This yields the following sum: $\sum_{p,q\geq 0} c_p c_q$

$$\left(\frac{2}{k}\right)\int_{D^2} z^{m_1+p}\bar{z}^{m_2+q}(1-z\bar{z})^{2/k+1}\,d\mu(z,\bar{z}) \times$$

$$\left(\frac{2}{k}\right)\int_{D^2} w^{n_1+q}\bar{w}^{n_2+p}(1-w\bar{w})^{2/k+1}\,d\mu(w,\bar{w}) =$$

$$= \sum_{p,q\geq 0} \delta_{m_1+p,\,m_2+q}\,\delta_{n_1+q,\,n_2+p}\cdot\frac{c_p\,c_q}{c_{m_1+p}\,c_{n_1+q}}$$

Thus the integral is nonzero iff

$$m_1+n_1 = m_2+n_2$$

and in the latter case we have a single summation.

Now we would like to consider a generalization of the quantization on the Lobachevskii plane to the symmetrized product of n-copies of the Lobachevskii plane. At this stage I have more questions than answers. Let $\mathcal{F}_k^{(N)}$ be the space of analytic functions on $(D^2)^N$ invariant with respect to the symmetric group S_N with the following scalar product

$$(f,g) = \left(\frac{2}{k}\right)^N \int_{(D^2)^N} f(z)\overline{g(z)}\, K(z,\bar{z})^{2/k+1}\, d\mu(z,\bar{z})$$

where $z = (z_1, z_2, \ldots z_N)$, $k \geq 2$,

$$K(z,\bar{z}) = \prod_{i,j=1}^{N} (1 - z_i \bar{z}_j)$$

$$d\mu(z,\bar{z}) = \det\left(\frac{\partial^2 \log K(z,\bar{z})}{\partial z_i \partial \bar{z}_j}\right) \prod_{j=1}^{N} \left(\frac{i}{2\pi}\right) dz_j \wedge d\bar{z}_j$$

cf. Berezin, Quantization, Izv. Akad. Nauk SSSR 38 (1974), 1109–1165. Clearly for $N=1$ we get the scalar product as on p. 29. We would like to repeat all the steps as we have outlined for $N=1$ case.

In particular, we need to find the reproduction kernel $L_k^{(N)}(z, \bar{w})$ and then deduce the multivariable version of the reproduction formula

$$f(z) = \left(\frac{2}{k}\right)^N \int_{(\mathbb{D}^2)^N} L_k^{(N)}(z, \bar{w}) f(w) K(z, \bar{z})^{2/k+1} \, d\mu(z, \bar{z})$$

In $N=1$ case we have derived the kernel $L_k(z, \bar{w})$ using the simple orthonormal basis, see p. 31. Thus the very first question is to find a natural orthonormal (or even first orthogonal) basis of symmetric polynomials in the variables $z_1, \ldots, z_N$. A natural conjecture is that the appropriate class of symmetric functions is given by the Jack polynomials

$$J_{2/k}(z_1, \ldots, z_N)$$

I cannot imagine any other natural class of symmetric functions in this case.

The literature on the N-th symmetric product of Lobachevskii planes is rather obscure, see e.g. A. Edigarian, W. Zwonek Geometry of the symmetrized polydisc math. CV/0402033, which contains a formula for the reproducing kernel for a special value of k. and also some references. However I believe that we should rather consider the quantization on the so-called matrix balls.

Let us consider the matrix ball of dimension N^2

$$B_N = \{ Z \in Mat_{N \times N}(\mathbb{C}) : I_N - ZZ^* > 0 \}$$

Then one can generalize the quantization of the Lobachevskii plane to the case of matrix balls or even more general symmetric spaces. Note

$$B_N \cong U(N,N) / U(N) \times U(N)$$

Berezin did this himself in the paper Quantization in complex symmetric spaces Izv. Akad. Nauk. SSSR 39 (1975), 363–402, 472

Then it has been developed by numerous people. I would like to mention a relatively recent paper by N. Reshetikhin and L. Takhtajan in arXiv and the paper by Y. Neretin also in arXiv, called Matrix balls, radial analysis of Berezin kernels and hypergeometric determinants, published in Moscow Math. J. V 1 (2001), 157-220. We might find some other relevant papers as well.

Here are some explicit formulas:

Let $\mathcal{F}_\alpha^N$ be the space of holomorphic functions on B_N with the scalar product

$$(f, g) = \frac{1}{C(\alpha)} \int_{B_N} f(z)\overline{g(z)} \det(1 - zz^*)^{\alpha - 2N} \{dz\}$$

where $\{dz\} = \prod_{i,j=1}^{N} dx_{ij}\, dy_{ij}$, $z_{ij} = x_{ij} + i y_{ij}$

Again we normalize the constant so that $(1, 1) = 1$, then $C(\alpha) = \mathrm{Vol}\, B_N \times$ $\prod_{k=1}^{N} \frac{\Gamma(\alpha - N - k + 1)}{\Gamma(\alpha - k + 1)}$, and the reproduction kernel is see Berezin

$$K_\alpha(z, w) = \det(1 - zw^*)^{-\alpha}$$

Neretin also has the nonholomorphic version of the space of functions on B_N and the reproduction kernel

$$L_\alpha(z,w) = \left[\frac{\det(1-zz^*)\ \det(1-ww^*)}{\det(1-zw^*)\ \det(1-wz^*)}\right]^\alpha$$

similar to what we have (see Neretin, Matrix balls, ... p. 175)

Next we want to consider the projection to the diagonal

$$z \sim \begin{pmatrix} z_1 & & 0 \\ & z_2 & \\ 0 & & \ddots \\ & & & z_N \end{pmatrix} \qquad 1 - z_i\bar{z}_i > 0$$

The projections of the Lebesgue measure on B_N and reproduction kernels are computed, though in terms of $\lambda_i = (z_i \bar{z}_i)^{1/2}$ rather than z_i, see again Neretin pp 165 and 204, and there is a big literature about this.

Now the basis of symmetric functions naturally given by the eigenfunctions of Laplacians on B_N twisted by α. The radial parts of Laplacians on B_N is nothing but quantum Calogero system and the eigenfunctions are given by the Jack polynomials with the parameter α. This particular setting has been studied by Oblomkov in math RT/0202076.

We can also review the literature on Jack's polynomials. It is well known that they form an orthogonal family of polynomials with respect to a certain density and we should be able to see the relation with our situation.

<u>Remark</u>: Another reason that the Jack polynomials seem so attractive is that their q-deformation is the Macdonald polynomials. In principle, the continuous series for $\widehat{sl}(2,\mathbb{R})$, should admit a q-deformation, and therefore all the correlation functions should have q-analogues. Therefore if Jack polynomials is the right orthogonal system for $\widehat{sl}(2,\mathbb{R})$, Macdonald polynomials will appear for $U_q(\widehat{sl}(2,\mathbb{R}))$.

To summarize my observations I note

1) 2-point functions can be studied naturally using <u>quantization of Lobachevskii plane</u> (in the sense of Berezin) with the central charge k as parameter of quant.

2) 2-point functions are most naturally interpreted as <u>traces of certain operators</u> in the spaces of functions on Lobachevskii plane depending on the parameter k

3) 2N-point functions can be interpreted similar to 2-point functions on <u>symmetrized products of N-copies of Lobachevskii plane</u>.

4) Symmetrized products of N-copies of Lobachevskii plane should be treated as eigenvalues of matrix balls $\{ Z \in Mat_{N \times N}(\mathbb{C}) : I_N - ZZ^* > 0 \}$. Thus one needs to use the <u>quantization of matrix balls</u> and interpret $2N$-point functions as traces of some operators in <u>the spaces of functions invariant under conjugation</u> on matrix balls depending on the parameter k.

5) <u>Jack polynomials</u> depending on param. k should form natural bases of invar. spaces of functions.