Generalized Teichmüller Spaces, Spin Structures and Ptolemy Transformations

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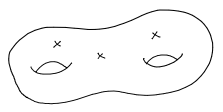
Outline

Let $F_s^g \equiv F$ be the Riemann surface of genus g and s punctures.
We assume $s > 0$ and $2 - 2g - s < 0$.



Teichmüller space $T(F)$ has many incarnations:

- {complex structures on F}/isotopy
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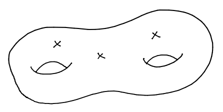

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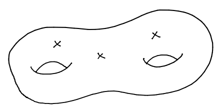

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Representation-theoretic definition:

$$T(F) \;=\; \mathrm{Hom}'(\pi_1(F), PSL(2,\mathbb{R}))/PSL(2,\mathbb{R}),$$

where *Hom'* stands for Homs such that the group elements corresponding to loops around punctures are parabolic ($|\mathrm{tr}| = 2$).

The image $\Gamma \in PSL(2,\mathbb{R})$ is a *Fuchsian group*.

By Poincaré uniformization we have $F = H^+/\Gamma$, where $PSL(2,\mathbb{R})$ acts on the hyperbolic upper-half plane H^+ as oriented isometries, given by fractional-linear transformations:

$$z \to \frac{az+b}{cz+d}.$$

The punctures of $\tilde{F} = H^+$ belong to the real line ∂H^+, which is called *absolute*.

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The primary object of interest in many areas of mathematics is the *moduli space*:

$$M(F) = T(F)/MC(F).$$

The *mapping class group* $MC(F)$: a group of the homotopy classes of orientation preserving homeomorphisms.

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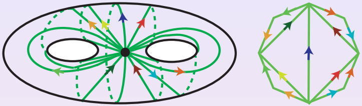

so that one assigns one positive number λ-length for every edge.

This provides coordinates for the decorated Teichmüller space:

$$\tilde{T}(F) = \mathbb{R}_+^s \times T(F)$$

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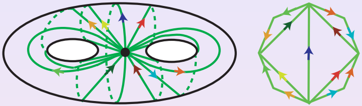

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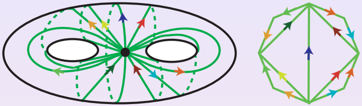

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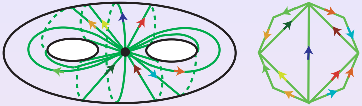

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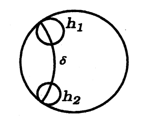

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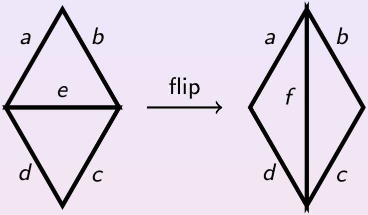

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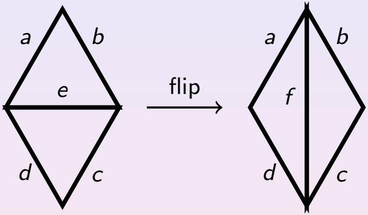

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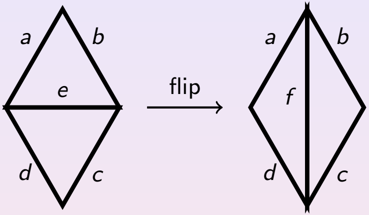

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Penner's coordinates can be used for the quantization of $T(F)$ (L. Chekhov, V. Fock, R. Kashaev, late 90s, early 2000s).

Higher Teichmüller spaces: $PSL(2, \mathbb{R})$ is replaced by some real Lie group G.

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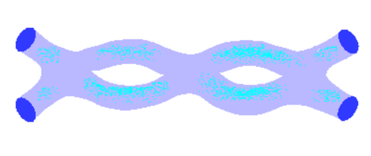

Superstrings, which, according to string theory, are the fundamental objects for the description of our world, carry extra anticommuting parameters θ^i, called *fermions*:

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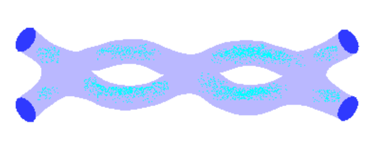

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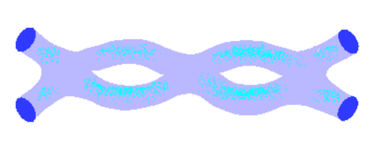

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That leads to generalizations of Teichmüller spaces, relevant for string theory, called $\mathcal{N} = 1$ and $\mathcal{N} = 2$ super-Teichmüller spaces $ST(F)$, depending on the number of extra fermionic degrees of freedom.

The corresponding supermoduli spaces were intensively studied by various scientists (E. D'Hocker, D. Phong, ...) in the low genus.

However, recently R. Donagi and E. Witten showed that in the higher genus supermoduli spaces are very much involved:

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These $\mathcal{N} = 1$ and $\mathcal{N} = 2$ super-Teichmüller spaces in the terminology of higher Teichmüller theory are related to *supergroups*

$$OSP(1|2), \quad OSP(2|2)$$

correspondingly.

In the late 80s the problem of construction of Penner's coordinates on $ST(F)$ was introduced on Yu.I. Manin's Moscow seminar.

The $\mathcal{N} = 1$ case was solved in:
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The $\mathcal{N} = 2$ case is solved recently in:
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Further directions of study:

- Cluster algebras with anticommuting variables (first attempt by V. Ovsienko, arXiv:1503.01894)
- Quantization of super-Teichmüller spaces (first attempt by J.Teschner et al., arXiv:1512.02617)
- Application to supermoduli theory and calculation of superstring amplitudes
- Higher super-Teichmüller theory for supergroups of higher rank

Cast of Characters

i) Superspaces and supermanifolds

Let $\Lambda(\mathbb{K}) = \Lambda^0(\mathbb{K}) \oplus \Lambda^1(\mathbb{K})$ be an exterior algebra over field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ with (in)finitely many generators 1, e_1, e_2,. . . , so that

$$a = a^{\#} + \sum_i a_i e_i + \sum_{ij} a_{ij} e_i \wedge e_j + \ldots, \quad \# : \Lambda(\mathbb{K}) \to \mathbb{K}$$

$a^{\#}$ is referred to as a *body* of a supernumber.

If $a \in \Lambda^0(\mathbb{K})$, it is called even (bosonic) number

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Note, that odd numbers anticommute.

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Superspace $\mathbb{K}^{(n|m)}$ is:

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One can define $(n|m)$ supermanifolds over $\Lambda(\mathbb{K})$ based on superspaces $\mathbb{K}^{(n|m)}$, where $\{z_i\}$ and $\{\theta_i\}$ serve as *even and odd coordinates*.

Special spaces:
- Upper $\mathcal{N} = N$ super-half-plane (we will need $\mathcal{N} = 1, 2$):

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$$\mathbb{R}_+^{(n|m)} = \{(z_1, z_2, \ldots, z_n|\theta_1, \theta_2, \ldots, \theta_m) \in \mathbb{R}^{(n|m)}|\ z_i^\# > 0, i = 1, \ldots, n\}$$

Superspace $\mathbb{K}^{(n|m)}$ is:

$$\mathbb{K}^{(n|m)} = \{(z_1, z_2, \dots, z_n|\theta_1, \theta_2, \dots, \theta_m) : z_i \in \Lambda^0(\mathbb{K}),\ \theta_j \in \Lambda^1(\mathbb{K})\}$$

One can define $(n|m)$ supermanifolds over $\Lambda(\mathbb{K})$ based on superspaces $\mathbb{K}^{(n|m)}$, where $\{z_i\}$ and $\{\theta_i\}$ serve as *even and odd coordinates*.

Special spaces:

• Upper $\mathcal{N} = N$ super-half-plane (we will need $\mathcal{N} = 1, 2$):

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ii) Supergroup $OSp(1|2)$

Definition:

$(2|1) \times (2|1)$ supermatrices g, obeying the relation

$$g^{st} J g = J,$$

where

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the supertranspose g^{st} of g is given by

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Some remarks:

• Lie superalgebra $osp(1|2)$:

Three even $h, X_\pm$ and two odd $v_\pm$ generators, satisfying the following commutation relations:

$$[h, v_\pm] = \pm v_\pm, \quad [v_\pm, v_\pm] = \mp 2X_\pm, \quad [v_+, v_-] = h.$$

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$OSp(1|2)$ acts on $\mathcal{N}=1$ super half-plane H^+, with the absolute $\partial H^+ = \mathbb{R}^{1|1}$ by superconformal fractional-linear transformations:

$$z \to \frac{az+b}{cz+d} + \eta\frac{\gamma z+\delta}{(cz+d)^2},$$

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iii) Ideal triangulations and trivalent fatgraphs

• Ideal triangulation of F: triangulation Δ of F with punctures at the vertices, so that each arc connecting punctures is not homotopic to a point rel punctures.

• Trivalent fatgraph: trivalent graph τ with cyclic orderings on half-edges about each vertex.

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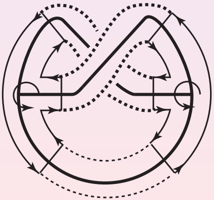
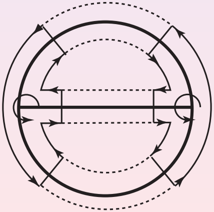


Fatgraph for F_1^1



Fatgraph for F_0^3

iv) ($\mathcal{N}=1$) Super-Teichmüller space

From now on let

$$ST(F) \;=\; \mathrm{Hom}'(\pi_1(F), OSp(1|2))/OSp(1|2).$$

Super-Fuchsian representations comprising Hom' are defined to be those whose projections

$$\pi_1 \to OSp(1|2) \to SL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$$

are Fuchsian group, corresponding to F.

Trivial bundle $S\tilde{T}(F) \;=\; \mathbb{R}_+^s \times ST(F)$ is called the decorated super-Teichmüller space.

Unlike (decorated) Teichmüller space, $ST(F)$ ($S\tilde{T}(F)$) has 2^{2g+s-1} connected components labeled by spin structures on F.

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v) Spin structures

Textbook definition:

Let M be an oriented n-dimensional Riemannian manifold, P_{SO} is an orthonormal frame bundle, associated with TM. A *spin structure* is a 2-fold covering map $P \to P_{SO}$, which restricts to $Spin(n) \to SO(n)$ on each fiber.

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There are several ways to describe spin structures on F:

- D. Johnson (1980):

Quadratic forms $q : H_1(F, \mathbb{Z}_2) \to \mathbb{Z}_2$, which are quadratic with respect to the intersection pairing $\cdot : H_1 \otimes H_1 \to \mathbb{Z}_2$, i.e. $q(a+b) = q(a) + q(b) + a \cdot b$ if $a, b \in H_1$.

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A spin structure on a uniformized surface $F = \mathcal{U}/\Gamma$ is determined by a lift $\tilde{\rho} : \pi_1 \to SL(2, \mathbb{R})$ of $\rho : \pi_1 \to PSL_2(\mathbb{R})$. Quadratic form q is computed using the following rules: $\mathrm{trace}\ \tilde{\rho}(\gamma) > 0$ if and only if $q([\gamma]) \neq 0$, where $[\gamma] \in H_1$ is the image of $\gamma \in \pi_1$ under the mod two Hurewicz map.

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Equivalence classes $\mathcal{O}(\tau)$ of all orientations on a trivalent fatgraph spine $\tau \subset F$, where the equivalence relation is generated by reversing the orientation of each edge incident on some fixed vertex, with the added bonus of a computable evolution under flips:

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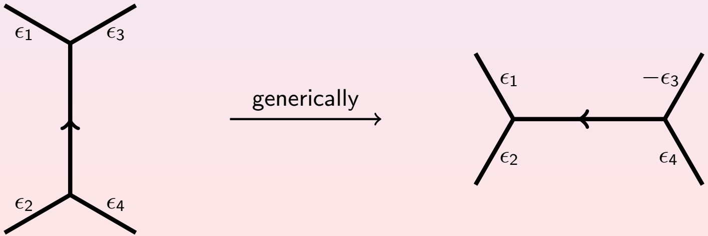

Coordinates on $S\tilde{T}(F)$

Fix a surface $F = F_g^s$ as above and

- $\tau \subset F$ is some trivalent fatgraph spine
- ω is an orientation on the edges of τ whose class in $\mathcal{O}(\tau)$ determines the component C of $S\tilde{T}(F)$

Then there are global affine coordinates on C:

- one even coordinate called a λ-length for each edge
- one odd coordinate called a μ-invariant for each vertex of τ,

the latter of which are taken modulo an overall change of sign.

Alternating the sign in one of the fermions corresponds to the reflection on the spin graph.

The above λ-lengths and μ-invariants establish a real-analytic homeomorphism

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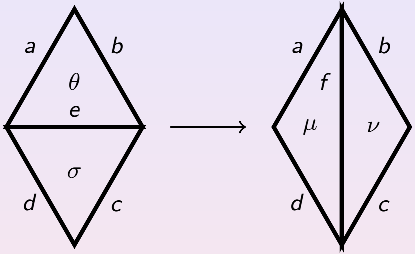
Superflips

When all a, b, c, d are different edges of the triangulations of F,



Ptolemy transformations are as follows:

$$ef = (ac + bd)\Big(1 + \frac{\sigma\theta\sqrt{\chi}}{1+\chi}\Big),$$

$$\nu = \frac{\sigma + \theta\sqrt{\chi}}{\sqrt{1+\chi}}, \quad \mu = \frac{\sigma\sqrt{\chi} - \theta}{\sqrt{1+\chi}}.$$

$\chi = \frac{ac}{bd}$ denotes the cross-ratio, and the evolution of spin graph follows from the construction associated to the spin graph evolution rule.

• These coordinates are natural in the sense that if $\varphi \in MC(F)$ has induced action $\tilde{\varphi}$ on $\tilde{\Gamma} \in S\tilde{T}(F)$, then $\tilde{\varphi}(\tilde{\Gamma})$ is determined by the orientation and coordinates on edges and vertices of $\varphi(\tau)$ induced by φ from the orientation ω, the λ-lengths and μ-invariants on τ.

• There is an even 2-form on $S\tilde{T}(F)$ which is invariant under super Ptolemy transformations, namely,

$$\omega = \sum_v d\log a \wedge d\log b + d\log b \wedge d\log c + d\log c \wedge d\log a - (d\theta)^2,$$

where the sum is over all vertices v of τ where the consecutive half edges incident on v in clockwise order have induced λ-lengths a, b, c and θ is the μ-invariant of v.

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Take instead of λ-lengths shear coordinates $z_e = \log\left(\frac{ac}{bd}\right)$ for every edge e, which are subject to linear relation: the sum of all z_e adjacent to a given vertex $= 0$.

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Sketch of construction via hyperbolic supergeometry

XIXth century perspective on hyperbolic (super)geometry:

$OSp(1|2)$ acts on super-Minkowski space $\mathbb{R}^{2,1|2}$ (in the bosonic case $PSL(2, \mathbb{R})$ acts on $\mathbb{R}^{2,1}$).

If $A = (x_1, x_2, y, \phi, \theta)$ and $A' = (x_1', x_2', y', \phi', \theta')$ in $\mathbb{R}^{2,1|2}$, the pairing is:

$$\langle A, A' \rangle = \frac{1}{2}(x_1 x_2' + x_1' x_2) - yy' + \phi\theta' + \phi'\theta.$$

Two surfaces of special importance for us are

- Superhyperboloid $\mathbb{H}$ consisting of points $A \in \mathbb{R}^{2,1|2}$ satisfying the condition $\langle A, A \rangle = 1$
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Sketch of construction via hyperbolic supergeometry

XIXth century perspective on hyperbolic (super)geometry:

$OSp(1|2)$ acts on super-Minkowski space $\mathbb{R}^{2,1|2}$ (in the bosonic case $PSL(2,\mathbb{R})$ acts on $\mathbb{R}^{2,1}$).

If $A=(x_1,x_2,y,\phi,\theta)$ and $A'=(x_1',x_2',y',\phi',\theta')$ in $\mathbb{R}^{2,1|2}$, the pairing is:

$$\langle A, A'\rangle = \frac{1}{2}(x_1x_2'+x_1'x_2) - yy' + \phi\theta' + \phi'\theta.$$

Two surfaces of special importance for us are

- Superhyperboloid $\mathbb{H}$ consisting of points $A\in\mathbb{R}^{2,1|2}$ satisfying the condition $\langle A, A\rangle = 1$
- Positive super light cone L^+ consisting of points $B\in\mathbb{R}^{2,1|2}$ satisfying $\langle B, B\rangle = 0$,

where $x_1^{\#}, x_2^{\#} \geq 0$.

There is an equivariant projection from $\mathbb{H}$ on the $\mathcal{N}=1$ super upper half-plane H^+.

The light cone

$OSp(1|2)$ does not act transitively on L^+:

The space of orbits is labelled by odd variable up to a sign.

We pick an orbit of the vector $(1, 0, 0, 0, 0)$ and denote it L_0^+.

There is an equivariant projection from L_0^+ to $\mathbb{R}^{1|1} = \partial H^+$.

Goal: Construction of the π_1-equivariant lift for all the data from the universal cover $\tilde{F}$, associated to its triangulation to L_0^+.

Such equivariant lift gives the representation of π_1 in $OSp(1|2)$.

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Orbits of 2 and 3 points in L_0^+

- There is a unique $OSp(1|2)$-invariant of two linearly independent vectors $A, B \in L_0^+$, and it is given by the pairing $\langle A, B\rangle$, the square root of which we will call λ-length.

Let $\zeta^b \zeta^e \zeta^a$ be a positive triple in L_0^+. Then there is $g \in OSp(1|2)$, which is unique up to composition with the fermionic reflection, and unique even r, s, t, which have positive bodies, and odd θ so that

$$g \cdot \zeta^e = t(1,1,1,\theta,\theta), \;\; g \cdot \zeta^b = r(0,1,0,0,0), \;\; g \cdot \zeta^a = s(1,0,0,0,0).$$

- The moduli space of $OSp(1|2)$-orbits of positive triples in the light cone is given by $(a, b, e, \theta) \in \mathbb{R}_+^{3|1}/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by fermionic reflection.

On the superline $\mathbb{R}^{1|1}$ the parameter θ is known as *Manin invariant*.

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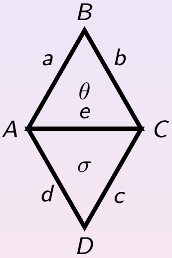
Orbits of 4 points in L_0^+: basic calculation

Suppose points A, B, C are put in the standard position.

The 4th point D, so that two new λ- lengths are c, d.



Fixing the sign of θ, we fix the sign of Manin invariant σ in terms of coordinates of D.

Important observation: if we turn the picture upside down, then

$$(\theta, \sigma) \to (\sigma, -\theta)$$

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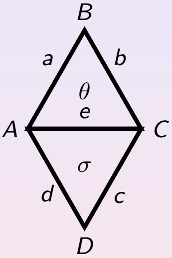

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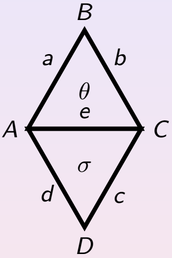

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The lift of ideal triangulation to super-Minkowski space

Denote:

- Δ is ideal trangulation of F, $\tilde{\Delta}$ is ideal triangulation of the universal cover $\tilde{F}$
- Δ_∞ ($\tilde{\Delta}_\infty$)-collection of ideal points of F ($\tilde{F}$).

Consider Δ together with:

• the orientation on the fatgraph $\tau(\Delta)$,

coordinate system $\tilde{C}(F, \Delta)$, i.e.

• positive even coordinate for every edge

• odd coordinate for every triangle

We call coordinate vectors $\vec{c}$, $\vec{c'}$ equivalent if they are identical up to overall reflection of sign of odd coordinates.

Let $C(F, \Delta) \equiv \tilde{C}(F, \Delta)/\sim$. This implies that

$$C(F, \Delta) \simeq \mathbb{R}_+^{6g+3s-6|4g+2s-4}/\mathbb{Z}_2$$

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Then there exists a lift for each $\vec{c} \in \ell : \tilde{\Delta}_\infty \to L_0^+$, with the property:

for every quadrilateral $ABCD$, if the arrow is pointing from σ to θ then the lift is given by the picture from the previous slide up to post-composition with the element of $OSp(1|2)$.

The construction of ℓ can be done in a recursive way:

Such lift is unique up to post-composition with $OSp(1|2)$ group element and it is π_1-equivariant. This allows us to construct representation of π_1 in $OSP(1|2)$, based on the provided data.

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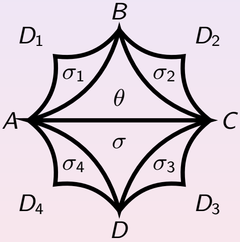

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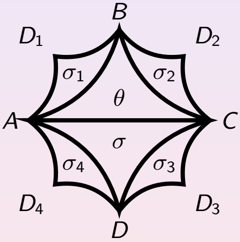

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Theorem

Fix $F, \Delta, \tau(\Delta)$ as before. Let ω be an orientation, corresponding to a specified spin structure s of F. Given a coordinate vector $\vec{c} \in \tilde{C}(F, \Delta)$, there exists a map called the lift,

$$\ell_\omega : \tilde{\Delta}_\infty \to L_0^+$$

which is uniquely determined up to post-composition by $OSp(1|2)$ under admissibility conditions discussed above, and only depends on the equivalent classes $C(F, \Delta)$ of the coordinates.

There is a representation $\hat{\rho} : \pi_1 := \pi_1(F) \to OSp(1|2)$, uniquely determined up to conjugacy by an element of $OSp(1|2)$ such that

(1) ℓ is π_1-equivariant, i.e. $\hat{\rho}(\gamma)(\ell(a)) = \ell(\gamma(a))$ for each $\gamma \in \pi_1$ and $a \in \tilde{\Delta}_\infty$;

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$$\rho : \pi_1 \xrightarrow{\hat{\rho}} OSp(1|2) \to SL(2, \mathbb{R}) \to PSL(2, \mathbb{R})$$

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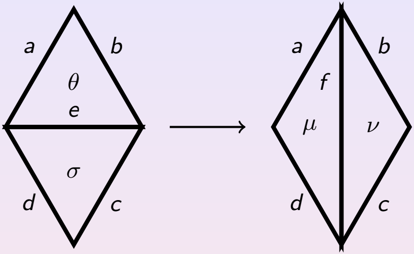
The super-Ptolemy transformations



$$ef = (ac+bd)\Big(1+\frac{\sigma\theta\sqrt{\chi}}{1+\chi}\Big),$$

$$\nu = \frac{\sigma+\theta\sqrt{\chi}}{\sqrt{1+\chi}}, \quad \mu = \frac{\sigma\sqrt{\chi}-\theta}{\sqrt{1+\chi}}$$

are the consequence of light cone geometry.

The space of all such lifts ℓ_ω coincides with the decorated super-Teichmüller space $S\tilde{T}(F) = \mathbb{R}_+^s \times ST(F)$.

In order to remove the decoration, one can pass to shear coordinates $z_e = \log\left(\frac{ac}{bd}\right)$.

It is easy to check that the 2-form

$$\omega = \sum_{\Delta} d\log a \wedge d\log b + d\log b \wedge d\log c + d\log c \wedge d\log a - (d\theta)^2$$

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$\mathcal{N}=2$ super-Teichmüller theory: prerequisites

$\mathcal{N}=2$ super-Teichmüller space is related to $OSP(2|2)$ supergroup of rank 2.

It is more useful to work with its 3×3 incarnation, which is isomorphic to $\Psi \ltimes SL(1|2)_0$, where Ψ is a certain automorphism of the Lie algebra $\mathfrak{sl}(1|2) \simeq \mathfrak{osp}(2|2)$.

$SL(1|2)_0$ is a supergroup, consisting of supermatrices

$$g = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix}$$

such that $f>0$ and their Berezinian $=1$.

This group acts on the space $\mathbb{C}^{1|2}$ as superconformal franctional-linear transformations.

As before, $\mathcal{N}=2$ super-Fuchsian groups are the ones whose projections

$$\pi_1 \to OSP(2|2) \to SL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$$

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As before, $\mathcal{N} = 2$ super-Fuchsian groups are the ones whose projections

$$\pi_1 \to OSP(2|2) \to SL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$$

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$\mathcal{N} = 2$ super-Teichmüller theory: prerequisites

$\mathcal{N} = 2$ super-Teichmüller space is related to $OSP(2|2)$ supergroup of rank 2.

It is more useful to work with its 3×3 incarnation, which is isomorphic to $\Psi \ltimes SL(1|2)_0$, where Ψ is a certain automorphism of the Lie algebra $\mathfrak{sl}(1|2) \simeq \mathfrak{osp}(2|2)$.

$SL(1|2)_0$ is a supergroup, consisting of supermatrices

$$g = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix}$$

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Note, that the pure bosonic part of $SL(1|2)_0$ is $GL^+(2,\mathbb{R})$.

Therefore, the construction of coordinates requires a new notion: $\mathbb{R}_+$-graph connection.

A G-graph connection on τ is the assignment $h_e \in G$ to each oriented edge e of τ so that $h_{\bar{e}} = h_e^{-1}$ if $\bar{e}$ is the opposite orientation to e. Two assignments $\{h_e\}, \{h'_e\}$ are equivalent iff there are $t_v \in G$ for each vertex v of τ such that $h'_e = t_v h_e t_w^{-1}$ for each oriented edge $e \in \tau$ with initial point v and terminal point w.

The moduli space of flat G-connections on F is isomorphic to the space of equivalent G-graph connections on τ.

By the way, spin structures can be identified with equivalence classes of $\mathbb{Z}_2$-graph connections.

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We say that two coordinate vectors of $\tilde{C}(F, \Delta)$ are equivalent if they are related by a finite number of such vertex rescalings (i.e. gauge transformations). In particular the underlying $\mathbb{R}_+$-graph connections on τ are equivalent.

Let $C(F, \Delta) := \tilde{C}(F, \Delta)/\sim$ be the equivalent classes of coordinate vectors. Then it can be represented by coordinates with $h_a h_b h_e = 1$ for the ratios of the same triangle. This implies that

$$C(F, \Delta) \simeq \mathbb{R}_+^{8g+4s-7|8g+4s-8}/\mathbb{Z}_2 \times \mathbb{Z}_2$$

Note, that two involutions we have, one corresponding to the fermion reflection and another one corresponding to Ψ give rise to two spin structures, which enumerate components of the $\mathcal{N} = 2$ super-Teichmüller space.

The light cone L_0^+ and upper sheet hyperboloid $\mathbb{H}_0^+$ in this case are certain orbits in a pseudo-euclidean superspace $\mathbb{R}^{2,2|4}$.

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Theorem

Fix F, Δ, τ as before. Let $\omega_{sign} := \omega_{s_{sign},\tau}$ be a representative, corresponding to a specified spin structure s_{sign} of F, and let $\omega_{inv} := \omega_{s_{inv},\tau}$ be the representative of another spin structure s_{inv}.

Given a coordinate vector $\vec{c} \in \tilde{C}(F,\Delta)$ there exists a map called the lift,

$$\ell_{\omega_{sign},\omega_{inv}} : \tilde{\Delta}_\infty \to L_0^+,$$

which is uniquely determined up to post-composition by $OSp(2|2)$ under some admissibility conditions, and only depends on the equivalent classes $C(F,\Delta)$ of the coordinates. Then there is a representation $\hat{\rho} : \pi_1 := \pi_1(F) \to OSp(2|2)$, uniquely determined up to conjugacy by an element of $OSp(2|2)$ such that

(1) ℓ is π_1-equivariant, i.e. $\hat{\rho}(\gamma)(\ell(a)) = \ell(\gamma(a))$ for each $\gamma \in \pi_1$ and $a \in \tilde{\Delta}_\infty$;

(2) $\hat{\rho}$ is a super-Fuchsian representation, i.e. the natural projection

$$\rho : \pi_1 \xrightarrow{\hat{\rho}} OSp(2|2) \to SL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$$

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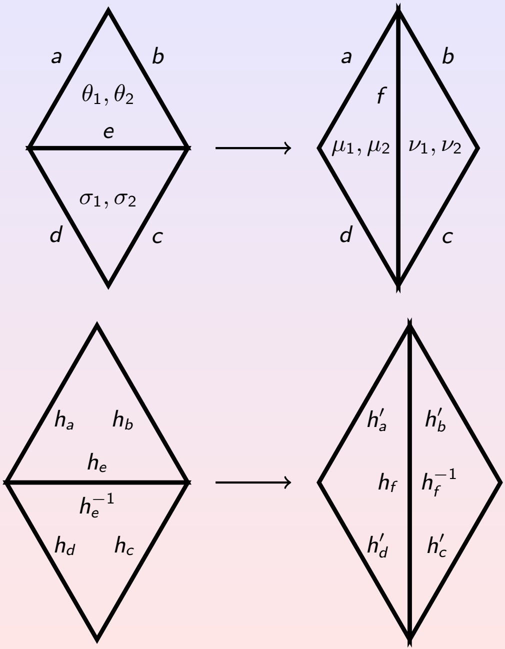
Generic Ptolemy transformations are:



and the transformation formulas are as follows:

$$ef = (ac+bd)\left(1+\frac{h_e^{-1}\sigma_1\theta_2}{2(\sqrt{\chi}+\sqrt{\chi}^{-1})}+\frac{h_e\sigma_2\theta_1}{2(\sqrt{\chi}+\sqrt{\chi}^{-1})}\right),$$

$$\mu_1=\frac{h_e\theta_1+\sqrt{\chi}\sigma_1}{\mathcal{D}},\quad \mu_2=\frac{h_e^{-1}\theta_2+\sqrt{\chi}\sigma_2}{\mathcal{D}},$$

$$\nu_1=\frac{\sigma_1-\sqrt{\chi}h_e\theta_1}{\mathcal{D}},\quad \nu_2=\frac{\sigma_2-\sqrt{\chi}h_e^{-1}\theta_2}{\mathcal{D}},$$

$$h'_a=\frac{h_a}{h_e c_\theta},\quad h'_b=\frac{h_b c_\theta}{h_e},\quad h'_c=h_c\frac{c_\theta}{c_\mu},\quad h'_d=h_d\frac{c_\nu}{c_\theta},\quad h_f=\frac{c_\sigma}{c_\theta^2},$$

where

$$\mathcal{D}:=\sqrt{1+\chi+\frac{\sqrt{\chi}}{2}(h_e^{-1}\sigma_1\theta_2+h_e\sigma_2\theta_1)},$$

$$c_\theta:=1+\frac{\theta_1\theta_2}{6}.$$

Remarks

The space of all lifts $\ell_{\omega_{sign},\omega_{inv}}$ is called decorated $\mathcal{N}=2$ super-Teichmüller space, which is again $\mathbb{R}^s_+$-bundle over $\mathcal{N}=2$ super-Teichmüller space.

Removal of the decoration is done using a similar procedure, using shear coordinates.

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Open problems/directions

1) Cluster superalgebras

2) Weil-Petersson-form in $\mathcal{N} = 2$ case

3) Duality between $\mathcal{N} = 2$ super Riemann surfaces and $(1|1)$-supermanifolds

4) Quantization of super-Teichmüller spaces

5) Weil-Petersson volumes

6) Application to supermoduli theory and calculation of superstring amplitudes

7) Higher super-Teichmüller theory for supergroups of higher rank

Thank you!