Algebraic and geometric structures of mathematical physics

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May 27, 2016

Outline

I work in Representation Theory with applications in Algebra, Geometry, Topology and Mathematical Physics.

The "gameplay" of modern mathematics is the interplay between algebra and geometry. In particular, this applies to mathematics emerging from mathematical physics problems.

The famous examples involve mirror symmetry, Khovanov homology, topological quantum field theories, geometric Langlands correspondence, umbral moonshine, which target a lot of areas of mathematics, from algebraic topology to number theory.

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In this presentation:

Coordinates on super-Teichmüller spaces

Homotopy algebras, Conformal Field Theory, and Einstein equations

Towards the continuous Kazhdan-Lusztig correspondence

Extra topics

Coordinates on Super-Teichmüller spaces

Let $F_s^g \equiv F$ be the Riemann surface of genus g and s punctures.

Teichmüller space $T(F)$ has many incarnations:

- {complex structures on F}/isotopy
- {conformal structures on F}/isotopy
- {hyperbolic structures on F}/isotopy

Representation-theoretic definition:
$T(F) = \mathrm{Hom}'(\pi_1(F), PSL(2,\mathbb{R}))/PSL(2,\mathbb{R})$.

The mapping class group $MC(F)$: group of homotopy classes of orientation preserving homeomorphisms. It acts on $T(F)$ by outer automorphisms of $\pi_1(F)$.

The primary object of interest is the *moduli space*:

$$M(F) = T(F)/MC(F).$$

The goal is to find system of coordinates on $T(F)$, so that $MC(F)$ is realized in the simplest possible way.

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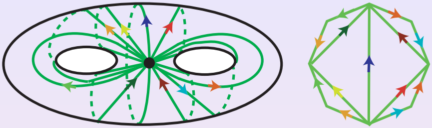

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The action of $MC(F)$ can be described combinatorially using elementary transformations called flips:

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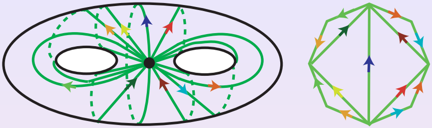

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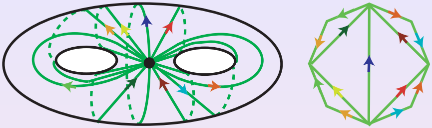

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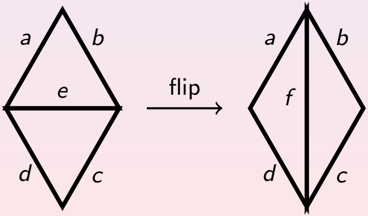

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Penner's coordinates can be used for the quantization of $T(F)$ (V. Fock, R. Kashaev, late 90s, early 2000s).

Higher (super)Teichmüller spaces: $PSL(2, \mathbb{R})$ is replaced by some reductive (super)group G. In the case of reductive groups G the construction of coordinates was given by V.Fock and A. Goncharov (2003).

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$N = 1$ and $N = 2$ Super-Teichmüller spaces $ST(F)$, related to supergroups $OSP(1|2), SL(1|2)$ correspondingly. In the late 80s the problem of construction of Penner's coordinates on $ST(F)$ was introduced on Yu.I. Manin's Moscow seminar.

The N=1 case was solved nearly 30 years later in:
R. Penner, A. Zeitlin, arXiv:1509.06302.
The N=2 case is solved in collaboration with
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Further directions of study:

- Cluster algebras with anticommuting variables
- Quantization of super-Teichmüller spaces (first attempt by J.Teschner et al. arXiv:1512.02617)
- Application to supermoduli theory and calculation of superstring amplitudes, which are highly nontrivial due to recent results of R. Donagi and E. Witten
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Homotopy algebras, Conformal Field Theory and Einstein equations

Early days of String Theory (the 1980s):

Einstein equations with matter fields emerged as consistency conditions for two-dimensional models.

At the same time it was discovered that linearized Einstein equations and their symmetries can be represented in terms of formalism of homological algebra:

$$Q\Psi = 0, \quad \Psi \to \Psi + Q\Lambda,$$

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It was conjectured in the 90s (A. Sen, B. Zwiebach,...) that Einstein equations and their symmetries are Generalized Maurer-Cartan equations and their symmetries:

$$Q\Psi + \frac{1}{2}[\Psi, \Psi] + \frac{1}{3!}[\Psi, \Psi, \Psi] + \ldots = 0$$
$$\Psi \to \Psi + Q\Lambda + [\Psi, \Lambda] + \frac{1}{2}[\Psi, \Psi, \Lambda] + \ldots,$$

where polylinear operations satisfy quadratic relations and generate L_∞-algebra.

The above L_∞ algebra structure involves perturbative expansion around flat metric (local, destroys geometry).

Claim:

There is a much deeper structure, called G_∞-algebra (defined by D. Tamarkin and B. Tsygan in the early 2000s), governing Einstein equations.
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The construction involves two important mathematical objects:

- Sheaves of vertex operator algebras
- Courant algebroids

Sheaves of vertex operator algebras (VOA) is a mathematical tool to describe certain two-dimensional physical models (2d conformal field theories) on a given manifold M (introduced by V. Gorbounov, F. Malikov, V, Schechtman, A. Vaintrob in the early 2000s).

Courant algebroids are now primarily related to the subject called "Generalized Geometry", introduced by N. Hitchin in the late 90s.

Important object is the Courant bracket:
in the simplest case it is the generalization of standard Lie bracket on the sections of direct sum of tangent and cotangent bundles of M. Unlike the Lie bracket, Courant bracket satisfies Jacobi identity only up to homotopy.

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Towards the continuous Kazhdan-Lusztig correspondence

Kazhdan-Lusztig correspondence (the early 90s).

Equivalence between

- Braided tesor category of finite-dimensional representations of quantum group $U_q(\mathfrak{g})$ (algebraic),
- Braided tensor category of standard highest weigth modules of affine Lie algebra $\mathfrak{g}$ (geometric).

Physical origins of this correspondence come from 3d Chern-Simons theory for Lie group G_c, corresponding to the compact form $\mathfrak{g}_c$ of $\mathfrak{g}$.

Correlation functions give knot polynomials (Jones polynomial for $\mathfrak{g}_c = \mathfrak{su}(2)$), which can be described via representation theory $U_q(\mathfrak{g}_c)$.

On the other hand, the boundary theory is Wess-Zumino-Witten (WZW) model, described by the representation theory of $\hat{\mathfrak{g}}_c$.

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This juxtaposition of theories found applications in many exciting topics in mathematics and physics like

- Topological Field Theories
- Mirror Symmetry
- Khovanov Homology
- Geometry of Hilbert schemes

Currently physicists are interested in the understanding of Chern-Simons theory for noncompact real reductive Lie groups $G_{\mathbb{R}}$, e.g. for $SL(2,\mathbb{R})$.

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One can use the following approach:

Construct WZW model on the boundary for $G_{\mathbb{R}}$. In other words, describe the braided tensor category of unitary modules for $\hat{\mathfrak{g}}_{\mathbb{R}}$.

i) Quantum group side of the story is known in this case. Modular double representations of $U_q(\mathfrak{g}_{\mathbb{R}})$: L. Faddeev and J. Teschner (for $U_q(sl(2,\mathbb{R}))$ in the early 2000s; I.B. Frenkel and I. Ip recently generalized these formulas for higher rank. This is the proper quantum analogue of principal series representations.

ii) The corresponding modules for $\widehat{\mathfrak{sl}}(2,\mathbb{R})$ were constructed in: I.B. Frenkel, A. Zeitlin, Comm.Math.Phys. 326, pp. 145-165, 2014 based on the earlier work A. Zeitlin, J.Funct.Anal. 263, pp. 529-548, 2012. The construction relies on the renormalization technique similar to the one used in field theory.
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Extra topics

- Quantum groups via semi-infininite cohomology,
  I. Frenkel, A. Zeitlin,
  J. Noncomm. Geom., 7, 1007-1026, 2013;
  Comm.Math.Phys., 297, 687-732, 2010
- Geometry of compactified moduli spaces
  and homotopy algebras of CFT,
  A. Zeitlin,
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  Int. J. Math., 23, 1250012, 2012
- Geometric Langlands correspondence for superalgebras.
  Simplest example considered in:
  A. Zeitlin, Lett. Math. Phys., 105, 149-167, 2015
- Integrable models of statistical physics via quantum K-theory,
  P. Pushkar, A. Smirnov, A. Zeitlin, to appear in 2016