6. Computation of 2-point correlation functions

Now we proceed to a direct calculation of 2-point correlators $(\mathrm{vac}, X^+_m X^-_n \,\mathrm{vac})$, which we have reduced to the 2-point correlators $(\mathrm{vac}, \alpha^-_m \alpha^+_n \mathrm{vac})$

For $m, n > 0$ we proved on page 26

$$(\mathrm{vac}, \alpha^-_m \alpha^+_n \,\mathrm{vac}) = -\left(\frac{2}{k}\right)^2 \iint_{D^2 D^2} (z-w)^2 z^{m-1} w^{n-1} \left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-\bar{z}w)}\right]^{2/k+1} d\mu(z,\bar{z})\, d\mu(w,\bar{w})$$

and one can show that the integral converges and one gets 0. This can be done as in the more complicated case of different signs.

We will compute directly a more complicated case $m < 0$, $n > 0$, and show that $(\text{vac}, \alpha^-_{-m} \alpha^+_n \text{vac}) = 0$, for $m \neq n$, and we will compute explicitely $\boxed{C_n(k) = (\text{vac}, \alpha^-_{-n} \alpha^+_n \text{vac})}$. Clearly, the cases when $m, n < 0$ or $m > 0$, $n < 0$, are exatly similar to $m, n > 0$; $m < 0$, $n > 0$.

We recall from page 26 that

$$\alpha^-_{-m} = \frac{i}{2\pi} \int_{D^2} \frac{d}{dz} \alpha^-(z, \bar{z})\, \bar{z}^{m-1} dz\, d\bar{z} \qquad m > 0$$

$$\alpha^+_n = \frac{i}{2\pi} \int_{D^2} \frac{d}{d\bar{w}} \alpha^+(w, \bar{w})\, w^{n-1} dw\, d\bar{w} \qquad n > 0$$

Thus we need to compute

$$(\mathrm{vac}, \alpha^-_{-m}\alpha^+_n \,\mathrm{vac}) =$$

$$= \left(\frac{i}{2\pi}\right)^2 \iint\limits_{D^2 D^2} \frac{d}{dz}\frac{d}{d\bar{w}} \left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-\bar{z}w)}\right]^{2/k} \bar{z}^{m-1} w^{n-1}\, dz\, d\bar{z}\, dw\, d\bar{w} =$$

p.27
$$= \left(\frac{i}{2\pi}\right)^2 \left(\frac{2}{k}\right) \iint\limits_{D^2 D^2} \frac{d}{dz}\left\{\frac{z-w}{(1-w\bar{w})(1-z\bar{w})}\left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-\bar{z}w)}\right]^{2/k}\right\} \bar{z}^{m-1} w^{n-1}\, dz\, d\bar{z}\, dw\, d\bar{w}$$

$$= \boxed{\left(\frac{i}{2\pi}\right)^2 \left(\frac{2}{k}\right) \iint\limits_{D^2 D^2} \frac{1}{(1-z\bar{w})^2}\left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-\bar{z}w)}\right]^{2/k} \bar{z}^{m-1} w^{n-1}\, dz\, d\bar{z}\, dw\, d\bar{w}}$$

denote I

$$- \boxed{\left(\frac{i}{2\pi}\right)^2 \left(\frac{2}{k}\right)^2 \iint\limits_{D^2 D^2} \frac{(z-w)(\bar{z}-\bar{w})}{(1-z\bar{z})(1-w\bar{w})}\,\frac{1}{(1-z\bar{w})^2}\left[\frac{(1-z\bar{z})(1-w\bar{w})}{(1-z\bar{w})(1-\bar{z}w)}\right]^{2/k} \bar{z}^{m-1} w^{n-1}\, dz\, d\bar{z}\, dw\, d\bar{w}}$$

denote II

Since $\dfrac{d}{dz}\dfrac{z-w}{(1-w\bar{w})(1-z\bar{w})} = \dfrac{1}{1-w\bar{w}}\left(\dfrac{1}{1-z\bar{w}} + \dfrac{(z-w)\bar{w}}{(1-z\bar{w})^2}\right) =$

$$= \frac{1}{1-w\bar{w}}\left(\frac{1-\cancel{z\bar{w}}+\cancel{z\bar{w}}-w\bar{w}}{(1-z\bar{w})^2}\right) = \frac{1}{(1-z\bar{w})^2}$$

Thus $(\mathrm{vac}, \alpha^-_{-m}\alpha^+_n \,\mathrm{vac}) = \mathrm{I} + \mathrm{II}$, and we will compute two integrals separately.

We denote (cf page 31)

$$(1-z\overline{w})^{-(\frac{2}{k}+1)} = \sum_{n\geq 0} \frac{(\frac{2}{k}+1)\cdots(\frac{2}{k}+n)}{n!}(z\overline{w})^n = \sum_{n\geq 0} c_n (z\overline{w})^n$$

$$(1-z\overline{w})^{-(\frac{2}{k}+2)} = \sum_{n\geq 0} \frac{(\frac{2}{k}+2)\cdots(\frac{2}{k}+n+1)}{n!}(z\overline{w})^n = \sum_{n\geq 0} \hat{c}_n (z\overline{w})^n$$

$$(1-z\overline{w})^{-\frac{2}{k}} = \sum_{n\geq 0} \frac{(\frac{2}{k})\cdots(\frac{2}{k}+n-1)}{n!}(z\overline{w})^n = \sum_{n\geq 0} \check{c}_n (z\overline{w})^n$$

Then we obtain

$$I = \left(\frac{i}{2\pi}\right)^2\left(\frac{2}{k}\right)\sum_{p,q\geq 0}\iint_{D^2 D^2} \hat{C}_p z^p \bar{w}^p \check{C}_q \bar{z}^q w^q \left[(1-z\bar{z})(1-w\bar{w})\right]^{2/k} \bar{z}^{m-1} w^{n-1}\, dz\,d\bar{z}\,dw\,d\bar{w}$$

$$= \left(\frac{2}{k}\right)\sum_{p,q\geq 0} \hat{C}_p \check{C}_q \times$$

$$\int_{D^2} z^p \bar{z}^{q+m-1}(1-z\bar{z})^{2/k}\left(\frac{i}{2\pi}\right)dz\,d\bar{z} \times$$

$$\int_{D^2} w^{q+n-1}\bar{w}^p (1-w\bar{w})^{2/k}\left(\frac{i}{2\pi}\right)dw\,d\bar{w}$$

$$= \left(\frac{2}{k}\right)\sum_{p,q\geq 0} \hat{C}_p \check{C}_q\, \delta_{p,q+m-1}\,\delta_{p,q+n-1} \times$$

$$\int_0^1 r^{2p}(1-r^2)^{2/k}\,2r\,dr \times \int_0^1 r^{2p}(1-r^2)^{2/k}\,2r\,dr$$

$$= \left(\frac{2}{k}\right) \sum_{p,q \geq 0} \hat{C}_p \check{C}_q \, \delta_{m,n} \, \delta_{p,q+n-1} \times \left(\int_0^1 r^p (1-r)^{2/k} dr \right)^2 =$$

$$= \left(\frac{2}{k}\right) \sum_{p,q \geq 0} \hat{C}_p \check{C}_q \, \delta_{m,n} \, \delta_{p,q+n-1} \left(\hat{C}_p\right)^{-2} \left(\int_0^1 (1-r)^{2/k} dr \right)^2 =$$

$$= \delta_{m,n} \left(\frac{2}{k}\right) \left(\frac{2}{k}+1\right)^{-2} \sum_{p,q \geq 0} \delta_{p,q+n-1} \, \check{C}_q \, \hat{C}_p^{-1} =$$

$$= \delta_{m,n} \left(\frac{2}{k}\right) \left(\frac{2}{k}+1\right)^{-2} \sum_{q \geq 0} \frac{\left(\frac{2}{k}\right) \cdots \left(\frac{2}{k}+q-1\right)}{q!} \cdot \frac{(q+n-1)!}{\left(\frac{2}{k}+2\right) \cdots \left(\frac{2}{k}+q+n\right)}$$

$$= \delta_{m,n} \left(\frac{2}{k}\right) \left(\frac{2}{k}+1\right)^{-2} \sum_{q \geq 0} \frac{(q+1) \cdots (q+n-1)}{\left(\frac{2}{k}+q\right) \cdots \left(\frac{2}{k}+q+n\right)} \left(\frac{2}{k}\right)\left(\frac{2}{k}+1\right) =$$

$$= \boxed{\delta_{m,n} \left(\frac{2}{k}\right)^2 \left(\frac{2}{k}+1\right)^{-1} \sum_{q \geq 0} \frac{(q+1) \cdots (q+n-1)}{\left(\frac{2}{k}+q\right) \cdots \left(\frac{2}{k}+q+n\right)}}$$

Similarly we compute

$$II = -\left(\frac{i}{2\pi}\right)^2\left(\frac{2}{k}\right)^2 \sum_{p,q\geq 0} \iint_{D^2 D^2} \left(z\bar z - z\bar w - \bar z w + w\bar w\right) \times$$

$$\hat c_p z^p \bar w^p \check c_q \bar z^q w^q \left[(1-z\bar z)(1-w\bar w)\right]^{2/k-1} \bar z^{m-1} w^{n-1} dz\,d\bar z\,dw\,d\bar w$$

$$= -\left(\frac{2}{k}\right)^2 \sum_{p,q\geq 0} \hat c_p \check c_q \times$$

$$\Big\{ \delta_{p+1,q+m}\,\delta_{p,q+n-1} \left(\int_0^1 r^{2p+2}(1-r^2)^{2/k-1}\,2r\,dr\right)\left(\int_0^1 r^{2p}(1-r^2)^{2/k-1}\,2r\,dr\right)$$

$$-\,\delta_{p+1,q+m-1}\,\delta_{p+1,q+n-1}\left(\int_0^1 r^{2p+2}(1-r^2)^{2/k-1}\,2r\,dr\right)\left(\int_0^1 r^{2p+2}(1-r^2)^{2/k-1}\,2r\,dr\right)$$

$$-\,\delta_{p,q+m}\,\delta_{p,q+n}\left(\int_0^1 r^{2p}(1-r^2)^{2/k-1}\,2r\,dr\right)\left(\int_0^1 r^{2p}(1-r^2)^{2/k-1}\,2r\,dr\right)$$

$$+\,\delta_{p,q+m-1}\,\delta_{p+1,q+n}\left(\int_0^1 r^{2p}(1-r^2)^{2/k-1}\,2r\,dr\right)\left(\int_0^1 r^{2p+2}(1-r^2)^{2/k-1}\,2r\,dr\right)\Big\}$$

Note that in all case m must be equal to n

We note that

$$\int_0^1 r^{2p}(1-r^2)^{2/k-1}\, 2r\,dr = \int_0^1 r^p(1-r)^{2/k-1}\,dr =$$

$$= c_p^{-1}\int_0^1 (1-r)^{2/k-1}\,dr = \left(\tfrac{2}{k}\right)^{-1} c_p^{-1}$$

Thus we obtain

$$\mathrm{II} = -\delta_{m,n}\sum_{p,q\geq 0}\hat{c}_p\,\check{c}_q\ \times$$

$$\Big\{\delta_{p,q+n-1}\,c_{p+1}^{-1}c_p^{-1} - \delta_{p,q+n-2}\,c_{p+1}^{-1}c_{p+1}^{-1}$$

$$-\delta_{p,q+n}\,c_p^{-1}c_p^{-1} + \delta_{p,q+n-1}\,c_p^{-1}c_{p+1}^{-1}\Big\}$$

Let us compute all four summands separately. We obtain

Second $\quad p = q+n-2$

$$\hat{c}_p\,\check{c}_q\,c_{p+1}^{-1}\,c_{p+1}^{-1} = \check{c}_q\,\cancel{\frac{\left(\frac{2}{k}+2\right)\cdots\left(\frac{2}{k}+p+1\right)}{p!}}\;\overset{p+1}{\frac{\cancel{(p+1)!}}{\left(\frac{2}{k}+1\right)\cancel{\cdots\left(\frac{2}{k}+p+1\right)}}}\;\frac{(p+1)!}{\left(\frac{2}{k}+1\right)\cdots\left(\frac{2}{k}+p+1\right)}$$

$$= \frac{\left(\frac{2}{k}\right)\cdots\left(\frac{2}{k}+q-1\right)}{q!}\cdot\frac{(p+1)!}{\left(\frac{2}{k}+1\right)\cdots\left(\frac{2}{k}+p+1\right)}\cdot\frac{(p+1)}{\left(\frac{2}{k}+1\right)} =$$

$$= \boxed{\frac{(q+1)\cdots(q+n-1)}{\left(\frac{2}{k}+q\right)\cdots\left(\frac{2}{k}+q+n-1\right)}\cdot\frac{\left(\frac{2}{k}\right)}{\left(\frac{2}{k}+1\right)}\,(q+n-1)}$$

Combining four terms we get $\delta_{m,n}\times$

$$-2\,\frac{(q+1)\cdots(q+n)}{\left(\frac{2}{k}+q\right)\cdots\left(\frac{2}{k}+q+n-1\right)}\;\frac{\left(\frac{2}{k}\right)}{\left(\frac{2}{k}+1\right)} +$$

$$+\frac{(q+1)\cdots(q+n)}{\left(\frac{2}{k}+q\right)\cdots\left(\frac{2}{k}+q+n\right)}\left(\frac{2}{k}+q+n+1\right)\cdot\frac{\left(\frac{2}{k}\right)}{\left(\frac{2}{k}+1\right)} +$$

$$+\frac{(q+1)\cdots(q+n-1)}{\left(\frac{2}{k}+q\right)\cdots\left(\frac{2}{k}+q+n-1\right)}\cdot(q+n-1)\,\frac{\left(\frac{2}{k}\right)}{\left(\frac{2}{k}+1\right)}$$

$$= \delta_{m,n} \times \frac{(q+1)\cdots(q+n-1)}{\left(\frac{2}{k}+q\right)\cdots\left(\frac{2}{k}+q+n\right)} \frac{\left(\frac{2}{k}\right)}{\left(\frac{2}{k}+1\right)} \times$$

$$\times \left\{ -2\left(\frac{2}{k}+q+n\right)(q+n) + (q+n)\left(\frac{2}{k}+q+n+1\right) + \left(\frac{2}{k}+q+n\right)(q+n-1) \right\}$$

$$= -\delta_{m,n} \left(\frac{2}{k}\right)^2 \left(\frac{2}{k}+1\right)^{-1} \cdot \frac{(q+1)\cdots(q+n-1)}{\left(\frac{2}{k}+q\right)\cdots\left(\frac{2}{k}+q+n\right)}$$

Thus we obtain

$$\boxed{\underline{II} = -\delta_{m,n} \left(\frac{2}{k}\right)^2 \left(\frac{2}{k}+1\right)^{-1} \sum_{q \geq 0} \frac{(q+1)\cdots(q+n-1)}{\left(\frac{2}{k}+q\right)\cdots\left(\frac{2}{k}+q+n\right)}}$$

which is precisely the negative of I computed on page 50 above!!

Computation of 2-point correlation functions II

Thus the previous calculations indicate that the 2-point correlation functions is a rather subtle object. There is still a very good chance that they are finite, since we obtained a difference of two identical but divergent series. Thus we might need to compute the improper integrals as on page 37 $\lim_{\varepsilon\to 0}\int_{D^2_{1-\varepsilon}}\int_{D^2_{1-\varepsilon}} \ldots d\mu(z\bar{z})\,d\mu(w\bar{w})$

It should not be very difficult to redo our calculations for finite ε and then take $\lim_{\varepsilon\to 0}$.

8. Virasoro algebra and irreducibility

We will start with a note that in this theory it will be more convinient to consider the "centered" normal ordering, the "centered" generating functions, etc. Namely we set

$$h^i(z) = \sum_{n\in\mathbb{Z}} h^i_n z^{-n}, \quad X^\pm(z) = \sum_{n\in\mathbb{Z}} X^\pm_n z^{-n}$$

$$:e^{\pm\frac{2}{k}\varphi(z)} h^0(z): = \frac{1}{2}\Big(e^{\pm\frac{2}{k}\varphi(z)} h^0_0 + h^0_0 e^{\pm\frac{2}{k}\varphi(z)}\Big)$$

$$+ e^{\pm\frac{2}{k}\varphi(z)}\Big(\sum_{n>0} h^0_n z^{-n}\Big) + \Big(\sum_{n<0} h^0_n z^{-n}\Big) e^{\pm\frac{2}{k}\varphi(z)}$$

Then the terms $\pm\frac{z^{-1}}{z}$ will be absorbed in the new normal ordering:

$$X^+(z) = :e^{-\frac{2}{k}\varphi(z)}(h^0(z) - ih^2(z)):$$

$$-X^-(z) = :e^{\frac{2}{k}\varphi(z)}(h^0(z) + ih^2(z)):$$

Note that the centered normal ordering is especially convinient for checking the unitarity. Also later on we will consider gener. funct.

$$X^\pm(z,\bar z) = \sum_{n>0} X^\pm_{-n} z^n + X^\pm_0 + \sum_{n>0} X^\pm_n \bar z^n$$

which coincide with $X^\pm(z)$ on the circle $|z|=1$.

We will start with the "standard" Virasoro algebra

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n} = \frac{1}{k} :h^0(z)^2: - \frac{1}{k} :h^1(z)^2: + \frac{1}{k'} :h^2(z)^2:$$

Then we know

$$[L_m\, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\cdot 3$$

$$[L_m, h^0_n] = -n\, h^0_{m+n}$$

$$[L_m, h^1_n] = -n\, h^1_{m+n}$$

$$[L_m, h^2_n] = -n\, h^2_{m+n}$$

The central charge 3 comes from 3 "bosons".

Then since $\alpha^{\pm}(z)$ have spin 0 we get the relation, which can also be verified directly

$$[L_m, \alpha^{\pm}_n] = (-m-n)\alpha^{\pm}_{m+n}$$

Next we compute

$$[L_m, X^+_n] = -nX^+_{m+n} - \frac{m^2}{2}\alpha^-_{m+n}$$

$$[L_m, X^-_n] = -nX^-_{m+n} + \frac{m^2}{2}\alpha^+_{m+n}$$

Thus $\{L_m\}_{m\in\mathbb{Z}}$ is "almost" the Virasoro algebra for $\widehat{SL}(2,\mathbb{R})$, but we need a correction term.

The verifications of the above commut. relat.

$$[L_m, X^+_n] = [L_m, \sum_r :\alpha^-_{n-r} h^0_r: - i\alpha^-_{n-r} h^2_r] =$$

$$= \sum_r (-m-n+r) :\alpha^-_{m+n-r} h^0_r: + (-r) :\alpha^-_{n-r} h^0_{m+r}:$$

$$-i \sum_r (-m-n+r) \alpha^-_{m+n-r} h^2_r + (-r) \alpha^-_{n-r} h^2_{m+r}$$

+ correction term from normal ordering in O

$$m > 0 \qquad \sum_{r=-1}^{-m'} (-r h^0_{m+r} \alpha^-_{n-r}) =$$

$$= \sum_{r=-1}^{-m'} (-r \alpha^-_{n-r} h^0_{m+r}) + \sum_{r=-1}^{-m'} r \alpha^-_{m+n}$$

$$m > 0 \qquad \sum_{r=1}^{m'} (-r \alpha^-_{n-r} h^0_{m+r}) =$$

$$= \sum_{r=1}^{m'} (-r) h^0_{m+r} \alpha^-_{n-r} - \sum_{r=1}^{m'} r \alpha^-_{m+n}$$

Here m' denotes that the last term is taken with the coefficient ½ according to the normal ordering $: \; :$ which gives the contribution as stated.

$$1+2+\dots+(m-1)+\frac{m}{2} = \frac{m(m-1)}{2}+\frac{m}{2} = \frac{m^2}{2}$$

The commutation relation $[L_m, X_n^-]$ is similar.

Now we define the corrected Virasoro

$$\mathcal{L}_n = L_n + \frac{i}{k'} n h_n^2 + \delta_{0,n}\frac{1}{4k'}$$

Then we obtain the relations

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

$$c = \frac{3k}{k-2}$$ as in physics literature

The verification is straightforward

$$[\mathcal{L}_m, \mathcal{L}_n] = [L_m + \frac{i}{k'} m h_m^2, L_n + \frac{i}{k'} n h_n^2] =$$

$$= (m-n) L_{m+n} + \frac{3}{12}(m^3 - m)\delta_{m+n,0}$$

$$- \frac{i}{k'} n^2 h^2_{m+n} + \frac{i}{k'} m^2 h^2_{m+n} + \frac{i^2}{(k')^2} m n m \delta_{m+n,0} \frac{k'}{2}$$

$$= (m-n)\left(L_{m+n} + \frac{i}{k'}(m+n) h^2_{m+n}\right) +$$

$$\delta_{m+n,0}\left(\frac{1}{4}(m^3 - m) + \frac{1}{2k'}(m^3 - m) + \frac{1}{2k'} m\right)$$

$$\frac{1}{2k'} m = \frac{1}{4k'}(m-n)$$

$$\frac{1}{4} + \frac{1}{2k'} = \frac{k'+2}{4k'} = \frac{k}{4(k-2)}$$

The corrected Virasoro satisfies

$$[\mathcal{L}_m, X^+_n] = -n X^+_{m+n}$$

$$[\mathcal{L}_m, X^-_n] = -n X^-_{m+n}$$

Since

$$\left[\frac{i}{k'} m h^2_m, -i\alpha^-_{n-r} h^2_r\right] =$$

$$= \frac{1}{k'} m\, m \frac{k'}{2} \alpha^-_{n+m} = \frac{m^2}{2} \alpha^-_{m+n}$$

in the first commutator and

similarly in the second commutator

Finally note that $\mathcal{L}^*_n = \mathcal{L}_{-n}$,

since $L^*_n = L_{-n}$, $\left(\frac{i}{k'} n h^2_n\right)^* = -\frac{i}{k'} n h^2_{-n}$

Applying $\exp(i\theta L_0')$ to $X^{\pm}_n$ will yield

$$X^{\pm}_n(\theta) = \sum_{r\in\mathbb{Z}} e^{ir\theta} : \alpha^{\mp}_r (h^0_{n-r} \mp i h^2_{n-r}):$$

Taking the Fourier components we get operators

$$X^+_{m,n} = :\alpha^-_m (h^0_n - i h^2_n):$$

$$X^-_{m,n} = :\alpha^+_m (h^0_n + i h^2_n):$$

which still preserve the subrepresentation $\tilde{F} \subset F$. Consider the commutators

$$[X^+_{m,n}, X^-_{p,q}] =$$

$$= [:\alpha^-_m h^0_n:, :\alpha^+_p h^0_q:] + [\alpha^-_m h^2_n, \alpha^+_p h^2_q]$$

$$+ i[:\alpha^-_m h^0_n:, \alpha^+_p h^2_q] - i[\alpha^-_m h^2_n, :\alpha^+_p h^0_q:]$$

$$= :\alpha^-_m \alpha^+_{n+p} h^0_q: + :\alpha^+_p \alpha^-_{m+q} h^0_n: + \alpha^+_p \alpha^-_m n \tfrac{k}{2} \delta_{n+q,0}$$

$$+ \alpha^-_m \alpha^+_p n \tfrac{k'}{2} \delta_{n+q,0} + i\alpha^-_m \alpha^+_{n+p} h^2_q - i\alpha^-_{m+q} \alpha^+_p h^2_n$$

Therefore the sum over $r \in \mathbb{Z}$ yields

$$\sum_{r \in \mathbb{Z}} [X^+_{m+r,n}, X^-_{p+r,q}] =$$

$$= \delta_{m+n+p,0} h^0_q + \delta_{p+m+q,0} h^0_n + n \tfrac{k'}{2} \delta_{n+q,0} \delta_{p+m,0}$$

$$+ n \tfrac{k}{2} \delta_{n+q,0} \delta_{p+m,0} + i\delta_{m+n+p,0} h^2_q - i\delta_{m+p+q,0} h^2_n$$

Set $m+n+p=0$ we get operator

$$h_q^0 + i h_q^2$$

set $m+p+q=0$ we get operator

$$h_n^0 - i h_n^2$$

Thus the algebra spanned by h_n^1, h_n^0, h_n^2 acts on $\tilde{F} \subset F$; we also obtain shifts $e^{\pm \frac{2}{k} h}$ from $X^{\pm}(z)$.

All these operators generate F and we obtained a contradiction with the assumption that $\tilde{F} \neq F$.