# Beltrami-Courant Differentials and Homotopy Gerstenhaber algebras

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# Outline

Sigma-models and conformal invariance conditions

Reformulation of Sigma-model in first-order form

Vertex algebroids, $G_\infty$-algebra and quasiclassical limit

Einstein Equations from $G_\infty$-algebras via Beltrami-Courant differential

# Sigma-models and conformal invariance conditions

Sigma-model:

$$S_{so} = \frac{1}{4\pi h}\int d^2z(G_{\mu\nu}+B_{\mu\nu})\partial X^{\mu}\bar{\partial}X^{\nu} + \int_{\Sigma}\sqrt{\gamma}R^{(2)}(\gamma)\Phi(X)$$

Symmetries: Diff symmetry and $B \to B + d\lambda$.
Conformal invariance conditions:

$$\mu\frac{d}{d\mu}G_{\mu\nu} = \beta^{G}_{\mu\nu}(G,B,\Phi,h) = 0, \quad \mu\frac{d}{d\mu}B_{\mu\nu} = \beta^{B}_{\mu\nu}(G,B,\Phi,h) = 0,$$
$$\mu\frac{d}{d\mu}\Phi = \beta^{\Phi}(G,B,\Phi,h) = 0$$

at the level $h^0$ turn out to be Einstein Equations:

$$R^{\mu\nu} = \frac{1}{4}H^{\mu\lambda\rho}H^{\nu}_{\lambda\rho} - 2\nabla^{\mu}\nabla^{\nu}\Phi,$$
$$\nabla_{\mu}H^{\mu\nu\rho} - 2(\nabla_{\lambda}\Phi)H^{\lambda\nu\rho} = 0,$$
$$4(\nabla_{\mu}\Phi)^2 - 4\nabla_{\mu}\nabla^{\mu}\Phi + R + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} = 0.$$

Early days of string theory: linearized Einstein Equations and their symmetries: ($G_{\mu\nu} = \eta_{\mu\nu} + s_{\mu\nu}$):

$$Q^{\eta}\Psi^s = 0, \quad \Psi^s \to \Psi^s + Q\Lambda$$

in a BRST complex associated to certain Virasoro module.
It was conjectured (Sen, Zwiebach,...) that Einstein equations with $h$-corrections and their symmetries are Generalized Maurer-Cartan (GMC) Equations and their symmetries:

$$Q\Psi + \frac{1}{2}[\Psi, \Psi]_h + \frac{1}{3!}[\Psi, \Psi, \Psi]_h + \ldots = 0$$

$$\Psi \to \Psi + Q\Lambda + [\Psi, \Lambda]_h + \frac{1}{2}[\Psi, \Psi, \Lambda]_h + \ldots,$$

where $[\cdot, \cdot, \ldots, \cdot]_h$ operations generate $L_\infty$-algebra.

We show that for a proper background, there is a richer structure, namely $G_\infty$-algebra, as well as a well-defined classical limit of the $L_\infty$-subalgebra, so that GMC equations are equivalent to Einstein Equations.

# First order version of sigma-model action

We start from free action:

$$S_0 = \frac{1}{2\pi i h}\int_\Sigma (\langle p \wedge \bar\partial X\rangle - \langle \bar p \wedge \partial X\rangle),$$

where $p \in X^*(\Omega^{(1,0)}(M)) \otimes \Omega^{(1,0)}(\Sigma)$, $\bar p \in X^*(\Omega^{(0,1)}(M)) \otimes \Omega^{(0,1)}(\Sigma)$.
Symmetries: $X^i \to X^i - v^i(X), X^{\bar i} \to X^{\bar i} - v^{\bar i}(\bar X)$,

$$p_i \to p_i + \partial_i v^k p_k, \quad p_{\bar i} \to p_{\bar i} + \partial_{\bar i} v^{\bar k} p_{\bar k}$$
$$p_i \to p_i - \partial X^k(\partial_k \omega_i - \partial_i \omega_k), \quad p_{\bar i} \to p_{\bar i} - \bar\partial X^{\bar k}(\partial_{\bar k}\omega_{\bar i} - \partial_{\bar i}\omega_{\bar k}).$$

Not invariant under general diffeomorphisms, i.e.

$$\delta S_0 = -\frac{1}{2\pi i h}\int (\langle \bar\partial v, p \wedge \bar\partial X\rangle + \langle \partial \bar v, \bar p \wedge \partial X\rangle).$$

It is necessary to add extra terms:

$$\delta S_\mu = -\frac{1}{2\pi i h}\int (\langle \mu, p \wedge \bar\partial X\rangle + \langle \bar\mu, \partial X \wedge \bar p\rangle),$$

where $\mu \in \Gamma(T^{(1,0)}M \otimes T^{*(0,1)}(M))$, $\bar\mu \in \Gamma(T^{(0,1)}M \otimes T^{*(1,0)}(M))$, so that: $\mu \to \mu - \bar\partial v + \dots$, $\bar\mu \to \bar\mu - \partial \bar v + \dots$

Continuing the procedure:

$$\tilde{S} = \frac{1}{2\pi i h}\int_{\Sigma}(\langle p \wedge \bar{\partial} X\rangle - \langle \bar{p} \wedge \partial X\rangle -$$
$$\langle \mu, p \wedge \bar{\partial} X\rangle - \langle \bar{\mu}, \partial X \wedge \bar{p}\rangle - \langle b, \partial X \wedge \bar{\partial} X\rangle),$$

where

$$\mu^i_{\bar j} \to$$
$$\mu^i_{\bar j} - \partial_{\bar j} v^i + v^k \partial_k \mu^i_{\bar j} + v^{\bar k}\partial_{\bar k}\mu^i_{\bar j} + \mu^i_{\bar k}\partial_{\bar j} v^{\bar k} - \mu^k_{\bar j}\partial_k v^i + \mu^i_{\bar l}\mu^k_{\bar j}\partial_k v^{\bar l},$$
$$b_{i\bar j} \to$$
$$b_{i\bar j} + v^k\partial_k b_{i\bar j} + v^{\bar k}\partial_{\bar k} b_{i\bar j} + b_{i\bar k}\partial_{\bar j} v^{\bar k} + b_{l\bar j}\partial_i v^l + b_{i\bar k}\mu^k_{\bar j}\partial_k v^{\bar k} + b_{l\bar j}\bar{\mu}^{\bar k}_i\partial_{\bar k} v^l,$$

so that:

$$X^i \to X^i - v^i(X,\bar X), \quad p_i \to p_i + p_k\partial_i v^k - p_k\mu^k_{\bar l}\partial_i v^{\bar l} - b_{j\bar k}\partial_i v^{\bar k}\partial X^j,$$
$$X^{\bar i} \to X^{\bar i} - v^{\bar i}(X,\bar X), \quad \bar p_{\bar i} \to \bar p_{\bar i} + \bar p_{\bar k}\partial_{\bar i} v^{\bar k} - \bar p_{\bar k}\bar\mu^{\bar k}_l\partial_i v^l - b_{\bar j k}\partial_{\bar i} v^k\bar\partial X^{\bar j}.$$

Similarly, the 1-form transformation:

$$b_{i\bar{j}} \to b_{i\bar{j}} + \partial_{\bar{j}}\omega_i - \partial_i\omega_{\bar{j}} + \mu^i_{\bar{j}}(\partial_i\omega_k - \partial_k\omega_i) +$$
$$\bar{\mu}^{\bar{s}}_i(\partial_{\bar{j}}\omega_{\bar{s}} - \partial_{\bar{s}}\omega_{\bar{j}}) + \bar{\mu}^{\bar{i}}_j\mu^s_{\bar{k}}(\partial_s\omega_{\bar{i}} - \partial_{\bar{i}}\omega_s)$$

such that

$$p_i \to p_i - \partial X^k(\partial_k\omega_i - \partial_i\omega_k) - \partial_{\bar{r}}\omega_i\partial X^{\bar{r}} - \bar{\mu}^{\bar{s}}_k\partial_i\omega_{\bar{s}}\partial X^k,$$
$$p_{\bar{i}} \to p_{\bar{i}} - \bar{\partial} X^{\bar{k}}(\partial_{\bar{k}}\omega_{\bar{i}} - \partial_{\bar{i}}\omega_{\bar{k}}) - \partial_r\omega_{\bar{i}}\bar{\partial} X^r - \mu^s_{\bar{k}}\partial_i\omega_s\bar{\partial} X^{\bar{k}}.$$

For simplicity: $E = TM \oplus T^*M$ and $\mathcal{E} = T^{(1,0)}M \oplus T^{*(1,0)}M$, $\bar{\mathcal{E}} = T^{(0,1)}M \oplus T^{*(0,1)}M$, i.e. $E = \mathcal{E} \oplus \bar{\mathcal{E}}$.
Let $\tilde{\mathbb{M}} \in \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$, such that

$$\tilde{\mathbb{M}} = \begin{pmatrix} 0 & \mu \\ \bar{\mu} & b \end{pmatrix}.$$

Introduce $\alpha \in \Gamma(E)$, i.e. $\alpha = (v, \bar{v}, \omega, \bar{\omega})$. Let $D : \Gamma(E) \to \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$, such that

$$D\alpha = \left( \begin{array}{cc} 0 & \bar{\partial} v \\ \partial \bar{v} & \partial \bar{\omega} - \bar{\partial} \omega \end{array} \right).$$

Then the transformation of $\tilde{\mathbb{M}}$ can be expressed:

$$\tilde{\mathbb{M}} \to \tilde{\mathbb{M}} - D\alpha + \phi_1(\alpha, \tilde{\mathbb{M}}) + \phi_2(\alpha, \tilde{\mathbb{M}}, \tilde{\mathbb{M}}).$$

Let us describe $\phi_1, \phi_2$ algebraically. In order to do that we need to pass to jet bundles, i.e.

$$\alpha \in J^\infty(\mathcal{O}_M) \otimes J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}})) \oplus J^\infty(\mathcal{O}(\mathcal{E})) \otimes J^\infty(\bar{\mathcal{O}}_M),$$

$$\tilde{\mathbb{M}} \in J^\infty(\mathcal{O}(\mathcal{E})) \otimes J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}}))$$

Then:

$$\alpha = \sum_J f^J \otimes \bar{b}^J + \sum_K b^K \otimes \bar{f}^K,$$
$$\tilde{\mathbb{M}} = \sum_I a^I \otimes \bar{a}^I,$$

where $a^I, b^J \in J^\infty(\mathcal{O}(\mathcal{E}))$, $f^I \in J^\infty(\mathcal{O}_M)$ and $\bar{a}^I, \bar{b}^J \in J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}}))$, $\bar{f}^I \in J^\infty(\bar{\mathcal{O}}_M)$. Then

$$\phi_1(\alpha, \tilde{\mathbb{M}}) = \sum_{I,J} [b^J, a^I]_D \otimes \bar{f}^J \bar{a}^I + \sum_{I,K} f^K a^I \otimes [\bar{b}^K, \bar{a}^I]_D,$$

where $[\cdot,\cdot]_D$ is a Dorfman bracket:

$$[v_1, v_2]_D = [v_1, v_2]^{Lie}, \quad [v, \omega]_D = L_v \omega,$$
$$[\omega, v]_D = -i_v d\omega, \quad [\omega_1, \omega_2]_D = 0.$$

Similarly:

$$\phi_2(\alpha, \tilde{\mathbb{M}}, \tilde{\mathbb{M}}) = \tilde{\mathbb{M}} \cdot D\alpha \cdot \tilde{\mathbb{M}}$$
$$\frac{1}{2} \sum_{I,J,K} \langle b^I, a^K \rangle a^J \otimes \bar{a}^J (\bar{f}^I) \bar{a}^K + \frac{1}{2} \sum_{I,J,K} a^J (f^I) a^K \otimes \langle \bar{b}^I, \bar{a}^K \rangle \bar{a}^J.$$

Relation to standard second order sigma-model: Let us fill in 0 in $\tilde{\mathbb{M}}$:

$$\mathbb{M} = \begin{pmatrix} g & \mu \\ \bar{\mu} & b \end{pmatrix}.$$

$$S_{fo} = \frac{1}{2\pi i h}\int_\Sigma (\langle p \wedge \bar{\partial} X\rangle + \langle \bar{p} \wedge \partial X\rangle -$$
$$-\langle g, p\wedge \bar{p}\rangle - \langle \mu, p\wedge\bar{\partial}X\rangle - \langle \bar{\mu}, \bar{p}\wedge \partial X\rangle - \langle b, \partial X\wedge\bar{\partial}X\rangle).$$

Same formulas express symmetries. If $\{g^{i\bar{j}}\}$ is nondegenerate, then :

$$S_{so} = \frac{1}{4\pi h}\int d^2z (G_{\mu\nu} + B_{\mu\nu})\partial X^\mu \bar{\partial} X^\nu,$$

$$\begin{aligned}
G_{s\bar{k}} &= g_{\bar{i}j}\bar{\mu}^{\bar{i}}_s \mu^j_{\bar{k}} + g_{s\bar{k}} - b_{s\bar{k}}, \quad B_{s\bar{k}} = g_{\bar{i}j}\bar{\mu}^{\bar{i}}_s \mu^j_{\bar{k}} - g_{s\bar{k}} - b_{s\bar{k}} \\
G_{si} &= -g_{i\bar{j}}\bar{\mu}^{\bar{j}}_s - g_{s\bar{j}}\bar{\mu}^{\bar{j}}_i, \quad G_{\bar{s}\bar{i}} = -g_{\bar{s}j}\mu^j_{\bar{i}} - g_{\bar{i}j}\mu^j_{\bar{s}} \\
B_{si} &= g_{s\bar{j}}\bar{\mu}^{\bar{j}}_i - g_{i\bar{j}}\bar{\mu}^{\bar{j}}_s, \quad B_{\bar{s}\bar{i}} = g_{\bar{i}j}\mu^j_{\bar{s}} - g_{\bar{s}j}\mu^j_{\bar{i}},
\end{aligned}$$

Symmetries: infinitesimal diffeomorphism transformations and the 2-form $B$ symmetry

$$G \to G - L_{\mathbf{v}} G, \quad B \to B - L_{\mathbf{v}} B$$
$$B \to B - 2d\boldsymbol{\omega}$$

if $\alpha = (\mathbf{v}, \boldsymbol{\omega})$, so that $\mathbf{v} \in \Gamma(TM)$, $\boldsymbol{\omega} \in \Omega^1(M)$.

# Vertex algebroids





The CFT corresponding to the chiral part of the free first order action

$$S_0 = \frac{1}{2\pi i h}\int_\Sigma \langle p \wedge \bar{\partial} X\rangle - \int R^{(2)}(\gamma)\phi'(X)$$

is locally described via VOA:

$$X^i(z)p_j(w) \sim \frac{h\delta^i_j}{z-w}$$

with Virasoro element

$$T(z) = \frac{1}{h} : \langle p(z)\partial X(z)\rangle : + \partial^2\phi'(X(z)).$$

The corresponding space of states $V = \sum_{n=0}^{+\infty} V_n$,

$$V_0 \to \mathcal{O}_M \otimes \mathbb{C}[h] = \mathcal{O}^h_M, \quad V_1 \to \mathcal{V} = \mathcal{O}(\mathcal{E}) \otimes \mathbb{C}[h] \equiv \mathcal{O}(\mathcal{E})^h$$

The structure of vertex algebra imposes algebraic relations on $V_0 \oplus V_1$ giving it a structure of a *vertex algebroid*.

A *vertex* $\mathcal{O}_M$*-algebroid* is a sheaf of $\mathbb{C}$-vector spaces $\mathcal{V}$ with a pairing $\mathcal{O}_M \otimes_{\mathbb{C}[h]} \mathcal{V} \to \mathcal{V}$, i.e. $f \otimes v \mapsto f * v$ such that $1 * v = v$, equipped with a structure of a Leibniz $\mathbb{C}[h]$-algebra $[\ ,\ ] : \mathcal{V} \otimes_{\mathbb{C}[h]} \mathcal{V} \to \mathcal{V}$, a $\mathbb{C}[h]$-linear map of Leibniz algebras $\pi : \mathcal{V} \to \Gamma(TM)$ usually referred to as an anchor, a symmetric $\mathbb{C}$-bilinear pairing $\langle\ ,\ \rangle : \mathcal{V} \otimes_{\mathbb{C}[h]} \mathcal{V} \to \mathcal{O}_M^h$ a $\mathbb{C}$-linear map $\partial : \mathcal{O}_M \to \mathcal{V}$ such that $\pi \circ \partial = 0$, which satisfy the relations

$$
\begin{aligned}
& f * (g * v) - (fg) * v = \pi(v)(f) * \partial(g) + \pi(v)(g) * \partial(f), \\
& [v_1, f * v_2] = \pi(v_1)(f) * v_2 + f * [v_1, v_2], \\
& [v_1, v_2] + [v_2, v_1] = \partial(\langle v_1, v_2 \rangle), \quad \pi(f * v) = f\pi(v), \\
& \langle f * v_1, v_2 \rangle = f \langle v_1, v_2 \rangle - \pi(v_1)(\pi(v_2)(f)), \\
& \pi(v)(\langle v_1, v_2 \rangle) = \langle [v, v_1], v_2 \rangle + \langle v_1, [v, v_2] \rangle, \\
& \partial(fg) = f * \partial(g) + g * \partial(f), \\
& [v, \partial(f)] = \partial(\pi(v)(f)), \quad \langle v, \partial(f) \rangle = \pi(v)(f),
\end{aligned}
$$

where $v, v_1, v_2 \in \mathcal{V}$, $f, g \in \mathcal{O}_M^h$.

$$
\begin{aligned}
&\partial f = df, \quad \pi(v)f = -hv(f), \quad \pi(\omega) = 0,\\
&f * v = fv + hdX^i\partial_i\partial_j fv^j, \quad f*\omega = f\omega,\\
&[v_1, v_2] = -h[v_1, v_2]_D - h^2 dX^i\partial_i\partial_k v_1^s\partial_s v_2^k,\\
&[v,\omega] = -h[v,\omega]_D, \quad [\omega, v] = -h[\omega, v]_D, \quad [\omega_1,\omega_2] = 0,\\
&\langle v,\omega\rangle = -h\langle v,\omega\rangle^s, \quad \langle v_1, v_2\rangle = -h^2\partial_i v_1^j\partial_j v_2^i, \quad \langle\omega_1.\omega_2\rangle = 0,
\end{aligned}
$$

Given a holomorphic volume form on open neighborhood $U$of $M$, one can associate a homotopy Gerstenhaber algebra to the vertex algebroid on $U$.

Consider the light modes of the corresponding BRST complex, (i.e. $L_0 = 0$). The resulting complex $(\mathcal{F}_h^{\cdot}, Q)$ is:

$$
\begin{tikzcd}
& \mathcal{V} \arrow[ddrr, "-\frac{1}{2}h\mathrm{div}"] & & \mathcal{V} \arrow[ddrr, "\frac{1}{2}h\mathrm{div}"] & & \\
& \bigoplus & & \bigoplus & & \\
\mathcal{O}_M^h \arrow[uur, "\partial"] & & \mathcal{O}_M^h \arrow[uur, "\partial"'] \arrow[r, "id"'] & \mathcal{O}_M^h & & \mathcal{O}_M^h.
\end{tikzcd}
$$

$\mathrm{div}$ stands for divergence operator with respect to the nonvanishing volume form applied to sections of $\Gamma(U, T^{(1,0)}(M))$.

## The homotopy Gerstenhaber algebra of Lian and Zuckerman

The homotopy associative and homotopy commutative product of Lian and Zuckerman:

$$(A, B) = Res_z \frac{A(z)B}{z}$$

$$\begin{aligned}
&Q(a_1, a_2)_h = (Qa_1, a_2)_h + (-1)^{|a_1|}(a_1, Qa_2)_h,\\
&(a_1, a_2)_h - (-1)^{|a_1||a_2|}(a_2, a_1)_h =\\
&Qm(a_1, a_2) + m(Qa_1, a_2) + (-1)^{|a_1|}m(a_1, Qa_2),\\
&Q(a_1, a_2, a_3)_h + (Qa_1, a_2, a_3)_h + (-1)^{|a_1|}(a_1, Qa_2, a_3)_h +\\
&(-1)^{|a_1|+|a_2|}(a_1, a_2, Qa_3)_h = ((a_1, a_2)_h, a_3)_h - (a_1, (a_2, a_3)_h)_h
\end{aligned}$$

Operator **b** of degree -1 on $(\mathcal{F}_h^{\cdot}, Q)$ which anticommutes with $Q$:

$$\begin{tikzcd}
& \mathcal{V} & \mathcal{V} \arrow[l, "-id"'] & \\
& \bigoplus & \bigoplus & \\
\mathcal{O}_M^h & \mathcal{O}_M^h \arrow[l, "id"'] & \mathcal{O}_M^h & \mathcal{O}_M^h \arrow[l, "-id"']
\end{tikzcd}$$

One can define a bracket:

$$(-1)^{|a_1|}\{a_1,a_2\}_h=\mathbf{b}(a_1,a_2)_h-(\mathbf{b}a_1,a_2)_h-(-1)^{|a_1|}(a_1\mathbf{b}a_2)_h,$$

$$\begin{aligned}
&\{a_1,a_2\}+(-1)^{(|a_1|-1)(|a_2|-1)}\{a_2,a_1\}=\\
&(-1)^{|a_1|-1}(Qm'_h(a_1,a_2)-m'_h(Qa_1,a_2)-(-1)^{|a_2|}m'_h(a_1,Qa_2)),\\
&\{a_1,(a_2,a_3)_h\}_h=(\{a_1,a_2\}_h,a_3)_h+(-1)^{(|a_1|-1)||a_2|}(a_2,\{a_1,a_3\}_h)_h,\\
&\{(a_1,a_2)_h,a_3\}_h-(a_1,\{a_2,a_3\}_h)_h-(-1)^{(|a_3|-1)|a_2|}(\{a_1,a_3\}_h,a_2)_h=\\
&(-1)^{|a_1|+|a_2|-1}(Qn'_h(a_1,a_2,a_3)-n'_h(Qa_1,a_2,a_3)-\\
&(-1)^{|a_1|}n'_h(a_1,Qa_2,a_3)-(-1)^{|a_1|+|a_2|}n'_h(a_1,a_2,Qa_3),\\
&\{\{a_1,a_2\}_h,a_3\}_h-\{a_1,\{a_2,a_3\}_h\}_h+\\
&(-1)^{(|a_1|-1)(|a_2|-1)}\{a_2,\{a_1,a_3\}_h\}_h=0.
\end{aligned}$$

The conjecture of Lian and Zuckerman, which was later proven by series of papers (Kimura, Zuckerman, Voronov; Huang, Zhao; Voronov) says that the symmetrized product and bracket of homotopy Gerstenhaber algebra constructed above can be lifted to $G_\infty$-algebra.

## Quasiclassical limit of LZ $G_\infty$ algebra

Let $\mathcal{V}|_{h=0} = \mathcal{V}^0$.
One can see that $(\mathcal{F}^{\cdot}, Q) \cong (\mathcal{F}^{\cdot}_1, Q)$ is a subcomplex of $(\mathcal{F}^{\cdot}_h, Q)$, which is:

$$
\begin{tikzcd}
 & \mathcal{V}^0 \arrow[ddr, "-\frac{1}{2}h\mathrm{div}"] & h\mathcal{V}^0 \arrow[ddr, "\frac{1}{2}h\mathrm{div}"] & \\
 & \oplus & \oplus & \\
\mathcal{O}_M \arrow[uur, "\partial"] & h\mathcal{O}_M \arrow[uur, "\partial"] \arrow[r, "id"'] & h\mathcal{O}_M & h^2\mathcal{O}_M
\end{tikzcd}
$$

Then

$$
\begin{aligned}
&(\cdot,\cdot)_h : \mathcal{F}^i \otimes \mathcal{F}^j \to \mathcal{F}^{i+j}[h], \quad \{\cdot,\cdot\} : \mathcal{F}^i \otimes \mathcal{F}^j \to h\mathcal{F}_{i+j-1}[h], \\
&\mathbf{b} : \mathcal{F}^i \to h\mathcal{F}^{i-1}[h],
\end{aligned}
$$

so that

$$
(\cdot,\cdot)_0 = \lim_{h\to 0}(\cdot,\cdot)_h, \quad \{\cdot,\cdot\}_0 = \lim_{h\to 0} h^{-1}\{\cdot,\cdot\}_h, \quad \mathbf{b}_0 = \lim_{h\to 0} h^{-1}\mathbf{b}
$$

are well defined.

The symmetrized operations $(\cdot,\cdot)_0$, $\{\cdot,\cdot\}_0$ satisfy the relations of homotopy Gerstenhaber algebra. The resulting $C_\infty$ and $L_\infty$ algebras are reduced to $C_3$ and $L_3$ algebras.

Conjecture: This $G_\infty$-algebra is $G_3$-algebra (no higher homotopies).

This is very close to the classical limit procedure for vertex algebroid: $[\cdot,\cdot]_0=\lim_{h\to 0}\frac{1}{h}[\cdot,\cdot]$, $\pi_0=\lim_{h\to 0}\frac{1}{h}\pi$, $\langle\cdot,\cdot\rangle_0=\frac{1}{h}\langle\cdot,\cdot\rangle$.
As a result we get a Courant algebroid:
A Courant $\mathcal{O}_M$-algebroid is an $\mathcal{O}_M$-module $\mathcal{Q}$ equipped with a structure of a Leibniz $\mathbb{C}$-algebra $[,]_0:\mathcal{Q}\otimes_{\mathbb{C}}\mathcal{Q}\to\mathcal{Q}$, an $\mathcal{O}_M$-linear map of Leibniz algebras (the anchor map) $\pi_0:\mathcal{Q}\to\Gamma(TM)$, a symmetric $\mathcal{O}_M$-bilinear pairing $\langle\cdot,\cdot\rangle:\mathcal{Q}\otimes_{\mathcal{O}_M}\mathcal{Q}\to\mathcal{O}_M$, a derivation $\partial:\mathcal{O}_M\to\mathcal{Q}$ which satisfy

$$\begin{aligned}
&\pi\circ\partial=0,\quad [q_1,fq_2]_0=f[q_1,q_2]_0+\pi_0(q_1)(f)q_2\\
&\langle[q,q_1],q_2\rangle+\langle q_1,[q,q_2]\rangle=\pi_0(q)(\langle q_1,q_2\rangle_0),\\
&[q,\partial(f)]_0=\partial(\pi_0(q)(f))\\
&\langle q,\partial(f)\rangle=\pi_0(q)(f)\quad [q_1,q_2]_0+[q_2,q_1]_0=\partial(\langle q_1,q_2\rangle_0)
\end{aligned}$$

for $f\in\mathcal{O}_M$ and $q,q_1,q_2\in\mathcal{Q}$. In our case $\mathcal{Q}\cong\mathcal{O}(\mathcal{E})$, $\pi_0$ is just a projection on $\mathcal{O}(TM)$

$$[q_1,q_2]_0=-[q_1,q_2]_D,\quad \langle q_1,q_2\rangle_0=-\langle q_1,q_2\rangle^s,\quad \partial=d.$$

Question: Is there a direct path (avoiding vertex algebra) from Courant algebroid to $G_3$-algebra?

## Baby version: Gerstenhaber algebra

Subcomplex $(\mathcal{F}^{\cdot}_{sm}, Q)$:

$$\begin{tikzcd}
 & \mathcal{O}(T^{(1,0)}M) \arrow[ddr, "-\frac{1}{2}\mathrm{div}"] & \mathcal{O}(T^{(1,0)}M) \arrow[ddr, "\frac{1}{2}\mathrm{div}"] & \\
 & \bigoplus & \bigoplus & \\
\mathbb{C} \arrow[uur, "0"] & \mathbb{C} \arrow[uur, "0"] \arrow[r, "i"'] & \mathcal{O}_M & \mathcal{O}_M
\end{tikzcd}$$

The $G_\infty$ algebra degenerates to G-algebra. Moreover, due to $\mathbf{b}_0$ it is a BV-algebra. Combine chiral and antichiral part:

$$\mathbf{F}^{\cdot}_{sm} = \mathcal{F}^{\cdot}_{sm} \otimes \bar{\mathcal{F}}^{\cdot}_{sm}$$

$$(-1)^{|a_1|}\{a_1, a_2\} = \mathbf{b}^{-}(a_1, a_2) - (\mathbf{b}^{-}a_1, a_2) - (-1)^{|a_1|}(a_1 \mathbf{b}^{-} a_2),$$

where $\mathbf{b}^{-} = \mathbf{b} - \bar{\mathbf{b}}$.

Maurer-Cartan elements, closed under $\mathbf{b}^-$:

$$\Gamma(T^{(1,0)}(M)\otimes T^{(0,1)}(M))\oplus \mathcal{O}(T^{(0,1)}(M)\oplus \mathcal{O}(T^{(1,0)}(M)\oplus \mathcal{O}_M\oplus \bar{\mathcal{O}}_M$$

Components:$(g, \bar{v}, v, \phi, \bar{\phi})$.

The Maurer-Cartan equation is equivalent to:

1). Vector field $div_{\Omega}g$, where $\Omega=\Omega' e^{-2\phi+2\bar{\phi}}$ is determined by $f\equiv -2\Phi_0=-2(\Phi_0'+\phi-\bar{\phi})$ and $\partial_i\partial_{\bar{j}}\Phi_0=0$, is such that its $\Gamma(T^{(1,0)}M)$, $\Gamma(T^{(0,1)}M)$ components are correspondingly holomorphic and antiholomorphic.
2). Bivector field $g\in\Gamma(T^{(1,0)}M\otimes T^{(0,1)}M)$ obeys the following equation:

$$[[g,g]]+\mathcal{L}_{div_{\Omega}(g)}g=0,$$

where $\mathcal{L}_{div_{\Omega}(g)}$ is a Lie derivative with respect to the corresponding vector fields and

$$[[g,h]]^{k\bar{l}}\equiv(g^{i\bar{j}}\partial_i\partial_{\bar{j}}h^{k\bar{l}}+h^{i\bar{j}}\partial_i\partial_{\bar{j}}g^{k\bar{l}}-\partial_i g^{k\bar{j}}\partial_{\bar{j}}h^{i\bar{l}}-\partial_i h^{k\bar{j}}\partial_{\bar{j}}g^{i\bar{l}})$$

3). $div_{\Omega}div_{\Omega}(g)=0$.
The infinitesimal symmetries of the Maurer-Cartan equation coincide with the holomorphic coordinate transformations of the volume form and tensor $\{g^{i\bar{j}}\}$.

These are Einstein equations with the following constraints:

$$G_{i\bar{k}} = g_{i\bar{k}}, \quad B_{i\bar{k}} = -g_{i\bar{k}}, \quad \Phi = \log\sqrt{g} + \Phi_0,$$
$$G_{ik} = G_{\bar{i}\bar{k}} = G_{ik} = G_{\bar{i}\bar{k}} = 0,$$

Physically:

$$\int [dp][d\bar{p}][dX][d\bar{X}]e^{-\frac{1}{2\pi i h}\int_\Sigma(\langle p\wedge\bar\partial X\rangle - \langle\bar{p}\wedge\partial X\rangle - \langle g, p\wedge\bar{p}\rangle) + \int_\Sigma R^{(2)}(\gamma)\Phi_0(X)} =$$
$$\int [dX][d\bar{X}]e^{\frac{-1}{4\pi h}\int d^2z (G_{\mu\nu}+B_{\mu\nu})\partial X^\mu\bar\partial X^\nu + \int R^{(2)}(\gamma)(\Phi_0(X)+\sqrt{g})}$$

# Main Conjecture



Consider

$$\mathbf{F}^{\cdot}_{b-} = \mathcal{F}^{\cdot} \otimes \bar{\mathcal{F}}^{\cdot}|_{b^-=0}$$

with the $L_\infty$-algebra structure given by Lian-Zuckerman construction. One can explicitly check that GMC symmetry

$$\Psi \to \Psi + Q\Lambda + [\Psi, \Lambda]_h + \frac{1}{2}[\Psi, \Psi, \Lambda]_h + \ldots,$$

reproduces

$$\mathbb{M} \to \mathbb{M} - D\alpha + \phi_1(\alpha, \mathbb{M}) + \phi_2(\alpha, \mathbb{M}, \mathbb{M}).$$

up to the second order in $\mathbb{M}$.

Conjecture: The corresponding Maurer-Cartan equation gives Einstein equations on $G, B, \Phi$ expressed in terms of Beltrami-Courant differential. The symmetries of the Maurer-Cartan equation reproduce mentioned above symmetries of Einstein equations.

Thank you!