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Braided Vertex Algebras, Semi-infinite Cohomology and Quantum Group

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based on

Igor Frenkel, AMZ
"Quantum group as semi-infinite cohomology", arXiv:0812.1620

AMS Sectional Meeting (Raleigh, NC)

2009

- **Motivation: equivalence of categories**

- $U_q(sl(2))$**: representations and intertwining operators**

- **Feigin-Fuks construction and the simplest braided VOA**

- $U_q(sl(2))$ **and homology of local systems**

- **Braided VOA on the space**

$$\mathbb{F}_\varkappa = \oplus_{\lambda \geqslant 0}(V_{\Delta(\lambda),\varkappa} \otimes V_\lambda)$$

- $SL_q(2)$ **as semi-infinite cohomology for** $\mathbb{F}_\varkappa \otimes \mathbb{F}_{-\varkappa}$

Motivation

Braided tensor categories:

$\mathcal{C}_q$ – **representations of** $U_q(sl(2))$

$\mathcal{C}_\varkappa$ – **homology of local systems on the configuration spaces**

$\mathcal{C}_c$ – **representations on the Virasoro algebra**

$\mathcal{C}_k$ – **representations of** $\hat{sl}(2)$

Equivalence:

$$\mathcal{C}_q \cong \mathcal{C}_k \qquad\qquad (\text{Kazhdan} - \text{Lusztig})$$

More explicit:

$$\mathcal{C}_q \cong^{(1)} \mathcal{C}_\varkappa \cong^{(2)} \mathcal{C}_c \cong^{(3)} \mathcal{C}_k$$

(1) Gomez-Sierra, Felder-Wiezerkowski, Varchenko

(2) Feigin-Fuks construction (implicitly)

(3) Quantum Drinfeld-Sokolov reduction

Equivalence of categories $\longrightarrow$
$\longrightarrow$ relation between $SL_q(2)$ and WZW conformal field theory

I. Frenkel, K. Styrkas, math/0409117
"Modified regular representations of affine and Virasoro algebras, VOA structure and semi-infinite cohomology"

Algebraic structure on the semi-infinite cohomology:

B.H. Lian, G.J. Zuckerman
Commun.Math.Phys. 154 (1993) 613,
"New Perspectives on the BRST-algebraic structure of string theory"

"Ground rings" in 2D gravity

Frenkel, Styrkas identified the center of $SL_q(2)$ with semi-infinite cohomology of modified regular VOA.

Here we reconstruct the full $SL_q(2)$ via the semi-infinite cohomology using braided VOA.

$U_q(sl(2))$: representations and intertwining operators

Notation:

M_λ – **Verma module with highest weight** λ

V_λ – **irreducible representation**

If $\lambda \in \mathbb{Z}_+$, **we have an exact sequence**

$$0 \to V_\lambda \to M_\lambda^c \to V_{-\lambda-2} \to 0$$

If $\lambda \in \mathbb{Z}_{\leqslant -1}$, **we have** $M_\lambda^c \cong V_\lambda$

λ **is generic if** $\lambda \notin \mathbb{Z}_+$

$\mu, \nu, \lambda \in \mathbb{C}$

$$\begin{aligned} \Phi^\nu_{\mu\lambda}(\cdot \otimes \cdot) &: M^c_\mu \otimes M^c_\lambda \to M^c_\nu \\ \Phi^{\mu\lambda}_\nu(\cdot) &: M_\nu \to M_\mu \otimes M_\lambda. \end{aligned}$$

$\mu, \nu, \lambda \in \mathbb{Z}_+$

$$\begin{aligned} \phi^\nu_{\mu\lambda}(\cdot \otimes \cdot) &: V^c_\mu \otimes V^c_\lambda \to V^c_\nu \\ \phi^{\mu\lambda}_\nu(\cdot) &: V_\nu \to V_\mu \otimes V_\lambda. \end{aligned}$$

Relations between intertwiners:

Proposition

a) λ_i **are generic**

$$\Phi^{\lambda_0}_{\rho\lambda_3}\Phi^{\rho}_{\lambda_1\lambda_2}(1\otimes PR)=\sum_{\xi}B^{M}_{\xi\rho}\begin{bmatrix}\lambda_0 & \lambda_1\\ \lambda_2 & \lambda_3\end{bmatrix}\Phi^{\lambda_0}_{\xi\lambda_2}\Phi^{\xi}_{\lambda_1\lambda_3}$$

b) $\lambda_i\in\mathbb{Z}_+$

$$\phi^{\lambda_0}_{\rho\lambda_3}\phi^{\rho}_{\lambda_1\lambda_2}(1\otimes PR)=\sum_{\xi}B^{V}_{\xi\rho}\begin{bmatrix}\lambda_0 & \lambda_1\\ \lambda_2 & \lambda_3\end{bmatrix}\phi^{\lambda_0}_{\xi\lambda_2}\phi^{\xi}_{\lambda_1\lambda_3}$$

M^c_λ **admits polynomial realization:** $F_\lambda=\mathbb{C}[\beta]\zeta^\lambda$

Proposition **Identification:**

$$E=q^H\gamma,\quad F=\beta[\zeta\partial_\zeta-N]q^{-H},\quad H=\zeta\partial_\zeta-2N,$$

where $N=\beta\partial_\beta$ **and** $\gamma=\partial^q_\beta$ **is a Jackson's q-derivative, gives a structure of** $U_q(sl(2))$**-module on** F_λ **and**

$$F_\lambda\cong M^c_\lambda$$

It leads to polynomial realization for intertwiners

$\Phi^{\nu}_{\mu\lambda}\in Hom(M^c_\mu\otimes M^c_\lambda,M^c_\nu)$**,** λ **is generic**

$\Phi'^{\nu}_{\mu\lambda}\in Hom(M^c_\mu\otimes V^c_\lambda,M^c_\nu)$**,** $\lambda\in\mathbb{Z}_+$

This allows us to prove

Proposition

Let $\lambda_i \in \mathbb{Z}_+$ $(i = 0, 1, 2, 3)$. There exists a continuation of the certain elements of the braiding matrix B^M such that

$$B^M_{\rho\xi}\left[\begin{array}{cc}\lambda_0 & \lambda_1\\ \lambda_2 & \lambda_3\end{array}\right] = B^V_{\rho\xi}\left[\begin{array}{cc}\lambda_0 & \lambda_1\\ \lambda_2 & \lambda_3\end{array}\right],$$

where $\rho, \xi \in \mathbb{Z}_+$ and

$$\lambda_1 + \lambda_2 \geqslant \rho \geqslant |\lambda_1 - \lambda_2|, \qquad \lambda_3 + \rho \geqslant \lambda_0 \geqslant |\lambda_3 - \rho|,$$
$$\lambda_1 + \lambda_3 \geqslant \xi \geqslant |\lambda_1 - \lambda_3|, \qquad \lambda_2 + \xi \geqslant \lambda_0 \geqslant |\lambda_2 - \xi|.$$

Feigin-Fuks construction and the simplest braided VOA

Feigin-Fuks:

$$a(z)a(w) \sim \frac{2\varkappa}{(z-w)^2}, \qquad L(z) = \frac{1}{4\varkappa} : a(z)^2 : + \frac{\varkappa - 1}{2\varkappa} a'(z)$$

$$c = 13 - 6(\varkappa + \frac{1}{\varkappa})$$

$$F_{\lambda,\varkappa} = S(a_{-1}, a_{-2}, \dots) \otimes \mathbf{1}_\lambda, \qquad a_n \mathbf{1}_\lambda = 0 \ (n > 0), \quad a_0 \mathbf{1}_\lambda = \lambda \mathbf{1}_\lambda$$

Let $\hat{F}_\varkappa = \oplus_{\lambda \in \mathbb{Z} \oplus \mathbb{Z}\varkappa} F_{\lambda,\varkappa}$ **and**

$$\mathbb{X}(\lambda, z) = \mathbf{1}_\lambda z^{\frac{\lambda a_0}{2\varkappa}} e^{\left(\frac{\lambda}{2\varkappa} \sum_{n>0} \frac{a_{-n}}{n} z^n\right)} e^{-\left(\frac{\lambda}{2\varkappa} \sum_{n>0} \frac{a_n}{n} z^{-n}\right)}$$

For $|z| > |w|$,

$$\mathbb{X}(\lambda, z)\mathbb{X}(\mu, w) = (z-w)^{\frac{\lambda\mu}{2\varkappa}} (\mathbb{X}(\lambda + \mu, w) + \dots)$$

$\mathscr{A}_{z,w}$ **is monodromy around the path**

$$w(t) = \frac{1}{2}\big((z+w) + (w-z)e^{\pi i t}\big),$$
$$z(t) = \frac{1}{2}\big((z+w) + (z-w)e^{\pi i t}\big), \quad t \in [0,1]$$

$$\mathscr{A}_{z,w}\big(\mathbb{X}(\lambda, z)\mathbb{X}(\mu, w)\big) = q^{\frac{\lambda\mu}{2}} \mathbb{X}(\mu, w)\mathbb{X}(\lambda, z), \qquad q = e^{\frac{\pi i}{\varkappa}}$$

Proposition

1) There exists a linear correspondence

$$v \longrightarrow Y(v,z) = \sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1}$$

such that $v \in \hat{F}_{\varkappa}$ **and** $v_{(n)} \in \mathrm{End}\hat{F}_{\varkappa}$**.**

2) Let $|z| > |w|$ **and** $v_\xi \in F_{\xi,\varkappa}, v_\eta \in F_{\eta,\varkappa}$**, where** $\xi, \eta \in \mathbb{C}$**. Then**

$$\mathscr{A}_{z,w}\big(Y(v_\xi,z)Y(v_\eta,w)\big) = q^{\xi\eta/2} Y(v_\eta,w)Y(v_\xi,z).$$

3) There is a vector $\mathbf{1} = \mathbf{1}_0$**, which satisfies**

$$Y(\mathbf{1},z) = \mathrm{Id}_{\hat{F}_{\varkappa}}, \qquad Y(v,z)\mathbf{1}|_{z=0} = v$$

for any $v \in \hat{F}_{\varkappa}$**.**

4) There exists an element $D \in \mathrm{End}(\hat{F}_{\varkappa})$ **such that**

$$D\mathbf{1} = 0, \quad [D, Y(v,z)] = \frac{\mathrm{d}}{\mathrm{d}z} Y(v,z), \quad \forall v \in \hat{F}_{\varkappa}.$$

5) There exists an element $\omega \in \hat{F}_{\varkappa}$ **such that**

$$Y(\omega,z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

and L_n **satisfy the relations of Virasoro algebra with** $L_{-1} = D$**.**

Denote: $\mathbb{X}_s^+(z) = \mathbb{X}(-2,z)$ **and** $\mathbb{X}_s^-(z) = \mathbb{X}(2\varkappa,z)$

Correlators:

$$\langle \mathbf{1}^*_\nu, \mathbb{X}(\mu_n, z_n)\dots\mathbb{X}^+_s(x_\ell)\dots\mathbb{X}^+_s(x_1)\dots\mathbb{X}(\mu_1, z_1)\mathbf{1}_{\mu_0}\rangle$$
$$= \Psi_{\vec{z}}(x_1,\dots,x_\ell)\delta_{\nu,\mu_n+\dots+\mu_1+\mu_0-2\ell}$$

$$\Psi_{\vec{z}}(x_1,\dots,x_\ell) =$$
$$\prod_{i<j}(x_i-x_j)^{2/\varkappa}\prod_{i,p}(x_i-z_p)^{-\lambda_p/\varkappa}\prod_{p<q}(z_p-z_q)^{\frac{\lambda_p\lambda_q}{2\varkappa}}$$

Screening charge:

$$Q^- v = \oint_{C_{z_2}}\frac{\mathrm{d}z}{2\pi i}\mathbb{X}^-_s(z_1)Y(z_2,v)\mathbf{1}|_{z_2=0}, \qquad v\in F_\lambda,\ \lambda\in\mathbb{Z}$$

Proposition **(Properties of Q^-)**

(i) $[Q^-, L_n] = 0$

(ii) Let $\lambda\in\mathbb{Z}$, then:

$$\ker Q^-|_{F_{\lambda,\varkappa}} \ = \ V_{\Delta(\lambda),\varkappa},\ \ \lambda\geqslant 0,\ \ \Delta(\lambda) = -\frac{\lambda}{2}+\frac{\lambda(\lambda+2)}{4\varkappa}$$
$$\ker Q^-|_{F_{\lambda,\varkappa}} \ = \ 0, \qquad \lambda<0$$

where $V_{\Delta(\lambda),\varkappa}$ is irreducible Virassoro module with highest weight $\Delta(\lambda)$

Corollary **The space $F_{\lambda,\varkappa}$ gives a realization for the dual Verma module**

$$0\longrightarrow V_{\Delta(\lambda),\varkappa}\longrightarrow\ F_{\lambda,\varkappa}\ \longrightarrow V_{\Delta(-\lambda-2),\varkappa}\longrightarrow 0 \qquad \lambda\geqslant 0$$
$$F_{\lambda,\varkappa}\ \cong V_{\Delta(\lambda),\varkappa} \qquad \lambda\in\mathbb{Z}_{\leqslant -1}$$

$U_q(sl(2))$ and homology of local systems

Gomez, Sierra (late 80s, early 90s)
Felder, Wiezerkowski (1991)
Varchenko et al. (90s)

$$\Psi_{\vec{z}}(x_1, \ldots, x_\ell) =$$
$$\prod_{i<j}(x_i - x_j)^{2/\varkappa} \prod_{i,p}(x_i - z_p)^{-\lambda_p/\varkappa} \prod_{p<q}(z_p - z_q)^{\frac{\lambda_p \lambda_q}{2\varkappa}},$$
$$\vec{z} = (z_1, \ldots, z_n)$$

defines 1-dimensional local system $\mathscr{S}$ on $\mathbb{C}^\ell \backslash \mathcal{C}$
$\mathcal{C}$ is a set of hyperplanes: $x_i = x_j, x_i = z_k$

Sections: $s(x) = \alpha\cdot$(univalent branch of $\Psi_{\vec{z}}(x)$)

One can define homology $H_\ell(\mathbb{C}^\ell \backslash \mathcal{C}, \mathscr{S})$

There is a natural action of a permutation group Σ, therefore one can define:

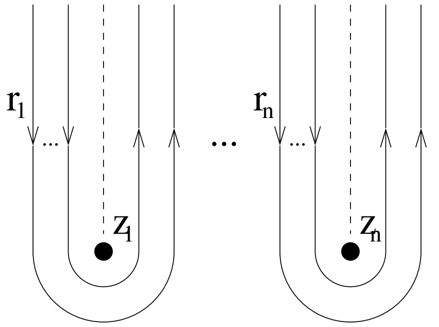
$$H_\ell^{-\Sigma}(z_1, \ldots, z_n, \lambda_1, \ldots, \lambda_n) = H_\ell^{-\Sigma}(\mathbb{C}^\ell \backslash \mathcal{C}, \mathscr{S})$$

$$Br\big((x_i - x_j)^\rho\big) = \begin{cases} e^{\rho \log(x_i - x_j)} & \mathrm{Re}\ x_i > \mathrm{Re}\ x_j \\ e^{\rho \log(x_j - x_i)} & \mathrm{Re}\ x_i < \mathrm{Re}\ x_j \end{cases}$$

$$\mathrm{Br}(\Psi) =$$
$$\prod \mathrm{Br}(x_i - x_j)^{2/\varkappa} \prod \mathrm{Br}(x_i - z_k)^{-\lambda_k/\varkappa} \prod \mathrm{Br}(z_i - z_j)^{\frac{\lambda_i \lambda_j}{2\varkappa}}$$

is a section of $\mathscr{S}$ over $D \subset \mathbb{C}^\ell$, if $z_i - z_j \notin i\mathbb{R}$ and $x_i - x_j \notin i\mathbb{R}$ in D.

Gomez-Sierra contours:

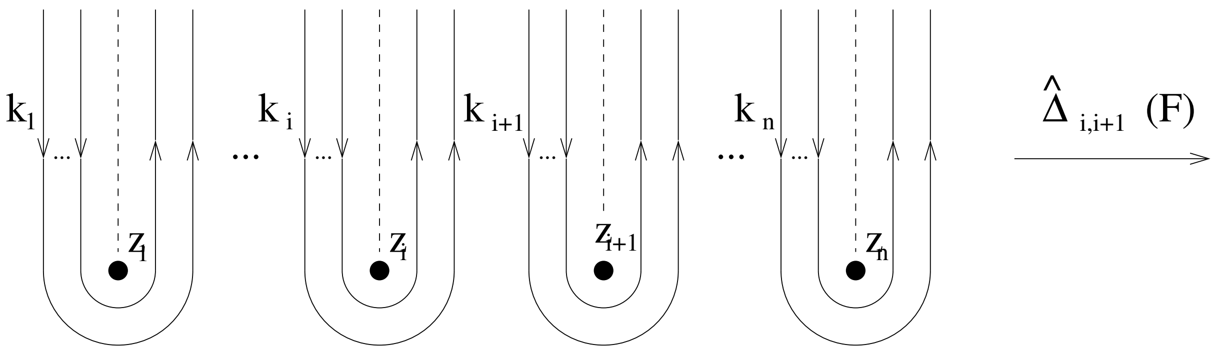


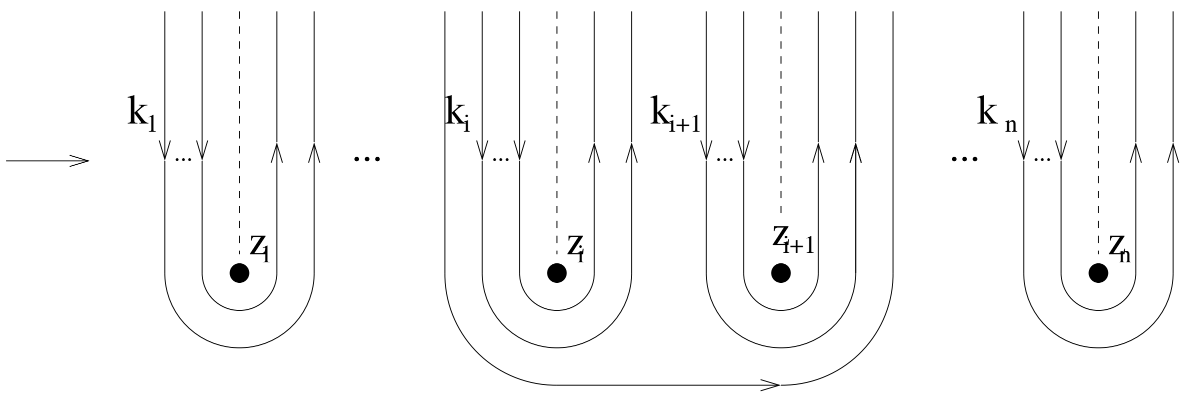
$$G^{\ell}_{r_1 r_2 \ldots, r_n}(z_1, \ldots, z_n, \lambda_1, \ldots, \lambda_n)$$

$$\varphi_{\vec{z}}: \ F^{k_1} v_{\lambda_1} \otimes \cdots \otimes F^{k_n} v_{\lambda_n} \longmapsto G^{\ell}_{k_1, \ldots, k_n}(z_1, \ldots, z_n; \lambda_1, \ldots, \lambda_\ell),$$

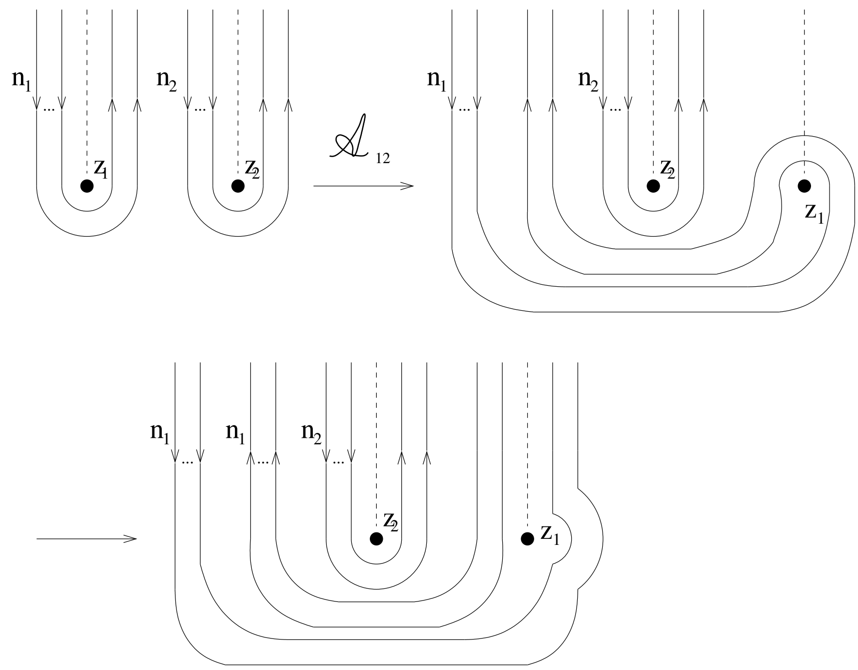
where $v_{\lambda_1}, \ldots, v_{\lambda_n}$ are highest weight vectors in Verma modules of $U_q(sl(z))$

One can obtain a geometric description of F-action:





Action of R-matrix as a monodromy operator:



Action of E-operator coincides up to a constant with the action of boundary operator.

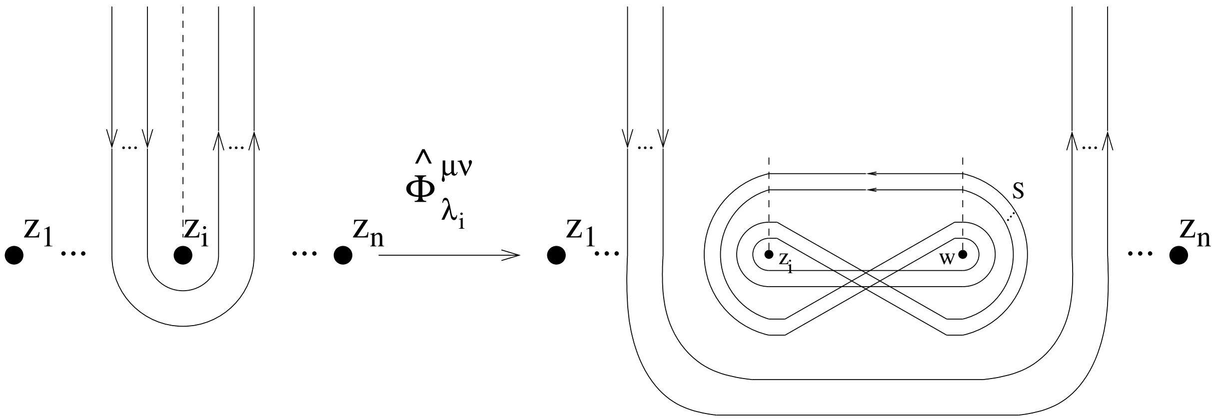
<u>**Theorem**</u>**: i) There is a natural isomorphism between $H_\ell^{-\Sigma}(z_1, z_2 ..., z_n;\ \lambda_1,\ \lambda_2\ ...\lambda_n)$ and singular vectors on the level $\lambda - 2\ell$ in the tensor product of n Verma modules**

$$Sing(M_{\lambda_1} \otimes ... \otimes M_{\lambda_n})[\lambda - 2\ell], \qquad \lambda = \lambda_1 + ... + \lambda_n$$

ii)The next diagram commutes:

$$\begin{array}{ccc}
\begin{array}{c} Sing_{\lambda-2\ell}(M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_i} \otimes \\ M_{\lambda_{i+1}} \otimes ... \otimes M_{\lambda_n}) \end{array} & \xrightarrow{i} & H_\ell^{-\Sigma}(z_1, \ldots, z_n; \lambda_1, \ldots, \lambda_n) \\
\big\downarrow \check{R}_{i,i+1} & & \big\downarrow \mathscr{A}_{i,i+1} \\
\begin{array}{c} Sing_{\lambda-2\ell}(M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_i+1} \otimes \\ M_{\lambda_{i+1}} \otimes ... \otimes M_{\lambda_n}) \end{array} & \xrightarrow{i} & H_\ell^{-\Sigma}(z_1, \ldots, z_n; \lambda_1, \ldots, \lambda_n)
\end{array}$$

Intertwiners also possess geometric realization:

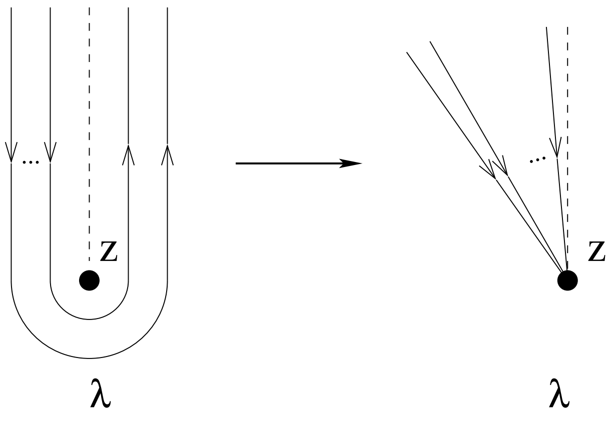


<u>Proposition</u> **Let** $\mathrm{Re}\ z_1 < \mathrm{Re}\ z_2 < \mathrm{Re}\ z_3$ **and** $\lambda_i\ (i = 0, 1, 2, 3)$ **be generic. Then**

$$\mathscr{A}_{2,3}\big(\hat{\Phi}^{\lambda_1\lambda_2}_{\rho}(z_1, z_2)\hat{\Phi}^{\rho\lambda_3}_{\lambda_0}(z_1, z_3)\big) =$$

$$= \sum_{\xi} B^M_{\rho\xi}\begin{bmatrix} \lambda_0 & \lambda_1 \\ \lambda_2 & \lambda_3 \end{bmatrix} \hat{\Phi}^{\lambda_1\lambda_3}_{\xi}(z_1, z_3)\hat{\Phi}^{\xi\lambda_2}_{\lambda_0}(z_1, z_2),$$

Irreducible representations: relative homology w.r.t. points $z_1, \ldots, z_n$



$$\mathscr{A}_{2,3}\big(\hat{\Phi}^{\lambda_1\lambda_2}_{\rho}(z_1, z_2)\hat{\Phi}^{\rho\lambda_3}_{\lambda_0}(z_1, z_3)\big) =$$

$$= \sum_{\xi} B^V_{\rho\xi}\begin{bmatrix} \lambda_0 & \lambda_1 \\ \lambda_2 & \lambda_3 \end{bmatrix} \hat{\Phi}^{\lambda_1\lambda_3}_{\xi}(z_1, z_3)\hat{\Phi}^{\xi\lambda_2}_{\lambda_0}(z_1, z_2), \qquad \lambda_i \in \mathbb{Z}_+$$

Braided VOA on the space

$$\mathbb{F}_\varkappa = \oplus_{\lambda\geqslant 0}(V_{\Delta(\lambda),\varkappa}\otimes V_\lambda)$$

Let $\lambda,\mu,\nu\in\mathbb{Z}$ such that $\nu\leq\lambda+\mu$. Then there exists an intertwining operator

$$\Phi^\nu_{\lambda\mu}(z): F_{\lambda,\varkappa}\otimes F_{\mu,\varkappa}\to F_{\nu,\varkappa}[[z,z^{-1}]]z^{\Delta(\nu)-\Delta(\mu)-\Delta(\lambda)},$$

i.e. the operator, such that

$$L_n\cdot\Phi^\nu_{\lambda\mu}(z)=\Phi^\nu_{\lambda\mu}(z)\Delta_{z,0}(L_n),$$

$$\Delta_{z,0}(L_n)=\oint_z\frac{\mathrm{d}\xi}{2\pi i}\xi^{n+1}\Big(\sum_m(\xi-z)^{-m-2}L_m\Big)\otimes 1+1\otimes L_n.$$

In particular case when the first argument is the highest weight vector $\mathbf{1}_\lambda\in F_{\lambda,\varkappa}$, the explicit expression for the matrix elements of an intertwiner are given by

$$\langle v^*,\Phi^\nu_{\lambda\mu}(z)(\mathbf{1}_\lambda\otimes v)\rangle=\int_{P^s_{\lambda\mu}}\Psi_{0,z}(x_1,\dots,x_s)$$
$$\langle v^*, :\mathbb{X}(\lambda,z)\mathbb{X}^+_s(x_1)\dots\mathbb{X}^+_s(x_s): v\rangle\mathrm{d}x^1\wedge\dots\wedge\mathrm{d}x^s,$$
$$v\in F_{\mu,\varkappa},\quad v^*\in F^*_{\lambda+\mu-2s,\varkappa}\ (s=\frac{\lambda+\mu-\nu}{2})$$

Proposition Let $z_1, z_2 \in \mathbb{R}$, such that $0 < z_1 < z_2$ and $\lambda_i \geq 0$ $(i = 0, 1, 2, 3)$. Then the following relation holds:

$$\mathscr{A}_{z_1,z_2}\big(\Phi^{\lambda_0}_{\lambda_3\rho}(z_2)\Phi^{\rho}_{\lambda_2\lambda_1}(z_1)\big)(P \otimes 1) =$$
$$\sum_{\xi} B^{V}_{\rho\xi}\begin{bmatrix} \lambda_0 & \lambda_1 \\ \lambda_2 & \lambda_3 \end{bmatrix}\Phi^{\lambda_0}_{\lambda_2\xi}(z_1)\Phi^{\xi}_{\lambda_3\lambda_1}(z_2).$$

We define a map:

$$Y : v \otimes a \to Y(v \otimes a, z) = \sum_{\nu,\mu} \Phi^{\nu}_{\lambda\mu}(z)(v \otimes \cdot) \otimes \phi^{\nu}_{\mu\lambda}(\cdot \otimes a).$$

Here $v \in F_{\lambda,\varkappa}$ **and** $a \in V_\lambda$ **for some** $\lambda \in \mathbb{Z}$.
One can show that $[Q^- \otimes 1, Y(v \otimes a, z)] = 0$ **if** $v \in V_{\Delta(\lambda),\varkappa}$. **Therefore,** Y **acts as follows:**

$$Y : \mathbb{F}_\varkappa \to End(\mathbb{F}_\varkappa)\{z\}.$$

Proposition (i) Let $z, w \in \mathbb{R}$ **and** $0 < z < w$, **then**

$$\mathscr{A}_{z,w}\big(Y(v_1 \otimes a_1, w)Y(v_2 \otimes a_2, z)\big) =$$
$$\sum_i Y(v_2 \otimes r_i^{(1)} a_2, z)Y(v_1 \otimes r_i^{(2)} a_1, w),$$

where $R = \sum_i r_i^{(1)} \otimes r_i^{(2)}$ **is the universal R-matrix for** $U_q(sl(2))$.
(ii) Let $t, w, z \in \mathbb{R}$, **such that** $0 < t < w < z$. **Then**

$$Y(v_1 \otimes a_1, z)Y\big(Y(v_2 \otimes a_2, w - t)v_3 \otimes a_3, t\big)\mathbf{1} =$$
$$Y\big(Y(v_1 \otimes a_1, z - w)v_2 \otimes a_2, w\big)Y(v_3 \otimes a_3, t)\mathbf{1}.$$

<u>**Definition**</u> **Let** $\mathbb{V} = \oplus_{\lambda \in I} \mathbb{V}_\lambda$ **be a direct sum of graded complex vector spaces, called sectors:** $\mathbb{V}_\lambda = \oplus_{n \in \mathbb{Z}_+} \mathbb{V}_\lambda[n]$**. Let** Δ_λ **be complex numbers (conformal weights).** $\mathbb{V}$ **is a braided VOA, if there are elements** $0 \in I$ **such that** $\Delta_0 = 0$**,** $\mathbf{1} \in \mathbb{V}_0[0]$**, linear maps** $D : \mathbb{V} \to \mathbb{V}$**,** $\mathcal{R} : \mathbb{V} \otimes \mathbb{V} \to \mathbb{V} \otimes \mathbb{V}$ **and the linear correspondence**

$$\mathbb{Y}(\cdot, z) \cdot : \mathbb{V} \otimes \mathbb{V} \to \mathbb{V}\{z\}, \quad \mathbb{Y} = \sum_{\lambda, \lambda_1, \lambda_2} \mathbb{Y}_\lambda^{\lambda_1 \lambda_2}(z),$$

$$\mathbb{Y}_\lambda^{\lambda_1 \lambda_2}(z) \in Hom(\mathbb{V}_{\lambda_1} \otimes \mathbb{V}_{\lambda_2}, \mathbb{V}_\lambda) \otimes z^{\Delta_\lambda - \Delta_{\lambda_1} - \Delta_{\lambda_2}} \mathbb{C}[[z, z^{-1}]],$$

such that the following properties are satisfied:

i) **Vacuum:** $\mathbb{Y}(\mathbf{1}, z)v = v$**,** $\mathbb{Y}(v, z)\mathbf{1}|_{z=0} = v$**.**

ii) **Complex analyticity: for any** $v_i \in \mathbb{V}_{\lambda_i}$**,** $(i = 1, 2, 3, 4)$ $\langle v_4^*, \mathbb{Y}(v_3, z_2)\mathbb{Y}(v_2, z_1)v_1 \rangle$ **converge in the domain** $|z_2| > |z_1|$ **to a complex analytic function**

$r(z_1, z_2) \in z_1^{h_1} z_2^{h_2} (z_1 - z_2)^{h_3} \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{-1}]$

where $h_1, h_2, h_3 \in \mathbb{C}$**.**

iii) **Derivation property:** $\mathbb{Y}(Dv, z)\mathbf{1} = \frac{d}{dz} \mathbb{Y}(v, z)$.

iv) **Braided commutativity (in a weak sense):**

$$\mathscr{A}_{z,w}(\mathbb{Y}(v, z)\mathbb{Y}(u, w)) = \sum_i \mathbb{Y}(u_i, w)\mathbb{Y}(v_i, z),$$

where $\mathcal{R}(u \otimes v) = \sum_i u_i \otimes v_i$**.**

v) There exists an element $\omega \in \mathbb{V}_0$, such that

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

and L_n satisfy the relations of Virasoro algebra with $L_{-1} = D$.

vi) Associativity (in a weak sense):

$$\mathbb{Y}(\mathbb{Y}(u, z-w)v, w) = \mathbb{Y}(u, z)\mathbb{Y}(v, w).$$

<u>Theorem</u>

The correspondence $Y : \mathbb{F}_{\varkappa} \to End(\mathbb{F}_{\varkappa})\{z\}$ gives a braided VOA structure on $\mathbb{F}_{\varkappa}$.

$SL_q(2)$ as semi-infinite cohomology for $\mathbb{F}_{\varkappa} \otimes \mathbb{F}_{-\varkappa}$

b-c ghost system: $b(z)c(w) \sim \frac{1}{z-w}$

$$b(z) = \sum_m b_m z^{-m-2}, \qquad c(z) = \sum_n c_n z^{-n+1}$$

Fock module

$$\Lambda = \{b_{-n_1} \dots b_{-n_k} c_{-m_1} \dots c_{-m_\ell} \mathbf{1}; \\ c_k \mathbf{1} = 0,\ k \geqslant 2; \quad b_k \mathbf{1} = 0,\ k \geqslant -1\}.$$

$$L^{\Lambda}(z) = 2 : \partial b(z) c(z) : + : b(z) \partial c(z) :$$

Grading is given by $N_g = \oint \frac{\mathrm{d}z}{2\pi i} : c(z) b(z) :$

Let V be a VOA with central charge 26, then holds:

Proposition **The operator of ghost number 1**

$$Q = \oint \frac{\mathrm{d}z}{2\pi i} J_B(z),$$
$$J_B(z) =: \big(L^V(z) + \frac{1}{2} L^{\Lambda}(z)\big) c(z) : + \frac{3}{2} \partial^2 c(z)$$

is nilpotent: $Q^2 = 0$ **on** $V \otimes \Lambda$.

One can define $H^{\frac{\infty}{2}+k}(Vir, \mathbb{C}\mathbf{c}, V)$

Proposition The space $\mathbb{F} = \mathbb{F}_{\varkappa} \otimes \mathbb{F}_{-\varkappa}$ possesses a structure of braided VOA such that the Virasoro algebra has central charge 26.

Lian-Zuckerman associative product on $H^{\frac{\infty}{2}+\cdot}(Vir, \mathbb{C}\mathbf{c}, \mathbb{F})$

$$\mu(U, V) = \mathrm{Res}_z\Big(\frac{U(z)V}{z}\Big)$$

U, V are representatives of $H^{\frac{\infty}{2}+\cdot}(Vir, \mathbb{C}\mathbf{c}, \mathbb{F})$

Proposition The operation μ being considered on $H^{\frac{\infty}{2}+\cdot}(Vir, \mathbb{C}\mathbf{c}, \mathbb{F})$ is associative and satisfies the following commutativity relation:

$$\mu(U, V) = \mu(\hat{r}_i^{(1)}V, \hat{r}_i^{(2)}U)(-1)^{|U||V|},$$

where $\hat{R} = \sum_i \hat{r}_i^{(1)} \otimes \hat{r}_i^{(2)} = R\bar{R}$ and $|\cdot|$ denotes the ghost number.

Proposition Let $\mathbf{F} = \oplus_{\lambda\in\mathbb{Z}_+}(V_{\Delta(\lambda),\varkappa} \otimes V_{\bar{\Delta}(\lambda),-\varkappa})$.
i) $\mathbf{F}$ has a VOA structure
ii) $H^{\frac{\infty}{2}+0}\big(Vir, \mathbb{C}\mathbf{c}, \mathbf{F}(\lambda)\big) = \mathbb{C}$, where
$\mathbf{F}(\lambda) = V_{\Delta(\lambda),\varkappa} \otimes V_{\bar{\Delta}(\lambda),-\varkappa}$

Proposition **The explicit form of the representatives of** $H^{\frac{\infty}{2}+0}\big(Vir, \mathbb{C}\mathbf{c}, \mathbf{F}(1)\big)$ **is:**

$$\Phi^0(z) = L^{\varkappa}_{-1}\Phi(z) - L^{-\varkappa}_{-1}\Phi(z) - \varkappa^{-1} : bc : (z)\Phi(z)$$

Theorem

(i)

$$H^{\frac{\infty}{2}+0}(Vir, \mathbb{C}\mathbf{c}, \mathbb{F}) \cong \bigoplus_{\lambda \geqslant 0} V^q_\lambda \otimes V^{q^{-1}}_\lambda$$

(ii) $H^{\frac{\infty}{2}+0}\big(Vir, \mathbb{C}\mathbf{c}, \mathbb{F}(1)\big)$ **generates all** $H^{\frac{\infty}{2}+0}(Vir, \mathbb{C}\mathbf{c}, \mathbb{F})$ **by means of multiplication** μ**, and the generating set**

$$A, B, C, D \in H^{\frac{\infty}{2}+0}\big(Vir, \mathbb{C}\mathbf{c}, \mathbb{F}(1)\big) \cong V^q_1 \otimes V^{q^{-1}}_1$$

satisfies the following relations:

$$\begin{aligned}
&AB = BAq^{-1}, \quad CB = BC, \quad DB = BDq, \\
&CA = ACq, \quad AD - DA = (q^{-1} - q)BC, \\
&CD = DCq^{-1}, \quad AD - q^{-1}BC = 1
\end{aligned}$$

or

$$\big(H^{\frac{\infty}{2}+0}(Vir, \mathbb{C}\mathbf{c}, \mathbb{F}), \mu\big) \cong SL_q(2)$$