Now we'll use KAN decomposition to construct the continuous series of $\widehat{SL}(2,\mathbb{R})$

First we recall the classical case

$$(\pi(g_1)f)(g) = f(gg_1)$$

$$(\pi(g_1)\underbrace{(\pi(g_2)f)}_{F})(g) = F(gg_1) = f(gg_1g_2)$$

$$F(g) = (\pi(g_2)f)(g) = f(gg_2)$$

$$(\pi(g_1g_2)f)(g) = f(gg_1g_2)$$

Invariance is the following

$$f(hg) = \rho(h)f(g)$$

$$\rho(h_1h_2)f(g) = f(h_1h_2g)$$

$$\shortparallel$$

$$\rho(h_1)\rho(h_2)f(g) = \rho(h_1)f(h_2g) = f(h_1h_2g)$$

$g = nak$

Thus functions on $K = S^1$ are $f(\theta)$

$$(\pi(\theta_1)f)(\theta) = f(\theta+\theta_1) \quad , \quad \pi(\theta_1) = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

Thus

$$\pi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{d}{d\theta}$$

Next we compute

$$\frac{d}{dt}\pi\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}\Bigg|_{t=0} = \pi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left(\pi\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}f\right)\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = f\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}\right)$$

$$= f\begin{pmatrix} \cos\theta\, e^t & \sin\theta\, e^{-t} \\ -\sin\theta\, e^t & \cos\theta\, e^{-t} \end{pmatrix} = a^{\mu+\rho} f\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix}$$

$$nak = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta\, e^{t} & \sin\theta\, e^{t} \\ -\sin\theta\, e^{-t} & \cos\theta\, e^{-t} \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\theta\, e^{t} - s\sin\theta\, e^{-t} & \sin\theta\, e^{t} + s\cos\theta\, e^{-t} \\ -\sin\theta\, e^{t} & \cos\theta\, e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Thus $\tan\theta = -\dfrac{c}{d}$, $\theta = \arctan\left(-\dfrac{c}{d}\right)$

and $e^{-2t} = c^2+d^2$, $t = -\dfrac{1}{2}\log(c^2+d^2)$

or $e^{t} = \dfrac{1}{\sqrt{c^2+d^2}}$

Therefore we obtain

$$\tan\theta_1 = -\frac{-\sin\theta e^{t}}{\cos\theta e^{-t}} = \tan\theta\cdot e^{2t}$$

We need to find

$$\frac{d}{dt} a(t)^{\mu+\rho} f\begin{pmatrix} \cos\theta_1(t) & \sin\theta_1(t) \\ -\sin\theta_1(t) & \cos\theta_1(t) \end{pmatrix}\Bigg|_{t=0}$$

$$= (\mu+\rho)\, a'(0)\, a(0)^{\mu+\rho-1} f\begin{pmatrix} \cos\theta_1(0) & \sin\theta_1(0) \\ -\sin\theta_1(0) & \cos\theta_1(0) \end{pmatrix}$$

$$+ a(0)^{\mu+\rho}\, \theta_1'(0) f'(\theta_1(0))$$

But we know $\theta_1(0) = \theta$

$$a(t) = \frac{1}{\sqrt{\sin^2\theta e^{2t} + \cos^2\theta e^{-2t}}}, \quad a(0) = 1$$

Thus we only need to find

$a'(0)$, $\theta_1'(0)$

$$a'(0) = -\frac{1}{2} \left. \frac{2e^{2t}\sin^2\theta - 2e^{-2t}\cos^2\theta}{1} \right|_{t=0} = \cos^2\theta - \sin^2\theta$$

$$\frac{d}{dt}\tan\theta_1(t) = \frac{1}{\cos^2\theta_1(t)}\,\theta_1'(t)$$

(recall $\frac{d}{dt}\frac{\sin t}{\cos t} = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$

$$\theta_1'(0) = \cos^2\theta \cdot \tan\theta \cdot 2 = 2\sin\theta\cos\theta$$

Thus we get

$$\pi\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = (\mu+\rho)(\cos^2\theta - \sin^2\theta) + 2\sin\theta\cos\theta\,\frac{d}{d\theta}$$

Finally we want to compute

$$\frac{d}{dt}\pi\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\Big|_{t=0} = \pi\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

$$\left(\pi\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}f\right)\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} = f\left(\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\right) =$$

$$= f\begin{pmatrix}\cos\theta & \sin\theta + t\cos\theta\\ -\sin\theta & \cos\theta - t\sin\theta\end{pmatrix} = a^{\mu+\rho} f\begin{pmatrix}\cos\theta_1 & \sin\theta_1\\ -\sin\theta_1 & \cos\theta_1\end{pmatrix}$$

$$\tan\theta_1 = \frac{\sin\theta}{\cos\theta - t\sin\theta}, \qquad \theta_1(0) = \theta$$

$$a(t) = \frac{1}{\sqrt{\sin^2\theta + (\cos\theta - t\sin\theta)^2}}, \qquad a(0) = 1$$

as before

Next we want to compute

$a'(0)$, $\theta_1'(0)$

$$a'(0) = \left(-\frac{1}{2}\right)\frac{2(\cos\theta - \cancel{t\sin\theta})(-\sin\theta)}{1}\bigg|_{t=0} = \sin\theta\cos\theta$$

$$\theta_1'(0) = \cos^2\theta \cdot -\frac{\sin\theta(-\sin\theta)}{(\cos\theta - t\sin\theta)^2}\bigg|_{t=0} = \sin^2\theta$$

Thus we get

$$\pi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (\mu+\rho)\sin\theta\cos\theta + \sin^2\theta\,\frac{d}{d\theta}$$

Thus finally

$$\pi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \frac{d}{d\theta}$$

$$\pi\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = (\mu+\rho)(\cos^2 - \sin^2\theta) + 2\sin\theta\cos\theta\frac{d}{d\theta}$$

$$\pi\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = (\mu+\rho)(\sin\theta\cos\theta) + \sin^2\theta\frac{d}{d\theta}$$

$$\Rightarrow \pi\begin{pmatrix}0 & 1/2\\ +1/2 & 0\end{pmatrix} = (\mu+\rho)(\sin\theta\cos\theta) + \frac{\sin^2\theta - \cos^2\theta}{2}\frac{d}{d\theta}$$

$$\sin^2\theta - 1/2 = \frac{1}{2}(\sin^2\theta - \cos^2\theta)$$

$$\Rightarrow \pi\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = (\mu+\rho)(2\sin\theta\cos\theta) + (\sin^2\theta - \cos^2\theta)\frac{d}{d\theta}$$

Recall $2\sin\theta\cos\theta = \sin 2\theta$, $\cos^2\theta - \sin^2\theta = \cos 2\theta$

Let us verify the commutation relations

$$\left[\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\right] = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -2\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

$$\left[\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 2\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$\left[\frac{d}{d\theta}, (2\bar{\ell})\cos 2\theta + \sin 2\theta \frac{d}{d\theta}\right] =$$

$$= -2(2\bar{\ell})\sin 2\theta + 2\cos 2\theta \frac{d}{d\theta}$$

$$\left[\frac{d}{d\theta}, (2\bar{\ell})\sin 2\theta - \cos 2\theta \frac{d}{d\theta}\right] =$$

$$= +2(2\bar{\ell})\cos 2\theta + 2\sin 2\theta \frac{d}{d\theta}$$

$$2(\cos^2 2\theta + \sin^2 2\theta)$$

$$\left[(2\bar{\ell})\cos 2\theta + \sin 2\theta \frac{d}{d\theta}, (2\bar{\ell})\sin 2\theta - \cos 2\theta \frac{d}{d\theta}\right] = 2\frac{d}{d\theta}$$

Finally we note the unitarity

$$\left(\sin 2\theta \frac{d}{d\theta}\right)^* = \left(\frac{d}{d\theta}\right)^* (\sin 2\theta)^* = -\frac{d}{d\theta}\sin 2\theta =$$

$$= -2\cos 2\theta - \sin 2\theta \frac{d}{d\theta}$$

Thus $\left(\cos 2\theta + \sin 2\theta \frac{d}{d\theta}\right)^* = -\cos 2\theta - \sin 2\theta \frac{d}{d\theta}$

i.e. $l = -\frac{1}{2} + i\lambda$ as before

We compare with Vilenkin p 298

$$A_1 = \frac{1}{2}\left(l e^{i\theta} + l e^{-i\theta} - 2\sin\theta \frac{d}{d\theta}\right) = l\cos\theta - \cancel{+}\sin\theta \frac{d}{d\theta}$$

$$A_2 = \frac{i}{2}\left(l e^{i\theta} - l e^{-i\theta} + 2i\cos\theta \frac{d}{d\theta}\right) = -l\sin\theta - \cos\theta \frac{d}{d\theta}$$

$$A_3 = \frac{d}{d\theta} \qquad l = -\frac{1}{2} + i\lambda$$

$$(\zeta(x), t)\,(\xi(x), s) = \left(\zeta(x) + \xi(x),\ t + s + \int \zeta'(x)\,\xi(x)\,dx \,/\, 2\right)$$

$$\gamma_{s(x)} = [\,\epsilon\, s(x),\ 0\,]$$

$$\mathcal{X}_{s(x)} f((\theta(x), t)) = \frac{d}{d\epsilon} f([\theta(x), t]\,[s(x), 0])\Big|_{\epsilon=0} =$$

$$= \frac{d}{d\epsilon} f\left(\theta(x) + \epsilon s(x),\ t + \epsilon \int \theta'(x)\, s(x)\right) =$$

$$= s(x) \frac{\partial f}{\partial \theta(x)} + \int \theta'(x)\, s(x) \frac{\partial}{\partial t}$$