Fourier Series in L^2

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April 29, 2019

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Let $\mathbb{T} = [0, 2\pi]$. Define

$$L^1(\mathbb{T}) = \{f|\ |f|_{L^1} = \frac{1}{2\pi}\int_{\mathbb{T}} |f| < \infty\}$$
$$L^2(\mathbb{T}) = \{f|\ |f|_{L^2} = \frac{1}{2\pi}\int_{\mathbb{T}} |f|^2 < \infty\}$$

In $L^2(\mathbb{T})$, the inner product is defined by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} f(\theta)\bar{g}(\theta)d\theta$$

If we approximate $f \in L^2$ with $g \in L^2$, we can measure the degree of approximation by the mean square distance

$$\|f-g\| = |f-g|_{L^2} = \langle f-g, f-g\rangle$$

We define the Fourier transform to be

$$\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(\theta)e^{-in\theta}d\theta \quad \text{where } \mathbb{T} = [0, 2\pi]$$

We would like to approximate any $f \in L^2$ with trigonometric polynomial $g(\theta) = \sum_{n=-N}^{N} b_n e^{in\theta}$. Then the following calculation shows that the approximation is best when $b_n = \hat{f}(n)$.

$$|f-g|_{L^2}^2 = |f|_{L^2}^2 - \langle f, g\rangle - \langle g, f\rangle + |g|_{L^2}^2 \tag{1}$$
$$= |f|_{L^2}^2 - \sum_{n=-N}^{N}(\hat{f}(n)\bar{b}_n + \bar{\hat{f}}(n)b_n - |b_n|_{L^2}^2) \tag{2}$$
$$= \sum_{n=-N}^{N}|b_n - \hat{f}(n)|^2 + |f|_{L^2}^2 - \sum_{n=-N}^{N}|\hat{f}(n)|_{L^2}^2 \tag{3}$$

Thus, we have the following proposition.

Proposition. Suppose that $f \in L^2(\mathbb{T})$. Then the minimum mean square is attained when b_n is the Fourier coefficient $b_n = \hat{f}(n)$. The mean square distance is given by:

$$|f-g|_{L^2}^2 = |f|_{L^2}^2 - \sum_{n=-N}^{N}|\hat{f}(n)|^2$$

In particular, for $N \in \mathbb{Z}^+$, we have the inequality

$$\sum_{n=-N}^{N} |\hat{f}(n)|^2 \leqslant |f|_{L^2}^2$$

In particular *Bessel's inequality*

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 \leqslant |f|_{L^2}^2 \tag{4}$$

Lemma. For any $f \in L^2(\mathbb{T})$, the Fourier series converges in $L^2(\mathbb{T})$. Let F be its limit, them $\hat{F} = \hat{f}$

Proof. We first show that the partial sums $S_N f = \sum_{-N}^{N} \hat{f}(n)e^{in\theta}$ is Cauchy by showing that $S_{N^+} f = \sum_0^N \hat{f}(n)e^{in\theta}$, thus converges in L^2.

$$\|S_{N^+}f - S_{M^+}f\|^2 = \frac{1}{2\pi}\int_{\mathbb{T}} |S_{N^+}f - S_{M^+}f|^2 = \sum_{M+1}^{N} |\hat{f}(n)|^2 \to\ 0^* \ \text{ M, N} \to \infty$$

Similarly we show that $S_{N^+}f$ is Cauchy. L^2 is complete $\Rightarrow$ The Fourier series $\sum \hat{f}(n)e^{inx}$ converges. Let F be its limit. It is left to show that $F = f$.

$$2\pi\hat{F}(n) = \int_{\mathbb{T}} (F(\theta) - S_N f(\theta))e^{-in\theta}d\theta + \int_{\mathbb{T}} S_N f(\theta)e^{-in\theta}d\theta$$

For $N > |n|$ $\int_{\mathbb{T}} S_N f(\theta)e^{-in\theta}d\theta = 2\pi\hat{f}(n)$. Thus,

$$|\hat{F}(n) - \hat{f}(n)| = \int_{\mathbb{T}} (F(\theta) - S_N f(\theta))e^{-in\theta}d\theta \leqslant \|F - S_N f\| \quad (N > |n|)$$

Let $N \to \infty$, we have $\hat{F} = \hat{f}$ □

Now the following discussions shows that we have $F = f$ a.e.

Proposition. Trigonometric polynomials are dense in $L^2([0.2\pi])$. (trig polynomials by def: $\sum_{-N}^{N} a_n e^{in\theta}$ $a \in \mathbb{C}$, dense: any $f \in L^2$ his a limit(L^2) of trig polynomial) (Hint: use Stone -Weierstrass Thm†)

Proof. First we notice that trigonometric polynomials do not separate points since $g(0) = g(2\pi)$ for all trigonometric polynomial g. Thus, we perform a change of variable and apply Stone - Weierstrass Thm to the unit circle.

Let $c(x) = \sum_{-N}^{N} b_n x^n$ where $x = e^{in\theta}$. $\theta \in [0, 2\pi]$. Then apply Stone-Weierstrass Theorem on c implies for every continuous function g defined on the unit circle, there exists a sequence of trigonometric polynomials f_n converges to g uniformly.

$$\|g - f_n\| \leqslant \varepsilon/2$$

Since continuous functions are dense in L^2. For any $f \in L^2([0, 2\pi])$

$$\|f - g\| \leqslant \varepsilon/2$$

Triangle inequality implies

$$\|f - f_n\| \leqslant \|f - g\| + \|g - f_n\| \leqslant \varepsilon$$

□

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*By Bessel's Inequality

†*page165 Rudin*, Walter *Principles of Mathematical Analysis*

Proposition. For $f \in L^2([0,2\pi])$, suppose $\hat{f}(n) = 0\ \forall n$, then $f \equiv 0$ in L^2. (using Trig polynomials are dense in $L^2([0,2\pi])$)

Proof. Since $\hat{f}(n) = 0$ for all n, $\hat{f}(n) = (f, e^{in\theta})_{L^2} = 0$ where $(\cdot,\cdot)$ is the inner product on L^2. For any $g \in L^2$, we can pick g_n be the sequence of trig polynomials that converges to g, then for any $\varepsilon > 0$, $\exists N$ st. for all $n > N$, $|g - g_n|_{L^2} < \varepsilon$, then

$$|(f, g - g_n)| \leqslant |f|_{L^2}|g - g_n|_{L^2} \leqslant \varepsilon|f|_{L^2}$$

So for all $g \in L^2$, $|(f, g - g_n)| = |(f,g) - (f,g_n)| = |(f,g)| \leqslant \varepsilon|f|_{L^2}$.
Take $g = f$, $|f|^2_{L^2} = (f,f) \leqslant \varepsilon|f|_{L^2} \Rightarrow |f|_{L^2} \leqslant \varepsilon$, Thus, $f \equiv 0$ a.e. □

If $\hat{F} = \hat{f}$, $\hat{F} - \hat{f} = \widehat{F - f} = 0 \Rightarrow F - f \equiv 0$ a.e. We conclude above discussion with the following main theorem.

Theorem. *Parseval's theorem* For any $f \in L^2(\mathbb{T})$, the Fourier series converges to f in $L^2(\mathbb{T})$ and we have Parseval's identity:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2 d\theta = \sum_{n\in\mathbb{Z}}|\hat{f}(n)|^2 \tag{5}$$

Proof. Since $S_N f \to f$, by definition $|S_N f - f|_{L^2} \to 0$. By reverse triangle inequality $||S_N f|_{L^2} - |f|_{L^2}| \leqslant |S_N f - f|_{L^2}$. Therefore, $|S_N f|_{L^2} \to |f|_{L^2}$. □

An alternative approach to $\hat{F} = \hat{f} \Rightarrow F = f$ a.e.

We first show that L^2 periodic functions is also L^1, then proposition implies the wanted result.

Proposition. If $f \in L^2([0,2\pi])$, then$f \in L^1([0,2\pi])$ (Hint: Cauchy-Schwartz)

Proof.

$$\|f\|_{L^1} = \int_0^{2\pi}|f| \leqslant \int_0^{2\pi}|1|\cdot|f| \leqslant \|1\|_{L^2}\cdot\|f\|_{L^2} = \|f\|_{L^2}$$

Remark: Is it true if $f \in L^2(\mathbb{R})$ then $f \in L^1(\mathbb{R})$? In general no, $\frac{1}{1+|x|}$ in L^2 but not in L^1. □

Proposition. Suppose that $f, g \in L^1(\mathbb{T})$ have the property that $\hat{f}(n) = \hat{g}(n)$, for all $n \in \mathbb{Z}$. Then $f = g$ a.e.

Proof. *p*age17, Pinsky □