Einstein equations, Beltrami-Courant differentials and Homotopy Gerstenhaber algebras

Anton M. Zeitlin

Columbia University, Department of Mathematics

University of Pennsylvania

Philadelphia

December 11, 2015

COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Outline

Sigma-models and conformal invariance conditions

Beltrami-Courant differential and first order sigma-models

Vertex/Courant algebroids, G_∞-algebra and quasiclassical limit

Einstein Equations from G_∞-algebras

Sigma-models and conformal invariance conditions

Sigma-models for string theory in curved spacetimes:

Let $X : \Sigma \to M$, where Σ is a compact Riemann surface (worldsheet) and M is a Riemannian manifold (target space).

Action functional of sigma model:

$$S_{so} = \frac{1}{4\pi h}\int_{\Sigma}(G_{\mu\nu}(X)dX^{\mu} \wedge *dX^{\nu} + X^{*}B)$$

where G is a metric on M, B is a 2-form on M.

Symetries:

i) conformal symmetry on the worldsheet,

ii) diffeomorphism symmetry and $B \to B + d\lambda$ on target space.

On the quantum level one can add one more term to the action:

$$S_{so} \to S_{so}^{\Phi} = S_{so} + \int_{\Sigma} \Phi(X) R^{(2)}(\gamma) \mathrm{vol}_{\Sigma},$$

where function Φ is called *dilaton*, γ is a metric on Σ.

In order to make sense of path integral

$$Z = \int DX \quad e^{-S_{so}^{\Phi}(X,\gamma)}$$

one has to apply renormalization procedure, so that G, B, Φ depend on certain *cutoff* parameter μ, so that in general quantum theory is not conformally invariant.

Conformal invariance conditions are:

$$\mu\frac{d}{d\mu}G_{\mu\nu}=\beta^G_{\mu\nu}(G,B,\Phi,h)=0,\quad \mu\frac{d}{d\mu}B_{\mu\nu}=\beta^B_{\mu\nu}(G,B,\Phi,h)=0,$$
$$\mu\frac{d}{d\mu}\Phi=\beta^\Phi(G,B,\Phi,h)=0$$

at the level h^0 turn out to be Einstein Equations with $Kalb-Ramond$ 2-form field B and dilaton Φ:

$$R_{\mu\nu}=\frac{1}{4}H_\mu^{\lambda\rho}H_{\nu\lambda\rho}-2\nabla_\mu\nabla_\nu\Phi,$$
$$\nabla^\mu H_{\mu\nu\rho}-2(\nabla^\lambda\Phi)H_{\lambda\nu\rho}=0,$$
$$4(\nabla_\mu\Phi)^2-4\nabla_\mu\nabla^\mu\Phi+R+\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}=0,$$

where 3-form $H=dB$, and $R_{\mu\nu}, R$ are Ricci and scalar curvature correspondingly.

In the early days of string theory:

Linearized Einstein Equations and their symmetries:
$(G_{\mu\nu} = \eta_{\mu\nu} + s_{\mu\nu},\ B_{\mu\nu} = b_{\mu\nu},\ \Phi = \phi)$:

$$Q^\eta \Psi(s, b, \phi) = 0, \quad \Psi^s(s, b, \phi) \to \Psi(s, b, \phi) + Q^\eta \Lambda$$

in a semi-infinite complex associated to Virasoro module of Hilbert space of states for the "free" theory, asssociated to flat metric.

It was conjectured (A. Sen, B. Zwiebach,...) in the early 90s that Einstein equations with h-corrections are Generalized Maurer-Cartan (GMC) Equations:

$$Q^\eta\Psi + \frac{1}{2}[\Psi, \Psi]_h + \frac{1}{3!}[\Psi, \Psi, \Psi]_h + ... = 0$$

$$\Psi \to \Psi + Q^\eta\Lambda + [\Psi, \Lambda]_h + \frac{1}{2}[\Psi, \Psi, \Lambda]_h + ...,$$

where $[\cdot, \cdot, ..., \cdot]_h$ operations, together with differential Q satisfy certain bilinear relations and generate L_∞-algebra (L stands for Lie).

In this talk:

i) Introducing complex structure:

Proper chiral "free action" $\to$ sheaves of vertex algebras/vertex algebroids.

Metric, B-field $\to$ Beltrami-Courant differential.

ii) Vertex algebroids $\to$ G_∞-algebras (G stands for Gerstenhaber).

Quasiclassical limit:
vertex algebroid $\to$ Courant algebroid, G_∞ algebra is truncated.

iii) Einstein equations and their h-corrections via Generalized Maurer-Cartan equation for L_∞-subalgebra of $G_\infty \otimes \bar{G}_\infty$.

First order version of sigma-model action

We start from the action functional:

$$S_0 = \frac{1}{2\pi i h}\int_\Sigma \mathcal{L}_0, \quad \mathcal{L}_0 = \langle p \wedge \bar\partial X\rangle - \langle \bar p \wedge \partial X\rangle,$$

where p, $\bar p$ are sections of $X^*(\Omega^{(1,0)}(M)) \otimes \Omega^{(1,0)}(\Sigma)$, $X^*(\Omega^{(0,1)}(M)) \otimes \Omega^{(0,1)}(\Sigma)$ correspondingly.

Infinitesimal local symmetries:

$$\mathcal{L}_0 \to \mathcal{L}_0 + d\xi$$

For holomorphic transformations we have:

$$X^i \to X^i - v^i(X), X^{\bar i} \to X^{\bar i} - v^{\bar i}(\bar X),$$
$$p_i \to p_i + \partial_i v^k p_k, \quad p_{\bar i} \to p_{\bar i} + \partial_{\bar i} v^{\bar k} p_{\bar k}$$
$$p_i \to p_i - \partial X^k(\partial_k\omega_i - \partial_i\omega_k), \quad p_{\bar i} \to p_{\bar i} - \bar\partial X^{\bar k}(\partial_{\bar k}\omega_{\bar i} - \partial_{\bar i}\omega_{\bar k}).$$

Not invariant under general diffeomorphisms, i.e.

$$\delta\mathcal{L}_0 = -\langle \bar\partial v, p \wedge \bar\partial X\rangle + \langle \partial \bar v, \bar p \wedge \partial X\rangle.$$

It is necessary to add extra terms:

$$\delta\mathcal{L}_\mu = -\langle\mu, p\wedge\bar\partial X\rangle - \langle\bar\mu, \partial X\wedge\bar p\rangle,$$

where $\mu\in\Gamma(T^{(1,0)}M\otimes T^{*(0,1)}(M))$, $\bar\mu\in\Gamma(T^{(0,1)}M\otimes T^{*(1,0)}(M))$, so that: $\mu\to\mu-\bar\partial v+\dots$, $\bar\mu\to\bar\mu-\partial\bar v+\dots$.

Continuing the procedure:

$$\begin{aligned}\tilde{\mathcal{L}} = {}&\langle p\wedge\bar\partial X\rangle - \langle\bar p\wedge\partial X\rangle - \\ &\langle\mu, p\wedge\bar\partial X\rangle - \langle\bar\mu, \partial X\wedge\bar p\rangle - \langle b, \partial X\wedge\bar\partial X\rangle,\end{aligned}$$

where

$$\begin{aligned}&\mu^i_{\bar j}\to\\ &\mu^i_{\bar j}-\partial_{\bar j}v^i+v^k\partial_k\mu^i_{\bar j}+v^{\bar k}\partial_{\bar k}\mu^i_{\bar j}+\mu^i_{\bar k}\partial_{\bar j}v^{\bar k}-\mu^k_{\bar j}\partial_k v^i+\mu^i_{\bar l}\mu^k_{\bar j}\partial_k v^{\bar l},\\ &b_{i\bar j}\to\\ &b_{i\bar j}+v^k\partial_k b_{i\bar j}+v^{\bar k}\partial_{\bar k}b_{i\bar j}+b_{i\bar k}\partial_{\bar j}v^{\bar k}+b_{l\bar j}\partial_i v^l+b_{i\bar k}\mu^k_{\bar j}\partial_k v^{\bar k}+b_{l\bar j}\bar\mu^{\bar k}_i\partial_{\bar k}v^l,\end{aligned}$$

so that the transformations of X- and p- fields are:

$$\begin{aligned}&X^i\to X^i-v^i(X,\bar X),\quad p_i\to p_i+p_k\partial_i v^k-p_k\mu^k_{\bar l}\partial_i v^{\bar l}-b_{j\bar k}\partial_i v^{\bar k}\partial X^j,\\ &X^{\bar i}\to X^{\bar i}-v^{\bar i}(X,\bar X),\quad \bar p_{\bar i}\to\bar p_{\bar i}+\bar p_{\bar k}\partial_{\bar i}v^{\bar k}-\bar p_{\bar k}\bar\mu^{\bar k}_l\partial_{\bar i}v^l-b_{\bar j k}\partial_{\bar i}v^k\bar\partial X^{\bar j}.\end{aligned}$$

Similarly, for the 1-form transformation we obtain:

$$b_{i\bar{j}} \to b_{i\bar{j}} + \partial_{\bar{j}}\omega_i - \partial_i\omega_{\bar{j}} + \mu^{i}_{\bar{j}}(\partial_i\omega_k - \partial_k\omega_i) +$$
$$\bar{\mu}^{\bar{s}}_{i}(\partial_{\bar{j}}\omega_{\bar{s}} - \partial_{\bar{s}}\omega_{\bar{j}}) + \bar{\mu}^{\bar{i}}_{j}\mu^{s}_{\bar{k}}(\partial_s\omega_{\bar{i}} - \partial_{\bar{i}}\omega_s)$$

and

$$p_i \to p_i - \partial X^k(\partial_k\omega_i - \partial_i\omega_k) - \partial_{\bar{r}}\omega_i\partial X^{\bar{r}} - \bar{\mu}^{\bar{s}}_{k}\partial_i\omega_{\bar{s}}\partial X^k,$$
$$p_{\bar{i}} \to p_{\bar{i}} - \bar{\partial} X^{\bar{k}}(\partial_{\bar{k}}\omega_{\bar{i}} - \partial_{\bar{i}}\omega_{\bar{k}}) - \partial_r\omega_{\bar{i}}\bar{\partial} X^r - \mu^{s}_{\bar{k}}\partial_{\bar{i}}\omega_s\bar{\partial} X^{\bar{k}}.$$

For simplicity:

$$E = TM \oplus T^*M, \quad E = \mathcal{E} \oplus \bar{\mathcal{E}},$$
$$\mathcal{E} = T^{(1,0)}M \oplus T^{*(1,0)}M, \quad \bar{\mathcal{E}} = T^{(0,1)}M \oplus T^{*(0,1)}M.$$

Let $\tilde{\mathbb{M}} \in \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$, such that

$$\tilde{\mathbb{M}} = \begin{pmatrix} 0 & \mu \\ \bar{\mu} & b \end{pmatrix}.$$

Introduce $\alpha \in \Gamma(E)$, i.e. $\alpha = (v, \bar{v}, \omega, \bar{\omega})$. Let $D : \Gamma(E) \to \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$, such that

$$D\alpha = \left(\begin{array}{cc} 0 & \bar{\partial} v \\ \partial \bar{v} & \partial \bar{\omega} - \bar{\partial} \omega \end{array} \right).$$

Then the transformation of $\tilde{\mathbb{M}}$ can be expressed:

$$\tilde{\mathbb{M}} \to \tilde{\mathbb{M}} - D\alpha + \phi_1(\alpha, \tilde{\mathbb{M}}) + \phi_2(\alpha, \tilde{\mathbb{M}}, \tilde{\mathbb{M}}).$$

Let us describe ϕ_1, ϕ_2 algebraically. In order to do that we need to pass to jet bundles, i.e.

$$\alpha \in J^\infty(\mathcal{O}_M) \otimes J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}})) \oplus J^\infty(\mathcal{O}(\mathcal{E})) \otimes J^\infty(\bar{\mathcal{O}}_M),$$

$$\tilde{\mathbb{M}} \in J^\infty(\mathcal{O}(\mathcal{E})) \otimes J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}}))$$

Then:

$$\alpha = \sum_J f^J \otimes \bar{b}^J + \sum_K b^K \otimes \bar{f}^K,$$
$$\tilde{\mathbb{M}} = \sum_I a^I \otimes \bar{a}^I,$$

where $a^I, b^J \in J^\infty(\mathcal{O}(\mathcal{E}))$, $f^I \in J^\infty(\mathcal{O}_M)$ and $\bar{a}^I, \bar{b}^J \in J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}}))$, $\bar{f}^I \in J^\infty(\bar{\mathcal{O}}_M)$. Then

$$\phi_1(\alpha, \tilde{\mathbb{M}}) = \sum_{I,J}[b^J, a^I]_D \otimes \bar{f}^J \bar{a}^I + \sum_{I,K} f^K a^I \otimes [\bar{b}^K, \bar{a}^I]_D,$$

where $[\cdot,\cdot]_D$ is a *Dorfman bracket*:

$$[v_1, v_2]_D = [v_1, v_2]^{Lie}, \quad [v, \omega]_D = L_v\omega,$$
$$[\omega, v]_D = -i_v d\omega, \quad [\omega_1, \omega_2]_D = 0.$$

Courant bracket is the antysymmetrized version of $[\cdot,\cdot]_D$.
Similarly:

$$\phi_2(\alpha, \tilde{\mathbb{M}}, \tilde{\mathbb{M}}) = \tilde{\mathbb{M}} \cdot D\alpha \cdot \tilde{\mathbb{M}}$$
$$\frac{1}{2}\sum_{I,J,K} \langle b^I, a^K \rangle a^J \otimes \bar{a}^J(\bar{f}^I)\bar{a}^K + \frac{1}{2}\sum_{I,J,K} a^J(f^I)a^K \otimes \langle \bar{b}^I, \bar{a}^K \rangle \bar{a}^J.$$

Relation to standard second order sigma-model: Let us fill in 0 in $\tilde{\mathbb{M}}$:

$$\mathbb{M} = \begin{pmatrix} g & \mu \\ \bar{\mu} & b \end{pmatrix}.$$

$$S_{fo} = \frac{1}{2\pi i h}\int_{\Sigma}(\langle p \wedge \bar{\partial} X\rangle - \langle \bar{p} \wedge \partial X\rangle -$$
$$-\langle g, p \wedge \bar{p}\rangle - \langle \mu, p \wedge \bar{\partial} X\rangle - \langle \bar{\mu}, \bar{p} \wedge \partial X\rangle - \langle b, \partial X \wedge \bar{\partial} X\rangle).$$

V.N. Popov, M.G. Zeitlin, Phys.Lett. B 163 (1985) 185, A. Losev, A. Marshakov, A.Z., Phys. Lett. B 633 (2006) 375

Same formulas express symmetries. If $\{g^{i\bar{j}}\}$ is nondegenerate, then :

$$S_{so} = \frac{1}{4\pi h}\int_{\Sigma}(G_{\mu\nu}(X)dX^{\mu} \wedge * dX^{\nu} + X^* B),$$

$$\begin{aligned}
G_{s\bar{k}} &= g_{\bar{i}j}\bar{\mu}^{\bar{i}}_s \mu^j_{\bar{k}} + g_{s\bar{k}} - b_{s\bar{k}}, \quad B_{s\bar{k}} = g_{\bar{i}j}\bar{\mu}^{\bar{i}}_s \mu^j_{\bar{k}} - g_{s\bar{k}} - b_{s\bar{k}} \\
G_{si} &= -g_{i\bar{j}}\bar{\mu}^{\bar{j}}_s - g_{s\bar{j}}\bar{\mu}^{\bar{j}}_i, \quad G_{\bar{s}\bar{i}} = -g_{\bar{s}j}\mu^j_{\bar{i}} - g_{\bar{i}j}\mu^j_{\bar{s}} \\
B_{si} &= g_{s\bar{j}}\bar{\mu}^{\bar{j}}_i - g_{i\bar{j}}\bar{\mu}^{\bar{j}}_s, \quad B_{\bar{s}\bar{i}} = g_{\bar{i}j}\mu^j_{\bar{s}} - g_{\bar{s}j}\mu^j_{\bar{i}}.
\end{aligned}$$

Symmetries $\mathbb{M} \to \mathbb{M} - D\alpha + \phi_1(\alpha, \mathbb{M}) + \phi_2(\alpha, \mathbb{M}, \mathbb{M})$ are equivalent to:

A.Z., Adv. Theor. Math. Phys. (2015), to appear

$$\begin{aligned}
&G \to G - L_{\mathbf{v}} G, \quad B \to B - L_{\mathbf{v}} B \\
&B \to B - 2d\boldsymbol{\omega} \\
&\alpha = (\mathbf{v}, \boldsymbol{\omega}), \quad \mathbf{v} \in \Gamma(TM), \boldsymbol{\omega} \in \Omega^1(M)
\end{aligned}$$

Vertex algebroids

The quantum theory, corresponding to the chiral part of the free first order Lagrangian $\mathcal{L}_0$ is described (under certain constraints on M) via sheaves of VOA on M (V. Gorbounov, F. Malikov, V. Schechtman, A. Vaintrob).

On the open set U of M we have VOA:

$$V = \sum_{n=0}^{\infty} V_n, \quad Y : V \to End(V)[[z, z^{-1}]],$$

generated by:

$$[X^i(z), p_j(w)] = h\delta^i_j \delta(z - w), \quad i, j = 1, 2, \dots, D/2$$
$$X^i(z) = \sum_{r\in\mathbb{Z}} X^i_r z^{-r}, p_j(z) = \sum_{s\in\mathbb{Z}} p_{j,s} z^{-s-1} \in End(V)[[z, z^{-1}]].$$

so that

$$V = \mathrm{Span}\{p_{j_1,-s_1}, \dots, p_{j_k,-s_k} X^{i_1}_{-r_1} \dots X^{i_l}_{-r_l}\} \otimes F(U) \otimes \mathbb{C}[h, h^{-1}],$$
$$r_m, s_n > 0,$$

$F(U)$ generated by X^i_0-modes.

The Virasoro element is:

$$T(z)=\sum_{n\in\mathbb{Z}} L_n z^{-n-2}=\frac{1}{h}:\langle p(z)\partial X(z)\rangle:+\partial^2\phi'(X(z)).$$

$$[L_n,L_m]=(n-m)L_{n+m}+\frac{D}{12}(n^3-n)\delta_{n,-m}$$

corresponding to correction:

$$\mathcal{L}_0\to\mathcal{L}_{\phi'}=\langle p\wedge\bar\partial X\rangle-2\pi ihR^{(2)}(\gamma)\phi'(X)$$

where $\phi'=\log\Omega$, where $\Omega(X)dX^1\wedge\cdots\wedge dX^n$ is a holomorphic volume form, i.e. for globally defined $T(z)$, M has to be Calabi-Yau.
The space V is a lowest weight module for the above Virasoro algebra.

V can be reproduced from V_0 and V_1 as a *vertex envelope*. The structure of vertex algebra imposes algebraic relations on $V_0\oplus V_1$ giving it a structure of a *vertex algebroid*.

In our case: $V_0\to\mathcal{O}^h_M=\mathcal{O}_M\otimes\mathbb{C}[h,h^{-1}]$,
$V_1\to\mathcal{V}^h=\mathcal{V}\otimes\mathbb{C}[h,h^{-1}],\quad \mathcal{V}=\mathcal{O}(\mathcal{E})$

A *vertex* $\mathcal{O}_M$*-algebroid* is a sheaf of $\mathbb{C}$-vector spaces $\mathcal{V}$ with a

i) $\mathbb{C}$-linear pairing $\mathcal{O}_M \otimes \mathcal{V} \to \mathcal{V}[h]$, i.e. $f \otimes v \mapsto f * v$ such that $1 * v = v$.

ii) $\mathbb{C}$-linear bracket, satisfying Leibniz algebra $[\ ,\ ] : \mathcal{V} \otimes \mathcal{V} \to h\mathcal{V}[h]$,

iii)$\mathbb{C}$-linear map of Leibniz algebras $\pi : \mathcal{V} \to h\Gamma(TM)[h]$ usually referred to as an anchor

iv) a symmetric $\mathbb{C}$-bilinear pairing $\langle\ ,\ \rangle : \mathcal{V} \otimes \mathcal{V} \to h\mathcal{O}_M[h]$,

v) a $\mathbb{C}$-linear map $\partial : \mathcal{O}_M \to \mathcal{V}$ such that $\pi \circ \partial = 0$,

naturally extending to $\mathcal{O}_M^h$ and $\mathcal{V}^h$, and satisfy the relations

$$\begin{aligned}
& f * (g * v) - (fg) * v = \pi(v)(f) * \partial(g) + \pi(v)(g) * \partial(f), \\
& [v_1, f * v_2] = \pi(v_1)(f) * v_2 + f * [v_1, v_2], \\
& [v_1, v_2] + [v_2, v_1] = \partial\langle v_1, v_2\rangle, \quad \pi(f * v) = f\pi(v), \\
& \langle f * v_1, v_2\rangle = f\langle v_1, v_2\rangle - \pi(v_1)(\pi(v_2)(f)), \\
& \pi(v)(\langle v_1, v_2\rangle) = \langle [v, v_1], v_2\rangle + \langle v_1, [v, v_2]\rangle, \\
& \partial(fg) = f * \partial(g) + g * \partial(f), \\
& [v, \partial(f)] = \partial(\pi(v)(f)), \quad \langle v, \partial(f)\rangle = \pi(v)(f),
\end{aligned}$$

where $v, v_1, v_2 \in \mathcal{V}^h$, $f, g \in \mathcal{O}_M^h$.

For our considerations $\mathcal{V} = \mathcal{O}(\mathcal{E})$:

$$\partial f = df, \quad \pi(v)f = -hv(f), \quad \pi(\omega) = 0,$$
$$f * v = fv + hdX^i \partial_i \partial_j f v^j, \quad f * \omega = f\omega,$$
$$[v_1, v_2] = -h[v_1, v_2]_D - h^2 dX^i \partial_i \partial_k v_1^s \partial_s v_2^k,$$
$$[v, \omega] = -h[v, \omega]_D, \quad [\omega, v] = -h[\omega, v]_D, \quad [\omega_1, \omega_2] = 0,$$
$$\langle v, \omega \rangle = -h\langle v, \omega \rangle^s, \quad \langle v_1, v_2 \rangle = -h^2 \partial_i v_1^j \partial_j v_2^i, \quad \langle \omega_1 . \omega_2 \rangle = 0,$$

where v and ω are vector fields and 1-forms correspondingly.

Together with $\mathrm{div}_{\phi'}$-the divergence operator with respect to ϕ' these operations generate vertex algebroid with Calabi-Yau structure.

Vertex algebra V is a Virasoro module. The corresponding semi-infinite complex V^{semi} (the analogue of Chevalley complex for Virasoro algebra) is a vertex algebra too:

$$V^{semi} = V \otimes \Lambda,$$
$$\Lambda \quad \text{generated by} \quad [b(z), c(w)]_+ = \delta(z-w).$$

The corresponding differential

$$Q = j_0, \quad j(z) = \sum_{n\in\mathbb{Z}} j_n z^{-n-1} = c(z)T(z) + :c(z)\partial c(z) b(z):$$

is nilpotent when $D = 26$ (famous dimension 26!). However, we will consider subcomplex of light modes (i.e. $L_0 = 0$) denoted in the following as $(\mathcal{F}^{\cdot}_h, Q)$, where we can drop this condition:

$$\begin{tikzcd}
 & \mathcal{V}^h \arrow[ddr, "-\frac{1}{2}h\mathrm{div}"] & \mathcal{V}^h \arrow[ddr, "\frac{1}{2}h\mathrm{div}"] & \\
 & \bigoplus & \bigoplus & \\
\mathcal{O}^h_M \arrow[uur, "\partial"] & \mathcal{O}^h_M \arrow[uur, "\partial"] \arrow[r, "id"'] & \mathcal{O}^h_M & \mathcal{O}^h_M.
\end{tikzcd}$$

The homotopy Gerstenhaber algebra of Lian and Zuckerman

The homotopy associative and homotopy commutative product of Lian and Zuckerman:

$$(A, B) = Res_z \frac{A(z)B}{z}$$

$$\begin{aligned}
&Q(a_1, a_2)_h = (Qa_1, a_2)_h + (-1)^{|a_1|}(a_1, Qa_2)_h,\\
&(a_1, a_2)_h - (-1)^{|a_1||a_2|}(a_2, a_1)_h =\\
&Qm(a_1, a_2) + m(Qa_1, a_2) + (-1)^{|a_1|}m(a_1, Qa_2),\\
&Q(a_1, a_2, a_3)_h + (Qa_1, a_2, a_3)_h + (-1)^{|a_1|}(a_1, Qa_2, a_3)_h +\\
&(-1)^{|a_1|+|a_2|}(a_1, a_2, Qa_3)_h = ((a_1, a_2)_h, a_3)_h - (a_1, (a_2, a_3)_h)_h
\end{aligned}$$

Operator **b** of degree -1 (0-mode of $b(z)$) on $(\mathcal{F}^{\cdot}_h, Q)$ which anticommutes with Q:

$$\begin{array}{ccccc}
 & \mathcal{V}^h & \xleftarrow{-id} & \mathcal{V}^h & \\
 & \bigoplus & & \bigoplus & \\
\mathcal{O}^h_M \xleftarrow{id} & \mathcal{O}^h_M & & \mathcal{O}^h_M & \xleftarrow{-id} \mathcal{O}^h_M
\end{array}$$

One can define a bracket:

$$(-1)^{|a_1|}\{a_1, a_2\}_h = \mathbf{b}(a_1, a_2)_h - (\mathbf{b}a_1, a_2)_h - (-1)^{|a_1|}(a_1\mathbf{b}a_2)_h,$$

$$\begin{aligned}
&\{a_1, a_2\}_h + (-1)^{(|a_1|-1)(|a_2|-1)}\{a_2, a_1\}_h = \\
&(-1)^{|a_1|-1}(Qm'_h(a_1, a_2) - m'_h(Qa_1, a_2) - (-1)^{|a_2|}m'_h(a_1, Qa_2)), \\
&\{a_1, (a_2, a_3)_h\}_h = (\{a_1, a_2\}_h, a_3)_h + (-1)^{(|a_1|-1)||a_2|}(a_2, \{a_1, a_3\}_h)_h, \\
&\{(a_1, a_2)_h, a_3\}_h - (a_1, \{a_2, a_3\}_h)_h - (-1)^{(|a_3|-1)|a_2|}(\{a_1, a_3\}_h, a_2)_h = \\
&(-1)^{|a_1|+|a_2|-1}(Qn'_h(a_1, a_2, a_3) - n'_h(Qa_1, a_2, a_3) - \\
&(-1)^{|a_1|}n'_h(a_1, Qa_2, a_3) - (-1)^{|a_1|+|a_2|}n'_h(a_1, a_2, Qa_3), \\
&\{\{a_1, a_2\}_h, a_3\}_h - \{a_1, \{a_2, a_3\}_h\}_h + \\
&(-1)^{(|a_1|-1)(|a_2|-1)}\{a_2, \{a_1, a_3\}_h\}_h = 0.
\end{aligned}$$

The conjecture of Lian and Zuckerman, which was later proven by series of papers (Kimura, Zuckerman, Voronov; Huang, Zhao; Voronov) says that the symmetrized product and bracket of homotopy Gerstenhaber algebra constructed above can be lifted to G_∞-algebra.

Homotopy algebras: G_∞, L_∞, C_∞

Let A be a graded vector space, consider free graded Lie algebra $Lie(A)$.

$$Lie^{k+1}(A) = [A, Lie^k A], \quad Lie^1(A) = A.$$

Consider free graded commutative algebra GA on the suspension $(Lie(A))[-1]$, i.e.

$$GA = \oplus_n \bigwedge^n Lie(A)[-n]$$

There are natural $[\cdot,\cdot]$, $\wedge$ operations on GA of degree -1, 0 correpondingly, generating a Gerstenhaber algebra.

A G_∞*-algebra* (Tamarkin, Tsygan, 2000) is a graded space V with a differential ∂ of degree 1 of $G(V[1]^*)$, such that ∂ is a derivation w.r.t bracket and the product.

Multiplicative Ideals, preserved by ∂: I_1-the commutant of $Lie(V[1]^*)$, $I_2 = \bigwedge_{n\geq 2}(Lie(V[1]^*)[-n]$. That induces differentials on corresponding factors: $\bigwedge_{n\geq 1}(V[1]^*)[-n]$ and $Lie(V[1]^*)[-1]$. The resulting structures on V are called L_∞*-algebra* and C_∞*-algebra* correspondingly.

Restriction of ∂ on $V[1]^*$:

$$V[1]^* \to Lie^{k_1}(V[1]^*) \wedge \cdots \wedge Lie^{k_n}(V[1]^*)$$

Conjugate map:

$$m_{k_1,k_2,\dots,k_n} : V^{\otimes^{k_1}} \otimes \cdots \otimes V^{\otimes^{k_n}} \to V.$$

of degree $3 - n - k_1 - \dots - k_n$, satisfying bilinear relations.

In our previous notation $m_1 = Q$, m_2-symmetrized LZ product, $m_{1,1}$–antisymmetrized LZ bracket.

L_∞ is generated by $m_1 \equiv Q$, $m_{1,1,\dots,1} \equiv [\cdot, \dots, \cdot]$ and C_∞ is generated by $m_1 \equiv Q$, $m_k \equiv (\cdot, \dots, \cdot)$.

An important feature of L_∞ algebra is a Maurer-Cartan equation (Φ is of degree 2) :

$$Q\Phi + \sum_{n\geq 2} \frac{1}{n!}[\underbrace{\Phi, \dots, \Phi}_{n}] + \cdots = 0,$$

which has infinitesimal symmetries:

$$\Phi \to \Phi + Q\Lambda + \sum_{n\geq 1} \frac{1}{n!}[\underbrace{\Phi \dots \Phi}_{n}, \Lambda]$$

Quasiclassical limit of LZ G_∞ algebra

The following complex $(\mathcal{F}^{\cdot}, Q)$:

$$
\begin{tikzcd}
 & \mathcal{V} \arrow[dr, "-\frac{1}{2}h\mathrm{div}"] \arrow[d, phantom, "\bigoplus"] & h\mathcal{V} \arrow[dr, "\frac{1}{2}h\mathrm{div}"] \arrow[d, phantom, "\bigoplus"] & \\
\mathcal{O}_M \arrow[ur, "\partial"] & h\mathcal{O}_M \arrow[ur, "\partial"] \arrow[r, "id"'] & h\mathcal{O}_M & h^2\mathcal{O}_M
\end{tikzcd}
$$

is a subcomplex of $(\mathcal{F}^{\cdot}_h, Q)$. Then

$$
\begin{aligned}
&(\cdot,\cdot)_h : \mathcal{F}^i \otimes \mathcal{F}^j \to \mathcal{F}^{i+j}[h], \quad \{\cdot,\cdot\} : \mathcal{F}^i \otimes \mathcal{F}^j \to h\mathcal{F}_{i+j-1}[h],\\
&\mathbf{b} : \mathcal{F}^i \to h\mathcal{F}^{i-1}[h],
\end{aligned}
$$

so that

$$
(\cdot,\cdot)_0 = \lim_{h\to 0}(\cdot,\cdot)_h, \quad \{\cdot,\cdot\}_0 = \lim_{h\to 0} h^{-1}\{\cdot,\cdot\}_h, \quad \mathbf{b}_0 = \lim_{h\to 0} h^{-1}\mathbf{b}
$$

are well defined.

The symmetrized operations $(\cdot,\cdot)_0$, $\{\cdot,\cdot\}_0, \dots$ satisfy the relations of the homotopy Gerstenhaber algebra, so that all non-covariant higher-order terms disappear from the multilinear operations.

The resulting C_∞ and L_∞ algebras are reduced to C_3 and L_3 algebras.

A.Z., Comm. Math. Phys. 303 (2011) 331-359.

Conjecture: This G_∞-algebra is the G_3-algebra (no homotopies beyond trilinear operations).

Classical limit procedure for vertex algebroid (due to P. Bressler):
$[\cdot,\cdot]_0 = \lim_{h\to 0}\frac{1}{h}[\cdot,\cdot]$, $\pi_0 = \lim_{h\to 0}\frac{1}{h}\pi$, $\langle\cdot,\cdot\rangle_0 = \lim_{h\to 0}\frac{1}{h}\langle\cdot,\cdot\rangle$.

The resulting operations form a Courant algebroid (Z.-J. Liu, A. Weinstein, P. Xu, 1997)

A Courant $\mathcal{O}_M$-algebroid is an $\mathcal{O}_M$-module $\mathcal{Q}$ equipped with a structure of a Leibniz $\mathbb{C}$-algebra $[\cdot,\cdot]_0 : \mathcal{Q}\otimes_{\mathbb{C}}\mathcal{Q}\to\mathcal{Q}$, an $\mathcal{O}_M$-linear map of Leibniz algebras (the anchor map) $\pi_0 : \mathcal{Q}\to\Gamma(TM)$, a symmetric $\mathcal{O}_M$-bilinear pairing $\langle\cdot,\cdot\rangle : \mathcal{Q}\otimes_{\mathcal{O}_M}\mathcal{Q}\to\mathcal{O}_M$, a derivation $\partial : \mathcal{O}_M\to\mathcal{Q}$ which satisfy

$$\begin{aligned}
&\pi\circ\partial = 0, \quad [q_1, fq_2]_0 = f[q_1,q_2]_0 + \pi_0(q_1)(f)q_2\\
&\langle[q,q_1],q_2\rangle + \langle q_1,[q,q_2]\rangle = \pi_0(q)(\langle q_1,q_2\rangle_0),\\
&[q,\partial(f)]_0 = \partial(\pi_0(q)(f))\\
&\langle q,\partial(f)\rangle = \pi_0(q)(f) \quad [q_1,q_2]_0 + [q_2,q_1]_0 = \partial\langle q_1,q_2\rangle_0
\end{aligned}$$

for $f\in\mathcal{O}_M$ and $q, q_1, q_2\in\mathcal{Q}$.
First it was obtained as an analogue of Manin's double for Lie bialgebroid.

In our case $\mathcal{Q}\cong\mathcal{O}(\mathcal{E})$, π_0 is just a projection on $\mathcal{O}(TM)$

$$[q_1,q_2]_0 = -[q_1,q_2]_D, \quad \langle q_1,q_2\rangle_0 = -\langle q_1,q_2\rangle^s, \quad \partial = d.$$

The corresponding L_3-algebra on the half-complex for Courant algebroid was constructed by D. Roytenberg and A. Weinstein (1998).

We show that it is a part of a more general structure, homotopy Gerstenhaber algebra.

<u>Question</u>: Is there a direct path (avoiding vertex algebra) from Courant algebroid to G_3-algebra? Odd analogue of Manin double?

<u>Remark</u>. C_3-algebra is related to gauge theory. The appropriate "metric" deformation gives a Yang-Mills C_3-algebra on a flat space.

A.Z., Comm. Math. Phys. 303 (2011) 331-359.

Simplest version: $G_\infty \to$ Gerstenhaber algebra

Subcomplex $(\mathcal{F}^{\cdot}_{sm}, Q)$:

$$
\begin{tikzcd}
 & \mathcal{O}(T^{(1,0)}M) \arrow[ddr, "-\frac{1}{2}\mathrm{div}"] & \mathcal{O}(T^{(1,0)}M) \arrow[ddr, "\frac{1}{2}\mathrm{div}"] & \\
 & \bigoplus & \overset{0}{\bigoplus} & \\
\mathbb{C} \arrow[uur, "0"] & \mathbb{C} \arrow[uur] \arrow[r, "i"'] & \mathcal{O}_M & \mathcal{O}_M
\end{tikzcd}
$$

The G_∞ algebra degenerates to G-algebra. Moreover, due to $\mathbf{b}_0$ it is a BV-algebra. Combine chiral and antichiral part:

$$\mathbf{F}^{\cdot}_{sm} = \mathcal{F}^{\cdot}_{sm} \otimes \bar{\mathcal{F}}^{\cdot}_{sm}$$

$$(-1)^{|a_1|}\{a_1, a_2\} = \mathbf{b}^{-}(a_1, a_2) - (\mathbf{b}^{-} a_1, a_2) - (-1)^{|a_1|}(a_1 \mathbf{b}^{-} a_2),$$

where $\mathbf{b}^{-} = \mathbf{b} - \bar{\mathbf{b}}$.

Maurer-Cartan elements, closed under $\mathbf{b}^-$:

$$\Gamma(T^{(1,0)}(M)\otimes T^{(0,1)}(M))\oplus \mathcal{O}(T^{(0,1)}(M)\oplus \mathcal{O}(T^{(1,0)}(M)\oplus \mathcal{O}_M\oplus \bar{\mathcal{O}}_M$$

Components:$(g,\bar{v},v,\phi,\bar{\phi})$.

The Maurer-Cartan equation is equivalent to:

A.Z., Nucl. Phys. B 794 (2008) 370-398; A.Z. ATMP, (2015), to appear.

1). Vector field $div_\Omega g$, where $\log\Omega=-2\Phi_0=-2(\phi'+\bar{\phi}'+\phi+\bar{\phi})$ and $\partial_i\partial_{\bar{j}}\Phi_0=0$, is such that its $\Gamma(T^{(1,0)}M)$, $\Gamma(T^{(0,1)}M)$ components are correspondingly holomorphic and antiholomorphic.
2). Bivector field $g\in\Gamma(T^{(1,0)}M\otimes T^{(0,1)}M)$ obeys the following equation:

$$[[g,g]]+\mathcal{L}_{div_\Omega(g)}g=0,$$

where $\mathcal{L}_{div_\Omega(g)}$ is a Lie derivative with respect to the corresponding vector fields and

$$[[g,h]]^{k\bar{l}}\equiv(g^{i\bar{j}}\partial_i\partial_{\bar{j}}h^{k\bar{l}}+h^{i\bar{j}}\partial_i\partial_{\bar{j}}g^{k\bar{l}}-\partial_i g^{k\bar{j}}\partial_{\bar{j}}h^{i\bar{l}}-\partial_i h^{k\bar{j}}\partial_{\bar{j}}g^{i\bar{l}})$$

3). $div_\Omega div_\Omega(g)=0$.

These are Einstein equations with the following constraints:

$$G_{i\bar{k}} = g_{i\bar{k}}, \quad B_{i\bar{k}} = -g_{i\bar{k}}, \quad \Phi = \log\sqrt{g} + \Phi_0,$$
$$G_{ik} = G_{\bar{i}\bar{k}} = G_{ik} = G_{\bar{i}\bar{k}} = 0,$$

Physically:

$$\int [dp][d\bar{p}][dX][d\bar{X}] e^{-\frac{1}{2\pi i h}\int_\Sigma (\langle p\wedge\bar{\partial}X\rangle - \langle\bar{p}\wedge\partial X\rangle - \langle g, p\wedge\bar{p}\rangle) + \int_\Sigma R^{(2)}(\gamma)\Phi_0(X)} =$$
$$\int [dX][d\bar{X}] e^{\frac{-1}{4\pi h}\int d^2z (G_{\mu\nu}+B_{\mu\nu})\partial X^\mu \bar{\partial} X^\nu + \int R^{(2)}(\gamma)(\Phi_0(X) + \log\sqrt{g})}$$

based on computations of
A. Tseytlin and A. Schwarz, Nucl.Phys. B399 (1993) 691-708.

Main Conjecture

Consider

$$\mathbf{F}^{\cdot}_{b-} = \mathcal{F}^{\cdot} \otimes \bar{\mathcal{F}}^{\cdot}|_{b^-=0}$$

with the L_∞-algebra structure given by Lian-Zuckerman construction. One can explicitly check that GMC symmetry
$(\Psi = \Psi(\mathbb{M}, \Phi, \mathrm{auxiliary\ \ fields})$

$$\Psi \to \Psi + Q\Lambda + [\Psi, \Lambda]_h + \frac{1}{2}[\Psi, \Psi, \Lambda]_h + ...,$$

reproduces

$$\mathbb{M} \to \mathbb{M} - D\alpha + \phi_1(\alpha, \mathbb{M}) + \phi_2(\alpha, \mathbb{M}, \mathbb{M}).$$

Conjecture: The corresponding Maurer-Cartan equation gives Einstein equations on G, B, Φ expressed in terms of Beltrami-Courant differential. The symmetries of the Maurer-Cartan equation reproduce mentioned above symmetries of Einstein equations.

Thank you!