Geometric wonders of classical and quantum integrable systems

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November 13, 2023

Example of integrable system: harmonic oscillator

2-dimensional phase space with coordinate q and momentum p:

$$\text{Hamiltonian:} \qquad H = \frac{p^2+q^2}{2},$$

$$\text{Poisson bracket:} \qquad \{F,G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial G}{\partial p}\frac{\partial F}{\partial q}.$$

$$\text{Equations of motion:} \qquad \left.\begin{array}{l} \frac{dq}{dt} = \{H,q\} = p \\ \frac{dp}{dt} = \{H,p\} = -q \end{array}\right\} \Rightarrow \frac{d^2q}{dt^2} + q = 0$$

Action-angle variables: polar coordinates in (q,p)-space.

Energy level set: $L_E = \{p^2+q^2 = E\}$ is a circle.

Equations of motion for action-angle variables (H,ϕ):

$$\frac{d\phi}{dt} = \omega, \qquad \frac{dH}{dt} = 0$$

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Classical integrable systems: what are they?

Symplectic $2n$-manifold M: phase space, which has information of coordinates and momenta of a physical system.

Equations of motion:

$$\frac{df}{dt} = \{H, f\}.$$

Integrability: family of conserved quantities: $\{F_i\}_{i=1}^n$:

$$\{F_i, F_j\} = 0, \quad F_1 = H.$$

Liouville-Arnold theorem:

- Compact connected components of $L_c = \{F_i = c_i\}_{i=1}^n$ are diffeomorphic to $\mathbb{T}^n$.
- Existence of action-angle variables $\{I_i\}_{i=1}^n$, $\{\phi^i\}_{i=1}^n$ in the neighborhood of L_c:

$$\frac{d\phi^i}{dt} = \omega^i, \quad \frac{dI_i}{dt} = 0.$$

Finding action/angle variables is a non-trivial problem.

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Explosion of interest in integrable systems: 60s -70s

Integrable soliton equations in (1+1)-dimensions,
e.g. Korteweg-de Vries (KdV) equation:

$$u_t = -u_{xxx} + 6uu_x.$$

C. S. Gardner, J. M. Greene, M. D. Kruskal, R. Miura'67; P. Lax'68;

L. Faddeev, V. Zakharov'71

Lie-theoretic methods through Lax pair formulation:

$$\frac{dL}{dt} = [A, L],$$

where $L = -\partial_x^2 + u(x,t)$ for KdV.

I. Gelfand, L. Dickey'76; V. Drinfeld, V. Sokolov'85

Inverse Scattering Method (ISM):

spectral data of $L \rightarrow$ action-angle variables

At the same time many finite-dimensional multiparticle integrable systems were discovered: Calogero-Moser, Toda, Ruijsennars-Schneider, etc.

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Quantum Integrable models: 70s-80s

Quantum integrability:

$$[H_i, H_j] = 0, \quad H_i : \mathcal{H} \to \mathcal{H}$$

Finding action/angle variables $\to$ simultaneous diagonalization of H_i.

Quantization of (1+1)-models? Put them on the lattice.

Lattice integrable models $\to$ new algebraic structures:

R-matrix and Yang-Baxter equation

accompanied with

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lead to the the discovery of Quantum inverse scattering method (QISM) developed by Leningrad School.

That eventually led to the discovery of quantum groups by Drinfeld and Jimbo.

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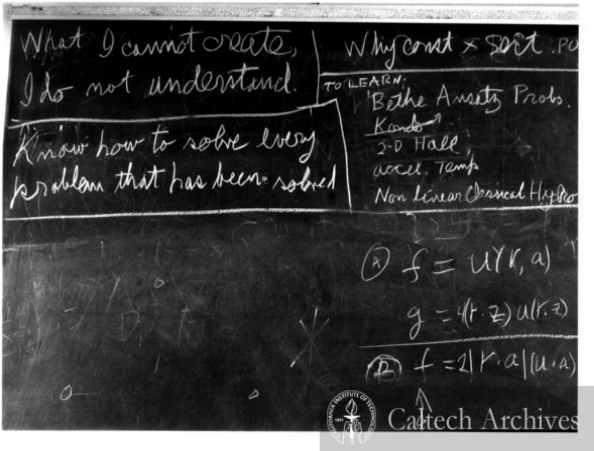

R.P. Feynman: "I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz and how mysteriously they jump out at you and work and you don't know why. I am trying to understand all this better."

90s: Geometrization era begins

- Dubrovin, Givental, Kontsevich, Witten established first relations with integrability in the context of enumerative geometry.

  Notable cases:

  - Witten's conjecture, proven by Kontsevich, relating intersection numbers on the moduli space of curves and the τ-function of KdV model.
  - Givental and collaborators: description of the enumerative geometry of flag varieties (quantum cohomology/quantum K-theory) via classical and quantum multiparticle systems of Toda type.

- Feigin, Frenkel, and collaborators, while studying conformal field theory/representation theory of affine Lie algebras, discovered the relation:

  Connections on $\mathbb{P}^1$ called opers $\leftrightarrow$ Gaudin integrable model

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  Hints from supersymmetric gauge theory $\rightarrow$ geometric realization of quantum integrable models solved by Bethe ansatz.

  Okounkov and his school: enumerative geometry of symplectic resolutions.

2000s – geometric representation theory and integrable models

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In this talk

Fusion of:

- Theory of integrable systems
- Geometric representation theory
- Enumerative geometry
- Supersymmetric gauge theories

More concretely, we will discuss the following:

- Nekrasov-Shatashvili conjectures:

  Bethe ansatz solution for quantum integrable systems encodes enumerative invariants of certain symplectic resolutions: quantum cohomology, quantum K-theory.

- On the other hand, geometrization of the relations in the corresponding rings lead to the deformation of the version of geometric Langlands correspondence by Feigin-Frenkel.

- Applications bring together many parts of theoretical physics and mathematics, such as quantum-classical duality, cluster algebras, and 3D mirror symmetry.

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In collaboration with:

P. Koroteev, P. Pushkar, E. Frenkel, D. Sage, A. Smirnov

- P. Koroteev, A.Z., *The Zoo of Opers and Dualities*, arXiv:2208.08031, to appear in Int. Math. Res. Not.
- P. Koroteev, A. Z., *3d Mirror Symmetry for Instanton Moduli Spaces*, Comm. Math. Phys.'23
- P. Koroteev, A.Z., *q-Opers, QQ-Systems, and Bethe Ansatz II: Generalized Minors*, J. Reine Angew. Math.'23
- P. Koroteev, A.Z.,*Toroidal q-Opers*, J. Inst. Math. Jussieu'23
- P. Koroteev, E. Frenkel, D. Sage, A.Z., *q-Opers, QQ-Systems, and Bethe Ansatz*, to appear in J. Eur. Math. Soc., arXiv:2002.07344
- P. Koroteev, D. Sage, A.Z., *(SL(N),q)-opers, the q-Langlands correspondence, and quantum/classical duality*, Comm. Math. Phys.'21
- P. Koroteev, P. Pushkar, A. Smirnov, A.Z., *Quantum K-theory of Quiver Varieties and Many-Body Systems*, Selecta Math. '21
- P. Koroteev, A.Z., *qKZ/tRS Duality via Quantum K-Theoretic Counts*, Math. Res. Lett.'21
- P. Pushkar, A. Smirnov, A.Z.,*Baxter Q-operator from quantum K-theory*, Adv. Math.'20

Affine algebras and finite-dimensional modules

Let us consider Lie algebra $\mathfrak{g}$.

The associated *loop algebra* is $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}]$ and t is known as *spectral parameter*.

The following representations, known as *evaluation modules*, form a tensor category of $\hat{\mathfrak{g}}$:

$$V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n),$$

where

- V_i are representations of $\mathfrak{g}$
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Quantum groups

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$$U_\hbar(\hat{\mathfrak{g}})$$

are deformations of $U(\hat{\mathfrak{g}})$, with a nontrivial intertwiner $R_{V_1,V_2}(a_1/a_2)$:

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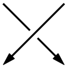
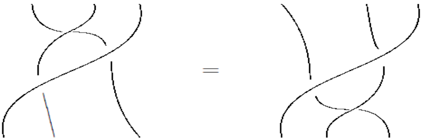


$$V_2(a_2) \otimes V_1(a_1)$$

which is a rational function of a_1, a_2, satisfying Yang-Baxter equation:



The generators of $U_\hbar(\hat{\mathfrak{g}})$ emerge as matrix elements of R-matrices: FRT construction.

Spin chain models and transfer matrices

Physical space:

$$\mathcal{H}_{\rm phys} = V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n).$$

Auxiliary spaces: $W(u)$.

Quantum monodromy matrix:

$$M(u) = (Z \otimes \mathrm{Id})\tilde{R}_{W(u),\mathcal{H}_{phys}} : W(u) \otimes \mathcal{H}_{\rm phys} \to W(u) \otimes \mathcal{H}_{\rm phys}$$

Here $\tilde{R}$ is the R-matrix, composed with permutation operator, $Z \in e^{\mathfrak{h}}$ - diagonal.

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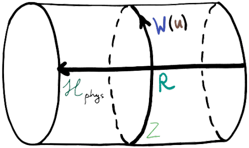
Transfer matrix:



$$T_{W(u)} = \mathrm{Tr}_{W(u)}\Big[M(u)\Big], \quad T_{W(u)} : \mathcal{H}_{\text{phys}} \to \mathcal{H}_{\text{phys}}$$

Integrability:

$$[T_{W'(u')}, T_{W(u)}] = 0$$

follows from Yang-Baxter relation.

Transfer matrices $T_{W(u)}$ generate Bethe algebra:

$$T_{W(u)} = \sum_n u^n I_n, \qquad [I_n, I_m] = 0.$$

Primary goal: diagonalize $\{T_{W(u)}\}$ simultaneously.

$\mathfrak{g} = \mathfrak{sl}(2)$: Heisenberg spin chain

$$\mathcal{H}_{\rm phys} = \mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) \otimes \cdots \otimes \mathbb{C}^2(a_n)$$

$$\text{States}: \quad \uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow$$

Here $\mathbb{C}^2(u)$ stands for 2-dimensional representation of $U_\hbar(\widehat{\mathfrak{sl}}_2)$.

Algebraic Bethe ansatz as a part of QISM:

$$T_{\mathbb{C}^2(u)} = {\rm Tr}_{\mathbb{C}^2(u)}\Big[(Z \otimes {\rm Id})\, \tilde{R}_{\mathbb{C}^2(u),\mathcal{H}_{\rm phys}}\Big] = {\rm Tr}\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = A(u)+D(u)$$

$$A(u), B(u), C(u), D(u) : \mathcal{H}_{\rm phys} \to \mathcal{H}_{\rm phys}$$

Bethe vectors:

$$|0\rangle = \uparrow\uparrow\uparrow \dots \uparrow\uparrow\uparrow$$

$$\{B(x_1)...B(x_k)|0\rangle; \quad C(x)|0\rangle = 0\}$$

Commutation relations between A, B, C, D: from Yang-Baxter equation.

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The eigenvalues are symmetric functions of Bethe roots $\{x_i\}$:

$$\prod_{j=1}^{n} \frac{x_i - a_j}{\hbar a_j - x_i} = z\,\hbar^{-n/2} \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{x_i \hbar - x_j}{x_i - x_j \hbar}\,, \quad i = 1, \ldots, k,$$

Special element in the Bethe algebra: Q-operator.

The eigenvalues $\mathcal{Q}(u)$ of the Q-operator are the generating functions for the elementary symmetric functions of Bethe roots:

$$\mathcal{Q}(u) = \prod_{i=1}^{k}(u - x_i)$$

A real challenge is to describe representation-theoretic meaning of Q-operator for general $\mathfrak{g}$ (possibly infinite-dimensional).

Bethe equations and Q-operator

The eigenvalues are symmetric functions of Bethe roots $\{x_i\}$:

$$\prod_{j=1}^{n} \frac{x_i - a_j}{\hbar a_j - x_i} = z\, \hbar^{-n/2} \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{x_i \hbar - x_j}{x_i - x_j \hbar}\,, \quad i = 1, \dots, k,$$

Special element in the Bethe algebra: Q-operator.

The eigenvalues $\mathcal{Q}(u)$ of the Q-operator are the generating functions for the elementary symmetric functions of Bethe roots:

$$\mathcal{Q}(u) = \prod_{i=1}^{k} (u - x_i)$$

A real challenge is to describe representation-theoretic meaning of Q-operator for general $\mathfrak{g}$ (possibly infinite-dimensional).

Quantum Knizhnik-Zamolodchikov (Frenkel-Reshetikhin) equations:

$$\Psi(a_1,\dots,qa_k,\dots,a_{i_n},\{z_i\}) = H_k^{(q)}\Psi(a_1,\dots,,a_n,\{z_i\}),$$
$$+$$
$$\text{commuting } q-\text{difference equations in } Z-\text{components (dynamical)}$$

- Ψ takes values in $\mathcal{H}_{\text{phys}} = V_1(a_1)\otimes V_2(a_2)\otimes\cdots\otimes V_n(a_n)$,
- operators $\{H_i^{(q)}\}$ are expressed in terms of products of R-matrices and twist parameter, e.g.,

$$\Psi(qa_1,\dots,a_n,\{z_i\}) = (Z\otimes 1\otimes\cdots\otimes 1)R_{V_1,V_n}\dots R_{V_1,V_2}\Psi$$

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Solutions to qKZ

Types of solutions:

- jointly analytic in a chamber of $\{a_i\}$:

  conformal blocks for $U_\hbar(\widehat{\mathfrak{g}})$, i.e. products of Intertwining operators for centrally extended $U_\hbar(\widehat{\mathfrak{g}})$, where central charge is related to q.

- jointly analytic in $\{z_i\}$:

  conformal blocks for deformed W-algebra: $W_{q,t}({}^L\mathfrak{g})$.

The relationship between this solutions is an essential part of

Quantum q-Langlands correspondence

M. Aganagic, E. Frenkel, A. Okounkov'17

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Geometrization: Nakajima quiver varieties

$\mathcal{M}$: affine space constructed via quiver (oriented graph) which is the representation of

$$G = GL(v_1) \times GL(v_2) \cdots \times GL(v_{\mathrm{rank}(\mathfrak{g})}),$$

We build $\mathcal{M}$ as a direct sum of:

- $\oplus_i Hom(V_i, W_i)$, where $dim(V_i) = v_i$, W_i is known as framing
- $\oplus_{i \to j} Hom(V_i, V_j)$

$T^*\mathcal{M}$: phase space with Poisson bracket.

Nakajima quiver variety is a "clever" quotient, called algebraic symplectic reduction:

$$N = T^*\mathcal{M} /\!\!/\!\!/\!\!/ G$$

Nakajima, Varagnolo-Vasserot, Maulik-Okounkov: Localized equivariant cohomology/K-theory of N has the structure of weight subspace for representations of $Y_\hbar(\mathfrak{g})/U_\hbar(\mathfrak{g})$. Weight is determined by a collection $v_1, \dots, v_{\mathrm{rank}(\mathfrak{g})}$

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Example: $N_{k,n} = T^*Gr(k,n)$

Let $dim(V) = k$, $dim(W) = n$,

$$\mathcal{M} = Hom(V, W),$$

$$T^*\mathcal{M} = Hom(V, W) \oplus Hom(W, V)$$

The moment map:

$$\mu : T^*\mathcal{M} \to End(V); \quad (A, B) \mapsto BA$$

generates the $GL(V)$-action.

Then

$$T^*Gr(k,n) = N_{k,n} = T^*\mathcal{M}////GL(V) = \mu^{-1}(0)//GL(V) = \mu^{-1}(0)_{ss}/GL(V).$$

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Equivariant K-theory

Torus action:

$$A = \mathbb{C}^\times_{a_1} \times \cdots \times \mathbb{C}^\times_{a_n} \circlearrowleft W,$$
$$\text{Full torus}: \quad T = A \times \mathbb{C}^\times_\hbar, \ \text{where } \mathbb{C}^\times_\hbar \text{ scales cotangent directions}$$

Tautological bundles:

$$\mathcal{V} = T^*\mathcal{M} \times V /\!\!/\!\!/\!\!/ GL(V), \quad \mathcal{W} = T^*\mathcal{M} \times W /\!\!/\!\!/\!\!/ GL(V)$$

For any $\tau \in S(x_1^{\pm 1}, x_2^{\pm 1}, \dots x_k^{\pm 1})$ we have:

$$\tau(\mathcal{V}) = T^*\mathcal{M} \times \tau(V) /\!\!/\!\!/\!\!/ GL(V)$$

Example: $\tau(V) = V^{\otimes 2} - \Lambda^3 V^*$ corresponds to

$$\tau(x_1, \dots, x_k) = (x_1 + \cdots + x_k)^2 - \sum_{1 \le i_1 \le i_2 \le i_3 \le k} x_{i_1}^{-1} x_{i_2}^{-1} x_{i_3}^{-1}.$$

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Tori, Fixed points and Bethe roots

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Fixed points: $\mathbf{p} = \{s_1, \dots, s_k\} \in \{a_1, \dots, a_n\}$

Let $N(n) = \sqcup_k N_{k,n} = \sqcup_k T^* Gr(k,n)$.

Localized K-theory: $K_T(N(n))_{loc}$ as a $\mathbb{Q}(a_1, \dots, a_n, \hbar)$-module is identified with:

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"Classical" Bethe equations: The eigenvalues of the operators of multiplication by τ are $\tau(x_1, \cdots, x_k)$ evaluated at the solutions of the following equations:

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Quasimaps

Quasimap f: $\mathcal{C} = \mathbb{P}^1 \dashrightarrow N_{k,n}$ is the following collection of data:

- vector bundle $\mathscr{V}$ on $\mathbb{P}^1$ of rank k.
- section $f \in H^0(\mathbb{P}^1, \mathscr{M} \oplus \mathscr{M}^* \otimes \hbar)$, satisfying the condition $\mu = 0$, where $\mathscr{M} = Hom(\mathscr{V}, \mathscr{W})$, so that $\mathscr{W}$ is a trivial bundle of rank n.

$$ev_p(f) = f(p) \in [\mu^{-1}(0)/GL(V)] \supset N_{k,n}$$

Quasimap is *stable* if $f(p) \in N_{k,n}$ for all but finitely many points, known as *singularities* of quasimap.

For the moduli space of stable quasimaps

$$QM(N_{k,n})$$

only $\mathscr{V}$ and f vary, while $\mathcal{C}$ and $\mathscr{W}$ remain the same.

$$deg(f) := deg(\mathscr{V}), \quad QM(N_{k,n}) = \sqcup_{d \geq 0} QM^d(N_{k,n}).$$

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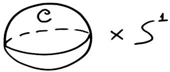
Nekrasov-Shatashvili ideas

In 2009 Nekrasov and Shatashvili looked at 3d SUSY gauge theories on $\mathcal{C}\times S^1$:



Gauge field theory with gauge group $G=\times_{i=1}^{rank\,\mathfrak{g}}U(v_i)$ defined by a certain action functional $S(\phi_{\{\alpha\}},A_{\{i\}})$.

- $A_{\{i\}}$: $U(v_i)$-connections (gauge fields)
- $\phi_{\{\alpha\}}$: sections of associated vector $U(v_i)$-bundles, corresponding to the quiver data (matter fields)

Physicists compute path integrals:

$$\langle F\rangle=\int[d\phi_{\{\alpha\}}][dA_{\{i\}}]e^{-S(\phi_{\{\alpha\}},A_{\{i\}})}F(\phi_{\{\alpha\}},A_{\{i\}})$$

Minima of S: Moduli of Higgs vacua $\longleftrightarrow$ Nakajima quiver variety:

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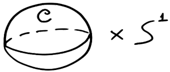

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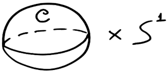

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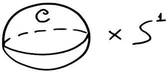


Gauge field theory with gauge group $G = \times_{i=1}^{rank\,\mathfrak{g}} U(v_i)$ defined by a certain action functional $S(\phi_{\{\alpha\}}, A_{\{i\}})$.

- $A_{\{i\}}$: $U(v_i)$-connections (gauge fields)
- $\phi_{\{\alpha\}}$: sections of associated vector $U(v_i)$-bundles, corresponding to the quiver data (matter fields)

Physicists compute path integrals:

$$\langle F \rangle = \int [d\phi_{\{\alpha\}}][dA_{\{i\}}] e^{-S(\phi_{\{\alpha\}}, A_{\{i\}})} F(\phi_{\{\alpha\}}, A_{\{i\}})$$

Minima of S: Moduli of Higgs vacua $\longleftrightarrow$ Nakajima quiver variety:

$$T^*\mathcal{M} /\!\!/\!\!/\!\!/ G = \mu^{-1}(0) /\!\!/ G = N$$

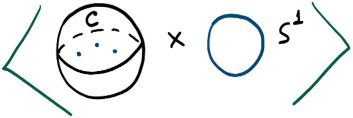
where $\mu = 0$ is a momentum map (low energy configuration) condition.

$\langle F \rangle$ corresponds to weighted K-theoretic counts of quasimaps:

Euler characteristics of equivariant pushforwards of sheaves of quasimap moduli space $QM(N_{k,n})$.

The weight (Kähler) parameter is $Z^{\deg(f)}$, which is exactly twist parameter Z we encountered before.

"Line operators"–traces of holonomy of gauge fields generate quantum K-theory ring, so that the structure constants of algebra are given by:



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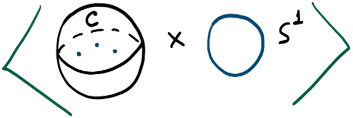


Conjecture of Nekrasov and Shatashvili '09 (through 3D gauge theory):

$$\text{Quantum equivariant K}-\text{theory ring of quiver variety} =$$

$$\text{Bethe algebra of related spin chain system}$$

Nekrasov-Shatashvili conjecture and enumerative computations

Conjecture of Nekrasov and Shatashvili '09 (through 3D gauge theory):

$$\text{Quantum equivariant K}-\text{theory ring of Nakajima variety} =$$

$$\text{Bethe algebra of related spin chain system}$$

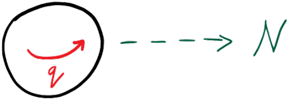
Okounkov'15:

$$\text{q}-\text{difference equations for counting (vertex) functions} =$$
$$\text{qKZ equations} + \text{ dynamical equations}$$

What are we counting?

Quasimaps to quiver varieties:

$$\begin{cases} q, \{a_i\} \rightarrow \text{equivariant parameters} \\ \{z_i\} \rightarrow \text{“counting” (Kähler) parameters} \end{cases}$$



Quantum tautological classes for $T^*Gr(k,n)$

Deformation of the product: $A \circledast B = A\otimes B + \sum_{d=1}^{\infty} A \otimes_d B\, z^d$.

Quantum tautological classes – deformations of $\tau = T^*\mathcal{M} \times \tau(V)$:

$$\hat{\tau}(z) = \tau + \sum_{d=1}^{\infty} \tau_d z^d \in K_T(N(n))[[z]]$$

Theorem.

- The eigenvalues of operators of quantum multiplication by $\hat{\tau}(z)$ are given by the values of the corresponding Laurent polynomials $\tau(x_1,\dots,x_k)$ evaluated at the solutions of Bethe equations:

$$\prod_{j=1}^{n} \frac{x_i - a_j}{\hbar a_j - x_i} = z\,\hbar^{-n/2} \prod_{\substack{j=1\\ j\neq i}}^{k} \frac{x_i\hbar - x_j}{x_i - x_j\hbar}, \quad i = 1,\dots,k,$$

- Baxter Q-operator: $Q(u) = \sum_{i=1}^{k} (-1)^i u^{k-i} \left[\Lambda^i V\right](z)\circledast$

P. Pushkar, A. Smirnov, A.Z.,
Baxter Q-operator from quantum K-theory, Adv. Math. '20

Further developments

- Generalization to type A_n, also connection to classical many body-systems for T^*Fl.

  P. Koroteev, P. Pushkar, A. Smirnov, A.Z.,
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QQ-system for T^*Gr

Short exact sequence of bundles:

$$0 \to V \to W \to \hbar \otimes V^\vee \to 0$$

$$Q(u) = \sum_{i=1}^{k} (-1)^i u^{k-i} \left[\Lambda^i V\right](z) \circledast$$

$$\widetilde{Q}(u) = \sum_{i=1}^{k} (-1)^i u^{k-i} \left[\Lambda^i V^\vee\right](z) \circledast$$

Eigenvalues of these two operators are related via QQ-system:

$$\widetilde{Q}(\hbar u) Q(u) - z Q(\hbar u) \widetilde{Q}(u) = \prod_i (u - a_i)$$

Equivalent to Bethe ansatz equations.

$(G, \hbar)$-connections

$$M_\hbar : \mathbb{P}^1 \to \mathbb{P}^1, \text{ such that } u \to \hbar u.$$

Let $\mathcal{F}_G$ be a G-bundle over $\mathbb{P}^1$:

- G - simple simply connected Lie group associated to Lie algebra $\mathfrak{g}$.
- $\mathcal{F}_G^\hbar$ is a pushforward w.r.t. $M_\hbar$.

$(G, \hbar)$-connection: a meromorphic section of $Hom_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{F}_G, \mathcal{F}_G^\hbar)$.

Locally:

$\hbar$-gauge transformations of $(G, \hbar)$-connection:

$$A(u) \to g(\hbar u)A(u)g^{-1}(u),$$

where $A(u) \in G(u) = G(\mathbb{C}(u))$.

Compare it with standard gauge transformations:

$$\partial_u + A(u) \to g(u)(\partial_u + A(u))g^{-1}(u),$$

where $A(u) \in \mathfrak{g}(u)$.

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Miura $(G, \hbar)$-opers

Miura $(G, \hbar)$-oper is a quadruple : $(\mathcal{F}_G, A, \mathcal{F}_{B_+}, \mathcal{F}_{B_-})$, such that:

- A is $(G, \hbar)$ - connection
- Oper condition: $\mathcal{F}_{B_+}$: A lies in the Coxeter cell $B_+ c B_+$
- Miura condition: $\mathcal{F}_{B_-}$: preserved by A

$(G, \hbar)$-oper is Z-twisted if it is gauge equivalent to $Z \in H$, namely

$$A(u) = v(\hbar u) Z v^{-1}(u), \text{ where } Z = \prod_i \zeta_i^{\check{\alpha}_i} \in H, \; v(u) \in G(u).$$

Example: $SL(r+1)$

$$A \in \begin{bmatrix} * & 0 & 0 & \cdots & 0 \\ * & * & 0 & \cdots & 0 \\ 0 & * & * & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * & * \end{bmatrix} \qquad Z - \text{diagonal}$$

Regular singularities: $\{\Lambda_i(u)\}_{i=1}^r$–polynomials on subdiagonal.

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There is a one-to-one correspondence between the set of nondegenerate Z-twisted Miura $(G,\hbar)$-opers and the set of nondegenerate polynomial solutions of the QQ-system:

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where $\widetilde{\xi}_i=\zeta_i^{-1}\prod_{j>i}\zeta_j^{-a_{ji}}$, $\xi_i=\zeta_i\prod_{j<i}\zeta_j^{a_{ji}}$.

P. Koroteev, D. Sage, A.Z.,

(SL(N),q) -opers, the q-Langlands correspondence, and quantum/classical duality, Comm. Math. Phys. '21

E. Frenkel, P. Koroteev, D. Sage, A.Z.,

q-Opers, QQ-Systems, and Bethe Ansatz, to appear in J. Eur. Math. Soc., arXiv:2002.07344

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