Enumerative geometry and quantum integrable systems

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Introduction

We will talk about the relationship between two seemingly independent areas of mathematics:

- Integrable systems based on quantum groups

  Exactly solvable models of statistical physics: spin chains, vertex models

  1930s: Hans Bethe: Bethe ansatz solution of Heisenberg model

  1960-70s: R.J. Baxter, C.N. Young: Yang-Baxter equation, Baxter operator

  1980s: Development of "QISM" by Leningrad school leading to the discovery of quantum groups by Drinfeld and Jimbo

  Since 1990s: textbook subject and an established area of mathematics and physics.

- Enumerative geometry: quantum equivariant K-theory

  Generalization of quantum cohomology in the early 2000s by A. Givental, Y.P. Lee and collaborators. Recently big progress in this direction by A. Okounkov and his school.

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  N. Nekrasov, S. Shatashvili, *Supersymmetric vacua and Bethe ansatz*, arXiv:0901.4744
  N. Nekrasov, S. Shatashvili, *Quantum integrability and supersymmetric vacua*, arXiv:0901.4748

- Subsequent work in geometric representation theory:

  A. Braverman, D. Maulik, A. Okounkov, *Quantum cohomology of the Springer resolution*, Adv. Math. 227 (2011) 421-458
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Understanding (enumerative) geometry of symplectic resolutions:

"Lie algebras of XXI century" (A. Okounkov' 2012)

Important examples: Springer resolution, Hilbert scheme of points in the plane, Hypertoric varieties,...

A large class of symplectic resolutions is provided by Nakajima quiver varieties (simplest subclass: $T^*Gr(k,n)$)

In this talk our main example will be $T^*Gr(k,n)$ and more generally (partial) flag varieties.

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Outline

Quantum groups, quantum integrability and quantum difference equations

Nekrasov-Shatashvili ideas

Quantum K-theory (via Okounkov) and integrability

Many-Body systems: back to Givental's ideas

Further directions

Based on:

- Petr P. Pushkar, Andrey Smirnov, A.Z., *Baxter Q-operator from quantum K-theory*, arXiv:1612.08723
- Peter Koroteev, Petr P. Pushkar, Andrey Smirnov, A.Z., *Quantum K-theory of Quiver Varieties and Many-Body Systems*, arXiv:1705.10419

Loop algebras and evaluation modules

Let us consider Lie algebra $\mathfrak{g}$. The associated loop algebra is $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}]$.

The following representations form a tensor category of $\hat{\mathfrak{g}}$:

$$V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n),$$

where

- V_i are representations of $\mathfrak{g}$
- a_i are values for *spectral parameter* t

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Quantum groups

Quantum group

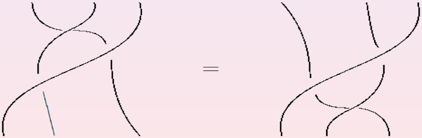
$$U_\hbar(\hat{\mathfrak{g}})$$

is a deformation of $U(\hat{\mathfrak{g}})$, with a nontrivial intertwiner $R_{V_1,V_2}(a_1/a_2)$:

$$V_1(a_1)\otimes V_2(a_2)$$

$$V_2(a_2)\otimes V_1(a_1)$$

which is a rational function of a_1, a_2, satisfying Yang-Baxter equation:



The generators of $U_\hbar(\hat{\mathfrak{g}})$ emerge as matrix elements of R-matrices (the so-called FRT construction).

Integrability and Baxter algebra

Source of integrability: commuting *transfer matrices*, generating *Baxter algebra* which are weighted traces of

$$\tilde{R}_{W(u),\mathcal{H}_{phys}} : W(u) \otimes \mathcal{H}_{\rm phys} \to W(u) \otimes \mathcal{H}_{\rm phys}$$

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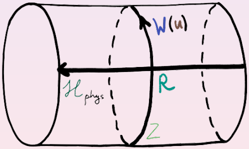
$$\tilde{R}_{W(u),\mathcal{H}_{phys}} : W(u) \otimes \mathcal{H}_{\mathrm{phys}} \to W(u) \otimes \mathcal{H}_{\mathrm{phys}}$$

over auxiliary $W(u)$ space:

$$T_W(u) = \mathrm{Tr}_{W(u)}\Big((Z \otimes 1)\ \tilde{R}_{W(u),\mathcal{H}_{phys}} \Big)$$



Here $Z \in e^{\mathfrak{h}}$, where $\mathfrak{h} \in \mathfrak{g}$ are diagonal matrices.

Integrability:

$$[T_{W'}(u'), T_W(u)] = 0$$

Special important case:

$W(u)$ can be certain infinite-dimensional representation of Borel subalgebra of $U_\hbar(\hat{\mathfrak{g}})$.

Then the transfer matrix is called *Baxter Q-operator*. Such operators generate all Baxter algebra.

Primary goal for physicists is to diagonalize $T_W(u)$ simultaneously.

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$\mathfrak{g} = \mathfrak{sl}(2)$: XXZ chain

Textbook example (and main example in this talk) is XXZ Heisenberg spin chain:

$$\mathcal{H}_{XXZ} = \mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) \otimes \cdots \otimes \mathbb{C}^2(a_n)$$

States:

$$\uparrow\uparrow\uparrow\uparrow \downarrow \uparrow\uparrow\uparrow \downarrow \uparrow\uparrow\uparrow\uparrow \downarrow \uparrow\uparrow\uparrow\uparrow\uparrow \downarrow\downarrow \uparrow\uparrow\uparrow$$

Here $\mathbb{C}^2$ stands for 2-dimensional representation of $U_\hbar(\widehat{\mathfrak{sl}}_2)$.

Algebraic method to diagonalize transfer matrices:

Algebraic Bethe ansatz

as a part of Quantum Inverse Scattering Method developed in the 1980s.

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The eigenvalues are generated by symmetric functions of Bethe roots $\{x_i\}$:

$$\prod_{j=1}^{n}\frac{x_i - a_j}{\hbar a_j - x_i} = z\,\hbar^{-n/2}\prod_{\substack{j=1\\ j\neq i}}^{k}\frac{x_i\hbar - x_j}{x_i - x_j\hbar}\,, \quad i = 1\cdots k,$$

so that the eigenvalues $\Lambda(u)$ of the Q-operator are the generating function for the elementary symmetric functions of Bethe roots:

$$\Lambda(u) = \prod_{i=1}^{k}(1 + u\cdot x_i)$$

A real challenge is to do it for general $\mathfrak{g}$ (possibly infinite-dimensional).

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q-difference equation

Modern way of looking at Bethe ansatz: solving q-difference equations for

$$\Psi(z_1,\dots,z_k;a_1,\dots,a_n)\in V_1(a_1)\otimes\cdots\otimes V_n(a_n)[[z_1,\dots,z_k]]$$

known as quantum Knizhnik-Zamolodchikov (aka Frenkel-Reshetikhin) equations:

$$\Psi(qa_1,\dots,a_n,\{z_i\})=(Z\otimes 1\otimes\cdots\otimes 1)R_{V_1,V_n}\dots R_{V_1,V_2}\Psi$$
$$+$$
$$\text{commuting difference equations in } z-\text{variables}$$

Here $\{z_i\}$ are the components of twist variable Z.

The latter series of equations are known as dynamical equations, studied by Etingof, Felder, Tarasov, Varchenko, ...

In $q\to 1$ limit we arrive to an eigenvalue problem.

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Solutions of qKZ equations and Bethe ansatz

$$\Psi_\mu = \int F_\mu(\{x_i\}, \{a_i\})\, \mathcal{K}(\{x_i\}, \{a_i\}, \{z_i\}, q)\, \{dx_i\}$$

Here index μ runs through the Hilbert space $\mathcal{H} = V_1(a_1) \otimes \cdots \otimes V_n(a_n)$

These were studied extensively by Tarasov and Varchenko in the case of $\mathfrak{g} = \mathfrak{gl}_n$.

Here F_μ are rational functions, of its variables and $\mathcal{K}$ is a fixed kernel, such that:

$$\mathcal{K} \sim e^{\frac{\mathcal{S}(\{x_i\}, \{a_i\}, \{z_i\})}{\ln(q)} + \cdots}$$

where $\mathcal{S}$ is known as Yang-Yang functional and

$$\frac{\partial \mathcal{S}}{\partial x_i} = 0 \text{ are Bethe equations for Bethe roots } \{x_i\}$$

$$e^{a_i \frac{\partial \mathcal{S}}{\partial a_i}} \text{ are the eigenvalues of qKZ operators}$$

In other words, $\mu \to F_\mu$ (off-shell Bethe function) is the diagonalization procedure.

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$$\frac{\partial \mathcal{S}}{\partial x_i} = 0 \text{ are Bethe equations for Bethe roots } \{x_i\}$$

$$e^{a_i \frac{\partial \mathcal{S}}{\partial a_i}} \text{ are the eigenvalues of qKZ operators}$$

In other words, $\mu \to F_\mu$ (off-shell Bethe function) is the diagonalization procedure.

$$\Psi_\mu = \int F_\mu(\{x_i\},\{a_i\})\, \mathcal{K}(\{x_i\},\{a_i\},\{z_i\},q)\, \{dx_i\}$$

Here index μ runs through the Hilbert space $\mathcal{H} = V_1(a_1)\otimes\cdots\otimes V_n(a_n)$

These were studied extensively by Tarasov and Varchenko in the case of $\mathfrak{g} = \mathfrak{gl}_n$.

Here F_μ are rational functions, of its variables and $\mathcal{K}$ is a fixed kernel, such that:

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Solutions of qKZ equations and Bethe ansatz

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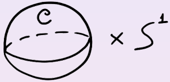
Nekrasov-Shatashvili ideas

Nekrasov and Shatashvili looked in 2009 at 3d SUSY gauge theories on $\mathcal{C} \times S^1$:



with gauge group

$$G = U(v_1) \times U(v_2) \times \dots U(v_{\mathrm{rank}\,\mathfrak{g}}),$$

so that the collection $\{v_i\}$ determines the weights of the corresponding subspace in $\mathcal{H}$.

In the simplest case of $\mathfrak{g} = \mathfrak{sl}(2)$ we just have one $U(v)$ and

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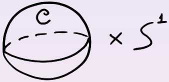

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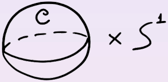

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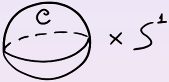

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Full Gauge/Bethe correspondence dictionary

Gauge group G: $U(v_1) \times U(v_2) \times \dots U(v_{rank\,\mathfrak{g}})$

The set $\{v_i\}$ determines the weight (i.e. number of inverted spins)

Maximal torus: $\{x_{i_1}, \dots, x_{i_{v_i}}\}$ — these are Bethe roots variables.

Matter Fields: affine space $\mathcal{M}$

- Standard matter fields: $\oplus_{i=1}^{rank\,\mathfrak{g}} V_i \otimes W_i$, s.t. $dim(V_i) = v_i$;

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  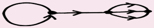

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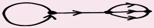

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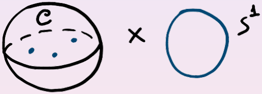

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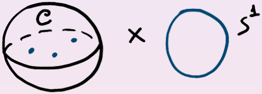

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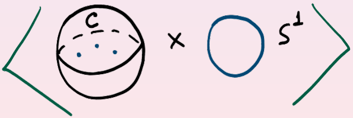
One can think of quantum K-theory ring:



Key Ideas

Nekrasov and Shatashvili:

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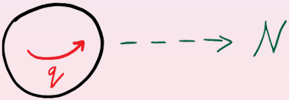
Further input by Okounkov:

$$q-\mathrm{difference\ equations}\ = \mathrm{qKZ\ equations} + \ \mathrm{dynamical\ equations}$$



In the following we will talk about this in the simplest case:

- Nakajima variety: $N = T^*Gr(k, n)$
- Quantum Integrable System: $\mathfrak{sl}(2)$ XXZ spin chain.

Notation

$$T^*Gr(k,n) = N_{k,n}, \quad \sqcup_k N_{k,n} = N(n).$$

As a Nakajima variety:

$$N_{k,n} = T^*\mathcal{M}/\!/\!/\!/GL(V) = \mu^{-1}(0)_s/GL(V), \text{where } \mathcal{M} = Hom(V,W)$$

Tautological bundles:

$$\mathcal{V} = T^*\mathcal{M} \times V/\!/\!/\!/GL(V), \quad \mathcal{W} = T^*\mathcal{M} \times W/\!/\!/\!/GL(V)$$

For any $\tau \in K_{GL(V)}(\cdot) = \Lambda(x_1^{\pm 1}, x_2^{\pm 1}, \ldots x_k^{\pm 1})$ we introduce a tautological bundle:

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Tori, Fixed points and Bethe roots

Torus action:

$$A = \mathbb{C}^\times_{a_1} \times \cdots \times \mathbb{C}^\times_{a_n} \circlearrowright W,$$

$$\text{Full torus}: \quad T = A \times \mathbb{C}^\times_{\hbar}, \text{ where } \mathbb{C}^\times_{\hbar} \text{ scales cotangent directions}$$

Fixed points: $\mathbf{p} = \{s_1, \dots, s_k\} \in \{a_1, \dots, a_n\}$

Denote $\mathcal{A} := \mathbb{Q}(a_1, \dots, a_n, \hbar)$, $R := \mathbb{Z}(a_1, \dots, a_n, \hbar)$, then localized K-theory is:

$$K_T(N(n))_{loc} = K_T(N(n)) \otimes_R \mathcal{A} = \sum_{k=0}^{n} K_T(N_{k,n}) \otimes_R \mathcal{A}$$

is a 2^n-dimensional $\mathcal{A}$-vector space, spanned by $\mathcal{O}_{\mathbf{p}}$.

Classical Bethe equations: The eigenvalues of the operators of multiplication by τ are $\tau(x_1, \cdots, x_k)$ evaluated at the solutions of the following equations:

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$$\text{Full torus}: \quad T = A \times \mathbb{C}^\times_{\hbar}, \ \text{where } \mathbb{C}^\times_{\hbar} \text{ scales cotangent directions}$$

Fixed points: $\mathbf{p} = \{s_1, \dots, s_k\} \in \{a_1, \dots, a_n\}$

Denote $\mathcal{A} := \mathbb{Q}(a_1, \dots, a_n, \hbar)$, $R := \mathbb{Z}(a_1, \dots, a_n, \hbar)$, then localized K-theory is:

$$K_T(N(n))_{loc} = K_T(N(n)) \otimes_R \mathcal{A} = \sum_{k=0}^{n} K_T(N_{k,n}) \otimes_R \mathcal{A}$$

is a 2^n-dimensional $\mathcal{A}$-vector space, spanned by $\mathcal{O}_{\mathbf{p}}$.

Classical Bethe equations: The eigenvalues of the operators of multiplication by τ are $\tau(x_1, \cdots, x_k)$ evaluated at the solutions of the following equations:

$$\prod_{j=1}^{n}(x_i - a_j) = 0, \quad i = 1, \dots, k, \ \textit{with } x_i \neq x_j$$

Quantum tautological classes and Bethe equations

We will use theory of quasimaps:

$$\mathcal{C} - - - \to N_{k,n}$$

in order to deform tensor product: $A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \otimes_d B \, z^d$.

We will also define quantum tautological classes:

$$\hat{\tau}(z) = \tau + \sum_{d=1}^{\infty} \tau_d z^d \in K_T(N(n))[[z]]$$

Theorem. [P. Pushkar, A. Smirnov, A.Z] The eigenvalues of operators of quantum multiplication by $\hat{\tau}(z)$ are given by the values of the corresponding Laurent polynomials $\tau(x_1, \dots, x_k)$ evaluated at the solutions of the following equations:

$$\prod_{j=1}^{n} \frac{x_i - a_j}{\hbar a_j - x_i} = z \, \hbar^{-n/2} \prod_{\substack{j=1 \\ j \neq i}}^{k} \frac{x_i \hbar - x_j}{x_i - x_j \hbar}, \quad i = 1 \cdots k,$$

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View $K_T(N(n))$ as $\mathcal{H}_{XXZ} = \mathbb{C}^2(a_1) \otimes \cdots \otimes \mathbb{C}^2(a_n)$, a tensor product of evaluation representations of $U_\hbar(\widehat{\mathfrak{sl}}(2))$.

Geometric realization of $U_\hbar(\hat{\mathfrak{sl}}(2))$?

Consider torus $A = \mathbb{C}_a^\times \circlearrowright W$, so that $W = a'W' + a''W''$.

Maulik and Okounkov:

$$K_T(N(n)^A) = K_T(N(n')) \otimes K_T(N(n'')) \xrightarrow{Stab_\pm} K_T(N(n)),$$

so that

$$R(a'/a'') = Stab_-^{-1} Stab_+ \ \text{is a trighonometric R} - \text{matrix}$$

which is a universal R-matrix considered in the corresponding evaluation representations.

K-theoretic realization of quantum groups

View $K_T(N(n))$ as $\mathcal{H}_{XXZ} = \mathbb{C}^2(a_1) \otimes \cdots \otimes \mathbb{C}^2(a_n)$, a tensor product of evaluation representations of $U_\hbar(\widehat{\mathfrak{sl}}(2))$.

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$$T_W(u) = Tr_{W(u)}[(Z \otimes 1)R_{W(u)\otimes \mathcal{H}_{XXZ}}]$$

Prefundamental representations $V_\pm(u)$:

(introduced by Bazhanov, Lukyanov, Zamolochikov, see also E. Frenkel and Hernandez)

- $V_\pm(u)$ generate the Grotendieck group of evaluation representations.
- $Q_+(u) = T_{V_+(u)}$ has eigenvalues $\prod_{i=1}^k (1 + u \cdot x_i)$, where x_i are solutions of the Bethe equations.
- Same story for $Q_-(u) = T_{V_-(u)}$, but Bethe equations change: $z \to z^{-1}$
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$$zK\,Q_+(\hbar u)Q_-(u) - Q_+(u)Q_-(\hbar u) = (zK-1)\prod_{i=1}^{n}(1 + a_i \cdot u)$$

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The quantum K-theoretic meaning of the Q-operator

Central result of our first paper:

Theorem. [P. Pushkar, A. Smirnov, A.Z.]

- The quantum multiplication on quantum tautological class corresponding to $\tau_u := \oplus_{m\geq 0} u^m \Lambda^m \mathcal{V}$ coincides with Q_+-operator, i..e
$$\hat{\tau}_u(z) = Q_+(u)$$
- Explicit universal formulas for quantum products::
$$\widehat{\Lambda^\ell \mathcal{V}}(z) = \Lambda^\ell \mathcal{V} + a_1(z)\ F_0 \Lambda^{\ell-1}\mathcal{V} E_{-1} + \cdots + a_\ell(z)\ F_0^\ell E_{-1}^\ell,$$
where $a_m(z) = \frac{(\hbar-1)^m\ \hbar^{\frac{m(m+1)}{2}} K^m}{(m)_\hbar! \prod_{i=1}^{m}(1-(-1)^n z^{-1}\hbar^i K)}$,
where K, F_0, E_{-1} are the generators of $U_\hbar(\widehat{\mathfrak{sl}_2})$.
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Quasimaps

Quasimap f: $\mathcal{C} \dashrightarrow N_{k,n}$ is the following collection of data:

- vector bundle $\mathscr{V}$ on $\mathcal{C}$ of rank k.
- section $f \in H^0(C, \mathscr{M} \oplus \mathscr{M}^* \otimes \hbar)$, satisfying the condition $\mu = 0$, where $\mathscr{M} = Hom(\mathscr{V}, \mathscr{W})$, so that $\mathscr{W}$ is a trivial bundle of rank n.

$$ev_p(f) = f(p) \in [\mu^{-1}(0)/GL(V)] \supset N_{k,n}$$

Quasimap is *stable* if $f(p) \in N_{k,n}$ for all but finitely many points, known as *singularities* of quasimap.

For the moduli space of quasimaps

$$QM(N_{k,n}) = \text{stable quasimaps to } N_{k,n}/\sim$$

only $\mathscr{V}$ and f vary, while $\mathcal{C}$ and $\mathscr{W}$ remain the same.

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Quasimaps

Quasimap $f\colon \mathcal{C} \dashrightarrow N_{k,n}$ is the following collection of data:

- vector bundle $\mathscr{V}$ on $\mathcal{C}$ of rank k.
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Relative quasimaps

Resolution, to make evaluation map proper:

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 & QM^d(N_{k,n})_{\text{relative } p} \arrow[dr, "\widetilde{\mathrm{ev}}_p"] & \\
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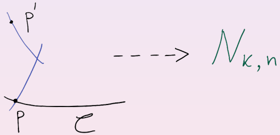
That allows the curve to break: emergence of "*accordeons*":



$$
\begin{tikzcd}
p' \arrow[r] \arrow[d] & \mathcal{C}' \arrow[r, dashed, "f'"] \arrow[d, "\pi"] & N_{k,n} \\
p \arrow[r] & \mathcal{C} &
\end{tikzcd}
$$

i) π is a stabilization of $(\mathcal{C}', \mathrm{p}')$

ii) f' : nonsing at p' and nodes of $\mathcal{C}'$

iii) $\mathrm{Aut}(\mathrm{f}')$ is finite

Virtual sheaves

$QM^d(N_{k,n})$ have perfect deformation-obstruction theory:

- If $(\mathscr{V}, \mathscr{W})$ defines quasimap nonsingular at p.

$$T^{\mathrm{vir}}_{(\mathscr{V},\mathscr{W})} QM^d{}_{\text{nonsing p}}(N_{k,n}) = H^{\bullet}(\mathscr{P} \oplus \hbar\,\mathscr{P}^*) - T_{f(p)} N_{k,n}$$

  where $T_{f(p)} N_{k,n}$ is a normalizing term, $\mathscr{P}$ is the polarization bundle on the curve $\mathcal{C}$:

$$\mathscr{P} = \mathscr{W}^* \otimes \mathscr{V} - \mathscr{V}^* \otimes \mathscr{V}.$$

- Virtual structure sheaf:

$$\hat{\mathscr{O}}_{\mathrm{vir}} = \mathscr{O}_{\mathrm{vir}} \otimes \mathscr{K}_{\mathrm{vir}}^{1/2} q^{\deg(\mathscr{P})/2},$$

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Pushforwards and degeneration formula

How to degenerate curve in a suitable way?

Blowup: $\pi : \mathbf{C} = \mathrm{Bl}_{c,0}\mathcal{C} \times \mathbb{C}_\epsilon \to \mathbb{C}_\epsilon$, so that $\mathcal{C}_\epsilon = \pi^{-1}(\epsilon) \cong \mathcal{C}$ for $\epsilon \neq 0$

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$$\chi(QM(\mathcal{C}_\epsilon \to N_{k,n}), \hat{\mathcal{O}}_{\mathrm{vir}} z^d) = (\mathbf{G}^{-1}\mathrm{ev}_{1,*}(\hat{\mathcal{O}}_{\mathrm{vir}} z^d), \mathrm{ev}_{2,*}(\hat{\mathcal{O}}_{\mathrm{vir}} z^d))$$

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$$\mathbf{G} = \sum_{d=0}^{\infty} z^d \mathrm{ev}_{p_1,p_2,*}\left(QM_{\text{relative } p_1, p_2}, \hat{\mathcal{O}}_{\mathrm{vir}}\right) \in K_T(N_{k,n})^{\otimes 2}[[z]]$$

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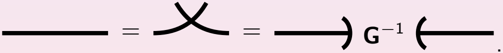

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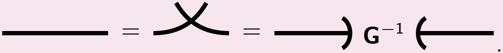

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Quantum multiplication, quantum tautological classes

We define the quantum product by means of the following element in $K_T(N_{k,n})^{\otimes^2}[[z]]$:

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represented by

$QK_T(N_{k,n}) = K_T(N_{k,n})[[z]]$ is a unital algebra, so that:

$$\hat{\mathbf{1}}(z) = \; 1 \bullet\!\!\longrightarrow \; = \sum_{d=0}^{\infty} z^d \mathrm{ev}_{p_2,*}\left(QM^d{}_{\text{relative } p_2}, \hat{\mathscr{O}}_{\text{vir}}\right)$$

Similarly, one defines quantum tautological classes:

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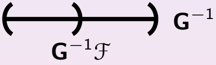

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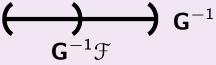

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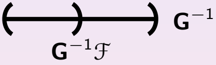

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Vertices and Fusion operator

Let us talk about $G = T \times \mathbb{C}_q^{\times}$-equivariant K-theory.

- Vertex, a class in $K_G(N_{k,n})_{loc}[[z]]$:

$$V^{(\tau)}(z) = \tau\bullet\!\!-\!\!\!-\!\!\!-\!\!\circ = \sum_{d=0}^{\infty} z^d \mathrm{ev}_{p_2,*}\Big(QM^d{}_{\mathrm{nonsing}\,p_2}, \hat{\mathcal{O}}_{\mathrm{vir}}\tau(\mathscr{V}|_{p_1})\Big)$$

  singular in $q \to 1$ limit.

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  Therefore, $\lim_{q\to 1} \hat{V}^{(\tau)}(z) = \hat{\tau}(z)$

Fusion operator is defined as the following class in $K_G^{\otimes 2}(N_{k,n})_{loc}[[z]]$:

$$\Psi(z) = \sum_{d=0}^{\infty} z^d \mathrm{ev}_{p_1,p_2,*}\Big(QM^d{}_{\substack{\mathrm{relative}\,p_1\\ \mathrm{nonsing}\,p_2}}, \hat{\mathcal{O}}_{\mathrm{vir}}\Big)$$

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Vertices and Fusion operator

Let us talk about $G = T \times \mathbb{C}_q^{\times}$-equivariant K-theory.

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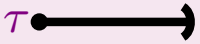
$$\hat{V}^{(\tau)}(z) =$$  $$= \sum_{d=0}^{\infty} z^d \mathrm{ev}_{p_2,*}\left(QM^d{}_{\text{relative } p_2}, \hat{\mathcal{O}}_{\mathrm{vir}} \tau(\mathscr{V}|_{p_1}) \right)$$

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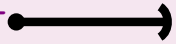
$\hat{V}^{(\tau)}(z) = \tau$ 

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Anton Zeitlin

Fusion relates two types of vertices:

$$\hat{V}^{(\tau)}(z) = \Psi(z) V^{(\tau)}(z)$$

Theorem. i)[A. Okounkov] Fusion operator satisfies q-difference equation:

$$\Psi(qz) = M(z)\Psi(z)\mathcal{O}(1)^{-1},$$

where $\mathcal{O}(1)$ is the operator of classical multiplication by the corresponding line bundle and

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where π is a projection from semistable curve $\mathcal{C}' \to \mathcal{C}$ and $\mathcal{O}_{p_1}$ is a class of point $p_1 \in \mathcal{C}$.

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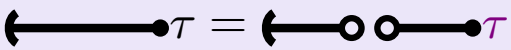

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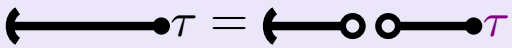

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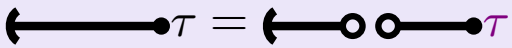

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where $\varphi(x)=\prod_{i=0}^{\infty}(1-q^ix)$, $z_\sharp=(-1)^n\hbar^{n/2}z$, α_p is a normalization parameter.

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Theorem. [P. Pushkar, A. Smirnov, A.Z.]

- The quantum multiplication on quantum tautological class corresponding to $\tau_u := \oplus_{m\geq 0} u^m \Lambda^m \mathcal{V}$ coincides with Q_+-operator, i..e
$$\hat{\tau}_u(z) = Q_+(u)$$

- Explicit universal formulas for quantum products:
$$\widehat{\Lambda^\ell \mathcal{V}}(z) = \Lambda^\ell \mathcal{V} + a_1(z)\; F_0 \Lambda^{\ell-1} \mathcal{V} E_{-1} + \cdots + a_\ell(z)\; F_0^\ell E_{-1}^\ell,$$
where $a_m(z) = \frac{(\hbar-1)^m\;\; \hbar^{\frac{m(m+1)}{2}}\; K^m}{(m)_\hbar! \prod_{i=1}^{m}(1-(-1)^n z^{-1} \hbar^i K)}$,
where K, F_0, E_{-1} are the generators of $U_\hbar(\widehat{\mathfrak{sl}}_2)$.

- $Q_-(u) = \widehat{\tau^\vee}_u(z)$, where $\tau_u^\vee = \oplus_{m\geq 0} u^m \Lambda^m \mathcal{V}^\vee$, $\mathcal{V}^\vee = \mathcal{W} - \mathcal{V}$.

Relation to Many-Body systems: (partial) flags

Givental and his collaborators (1990s and early 2000s): relation between quantum geometry of flag varieties and many body systems.

Cotangent bundle to partial flag variety is a

Nakajima variety of type A:

Stability condition: maps $\mathbf{W}_{n-1} \to \mathbf{V}_{n-1}$ and $\mathbf{V}_i \to \mathbf{V}_{i-1}$ are surjective.

The fixed points: chains of subsets $V_1 \subset \dots \subset V_{n-1} \subset W_{n-1}$, where $|V_i| = \mathbf{v}_i$, $W_{n-1} = \{a_1, \dots, a_{\mathbf{w}_{n-1}}\}$.

Special case when $\mathbf{v}_i = i$, $\mathbf{w}_{n-1} = n$ is known as complete flag variety, which we denote as $T^*\mathbb{F}\ell_n$.

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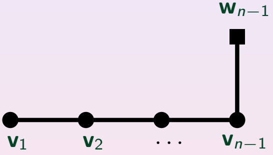

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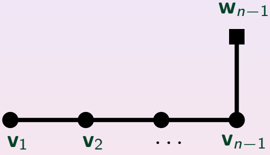

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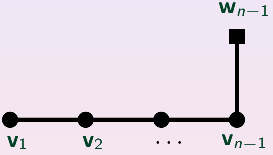

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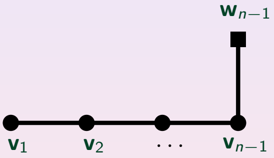

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Proceeding as before, we obtain

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The Bethe equations at hand are the equations for $\mathfrak{gl}(n)$ spin chain.

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Relation to Ruijsenaars-Schneider and Toda systems

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Gaiotto and Koroteev indicated that in the context of Gauge/Bethe correspondence of Nekrasov and Shatashvili.

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Theorem.[KPSZ] Quantum equivariant K-theory of the cotangent bundle to complete n-flag variety is given by:

$$QK_T(T^*\mathbb{F}\ell_n) = \frac{\mathbb{C}[\zeta_1^{\pm 1},\dots,\zeta_n^{\pm 1}; a_1^{\pm 1},\dots,a_n{}^{\pm 1},\hbar^{\pm 1}; p_1^{\pm 1},\dots,p_n^{\pm 1}]}{\{H_r(\zeta_i,p_i,\hbar)=e_r(\alpha_1,\dots,\alpha_n)\}},$$

so that RS Hamiltonians H_r are given by:

$$H_r = \sum_{\substack{\mathcal{I}\subset\{1,\dots,n\}\\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I}\\ j\notin\mathcal{I}}} \frac{\zeta_i\,\hbar^{-1/2}-\zeta_j\,\hbar^{1/2}}{\zeta_i-\zeta_j}\prod_{k\in\mathcal{I}} p_k\,,$$

where $r=0,1,\dots,n$, so that $z_i=\frac{\zeta_i}{\zeta_{i+1}}$ for $i=1,\dots,n-2$, $\alpha_k=\hbar^{n/2}a_k$ and

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Fiber removal

Special limiting procedure $\hbar \to \infty$ and rescaling Kähler parameters, which correspond to the removal of the fiber contributions in localization theorem leads to the following result [KPSZ]:

$$QK_T(\mathbb{F}\ell_n) = \frac{\mathbb{C}[\mathfrak{z}_1^{\pm 1}, \dots, \mathfrak{z}_n^{\pm 1}; \mathfrak{a}_1^{\pm 1}, \dots, \mathfrak{a}_n^{\pm 1}; \mathfrak{p}_1^{\pm 1}, \dots, \mathfrak{p}_n^{\pm 1}]}{\{H_r^{\text{q-Toda}}(\mathfrak{z}_i, \mathfrak{p}_i) = e_r(\mathfrak{a}_1, \dots, \mathfrak{a}_n)\}},$$

where H_r are given by:

$$H_r^{\text{q-Toda}} = \sum_{\substack{\mathcal{I}=\{i_1<\dots<i_r\} \\ \mathcal{I}\subset\{1,\dots,n\}}} \prod_{\ell=1}^{r} \left(1 - \frac{\mathfrak{z}_{i_\ell - 1}}{\mathfrak{z}_{i_\ell}}\right)^{1-\delta_{i_\ell - i_{\ell-1},1}} \prod_{k\in\mathcal{I}} \mathfrak{p}_k,$$

so, e.g.

$$H_1^{\text{q-Toda}} = \mathfrak{p}_1 + \sum_{i=2}^{n} \mathfrak{p}_i \left(1 - \frac{\mathfrak{z}_{i-1}}{\mathfrak{z}_i}\right).$$

That limiting procedure

$$\text{Ruijsenaars} - \text{Schneider} \to \text{Toda}$$

was widely discussed in integrable systems community (Etingof; Gerasimov, Lebedev, Oblezin, ...).

Further directions

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  What we obtained so far is $q \to 1$ limit.

  q-difference Hamiltonians$\leftrightarrow$ q-difference equation for the vertex

  Recent work of Zabrodin and Zotov connects qKZ equations and eigenfunction problem for RS Hamiltonians. Geometric meaning?

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- New kinds of integrable systems. Simplest example: Hilbert scheme of points on a plane.
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