On the BV double of the Courant algebroid

Anton M. Zeitlin

Louisiana State University, Department of Mathematics

University of Notre Dame

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Outline

Homotopy algebras related to vertex algebras

Vertex/Courant algebroids and BV double

Flat metric deformation and YM C_∞-algebra

"Doubling": Gravity and Double Field Theory

Motivation

- Yang-Mills^2=Gravity: homotopical interpretation.
- Homotopical algebras of Field Theories (A_∞, L_∞, G_∞, $BV_\infty^\square$, ...): where do they come from?
- Relationship between open and closed strings.

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- Homotopical algebras of Field Theories (A_∞, L_∞, G_∞, $BV_\infty^\Box$, $\dots$): where do they come from?
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Vertex/Courant algebroids and their homotopy algebras: BV double

Flat metric deformation and Yang-Mills C_∞-algebra

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Vertex algebras: a reminder

Vertex operator (super)algebra :

- Graded vector space $V = \sum_{n,m} V_n[m]$,
- Vertex operators $Y : V \to \mathrm{End}(V)[[z^{\pm 1}]]$:

$$Y : A \mapsto A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1},$$

- A vector $|0\rangle \in V_0[0]$, such that:

$$\lim_{z\to 0} A(z)|0\rangle = A; \quad Y(|0\rangle, z) = Id_V$$

- Locality property: $(z-w)^N[A(z), B(w)] = 0$
- Virasoro element $|L\rangle \in V_2[0]$, such that $L(z) = \sum_n L_n z^{-n-2}$ satisfy the relations of Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}$$

L_0 provides grading and L_{-1} is a translation operator:

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Topological vertex algebra and Lian-Zuckerman operations

V: topological vertex operator algebra (TVOA), if there exist four elements: $J \in V_1[1]$, $b \in V_2[-1]$, $F \in V_1[0]$, $L \in V_2[0]$, such that

$$[Q, b(z)] = L(z), \quad Q^2 = 0, \quad b_0^2 = 0,$$

where

$$Q = J_0, \; J(z) = \sum_n J_n z^{-n-1},$$
$$b(z) = \sum_n b_n z^{-n-2}, \quad L(z) = \sum_n L_n z^{-n-2}, \; F(z) = \sum_n F_n z^{-n-1}$$

and F_0, L_0 commute, so that F_0 gives fermionic grading.

Lian-Zuckerman operations:

$$\mu(a_1, a_2) = Res_z \frac{a_1(z)a_2}{z}, \quad \{a_1, a_2\} = (-1)^{|a_1|} Res_z(b_{-1}a_1)(z)a_2$$

satisfy the relations of a homotopy BV algebra.

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BV-LZ algebra

- $Q\mu(a_1, a_2) = \mu(Qa_1, a_2) + (-1)^{|a_1|}\mu(a_1, Qa_2)$,
- Homotopy commutativity:

$$\mu(a_1, a_2) - (-1)^{|a_1||a_2|}\mu(a_2, a_1) = Qm(a_1, a_2) + m(Qa_1, a_2) + (-1)^{|a_1|}m(a_1, Qa_2)$$

- Homotopy associativity:

$$\mu(\mu(a_1, a_2), a_3) - \mu(a_1, \mu(a_2, a_3)) = Q\nu(a_1, a_2, a_3) + \nu(Qa_1, a_2, a_3) + (-1)^{|a_1|}\nu(a_1, Qa_2, a_3) + (-1)^{|a_1|+|a_2|}\nu(a_1, a_2, Qa_3),$$

- $\{a_1, a_2\} = (-1)^{|a_1|}(\mathbf{b}\mu(a_1, a_2) - \mu(\mathbf{b}a_1, a_2) - (-1)^{|a_1|}\mu(a_1, \mathbf{b}a_2)$, where $\mathbf{b} = b_0$.
- Operation $\{a\ ,\ \cdot\}$ is a derivation of μ and Q is a derivation of $\{\cdot\ ,\ \cdot\}$, namely:

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"Doubling": Gravity and Double Field Theory

Properties:

- **b** is a derivation of degree -1 for $\{\cdot\ ,\ \cdot\}$:

$$\mathbf{b}\{a_1, a_2\} = \{\mathbf{b}a_1, a_2\} + (-1)^{|a_1|-1}\{a_1, \mathbf{b}a_2\}.$$

- $\{\cdot\ ,\ \cdot\}$ is graded-antisymmetric up to Q-homotopy:

$$\{a_1, a_2\} + (-1)^{(|a_1|-1)(|a_2|-1)}\{a_2, a_1\} = (-1)^{|a_1|-1}(Qn(a_1, a_2) - n(Qa_1, a_2) - (-1)^{|a_2|}n(a_1, Qa_2)),$$

  where $n = [\mathbf{b}, m]$.

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Kimura, Voronov, Zuckerman'96, Huang, Zhao'99 and Voronov'99: symmetrized versions of these operations can be extended to a "weak" G_∞-algebra.

I. Gálvez, V. Gorbounov, A. Tonks'06 proved that it has the structure of G_∞-algebra as defined by Tamarkin and Tsygan'00 in the case of TVOA with conformal grading in $\mathbb{N}$.

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Semi-infinite (BRST) complex

Consider super VOA Λ, obtained from the following super Heisenberg algebra:

$$[b_n, c_m] = \delta_{n+m,0}, \quad n, m \in \mathbb{Z}.$$

One can construct the space of Λ as a Fock module:

$$\Lambda = \{b_{-n_1} \dots b_{-n_k} c_{-m_1} \dots c_{-m_l} \mathbf{1}, n_1, \dots, n_k > 1, m_1, \dots, m_l > -1; \\ c_k \mathbf{1} = 0,\ k \geqslant 2; \quad b_k \mathbf{1} = 0,\ k \geqslant -1\}.$$

I. Frenkel, H. Garland, G. Zuckerman'86: For any VOA with $c = 26$ $V \otimes \Lambda$ is a TVOA, however Q^2 is nilpotent if $c \neq 26$.

Nevertheless, if VOA is positively graded w.r.t. conformal weight, there is always a *subcomplex of light modes*, annihilated by:

$$\mathcal{L}_0 = L_0^V + L_0^{b,c}$$

which is preserved by Lian-Zuckerman operations.

Only elements from V_0 and V_1 participate!

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Complex of light modes

$$0 \to \mathcal{F}^0 \xrightarrow{Q} \mathcal{F}^1 \xrightarrow{Q} \mathcal{F}^2 \xrightarrow{Q} \mathcal{F}^3 \to 0$$

There exist two operators $\mathbf{b} = b_0$, $\mathbf{c} = c_0$:

$$[Q, \mathbf{b}] = 0, \quad [\mathbf{b}, \mathbf{c}] = 1$$

$$\begin{tikzcd}
 & V_1' & V_1'' \arrow[l, "\mathbf{b}"'] & \\
 & \bigoplus & \bigoplus & \\
V_0 & V_0' \arrow[l, "\mathbf{b}"'] & V_0'' & V_0''' \arrow[l, "\mathbf{b}"']
\end{tikzcd}$$

Action of Q: here $\mathrm{d} = L_{-1}[-1]$, $\mathrm{d}^* = \frac{1}{2}L_1[-1]$, $\tilde{Q} = [Q, \mathbf{cb}]$.

$$\begin{tikzcd}
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Operations on V_0, V_1, the BV-LZ algebra is made of on $(\mathcal{F}^\bullet, Q)$:

$$u_1 u_2 = Res_z\Big(\frac{u_1(z)u_2}{z}\Big), \quad u * A = Res_z\Big(\frac{u(z)A}{z}\Big),$$
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where $u, u_1, u_2 \in V_0$, $A \in V_1$ generate a *vertex algebroid* V. Gorbounov, F. Malikov, V. Schechtman, A. Vaintrob'00.

Classical limit: (P. Bressler'05) assume $V_1 = \mathcal{V}_1[[h]]$, where h is a formal parameter $h \to 0$, so that $\mathcal{V}_1 = V_1/hV_1$ is a commutative vertex algebroid. Consider the following limit:

$$\langle \bar{A}, \partial u\rangle = \overline{\frac{1}{h}[A, u]}, \quad [\bar{A}_1, \bar{A}_2] = \overline{\frac{1}{h}[A_1, A_2]},$$
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Vertex algebroid and its classical limit

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Courant algebroid with a Calabi-Yau structure

We say that there is a a structure of $\mathcal{V}_0$-*Courant algebroid* on $\mathcal{V}_1$ is the following data:

- $\mathcal{V}_0$ is a commutative $\mathbb{K}$-algebra; $\mathcal{V}_1$ is a $\mathcal{V}_0$-module,
- There is a symmetric bilinear form $\langle\cdot,\cdot\rangle : \mathcal{V}_1 \otimes_{\mathcal{V}_0} \mathcal{V}_1 \to \mathcal{V}_0$,
- derivation $\partial : \mathcal{V}_0 \to \mathcal{V}_1$,
- Dorfman bracket $[\cdot,\cdot] : \mathcal{V}_1 \otimes \mathcal{V}_1 \to \mathcal{V}_1$,

These data satisfy the following conditions:

1. $[A_1, uA_2] = u[A_1, A_2] + \langle A_1, \partial u\rangle A_2$
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3. $[A_1, A_2] + [A_2, A_1] = \partial\langle A_1, A_2\rangle$
4. $[A_1, [A_2, A_3]] = [[A_1, A_2], A_3] + [A_2, [A_1, A_3]]$
5. $[\partial u, A] = 0$
6. $\langle \partial u_1, \partial u_2\rangle = 0$,

where $A, A_1, A_2, A_3 \in V_1$ and $u_1, u_2 \in V_0$.

Calabi-Yau structure $\mathrm{div} : \mathcal{V}_1 \to \mathcal{V}_0$:

$$\mathrm{div}\ \partial = 0,$$
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Classical limit of BV-LZ algebra

Consider the h-twisted complex:

$$
\begin{tikzcd}
V_0 \arrow[r, "\mathrm{d}"] & V_1' \arrow[r, "\mathrm{d}^*"] & hV_0'' & \\
& \bigoplus & \bigoplus & \\
& hV_0' \arrow[r, "\mathrm{d}"] \arrow[uur, "\tilde{Q}"] & hV_1'' \arrow[r, "\mathrm{d}^*"] & h^2 V_0'''
\end{tikzcd}
$$

which is $\mathbf{b} = \frac{b_0}{h}$-invariant. Quasiclassical limit:

$$\mu_h(\cdot,\cdot) = \mu(\cdot,\cdot) + O(h), \quad m_h(\cdot,\cdot) = m(\cdot,\cdot) + O(h),$$
$$\nu_h(\cdot,\cdot,\cdot) = \nu(\cdot,\cdot,\cdot) + O(h), \quad \{\cdot,\cdot\}_h = h\{\cdot,\cdot\} + O(h^2),$$

gives a BV-LZ algebra structure for Courant algebroid:

$$
\begin{tikzcd}
\mathcal{V}_0 \arrow[r, "\partial{[-1]}"] & \mathcal{V}_1' \arrow[r, "\frac{1}{2}\mathrm{div}{[-1]}"] & \mathcal{V}_0'' & \\
& \bigoplus & \bigoplus & \\
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A.Z., Comm. Math. Phys. 303 (2011) 1005-1068

Courant algebroid with CY structure as the BV-LZ algebra

Start with complex $(\mathcal{F}^\bullet, Q)$:

$$0 \to \mathcal{F}^0 \xrightarrow{Q} \mathcal{F}^1 \xrightarrow{Q} \mathcal{F}^2 \xrightarrow{Q} \mathcal{F}^3 \to 0$$

with operators $\mathbf{b}, \mathbf{c}$ of degree -1 and 1 correspondingly.

$$[Q, \mathbf{b}] = 0, \quad [\mathbf{b}, \mathbf{c}] = 1, \quad \mathbf{b}^2 = 0, \quad \mathbf{c}^2 = 0.$$

Half-complex and additional "conformal" grading using $\mathbf{b}$-operator:

$$\mathscr{V}_0 = \mathscr{V}_0^{1/2} \oplus \tilde{\mathscr{V}}_0^{1/2}, \quad \mathscr{V}_1 = \mathscr{V}_1^{1/2} \oplus \tilde{\mathscr{V}}_1^{1/2},$$

$$\begin{tikzcd}
\mathcal{V}_0 \arrow[r, "\mathrm{d}"] & \mathcal{V}'_1 \arrow[r, "\mathrm{d}^*"] & \mathcal{V}''_0 & \\
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& \mathcal{V}'_0 \arrow[r, "\mathrm{d}"] \arrow[uur, "\tilde{Q}"] & \mathcal{V}''_1 \arrow[r, "\mathrm{d}^*"] & \mathcal{V}'''_0,
\end{tikzcd}$$

where:

$$\mathscr{V}_0^{1/2} = \mathcal{V}'_0 \oplus \mathcal{V}''_0, \quad \tilde{\mathscr{V}}_0^{1/2} = \mathcal{V}''_0 \oplus \mathcal{V}'''_0$$
$$\mathscr{V}_1^{1/2} = \mathcal{V}'_1, \quad \tilde{\mathscr{V}}_1^{1/2} = \mathcal{V}''_1.$$

Outline

Homotopy algebras related to vertex algebras

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Flat metric deformation and YM C_∞-algebra

"Doubling": Gravity and Double Field Theory

Theorem. A.Z.'24

The BV-LZ algebra the complex $(\mathcal{F}^\bullet, Q)$ satisfying:

1. The action $\mu(\mathcal{F}^0\ ,\ \cdot) : \mathcal{F}^i \to \mathcal{F}^i$ gives $\mathcal{F}^0$ an associative algebra structure and an $\mathcal{F}^0$-module structure on $\mathcal{F}^i$ for all i.
2. $\{\mathscr{V}_i, \mathscr{V}_j\} \subset \oplus_{k\geq 1}\mathscr{V}_{i+j-k}$.
3. $\mu(\mathscr{V}_i, \mathscr{V}_j) \subset \oplus_{k\geq 0}\mathscr{V}_{i+j-k}$, while the restriction $\mu(\mathscr{V}_1, \mathscr{V}_1)|_{\mathscr{V}_0}$ is a symmetric bilinear form.
4. $\mathbf{c}\ \mu(a_1, a_2) = (-1)^{|a_1|}\mu(a_1, \mathbf{c}\ a_2)$,
5. The homotopy of the product m is non-vanishing only on a half-complex $m : \mathscr{V}_i^{1/2} \otimes_{\mathcal{F}^0} \mathscr{V}_j^{1/2} \to \mathscr{V}_{i+j-2}^{1/2}$ is a bilinear form,

is equivalent to $\mathcal{V}_0$-Courant algebroid structure on $\mathcal{V}_1$ with the CY structure given by $\tilde{Q}^{-1}\mathrm{d}^*[1]|_{V_1'} = \frac{1}{2}\mathrm{div}$.

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L_3- and C_3-algebras

The associativity homotopy ν is nontrivial only only on the spaces:

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Both these L_∞- and C_∞-subalgebras are really L_3- and C_3-algebras.

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Example: Courant algebroid on $\Gamma(TM \oplus T^*M)$

$$
\begin{tikzcd}
 & \Gamma(TM) \arrow[ddr, "\frac{1}{2}\mathrm{div}_\omega"] & \Gamma(TM) \arrow[ddr, "\frac{1}{2}\mathrm{div}_\omega"] & \\
 & \bigoplus & \bigoplus & \\
\mathcal{F}(M) \arrow[ddr, "d"'] & \mathcal{F}(M) \arrow[r, "\mathrm{id}"] \arrow[ddr, "d"'] & \mathcal{F}(M) & \mathcal{F}(M) \\
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 & \Gamma(T^*M) & \Gamma(T^*M) &
\end{tikzcd}
$$

Corresponding vertex algebra is the family of β-γ-systems, generated by:

$$X^\mu(z)p_\nu(w) \sim \frac{h\delta^\mu_\nu}{z-w}; \quad X^\mu(z)X^\nu(w) \sim 0; \quad p_\mu(z)p_\nu(w) \sim 0$$

with the vector space being a Fock space for: $[x^\mu_n, p_{\nu,m}] = h\delta^i_j\delta_{n,-m}$.

The Virasoro element is given by:

$$L(z) = -\frac{1}{h}\sum_\mu : p_\mu(z)\partial X^\mu(z) : + \partial^2_z \log(\omega(X(z)))$$

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Example: Courant algebroid on $\Gamma(TM \oplus T^*M)$

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 & \Gamma(TM) \arrow[ddr, "\frac{1}{2}\mathrm{div}_\omega"] & \Gamma(TM) \arrow[ddr, "\frac{1}{2}\mathrm{div}_\omega"] & \\
 & \bigoplus & \bigoplus & \\
\mathcal{F}(M) \arrow[ddr, "d"'] & \mathcal{F}(M) \arrow[r, "\mathrm{id}"] \arrow[ddr, "d"'] & \mathcal{F}(M) & \mathcal{F}(M) \\
 & \bigoplus & \bigoplus & \\
 & \Gamma(T^*M) & \Gamma(T^*M) &
\end{tikzcd}$$

Corresponding vertex algebra is the family of β-γ-systems, generated by:

$$X^\mu(z)p_\nu(w) \sim \frac{h\delta^\mu_\nu}{z-w}; \quad X^\mu(z)X^\nu(w) \sim 0; \quad p_\mu(z)p_\nu(w) \sim 0$$

with the vector space being a Fock space for: $[x^\mu_n, p_{\nu,m}] = h\delta^i_j\delta_{n,-m}$.

The Virasoro element is given by:

$$L(z) = -\frac{1}{h}\sum_\mu : p_\mu(z)\partial X^\mu(z) : + \partial_z^2 \log(\omega(X(z))$$

Vertex algebroid $\to$ Courant algebroid: annihilating all non-covariant terms.

Flat metric deformation

Consider the elements $\{f_i\}_{i=1}^d \in V_1'$ such that

$$Qf_i = 0, \quad \mu(f_i, f_j) = 0, \quad \forall i, j.$$

and introduce the following operator: $R^\eta = \sum_{i,j} \eta^{ij}\mu(f_i, \{f_j, \cdot\})$,

Explicitly:

$$\begin{tikzcd}
V_0 \arrow[r, "\hat{\mathrm{d}}"] \arrow[dr, "-\mathbf{c}\Delta"'] & V_1' \arrow[r, "\hat{\mathrm{d}}^*"] \arrow[dr, "-\mathbf{c}\Delta"'] & V_0'' \arrow[dr, "-\mathbf{c}\Delta"'] & \\
& V_0' \arrow[u, phantom, "\bigoplus"] \arrow[r, "\hat{\mathrm{d}}"] & V_1'' \arrow[u, phantom, "\bigoplus"] \arrow[r, "\hat{\mathrm{d}}^*"] & V_0''',
\end{tikzcd}$$

where:

$$\Delta\ \cdot = \sum_{i,j} \eta^{ij}\{f_i, \{f_j, \cdot\}\}, \quad \hat{\mathrm{d}}\ \cdot = (-1)^{|\cdot|}\sum_{i,j}\eta^{ij}\mu(\{f_j,\ \cdot\}, f_i),$$

$$\hat{\mathrm{d}}^*\ \cdot = \frac{1}{2}\sum_{i,j}\eta^{ij}\tilde{Q}m(f_i, \{f_j,\ \cdot\}) \text{ on } V_1',$$

$$\hat{\mathrm{d}}^*\ \cdot = -\frac{1}{2}\sum_{i,j}\eta^{ij}\mathbf{c}\tilde{Q}m(f_i, \mathbf{b}\{f_j,\ \cdot\}) \text{ on } V_1''.$$

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Deformation of the C_∞ algebra and $BV_\infty^\square$-algebra

- Deformation of the differential:

$$Q \to Q^\eta = Q + R^\eta$$

- Deformation is linear in η^{ij}: $\mu \to \mu + \bar{\mu}^\eta$:

$$\bar{\mu}^\eta(a_1, a_2) = \sum_{i,j} \nu(f_i, \{f_j, a_1\}, a_2)\eta^{ij} - \sum_{i,j} \mu(m(f_i, a_1), \{f_j, a_2\})\eta^{ij}$$

- Trilnear operation is not deformed

The relation

$$[Q^\eta, \mathbf{b}] = -\Delta$$

destroys the rest of homotopy Gerstenhaber algebra structure and leads to $BV_\infty^\square$-algebra, a notion due to M. Reiterer'19.

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Maurer-Cartan equation

There is an A_∞-algebra on $(\mathcal{F}^\bullet \otimes U(\mathfrak{g}), Q)$.

The Maurer-Cartan equation:

$$Q^\eta \Psi + \mu^\eta(\Psi, \Psi) + \nu^\eta(\Psi, \Psi, \Psi) = 0$$

and its symmetries:

$$\Psi \to \Psi + Q\lambda + \mu^\eta(\Psi, \lambda) - \mu^\eta(\lambda, \Psi),$$

where $\Psi \in \mathcal{F}^1 \otimes \mathfrak{g}$, $\lambda \in \mathcal{F}^0 \otimes \mathfrak{g}$.

$\Gamma(TM \oplus T^*M)$ and Yang-Mills

Take $M = \mathbb{R}^N$, and we obtain that η-deformed C_∞-algebra is the C_∞-algebra of Yang-Mills theory

A.Z., JHEP 2007 (09), 068

A.Z., JHEP 2010 (3), 1-32

and Maurer-Cartan equations is equivalent to:

$$\sum_{i,j} \eta^{ij}[\nabla_i, [\nabla_j, \nabla_k]] = \sum_{i,j} \eta^{ij}[[\nabla_k, \Phi_i], \Phi_j],$$
$$\sum_{i,j} \eta^{ij}[\nabla_i, [\nabla_j, \Phi_k]] = \sum_{i,j} \eta^{ij}[\Phi_i, [\Phi_j, \Phi_k]],$$

where $\Phi_i = B_i - \sum_j A^j \eta_{ij}$, $\mathcal{A}_i = B_i + \sum_j A^j \eta_{ij}$ and $\nabla_i = \partial_i + \mathcal{A}_i$.

Here B_i are the components of $\mathbf{B} \in \Gamma(T^*M) \otimes \mathfrak{g}$ and A_i are the components of $\mathbf{A} \in \Gamma(TM) \otimes \mathfrak{g}$, constituting the components of the Maurer-Cartan element.

The gauge symmetries correspond to the following transformation of fields:

$$\mathcal{A}_i \to \mathcal{A}_i + \partial_i u + [\mathcal{A}_i, u], \quad \Phi_i \to \Phi_i + [\Phi_i, u].$$

Physical interpretation

Some CFT on the half-plane with action S_0:

$$S = S_0 + \int_{H^+} \phi^{(2)}$$

where

$$\phi^{(2)} = dz \wedge d\bar{z}(\sum_{i,j} \eta^{ij}[b_{-1}, f_i](z)[b_{-1}, f_j](\bar{z}))$$

Deformation of the BRST differential:

$$Q \to Q^{\eta} = Q + \int \phi^{(1)}$$

where $Q\phi^{(2)} = d\phi^{(1)}$:

$$\phi^{(1)} = d\bar{z} \sum_{i,j} \eta^{ij}[b_{-1}, f_i(z)]f_j(\bar{z}) - dz \sum_{i,j} \eta^{ij} f_i(z)[b_{-1}, f_j(\bar{z})]$$

Then

$$R^{\eta} a = P_0 \int_{C_{\epsilon,0}} \phi^{(1)} a$$

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Doubling: $G_\infty \otimes \bar{G}_\infty$ algebra

Naturally there is a homotopy Gerstenhaber algebra on the product. We choose the bracket structure this way:

$$(-1)^{|a_1|}\{a_1, a_2\} = \mathbf{b}^- \mu(a_1, a_2) - \mu(\mathbf{b}^- a_1, a_2) - (-1)^{|a_1|}\mu(a_1 \mathbf{b}^- a_2),$$

where $\mathbf{b}^- = \mathbf{b} - \bar{\mathbf{b}}$ and $\mathcal{Q} = Q + \bar{Q}$.

Consider the case of just G-algebra in the concrete example $(\mathcal{F}^\bullet_{sm}, Q)$:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "0"] & \mathcal{O}(T^{(1,0)}M) \arrow[r, "\frac{1}{2}\mathrm{div}_\omega"] & \mathcal{O}_M & \\
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Maurer-Cartan equation

Maurer-Cartan elements, closed under $\mathbf{b}^-$:

$$\Gamma(T^{(1,0)}(M) \otimes T^{(0,1)}(M)) \oplus \mathcal{O}(T^{(0,1)}(M)) \oplus \mathcal{O}(T^{(1,0)}(M)) \oplus \mathcal{O}_M \oplus \bar{\mathcal{O}}_M$$

Components:$(g, \bar{v}, v, \phi', \bar{\phi}')$.

The Maurer-Cartan equation $\mathcal{Q}\Psi + \frac{1}{2}\{\Psi, \Psi\} = 0$ is equivalent to:

A.Z., Nucl. Phys. B 794 (2008) 370-398; A.Z., Adv. Theor. Math. Phys. 19 (2015) 1249-1275

- $div_\Omega g \in \mathcal{O}(T^{(1,0)}M) \oplus \bar{\mathcal{O}}(T^{(0,1)}M)$,

  where $\log \Omega = -2\Phi_0 = \log \omega - 2(\phi' + \bar{\phi}')$.

- Bivector field $g \in \Gamma(T^{(1,0)}M \otimes T^{(0,1)}M)$ obeys the following equation:

$$[[g, g]] + \mathcal{L}_{div_\Omega(g)} g = 0,$$

  where $\mathcal{L}_{div_\Omega(g)}$ is a Lie derivative with respect to the corresponding vector field and

$$[[g, h]]^{k\bar{l}} \equiv (g^{i\bar{j}}\partial_i\partial_{\bar{j}}h^{k\bar{l}} + h^{i\bar{j}}\partial_i\partial_{\bar{j}}g^{k\bar{l}} - \partial_i g^{k\bar{j}}\partial_{\bar{j}}h^{i\bar{l}} - \partial_i h^{k\bar{j}}\partial_{\bar{j}}g^{i\bar{l}})$$

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"Doubling": Gravity and Double Field Theory

Maurer-Cartan equations are Einstein equations

These are the Einstein equations with the B-field and dilaton:

$$R_{\mu\nu} = \frac{1}{4} H_{\mu}^{\lambda\rho} H_{\nu\lambda\rho} - 2\nabla_\mu \nabla_\nu \Phi,$$
$$\nabla^\mu H_{\mu\nu\rho} - 2(\nabla^\lambda \Phi) H_{\lambda\nu\rho} = 0,$$
$$4(\nabla_\mu \Phi)^2 - 4\nabla_\mu \nabla^\mu \Phi + R + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} = 0,$$

where 3-form $H = dB$, and $R_{\mu\nu}, R$ are Ricci and scalar curvature correspondingly.

with the following constraints:

$$G_{i\bar{k}} = g_{i\bar{k}}, \quad B_{i\bar{k}} = -g_{i\bar{k}}, \quad \Phi = \log\sqrt{g} + \Phi_0,$$
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Physical explanation

- 1980s: D. Friedan, E. Fradkin, A. Tseytlin, ...
  Einstein equations emerge in sigma model

$$S_{so} = \frac{1}{4\pi h}\int_\Sigma (\langle dX \wedge * dX\rangle_G + X^* B) + \int_\Sigma \Phi(X) R^{(2)}(\gamma)\mathrm{vol}_\Sigma$$

  as the conformal invariance conditions. Here $X : \Sigma \to M$, where Σ is a Riemann surface (worldsheet) and M is a Riemannian manifold (target space).

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First order action

Deformation of the first order action described by vertex algebras:

$$S_{fo}^{free} = \frac{1}{2\pi ih}\int_\Sigma(\langle p\wedge\bar\partial X\rangle - \langle\bar p\wedge\partial\bar X\rangle) + \int_\Sigma R^{(2)}(\gamma)\Phi_0(X),$$

where

$$p\in X^*(\Omega^{(1,0)}(M))\otimes\Omega^{(1,0)}(\Sigma),\quad \bar p\in X^*(\Omega^{(0,1)}(M))\otimes\Omega^{(0,1)}(\Sigma),$$

namely:

$$S_{fo} = S_{fo}^{free} - \frac{1}{2\pi ih}\int_\Sigma\langle g, p\wedge\bar p\rangle,$$

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From the path integral perspective:

$$\int[dp][d\bar p][dX][d\bar X]e^{\frac{-1}{2\pi ih}\int_\Sigma(\langle p\wedge\bar\partial X\rangle - \langle\bar p\wedge\partial X\rangle - \langle g, p\wedge\bar p\rangle)+\int_\Sigma R^{(2)}(\gamma)\Phi_0(X)} =$$

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based on computations of A. Tseytlin, A. Schwarz'93.

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Full action and the Beltrami-Courant differential

Let's introduce

$$\mathbb{M} = \begin{pmatrix} g & \mu \\ \bar{\mu} & b \end{pmatrix} \in \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}}),$$

where $\mathcal{E} = T^{(1,0)}M \oplus T^{*(1,0)}M, \quad \bar{\mathcal{E}} = T^{(0,1)}M \oplus T^{*(0,1)}M.$

V. Popov, M. Zeitlin, Phys.Lett. B 163 (1985) 185-188

$$S_{fo} = \frac{1}{2\pi i h} \int_\Sigma \Big(\langle p \wedge \bar{\partial} X \rangle - \langle \bar{p} \wedge \partial \bar{X} \rangle - \langle \bar{v} \wedge \mathbb{M} v \rangle \Big) + \int_\Sigma R^{(2)}(\gamma) \Phi_0(X),$$

A. Losev, A. Marshakov, A.Z., Phys. Lett. B 633 (2-3) (2006) 375-381

where $v = (p, \partial X)$, $\bar{v} = (\bar{p}, \bar{\partial}\bar{X})$.

Integrating over $p, \bar{p}$ we obtain:

$$S^{full}_{so} = \frac{1}{4\pi h} \int_\Sigma (G_{\mu\nu}(X) dX^\mu \wedge * dX^\nu + X^* B) + \int_\Sigma R^{(2)}(\gamma) \Phi(X),$$

where

$$\begin{aligned}
G_{s\bar{k}} &= g_{\bar{i}j} \bar{\mu}^{\bar{i}}_s \mu^j_{\bar{k}} + g_{s\bar{k}} - b_{s\bar{k}}, \quad B_{s\bar{k}} = g_{\bar{i}j} \bar{\mu}^{\bar{i}}_s \mu^j_{\bar{k}} - g_{s\bar{k}} - b_{s\bar{k}} \\
G_{si} &= -g_{i\bar{j}} \bar{\mu}^{\bar{j}}_s - g_{s\bar{j}} \bar{\mu}^{\bar{j}}_i, \quad G_{\bar{s}\bar{i}} = -g_{\bar{s}j} \mu^j_{\bar{i}} - g_{\bar{i}j} \mu^j_{\bar{s}} \\
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$$\mathbb{M} = \begin{pmatrix} g & \mu \\ \bar{\mu} & b \end{pmatrix} \in \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}}),$$

where $\mathcal{E} = T^{(1,0)}M \oplus T^{*(1,0)}M, \quad \bar{\mathcal{E}} = T^{(0,1)}M \oplus T^{*(0,1)}M.$

V. Popov, M. Zeitlin, Phys.Lett. B 163 (1985) 185-188

$$S_{fo} = \frac{1}{2\pi i h} \int_\Sigma \Big(\langle p \wedge \bar{\partial} X \rangle - \langle \bar{p} \wedge \partial \bar{X} \rangle - \langle \bar{v} \wedge \mathbb{M} v \rangle \Big) + \int_\Sigma R^{(2)}(\gamma) \Phi_0(X),$$

A. Losev, A. Marshakov, A.Z., Phys. Lett. B 633 (2-3) (2006) 375-381

where $v = (p, \partial X)$, $\bar{v} = (\bar{p}, \bar{\partial}\bar{X})$.

Integrating over $p, \bar{p}$ we obtain:

$$S_{so}^{full} = \frac{1}{4\pi h} \int_\Sigma (G_{\mu\nu}(X) dX^\mu \wedge * dX^\nu + X^* B) + \int_\Sigma R^{(2)}(\gamma) \Phi(X),$$

where

$$\begin{aligned}
G_{s\bar{k}} &= g_{\bar{i}j} \bar{\mu}^{\bar{i}}_s \mu^j_{\bar{k}} + g_{s\bar{k}} - b_{s\bar{k}}, \quad B_{s\bar{k}} = g_{\bar{i}j} \bar{\mu}^{\bar{i}}_s \mu^j_{\bar{k}} - g_{s\bar{k}} - b_{s\bar{k}} \\
G_{si} &= -g_{i\bar{j}} \bar{\mu}^{\bar{j}}_s - g_{s\bar{j}} \bar{\mu}^{\bar{j}}_i, \quad G_{\bar{s}\bar{i}} = -g_{\bar{s}j} \mu^j_{\bar{i}} - g_{\bar{i}j} \mu^j_{\bar{s}} \\
B_{si} &= g_{s\bar{j}} \bar{\mu}^{\bar{j}}_i - g_{i\bar{j}} \bar{\mu}^{\bar{j}}_s, \quad B_{\bar{s}\bar{i}} = g_{\bar{i}j} \mu^j_{\bar{s}} - g_{\bar{s}j} \mu^j_{\bar{i}}, \quad \Phi = \Phi_0(X) + \log \sqrt{g}
\end{aligned}$$

Symmetries and Beltrami-Courant differential

Diffeomorphism symmetry and $B \to B + d\lambda$ on target space:

Introduce $\alpha \in \Gamma(\mathcal{E} \oplus \bar{\mathcal{E}})$, i.e. $\alpha = (v, \omega, \bar{v}, \bar{\omega})$.

Let $D : \Gamma(\mathcal{E} \oplus \bar{\mathcal{E}}) \to \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$, such that

$$D\alpha = \begin{pmatrix} 0 & -\bar{\partial} v \\ -\partial \bar{v} & \bar{\partial}\omega - \partial\bar{\omega} \end{pmatrix}.$$

Then the transformation of $\mathbb{M}$ is: A.Z., Adv. Theor. Math. Phys. 19 (2015) 1249-1275

$$\mathbb{M} \to \mathbb{M} + D\alpha + \phi_2(\alpha, \mathbb{M}) + \phi_3(\alpha, \mathbb{M}, \mathbb{M}).$$

Decomposing into holomorphic and antiholomorphic parts:

$$\alpha = \sum_J f^J \otimes \bar{b}^J + \sum_K b^K \otimes \bar{f}^K,$$

$$\tilde{\mathbb{M}} = \sum_I a^I \otimes \bar{a}^I,$$

Then

$$\phi_2(\alpha, \tilde{\mathbb{M}}) = \sum_{I,J} [b^J, a^I]_D \otimes \bar{f}^J \bar{a}^I + \sum_{I,K} f^K a^I \otimes [\bar{b}^K, \bar{a}^I]_D,$$

$$\phi_3(\alpha, \mathbb{M}, \mathbb{M}) = \frac{1}{2} \sum_{I,J,K} \langle b^I, a^K \rangle a^J \otimes \bar{a}^J(\bar{f}^I)\bar{a}^K + \frac{1}{2} \sum_{I,J,K} a^J(f^I) a^K \otimes \langle \bar{b}^I, \bar{a}^K \rangle \bar{a}^J.$$

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Full Maurer-Cartan for $L_\infty \subset G_\infty \otimes G_\infty$

Consider

$$(\mathbf{F}^{\bullet}_{b^-}, \mathcal{Q}) = (\mathcal{F}^{\bullet}, Q) \otimes (\bar{\mathcal{F}}^{\bullet}, \bar{Q})|_{b^-=0}$$

Observation:

$$\Psi \to \Psi + \mathcal{Q}\Lambda - \{\Lambda, \Psi\} + \frac{1}{2}\{\Lambda, \Psi, \Psi\},$$

where $\{\cdot, \cdot, \cdot\}$ is a homotopy for Jacobi identity (non-symmetric bracket) reproduces symmetries

$$\mathbb{M} \to \mathbb{M} + D\alpha + \phi_2(\alpha, \mathbb{M}) + \phi_3(\alpha, \mathbb{M}, \mathbb{M})$$

A.Z., Adv. Theor. Math. Phys. 19 (2015) 1249-1275

Conjecture: The corresponding Maurer-Cartan equation gives Einstein equations on G, B, Φ expressed in terms of Beltrami-Courant differential. The symmetries of the Maurer-Cartan equation reproduce mentioned above symmetries of Einstein equations.

Outline

Homotopy algebras related to vertex algebras

Vertex/Courant algebroids and BV double

Flat metric deformation and YM C_∞-algebra

"Doubling": Gravity and Double Field Theory

Another doubling: $BV_\infty^\square \otimes \overline{BV}_\infty^\square$-algebra

Flat metric deformation of the BV double leads to the $BV_\infty^\square$ algebra, where in addition to C_∞ structure there is a relation

$$[Q, \mathbf{b}] = -\Delta = -\sum_{ij} \eta^{ij}\{f_i\{f_j, \cdot\}\}$$

so that the bracket structure satisfies the relations of G_∞-algebra up to certain corrections.

What structure exists on $(\mathcal{F}^\bullet, Q^\eta) \otimes (\mathcal{F}^\bullet, \bar{Q}^\eta)|_{\mathbf{b}_-=0}$?

$$[Q, \frac{1}{2}\mathbf{b}_\pm] = -\Delta_\pm, \quad \Delta_+ = \frac{1}{2}\Delta + \frac{1}{2}\bar{\Delta}, \quad \Delta_- = \frac{1}{2}\Delta - \frac{1}{2}\bar{\Delta}$$

The bracket based on $\mathbf{b}_-$ gives a "kind of" homotopy Lie algebra (see next slide), which leads to the homotopy Lie algebra on the diagonal

$$\delta : M \to M \times \bar{M}$$

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Double Field Theory

For the standard example of Courant algebroid, let's introduce cooordinates and the *T-dual coordinates*:

$$x^i = X^i + \bar{X}^i, \quad \tilde{x}_i = X^i - \bar{X}^i$$

and therefore

$$\Delta_- = -2\sum_i \partial_i \tilde{\partial}^i.$$

Strongly constrained Double Field Theory C. Hull, B. Zwiebach'09:

$$\Delta_- A = 0, \quad \sum_i \partial_i A \tilde{\partial}^i B + \tilde{\partial}^i A \partial_i B = 0$$

for any two fields in $(\mathcal{F}^\bullet, Q^\eta) \otimes (\mathcal{F}^\bullet, \bar{Q}^\eta)|_{\mathbf{b}_-=0}$.

The Maurer-Cartan equation (under above condition) reproduces the action of Double Field Theory (O. Hohm et al.'24), where $\{x^i\}$ and $\{\tilde{x}_i\}$ are the coordinates on the torus and the T-dual torus.

Outline

Homotopy algebras related to vertex algebras

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Thank you!