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Homotopy BV algebras, Courant algebroids and String Field Theory

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AMS Sectional Meeting

Richmond, VA

November 6-7, 2010

- **Introduction/Motivations**

- **Vertex/Courant algebroids with Calabi-Yau structure and associated homotopy BV algebras of Lian-Zuckerman**

- **Application: Einstein equations from the tensor product $BV \otimes \overline{BV}$**

- **A_∞-algebras. $A_{(3)}$-algebra of Courant algebroid**

- **Yang-Mills equations from $A_{(3)}$-algebra and their relation to Courant algebroid**

- **Open questions**

Motivations from PHYSICS

String Theory in background fields (perturbed 2D CFT)

β-function: $\beta(\phi_{\{v\}}, h) = \sum_{i\in\mathbb{N}} h^i \beta_i(\phi_{\{v\}})$

Condition of conformal invariance:

$$\beta(\phi_{\{v\}}, h) = 0$$

$\beta_1(\phi_{\{v\}}) = 0 \quad \Longleftrightarrow \quad$ **Classical Field Equations (Yang-Mills, Einstein)**

String Field Theory suggests an algebraic meaning of β-function

$\beta(\phi_{\{v\}}, h) = 0 \Longleftrightarrow Q\Phi + \mu_2(\Phi,\Phi) + \mu_3(\Phi,\Phi,\Phi) + \cdots = 0,$

$\Phi = \Phi(\phi_{\{v\}}, h)$.

μ_i **generate** A_∞**- or** L_∞**- algebras: homotopy generalizations of DGA and DGLA.**

Motivations from MATHEMATICS

- **Relations between vertex algebras and homotopy algebras.**

vertex algebroids $\longleftrightarrow$ **vertex algebras**
$(V = \oplus_{n \geqslant 0} V_n)$

vertex algebroids with CY structure $\longrightarrow$ **homotopy BV algebras**

vertex algebras **?** **homotopy BV algebras**

- **Geometry:**

Vertex algebroids $\longrightarrow$ **Courant algebroids**

1) Homotopy algebras in generalized complex geometry

2) Relations between classical field equations and Courant algebroids

Vertex algebroids

Def. (*P. Bressler*) **Vertex** A**-algebroid when** A **is a commutative** k**-algebra is an** A**-module with pairing:** $A \otimes V \to V, f \otimes v \mapsto f * v$**, such that** $1 * v = v$

i) **Leibniz** k**-algebra:** $[,] : V \otimes_k V \to V$

ii) **a** k**-linear map of Leibniz algebras:** $\pi : V \to \mathrm{Der} A$**(the anchor)**

iii) **a symmetric** k**-bilinear pairing** $<,>: V \otimes_k V \to A$

iv) **a** k**-linear map** $\partial : A \to V$ **s.t.** $\pi \circ \partial = 0$

wich satisfy

$$\begin{aligned}
f * (g * v) - (fg) * v &= \pi(v)(f) * \partial(g) + \pi(v)(g) * \partial f \\
[v_1, f * v_2] &= \pi(v_1)(f) * v_2 + f * [v_1, v_2] \\
[v_1, v_2] + [v_2, v_1] &= \partial(< v_1, v_2 >) \\
\pi(f * v) &= f\pi(v) \\
< f * v_1, v_2 > &= f < v_1, v_2 > -\pi(v_1)\big(\pi(v_2)(f)\big) \\
\pi(v)(< v_1, v_2 >) &= < [v, v_1], v_2 > + < v_1, [v, v_2] > \\
\partial(fg) &= f * \partial(g) + g * \partial(f) \\
[v, \partial(f)] &= \partial\big(\pi(v)(f)\big) \\
< v, \partial(f) > &= \pi(v)(f)
\end{aligned}$$

We will le also interested in $\mathcal{O}_X$**-vertex algebroid when** A **is replaced by** $\mathcal{O}_X$ **and** V **is replaced by a sheaf of** k**-vector spaces.**

Vertex algebroid $\longrightarrow$ Vertex algebra

$$f_{(-1)}g = fg, \quad f_{(-1)}v = f * v,$$
$$v_{(-1)}f = f * v - \partial\pi(v)(f),$$
$$v_{(0)}f = -f_{(0)}v = \pi(v)(f),$$
$$v_{(0)}w = [v, w], \quad v_{(1)}w =< v, w >$$

This gives a "truncated" vertex algebras. Then one can construct a vertex algebras

$$\mathrm{Vert} = \oplus_{n\geqslant 0} V_n, \quad \text{s.t.} \quad V_0 = A, \quad V_1 = V$$

The inverse statement is also true: for a given vertex algebra, s.t. $V_0 = A$, $V_1 = V$, one can construct a vertex algebroid.

V. Gorbunov, F. Malikov, V. Schechtman,
Invent. Math. 155 (2004), 605-680

Calabi-Yau structure

Def. Operator $\widehat{\mathrm{div}} : V \to A$ is called a Calabi-Yau structure on V if $\widehat{\mathrm{div}} \circ \partial = 0$

$$\widehat{\mathrm{div}}(f * v) \ = \ f\widehat{\mathrm{div}}v + < \partial f, v >$$
$$\widehat{\mathrm{div}}[v_1, v_2] \ = \ \pi(v_1)(\widehat{\mathrm{div}}v_2) - \pi(v_2)(\widehat{\mathrm{div}}v_1)$$

In the case of $\mathcal{O}_X$-vertex algebroids, on a manifold X with a volume form Φ, one can construct $\widehat{\mathrm{div}} \equiv \mathrm{div}_\Phi \circ \pi$, where div_Φ is a divergence operator for a vector field w.r.t. the volume form Φ.

Example: Let Vert be VOA, then L_1 gives a Calabi-Yau structure.

Courant algebroids

Def. (*Z.-J.Liu, A. Weinstein, P. Xu; P. Bressler*)
Courant A-algebroid is an A-module Q equipped with

i) **a structure of Leibniz k-algebra**

$$[,] : Q \otimes_k Q \to Q$$

ii) **anchor map** $\pi : Q \to \mathrm{Der}A$

iii) **pairing** $<,>: Q \otimes_A Q \to A$

iv) **derivation** $\partial : A \to Q$

They satisfy:

$$\begin{aligned}
\pi \circ \partial &= 0 \\
[q_1, f q_2] &= f[q_1, q_2] + \pi(q_1)(f) q_2 \\
< [q, q_1], q_2 > + < q_1, [q, q_2] > &= \pi(q) < q_1, q_2 > \\
[q, \partial(f)] &= \partial\big(\pi(q)(f)\big) \\
< q, \partial(f) > &= \pi(q)(f) \\
[q_1, q_2] + [q_2, q_1] &= \partial(< q_1, q_2 >)
\end{aligned}$$

Courant algebroids $\longleftrightarrow$ Poisson vertex algebras

Classical limit

Suppose $V = Q[h]$, such that $Q = V/hV$ is commutative, i.e. $*$ is commutative on Q, Leibniz bracket, pairing, anchor, and $\widehat{\mathrm{div}}$ take values in hV and $\partial Q \subset Q$. Then one can define the Courant A-algebroid structure on Q as follows:

$$\pi_Q(\cdot) = \lim_{h\to 0}\frac{1}{h}\pi_V(\cdot), \quad [\cdot,\cdot]_Q = \lim_{h\to 0}\frac{1}{h}[\cdot,\cdot]_V,$$
$$\langle\cdot,\cdot\rangle_Q = \lim_{h\to 0}\frac{1}{h}\langle\cdot,\cdot\rangle_V, \quad \widehat{\mathrm{div}}_Q = \lim_{h\to 0}\frac{1}{h}\widehat{\mathrm{div}}_V$$

Later we will interpret this classical limit on the level of the homotopy BV algebras.

Example. **Vertex algebra, generated by the quantum fields $p_i(z)$ and $X^i(z)$ $(i = 1, ..., D)$ of conformal weights 1 and 0 correspondingly, and their OPE:**

$$p_i(z)X^k(w) \sim \frac{h\delta_i^k}{z-w}$$

Then, V_0 is given by the states are generated by $F(X(z))$, while V_1 is spanned by $\sum_i :v^i(X)p_i:(z)$ (vector fields), $\sum_k w_k(X)\partial X^k(z)$ (1-forms). In fact, we have a VOA, where $L(z) = -\frac{1}{h}\sum_i :p_i\partial X^i:(z) + \partial^2\phi(X)(z)$. The Calabi-Yau structure is given by the action of L_1-operator.

Lian-Zuckerman operations

Let $\mathrm{Vert} = \oplus_{n \geqslant 0} V_n$ **be a VOA with Virasoro element** $L^V(z)$**.**
Consider the corresponding semi-infinite complex $C = \mathrm{Vert} \otimes \Lambda$**, where** Λ **is a VOA generated by** $b(z)c(w) \sim \frac{1}{z-w}$

Theorem(*I. Frenkel, H. Garland, G. Zuckerman*)
The operator $\mathcal{D}$**, such that**

$$\mathcal{D} = \oint (c(z)L(z) + :c(z)\partial c(z)b(z):)\mathrm{d}z$$

is nilpotent on $\mathrm{Vert} \otimes \Lambda$ **iff the central charge of the Virasoro algebra** L_n^V **is equal to 26.**

Proposition **The operator** $\mathcal{D}$ **is nilpotent on the space** $C_{L_0} = \ker L_0$**, where** $L_0 = L_0^V + L_0^\Lambda$ **for all values of central charge.**

C_{L_0} **is spanned by:**

$$u(z), \quad c(z)A(z), \quad \partial c(z)a(z), \quad c(z)\partial c(z)\tilde{A}(z),$$
$$c(z)\partial^2 c(z)\tilde{a}(z), \quad c(z)\partial c(z)\partial^2 c(z)\tilde{u}(z),$$

where $u, \tilde{u}, a, \tilde{a} \in V_0; \quad A, \tilde{A} \in V_1$

Lian and Zuckerman introduced operations:

$$\mu(a_1,a_2)=\mathrm{Res}\frac{a_1(z)a_2}{z},\qquad \{a_1,a_2\}=\mathrm{Res}_z(b_{-1}a_1)(z)a_2$$

B.H. Lian, G.J. Zuckerman
Commun.Math.Phys.154 (1993) 613

These operations satisfy the relations of the homotopy BV algebra when central charge of $\{L_n^V\}$ is equal to 26.

__**Theorem**__ (*B.H. Lian-G.J. Zuckerman*)
i) *The operation μ is homotopy commutative and homotopy associative:*

$$\mathcal{D}\mu(a_1,a_2)=\mu(\mathcal{D}a_1,a_2)+(-1)^{|a_1|}\mu(a_1,\mathcal{D}a_2),$$

$$\begin{aligned}&\mu(a_1,a_2)-(-1)^{|a_1||a_2|}\mu(a_2,a_1)=\\&=\mathcal{D}m(a_1,a_2)+m(\mathcal{D}a_1,a_2)+(-1)^{|a_1|}m(a_1,\mathcal{D}a_2),\end{aligned}$$

$$\begin{aligned}&\mathcal{D}n(a_1,a_2,a_3)+n(\mathcal{D}a_1,a_2,a_3)+(-1)^{|a_1|}n(a_1,\mathcal{D}a_2,a_3)+\\&+(-1)^{|a_1|+|a_2|}n(a_1,a_2,\mathcal{D}a_3)=\\&=\mu(\mu(a_1,a_2),a_3)-\mu(a_1,\mu(a_2,a_3))\end{aligned}$$

ii) *The operations* μ *and* $\{\cdot,\cdot\}$ *are related in the following way:*

$$\{a_1, a_2\} = b_0\mu(a_1, a_2) - \mu(b_0a_1, a_2) - (-1)^{|a_1|}\mu(a_1, b_0a_2).$$

iii) *The operations* μ *and* $\{\cdot,\cdot\}$ *satisfy the relations:*

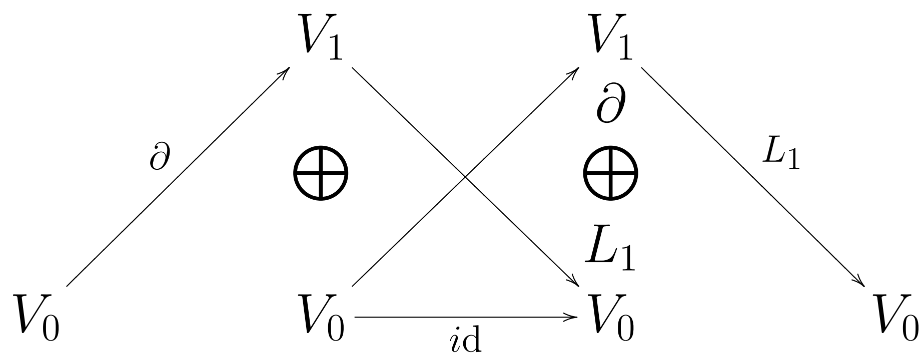
$$\begin{aligned}
&\{a_1, a_2\} + (-1)^{(|a_1|-1)(|a_2|-1)}\{a_2, a_1\} = \\
&(-1)^{|a_1|-1}(\mathcal{D}m'(a_1, a_2) - m'(\mathcal{D}a_1, a_2) - \\
&(-1)^{|a_2|}m'(a_1, \mathcal{D}a_2)),
\end{aligned}$$

$$\begin{aligned}
&\{a_1, \mu(a_2, a_3)\} = \\
&\mu(\{a_1, a_2\}, a_3) + (-1)^{(|a_1|-1)||a_2|}\mu(a_2, \{a_1, a_3\}),
\end{aligned}$$

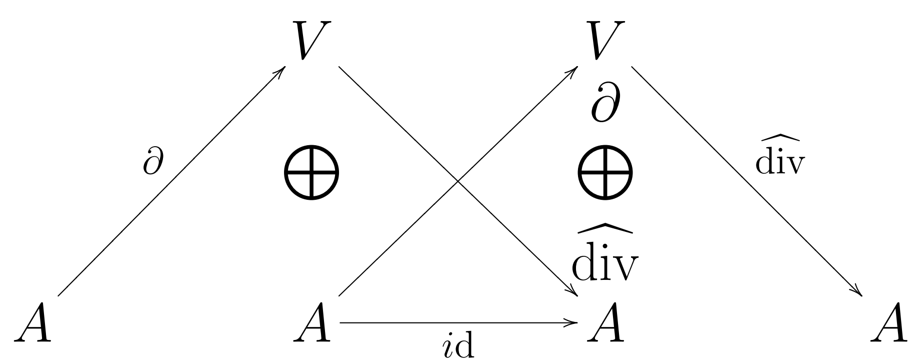
$$\begin{aligned}
&\{\mu(a_1, a_2), a_3\} - \mu(a_1, \{a_2, a_3\}) - \\
&(-1)^{(|a_3|-1)|a_2|}\mu(\{a_1, a_3\}, a_2) = \\
&(-1)^{|a_1|+|a_2|-1}(\mathcal{D}n'(a_1, a_2, a_3) - n'(\mathcal{D}a_1, a_2, a_3) - \\
&(-1)^{|a_1|}n'(a_1, \mathcal{D}a_2, a_3) - (-1)^{|a_1|+|a_2|}n'(a_1, a_2, \mathcal{D}a_3),
\end{aligned}$$

$$\begin{aligned}
&\{\{a_1, a_2\}, a_3\} - \{a_1, \{a_2, a_3\}\} + \\
&(-1)^{(|a_1|-1)(|a_2|-1)}\{a_2, \{a_1, a_3\}\} = 0.
\end{aligned}$$

<u>Proposition</u> **There is a structure of the homotopy BV algebra on C_{L_0}, such that the differential acts follows:**



<u>Theorem</u> **Let V be a vertex algebroid with a Calabi-Yau structure, then there exists a structure of a homotopy BV algebra on the complex $\mathcal{F}_V$:**



<u>Corollary</u> **The homotopy Lie algebra on the $\mathcal{F}_V^1$ subspace coincides with the homotopy Lie algebra of Roytenberg-Weinstein in the case of Courant algebroid.**

D. Roytenberg, A. Weinstein,
Lett. Math. Phys. 46, 81-93

In the following we will call this homotopy algebra $LZ(V)$ or $LZ(\mathrm{Vert})$.

Let $V = Q[h]$, and all the conditions of the classical limit are satisfied (see above). Then Q has a structure of a Courant algebroid.

Question:

Is it possible to describe classical limit

$$LZ(V) \to LZ(Q)$$

in terms of LZ operations?

One can find a subcomplex $(\mathcal{F}_{h,Q}, \mathcal{D}_V)$ in $(\mathcal{F}_V, \mathcal{D}_V)$, which is isomorphic to $(\mathcal{F}_Q, \mathcal{D}_Q)$ on the level of k-vector spaces, such that on $(\mathcal{F}_{h,Q}, \mathcal{D}_V)$ we have:

$$\mathcal{D}_Q = \lim_{h\to 0} \mathcal{D}_V, \quad \mu_Q(\cdot,\cdot) = \lim_{h\to 0} \mu_V(\cdot,\cdot),$$
$$\{\cdot,\cdot\}_Q = \lim_{h\to 0} \frac{1}{h}\{\cdot,\cdot\}_V$$

A.M.Z.
Quasiclassical Lian-Zuckerman Homotopy Algebras, Courant Algebroid and Gauge Theory, Comm.Math.Phys., in press (arXiv:0910.3652)

Equations of Field Theory and LZ algebra

From now on we assume that all Courant algebroids we consider are Courant $\mathcal{O}_X$-algebroids.

Closed strings $\longrightarrow \mathrm{Vert} \otimes \overline{\mathrm{Vert}}$

Claim 1

"Einstein equations"
(β_1-function for closed strings) are equivalent to generalized Maurer-Cartan equations for the homotopy Lie algebra of

$$LZ(Q_{\mathrm{Vert}}) \otimes LZ(\overline{Q_{\mathrm{Vert}}})$$

Open strings $\longrightarrow (\mathrm{Vert}) \otimes \mathfrak{U}(\mathfrak{g})$

Claim 2

"Yang-Mills equations"
(β_1-function for open strings) are equivalent to generalized Maurer-Cartan equations for the homotopy associative algebra of

$$LZ(Q_{\mathrm{Vert}}) \otimes \mathfrak{U}(\mathfrak{g})$$

Split Courant algebroids and Einstein equations

Let Q be a Courant algebroid with Calabi-Yau structure and $Q = T \oplus \Omega$, such that T is a Lie algebroid w.r.t. $[,]$, $\mathrm{Im}\partial \in \Omega$.
We refer to such Courant algebroid as split.

Proposition **The homotopy BV algebra $LZ(Q)$ has a BV subalgebra on the subcomplex:**

$$\begin{tikzcd} & T \arrow[ddr, "\widehat{\mathrm{div}}"] & T \arrow[ddr, "\widehat{\mathrm{div}}"] & \\ & \oplus & \oplus & \\ \mathbb{C} \arrow[uur, "0"] & \mathbb{C} \arrow[r, "id"'] & A & A \end{tikzcd}$$

Let T be the sheaf of the holomorphic sections of the tangent bundle. Let us denote the corresponding BV algebra BV^h. Let BV^a denote its antiholomorphic analogue.
Let $BV^E \equiv BV^h \otimes BV^a$.

Consider the Maurer-Cartan equation:

$$\mathcal{D}\Phi + \{\Phi, \Phi\} = 0,$$

where Φ is an element of BV^E of degree 2

$$\Phi \in T \otimes \bar{T} \oplus T \oplus \bar{T} \oplus \mathcal{O}_X \oplus \bar{\mathcal{O}}_X$$

The resulting equations (system of linear and bilinear equations) turn out to be

Einstein Equations with dilaton and a B-field:

$$R_{\mu\nu} = -\frac{1}{4} H_{\mu\lambda\rho} H_{\nu}^{\lambda\rho} + 2\nabla_{\mu}\nabla\nu\Phi,$$
$$\nabla_{\mu} H^{\mu\nu\rho} - 2(\nabla_{\lambda}\Phi) H^{\lambda\nu\rho} = 0,$$
$$4(\nabla_{\mu}\Phi)^2 - 4(\nabla_{\mu}\nabla^{\mu}\Phi) + R + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} = 0,$$

where $\{G_{\mu\nu}\} = \{g_{i\bar{j}}\}$ (hermitian), $\{B_{\mu\nu}\} = \{-g_{i\bar{j}}\}$ (associated 2-form) and $H = dB$. Tensor $\{g^{i\bar{j}}\}$ comes from $T \otimes \bar{T}$.

These equations are very interesting on their own. In particular they imply that the corresponding manifold X is "almost" Calabi-Yau:

$\tilde{\Phi} = \log(\sqrt{g} e^{-\Phi})$ **satisfies the equation $\partial_i \partial_{\bar{k}} \tilde{\Phi} = 0$, i.e. volume form defined by $\tilde{\Phi}$ is pluriharmonic.**

A.M.Z.
Perturbed Beta-Gamma Systems and Complex Geometry,
Nucl. Phys. B 794 (2008) 381-401.

A_∞-algebras

Consider the chain complex $(V, \mathcal{D})$. Consider multilinear operations $\mu_i : V^{\otimes^i} \to V$ of degree $2-i$ such that $\mu_1 = \mathcal{D}$.

Def. The space V is an A_∞-algebra if the operations μ_n satisfy the bilinear identity

$$\sum_{i=1}^{n-1} (-1)^i M_i \circ M_{n-i+1} = 0$$

on $V^{\otimes^n}$, where M_s acts on V^m for any $m \geqslant s$ as follows:

$$M_s = \sum_{\ell=0}^{n-s} (-1)^{\ell(s+1)} \mathbf{1}^{\otimes^\ell} \otimes \mu_s \otimes \mathbf{1}^{\otimes^{m-s-\ell}}$$

Relations between $\mathcal{D}, \mu_2, \mu_3$:

$$\begin{aligned}
&\mathcal{D}^2 = 0,\\
&\mathcal{D}\mu_2(a_1, a_2) = \mu_2(\mathcal{D}a_1, a_2) + (-1)^{|a_2|}\mu_2(a_1, \mathcal{D}a_2),\\
&\mathcal{D}\mu_3(a_1, a_2, a_3) + \mu_3(\mathcal{D}a_1, a_2, a_3) +\\
&(-1)^{|a_2|}\mu_1(a_1, \mathcal{D}a_2, a_3) + (-1)^{|a_2|}\mu_3(a_1, a_2, \mathcal{D}a_3) =\\
&\mu_2\big(\mu_2(a_1, a_2), a_3\big) - \mu_2\big(a_1, \mu_2(a_2, a_3)\big).
\end{aligned}$$

For $\mu_n = 0, n \geqslant 3$ it is just a DGA.

A_∞ **is called** $A_{(k)}$ **algebra if** $\mu_n = 0, n > k$**.**

Generalized Maurer-Cartan equation

Consider $X \in V$ of degree 1.

Then the equation

$$\boxed{\mathcal{D}X + \sum_{n \geqslant 2} \mu_n(X, \ldots, X) = 0}$$

is called
Generalized Maurer-Cartan equation.

Symmetries:

$$X \to X + \varepsilon\left(\mathcal{D}\alpha + \sum_{n \geqslant 2, k} (-1)^{n-k} \mu_n(X, \ldots, \alpha, \ldots, X)\right),$$

α is an element of degree 0.

Yang-Mills equations from $A_{(3)}$-algebra of $LZ(Q)$

$$\sum_{i,j} \eta^{ij}\left[\nabla_i, [\nabla_j, \nabla_k]\right] = 0, \qquad \nabla_i = \partial_i + A_i$$

Two objects: $\{A_i\}$, $\quad \{\eta^{ij}\}$

A_i **comes from Maurer-Cartan element**
Where η^{ij} **comes from?**

Flat background deformation of $A_{(3)}$**:**

$$\mathcal{D} \to \mathcal{D}^{\eta} = \mathcal{D} + \sum_{i,j} \eta^{ij}\mu(s_i\{s_j, \cdot\})$$

where $\{s_i, s_j\} = 0$**, such that** $s_i \in \mathcal{F}^1$ **and** $\mathcal{D}s_i = 0$

Theorem **i) Operations** $\mathcal{D}$**,** μ**,** n **generate** $A_{(3)}$**-algebra on** $\mathcal{F}_Q$**.**
ii) There exists a deformation $A^{\eta}_{(3)}(Q)$ **of** $A_{(3)}(Q)$**, associated with** $\mathcal{D}^{\eta}$**, such that** $\mu^{\eta} = \mu + \nu^{\eta}$**,** $n^{\eta} = n$**.**

A.M.Z.
Quasiclassical Lian-Zuckerman Homotopy Algebras, Courant Algebroid and Gauge Theory, Comm.Math.Phys., in press (arXiv:0910.3652)

The corresponding GMC equation

$$\boxed{\mathcal{D}^\eta\Phi + \mu^\eta(\Phi,\Phi) + n^\eta(\Phi,\Phi,\Phi) = 0}$$

where $\Phi \in \mathcal{F}^1 \otimes \mathfrak{g}$ **will be called generalized Yang-Mills equation associated to Courant algebroid** Q **and Lie algebra** $\mathfrak{g}$**.**

Symmetries:

$$\Phi \to \Phi + \varepsilon\big(\mathcal{D}^\eta\alpha + \mu^\eta(\Phi,\alpha) - \mu^\eta(\alpha,\Phi)\big)$$

where $\alpha \in \mathcal{F}^0 \otimes \mathfrak{g}$

Special case: $V = T\mathbb{R}^n \oplus T^*\mathbb{R}^n$

$$\begin{aligned}\sum_{i,j}\eta^{ij}\big[\nabla_i,[\nabla_j,\nabla_k]\big] &= \sum_{i,j}\eta^{ij}\big[[\nabla_k,\phi_i],\phi_j\big]\\ \sum_{i,j}\eta^{ij}\big[\nabla_i,[\nabla_j,\phi_k]\big] &= \sum_{i,j}\eta^{ij}\big[\phi_i,[\phi_j,\phi_k]\big]\end{aligned}$$

Symmetries:

$$A_i \to \varepsilon(\partial_i u + [A_i,u]), \qquad \phi_i \to \phi_i + \varepsilon[\phi_i,u]$$

where ϕ_i **are matter fields.**

$A_{(3)}$ algebra of pure YM theory

$\mathcal{D}$**:**

$$0 \to \Omega^0(\mathbb{R}^D) \xrightarrow{\mathrm{d}} \Omega^1(\mathbb{R}^D) \xrightarrow{\mathrm{d}{*}\mathrm{d}} \Omega^{D-1}(\mathbb{R}^D) \xrightarrow{\mathrm{d}} \Omega^D(\mathbb{R}^D) \to 0$$

$\mu_2(f_1, f_2)$**:**

$f_2 \backslash f_1$	v	$\mathbf{A}$	$\mathbf{V}$	a
w	vw	$\mathbf{A}w$	$\mathbf{V}w$	aw
$\mathbf{B}$	$v\mathbf{B}$	$(\mathbf{A}, \mathbf{B})$	$\mathbf{B} \wedge \mathbf{V}$	$\mathbf{0}$
$\mathbf{W}$	$v\mathbf{W}$	$\mathbf{A} \wedge \mathbf{W}$	$\mathbf{0}$	$\mathbf{0}$
b	vb	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

f_1 takes values in $\{v, \mathbf{A}, \mathbf{V}, a\}$
f_2 takes values in $\{w, \mathbf{B}, \mathbf{W}, b\}$
$u, v \in \Omega^0(\mathbb{R}^D)$, $\quad \mathbf{A}, \mathbf{B} \in \Omega^1(\mathbb{R}^D)$,
$\mathbf{V}, \mathbf{W} \in \Omega^{D-1}(\mathbb{R}^D)$, $\quad a, b \in \Omega^D(\mathbb{R}^D)$

Trilinear operation μ_3 is defined to be nonzero only when all arguments belong to $\mathcal{F}^1$.
For A, B, C$\in \mathcal{F}^1$ we have

$$\mu_3(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \mathbf{A} \wedge *(\mathbf{B} \wedge \mathbf{C}) - (-1)^{D-2}\big(*(\mathbf{A} \wedge \mathbf{B})\big) \wedge \mathbf{C}$$

A.M.Z.
Conformal Field Theory and Algebraic Structure of Gauge Theory, JHEP03(2010)056, arXiv:0812.1840

Open questions

- **Vertex algebras $\longleftrightarrow$ homotopy BV algebras**

  What is the precise relation?

- **Canonical string theory is not a vertex algebra:**

  $$\text{Closed}: \quad X^\mu(z,\bar{z})X^\nu(w,\bar{w}) \sim \eta^{\mu\nu}\log|z-w|^2$$

  (two series of oscillators)

  $$\text{Open}: \quad X^\mu(t)X^\nu(s) \sim \eta^{\mu\nu}\log|t-s|^2; \quad t,s\in\mathbb{R}$$

  (one series of oscillators)

<u>Claim</u> **One can generalize the Lian-Zuckerman operations in such a way that closed strings will generate homotopy Lie algebra, while open strings will generate homotopy associative algebra.**