathrm{tr}}} \|\breve{\lambda}(\mu;\cdot)\|_{\ell^\infty(\breve{\Gamma}_1^c)}$

3: **while** $(r_n > \epsilon_{\mathrm{du}})$ **do**

4: Search $\mu_n \in \mathrm{argmax}_{\mu\in\mathcal{P}^{\mathrm{tr}}} \|\breve{\lambda}(\mu;\cdot) - \Pi_{\breve{K}_{n-1}}(\breve{\lambda}(\mu;\cdot))\|_{\ell^\infty(\breve{\Gamma}_1^{c,\mathrm{tr}})}$

5: Set $\breve{K}_n := \mathrm{span}_+\{\breve{K}_{n-1}, \breve{\lambda}(\mu_n;\cdot)\}$

6: Set $n = n + 1$

7: Set $r_n := \max_{\mu\in\mathcal{P}^{\mathrm{tr}}} \|\breve{\lambda}(\mu;\cdot) - \Pi_{\breve{K}_{n-1}}(\breve{\lambda}(\mu;\cdot))\|_{\ell^\infty(\breve{\Gamma}_1^c)}$

8: **end while**

9: Set $R := n - 1$

Output : $\breve{W}_R := \breve{K}_R$

→ Python cvxopt library for positive projections