Continuous Series for $\widehat{sl}(2,\mathbb{R})$

Formulas

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1. Finite-dimensional Lie algebra

$$sl(2,\mathbb{C}) = \mathbb{C}X^+ \oplus \mathbb{C}X^- \oplus \mathbb{C}X^0$$

$$[X^0, X^\pm] = \pm X^\pm$$

$$[X^+, X^-] = 2X^0$$

Killing form

$$\langle X^+, X^- \rangle = 1$$

$$\langle X^0, X^0 \rangle = \frac{1}{2}$$

Hermitian structure

$$(X^\pm)^* = -X^\mp$$

$$(X^0)^* = X^0$$

2. Affine Lie algebra

$$\hat{sl}(2,\mathbb{C}) = \bigoplus_{n\in\mathbb{Z}} (\mathbb{C}X_n^+ \oplus \mathbb{C}X_n^- \oplus \mathbb{C}X_n^0)$$

$$[X_m^0, X_n^0] = -\frac{k}{2} m\,\delta_{m+n,0}$$

$$[X_m^0, X_n^\pm] = \pm X_{m+n}^\pm$$

$$[X_m^+, X_n^-] = -k\,m\,\delta_{m+n,0} + 2X_{m+n}^0$$

Killing form

$$\langle X_m^+, X_n^- \rangle = \delta_{m+n,0}$$

$$\langle X_m^0, X_n^0 \rangle = \frac{1}{2}\delta_{m+n,0}$$

Hermitian structure

$$(X_n^\pm)^* = -X_{-n}^\mp$$

$$(X_n^0)^* = X_{-n}^0$$

$$k^* = k$$

3. Representations of affine Lie algebra

Heisenberg algebra

$$[h^0_m, h^0_n] = m\delta_{m+n,0}\, k/2$$

$$[h^1_m, h^1_n] = -m\delta_{m+n,0}\, k/2$$

$$[h^2_m, h^2_n] = m\delta_{m+n,0}\, k'/2 \qquad k' = k-2$$

Generating functions

$$h^0(z) = \sum_{n\in\mathbb{Z}} h^0_n z^{-n-1}$$

$$h^1(z) = \sum_{n\in\mathbb{Z}} h^1_n z^{-n-1}$$

$$h^2(z) = \sum_{n\in\mathbb{Z}} h^2_n z^{-n-1}$$

3.1 Space of representation, $k>2$

$$F = F^0 \otimes F^1 \otimes F^2 \otimes \mathbb{C}[\mathbb{Z}]$$

$$F^0 = \mathrm{Funct}\,(h^0_{-n},\ n\geq 1)$$

$$F^1 = \mathrm{Funct}\,(h^1_{n},\ n\geq 1)$$

$$F^2 = \mathrm{Funct}\,(h^2_{-n},\ n\geq 1)$$

$$\mathrm{vac} = 1\otimes 1 \otimes 1 \otimes 1$$

$$h^0_0\,\mathrm{vac} = h^1_0\,\mathrm{vac} = 0 \qquad h^2_0\,\mathrm{vac} = s\,\mathrm{vac}$$

$s \in \mathbb{R}$ parameter of representation

$\mathbb{C}[\mathbb{Z}]$ has a basis $e^{n(\frac{2}{k}h)}$, $n\in\mathbb{Z}$

$$h^0_0\, e^{n(\frac{2}{k}h)} = n e^{n(\frac{2}{k}h)}, \quad h^1_0\, e^{n(\frac{2}{k}h)} = -n e^{n(\frac{2}{k}h)}$$

Space of representation in Gaussian picture

$$F^0 \otimes F^1 \cong \mathrm{Funct}(h_n, n \neq 0) \underbrace{e^{-\frac{1}{k}\sum_{n\geq 1}\frac{1}{n}h_n h_{-n}}}_{\mathrm{vac}}$$

We have a representation of the Heisenberg

$$[h^0_m - h^1_m, \overset{h_n}{\overset{\parallel}{h^0_n + h^1_n}}] = m\,\delta_{m+n,0}\,k$$

We identify $h^0_m - h^1_m = m\,k\,\frac{\partial}{\partial h_{-m}}$, then

$$h^0_n = \frac{1}{2}\left(h_n + nk\frac{\partial}{\partial h_{-n}}\right)$$

$$h^1_n = \frac{1}{2}\left(h_n - nk\frac{\partial}{\partial h_{-n}}\right)$$

$$\left(h_n + nk\frac{\partial}{\partial h_{-n}}\right)e^{-\frac{1}{kn}h_n h_{-n}} = \left(h_n + nk\left(-\frac{1}{nk}\right)h_n\right)e^{\cdots} = 0$$

$$\left(h_{-n} + nk\frac{\partial}{\partial h_{-n}}\right)e^{-\frac{1}{kn}h_n h_{-n}} = \left(h_{-n} + nk\left(-\frac{1}{nk}\right)h_n\right)e^{\cdots} = 0$$

Thus $h^0_n\,\mathrm{vac} = h^1_0\,\mathrm{vac} = 0$ as in $F^0 \otimes F^1$

3.2 Vertex operators

$$\alpha^{\pm}(z) = e^{\pm\frac{2}{k}\varphi(z)} = \sum_{n\in\mathbb{Z}} \alpha^{\pm}_{n} z^{-n}$$

$$\varphi(z) = \sum_{n\neq 0} \frac{h_{-n}}{n} z^{n} + h_0 \log z + h$$

$$\frac{d}{dz}\varphi(z) = \sum_{n\in\mathbb{Z}} h_{-n} z^{n-1} = h(z)$$

$$h(z) \overset{\text{def}}{=} h^0(z) + h'(z), \qquad h_n = h^0_n + h'_n$$

We also assume as usual

$$[h^0_0, h^0] = \frac{k}{2}$$

$$[h'_0, h'] = -\frac{k}{2}$$

Remark: Since h_0 acts by 0 in the representation the term $h_0 \log z$ can be omitted from $\varphi(z)$.

3.3 Operators of representation, $k > 2$

$$X^{+}(z) = :e^{-\frac{2}{k}\varphi(z)}\left(h^{0}(z) - ih^{2}(z) - \frac{z^{-1}}{2}\right):$$

$$-X^{-}(z) = :e^{\frac{2}{k}\varphi(z)}\left(h^{0}(z) + ih^{2}(z)) + \frac{z^{-1}}{2}\right):$$

$$X^{0}(z) = h^{1}(z)$$

where the normal ordering is

$$:e^{\pm\frac{2}{k}\varphi(z)}h^{0}(z): \stackrel{\text{def}}{=} e^{\pm\frac{2}{k}\varphi(z)}\left(\sum_{n\geq 0} h^{0}_{n} z^{-n-1}\right) +$$

$$+\left(\sum_{n<0} h^{0}_{n} z^{-n-1}\right) e^{\pm\frac{2}{k}\varphi(z)}$$

3.4 Verification of commutation relations

$[X^0_m, X^0_n] = -\frac{k}{2} m \delta_{m+n,0}$ by definition

$[X^0_m, X^\pm_n] = \pm X^\pm_{m+n}$ follows from

$[h'_m, \alpha^\mp_n] = \pm \alpha^\mp_{m+n}$ which is equivalent to

$[h'_m, \alpha^\mp(w)] = \pm w^m \alpha^\mp(w)$

$[h'_m, e^{\mp\frac{2}{k}\frac{h_{-m}}{m} w^m}] = \mp w^m [h'_m, \frac{2}{k}\frac{h_{-m}}{m}] e^{\mp\frac{2}{k} h_{-m} w}$

Thus the only nontrivial commutation

$[X^+_m, -X^-_n] = k m \delta_{m+n,0} - 2X^0_{m+n}$

In what follows we will absorb

$\pm\frac{1}{2}$ in $\pm i h^2_0$

We have the following nontrivial OPE

$$:e^{-\frac{2}{k}\varphi(z)}\left(h^0(z)-ih^2(z)\right): \; :e^{+\frac{2}{k}\varphi(w)}\left(h^0(w)+ih^2(w)\right):$$

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$$:e^{-\frac{2}{k}\varphi(z)}\left(h^0(z)-ih^2(z)\right): \; :e^{\frac{2}{k}\varphi(w)}\left(h^0(w)+ih^2(w)\right):$$

We have the following facts

$$h^0(z)h^0(w) = :h^0(z)h^0(w): + \frac{k/2}{(z-w)^2}$$

$$h^2(z)h^2(w) = :h^2(z)h^2(w): + \frac{k'/2}{(z-w)^2}$$

where normal ordering is for $k' > 2$

$$h^0(z)h^0(w) - :h^0(z)h^0(w): = \sum_{n\geq 0} [h^0(n), h^0(-n)] z^{-n-1} w^{n-1}$$

$$= \frac{k}{2} \sum_{n\geq 0} n \frac{w^{n-1}}{z^{n+1}} = \frac{k}{2} \frac{1}{(z-w)^2} \quad , \quad \text{since}$$

$$\frac{1}{z-w} = \frac{1}{z}\left(\frac{1}{1-\frac{w}{z}}\right) = \frac{1}{z} + \frac{w}{z^2} + \frac{w^2}{z^3} + \cdots$$

$$\frac{d}{dw}\frac{1}{z-w} = \frac{1}{(z-w)^2} = \frac{1}{z^2} + 2\frac{w}{z^3} + 3\frac{w^2}{z^4} + \cdots$$

The second fact is similar

We have the following facts

$$h^0(z)\, e^{\frac{2}{k}\varphi(w)} = :h^0(z) e^{\frac{2}{k}\varphi(w)}: + \frac{1}{z-w}\, e^{\frac{2}{k}\varphi(w)}$$

$$e^{-\frac{2}{k}\varphi(z)} h^0(w) = :e^{-\frac{2}{k}\varphi(z)} h^0(w): + \frac{1}{z-w}\, e^{-\frac{2}{k}\varphi(z)}$$

where normal ordering is for $k' > 2$

$$h^0(z) e^{\frac{2}{k}\varphi(w)} - :h^0(z) e^{\frac{2}{k}\varphi(w)}: =$$

$$= \Big[\sum_{n\geq 0} h^0_n z^{-n-1} ,\ e^{\frac{2}{k}\sum \frac{h^0_{-n}}{n} w^n} \Big] =$$

$$= \Big(\sum_{n\geq 0} \frac{w^n}{z^{n+1}} \Big) e^{\frac{2}{k}\sum \frac{h^0_{-n}}{n} w^n} = \frac{1}{z-w}\, e^{\frac{2}{k}\varphi(w)}$$

$$e^{-\frac{2}{k}\varphi(z)} h^0(w) - :e^{-\frac{2}{k}\varphi(z)} h^0(w): =$$

$$= \Big[e^{-\frac{2}{k}\sum \frac{h^0_n}{n} z^{-n}} ,\ \sum_{n\geq 1} h^0_{-n} w^{n-1} \Big] =$$

$$= \Big(\sum_{n\geq 1} \frac{w^{n-1}}{z^n} \Big) e^{-\frac{2}{k}\sum \frac{h^0_n}{n} z^{-n}} = \frac{1}{z-w}\, e^{-\frac{2}{k}\varphi(z)}$$

Thus we obtain the following OPE

$$e^{-\frac{2}{k}\varphi(z)} \quad e^{\frac{2}{k}\varphi(w)} \quad \frac{k/2}{(z-w)^2}$$

$$e^{-\frac{2}{k}\varphi(z)} \quad e^{\frac{2}{k}\varphi(w)} \quad \frac{k/2}{(z-w)^2}$$

$$e^{-\frac{2}{k}\varphi(z)} \quad e^{\frac{2}{k}\varphi(w)} \quad \frac{1}{(z-w)^2}$$

$$e^{-\frac{2}{k}\varphi(z)} \quad e^{\frac{2}{k}\varphi(w)} \quad \frac{1}{z-w}\left(h^0(w)+ih^2(w)\right)$$

$$e^{-\frac{2}{k}\varphi(z)} \quad e^{\frac{2}{k}\varphi(w)} \quad \frac{1}{z-w}\left(h^0(z)-ih^2(z)\right)$$

The first three terms are similar

$$k/2 + k/2 + 1 = k$$

Finally we can compute the relation

$$[X^+_m, -X^-_n] = \iint e^{-\frac{2}{k}\varphi(z)} e^{\frac{2}{k}\varphi(w)} \frac{k}{(z-w)^2} z^m w^n \, dz \, dw$$

$$+ \iint e^{-\frac{2}{k}\varphi(z)} e^{\frac{2}{k}\varphi(w)} \frac{1}{(z-w)} \left(h^0(w) + \cancel{i h^2(w)}\right) z^m w^n \, dz \, dw$$

$$+ \iint e^{-\frac{2}{k}\varphi(z)} e^{\frac{2}{k}\varphi(w)} \frac{1}{z-w} \left(h^0(z) - \cancel{i h^2(z)}\right) z^m w^n \, dz \, dw$$

$$= \int k m \, w^{m+n-1} \, dw + \int \cancel{k}\left(-\frac{2}{\cancel{k}}\right)\left(\cancel{h^0(w)} + h^1(w)\right) w^{m+n} \, dw$$

$$+ \int 2 \cancel{h^0(w)} \, w^{m+n} \, dw = k m \delta_{m+n,0} - 2 h^1_{m+n}$$

$$= k m \delta_{m+n,0} - 2 X^0_{m+n}$$

3.5 Hermitian structure on representations

We define

$$(h^0_m)^* = h^0_{-m}, \quad (h^1_m)^* = h^1_{-m}, \quad (h^2_m)^* = h^2_{-m}$$

$$(h^0)^* = -h^0, \quad (h^1)^* = -h^1$$

Then we have

$$(\alpha^{\pm}_n)^* = \alpha^{\mp}_{-n},$$ which is equivalent

$$\Big(\sum_{n\in\mathbb{Z}} \alpha^{\pm}_n z^{-n}\Big)^* = \sum_{n\in\mathbb{Z}} \alpha^{\mp}_{-n} \bar{z}^{-n}$$

$$\Big(e^{\pm\frac{2}{k}\varphi(z)}\Big)^* = e^{\mp\frac{2}{k}\varphi(\bar{z}^{-1})}$$

$$\varphi(z)^* = -\varphi(\bar{z}^{-1})$$

$$\Big(\sum_{n\neq 0}\frac{h_{-n}}{n} z^n + h_0 \log z + h\Big)^* = -\Big(\sum_{n\neq 0}\frac{h_{-n}}{n}\bar{z}^{-n} + h_0 \log \bar{z}^{-1} + h\Big)$$

3.6 Verification of Hermitian structure

$(X^0_n)^* = X^0_{-n}$ by definition

$(X^+_n)^* = -X^-_{-n}$ is equivalent to

$$\left(\sum X^+_n z^{-n}\right)^* = -\sum X^-_{-n} \bar{z}^{-n}$$

We need to verify

$$\left[e^{-\frac{2}{k}\varphi(z)}\left(\sum_{n\geq 0} h^0_n z^{-n} - \frac{1}{2}\right) + \left(\sum_{n<0} h^0_n z^{-n}\right) e^{-\frac{2}{k}\varphi(z)}\right]^*$$

$$= \left(\sum_{n\geq 0} h^0_{-n} \bar{z}^{-n} - \frac{1}{2}\right) e^{\frac{2}{k}\varphi(\bar{z}^{-1})} + e^{\frac{2}{k}\varphi(\bar{z}^{-1})}\left(\sum_{n<0} h^0_{-n} \bar{z}^{-n}\right)$$

$$= e^{\frac{2}{k}\varphi(\bar{z}^{-1})}\left(\sum_{n\geq 0} h^0_n \cancel{z^{-n}} + \frac{1}{2}\right) + \left(\sum_{n<0} h^0_n \bar{z}^{n}\right) e^{\frac{2}{k}\varphi(\bar{z}^{-1})}$$

$$\left(h^0_0 - \frac{1}{2}\right) e^{\frac{2}{k}\bar{\varphi}(\bar{z}^{-1})} = e^{\frac{2}{k}\varphi(\bar{z}^{-1})}\left(h^0_0 + \frac{1}{2}\right) \qquad \text{Q.E.D.}$$

4. Correlation functions: 2 point

$$\boxed{\begin{aligned}(\mathrm{vac}, X^+_m(-X^-_n)\mathrm{vac}) &= \left(s^2+\tfrac{1}{4}\right)(\mathrm{vac}, \alpha^-_m \alpha^+_n \mathrm{vac}) + \\ &+ k\sum_{r\geq 1} r\, (\mathrm{vac}, \alpha^-_{m-r}\alpha^+_{n+r}\mathrm{vac})\end{aligned}}$$

First proof from OPE

$$(\mathrm{vac}, X^+(z)(-X^-(w))\mathrm{vac}) = \frac{k}{(z-w)^2}(\mathrm{vac}, e^{-\frac{2}{k}\varphi(z)} e^{\frac{2}{k}\varphi(w)} \mathrm{vac})$$

$$+\frac{1}{zw}\left(-is+\tfrac{1}{2}\right)\left(is+\tfrac{1}{2}\right)(\mathrm{vac}, e^{-\frac{2}{k}\varphi(z)} e^{\frac{2}{k}\varphi(w)} \mathrm{vac})$$

Since $\left[e^{-\frac{2}{k}\varphi(z)}, h_0^0 z^{-1}\right] = e^{-\frac{2}{k}\varphi(z)} z^{-1}$

gives an extra contribution: $\left(-is-\frac{1}{2}\right)+1$

Second proof in the component form

$$X^+_m = \sum_j :\alpha^-_{m-j} h^0_j: - i\alpha^-_{m-j} h^2_j - \frac{1}{2}\alpha^-_m$$

$$-X^-_n = \sum_j :\alpha^+_{m-j} h^0_j: + i\alpha^+_{m-j} h^2_j + \frac{1}{2}\alpha^+_m$$

4.2 Correlation functions: general

We would like to find a general correlation function

$$(vac, X^+(z_1) \dots X^+(z_n)(-X^-(w_1)) \dots (-X^-(w_n))\, vac)$$

Our conjecture is the following: we obtain a rational function in z_i, w_i

$$\times\ (vac, \alpha^-(z_1) \dots \alpha^-(z_n)\, \alpha^+(w_1) \dots \alpha^+(w_n)\, vac)$$

Rational function = Basic rational function +

$$+ \sum_{i,j} \frac{S^2 + 1/4}{z_i w_j} \text{ Basic rational function without } z_i, w_j$$

+ lower terms of the same type.

Basic rational function is precisely the same as in the vacuum repres

$$(v_0, X^+(z_1)\dots X^+(z_n)X^-(w_1)\dots X^-(w_n)v_0)$$

and are given by the explicit formulas as in F-Zhu DMJ 66 (1992)

Note that all the rational functions are understood as power series for

$$|z_1|>\dots>|z_n|>|w_1|>\dots>|w_n|$$

The conjecture should be verified by computing the commutation relations.

4.3 Regularization

We define regularized vertex operators

$$\alpha^{\pm}(z,\bar z) = e^{\pm \frac{2}{k}\varphi(z,\bar z)}$$

$$\varphi(z,\bar z) = \sum_{n\geq 1} \frac{h_{-n} z^n}{n} + \frac{h_n \bar z^n}{-n} + h$$

We omit the term with h_0 since it acts trivially in the representation

$$(\mathrm{vac},\ \alpha^-(z,\bar z)\,\alpha^+(w,\bar w)\,\mathrm{vac}) = \left[\frac{(1-z\bar z)(1-w\bar w)}{(1-z\bar w)(1-\bar z w)}\right]^{2/k}$$

More generally we have

$$(\mathrm{vac},\ \alpha^-(z_1,\bar z_1)\ldots\alpha^-(z_n,\bar z_n)\alpha^+(w_1,\bar w_1)\ldots\alpha^+(w_n,\bar w_n)\mathrm{vac}) =$$

$$= \prod_{i,j}\left[\frac{(1-z_i\bar z_j)(1-w_i\bar w_j)}{(1-z_i\bar w_j)(1-\bar z_i w_j)}\right]^{2/k}$$

Verification of regularized correlator: Heisenberg

$$e^{A+B} = e^{A} e^{B} e^{-\frac{1}{2}[AB]}, \quad \text{if } [A,B] \text{ commutes w. } A,B$$

$$\left(1+(A+B)+\tfrac{1}{2}(A+B)^2\right) = \left(1+A+\tfrac{1}{2}A^2\right)\left(1+B+\tfrac{1}{2}B^2\right)\left(1-\tfrac{1}{2}[AB]\right)$$

$$\tfrac{1}{2}(A+B)^2 = \tfrac{1}{2}A^2 + \tfrac{1}{2}B^2 + AB - \tfrac{1}{2}(AB-BA)$$

$$\frac{2}{k}\left(-\varphi(z,\bar z)+\varphi(w,\bar w)\right) = \frac{2}{k}\sum_{n\geq 1} \frac{h_{-n}(-z^n+w^n)}{n} + \frac{h_n(-\bar z^n+\bar w^n)}{-n}$$

$$= \frac{2}{k}\sum_{n\geq 1} \frac{h^0_{-n}(-z^n+w^n)}{n} + \frac{h^0_n(-\bar z^n+\bar w^n)}{-n} + \frac{h^1_{-n}(-z^n+w^n)}{n} + \frac{h^1_n(-\bar z^n+\bar w^n)}{-n}$$

$$A = \frac{2}{k}\sum_{n\geq 1} \frac{h^0_{-n}(-z^n+w^n)}{n} + \frac{h^1_n(-\bar z^n+\bar w^n)}{-n} \qquad \text{"creation"}$$

$$B = \frac{2}{k}\sum_{n\geq 1} \frac{h^0_n(-\bar z^n+\bar w^n)}{-n} + \frac{h^1_{-n}(-z^n+w^n)}{n} \qquad \text{"annihilation"}$$

$$[A,B] = \frac{2}{k}\left(\sum_{n\geq 1} \frac{(-z^n+w^n)(-\bar z^n+\bar w^n)}{n} + \sum_{n\geq 1} \frac{(-z^n+w^n)(-\bar z^n+\bar w^n)}{n}\right)$$

Thus we obtained

$$(\mathrm{vac}, \alpha^-(z,\bar z)\alpha^+(w,\bar w)\,\mathrm{vac}) = e^{-\frac{1}{2}[A,B]} =$$

$$= \exp\left(-\frac{2}{k}\right)\sum_{n\geq 1}\frac{z^n\bar z^n + w^n\bar w^n - z^n\bar w^n - \bar z^n w^n}{n} =$$

$$\left(\sum_{n\geq 1}\frac{t^n}{n} = -\log(1-t)\right)$$

$$= \exp\frac{2}{k}\left(\log(1-z\bar z)+\log(1-w\bar w)-\log(1-z\bar w)-\log(1-\bar z w)\right)$$

$$= \left[\frac{(1-z\bar z)(1-w\bar w)}{(1-z\bar w)(1-\bar z w)}\right]^{2/k}$$

Verification of regularized correlator: Gaussian

$$(\mathrm{vac},\ \alpha^-(z,\bar z)\alpha^+(w,\bar w)\,\mathrm{vac}) =$$

$$\int e^{+\frac{2}{k}\left(\sum_{n\geq 1}\frac{h_{-n}(-z^n+w^n)}{n}+\frac{h_n(\bar z^n+\bar w^n)}{-n}\right)} e^{-\frac{2}{k}\sum_{n\geq 1}\frac{1}{n}h_n h_{-n}} \prod dh_n dh_{-n}$$

$$-\frac{2}{k}\sum_{n\geq 1}\frac{1}{n}(h_n+z^n-w^n)(h_{-n}-\bar z^n+\bar w^n) =$$

$$= -\frac{2}{k}\sum_{n\geq 1}\frac{1}{n}\left(h_n h_{-n}+h_{-n}(z^n-w^n)+h_n(-\bar z^n+\bar w^n)\right)$$

$$+\frac{2}{k}\sum_{n\geq 1}\frac{1}{n}(z^n-w^n)(\bar z^n-\bar w^n)$$

Therefore we obtain

$$(\mathrm{vac},\ \alpha^-(z,\bar z)\alpha^+(w,\bar w)\,\mathrm{vac}) = e^{-\frac{2}{k}\sum_{n\geq 1}\frac{1}{n}(z^n-w^n)(\bar z^n-\bar w^n)}$$

$$= e^{\frac{2}{k}\left(\log(1-z\bar z)+\log(1-w\bar w)-\log(1-z\bar w)-\log(1-\bar z w)\right)} = \left[\frac{(1-z\bar z)(1-w\bar w)}{(1-z\bar w)(1-\bar z w)}\right]^{2/k}$$

4.4 Calculation of correlators

We want to compute

$$(\mathrm{vac},\ \alpha^-_m \alpha^+_n\ \mathrm{vac})$$

where

$$\alpha^{\pm}_m = \frac{1}{2\pi i}\int_{S^1} \alpha^{\pm}(z)\, z^{m-1} dz \quad , \quad m\in\mathbb{Z}$$

We will express $\alpha^{\pm}_m$ via $\alpha^{\pm}(z,\bar{z})$

Note that

$$\int_{D^2} z^n \bar{z}^n \frac{dz\, d\bar{z}}{z\bar{z}} = \frac{2}{i}\int_{D^2} (z\bar{z})^{2n-2} dx\, dy =$$

$$= \frac{4\pi}{i}\int_0^1 r^{2n-2}\, r\, dr = \frac{2\pi}{i}\int_0^1 r^{n-1} dr = \frac{2\pi}{i}\cdot\frac{1}{n}$$

Thus we obtain

$$\alpha^{\pm}_{m} = \frac{i}{2\pi} \int_{D^2} \bar{z} \frac{d}{d\bar{z}} \alpha^{\pm}(z,\bar{z}) \cdot z^m \frac{dz}{z} \frac{d\bar{z}}{\bar{z}} \qquad m \geq 0$$

$$\alpha^{\pm}_{-m} = \frac{i}{2\pi} \int_{D^2} z \frac{d}{dz} \alpha^{\pm}(z,\bar{z}) \; \bar{z}^m \frac{dz}{z} \frac{d\bar{z}}{\bar{z}} \qquad m \geq 0$$

Let $m, n \geq 0$, then

$(\text{vac}, \alpha^{-}_{m} \alpha^{+}_{n} \text{ vac}) =$

$$\left(\frac{i}{2\pi}\right)^2 \int_{D^2}\int_{D^2} \frac{d}{d\bar{z}} \frac{d}{d\bar{w}} \left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-\bar{z}w)}\right]^{2/k} z^{m-1} dz d\bar{z} \; w^{n-1} dw d\bar{w}$$

$$= \left(\frac{2}{k}\right)^2 \int_{D^2}\int_{D^2} \left(-(z-w)^2\right) z^{m-1} w^{n-1} \left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-\bar{z}w)}\right]^{2/k+1} d\mu(z,\bar{z}) d\mu(w,\bar{w})$$

where $d\mu(z,\bar{z}) = \frac{i}{2\pi} \frac{dz \wedge d\bar{z}}{(1-z\bar{z})^2}$ invariant on D^2 measure

We have used the following

$$\frac{d}{d\bar{z}}\frac{d}{d\bar{w}}\left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-\bar{z}w)}\right]^{2/k} =$$

$$= \frac{d}{d\bar{z}}\left(\frac{2}{k}\right)\left(\frac{-w}{1-w\bar{w}} - \frac{-z}{1-z\bar{w}}\right)\left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-\bar{z}w)}\right]^{2/k} =$$

$$= \left(\frac{2}{k}\right)\left(\frac{z-w}{(1-w\bar{w})(1-z\bar{w})}\right)\frac{d}{d\bar{z}}\left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-\bar{z}w)}\right]^{2/k} =$$

$$= \left(\frac{2}{k}\right)^2 \frac{-(z-w)^2}{(1-w\bar{w})(1-z\bar{w})(1-z\bar{z})(1-\bar{z}w)}\left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-\bar{z}w)}\right]^{2/k} =$$

$$= \left(\frac{2}{k}\right)^2\left(\frac{2\pi}{i}\right)^2\left(-(z-w)^2\right)\left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-w\bar{z})}\right]^{2/k+1} d\mu(z,\bar{z})\, d\mu(w,\bar{w})$$

where $d\mu(z,\bar{z}) = \dfrac{i}{2\pi}\dfrac{dz\wedge d\bar{z}}{(1-z\bar{z})^2}$ an invariant

measure on the Lobachevskii plane

Next we would like to compute various correlators in $\alpha_n^{\pm}$. It turns out that it can efficiently achieved using the Berezin quantization theory in Lobachevskii plane, see CMP 40 (1975), 153-174, F. A. Berezin, General concept of quantization. We will review the relevant formulas from this paper and then complete the calculation of the correlation functions.

5. Quantization on the Lobachevskii plane

Let $\mathcal{F}_k$ be the space of analytic functions on D^2, with the scalar prod.

$$(f,g) = \frac{2}{k} \int_{D^2} f(z)\overline{g(z)}(1-z\bar{z})^{2/k+1}\, d\mu(z,\bar{z})$$

where $d\mu(z,\bar{z}) = \frac{i}{2\pi} \frac{dz \wedge d\bar{z}}{(1-z\bar{z})^2}$ invar. measure

In particular, we have

$$(\mathbb{1},\mathbb{1}) = 1 \qquad \mathbb{1} \text{ - constant function}$$

Since $(\mathbb{1},\mathbb{1}) = \frac{2}{k}\int_{D^2}(1-z\bar{z})^{2/k-1}\frac{2dxdy}{2\pi} =$

$$= \frac{2}{k}\int_0^1 (1-r^2)^{2/k-1}\, 2r\, dr = \frac{2}{k}\int_0^1 (1-r)^{2/k-1} dr =$$

$$= \frac{2}{k} \cdot \frac{(1-r)^{2/k}}{2/k}\Big|_1^0 = 1$$

Next we will consider symbols of bounded linear operators in $\mathcal{F}_k$.

$$(\hat{A} f)(z) = \frac{2}{k} \int_{D^2} A(z, \bar{w}) f(w) \left(\frac{1 - w\bar{w}}{1 - z\bar{w}} \right)^{\frac{2}{k}+1} d\mu(w, \bar{w})$$

$$\hat{A} \longleftrightarrow A(z, \bar{w})$$

We note that under this correspondence we have

$$Id \longleftrightarrow \mathbb{1}$$

Thus we need to prove an identity

$$f(z) = \frac{2}{k} \int_{D^2} f(w) \left(\frac{1 - w\bar{w}}{1 - z\bar{w}} \right)^{\frac{2}{k}+1} d\mu(w, \bar{w})$$

We will call it quantum Cauchy formula

Verification of quantum Cauchy formula

Consider the following basis

$$f_n(z) = \left[\frac{\left(\frac{2}{k}+1\right) \cdots \left(\frac{2}{k}+n\right)}{1 \cdot 2 \cdots n} \right]^{1/2} z^n = C_n^{1/2} z^n$$

Then $\{f_n(z)\}_{n=0,1,2,\ldots}$ form an orthonormal basis in $\mathcal{F}_k$. Orthogonality is obvious.

To prove the orthonormality we note

$$\int_0^1 r^n (1-r)^{2/k-1} dr = \left(\frac{n}{\frac{2}{k}+n} \right) \int_0^1 r^{n-1} (1-r)^{2/k-1} dr$$

and

$$\int_0^1 (1-r)^{2/k-1} dr = \left. \frac{(1-r)^{2/k}}{2/k} \right|_1^0 = \frac{1}{2/k}$$

In fact we have

$$\frac{2}{k}\int_0^1 r^n (1-r)^{\frac{2}{k}-1}\, dr = -\cancel{\frac{2}{k}}\int_0^1 r^n \,\frac{d(1-r)^{2/k}}{\cancel{2/k}} =$$

$$= \int_0^1 (1-r)^{2/k}\, dr^n = n\int_0^1 r^{n-1}(1-r)(1-r)^{2/k-1}\, dr$$

thus $\left(\frac{2}{k}+n\right)\int_0^1 r^n(1-r)^{2/k-1}\, dr = n\int_0^1 r^{n-1}(1-r)^{2/k-1}\, dr$

Now we compute the square norm

$$(f_n, f_n) = C_n\,\frac{2}{k}\int_{D^2} z^n\bar{z}^n(1-z\bar{z})^{\frac{2}{k}+1}\frac{1}{2\pi}\,\frac{2\,dx\,dy}{(1-z\bar{z})^2} =$$

$$= C_n\,\frac{2}{k}\int_0^1 r^{2n}(1-r^2)^{\frac{2}{k}-1}\,2r\,dr =$$

$$= C_n\,\frac{2}{k}\int_0^1 r^n(1-r)^{\frac{2}{k}-1}\, dr = \frac{2}{k}\int_0^1 (1-r)^{2/k-1}\, dr = 1$$

Using the orthonormal basis we form the reproduction kernel

$$L_k(z,\bar{z}) = \sum_{n\geq 0} f_n(z)\overline{f_n(w)} =$$

$$= \sum_{n\geq 0} \frac{(\frac{2}{k}+1)\cdots(\frac{2}{k}+n)}{n!}(z\bar{w})^n = (1-z\bar{w})^{-(\frac{2}{k}+1)}$$

This immediately implies the quantum Cauchy formula, since

$$f(z) = \frac{2}{k}\int_{D^2} L_k(z,\bar{w})\, f(w)\,(1-w\bar{w})^{2/k+1}\, d\mu(w,\bar{w})$$

$$= \frac{2}{k}\int_{D^2} f(w)\left(\frac{1-w\bar{w}}{1-z\bar{w}}\right)^{2/k+1} d\mu(w,\bar{w})$$

Next we will find the symbol for the product of two operators $\hat{A} = \hat{A}_1 \cdot \hat{A}_2$

In fact it is sufficient to compare

$$(\hat{A}f)(z) = \left(\frac{2}{k}\right)^2 \int_{D^2}\int_{D^2} A_1(z,\bar{w}) A_2(w,\bar{\zeta}) f(\zeta) \left(\frac{1-w\bar{w}}{1-z\bar{w}}\right)^{2/k+1} \left(\frac{1-\zeta\bar{\zeta}}{1-w\bar{\zeta}}\right)^{2/k+1} d\mu(\zeta,\bar{\zeta})\, d\mu(w,\bar{w})$$

$$(\hat{A}f)(z) = \left(\frac{2}{k}\right) \int_{D^2} A(z,\bar{\zeta}) f(\zeta) \left(\frac{1-\zeta\bar{\zeta}}{1-z\bar{\zeta}}\right)^{2/k+1} d\mu(\zeta,\bar{\zeta})$$

$$A(z,\bar{\zeta}) = \left(\frac{2}{k}\right) \int_{D^2} A_1(z,\bar{w}) A_2(w,\bar{\zeta}) \left(\frac{1-w\bar{w}}{1-z\bar{w}}\right)^{2/k+1} \left(\frac{1-z\bar{\zeta}}{1-w\bar{\zeta}}\right)^{2/k+1} d\mu(w\bar{w})$$

In particular, we obtain

$$A(z,\bar{z}) = \left(\frac{2}{k}\right) \int_{D^2} A_1(z,\bar{w}) A_2(w,\bar{z}) \left[\frac{(1-w\bar{w})(1-z\bar{z})}{(1-z\bar{w})(1-w\bar{z})}\right]^{2/k+1} d\mu(w,\bar{w})$$

Now we can deduce the quantum Cauchy formula for harmonic functions

$$\boxed{f(z) = \frac{2}{k}\int_{D^2} f(w)\left[\frac{(1-w\overline{w})(1-z\overline{z})}{(1-z\overline{w})(1-w\overline{z})}\right]^{2/k+1} d\mu(w,\overline{w})}$$

In fact let f be holomorphic then we set $A_1(z,\overline{w}) = 1$, $A_2(w,\overline{z}) = f(w)$. Thus $\hat{A}_1 = Id$ and $\hat{A}(z,\overline{z}) = f(z)$ and we get the formula using the composition of symbols.

If $f(\overline{z})$ is holomorphic, then we set $A_1(z,\overline{w}) = f(\overline{w})$, $A_2(w,\overline{z}) = 1$, thus $\hat{A}_2 = Id$ and $\hat{A}(z,\overline{z}) = f(\overline{z})$ and again we get the above identity

One can try to apply the above formula for the calculations of the correlation functions $(\mathrm{vac}, \alpha_m^- \alpha_n^+ \mathrm{vac})$ where we get expressions like

$$\left(\frac{2}{k}\right)^2 \iint_{D^2 D^2} (z-w)^2 z^{m-1} w^{n-1} \left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-\bar{z}w)}\right]^{2/k+1} d\mu(z,\bar{z})\, d\mu(w,\bar{w}),$$

for $m \geq 0$, $n \geq 0$, or

$$\left(\frac{2}{k}\right)^2 \iint_{D^2 D^2} (z-w)^2 \bar{z}^{|m|-1} w^{n-1} \left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-\bar{z}w)}\right]^{2/k+1} d\mu(z,\bar{z})\, d\mu(w,\bar{w})$$

for $m \leq 0$, $n \geq 0$, etc. If we integrate over w we get formally

$$\left(\frac{2}{k}\right) \int_{D^2} (z-\bar{z})^2 z^{m+n-2}\, d\mu(z,\bar{z}) \quad \text{or} \quad \left(\frac{2}{k}\right) \int_{D^2} (z-\bar{z})^2 \bar{z}^{|m|-1} z^{n-1}\, d\mu(z,\bar{z})$$

Thus formally these integrals are 0, however the integrals such as

$$\left(\frac{2}{k}\right)\int_{D^2} (z\bar{z})^{n-1}\, d\mu(z,\bar{z})$$

diverge. Thus presumably our correlation functions in α's should be understood via improper integrals

$$\lim_{\varepsilon\to 0} \int_{D^2_{1-\varepsilon}} \int_{D^2_{1-\varepsilon}} \cdots \, d\mu(z,\bar{z})\, d\mu(w,\bar{w})$$

where $D^2_{1-\varepsilon}$ are disks of radius $1-\varepsilon$. But then it is not clear how to compute these integrals, since we cannot apply the above Cauchy formulas any more.

Another interpretation of various correlators in $\alpha_n^{\pm}$ can be achieved via traces of certain "simple" operators in $\mathcal{F}_k$. It is based on the following formula

$$\boxed{\operatorname{tr}_{\mathcal{F}_k}(\hat{A}) = \left(\frac{2}{k}\right)^2 \int_{D^2}\int_{D^2} A(z,\bar{w}) \left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-w\bar{z})}\right]^{2/k+1} d\mu(z,\bar{z})\, d\mu(w,\bar{w})}$$

where $A(z,\bar{w})$ is the symbol of $\hat{A}$ defined above on page 30. In the case of $(\text{vac}, \alpha_m^- \alpha_n^+ \text{vac})$ $A(z,\bar{w})$ is a sum of 3 simple monomials, see page 36

Verification of the trace formula

$$\operatorname{tr}_{\mathcal{F}_k}(\hat{A}) = \sum_{n\geq 0} (\hat{A} f_n, f_n)_{\mathcal{F}} \qquad \{f_n\}\ \text{page 31}$$

$$= \sum_{n\geq 0} \left(\tfrac{2}{k}\right) \int_{D^2} (\hat{A} f_n)(z)\, \overline{f_n(z)}\, (1 - z\bar{z})^{\frac{2}{k}+1}\, d\mu(z,\bar{z}) =$$

$$= \sum_{n\geq 0} \left(\tfrac{2}{k}\right)^2 \int_{D^2}\int_{D^2} A(z,\bar{w})\, f_n(w)\, \overline{f_n(z)} \left(\frac{1-w\bar{w}}{1-z\bar{w}}\right)^{\frac{2}{k}+1} (1-z\bar{z})^{\frac{2}{k}+1}\, d\mu(z,\bar{z})\, d\mu(w,\bar{w})$$

$$= \left(\tfrac{2}{k}\right)^2 \int_{D^2}\int_{D^2} A(z,\bar{w})\, (1-w\bar{z})^{-\left(\frac{2}{k}+1\right)} \left(\frac{1-w\bar{w}}{1-z\bar{w}}\right)^{\frac{2}{k}+1} (1-z\bar{z})^{\frac{2}{k}+1}\, d\mu(z,\bar{z})\, d\mu(w,\bar{w})$$

$$= \left(\tfrac{2}{k}\right)^2 \int_{D^2}\int_{D^2} A(z,\bar{w}) \left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-w\bar{z})}\right]^{2/k+1} d\mu(z,\bar{z})\, d\mu(w,\bar{w})$$

We can now apply the formula for the trace of an operator given by a symbol to the computation of our correlators such as

$$\left(\frac{2}{k}\right)^2 \int_{D^2}\int_{D^2} z^n \bar{w}^n \left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-w\bar{z})}\right]^{2/k+1} d\mu(z,\bar{z})\, d\mu(w,\bar{w})$$

First we will determine the operator with the symbol $A(z,\bar{w}) = z^n \bar{w}^n$ and then we will determine its trace, which will yield the above integral.

Let $f(z) = z^m$, then we have

$$(\hat{A} f)(z) = \frac{2}{k} \int_{D^2} z^n \bar{w}^n w^m \left(\frac{1 - w\bar{w}}{1 - z\bar{w}} \right)^{\frac{2}{k}+1} d\mu(w, \bar{w})$$

We present $d\mu(w, \bar{w}) = \frac{i}{2\pi} \frac{dw \wedge d\bar{w}}{(1 - w\bar{w})^2}$ and

$$(1 - z\bar{w})^{-(\frac{2}{k}+1)} = \sum_{j \geq 0} f_j(z) \overline{f_j(w)}$$

and we get

$$z^n \left(\frac{2}{k}\right) \int_{D^2} \bar{w}^n w^m \sum_{j \geq 0} f_j(z) \overline{f_j(w)} \; \frac{i}{2\pi} \frac{dw \wedge d\bar{w}}{(1 - w\bar{w})^{1 - 2/k}}$$

thus only term with $j = m - n$ contrib.

$$C_{m-n} z^m \left(\frac{2}{k}\right) \int_{D^2} \bar{w}^m w^m \frac{i \, dw \wedge d\bar{w}}{2\pi (1 - w\bar{w})^{1 - 2/k}} =$$

$$= C_{m-n} z^m \left(\frac{2}{k}\right) \int_0^1 r^{2m} \frac{2r\,dr}{(1 - r^2)^{1 - 2/k}} = \frac{C_{m-n}}{C_m} z^m$$

Thus the trace formally is given by

$$tr(\hat{A}) = \sum_{m \geq n} \frac{c_{m-n}}{c_m}$$

The sum certainly does not converge since $\lim_{m\to\infty} \frac{c_{m-n}}{c_m} = 1$, however we actually have differences of the above and double differences integrals, e.g.

$$\left(\frac{2}{k}\right)^2 \int_{D^2}\int_{D^2} \left(z^{n_1}\bar{w}^{n_1} - z^{n_2}\bar{w}^{n_2}\right)\left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-w\bar{z})}\right]^{2/k+1} d\mu(z,\bar{z})\,d\mu(w,\bar{w})$$

$$= \sum_{m \geq 0} \frac{c_{m-n_1} - c_{m-n_2}}{c_m}, \quad (c_{negative} = 0)$$

which has only logarithmic divergence.

Let us now compute a more general integral than on page 40 directly

$$\left(\frac{2}{k}\right)^2 \int\limits_{D^2}\int\limits_{D^2} z^{m_1}\bar{z}^{m_2} w^{n_1}\bar{w}^{n_2} \left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-w\bar{z})}\right]^{2/k+1} d\mu(z,\bar{z})\, d\mu(w,\bar{w})$$

We will use the kernel decompositions

$$(1-z\bar{w})^{-(2/k+1)} = \sum_{p\geq 0} f_p(z) f_p(\bar{w})$$

$$(1-w\bar{z})^{-(2/k+1)} = \sum_{q\geq 0} f_q(w) f_q(\bar{z})$$

Then we can rewrite the integral: $\sum\limits_{p,q\geq 0}$

$$\left(\frac{2}{k}\right) \int\limits_{D^2} z^{m_1}\bar{z}^{m_2} f_p(z) f_q(\bar{z}) (1-z\bar{z})^{2/k+1} d\mu(z,\bar{z}) \;\times$$

$$\times \left(\frac{2}{k}\right) \int\limits_{D^2} w^{n_1}\bar{w}^{n_2} f_q(w) f_p(\bar{w}) (1-w\bar{w})^{2/k+1} d\mu(w,\bar{w})$$