Point-wise Convergence

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Fourier series is defined to be

$$\sum \hat{f}(\theta)e^{in\theta}$$

with partial sums

$$S_N f = \sum_{-N}^{N} \hat{f}(\theta)e^{in\theta}$$

Where the Fourier coefficient for $f \in L^1(\mathbb{T})$, $\mathbb{T} = [-\pi, \pi]$ is defined by

$$\hat{f} = \frac{1}{2\pi}\int_{\mathbb{T}} f(\theta)e^{-in\theta}d\theta$$

To have a nicer form of the $S_N f$ to work with, we want write it as the convolution of f with Dirichlet kernel $D_N(\theta) = \sum_{-N}^{N} e^{in\theta}$ by next proposition.

The convolution of two L^1 function is defined by

$$(f * g)(\theta) = \frac{1}{2\pi}\int_{\mathbb{T}} f(\phi)g(\theta - \phi)d\phi$$

Following discussions will use Lebesgue's Dominated Convergence Theorem which is stated here without providing proof.

Theorem. *Dominated convergence theorem.* Let f_n be a sequence in L^1 such that $f_n \to f$ a.e., and there exists a non-negative $g \in L^1$ such that $|f_n| \leqslant g$ for all n. Then $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$.

Proposition. Suppose that $\sum_{n\in\mathbb{Z}} |A_n| < \infty$ and we have an absolutely convergent Fourier series $K(\theta) = \sum_{n\in\mathbb{Z}} A_n e^{in\theta}$. Then for any $f \in L^1(\mathbb{T})$ we have

$$(f * K)(\theta) = \sum_{n\in\mathbb{Z}} A_n \hat{f}(n)e^{in\theta}$$

Proof.

$$f(\theta - \phi)K(\phi) = \sum A_n e^{in\phi} f(\theta - \phi)$$

To apply dominated convergence theorem, we need to check the RHS converges a.e. and bounded by a integrable function. Since $f \in L^1$, $|f| \in L^1$, K is bounded by $|\sum A_n|$, this series is bounded by $|f||\sum A_n|$ and converges by assumption. Thus,

$$\begin{aligned}
\int_{\mathbb{T}} \sum A_n e^{in\phi} f(\theta - \phi) &= \sum A_n \int_{\mathbb{T}} e^{in\phi} f(\theta - \phi)d\phi \\
&= \sum A_n \int_{\mathbb{T}} e^{in\phi} f(\theta - \phi)d\phi \\
&= \sum A_n \int_{\mathbb{T}} e^{in(\theta - \psi)} f(\psi)d\psi \\
&= \sum A_n \int_{\mathbb{T}} e^{in\theta}e^{-in\psi} f(\psi)d\psi \\
&= 2\pi \sum A_n \hat{f}(n)e^{in\theta}
\end{aligned}$$

$\square$

Let $1_{[-N,N]}(n)$ be the indicator function, the Dirichlet Kernel $(D_N f) = \sum_{-N}^{N}$, above proposition applied to $S_N f = \sum 1_{[-N,N]}(n)\hat{f}(n)e^{in\theta}$, we have

$$S_N f(\theta) = (D_N * f)(\theta)$$

Lemma. Dirichlet kernel can be equivalently expressed as

$$D_N(\theta) = \frac{\sin(N+\frac{1}{2})(\theta)}{\sin(\theta/2)} \quad \theta \neq 0, \pm 2\pi ...$$

Proof. This is proved by calculating the finite sum. Recall that $\sum_0^N ar^n = a\frac{1-r^{N+1}}{1-r}$ Let $r = e^{i\theta}$, $a = r^{-n}$

$$\begin{aligned}
\sum_{-N}^{N} r^n &= r^{-N}\frac{1-r^{2N+1}}{1-r} \\
&= \frac{r^{-N-1/2}}{r^{-N-1/2}} r^{-N}\frac{1-r^{2N+1}}{1-r} \\
&= \frac{r^{-N-1/2} - r^{N+1/2}}{r^{-1/2} - r^{1/2}}
\end{aligned}$$

Substitute $e^{i\theta}$ back, and use $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$ we have

$$\begin{aligned}
\sum_{-N}^{N} e^{in\theta} &= \frac{e^{(-N-1/2)i\theta} - e^{(N+1/2)i\theta}}{e^{(-1/2)i\theta} - e^{(1/2)i\theta}} \\
&= \frac{-2i\sin((N+1/2)\theta)}{-2i\sin(\theta/2)} \\
&= \frac{\sin(N+1/2)(\theta)}{\sin(\theta/2)}
\end{aligned}$$

$\square$

Thus, partial sums can be rewritten as

$$S_N f = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\sin(N+\frac{1}{2})(\phi)}{\sin(\phi/2)} f(\theta-\phi)d\phi$$

A fundamental tool is given by the next theorem.

Theorem. *Riemann Lebesgue Lemma.* For any $f \in L^1(\mathbb{T})$, $\lim_{|n|\to\infty} \hat{f}(n) = 0$.

Proof. We first make the change of variable $\phi = \theta + \pi/n$, the interval doesn't change since for periodic functions $\int_a^{a+2\pi} f(x)dx = \int_0^{2\pi} f(x)dx$. Thus, we have

$$\begin{aligned}2\pi\hat{f}(n) &= \int_{\mathbb{T}} f(\theta)e^{-in\theta}d\theta \\ &= \int_{\mathbb{T}} f(\phi - \pi/n)e^{-in\phi-\pi/n}d\phi \\ &= \int_{\mathbb{T}} f(\phi - \pi/n)e^{-in\phi}e^{-i\pi}d\phi \\ &= -\int_{\mathbb{T}} f(\phi - \pi/n)e^{-in\phi}d\phi\end{aligned}$$

Adding above equation to $2\pi\hat{f}(n) = \int_{\mathbb{T}} f(\phi)e^{-in\phi}d\phi$, we have

$$4\pi\hat{f}(n) = \int_{\mathbb{T}} (f(\phi - \pi/n) - f(\phi))e^{-in\phi}d\theta$$

If f is continuous on $\mathbb{T}$, f is uniformly continuous on $\mathbb{T}$. Thus $f(\phi - \pi/n) - f(\phi) \to 0$ as $|n| \to 0$ uniformly, hence $\hat{f}(n) \to 0$.
For any $f \in L^1$, we find g continuous such that $|f-g|_{L^1} = \frac{1}{2\pi}\int |f(x) - g(x)|dx < \varepsilon$ for any $\varepsilon > 0$. By linearity, we have

$$\hat{f}(n) = \hat{g}(n) + (\widehat{f-g})(n)$$

We already showed $\hat{g}(n) \to 0$. Also

$$\begin{aligned}|(\widehat{f-g})(n)| &= |\frac{1}{2\pi}\int_{\mathbb{T}} (f(\theta) - g(\theta))e^{-in\theta}d\theta| \\ &\leqslant \frac{1}{2\pi}\int_{\mathbb{T}} |f(\theta) - g(\theta)||e^{-in\theta}|d\theta \\ &\leqslant \varepsilon\end{aligned}$$

Thus $\hat{f}(n) \leqslant \varepsilon$ completing this proof. □

Now we are in the position to prove point-wise convergence theorems. First, recall that

$$S_N f = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\sin(N + \frac{1}{2})(\phi)}{\sin(\phi/2)} f(\theta - \phi)d\phi$$

We will simplify this equation in two steps: we first replace $\sin(\phi/2)$ by the function $\phi/2$, then replace the domain of integration $\mathbb{T}$ by a small interval around $\phi = 0$. Since Dirichlet Kernel is even, we have

$$S_N f = \frac{1}{2\pi}\int_0^{\pi} (f(\theta - \phi) + f(\theta + \phi))\frac{\sin(N + \frac{1}{2})(\phi)}{\sin(\phi/2)}d\phi$$

Since $\frac{1}{2\pi}\int_{\mathbb{T}} \frac{\sin(N+\frac{1}{2})(\phi)}{\sin(\phi/2)}d\phi = 1$, for any constant S,

$$S_N f - S = \frac{1}{2\pi}\int_0^{\pi} (f(\theta - \phi) + f(\theta + \phi) - 2S)\frac{\sin(N + \frac{1}{2})(\phi)}{\sin(\phi/2)}d\phi$$

First consider the function

$$\phi \to \frac{1}{\sin(\phi/2)} - \frac{2}{\phi}$$

This is bounded and continuous on $[-\pi, \pi]$. Since it is the sum of two continuous functions on $[-\pi, 0)$ and $(0, \pi]$. Also, by l'Hospital's rule, the difference tends to zero at $\phi = 0$, thus it is also continuous at $\phi = 0$. Then boundedness follows.

Proposition. The following integral goes to zero as $n \to \infty$

$$\int_0^\pi (f(\theta - \phi) + f(\theta + \phi))(\frac{1}{\sin(\phi/2)} - \frac{2}{\phi})(\sin(N + \frac{1}{2}))\phi d\phi$$

Proof. First since $f \in L^1$, $(\frac{1}{sin(2/\pi)} - \frac{2}{\phi})$ is bounded, $F_\theta(\phi)$ is L^1. Also notice that

$$\sin(N + \frac{1}{2})\phi = \frac{e^{i(N+\frac{1}{2})\phi} - e^{-i(N+\frac{1}{2})\phi}}{2i} = \frac{e^{iN\phi} e^{\frac{i\phi}{2}} - e^{-iN\phi} e^{\frac{i\phi}{2}}}{2i}$$

Riemann Lebesgue Lemma implies

$$2\pi \hat{F}_\theta(\phi) = \int_{\mathbb{T}} f(\theta - \phi) + f(\theta + \phi))(\frac{1}{\sin(\phi/2)} - \frac{2}{\phi})e^{-iN\phi} d\phi \to 0$$

Which implies

$$\int_{\mathbb{T}} (f(\theta - \phi) + f(\theta + \phi))(\frac{1}{\sin(\phi/2)} - \frac{2}{\phi})(\sin(N + \frac{1}{2}))\phi d\phi \to 0$$

since the integrand is an odd function, it tends to zero on $[0, \pi]$, which finishes the proof. □

Thus, we have reduced to proving the pointwise convergence of

$$\lim_N \int_0^\pi (f(\theta - \phi) + f(\theta + \phi) - 2S) \frac{\sin(N + \frac{1}{2})(\phi)}{\phi} d\phi$$

Lemma.

$$\lim_N \int_\delta^\pi (f(\theta - \phi) + f(\theta + \phi) - 2S) \frac{\sin(N + \frac{1}{2})(\phi)}{\phi} d\phi = 0, \quad \text{for small } \delta > 0$$

Proof. Since $\frac{f(\theta - \phi) + f(\theta + \phi) - 2S}{\phi} \in L^1$, Riemann Lebesgue Lemma implies the desired result. □

Theorem. Let $f \in L^1(\mathbb{T})$, the partial sums $S_N f(\theta)$ converges to a limit S as $N \to \infty \Leftrightarrow$ for some $\delta \in (0, \pi)$, we have

$$\lim_N \int_0^\delta (f(\theta - \phi) + f(\theta + \phi) - 2S) \frac{\sin(N + \frac{1}{2})(\phi)}{\phi} d\phi = 0 \qquad (*)$$

Note: if there exists one such δ_0, then it above holds for all $\delta \in (0, \pi)$ since the integral on the interval from δ_0, δ tends to zero by Riemann Lebesgue lemma.

Theorem. *Dini's theorem.* Suppose that f satisfies a Dini condition at θ, meaning that for some $\delta > 0$ and some real number S

$$\int_0^\delta \frac{|f(\theta+\phi)+f(\theta-\phi)-2S|}{\phi} d\phi < \infty$$

Proof. If Dini's condition is satisfied, we apply Riemann Lebesgue lemma to $\frac{|f(\theta+\phi)+f(\theta-\phi)-2S|}{\phi}$ to get

$$\lim_N \int_0^\delta (f(\theta-\phi)+f(\theta+\phi)-2S)\frac{\sin(N+\frac{1}{2})(\phi)}{\phi} d\phi = 0$$

which by previous theorem implies the pointwise convergence of $S_N f$. □

Corollary. Suppose that f satisfies a symmetric Hölder condition at θ, meaning

$$|f(\theta-\phi)+f(\theta+\phi)-2f(\theta)| \leqslant C|\phi|^\alpha$$

for $0 < \phi < \delta$, where $0 < \alpha < 1$. Then $\lim_N S_N f(\theta) = f(\theta)$.

Proof.

$$\begin{aligned}
\int_0^\delta \frac{|f(\theta-\phi)+f(\theta+\phi)-2f(\theta)|}{\phi} d\phi &\leqslant \int_0^\delta C\phi^{\alpha-1} d\phi \\
&= \frac{C}{\alpha}\phi^\alpha \mid_0^\delta \\
&= \frac{C}{\alpha}\delta^\alpha \\
&\leqslant \frac{C\pi}{\alpha} \\
&< \infty
\end{aligned}$$

This shows that Hölder condition implies Dini condition. □

Before stating the main theorem, we need to introduce some tools.

Sine Integral

Sine integral is defined to be

$$Si(x) = \frac{2}{\pi}\int_0^x \frac{\sin t}{t} dt \quad 0 \leqslant x < \infty$$

Sine integral has three properties:

- $Si(0) = 0$, $\lim_{x\to\infty} Si(x) = 1$.
- $Si(x) \leqslant Si(\pi) = 1.18...$ for all $x \geqslant 0$.

- $Si(x)$ has local maxima at π, 3π, ... and local minima at 2π, 4π, ...

Proof. First note that $\lim_{n\to\infty} Si(x)$ exists since

$$\frac{\pi}{2}\lim_{n\to\infty} Si(x) = \sum_{0}^{\infty}\int_{n\pi}^{(n+1)\pi}\frac{\sin t}{t} \quad \text{converges}$$

To find its value, by Riemann Lebesgue Lemma

$$\lim_{n\to\infty}\int_{\mathbb{T}}(\frac{1}{\sin(\phi/2)}-\frac{2}{\phi})\sin(n+\frac{1}{2})\phi d\phi = 0$$

And since

$$\lim_{n\to\infty}\int_{\mathbb{T}}\frac{\sin(n+\frac{1}{2})\phi}{\sin(\phi/2)}d\phi = \lim_{n\to\infty}\int_{\mathbb{T}} D_n(\phi)d\phi = 2\pi$$

Also, since the integrand is even, we have

$$\frac{2}{\pi}\lim_{n\to\infty}\int_0^{\pi}\frac{\sin(n+\frac{1}{2})\phi}{\phi}d\phi = 1$$

The left hand side is equivalent to $Si(n\pi+\frac{\pi}{2})$ by a change of variable $t = x\sin(n+1/2)$, and thus $\lim_{n\to\infty} Si(n\pi+\frac{\pi}{2}) = 1$.
The second and third properties is simply proved by finding where the derivative equals zero and compare local maxima values. □

Facts on Measure Theory

We consider half intervals and one side continuous functions in the following discussion. Let h-intervals denote the half intervals. Here we choose to use $(a,b]$ and right continuous functions. Note that the whole theory is the same if we choose to use $[a,b)$ and left continuous functions.

Definition. We define **Lebesgue-Stieltjes measure** associated to F, usually denote this by μ_F, to be

$$\mu(E) = \inf\{\sum_{1}^{n}[F(b_j)-F(a_j)] : E \subset \bigcup_{1}^{\infty}(a_j,b_j]\}$$

An important measure, **Lebesgue measure** is the complete measure μ_F associated to the function $F(x) = x$.

Definition. If $F:\mathbb{R}\to\mathbb{C}$ and $x\in\mathbb{R}$. The **total variation function** of f is defined to be

$$T_F(x) = \sup\sum_{i=1}^{N}|F(x_i)-F(x_{i-1})| \quad -\infty = x_0 < x_1 < ... < x_n = x \;\; n\in\mathbb{N}$$

If $T_F(\infty) = \lim_{N\to\infty} T_F(x)$ is finite, we say f is of **bounded variation** (on $\mathbb{R}$). We denote the space of all such F by BV. $\sup\sum_{i=1}^{N}|F(x_i)-F(x_{i-1})| \quad a = x_0 < x_1 < ... < x_n = b \;\; n\in\mathbb{N}$ is called the **total variation** of F on $[a,b]$.

Theorem. If $F : \mathbb{R} \to \mathbb{R}$ $F \in BV \Leftrightarrow F$ is the difference of two bounded increasing functions; in particular if $F \in BV$, $F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$

Since the discontinuities of increasing functions is at most countable, thus we have the following proposition.

Proposition. If $F \in BV$, then the discontinuity of F is at most countable.

For F of bounded variations, $F = f^+ + f^-$, we assign a signed measure to F by $\mu_F = \mu_{f^+} + \mu_{f^-}$. We denote the integral of a function g with respect to the measure μ_F by $\int gdF$ or $\int g(x)dF(x)$; such integrals are called **Lebesgue-Stieltjes** integrals.

Theorem. *Integral by part formula*. If F and G are in NBV and at least on of them is continuous, then for $-\infty < a < b < infty$,

$$\int_{(a,b]} FdG + \int_{(a,b]} GdF = F(b)G(b) - F(a)G(a)$$

Main Theorem

Theorem. Suppose that f is of bounded variation on $[\theta - \delta, \theta + \delta]$ for some $\delta > 0$. Then $\lim_N S_N f(\theta) = \frac{1}{2}[f(\theta + 0) + f(\theta - 0)]$. ($f(\theta + 0) \equiv \lim_{x \to \theta^+} f(\theta)$).

Proof. We write $m = N + \frac{1}{2}$, $F(\phi) = \frac{1}{2}(f(\theta + \phi) + f(\theta - \phi))$. $S = F(0+0)$; from equation (*) we have

$$S_N - S = \frac{2}{\pi} \int_0^\delta (\frac{f(\theta + \phi) + f(\theta - \phi)}{2} - S) \frac{\sin m\phi}{\phi} d\phi + o$$

Where o denotes the error term when we performed reductions. Also $\frac{2}{\pi} \frac{\sin m\phi}{\phi} d\phi = dSi(m\phi)$, above equation becomes

$$S_N - S = \int_0^\delta (F(\phi) - S) dSi(m\phi) + o$$

On the interval $[0, \delta]$ $Si(x)$ and f is of bounded variation, and $Si(x)$ is continuous. We set $F(\phi) = F(\phi - 0)$ where F is discontinuous at ϕ. This makes F left continuous. We could do this because F is of bounded variation, and we only redefined countable many points, so set of redefined points has measure zero in $\mu_{Si(m\phi)} = \frac{\sin m\phi}{\phi} \mu_\phi$. Integral by part implies:

$$S_N - S = [F(\delta - 0) - S] Si(m\delta) - \int_0^\delta Si(m\phi) dF(\phi) + o$$

$Si(\infty) = 1$ and $Si(x)$ bounded by $Si(\pi)$, so dominated convergence theorem implies:

$$\lim_N \int_0^\delta Si(m\phi) dF(\phi) = \int_0^\delta dF(\theta) = F(\delta) - F(0)$$

Since $F(\phi)$ is continuous at $\phi = 0$, $F(0) = F(0+0) = S$, $\lim_N o = 0$.

$$\lim_N S_N - S = [F(\delta - 0) - S] - [F(\delta - 0) - S] = 0$$

$\square$