On the First Order Formalism in String Theory

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Contents

1 Simple model

Consider the two-dimensional CFT with the following action:

$$S_0 = \frac{1}{2\pi\alpha'} \int d^2z (p_i \bar{\partial} X^i + p_{\bar{i}} \partial X^{\bar{i}}), \tag{1}$$

where $p, \bar{p}$-fields are (1,0)- and (0,1)-forms correspondingly, while the $X, \bar{X}$-fields are of the weights (0,0) or scalars, the volume element is $d^2z = idz \wedge d\bar{z}$. This Lagrangian provides the following operator product expansions:

$$\begin{aligned}
& X^i(z_1)p_j(z_2) \sim \frac{\alpha' \delta^i_j}{z_1 - z_2}, \\
& X^{\bar{i}}(\bar{z}_1)p_{\bar{j}}(\bar{z}_2) \sim \frac{\alpha' \delta^{\bar{i}}_{\bar{j}}}{\bar{z}_1 - \bar{z}_2}, \\
& X^i(z_1)X^k(z_2) \sim 0, \quad p_i(z_1)p_j(z_2) \sim 0 \\
& X^{\bar{i}}(z_1)X^{\bar{k}}(z_2) \sim 0, \quad p_{\bar{i}}(z_1)p_{\bar{j}}(z_2) \sim 0
\end{aligned} \tag{2}$$

i.e. there are no contractions between the X- and p-fields themselves. Let us, first, perturb this theory by the following vertex operator:

$$V_g = \frac{1}{2\pi\alpha'} g^{i\bar{j}}(X, \bar{X}) p_i p_{\bar{j}} \tag{3}$$

That is the full action is:

$$S_{F.O.} = \frac{1}{2\pi\alpha'} \int d^2z (p_i \bar{\partial} X^i + p_{\bar{i}} \partial X^{\bar{i}} - g^{i\bar{j}} p_i p_{\bar{j}}) \tag{4}$$

On the classical level, solving equations of motion for $p, \bar{p}$, we find that this action is equivalent to:

$$\mathcal{S} = \frac{1}{2\pi\alpha'} \int d^2z g_{i\bar{j}} \bar{\partial} X^i \partial X^{\bar{j}} \tag{5}$$

which can be rewritten in more familiar form:

$$\mathcal{S}=\frac{1}{4\pi\alpha'}\int d^2z(G_{\mu\nu}+B_{\mu\nu})\partial X^\mu\bar{\partial}X^\nu \tag{6}$$

where μ,ν run now over both holomorphic and anti-holomorphic indices, while G, and B are the symmetric Riemann metric and antisymmetric Kalb-Ramond B-field correspondingly, obeying the constraint $G_{i\bar{j}}=-B_{i\bar{j}}$, or

$$G_{s\bar{k}}=g_{s\bar{k}},\quad B_{s\bar{k}}=-g_{s\bar{k}} \tag{7}$$

However, in quantum case the naive functional integral with the action $S_{F.O.}$ over the $p,\bar{p}$-variables:

$$\int[dX][d\bar{X}][dp][d\bar{p}]e^{-S_{F.O.}[X,\bar{X},p,\bar{p}]}=$$
$$\int[dX][d\bar{X}]e^{-\mathcal{S}[X,\bar{X}]}\det(A) \tag{8}$$

gives rise to additional term in the action, which is related to the determinant of the operator:

$$A_{mz,nz'}=V^{-1}g_{mn}(z)\delta_{z,z'} \tag{9}$$

(here we have introduced the discretized world-sheet delta function). Using the relation $\det(A)=e^{\mathrm{Tr}\log A}$, we write

$$\log A_{mz,nz'}=\log\left(g_{m,n}(z)\delta_{z,z'}\right)-\delta_{mz,nz'}\log V \tag{10}$$

and take the trace by $\sum_z=V^{-1}\int d^2z\sqrt{h}$, to make the sum invariant under the change of world-sheet coordinates, where h is determinant of metric on CP^1. Up to a field independent factor, this can be rewritten as

$$\det(A)\sim e^{V^{-1}\int d^2z\sqrt{h}\log g}=e^{\frac{1}{2\pi}\int d^2z\sqrt{h}R\log\sqrt{g}} \tag{11}$$

where R is the world-sheet curvature on CP^1 and g is the determinant of the target-space metric $g_{i\bar{j}}$. Such "phenomenological" calculation shows that the integration over $p, \bar{p}$-variables leads to an extra *dilatonic* term in the action. For the Kähler target-space metric $g^{i\bar{j}}$ the conditions of conformal invariance leads to the vanishing Ricci tensor, and the gauge condition (see below) leads to the constant determinant. In the case of general Hermitean metric g we will show that the conditions of conformal invariance for the first order model (4) leads to the background Einstein equations with a dilaton, thus confirming the phenomenological calculation above.

Now let us analyse divergences arizing in this theory in order to establish more strictly the conformal invariance. First, we encounter the linear divergence (of the order of $(\alpha')^0$) arizing from the contraction of p_i and X^j inside the operator (3) $g^{i\bar{j}}p_i p_{\bar{j}}$ itself. As in the case of vector field, these lead to the transversality, i.e.:

$$\partial_i g^{i\bar{j}} = 0, \quad \partial_{\bar{j}} g^{i\bar{j}} = 0. \tag{12}$$

Note, that in contrast to conventional second-order formalism, where the transversality constrains on-shell vertex operators of the massless photons or gravitons, here the constraint (12) appeares before any mass-shell condition.

Another way to see this, is to study the OPE of (3) with the stress- energy tensor $T = -(\alpha')^{-1} p_i \partial X^i$ (and its counterpart of opposite chirality $\tilde{T} = -(\alpha')^{-1} p_{\bar{i}} \bar{\partial} X^{\bar{i}}$):

$$-(\alpha')^{-1}p_i\partial X^i(z)(2\pi\alpha')^{-1}g^{i\bar{j}}p_ip_{\bar{j}}(z') = \tag{13}$$
$$\frac{-1}{(z-z')^3}(2\pi)^{-1}\partial_i g^{i\bar{j}}p_{\bar{j}}(z') + \frac{1}{(z-z')^2}$$
$$(2\pi\alpha')^{-1}g^{i\bar{j}}p_ip_{\bar{j}}(z') +$$
$$\frac{1}{z-z'}(2\pi\alpha')^{-1}\partial_{z'}g^{i\bar{j}}p_ip_{\bar{j}}(z') + \dots$$

Two last terms in the r.h.s. of (13) are standard singular terms from the OPE of the stress-tensor with primary field of unit dimension (so that, integrated over the world-sheet it becomes co-ordinate invariant). However the first singular term in the r.h.s. (the action of the L_1-Virasoro operator) deviates it from primary operator, unless (12) is satisfied.

Consider now the OPE of two operators (3):

$$V(z_1)V(z_2) \sim \frac{a^{(0,0)}(z_2)}{|z_1-z_2|^4} + \frac{a^{(0,1)}(z_2)}{|z_1-z_2|^2(z_1-z_2)} +$$
$$\frac{a^{(1,0)}(z_2)}{|z_1-z_2|^2(\bar{z}_1-\bar{z}_2)} + \frac{a^{(1,1)}(z_2)}{|z_1-z_2|^2} + \tag{14}$$
$$\frac{a^{(0,2)}(z_2)}{(z_1-z_2)^2} + \frac{a^{(2,0)}(z_2)}{(\bar{z}_1-\bar{z}_2)^2} + \frac{a^{(2,1)}(z_2)}{\bar{z}_1-\bar{z}_2} + \frac{a^{(1,2)}(z_2)}{z_1-z_2}$$

The $a^{(i,j)}$ coefficients in the OPE are not independent and satisfy the following conditions:

$$a^{(1,0)} = 1/2\partial a^{(0,0)}, \quad a^{(0,1)} = 1/2\bar{\partial}a^{(0,0)}, \tag{15}$$
$$a^{(1,2)} = 1/2(\bar{\partial}a^{(1,1)} - 1/2\partial\bar{\partial}^2 a^{(0,0)} + \partial a^{(0,2)}),$$
$$a^{(2,1)} = 1/2(\partial a^{(1,1)} - 1/2\partial^2\bar{\partial}a^{(0,0)} + \bar{\partial}a^{(2,0)})$$

We will consider the point splitting regularization, that is the second order of the perturbation series in

V has the following form:

$$\frac{1}{2}\int_{|z_1-z_2|>\epsilon} d^2z_1 d^2z_2 V(z_1)V(z_2) \tag{16}$$

We can write the corresponding OPE as follows, leaving only the terms at the level $(\alpha')^0$ giving the singularities in the limit $\epsilon \to 0$:

$$V(z_1)V(z_2) \sim \frac{a^{(1,1)}(z_2)}{|z_1-z_2|^2} + O(\alpha'), \tag{17}$$
$$a^{(1,1)} = (2\pi^2)^{-1}(g^{\bar{i}j}\partial_{\bar{i}}\partial_j g^{\bar{k}l} - \partial_{\bar{s}} g^{\bar{k}r}\partial_r g^{\bar{s}l})p_{\bar{k}}p_l + O(\alpha')$$

Now, let us turn to (the only at the level $(\alpha')^0$) logarithmic divergence, coming from the double p, X contractions of these two operators. To make the theory conformally invariant we need to put it to zero:

$$g^{\bar{i}j}\partial_{\bar{i}}\partial_j g^{\bar{k}l} - \partial_{\bar{p}} g^{\bar{k}r}\partial_r g^{\bar{p}l} = 0 \tag{18}$$

This quadratic system of equations with the gauge condition (12) is equivalent to the sytem of the gauged Einstein equations with a Kalb-Ramond field and a dilaton:

$$R^{\mu\nu} = -1/4 H^{\mu\lambda\rho}H^{\nu}_{\lambda\rho} + 2\nabla^{\mu}\nabla^{\nu}\Phi, \tag{19}$$
$$\nabla_{\mu}H^{\mu\nu\rho} - 2(\nabla_{\lambda}\Phi)H^{\lambda\nu\rho} = 0, \tag{20}$$
$$4(\nabla_{\mu}\Phi)^2 - 4\nabla_{\mu}\nabla^{\mu}\Phi + R + 1/12 H_{\mu\nu\rho}H^{\mu\nu\rho} = 0. \tag{21}$$

where the change of variables from g to the physical ones G, B, Φ is given by the following expressions:

$$G_{s\bar{k}} = g_{s\bar{k}}, \quad B_{s\bar{k}} = -g_{s\bar{k}}, \quad \Phi = ln\sqrt{g}. \tag{22}$$

Proof:

We will use the following formulae (see appendix Γ of the book of V.A. Fock "Theory of Space, Time and Gravity"):

$$R^{\mu\nu} = -1/2 G^{\alpha\beta}\partial_\alpha\partial_\beta G^{\mu\nu} - \Gamma^{\mu\nu} + \Gamma^{\mu,\alpha\beta}\Gamma^{\nu}_{\alpha\beta}, \qquad (23)$$

where

$$\begin{aligned} \Gamma^{\mu\nu} &= G^{\mu\rho}G^{\nu\sigma}\Gamma_{\rho\sigma}, \\ \Gamma_{\rho\sigma} &= 1/2(\partial_\rho\Gamma_\sigma + \partial_\sigma\Gamma_\rho) - \Gamma^{\nu}_{\rho\sigma}\Gamma_\nu \\ \Gamma_\nu &= G^{\alpha\beta}\partial_\beta G_{\alpha\nu} - 1/2\partial_\nu ln(G) \qquad (24) \end{aligned}$$

Now let's remind that in our case $\partial_\mu G^{\mu\rho} = 0$. This leads to the simple relation: $\Gamma_\nu = -1/2\partial_\nu ln(G)$. Therefore $\Gamma_{\mu\nu} = -2\nabla_\mu\nabla_\nu\Phi$, $\Phi = ln\sqrt{g}$, where g is the determinant of the matrix $g_{i\bar{j}}$. Now let's study the third term in (23). At first we consider the component form of the $\Gamma^{\nu}_{\alpha\beta}$:

$$\begin{aligned} \Gamma^{i}_{rs} &= 1/2 g^{i\bar{k}}(\partial_r g_{\bar{k}s} + \partial_s g_{\bar{k}r}), \qquad (25) \\ \Gamma^{i}_{r\bar{s}} &= 1/2 g^{i\bar{k}}(\partial_{\bar{s}} g_{r\bar{k}} - \partial_{\bar{k}} g_{r\bar{s}}) \quad and \quad c.c. \qquad (26) \end{aligned}$$

All other components are equal to zero. One can see that $\Gamma_{\bar{i},r\bar{s}} = 1/2 H_{\bar{s}\bar{i}r}$. Thus the third term from (23) will provide contribution of H^2 type. The additional summands in $\Gamma\Gamma$ product appears, when we take $\mu = \bar{i}$ and $\nu = j$:

$$\begin{aligned} \Gamma^{\bar{i},kl}\Gamma^{j}_{kl} &= -1/4(g^{k\bar{r}}\partial_{\bar{r}}g^{l\bar{i}} + g^{l\bar{r}}\partial_{\bar{r}}g^{k\bar{i}})g^{j\bar{p}}(\partial_k g_{\bar{p}l} + \partial_l g_{\bar{p}k}) \\ &= -1/4(g^{k\bar{r}}\partial_{\bar{r}}g^{l\bar{i}} - g^{l\bar{r}}\partial_{\bar{r}}g^{k\bar{i}})g^{j\bar{p}}(\partial_k g_{\bar{p}l} - \partial_l g_{\bar{p}k}) - \\ & g^{k\bar{r}}\partial_{\bar{r}}g^{l\bar{i}}g^{j\bar{p}}\partial_l g_{\bar{p}k} = -1/4 H^{\bar{i}kl}H^{j}_{kl} + \partial_{\bar{r}}g^{\bar{i}k}\partial_k g^{\bar{r}j}. \qquad (27) \end{aligned}$$

But this additional term $\partial_{\bar{r}}g^{\bar{i}k}\partial_k g^{\bar{r}j}$ cancel with the first term from RHS of (23) due to the equation (18).

Thus, unifying all the information we have got the relation (23) can be rewritten as:

$$R^{\mu\nu} = -1/4 H^{\mu\lambda\rho} H^{\nu}_{\lambda\rho} + 2\nabla^{\mu}\nabla^{\nu}\Phi. \tag{28}$$

So, the first desired equation is obtained. Similarly, one can prove the following relation:

$$4(\nabla_{\mu}\Phi)^2 - 2\nabla_{\mu}\nabla^{\mu}\Phi - 1/6 H_{\mu\nu\rho}H^{\mu\nu\rho} = 0 \tag{29}$$

Namely, $\partial_{\mu}\Phi = 1/2 g^{\bar{i}k}\partial_{\mu} g_{\bar{i}k}$ That is

$$4(\nabla_{\mu}\Phi)^2 = 2 g^{\bar{l}k}\partial_i g_{\bar{l}k} g^{i\bar{j}} g^{\bar{s}r}\partial_{\bar{j}} g_{\bar{s}r} \tag{30}$$

and

$$-2\nabla_{\mu}\nabla^{\mu}\Phi = -2 g^{\bar{i}j}\partial_{\bar{i}}(g^{\bar{l}k}\partial_j g_{\bar{l}k}) + 2 g^{\bar{i}j}\Gamma^{r}_{\bar{i}j} g^{\bar{l}k}\partial_r g_{\bar{l}k} + 2 g^{\bar{i}j}\Gamma^{\bar{r}}_{\bar{i}j} g^{\bar{l}k}\partial_{\bar{r}} g_{\bar{l}k} \tag{31}$$

Using (25) we arrive to

$$g^{\bar{i}j}\Gamma^{r}_{\bar{i}j} = -1/2 g^{\bar{i}j}\partial_{\bar{l}} g_{\bar{i}j} g^{\bar{l}r}, \quad g^{\bar{i}j}\Gamma^{\bar{r}}_{\bar{i}j} = -1/2 g^{\bar{i}j}\partial_{l} g_{\bar{i}j} g^{\bar{r}l}. \tag{32}$$

The sum of (30) and (31) can be rewritten in the following form:

$$4(\nabla_{\mu}\Phi)^2 - 2\nabla_{\mu}\nabla^{\mu}\Phi = -2 g^{\bar{i}j}\partial_{\bar{i}}(g^{\bar{l}k}\partial_j g_{\bar{l}k}) \tag{33}$$

The H^2 term is equal to:

$$1/6 H_{\mu\nu\rho}H^{\mu\nu\rho} = H_{i\bar{j}\bar{k}}H^{i\bar{j}\bar{k}} = (-\partial_{\bar{k}} g_{i\bar{j}} + \partial_{\bar{j}} g_{i\bar{k}}) (-\partial_s g^{i\bar{j}} g^{s\bar{k}} + \partial_s g^{i\bar{k}} g^{s\bar{j}}) = 2 g^{s\bar{k}}\partial_{\bar{k}} g_{i\bar{j}}\partial_s g^{i\bar{j}} - 2\partial_{\bar{j}} g_{i\bar{k}} g^{s\bar{k}}\partial_s g^{i\bar{j}} \tag{34}$$

Due to the relation (18) we find that the equation (29) is satisfied. Combining (19) and (29) one obtains more convinient one (see (21)):

$$4(\nabla_{\mu}\Phi)^2 - 4\nabla_{\mu}\nabla^{\mu}\Phi + R + 1/12 H_{\mu\nu\rho}H^{\mu\nu\rho} = 0 \tag{35}$$

The third equation one can get by means of simple analysis of (18):

$$\partial_{\bar{i}}(g^{\bar{i}j}\partial_j g^{\bar{k}l} - g^{\bar{k}r}\partial_r g^{\bar{i}l}) = 0 \quad and \quad c.c. \tag{36}$$

this leads to the relation:

$$\partial_{\bar{i}} H^{\bar{i}\bar{k}l} = 0 \quad and \quad c.c. \tag{37}$$

Also the identity

$$\partial_l(g^{\bar{i}j}\partial_j g^{\bar{k}l} - g^{\bar{k}r}\partial_r g^{\bar{i}l}) = 0 \quad and \quad c.c. \tag{38}$$

yields:

$$\partial_l H^{l\bar{i}\bar{k}} = 0 \quad and \quad c.c. \tag{39}$$

These all the relations can be summarised in the following one:

$$\nabla_\mu H^{\mu\nu\rho} - 2(\nabla_\lambda \Phi) H^{\lambda\nu\rho} = 0 \tag{40}$$

We note here that in the case when metric g is Kahler it is not hard to show, that there is no need in the additional gauge constraint (12) to prove the coincidence of the equation (18) with Einstein equation in the vacuum $R_{i\bar{j}} = 0$. However, the gauge conditions are not unnecessary, they lead to the constant determinant of $g^{i\bar{j}}$:

$$0 = \partial_i g^{i\bar{j}} g_{k\bar{j}} = -g^{i\bar{j}}\partial_i g_{k\bar{j}} = -g^{i\bar{j}}\partial_k g_{i\bar{j}} = -\partial_k \log g \quad and \quad c.c. \tag{41}$$

2 The complete action and the precise analysis of the conformal invariance on the infinitesimal level

Now we perturb our free action by all possible "massless" operators, corresponding to more general deformation of metric, B-field as well as the complex structure. The "complete" perturbed action is now:

$$S_{F.O.} = \frac{1}{2\pi\alpha'}\int d^2z(p_i\bar{\partial}X^i + p_{\bar{i}}\partial X^{\bar{i}}$$
$$-g^{i\bar{j}}p_ip_{\bar{j}} - \bar{\mu}_i^{\bar{j}}\partial X^i p_{\bar{j}} - \mu_{\bar{i}}^{j}\bar{\partial}X^{\bar{i}}p_j - b_{i\bar{j}}\partial X^i\bar{\partial}X^{\bar{j}}) \tag{42}$$

On the classical level excluding $p, \bar{p}$ from the equations of motion we find that this action is equivalent to the following one:

$$\mathcal{S} = \frac{1}{2\pi\alpha'}\int d^2z(g_{i\bar{j}}(\bar{\partial}X^i - \mu^i_{\bar{k}}\bar{\partial}X^{\bar{k}})(\partial X^{\bar{j}} - \bar{\mu}^{\bar{j}}_k\partial X^k)$$
$$-b_{i\bar{j}}\partial X^i\bar{\partial}X^{\bar{j}}), \tag{43}$$

which can be rewritten in the usual sigma model form (6) where G and B are now, compare to the previous section, defined as follows:

$$\begin{aligned}
G_{s\bar{k}} &= g_{\bar{i}j}\bar{\mu}^{\bar{i}}_s\mu^j_{\bar{k}} + g_{s\bar{k}} - b_{s\bar{k}}, \quad B_{s\bar{k}} = g_{\bar{i}j}\bar{\mu}^{\bar{i}}_s\mu^j_{\bar{k}} - g_{s\bar{k}} - b_{s\bar{k}} \\
G_{si} &= -g_{i\bar{j}}\bar{\mu}^{\bar{j}}_s - g_{s\bar{j}}\bar{\mu}^{\bar{j}}_i, \quad G_{\bar{s}\bar{i}} = -g_{\bar{s}j}\mu^j_{\bar{i}} - g_{\bar{i}j}\mu^j_{\bar{s}} \\
B_{si} &= g_{s\bar{j}}\bar{\mu}^{\bar{j}}_i - g_{i\bar{j}}\bar{\mu}^{\bar{j}}_s, \quad B_{\bar{s}\bar{i}} = g_{\bar{i}j}\mu^j_{\bar{s}} - g_{\bar{s}j}\mu^j_{\bar{i}}
\end{aligned} \tag{44}$$

We claim that as for the simple model considered in the previous section the conformal invariance conditions at the order $(\alpha')^0$ are governed by the Einstein equations with a dilaton $\Phi = \sqrt{g}$. Below we are going show this at the infinitesimal level. Namely, consider the case when $g_{i\bar{k}} = \delta_{i\bar{k}} + \alpha_{i\bar{k}}$ with α, as well as all μ, $\bar{\mu}$, b taken infinitesimally small. Here we will be more precise remembering that we are not just seeking the divergences but we need to find how the regularized divergent terms transform under the conformal map and then we will seek the conditions when the anomalus terms cancel providing the conformal invariance of the theory. First, let's look on the transformation of the regularized operators incerted in the action. We will use the standard "holomorphic ordering" regularization, that is there are no contractions of p, X inside the operators $a, \mu, \bar{\mu}$. However, these operators are not primary in general, as it can be easily seen from the OPE with the holomorphic (and antiholomorphic) parts of stress-energy tensor:

$$\begin{aligned} T(z_1)V'(z_2) &= \frac{1}{(z_1-z_2)^3}b^{(0,1)}(z_2) + \frac{1}{(z_1-z_2)^2}V'(z_2) \\ &+ \frac{1}{z_1-z_2}\partial_{z_2}V'(z_2), \\ V' &= (2\pi\alpha')^{-1}(a^{i\bar{j}}p_i p_{\bar{j}} + \bar{\mu}_i^{\bar{j}}\partial X^i p_{\bar{j}} \\ &+ \mu_{\bar{i}}^{j}\bar{\partial}X^{\bar{i}}p_j + b_{i\bar{j}}\partial X^i \bar{\partial}X^{\bar{j}}) \\ b^{(0,1)} &= -\partial_i g^{i\bar{j}}p_{\bar{j}} - \partial_i \mu_{\bar{j}}^{i}\bar{\partial}X^{\bar{j}} \end{aligned} \tag{45}$$

That is, the L_1 Virasoro generator is no longer nilpotent and therefore under the infinitesimal transformation generated by $-a_0L_0 - a_1L_1$ applied to these operators (accompanied with the same one of the opposite chirality) results in the following expression:

$$\begin{aligned} &V' \to V' - (a_0+\bar{a}_0)V' + a_1\partial_i g^{i\bar{j}}p_{\bar{j}} + \bar{a}_1\partial_{\bar{j}}g^{i\bar{j}}p_i + \\ &a_1\partial_i\mu_{\bar{j}}^{i}\bar{\partial}X^{\bar{j}} + \bar{a}_1\partial_{\bar{i}}\bar{\mu}_j^{\bar{i}}\partial X^j \end{aligned} \tag{46}$$

thus giving anomalous terms with the coefficient a_1. Next anomalous terms arize when we consider the perturbation series, namely (we are seeking the terms of the order $(\alpha')^0$) from the second order of the perturbation theory in V. As in the previous section we use here the point splitting regularization. The terms needed to be regularized come from the contractions of $\delta^{i\bar{j}}p_i p_{\bar{j}}$ operator with V', namely:

$$
\int_{|z_1-z_2+a_0(z_1-z_2)+a_1(z_1-z_2)^2|>\epsilon} d^2z_1 d^2z_2 \delta^{i\bar{j}} p_i p_{\bar{j}}(z_1) V'(z_2) - \int_{|z_1-z_2|>\epsilon} d^2z_1 d^2z_2 \delta^{i\bar{j}} p_i p_{\bar{j}}(z_1) V'(z_2) \tag{47}
$$

The contribution of the order $(\alpha')^0$ is given by the double contractions of X and p thus leading to the poles of the second

$$
\begin{aligned}
&\Big(\int_{|z_1-z_2+a_0(z_1-z_2)+a_1(z_1-z_2)^2|>\epsilon} d^2z_1 d^2z_2 \frac{1}{|z_1-z_2|^2} \\
&- \int_{|z_1-z_2|>\epsilon} d^2z_1 d^2z_2 \frac{1}{|z_1-z_2|^2}\Big) \qquad (48)\\
&(\delta^{i\bar{j}} \partial_i \partial_{\bar{j}} g^{k\bar{l}} p_k p_{\bar{l}}(z_2) + \delta^{k\bar{l}} \partial_k \partial_{\bar{l}} \bar{\mu}^{\bar{j}}_i \partial X^i p_{\bar{j}}(z_2) \\
&+ \delta^{k\bar{l}} \partial_k \partial_{\bar{l}} \mu^j_{\bar{i}} \bar{\partial} X^{\bar{i}} p_j(z_2) + \delta^{k\bar{l}} \partial_k \partial_{\bar{l}} b_{i\bar{j}} \partial X^i \bar{\partial} X^{\bar{j}}(z_2)
\end{aligned}
$$

and the third order:

$$
\begin{aligned}
&\Big(\int_{|z_1-z_2+a_0(z_1-z_2)+a_1(z_1-z_2)^2|>\epsilon} d^2z_1 d^2z_2 \frac{1}{(z_1-z_2)|z_1-z_2|^2} - \\
&\int_{|z_1-z_2|>\epsilon} d^2z_1 d^2z_2 \frac{1}{(z_1-z_2)|z_1-z_2|^2}\Big) \qquad (49)\\
&(\delta^{k\bar{l}} \partial_{\bar{l}} \bar{\mu}^{\bar{j}}_k p_{\bar{j}}(z_2) + \delta^{k\bar{l}} \partial_{\bar{l}} b_{k\bar{j}} \bar{\partial} X^{\bar{j}}(z_2)) \\
&+\Big(\int_{|z_1-z_2+a_0(z_1-z_2)+a_1(z_1-z_2)^2|>\epsilon} d^2z_1 d^2z_2 \frac{1}{(\bar{z}_1-\bar{z}_2)|z_1-z_2|^2} - \\
&\int_{|z_1-z_2|>\epsilon} d^2z_1 d^2z_2 \frac{1}{(\bar{z}_1-\bar{z}_2)|z_1-z_2|^2}\Big) \\
&(\delta^{k\bar{l}} \partial_k \mu^j_{\bar{l}} p_j(z_2) + \delta^{k\bar{l}} \partial_k b_{j\bar{l}} \partial X^j(z_2))
\end{aligned}
$$

To calculate the difference of the two integrals we will make the change of variables: $z_1 - z_2 = z$ and then go to the polar coordinates and expand $z = \epsilon e^{i\phi}(1+\rho(\phi))$, to the next order in ϵ, $\rho = -1/2(a_0+\bar{a}_0)-1/2\epsilon(a_1e^{i\phi}+\bar{a}_1e^{-i\phi})$, thus:

$$
\begin{aligned}
&\Big(\int_{|z+a_0z+a_1z^2|>\epsilon} d^2z\frac{1}{z|z^2|} - \int_{|z|>\epsilon} d^2z\frac{1}{z|z^2|}\Big) = \\
&\int_0^{2\pi} d\phi\frac{2\epsilon^2\rho(\phi)}{\epsilon^3e^{i\phi}} = 2\pi a_1
\end{aligned}
\tag{50}
$$

Similarly

$$
\begin{aligned}
&\Big(\int_{|z+a_0z+a_1z^2|>\epsilon} d^2z\frac{1}{\bar{z}|z^2|} - \int_{|z|>\epsilon} d^2z\frac{1}{\bar{z}|z^2|}\Big) = \\
&\int_0^{2\pi} d\phi\frac{2\epsilon^2\rho(\phi)}{\epsilon^3e^{-i\phi}} = 2\pi \bar{a}_1
\end{aligned}
\tag{51}
$$

and

$$
\begin{aligned}
&\Big(\int_{|z+a_0z+a_1z^2|>\epsilon} d^2z\frac{1}{|z^2|} - \int_{|z|>\epsilon} d^2z\frac{1}{|z^2|}\Big) = \\
&\int_0^{2\pi} d\phi\frac{2\epsilon^2\rho(\phi)}{\epsilon^2} = 2\pi (a_0+\bar{a}_0)
\end{aligned}
\tag{52}
$$

We also need to consider the pole of the fourth order from the product of b and $p\bar{p}$ terms and in this case it is easy to see that the difference is again with the a_0 coefficient but with the quadratic divergence $(\frac{1}{\epsilon^2})$. Thus the coefficient of a_0 leads to the following equations:

$$
\begin{aligned}
&\delta^{k\bar{l}}\partial_k\partial_{\bar{l}}a^{i\bar{j}} = 0, \quad \delta^{k\bar{l}}\partial_k\partial_{\bar{l}}\bar{\mu}_i^{\bar{j}} = 0, \\
&\delta^{k\bar{l}}\partial_k\partial_{\bar{l}}\mu_{\bar{i}}^{j} = 0, \quad \delta^{k\bar{l}}\partial_k\partial_{\bar{l}}b_{i\bar{j}} = 0.
\end{aligned}
\tag{53}
$$

the a_1 contribution consists of the two parts (49), (46):

$$
\begin{aligned}
&\partial_i g^{i\bar{j}} + \delta^{k\bar{l}}\partial_{\bar{l}}\bar{\mu}_k^{\bar{j}} = 0, \quad \partial_{\bar{j}} g^{i\bar{j}} + \delta^{k\bar{l}}\partial_k\mu_{\bar{l}}^{i} = 0 \\
&\partial_i\mu_{\bar{j}}^{i} + \delta^{k\bar{l}}\partial_{\bar{l}}b_{k\bar{j}} = 0, \quad \partial_{\bar{i}}\mu_{j}^{\bar{i}} + \delta^{k\bar{l}}\partial_k b_{j\bar{l}} = 0.
\end{aligned}
\tag{54}
$$

And the contribution of the quadratic diverging term is very simple:

$$\delta^{k\bar{l}} b_{k\bar{l}} = 0 \tag{55}$$

Now these relations are the same which we get in the "integrated over $p, \bar{p}$" theory. The corresponding vertex operator there is:

$$V = \frac{1}{4\pi\alpha'}(G_{\mu\nu} + B_{\mu\nu} - \delta_{\mu\nu})\partial X^{\mu}\bar{\partial}X^{\nu} + \delta^2(0)\delta_{i\bar{j}}a^{i\bar{j}} \tag{56}$$

The absence of linear divegences lead to the gauge

$$\partial_{\mu}\delta^{\mu\nu}G_{\nu\chi} = 0, \quad \partial_{\mu}\delta^{\mu\nu}B_{\nu\chi} = 0 \tag{57}$$

which coincide with (12). The logarithmic divergences lead to

$$\partial_{\mu}\partial_{\nu}\delta^{\mu\nu}G_{\lambda\chi} = 0, \quad \partial_{\mu}\partial_{\nu}\delta^{\mu\nu}B_{\lambda\chi} = 0. \tag{58}$$

The quadratic divegences arising both from the third (determinant term) from (56) and the contraction of $\partial X^{\mu}\bar{\partial}X^{\nu}$ give in the result the familiar condition:

$$\delta^{k\bar{l}} b_{k\bar{l}} = 0 \tag{59}$$

Thus, on the infinitesimal level both models agree.

3 The complete system of equations leading to conformal invariance at one loop

In the previous section we have considered the infinitesimal version of equations providing the conformal invariance at the level α'^0. Now we will try to find their global form. Let's consider some point X_0 and expand all fields (g, μ, b) around this point($X^i =$

$X_0^i + \chi^i$). One can get rid of the constant part of μ by the change of the coordinates

$$\chi^i = \xi^i + \mu^i_{\bar{j}}(X_0)\xi^{\bar{j}} \tag{60}$$

and the constant part of b because it is an exact form. Thus the action has the folowing form:

$$\begin{aligned} S_{F.O.} &= \frac{1}{2\pi\alpha'}\int d^2z(p_i\bar{\partial}\xi^i + p_{\bar{i}}\partial\xi^{\bar{i}} \\ &- g^{i\bar{j}}(\chi)p_ip_{\bar{j}} - \hat{\bar{\mu}}^{\bar{j}}_i(\chi)\partial\xi^i p_{\bar{j}} \\ &- \hat{\mu}^j_{\bar{i}}(\chi)\bar{\partial}\xi^{\bar{i}}p_j - \hat{b}_{i\bar{j}}(\chi)\partial\xi^i\bar{\partial}\xi^{\bar{j}}) \end{aligned} \tag{61}$$

the fields have the following expansions:

$$\begin{aligned} g^{i\bar{j}}(\chi) &= g^{i\bar{j}}(X_0) + \chi^\mu\partial_\mu g^{i\bar{j}}(X_0) \\ &+ 1/2\chi^\mu\chi^\nu\partial_\mu\partial_\nu g^{i\bar{j}}(X_0) + O(\chi^3) \\ \hat{b}_{i\bar{j}}(\chi) &= \chi^\mu\partial_\mu b_{i\bar{j}}(X_0) + 1/2\chi^\mu\chi^\nu\partial_\mu\partial_\nu b_{i\bar{j}}(X_0) + O(\chi^3) \\ \hat{\mu}^i_{\bar{j}}(\chi) &= \chi^\mu\partial_\mu\mu^i_{\bar{j}}(X_0) + 1/2\chi^\mu\chi^\nu\partial_\mu\partial_\nu\mu^i_{\bar{j}}(X_0) \\ &+ O(\chi^3) \quad and \quad c.c. \end{aligned} \tag{62}$$

In order to analyse the conditions of conformal invariance it will be more useful to consider the free action with $g^{i\bar{j}}(X_0)p_ip_{\bar{j}}$ included (in such a way all perturbing operators are linear in ξ). The operator products then change in the following way:

$$\begin{aligned} &X^i(z_1)p_j(z_2) \sim \frac{\alpha'\delta^i_j}{z_1 - z_2}, \quad X^{\bar{i}}(\bar{z}_1)p_{\bar{j}}(\bar{z}_2) \sim \frac{\alpha'\delta^{\bar{i}}_{\bar{j}}}{\bar{z}_1 - \bar{z}_2} \\ &X^{\bar{i}}(z_1)X^{\bar{k}}(z_2) \sim -\alpha' ln|z_1 - z_2|^2 g^{i\bar{j}}(X_0) \\ &X^i(z_1)X^k(z_2) \sim 0, \\ &p_i(z_1)p_j(z_2) \sim 0, \quad p_{\bar{i}}(z_1)p_{\bar{j}}(z_2) \sim 0 \end{aligned} \tag{63}$$

One can easily see that diverging terms (which we are intersted in) with one or two derivatives at the level

α'^0 come from the first and the second order of the perturbation series. So, analysing the transformation properties of the corresponding operators and their products we can go through the approach introduced in the previous section. To describe the resulting condition we will introduce the following operators:

$$
\begin{aligned}
&d_{\bar{i}} = \partial_{\bar{i}} + \mu_{\bar{i}}^{k}\partial_k \quad and \quad c.c.\\
&d_{\bar{i}j} = \partial_{\bar{i}}\partial_j + \mu_{\bar{i}}^{k}\partial_k\partial_j + \bar{\mu}_j^{\bar{l}}\partial_{\bar{i}}\partial_{\bar{l}} + \bar{\mu}_j^{\bar{l}}\mu_{\bar{i}}^{k}\partial_{\bar{l}}\partial_k\\
&d_{\bar{i}\bar{j}} = \partial_{\bar{i}}\partial_{\bar{j}} + \mu_{\bar{i}}^{k}\mu_{\bar{j}}^{l}\partial_k\partial_l \quad and \quad c.c. \qquad (64)
\end{aligned}
$$

To put the anomalous terms to zero we need to impose the following conditions:

$$
\begin{aligned}
&d_i g^{i\bar{j}} + g^{\bar{k}l} d_{\bar{k}}\bar{\mu}_l^{\bar{j}} = 0, \qquad (65)\\
&d_i \mu_{\bar{j}}^{i} + g^{\bar{k}l} d_{\bar{k}} b_{l\bar{j}} = 0 \quad and \quad c.c.\\
&g^{j\bar{i}} d_{\bar{i}j} g^{r\bar{s}} - d_k g^{r\bar{n}} d_{\bar{n}} g^{k\bar{s}} - d_{\bar{m}}\mu_{\bar{n}}^{r} d_k \bar{\mu}_l^{\bar{s}} g^{l\bar{m}} g^{k\bar{n}}\\
&-d_l g^{p\bar{s}} d_p \mu_{\bar{k}}^{r} g^{l\bar{k}} - d_{\bar{l}} g^{r\bar{p}} d_{\bar{p}}\bar{\mu}_k^{\bar{s}} g^{\bar{l}k} = 0 \qquad (66)\\
&g^{j\bar{i}} d_{\bar{i}j}\mu_{\bar{s}}^{k} - g^{n\bar{p}} d_n \mu_{\bar{s}}^{r} d_r \mu_{\bar{p}}^{k}\\
&-g^{m\bar{p}} d_{\bar{p}}\mu_{\bar{r}}^{k} d_l b_{m\bar{s}} g^{l\bar{r}} - d_{\bar{n}} g^{k\bar{r}} d_{\bar{r}} b_{m\bar{s}} g^{m\bar{n}} -\\
&d_p g^{k\bar{i}} d_{\bar{i}}\mu_{\bar{s}}^{p} + d_{\bar{s}} g^{n\bar{i}} d_n \mu_{\bar{i}}^{k} + d_{\bar{s}}\mu_{\bar{i}}^{n} d_n g^{k\bar{i}} = 0 \qquad (67)\\
&g^{j\bar{i}} d_{\bar{i}j} b_{r\bar{s}} + d_{\bar{p}} b_{n\bar{s}} d_r g^{n\bar{p}} + d_k b_{r\bar{i}} d_{\bar{s}} g^{k\bar{i}} - d_k \bar{\mu}_r^{\bar{i}} d_{\bar{i}}\mu_{\bar{s}}^{k} +\\
&d_r \bar{\mu}_k^{\bar{i}} d_{\bar{i}}\mu_{\bar{s}}^{k} + d_{\bar{s}}\mu_{\bar{p}}^{n} d_n \bar{\mu}_r^{\bar{p}} - d_{\bar{p}} b_{r\bar{j}} d_n b_{l\bar{s}} g^{n\bar{j}} g^{l\bar{p}} = 0 \qquad (68)\\
&d_{\bar{j}\nu} g^{i\bar{j}} + g^{k\bar{l}} d_{k\nu}\mu_{\bar{l}}^{i} = 0, \qquad (69)\\
&d_i d_\nu \mu_{\bar{j}}^{i} + g^{\bar{k}l} d_{\bar{k}\nu} b_{l\bar{j}} = 0 \quad and \quad c.c.\\
&d_{\nu\nu'} b_{r\bar{s}} g^{r\bar{s}} + d_\nu b_{r\bar{s}} d_{\nu'} g^{r\bar{s}} + d_{\nu'} b_{r\bar{s}} d_\nu g^{r\bar{s}} \qquad (70)\\
&+d_\nu b_{i\bar{j}} d_{\nu'} b_{p\bar{s}} g^{i\bar{s}} g^{p\bar{j}} + d_\nu \mu_{\bar{j}}^{i} d_{\nu'}\bar{\mu}_i^{\bar{j}} +\\
&d_{\nu'}\mu_{\bar{j}}^{i} d_\nu \bar{\mu}_i^{\bar{j}} = 0\\
&d_\nu b_{r\bar{s}} g^{r\bar{s}} = 0 \quad and \quad c.c. \qquad (71)
\end{aligned}
$$

The equations (65), (69) correspond to linear divergences, (66-68) to the logarithmic divergences and (70-71) to quadratic ones.

We hope that these constraints on μ, b and g as in the previous section lead to the gauged Einstein equations. At least one can easily prove this for constant μ and $b = 0$ by the coordinates change. We can also prove that for infinitesimal μ giving contribution only to the Kalb-Ramond field (that is with additional constraint $g_{i\bar{j}}\bar{\mu}_s^{\bar{j}} + g_{s\bar{j}}\bar{\mu}_i^{\bar{j}} = 0$) and $b = 0$.

4 Symmetries, algebraic constructions and BRST operator

In this section we return back to the simple first order model with only g-field nonzero. Let's consider the following operators:

$$\begin{aligned} \mathcal{N}_v &= n_v + \bar{n}_{\bar{v}} = \frac{1}{2\pi i\alpha'}\int_{S^1} dz v^i(X)p_i \\ &- \frac{1}{2\pi i\alpha'}\int_{S^1} d\bar{z}\bar{v}^{\bar{i}}(\bar{X})p_{\bar{i}} \\ \mathcal{R}_\omega &= r_\omega + \bar{r}_{\bar{\omega}} = \frac{1}{2\pi i\alpha'}\int_{S^1} dz\omega_i(X)\partial X^i \\ &- \frac{1}{2\pi i\alpha'}\int_{S^1} d\bar{z}\bar{\omega}_{\bar{i}}(\bar{X})\bar{\partial}\bar{X}^{\bar{i}} \end{aligned} \tag{72}$$

Again here the singularities should be avoided similar to (12) that is $\partial_i v^i = \partial_{\bar{i}} v^{\bar{i}} = 0$. Acting by $\mathcal{N}$ and $\mathcal{R}$ charges on V-operator, we find that the transformation of the fields up to the order α' is:

$$\begin{aligned} &\delta_v g^{\bar{i}j} = -v^k\partial_k g^{\bar{i}j} - \bar{v}^{\bar{k}}\partial_{\bar{k}} g^{\bar{i}j} + \partial_{\bar{k}}\bar{v}^{\bar{i}} g^{\bar{k}j} + \partial_k v^j g^{\bar{i}k} + O(\alpha') \\ &\delta_\omega g^{\bar{i}j} = 0 \end{aligned} \tag{73}$$

$$\delta_\omega \mu^j_{\bar{i}} = \partial_{\bar{i}}\bar{\omega}_{\bar{k}} g^{\bar{k}j} - \partial_{\bar{k}}\bar{\omega}_{\bar{i}} g^{\bar{k}j} + O(\alpha')$$
$$\delta_\omega \bar{\mu}^{\bar{j}}_i = \partial_i \omega_k g^{k\bar{j}} - \partial_k \omega_i g^{k\bar{j}} + O(\alpha')$$

Thus δ_v transformation can be identified (up to the next order in α') with the holomorphic coordinate transformations. The symmetry related with δ_ω is a litle bit more complicated. Let's look on how the G- and B -fields from (6) transform under δ_ω:

$$\delta_\omega B_{\mu\nu} = \partial_\nu \omega_\mu - \partial_\mu \omega_\nu + O(\alpha'), \quad \delta_\omega G_{\mu\nu} = O(\alpha') \quad (74)$$

Thus it is a well known symmetry $B \to B + \partial\omega + \bar{\partial}\bar{\omega}$. Now let's rewrite the equations (18) in the chiral-algebraic form, namely the operator V (3) can be written in the following way:

$$V(X, p, \bar{X}, \bar{p}) = \sum_i \mathcal{U}_i(X, p) \otimes \bar{\mathcal{U}}_i(\bar{X}, \bar{p}), \quad (75)$$

where we divided V on left- and right-chiral parts ($\mathcal{U}_i$ and $\bar{\mathcal{U}}_i$ have conformal weights (1,0) and (0,1) correspondingly). Now the equations (18) appear as coefficients of p, $\bar{p}$, ∂X, $\bar{\partial}\bar{X}$ in the following expression:

$$[[V, V]](X, p, \bar{X}, \bar{p}) = \quad (76)$$
$$lim_{\alpha' \to 0} \sum_i \sum_j [\mathcal{U}_i, \mathcal{U}_j](X, p) \otimes [\bar{\mathcal{U}}_i, \bar{\mathcal{U}}_j](\bar{X}, \bar{p}) = 0$$

Where the commutators are defined in the usual way in terms of residues, that is:

$$[\mathcal{U}_i, \mathcal{U}_j](z_2) = res_{z_1 \to z_2} \mathcal{U}_i(z_1)\mathcal{U}_j(z_2) \quad (77)$$

One can easily verify that the relation (76) is independent on the choice of the expansion (75). Now let's look on an algebra of symmetries of these equations. One should remember that we have already defined charges (72). One can rewrite them in the above notation as follows:

$$\mathcal{N} = n_v \otimes 1 + 1 \otimes \bar{n}_{\bar{v}}, \quad \mathcal{R}_\omega = r_\omega \otimes 1 + 1 \otimes \bar{r}_{\bar{\omega}}, \tag{78}$$

The algebra of these charges is the following one:

$$\begin{aligned} &[n_{v_1}, n_{v_2}] = n_{[v_2,v_1]} + \alpha' r_{\omega(v_1,v_2)} \\ &[r_\omega, n_v] = r_{\mathcal{L}_v \omega}, \quad [r_{\omega_1}, r_{\omega_2}] = 0 \end{aligned} \tag{79}$$

Here $\omega_n(v_1, v_2) = \frac{1}{2}(\partial_k v_2^l \partial_n \partial_l v_1^k - \partial_n \partial_k v_1^l \partial_l v_2^k)$ and $\mathcal{L}_v \omega_k = \partial_i \omega_k v^i + \omega_i \partial_k v^i$ is a Lie derivative. The same algebraic relations hold for the charges of the opposite chirality. One should note that it is a deformation via the extension of the semidirect product of the algebra of conformal coordinate transformations and the algebra of 1-forms (but in the limit $\alpha' \to 0$ the extension disappear!). It should be noted also that there exist a nondegenerate inner product on such an algebra, invariant under the adjoint action:

$$\begin{aligned} &(n_{v_1}, n_{v_2}) = 0, \quad (r_{\omega_1}, r_{\omega_2}) = 0, \\ &(n_v, r_\omega) = \int v^i(X) \omega_i(X) d\Omega(X), \end{aligned} \tag{80}$$

where $d\Omega$ is a holomorphic volume form. In this formalism the transformation formulae (73) appear as an adjoint action of the composite generators $\mathcal{N}$, $\mathcal{R}$:

$$\begin{aligned} \delta_v V &= \sum_i [n_v, \mathcal{U}_i] \otimes \bar{\mathcal{U}}_i + \sum_\alpha \mathcal{U}_i \otimes [\bar{n}_{\bar{v}}, \bar{\mathcal{U}}_i], \\ \delta_\omega V &= \sum_i [r_\omega, \mathcal{U}_i] \otimes \bar{\mathcal{U}}_i + \sum_i \mathcal{U}_i \otimes [\bar{r}_{\bar{\omega}}, \bar{\mathcal{U}}_i]. \end{aligned} \tag{81}$$

Defining the BRST operator for the free theory (1) in the usual way

$$Q = \int_{S^1} \mathcal{J}, \quad \mathcal{J} = j dz - \tilde{j} d\bar{z}, \tag{82}$$

$$j \;=\; cT^m + :bc\partial c: + \frac{3}{2}\partial^2 c,$$
$$\tilde{j} \;=\; \tilde{c}\tilde{T}^m + :\tilde{b}\tilde{c}\bar{\partial}\tilde{c}: + \frac{3}{2}\bar{\partial}^2\tilde{c},$$

where the T^m and $\tilde{T}^m$ are correspondingly holomorphic and antiholomorphic components of the energy-momentum tensor, one can see that our gauge conditions are:

$$[Q, c\tilde{c}V] = 0, \quad [Q, cn_v] = [Q, \tilde{c}\bar{n}_{\bar{v}}] = 0, \tag{83}$$
$$[Q, cr_\omega] = [Q, \tilde{c}\bar{r}_{\bar{\omega}}] = 0. \tag{84}$$

The equation (76) can be rewritten in the following way:

$$lim_{\epsilon,\alpha'\to 0}(\int_{C_{\epsilon,z}} dz'\tilde{c}(\bar{z}')V(z') \tag{85}$$
$$-d\bar{z}'cV(z'))c(z)\tilde{c}(\bar{z})V(z) = 0$$

where $C_{\epsilon,z}$ is a small contour around the point z. More generally, denoting

$$c\tilde{c}V = \phi^{(0)}, \quad V = \phi^{(2)}, \quad cV - \tilde{c}V = \phi^{(1)}, \tag{86}$$

such that

$$[Q, \phi^{(2)}] = d\phi^{(1)}, [Q, \phi^{(1)}] = d\phi^{(0)} and [Q, \phi^{(0)}] = 0 \tag{87}$$

the full system of gauge conditions and Einstein equations appear to be somewhat like Maurer-Cartan equations:

$$lim_{\epsilon,\alpha'\to 0}([Q, \phi^{(0)}](z) + \int_{C_{\epsilon,z}} \phi^{(1)}\phi^{(0)}(z)) = 0 \tag{88}$$

5 Dilaton term

The construction introduced in the previous section motivate us to introduce the additional degree of freedom, corresponding to dilaton in the following way.

The appearance of the terms $\int \sqrt{h} R\Phi_0(X)$ in the action lead to the deformation of the energy-momentum tensor:

$$T \to T + \partial^2\Phi_0, \quad \bar{T} \to \tilde{T} + \bar{\partial}^2\Phi_0, \quad \Theta \to \partial\bar{\partial}\Phi_0 \tag{89}$$

where Θ is the trace of $T_{\alpha\beta}$. To make the theory conformal invariant on need to put Θ to zero. That is we need to put the following constraint: $\partial_k\partial_{\bar{i}}\Phi_0 = 0$. THe deformation of the energy-momentum tensor leads to the deformation of the corresponding BRST operator. Now let's consider the condition $[Q^{def}, c\tilde{c}V] = 0$ where $V = \frac{1}{2\pi\alpha'} g^{i\bar{j}}(X, \bar{X}) p_i p_{\bar{j}}$. Then we will find the new gauge condition:

$$\partial_i g^{i\bar{j}} - 2\partial_i\Phi_0 g^{i\bar{j}} = 0 \tag{90}$$

which can be also rewritten simply as $\partial_i(e^{-2\Phi_0} g^{i\bar{j}}) = 0$. Let's consider now the equations (18) accompanied by the relations, obtained above, that is, let's study the following system of equations:

$$\partial_i(e^{-2\Phi_0} g^{i\bar{j}}) = 0 \tag{91}$$
$$g^{\bar{i}j}\partial_{\bar{i}}\partial_j g^{\bar{k}l} - \partial_{\bar{p}} g^{\bar{k}r}\partial_r g^{\bar{p}l} = 0 \tag{92}$$
$$\partial_k\partial_{\bar{i}}\Phi_0 = 0 \tag{93}$$

We will now show that these equations lead to the Einstein equations with the B-field and a dilaton where

$$G_{i\bar{j}} = -B_{i\bar{j}} = g_{i\bar{j}}, \quad \Phi = \sqrt{g} + \Phi_0 \tag{94}$$

Proof:

Recall that in the derivation of the first of Einstein equations for simple model (19) we have used the gauge condition only once, when put $G^{\alpha\beta}\partial_\alpha G_{\beta\nu}=0$. In the present case we have $G^{\alpha\beta}\partial_\alpha G_{\beta\nu} = -2\partial_\nu\Phi_0$ therefore

$$\Gamma_{\mu\nu}=-2\nabla_\mu\nabla_\nu(\Phi_0+\sqrt{g})=-2\nabla_\mu\nabla_\nu(\Phi) \tag{95}$$

Therefore the equation (19) is valid in our case. Let's derive the second one:

$$\begin{aligned}\partial_{\bar{i}}H^{\bar{i}\bar{k}l} &= \partial_{\bar{i}}(g^{\bar{i}j}\partial_j g^{\bar{k}l}-g^{\bar{k}r}\partial_r g^{\bar{i}l}) \\ &= 2\partial_{\bar{s}}\Phi_0 g^{\bar{s}j}\partial_j g^{\bar{k}l}-2g^{\bar{k}r}\partial_r(\partial_{\bar{s}}\Phi_0 g^{\bar{s}l}) \\ &= 2\partial_{\bar{s}}\Phi_0(g^{\bar{s}j}\partial_j g^{\bar{k}l}-g^{\bar{k}j}\partial_j g^{\bar{s}l})=2\partial_{\bar{s}}\Phi_0 H^{\bar{s}\bar{k}l}\end{aligned} \tag{96}$$

Where we have used in the first line the relations (91) and (92) and in the second one (93). Considering the conjugate equation and summing them we arrive to (20).

In a similar way one can derive the third "dilaton" equation (21). Recall that in the case of simple model we have used the gauge condition in the derivation of this equation in (32) and then putting that in (31). Therefore in our case this lead to additional terms, that is we have:

$$\begin{aligned}&8\partial_{\bar{i}}\sqrt{g}\partial_j\sqrt{g}g^{j\bar{i}}-2\nabla^\mu\nabla_\mu\sqrt{g}-1/6H^{\mu\nu\rho}H_{\mu\nu\rho} \\ &=2g^{\bar{i}j}g^{\bar{k}r}\partial_{\bar{i}}g_{\bar{k}j}\partial_r\sqrt{g}+2g^{\bar{i}j}g^{\bar{r}k}\partial_j g_{\bar{i}k}\partial_{\bar{r}}\sqrt{g} \\ &=-4g^{r\bar{k}}\partial_{\bar{k}}\Phi_0\partial_r\sqrt{g}-4g^{r\bar{k}}\partial_r\Phi_0\partial_{\bar{k}}\sqrt{g}\end{aligned} \tag{97}$$

Let's consider the following combination:

$$\begin{aligned}&8(\partial_i\Phi_0\partial_{\bar{j}}\Phi_0 g^{\bar{j}i}+g^{r\bar{k}}\partial_{\bar{k}}\Phi_0\partial_r\sqrt{g} \\ &+g^{r\bar{k}}\partial_r\Phi_0\partial_{\bar{k}}\sqrt{g})-4g^{\bar{j}i}\nabla_{\bar{j}}\nabla_i\Phi_0 \\ &=8(\partial_i\Phi_0\partial_{\bar{j}}\Phi_0 g^{\bar{j}i}+g^{r\bar{k}}\partial_{\bar{k}}\Phi_0\partial_r\sqrt{g}+g^{r\bar{k}}\partial_r\Phi_0\partial_{\bar{k}}\sqrt{g}) \\ &-8\partial_i\Phi_0\partial_{\bar{j}}\Phi_0 g^{\bar{j}i}-4g^{r\bar{k}}\partial_{\bar{k}}\Phi_0\partial_r\sqrt{g}-4g^{r\bar{k}}\partial_r\Phi_0\partial_{\bar{k}}\sqrt{g} \\ &=4g^{r\bar{k}}\partial_{\bar{k}}\Phi_0\partial_r\sqrt{g}+4g^{r\bar{k}}\partial_r\Phi_0\partial_{\bar{k}}\sqrt{g}\end{aligned} \tag{98}$$

During these calculations we have used that $g^{\bar{j}i}\Gamma^{\mu}_{i\bar{j}} = -G^{\mu\nu}\partial_\nu\sqrt{g} - \partial_\nu\Phi_0 G^{\mu\nu}$.
Now summing (97) and (98) we get the desired equation (21).

6 Conclusion and Main Results

1) A new description of nontrivial backgrounds for String Theory is given.

2) The conditions of conformal invariance for the first order sigma models are studied and shown to coincide with the gauged Einstein equations.

3) The relation to the theory of deformed BRST complex is established. In some particular case the Einstein equations are written as generalized Maurer-Cartan equations.