Superopers on Supercurves

Anton M. Zeitlin

Columbia University, Department of Mathematics

567th WE-Heraeus Seminar "Integrable Lattice Models and Quantum Field Theories"

University of Wuppertal, Germany,

June 28 - July 2, 2014

Outline

Reminder: Opers and Gaudin models

Supercurves and $osp(2|1)$ superopers

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Relation to Gaudin Models: $osp(2|1)$

Reminder: Opers and Gaudin models

G-oper on a smooth curve, formal disc or punctured formal disc.
Triple: $(\mathcal{F}, \nabla, \mathcal{F}_B)$
$\mathcal{F}$ - G-bundle, ∇ - connection on $\mathcal{F}$, $\mathcal{F}_B$ is a reduction to B (Borel supgroup)
Locally (for $\mathfrak{g} = \mathfrak{sl}(n)$) Op_G are equivalence classes of:

$$\partial_z + \begin{pmatrix} * & * & * & \dots & * & * \\ 1 & * & * & \dots & * & * \\ 0 & 1 & * & \dots & * & * \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 & * \end{pmatrix}$$

with respect to N.
Equivalent description:

$$\begin{aligned} L &= \partial_z^n + u_1(z)\partial_z^{n-2} + \dots + u_{n-1}(z) \\ L &: \Omega^{-(n-1)/2} \longrightarrow \Omega^{(n+1)/2} \end{aligned}$$

Local description: Adler, Gelfand-Dickey, Drinfeld-Sokolov
Coordinate-indepedent: Beilinson-Drinfeld

Gaudin model

$\mathfrak{g}$ is simple finite dimensional Lie algebra

$$V_{\{\lambda_i\}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}$$

Gaudin hamiltonians:

$$H_i = \sum_{j \neq i} \sum_{a=1}^{d} \frac{T_a^{(i)} T^{a(j)}}{z_i - z_j} \quad i = 1, \dots, N$$

$$V_{\{\lambda_i\}} = \oplus_\mu V_\mu \otimes Hom_{\mathfrak{g}}(V_\mu, V_{\{\lambda_i\}})$$

$$Hom_{\mathfrak{g}}(V_\mu, V_{\{\lambda_i\}}) = (V_{\{\lambda_i\}} \otimes V_{\lambda_\infty})^G$$

where $\lambda_\infty = -\omega_0(\mu)$. Diagonalization via Bethe ansatz (sl_2)

$$|0>= v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_N}, \quad f(w) = \sum_{i=1}^{N} \frac{f^{(i)}}{w - z_i}$$

$$|w_1, \dots, w_m >= f(w_1) f(w_2) \dots f(w_m)|0> \quad \text{(Bethe vectors)}$$

$$\sum_{i=1}^{N} \frac{\lambda_i}{w_j - z_i} - \sum_{s \neq j} \frac{2}{w_j - w_s} = 0, \quad j = 1, \dots, m$$

$$\sum_{i=1}^{N} \lambda_i - m = \mu \quad \text{(Bethe equations)}$$

The relation: There is an injective map from the spectrum of Gaudin hamiltonians acting on

$$V^G_{\{\lambda_i\},\lambda_\infty} = (V_{\{\lambda_i\}} \otimes V_{\lambda_\infty})^G$$

to the set of ${}^L G$-opers on $\mathbb{P}^1$ with regular singularities at $(z_1, \dots, z_N, \infty)$ that have residues $\lambda_1, \dots, \lambda_N, \lambda_\infty$ and trivial monodromy representation.

E. Frenkel, *Opers on the projective line, Flag Manifolds and Bethe Ansatz*, Mosc. Math. J. **4** (2004) 655-705.

B. Feigin, E. Frenkel, N. Reshetikhin, *Gaudin Model, Bethe Ansatz and critical level*, Comm. Math. Phys. **166** (1994) 27-62.

Super Riemann surfaces

Σ is complex supermanifold od dimension $(1|1)$

$$
\begin{aligned}
z_\alpha &= F_{\alpha\beta}(z_\beta, \theta_\beta)\\
\theta_\alpha &= \Psi_{\alpha\beta}(z_\beta, \theta_\beta)
\end{aligned}
$$

Super Riemann surface:
$\mathcal{D}$ is subbundle of dimension $(0|1)$

$$0 \longrightarrow \mathcal{D} \longrightarrow T\Sigma \longrightarrow \mathcal{D}^2 \longrightarrow 0$$

For any nonzero section $D \in \Gamma(\mathcal{D}) \quad D^2 \neq 0(\mathrm{mod} D)$
Superconformal coordinates:

$$D = f(z,\theta)D_\theta, \quad D_\theta = \partial_\theta + \theta\partial_z$$

$D_\theta = (D_\theta\theta')D_{\theta'}$ is superconformal transformation.
Example: $\quad SC^*: \quad z' = -\frac{1}{z}, \quad \theta' = \frac{\theta}{z}$

$$
\begin{aligned}
SPL_2: \quad z &\longrightarrow \frac{az+b}{cz+d} + \theta\frac{\gamma z+\delta}{(cz+d)^2}\\
\theta &\longrightarrow \frac{\gamma z+\delta}{cz+d} + \theta\frac{1+1/2\delta\gamma}{cz+d}
\end{aligned}
$$

Superprojective structures, Superprojective connections, and $osp(2|1)$ superopers

(s)Projective structure on Σ:
Covering of Σ by $\{U_\alpha\}$, such that transition functions are given by SPL_2 (PGL_2)
(s)Projective structures $\longleftrightarrow$ (s)Projective connections
Superprojective connection:

$$L : \mathcal{D}^{-1} \longrightarrow \mathcal{D}^2, \quad L = D_\theta^3 - \omega(z,\theta)$$
$$\omega = \omega'(D_\theta\theta')^3 + \{\theta';\ z,\theta\}$$
$$(cf.\ L = \partial_z^2 - u(z))$$

Flat superholomorphic connections:

$$d_A(fs) = df \otimes s + (-1)^{|f|} f d_A s$$
$$d_A^2 = 0$$
$$d_A = \partial + (\eta A_z + d\theta A_\theta) = \eta D_\theta^{A^2} + d\theta D_\theta^A$$
$$\eta = dz - \theta d\theta$$
$$\nabla : \mathcal{D} \longrightarrow End(V) \quad D_\theta^A = D_\theta + A_\theta$$
$$A_\theta \longrightarrow g A_\theta g^{-1} - D_\theta g g^{-1}$$

$osp(2|1)$ superoper

Useful exercise: superprojective structure $\longrightarrow$ flat connection
SPL_2 - bundle $\mathcal{F}$ over $\Sigma \longrightarrow SC^*_{\mathcal{F}} = \mathcal{F} \times_{SPL_2} SC^*$
$SC^*_{\mathcal{F}}$ has global section with nonvanishing derivative.
$SC^* = G/B$.
One can show that in B-trivialization the connection ∇ is (local coordinates)

$$\Delta = D_\theta + \begin{pmatrix} \alpha & b & \beta \\ -a & 0 & b \\ 0 & a & -\alpha \end{pmatrix} \qquad a \neq 0$$

modulo B -gauge transformations
In other words we have a triple $\quad (\mathcal{F}, \mathcal{F}_B, \nabla)$
Superprojective connections $\longleftrightarrow$

$$D_\theta + \begin{pmatrix} 0 & 0 & \omega(z,\theta) \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

There are bijections between:

i) Superprojective structures on Σ

ii) Superprojective connections on Σ

iii) SPL_2 - opers

Superopers for higher rank

Superalgebras with pure odd family of simple roots:
$sl(n+1|n),\ osp(2n\pm 1|2n),\ osp(2n|2n),\ osp(2n+2|2n),\ n\geqslant 0,$
$D(2,1;\alpha) \quad \alpha\neq 0,\pm 1$

$$\chi_{-1}=\Sigma_i f_i, \qquad \check{\rho}=\sum_i \check{\omega}_i$$

$$[\check{\rho},\chi_{\pm 1}]=\pm\chi_{\pm 1} \qquad [\chi_{\pm 1},\chi_{\mp 1}]=\check{\rho}$$

generate $osp(2|1)$ triple.
$\mathbf{O}$ is an open B-orbit $\subset [\mathfrak{n},\mathfrak{n}]^{\perp}/\mathfrak{b}\subset\mathfrak{g}/\mathfrak{b}$ s.t. vectors are stabilized by N and their negative simple root components w.r.t. $H=B/N$ are non-zero.
Suppose we have G-bundle $\mathcal{F}$ with connection ∇ and a reduction $\mathcal{F}_B$, Consider any ∇' on $\mathcal{F}$ preserving $\mathcal{F}_B$. $\nabla-\nabla'$ gives a section of $(\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B}\otimes\Omega$, which we denote as $\nabla/\mathcal{F}_B$

G-superoper is

$$(\mathcal{F}, \mathcal{F}_B, \nabla)$$

$\mathcal{F}$ - principle G-bundle, $\mathcal{F}_B$ - B-reduction, ∇ - connection, such that $\nabla/\mathcal{F}_B$ takes values in $\mathbf{O}_{\mathcal{F}_B} \subset (\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B}$

Locally

$$D_\theta + \sum_{i=1}^{\ell} a_i(z,\theta) f_i + \mu(z,\theta)$$

modulo $B(R) \quad (a_i \neq 0)$

$\check{\rho}$ gives a principal gradation $\mathfrak{b} = \oplus_{i\geqslant 0}\mathfrak{b}_i$.

$$D_\theta + \chi_{-1} + \sum_{i=1}^{\ell} g_i(z,\theta)\tilde{\chi}_i$$

$\tilde{\chi}_i$ are ad_{χ_1} invariants.

Coordinate transformations:

$$D_\xi + D_\xi\alpha \cdot \chi_{-1} + D_\xi\alpha \cdot \mu \quad \xrightarrow{\check\rho(D_\xi\alpha)}$$
$$D_\xi + \chi_{-1} + D_\xi\alpha \cdot Ad_{\check\rho(D_\xi\alpha)} \cdot \mu - \frac{\partial_\omega\alpha}{D_\xi\alpha}\check\rho$$
$$(z,\theta) = (f(w,\xi), \alpha(w,\xi))$$

Transformation for $g_1, \ldots, g_n$:

$$\exp\left(\kappa\chi_1 - \frac{1}{2}(D\kappa)[\chi_1,\chi_1]\right)\check\rho(D_\xi\alpha), \quad \kappa = \frac{\partial_w\alpha}{D_\xi\alpha}$$

$$s\mathrm{Op}_G(\Sigma) \cong sProj(\Sigma) \times \oplus_{j=1, j\neq 2}^{\ell}\Gamma(\Sigma, \mathcal{D}^{-d_j-1})$$

and

$$\mathcal{F}_H = \mathcal{F}_B \times_B H = \mathcal{F}_{B\backslash N} \cong \mathcal{D}^{-\check\rho}$$

where $\mathcal{D}^{-\check\rho} \times_H \mathbb{C}_\lambda = \mathcal{D}^{-\langle\check\rho,\lambda\rangle}$.

Superopers $\longrightarrow$ Opers: $\nabla \longrightarrow \overline{\nabla^2}$

Superopers with regular singularities and Miura superopers

Beilinson-Drinfeld operator on a disc D_x:

$$\nabla = D_\theta + \sum_{i+1}^{\ell} z^{<\alpha_i,\check{\lambda}>} f_i + \mu(z,\theta)$$

$a_j \neq 0$ modulo $N(O_x)$. Denote $sOp_G(SD_x)_{\check{\lambda}}$
Corresponding oper with regular singularity:

$$\Big(\frac{\check{\rho}}{2} + \check{\lambda}\Big)(z)\overline{\nabla^2}\Big(-\frac{\check{\rho}}{2} - \check{\lambda}\Big)(z) = \partial_z + \frac{1}{z}\Big(\chi_{-1}^2 - \check{\lambda} - \frac{\check{\rho}}{2}\Big) + v(z)$$

Residue: $(-\check{\lambda} - \frac{\check{\rho}}{2})/W$

Miura superopers

$(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B)$ is Miura superoper if $(\mathcal{F}, \nabla, \mathcal{F}_B)$ is a superoper and $\mathcal{F}'_B$ is preserved by ∇.
The set of all reductions

$$(G/B)_{\mathcal{F}_x} = \mathcal{F}_x \times G/B = \mathcal{F}'_{B,x} \times G/B$$

If $\tau \in sOp_G(\Sigma)$ (τ has regular singularity and trivial monodromy), then for any $x \in \Sigma$

$$sMOp_G^{\tau}(\Sigma) \cong (G/B)_{\mathcal{F}'_{B,x}}$$

${}^0G = \cup_\omega S_\omega = \cup_\omega {}^0B\omega_0\omega\, {}^0B$ — Bruhat decomposition for the underlying semisimple Lie group ${}^0G$.

$$\mathcal{F}'_{B,x} \in S_{\omega_0}$$

If $\mathcal{F}_{B,x} \in S_\omega$ then $\mathcal{F}'_{B,x}$ is in "relative position ω" at x, if $\omega = 1$ then they are in "generic" position.

Every (super)oper on the punctured (super)disc is generic.

H-connections on $\mathcal{D}^{-\check{\rho}}$

$$\nabla = D_\theta + u \longrightarrow D_\xi + D_\xi\alpha \cdot u - \check{\rho}\frac{D_\xi \alpha}{\partial_\xi \alpha}$$

Isomorphism of algebraic supervarieties:

$$sMOp_{gen}(U) \cong Conn_U^H$$
$$\nabla \longrightarrow \nabla + \chi_{-1}$$

For a given $\tau \in sMOp_G(SD_x)_{\check{\lambda}}$
H-connection is

$$D_\theta + \frac{\theta}{z}\check{\mu} + f(z,\theta)$$
$$\check{\mu} = \frac{\check{\rho}}{2} - \omega(\check{\lambda} + \frac{\check{\rho}}{2})$$

and $\overline{D_\theta f(z,\theta)}$ is regular in z.

Superopers on SC^*

$$sMOp_G(SC^*)_{\{Z_i\},(\infty,0);(\check\lambda_i),\check\lambda_\infty} \qquad Z_i=(z_i,\theta_i)$$

$$\begin{aligned} D_\theta \quad &- \quad \sum_{i=1}^{n}\frac{\Big(y_i(\check\lambda_i+\frac{\check\rho}{2})-\frac{\check\rho}{2}\Big)(\theta-\theta_i)}{z-z_i+\theta\theta_i} - \\ & \quad\; \sum_{j=1}^{m}\frac{\Big(y_j'(\frac{\check\rho}{2})-\frac{\check\rho}{2}\Big)(\theta-\xi_j)}{z-\omega_j+\theta\xi_j}+f(z,\theta) \\ \overline{f(z,\theta)} \quad &= \quad \overline{D_\theta f(z,\theta)}=0 \end{aligned}$$

Changing coordinates $(u,\eta)=(-\frac{1}{z},\frac{\theta}{z})$ and looking at η/u coefficient:

$$\sum_{i=1}^{N}\Big(y_i(\check\lambda_i+\frac{\check\rho}{2})-\frac{\check\rho}{2}\Big)+\sum_{j=1}^{m}\Big(y_j'(\frac{\check\rho}{2})-\frac{\check\rho}{2}\Big)=y_\infty\omega_0\Big(-\omega_0(\check\lambda_\infty)+\frac{\check\rho}{2}\Big)-\frac{\check\rho}{2}$$

Bethe anzatz equations of $osp(2|1)$

For a given superoper τ: $\Phi_\tau : SC^* \longrightarrow G/B$, $\Phi_\tau : (\infty, 0) \longrightarrow B$

τ is nondegenerate if

i) $\Phi_\tau(Z_i)$ is in generic position with B

ii) The relative position for all other points is either reflection w.r.t. simple root or generic

iii) no "extra terms" for H-connections

$osp(2|1)$ case:

$$\nabla = D + \sum_{i=1}^{N} \frac{\lambda_i(\theta - \theta_i)}{z - z_i + \theta\theta_i} - \sum_{j=1}^{m} \frac{\theta - \xi_j}{z - w_j + \theta\xi_j}$$

$\lambda_\infty = \sum_{i=1}^N \lambda_i - m, \qquad \lambda_i \in \mathbb{Z}_{\geqslant 0}$

$$\sum_{i=1}^{N} \frac{\lambda_i}{w_k - z_i + \xi_k\theta_i} - \sum_{j=1}^{m} \frac{1}{w_k - w_j + \xi_k\xi_j} = 0$$

$$\sum_{i=1}^{N} \frac{\lambda_i\theta_i}{w_k - z_i + \xi_k\theta_i} + \sum_{j=1}^{m} \frac{\xi_j}{w_k - w_j + \xi_k\xi_j} = 0 \qquad k = 1, \dots, m$$

The body of the first family of equations gives Bethe anzatz equations for $osp(2|1)$ Gaudin model.

Thank you!