

# Einstein equations, Beltrami-Courant differentials and Homotopy Gerstenhaber algebras

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May 25, 2016



## Outline

Sigma-models and  
conformal invariance  
conditions

Beltrami-Courant  
differential

Vertex/Courant  
algebroids,  
 $G_\infty$ -algebras and  
quasiclassical limit

Einstein Equations

Sigma-models and conformal invariance conditions

Beltrami-Courant differential and first order sigma-models

Vertex/Courant algebroids,  $G_\infty$ -algebras and quasiclassical limit

Einstein Equations from  $G_\infty$ -algebras

# Sigma-models and conformal invariance conditions

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Sigma-models for string theory in curved spacetimes:

Let  $X : \Sigma \rightarrow M$ , where  $\Sigma$  is a compact Riemann surface (worldsheet) and  $M$  is a Riemannian manifold (target space).

Action functional of sigma model:

$$S_{so} = \frac{1}{4\pi h} \int_{\Sigma} (G_{\mu\nu}(X) dX^{\mu} \wedge *dX^{\nu} + X^* B)$$

where  $G$  is a metric on  $M$ ,  $B$  is a 2-form on  $M$ .

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- i) conformal symmetry on the worldsheet,
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On the quantum level one can add one more term to the action (due to E. Fradkin and A. Tseytlin):

$$S_{so} \rightarrow S_{so}^\Phi = S_{so} + \int_\Sigma \Phi(X) R^{(2)}(\gamma) \text{vol}_\Sigma,$$

where function  $\Phi$  is called *dilaton*,  $\gamma$  is a metric on  $\Sigma$ .

In order to make sense of path integral

$$Z = \int DX \ e^{-S_{so}^\Phi(X, \gamma)}$$

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## Conformal invariance conditions

$$\mu \frac{d}{d\mu} G_{\mu\nu} = \beta_{\mu\nu}^G(G, B, \Phi, h) = 0, \quad \mu \frac{d}{d\mu} B_{\mu\nu} = \beta_{\mu\nu}^B(G, B, \Phi, h) = 0,$$

$$\mu \frac{d}{d\mu} \Phi = \beta^\Phi(G, B, \Phi, h) = 0$$

at the level  $h^0$  turn out to be Einstein Equations with 2-form field  $B$  and dilaton  $\Phi$ :

$$R_{\mu\nu} = \frac{1}{4} H_\mu^{\lambda\rho} H_{\nu\lambda\rho} - 2\nabla_\mu \nabla_\nu \Phi,$$

$$\nabla^\mu H_{\mu\nu\rho} - 2(\nabla^\lambda \Phi) H_{\lambda\nu\rho} = 0,$$

$$4(\nabla_\mu \Phi)^2 - 4\nabla_\mu \nabla^\mu \Phi + R + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} = 0,$$

where 3-form  $H = dB$ , and  $R_{\mu\nu}$ ,  $R$  are Ricci and scalar curvature correspondingly.



In the early days of string theory:

Linearized Einstein Equations and their symmetries:

$(G_{\mu\nu} = \eta_{\mu\nu} + s_{\mu\nu}, B_{\mu\nu} = b_{\mu\nu}, \Phi = \phi)$ :

$$Q^\eta \Psi(s, b, \phi) = 0, \quad \Psi^s(s, b, \phi) \rightarrow \Psi(s, b, \phi) + Q^\eta \Lambda$$

in a semi-infinite complex associated to Virasoro module of Hilbert space of states for the "free" theory, associated to flat metric.

It was conjectured (A. Sen, B. Zwiebach,...) in the early 90s that Einstein equations with  $h$ -corrections are Generalized Maurer-Cartan (GMC) Equations:

$$Q^\eta \Psi + \frac{1}{2}[\Psi, \Psi]_h + \frac{1}{3!}[\Psi, \Psi, \Psi]_h + \dots = 0$$

$$\Psi \rightarrow \Psi + Q^\eta \Lambda + [\Psi, \Lambda]_h + \frac{1}{2}[\Psi, \Psi, \Lambda]_h + \dots,$$

where  $[\cdot, \cdot, \dots, \cdot]_h$  operations, together with differential  $Q$  satisfy certain bilinear relations and generate  $L_\infty$ -algebra ( $L$  stands for Lie).

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## In this talk:

### i) Introducing complex structure:

Proper chiral "free action"  $\rightarrow$  sheaves of vertex algebras/vertex algebroids.

Metric,  $B$ -field  $\rightarrow$  Beltrami-Courant differential.

### ii) Vertex algebroids $\rightarrow G_\infty$ -algebras ( $G$ stands for Gerstenhaber).

Quasiclassical limit:

vertex algebroid  $\rightarrow$  Courant algebroid,  $G_\infty$  algebra is truncated.

### iii) Einstein equations and their $\hbar$ -corrections via Generalized Maurer-Cartan equation for $L_\infty$ -subalgebra of $G_\infty \otimes \bar{G}_\infty$ .

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# First order version of sigma-model action

We start from the action functional:

$$S_0 = \frac{1}{2\pi i h} \int_{\Sigma} \mathcal{L}_0, \quad \mathcal{L}_0 = \langle p \wedge \bar{\partial} X \rangle - \langle \bar{p} \wedge \partial X \rangle,$$

where  $p, \bar{p}$  are sections of  $X^*(\Omega^{(1,0)}(M)) \otimes \Omega^{(1,0)}(\Sigma)$ ,  
 $X^*(\Omega^{(0,1)}(M)) \otimes \Omega^{(0,1)}(\Sigma)$  correspondingly.

Infinitesimal local symmetries:

$$\mathcal{L}_0 \rightarrow \mathcal{L}_0 + d\xi$$

For holomorphic transformations we have:

$$\begin{aligned} X^i &\rightarrow X^i - v^i(X), & X^{\bar{i}} &\rightarrow X^{\bar{i}} - \bar{v}^{\bar{i}}(\bar{X}), \\ p_i &\rightarrow p_i + \partial_i v^k p_k, & p_{\bar{i}} &\rightarrow p_{\bar{i}} + \partial_{\bar{i}} \bar{v}^{\bar{k}} p_{\bar{k}} \\ p_i &\rightarrow p_i - \partial X^k (\partial_k \omega_i - \partial_i \omega_k), & p_{\bar{i}} &\rightarrow p_{\bar{i}} - \bar{\partial} X^{\bar{k}} (\partial_{\bar{k}} \omega_{\bar{i}} - \partial_{\bar{i}} \omega_{\bar{k}}). \end{aligned}$$

Not invariant under general diffeomorphisms, i.e.

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It is necessary to add extra terms:

$$\delta \mathcal{L}_\mu = -\langle \mu, p \wedge \bar{\partial} X \rangle - \langle \bar{\mu}, \partial X \wedge \bar{p} \rangle,$$

where  $\mu \in \Gamma(T^{(1,0)}M \otimes T^{*(0,1)}(M))$ ,  $\bar{\mu} \in \Gamma(T^{(0,1)}M \otimes T^{*(1,0)}(M))$ , so that:  $\mu \rightarrow \mu - \bar{\partial} v + \dots$ ,  $\bar{\mu} \rightarrow \bar{\mu} - \partial \bar{v} + \dots$

Continuing the procedure:

$$\begin{aligned} \tilde{\mathcal{L}} = & \langle p \wedge \bar{\partial} X \rangle - \langle \bar{p} \wedge \partial X \rangle - \\ & \langle \mu, p \wedge \bar{\partial} X \rangle - \langle \bar{\mu}, \partial X \wedge \bar{p} \rangle - \langle b, \partial X \wedge \bar{\partial} X \rangle, \end{aligned}$$

where

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so that the transformations of  $X$ - and  $p$ - fields are:

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Similarly, for the 1-form transformation we obtain:

$$b_{i\bar{j}} \rightarrow b_{i\bar{j}} + \partial_{\bar{j}}\omega_i - \partial_i\omega_{\bar{j}} + \mu_{\bar{j}}^i(\partial_i\omega_k - \partial_k\omega_i) + \\ \bar{\mu}_{\bar{i}}^{\bar{s}}(\partial_{\bar{j}}\omega_{\bar{s}} - \partial_{\bar{s}}\omega_{\bar{j}}) + \bar{\mu}_{\bar{j}}^{\bar{i}}\mu_{\bar{k}}^{\bar{s}}(\partial_s\omega_{\bar{i}} - \partial_{\bar{i}}\omega_s)$$

and

$$p_i \rightarrow p_i - \partial X^k(\partial_k\omega_i - \partial_i\omega_k) - \partial_{\bar{r}}\omega_i\partial X^{\bar{r}} - \bar{\mu}_{\bar{k}}^{\bar{s}}\partial_i\omega_{\bar{s}}\partial X^k, \\ p_{\bar{i}} \rightarrow p_{\bar{i}} - \bar{\partial} X^{\bar{k}}(\partial_{\bar{k}}\omega_{\bar{i}} - \partial_{\bar{i}}\omega_{\bar{k}}) - \partial_r\omega_{\bar{i}}\bar{\partial} X^r - \mu_k^s\partial_i\omega_s\bar{\partial} X^{\bar{k}}.$$

For simplicity:

$$E = TM \oplus T^*M, \quad E = \mathcal{E} \oplus \bar{\mathcal{E}}, \\ \mathcal{E} = T^{(1,0)}M \oplus T^{*(1,0)}M, \quad \bar{\mathcal{E}} = T^{(0,1)}M \oplus T^{*(0,1)}M.$$



Let  $\tilde{M} \in \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$ , such that

$$\tilde{M} = \begin{pmatrix} 0 & \mu \\ \bar{\mu} & b \end{pmatrix}.$$

Introduce  $\alpha \in \Gamma(E)$ , i.e.  $\alpha = (v, \bar{v}, \omega, \bar{\omega})$ . Let  $D : \Gamma(E) \rightarrow \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$ , such that

$$D\alpha = \begin{pmatrix} 0 & \bar{\partial}v \\ \partial\bar{v} & \partial\bar{\omega} - \bar{\partial}\omega \end{pmatrix}.$$

Then the transformation of  $\tilde{M}$  is:

$$\tilde{M} \rightarrow \tilde{M} - D\alpha + \phi_1(\alpha, \tilde{M}) + \phi_2(\alpha, \tilde{M}, \tilde{M}).$$

Let us describe  $\phi_1, \phi_2$  algebraically. In order to do that we need to pass to jet bundles, i.e.

$$\alpha \in J^\infty(\mathcal{O}_M) \otimes J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}})) \oplus J^\infty(\mathcal{O}(\mathcal{E})) \otimes J^\infty(\bar{\mathcal{O}}_M),$$

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One can write formally:

$$\alpha = \sum_J f^J \otimes \bar{b}^J + \sum_K b^K \otimes \bar{f}^K,$$

$$\tilde{\mathbb{M}} = \sum_I a^I \otimes \bar{a}^I,$$

where  $a^I, b^J \in J^\infty(\mathcal{O}(\mathcal{E}))$ ,  $f^I \in J^\infty(\mathcal{O}_M)$  and  $\bar{a}^I, \bar{b}^J \in J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}}))$ ,  $\bar{f}^I \in J^\infty(\bar{\mathcal{O}}_M)$ . Then

$$\phi_1(\alpha, \tilde{\mathbb{M}}) = \sum_{I,J} [b^J, a^I]_D \otimes \bar{f}^J \bar{a}^I + \sum_{I,K} f^K a^I \otimes [\bar{b}^K, \bar{a}^I]_D,$$

where  $[\cdot, \cdot]_D$  is a *Dorfman bracket*:

$$[v_1, v_2]_D = [v_1, v_2]^{Lie}, \quad [v, \omega]_D = L_v \omega,$$

$$[\omega, v]_D = -i_v d\omega, \quad [\omega_1, \omega_2]_D = 0.$$

Courant bracket is the antisymmetrized version of  $[\cdot, \cdot]_D$ .

Similarly:

$$\phi_2(\alpha, \tilde{\mathbb{M}}, \tilde{\mathbb{M}}) = \tilde{\mathbb{M}} \cdot D\alpha \cdot \tilde{\mathbb{M}}$$

$$\frac{1}{2} \sum_{I,J,K} \langle b^I, a^K \rangle a^J \otimes \bar{a}^J (\bar{f}^I) \bar{a}^K + \frac{1}{2} \sum_{I,J,K} a^J (f^I) a^K \otimes \langle \bar{b}^I, \bar{a}^K \rangle \bar{a}^J.$$

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$$B \rightarrow B - 2d\omega$$

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Relation to standard second order sigma-model: Let us fill in 0 in  $\tilde{\mathbb{M}}$ :

$$\mathbb{M} = \begin{pmatrix} g & \mu \\ \bar{\mu} & b \end{pmatrix}.$$

$$S_{fo} = \frac{1}{2\pi i\hbar} \int_{\Sigma} (\langle p \wedge \bar{\partial} X \rangle - \langle \bar{p} \wedge \partial X \rangle - \\ - \langle g, p \wedge \bar{p} \rangle - \langle \mu, p \wedge \bar{\partial} X \rangle - \langle \bar{\mu}, \bar{p} \wedge \partial X \rangle - \langle b, \partial X \wedge \bar{\partial} X \rangle).$$

V.N. Popov, M.G. Zeitlin, Phys.Lett. B 163 (1985) 185, A. Losev, A. Marshakov, A.Z., Phys. Lett. B 633 (2006) 375

Same formulas express symmetries. If  $\{g^{i\bar{j}}\}$  is nondegenerate, then :

$$S_{so} = \frac{1}{4\pi\hbar} \int_{\Sigma} (G_{\mu\nu}(X) dX^{\mu} \wedge *dX^{\nu} + X^* B),$$

$$G_{s\bar{k}} = g_{i\bar{j}} \bar{\mu}_s^i \mu_k^j + g_{s\bar{k}} - b_{s\bar{k}}, \quad B_{s\bar{k}} = g_{i\bar{j}} \bar{\mu}_s^i \mu_k^j - g_{s\bar{k}} - b_{s\bar{k}}$$

$$G_{si} = -g_{i\bar{j}} \bar{\mu}_s^j - g_{s\bar{j}} \bar{\mu}_i^j, \quad G_{\bar{s}\bar{i}} = -g_{s\bar{j}} \mu_i^j - g_{i\bar{j}} \mu_s^j$$

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Symmetries  $\mathbb{M} \rightarrow \mathbb{M} - D\alpha + \phi_1(\alpha, \mathbb{M}) + \phi_2(\alpha, \mathbb{M}, \mathbb{M})$  are equivalent to:

A.Z., Adv. Theor. Math. Phys. (2015), to appear

$$G \rightarrow G - L_v G, \quad B \rightarrow B - L_v B$$

$$B \rightarrow B - 2d\omega$$

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$$B_{si} = g_{s\bar{i}}\bar{\mu}_i^j - g_{ij}\bar{\mu}_s^j, \quad B_{\bar{s}\bar{i}} = g_{\bar{i}\bar{j}}\mu_{\bar{s}}^j - g_{\bar{s}\bar{j}}\mu_{\bar{i}}^j.$$

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The quantum theory, corresponding to the chiral part of the free first order Lagrangian  $\mathcal{L}_0$  is described (under certain constraints on  $M$ ) via sheaves of VOA on  $M$  (V. Gorbounov, F. Malikov, V. Schechtman, A. Vaintrob).

On the open set  $U$  of  $M$  we have VOA:

$$V = \sum_{n=0}^{\infty} V_n, \quad Y : V \rightarrow \text{End}(V)[[z, z^{-1}]],$$

generated by:

$$[X^i(z), p_j(w)] = h \delta_j^i \delta(z-w), \quad i, j = 1, 2, \dots, D/2$$
$$X^i(z) = \sum_{r \in \mathbb{Z}} X_r^i z^{-r}, \quad p_j(z) = \sum_{s \in \mathbb{Z}} p_{j,s} z^{-s-1} \in \text{End}(V)[[z, z^{-1}]],$$

so that

$$V = \text{Span}\{p_{j_1, -s_1}, \dots, p_{j_k, -s_k} X_{-r_1}^{i_1} \dots X_{-r_l}^{i_l}\} \otimes F(U) \otimes \mathbb{C}[h, h^{-1}],$$
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The Virasoro element is:

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = \frac{1}{h} : \langle p(z) \partial X(z) \rangle : + \partial^2 \phi'(X(z)).$$

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{D}{12}(n^3 - n)\delta_{n,-m}$$

corresponding to correction:

$$\mathcal{L}_0 \rightarrow \mathcal{L}_{\phi'} = \langle p \wedge \bar{\partial} X \rangle - 2\pi i h R^{(2)}(\gamma) \phi'(X)$$

where  $\phi' = \log \Omega$ , where  $\Omega(X) dX^1 \wedge \cdots \wedge dX^n$  is a holomorphic volume form, i.e. for globally defined  $T(z)$ ,  $M$  has to be Calabi-Yau.

The space  $V$  is a lowest weight module for the above Virasoro algebra.

$V$  can be reproduced from  $V_0$  and  $V_1$  as a *vertex envelope*. The structure of vertex algebra imposes algebraic relations on  $V_0 \oplus V_1$  giving it a structure of a *vertex algebroid*.

In our case:  $V_0 \rightarrow \mathcal{O}_M^h = \mathcal{O}_M \otimes \mathbb{C}[h, h^{-1}]$ ,  
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The Virasoro element is:

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = \frac{1}{h} : \langle p(z) \partial X(z) \rangle : + \partial^2 \phi'(X(z)).$$

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{D}{12}(n^3 - n)\delta_{n,-m}$$

corresponding to correction:

$$\mathcal{L}_0 \rightarrow \mathcal{L}_{\phi'} = \langle p \wedge \bar{\partial} X \rangle - 2\pi i h R^{(2)}(\gamma) \phi'(X)$$

where  $\phi' = \log \Omega$ , where  $\Omega(X) dX^1 \wedge \cdots \wedge dX^n$  is a holomorphic volume form, i.e. for globally defined  $T(z)$ ,  $M$  has to be Calabi-Yau.

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$$f * (g * v) - (fg) * v = \pi(v)(f) * \partial(g) + \pi(v)(g) * \partial(f),$$

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where  $v, v_1, v_2 \in \mathcal{V}^h$ ,  $f, g \in \mathcal{O}_M^h$ .

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i)  $\mathbb{C}$ -linear pairing  $\mathcal{O}_M \otimes \mathcal{V} \rightarrow \mathcal{V}[h]$ , i.e.  $f \otimes v \mapsto f * v$  such that  $1 * v = v$ .

ii)  $\mathbb{C}$ -linear bracket, satisfying Leibniz algebra  $[ , ] : \mathcal{V} \otimes \mathcal{V} \rightarrow h\mathcal{V}[h]$ ,

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For our considerations  $\mathcal{V} = \mathcal{O}(\mathcal{E})$ :

$$\partial f = df, \quad \pi(v)f = -hv(f), \quad \pi(\omega) = 0,$$

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where  $v$  and  $\omega$  are vector fields and 1-forms correspondingly.

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Einstein equations,  
Beltrami-Courant  
differentials and  
Homotopy  
Gerstenhaber algebras

Anton Zeitlin

## Outline

## Sigma-models and conformal invariance conditions

### Beltrami-Courant differential

Vertex/Courant  
algebroids,  
 $G_\infty$ -algebras and  
quasiclassical limit

## Einstein Equations

Vertex algebra  $V$  is a Virasoro module. The corresponding semi-infinite complex  $V^{semi}$  (the analogue of Chevalley complex for Virasoro algebra) is a vertex algebra too:

$$V^{semi} = V \otimes \Lambda,$$

$$\Lambda \text{ generated by } [b(z), c(w)]_+ = \delta(z - w).$$

The corresponding differential

$$Q = j_0, \quad j(z) = \sum_{n \in \mathbb{Z}} j_n z^{-n-1} = c(z)T(z) + :c(z)\partial c(z)b(z):$$

is nilpotent when  $D = 26$  (famous dimension 26!). However, we will consider subcomplex of light modes (i.e.  $L_0 = 0$ ) denoted in the following as  $(\mathcal{F}_h, Q)$ , where we can drop this condition:

$$\begin{array}{ccccc}
 & \mathcal{V}^h & & \mathcal{V}^h & \\
 \nearrow \partial & & \searrow \partial & \nearrow \partial & \searrow \frac{1}{2}h\text{div} \\
 & \oplus & & \oplus & \\
 \mathcal{O}_M^h & & \mathcal{O}_M^h & \xrightarrow{id} & \mathcal{O}_M^h & \xrightarrow{\frac{1}{2}h\text{div}} & \mathcal{O}_M^h
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 & & & \nwarrow \frac{1}{2}h\text{div} & \\
 & & & & \mathcal{O}_M^h
 \end{array}$$

Diagram illustrating the relationship between the vertex algebra  $\mathcal{V}^h$  and the space of light modes  $\mathcal{O}_M^h$ . The diagram shows two copies of  $\mathcal{V}^h$  at the top, connected by a direct sum  $\oplus$ . Below them are two copies of  $\mathcal{O}_M^h$ . The differential  $\partial$  maps  $\mathcal{O}_M^h$  to  $\mathcal{V}^h$  and  $\mathcal{V}^h$  to  $\mathcal{O}_M^h$ . The identity map  $id$  maps  $\mathcal{O}_M^h$  to  $\mathcal{O}_M^h$ . The divergence operator  $\frac{1}{2}h\text{div}$  maps  $\mathcal{V}^h$  to  $\mathcal{O}_M^h$ .

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Operator  $\mathbf{b}$  of degree -1 (0-mode of  $b(z)$ ) on  $(\mathcal{F}_h, Q)$  which anticommutes with  $Q$ :

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One can define a bracket:

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so that together with  $Q$ ,  $(\cdot, \cdot)_h$  it satisfies the relations of homotopy Gerstenhaber algebra:

$$\begin{aligned} & \{a_1, a_2\}_h + (-1)^{(|a_1|-1)(|a_2|-1)} \{a_2, a_1\}_h = \\ & (-1)^{|a_1|-1} (Qm'_h(a_1, a_2) - m'_h(Qa_1, a_2) - (-1)^{|a_2|} m'_h(a_1, Qa_2)), \\ & \{a_1, (a_2, a_3)_h\}_h = (\{a_1, a_2\}_h, a_3)_h + (-1)^{(|a_1|-1)|a_2|} (a_2, \{a_1, a_3\}_h)_h, \\ & \{ (a_1, a_2)_h, a_3 \}_h - (a_1, \{a_2, a_3\}_h)_h - (-1)^{(|a_3|-1)|a_2|} (\{a_1, a_3\}_h, a_2)_h = \\ & (-1)^{|a_1|+|a_2|-1} (Qn'_h(a_1, a_2, a_3) - n'_h(Qa_1, a_2, a_3) - \\ & (-1)^{|a_1|} n'_h(a_1, Qa_2, a_3) - (-1)^{|a_1|+|a_2|} n'_h(a_1, a_2, Qa_3), \\ & \{ \{a_1, a_2\}_h, a_3 \}_h - \{a_1, \{a_2, a_3\}_h\}_h + \\ & (-1)^{(|a_1|-1)(|a_2|-1)} \{a_2, \{a_1, a_3\}_h\}_h = 0. \end{aligned}$$

The conjecture of Lian and Zuckerman, which was later proven by series of papers (Kimura, Zuckerman, Voronov; Huang, Zhao; Voronov) says that the symmetrized product and bracket of homotopy Gerstenhaber algebra constructed above can be lifted to  $G_\infty$ -algebra.

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# Homotopy algebras: $G_\infty$ , $L_\infty$ , $C_\infty$

Let  $A$  be a graded vector space, consider free graded Lie algebra  $Lie(A)$ .

$$Lie^{k+1}(A) = [A, Lie^k A], \quad Lie^1(A) = A.$$

Consider free graded commutative algebra  $GA$  on the suspension  $(Lie(A))[-1]$ , i.e.

$$GA = \bigoplus_n \bigwedge^n Lie(A)[-n]$$

There are natural  $[\cdot, \cdot]$ ,  $\wedge$  operations on  $GA$  of degree -1, 0 correspondingly, generating a Gerstenhaber algebra.

A  $G_\infty$ -algebra (Tamarkin, Tsygan, 2000) is a graded space  $V$  with a differential  $\partial$  of degree 1 of  $G(V[1]^*)$ , such that  $\partial$  is a derivation w.r.t bracket and the product.

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Gerstenhaber algebras

Anton Zeitlin

Outline

Sigma-models and  
conformal invariance  
conditions

Beltrami-Courant  
differential

Vertex/Courant  
algebroids,  
 $G_\infty$ -algebras and  
quasiclassical limit

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of degree  $3 - n - k_1 - \dots - k_n$ , satisfying bilinear relations.

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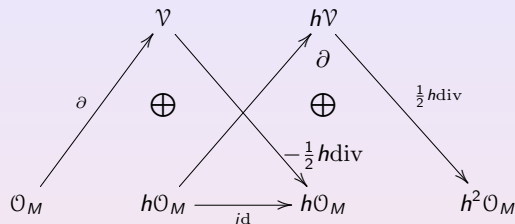
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# Quasiclassical limit of LZ $G_\infty$ algebra

The following complex  $(\mathcal{F}, Q)$ :



is a subcomplex of  $(\mathcal{F}_h, Q)$ . Then

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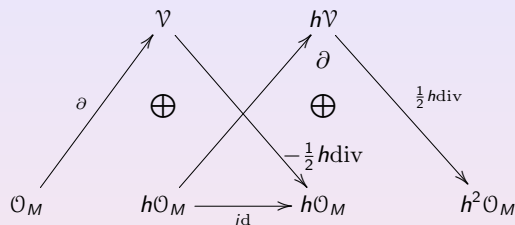
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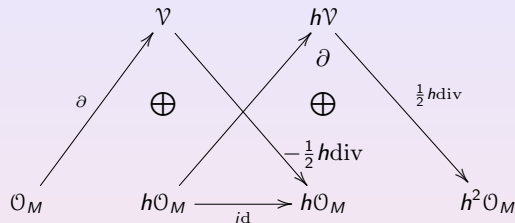
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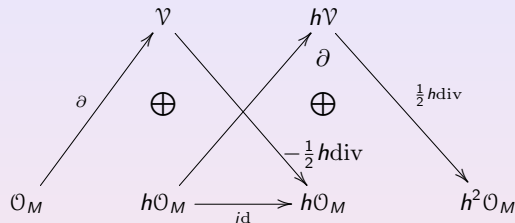
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The symmetrized operations  $(\cdot, \cdot)_0, \{\cdot, \cdot\}_0, \dots$  satisfy the relations of the homotopy Gerstenhaber algebra, so that all non-covariant higher-order terms disappear from the multilinear operations.

The resulting  $C_\infty$  and  $L_\infty$  algebras are reduced to  $C_3$  and  $L_3$  algebras.

A.Z., Comm. Math. Phys. 303 (2011) 331-359.

Conjecture: This  $G_\infty$ -algebra is the  $G_3$ -algebra (no homotopies beyond trilinear operations).

Classical limit procedure for vertex algebroid (due to P. Bressler):

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A Courant  $\mathcal{O}_M$ -algebroid is an  $\mathcal{O}_M$ -module  $\mathcal{Q}$  equipped with a structure of a Leibniz  $\mathbb{C}$ -algebra  $[\cdot, \cdot]_0 : \mathcal{Q} \otimes_{\mathbb{C}} \mathcal{Q} \rightarrow \mathcal{Q}$ , an  $\mathcal{O}_M$ -linear map of Leibniz algebras (the anchor map)  $\pi_0 : \mathcal{Q} \rightarrow \Gamma(TM)$ , a symmetric  $\mathcal{O}_M$ -bilinear pairing  $\langle \cdot, \cdot \rangle : \mathcal{Q} \otimes_{\mathcal{O}_M} \mathcal{Q} \rightarrow \mathcal{O}_M$ , a derivation  $\partial : \mathcal{O}_M \rightarrow \mathcal{Q}$  which satisfy

$$\begin{aligned}\pi_0 \circ \partial &= 0, & [q_1, f q_2]_0 &= f [q_1, q_2]_0 + \pi_0(q_1)(f) q_2 \\ \langle [q, q_1], q_2 \rangle + \langle q_1, [q, q_2] \rangle &= \pi_0(q)(\langle q_1, q_2 \rangle_0), \\ [q, \partial(f)]_0 &= \partial(\pi_0(q)(f)) \\ \langle q, \partial(f) \rangle &= \pi_0(q)(f) & [q_1, q_2]_0 + [q_2, q_1]_0 &= \partial \langle q_1, q_2 \rangle_0\end{aligned}$$

for  $f \in \mathcal{O}_M$  and  $q, q_1, q_2 \in \mathcal{Q}$ .

First it was obtained as an analogue of Manin's double for Lie bialgebroid by Z-J. Liu, A. Weinstein, P. Xu.

In our case  $\mathcal{Q} \cong \mathcal{O}(\mathcal{E})$ ,  $\pi_0$  is just a projection on  $\mathcal{O}(TM)$

$$[q_1, q_2]_0 = -[q_1, q_2]_D, \quad \langle q_1, q_2 \rangle_0 = -\langle q_1, q_2 \rangle^s, \quad \partial = d.$$

A Courant  $\mathcal{O}_M$ -algebroid is an  $\mathcal{O}_M$ -module  $\mathcal{Q}$  equipped with a structure of a Leibniz  $\mathbb{C}$ -algebra  $[\cdot, \cdot]_0 : \mathcal{Q} \otimes_{\mathbb{C}} \mathcal{Q} \rightarrow \mathcal{Q}$ , an  $\mathcal{O}_M$ -linear map of Leibniz algebras (the anchor map)  $\pi_0 : \mathcal{Q} \rightarrow \Gamma(TM)$ , a symmetric  $\mathcal{O}_M$ -bilinear pairing  $\langle \cdot, \cdot \rangle : \mathcal{Q} \otimes_{\mathcal{O}_M} \mathcal{Q} \rightarrow \mathcal{O}_M$ , a derivation  $\partial : \mathcal{O}_M \rightarrow \mathcal{Q}$  which satisfy

$$\pi_0 \circ \partial = 0, \quad [q_1, f q_2]_0 = f [q_1, q_2]_0 + \pi_0(q_1)(f) q_2$$

$$\langle [q, q_1], q_2 \rangle + \langle q_1, [q, q_2] \rangle = \pi_0(q)(\langle q_1, q_2 \rangle_0),$$

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for  $f \in \mathcal{O}_M$  and  $q, q_1, q_2 \in \mathcal{Q}$ .

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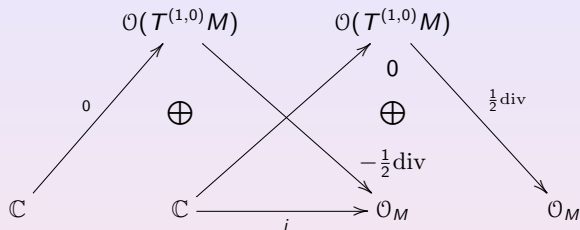






# Simplest version: $G_\infty \rightarrow$ Gerstenhaber algebra

Subcomplex  $(\mathcal{F}_{sm}, Q)$ :



The  $G_\infty$  algebra degenerates to G-algebra. Moreover, due to  $\mathbf{b}_0$  it is a BV-algebra. Combine chiral and antichiral part:

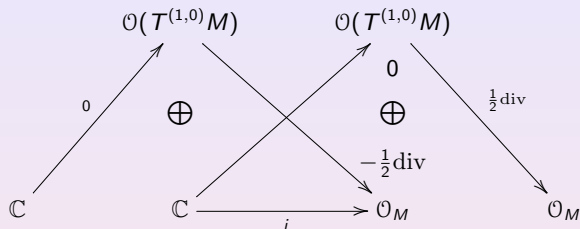
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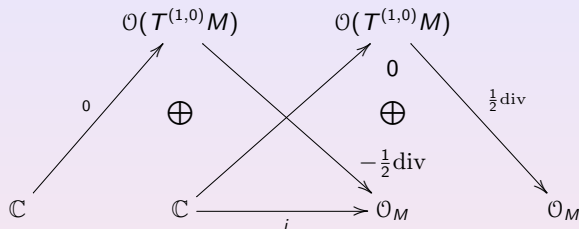
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Components:  $(g, \bar{v}, v, \phi, \bar{\phi})$ .

The Maurer-Cartan equation is equivalent to:

A.Z., Nucl. Phys. B 794 (2008) 370-398; A.Z., Adv. Theor. Math. Phys. 19 (2015) 1249-1275

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These are Einstein equations with the following constraints:

$$\begin{aligned} G_{i\bar{k}} &= g_{i\bar{k}}, & B_{i\bar{k}} &= -g_{i\bar{k}}, & \Phi &= \log \sqrt{g} + \Phi_0, \\ G_{ik} &= G_{\bar{i}\bar{k}} = G_{ik} = G_{\bar{i}\bar{k}} = 0, \end{aligned}$$

Physically:

$$\begin{aligned} \int [dp][d\bar{p}][dX][d\bar{X}] e^{-\frac{1}{2\pi i\hbar} \int_{\Sigma} (\langle p \wedge \bar{\partial} X \rangle - \langle \bar{p} \wedge \partial X \rangle - \langle g, p \wedge \bar{p} \rangle) + \int_{\Sigma} R^{(2)}(\gamma) \Phi_0(X)} = \\ \int [dX][d\bar{X}] e^{\frac{-1}{4\pi\hbar} \int d^2z (G_{\mu\nu} + B_{\mu\nu}) \partial X^\mu \bar{\partial} X^\nu + \int R^{(2)}(\gamma) (\Phi_0(X) + \log \sqrt{g})} \end{aligned}$$

based on computations of

A. Tseytlin and A. Schwarz, Nucl.Phys. B399 (1993) 691-708.



# Main Conjecture

Einstein equations,  
Beltrami-Courant  
differentials and  
Homotopy  
Gerstenhaber algebras

Anton Zeitlin

Consider

$$\mathbf{F}_{b^-} = \mathcal{F} \otimes \bar{\mathcal{F}}|_{b^-=0}$$

with the  $L_\infty$ -algebra structure given by Lian-Zuckerman construction.

One can explicitly check that GMC symmetry

( $\Psi = \Psi(\mathbb{M}, \Phi, \text{auxiliary fields})$ )

$$\Psi \rightarrow \Psi + Q\Lambda + [\Psi, \Lambda]_h + \frac{1}{2}[\Psi, \Psi, \Lambda]_h + \dots,$$

reproduces

$$\mathbb{M} \rightarrow \mathbb{M} - D\alpha + \phi_1(\alpha, \mathbb{M}) + \phi_2(\alpha, \mathbb{M}, \mathbb{M}).$$

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Conjecture: The corresponding Maurer-Cartan equation gives Einstein equations on  $G, B, \Phi$  expressed in terms of Beltrami-Courant differential. The symmetries of the Maurer-Cartan equation reproduce mentioned above symmetries of Einstein equations.

Outline

Sigma-models and  
conformal invariance  
conditions

Beltrami-Courant  
differential

Vertex/Courant  
algebroids,  
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# Thank you!