Quantum KdV Hierarchies Based on Affine Lie Superalgebras

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Contents

1 Introduction

1.1 Quantization of the Integrable Systems

The integrable nonlinear systems under consideration (e.g. sine-Gordon, Nonlinear Schroedinger, Korteweg-de Vries) can be written as a compatibility condition of the system of linear equations:

$$\partial_x \Psi = U(x,t,\lambda)\Psi \tag{1}$$

$$\partial_t \Psi = V(x,t,\lambda)\Psi, \tag{2}$$

the so-called zero-curvature condition:

$$\partial_x V - \partial_t U + [V,U] = 0, \tag{3}$$

where functions U and V take their values in some Lie algebra g. Also it can be rewritten in the so-called L-A pair form:

$$\dot{\mathcal{L}} = [\mathcal{A},\mathcal{L}], \quad \mathcal{L} \equiv \partial_x - U, \quad \mathcal{A} \equiv V \tag{4}$$

The Hamiltonian interpretation allowed to consider the transformation to the scattering data of the linear problem (1), as a transformation to the "action-angle" variables in terms of which the problem is reduced to the system of linear equations. This transform with its inverse give the solution of the Cauchy problem and is called "Inverse Scattering Method".

The corresponding infinite family of the integrals of motion can be extracted from the spectral parameter expansion of the trace of the monodromy matrix of the equation (1). The Poisson brackets of the elements of the monodromy matrix $T(\lambda)$ for different values of the spectral parameter has the following form (usually):

$$\{T(\lambda)\otimes, T(\mu)\} = [r(\lambda\mu^{-1}), T(\lambda)\otimes T(\mu)], \tag{5}$$

where r is called "classical r- matrix". Using this relation one can easily show that

$$\{t(\lambda), t(\mu)\} = 0, \tag{6}$$

where $t(\lambda) = trT(\lambda)$, i.e. integrability condition as an existence of the involutive family of the integrals of motion.

After the quantization (4) transforms to the RTT-relation:

$$\begin{aligned} &R(\lambda\mu^{-1})(T^{(q)}(\lambda) \otimes I)(I \otimes T^{(q)}(\mu)) = \\ &(I \otimes T^{(q)}(\mu))(T^{(q)}(\lambda) \otimes I)R(\lambda\mu^{-1}), \end{aligned} \tag{7}$$

where R is called "quantum R-matrix". The correspondence between classical and quantum R-matrices is the following one:

$$R(\lambda) = 1 - hr(\lambda) + O(h^2), \tag{8}$$

where h is a deformation parameter. One can easily show that

$$[t^{(q)}(\lambda), t^{(q)}(\mu)] = 0 \tag{9}$$

as it was on the classical level, i.e. we get the quantum integrability, the existence of infinite family of mutually commuting operators. The relation (7) is a starting point of the so-called "Quantum Inverse Scattering Method (QISM)". Using the RTT-relation with the use of different methods, for example Algebraic Bethe Ansatz, one can find the spectrum of the transfer-matrix $t^{(q)}(\lambda)$ and different correlators. In order to obtain (7) for the two-dimensional field theory systems it is often necessary to consider them on a lattice.

But for some systems, such as (modified) Korteweg-de Vries ((m)KdV) hierarchy it is possible to construct the RTT-relation and find the explicit form of the monodromy matrix using the continuous field theory.

1.2 (m)KdV Hierarchy, its Quantization and the Integrable Structure of the Conformal Field Theory

1.2.1 Classical case

$$\mathcal{L}_{mKdV} = \partial_u - \phi'(u)h - \lambda(X_+ + X_-), \tag{10}$$

where u is a variable on a cylinder of a circumference 2π and where $\phi(u)$ is a scalar field with the following Poisson brackets on a circle:

$$\{\phi'(u), \phi'(v)\} = -\delta'(u-v) \tag{11}$$

and periodicity condition:

$$\phi(u + 2\pi) = \phi(u) + 2\pi i p \tag{12}$$

The generators $h, X_\pm$ form an $sl(2)$ algebra:

$$[h, X_\pm] = \pm 2X_\pm. \tag{13}$$

The usual KdV operator can be obtained by the gauge transformation (with the trace of the monodromy matrix remains unchanged):

$$\mathcal{L}_{KdV} = \partial_u - (X_+ + (\lambda^2 - U(u))X_-), \tag{14}$$

where the relation between $U(u)$ and $\phi'(u)$ is known as Miura transformation:

$$U(u) = -\phi'(u)^2 - \phi''(u) \tag{15}$$

and the Poisson brackets

$$\begin{aligned} &\{U(u), U(v)\} = \\ &\delta'''(u-v) + 2U'(u)\delta(u-v) + 4U(u)\delta'(u-v), \end{aligned} \tag{16}$$

gives the functional realization of the Virasoro algebra. The scalar linear problem is the same for both operators:

$$L\psi = \lambda^2\psi, \quad L = \partial_u^2 + U(u). \tag{17}$$

The formal solution to the linear problem $\mathcal{L}_{mKdV}\chi(u)=0$ is:

$$\chi(u) = \pi_s(e^{\phi(u)h} Pexp(\int_0^u du'(\lambda X_+ e^{-2\phi(u')} + \lambda X_- e^{2\phi(u')})))\chi_0 \tag{18}$$

where π_s is some representation of $sl(2)$ labeled by integer or half-integer number s. Then the monodromy matrix can be defined in the following way:

$$\mathrm{M}_s(\lambda) = \pi_s(e^{2\pi iph} Pexp(\int_0^{2\pi} du(\lambda X_+ e^{-2\phi} + \lambda X_- e^{2\phi}))) \tag{19}$$

Next let's define the auxiliary L-matrices:

$$\mathrm{L}_s(\lambda) = \pi_s(e^{-\pi iph})\mathrm{M}_s(\lambda) = \pi_s(e^{\pi iph} Pexp(\int_0^{2\pi} du(\lambda X_+ e^{-2\phi} + \lambda X_- e^{2\phi}))) \tag{20}$$

which satisfy the quadratic r-matrix Poisson bracket relation:

$$\{\mathrm{L}_s(\lambda)\otimes, \mathrm{L}_{s'}(\mu)\} = [\mathrm{r}_{ss'}(\lambda\mu^{-1}), \mathrm{L}_s(\lambda)\otimes \mathrm{L}_{s'}(\mu)], \tag{21}$$

where the classical r-matrix is ($\mathrm{r}_{ss'} = \pi_s \otimes \pi_{s'}(\mathrm{r})$):

$$\begin{aligned}\mathrm{r}(\lambda\mu^{-1}) \ &= \ \frac{1}{2}\frac{\lambda\mu^{-1}+\lambda^{-1}\mu}{\lambda\mu^{-1}-\lambda^{-1}\mu} h\otimes h + \\ &\quad \frac{2}{\lambda\mu^{-1}-\lambda^{-1}\mu}(X_+\otimes X_- + X_-\otimes X_+)\end{aligned}$$

The traces of the monodromy matrices $tr\mathrm{M}_s(\lambda) = \mathrm{t}_s(\lambda)$ for different values of λ commute:

$$\{\mathrm{t}_s(\lambda), \mathrm{t}_{s'}(\mu)\} = 0. \tag{22}$$

The expansion of $\mathrm{t}_{1/2}(\lambda)$ in the series of the integrals of motion is:

$$\frac{1}{2\pi}\log(\mathrm{t}_{1/2}(\lambda)) = \lambda - \sum_{n=1}^{\infty} c_n I_{2n-1}^{(cl)} \lambda^{-4n+2}, \ \ \lambda \to \infty \tag{23}$$

where $c_1 = \frac{1}{2}$, $c_n = \frac{(2n-3)!!}{2^n n!}$ for $n > 1$.

$$\begin{aligned} I_1^{(cl)} &= \int \frac{du}{2\pi} U(u), \\ I_3^{(cl)} &= \int \frac{du}{2\pi} U^2(u), \\ I_5^{(cl)} &= \int \frac{du}{2\pi}((U')^2(u) - 2U^3(u)). \\ &\cdot \quad \cdot \quad \cdot \end{aligned} \tag{24}$$

Therefore in quantum case the integrals of motion should be expressed in terms of Virasoro generators.

1.2.2 Quantization

$$\begin{aligned} &\mathrm{L}_s(\lambda) = \pi_s(e^{-\pi i p h})\mathrm{M}_s(\lambda) = \\ &e^{\pi i p h} Pexp\left(\int_0^{2\pi} du(\lambda X_+ e^{-2\phi} + \lambda X_- e^{2\phi})\right) \end{aligned} \tag{25}$$

Let's rewrite it in a more canonical form

$$\begin{aligned} &\mathrm{L} = e^{-\pi i p h_{\alpha_1}}\mathrm{M} = \\ &e^{\pi i p h_{\alpha_1}} Pexp\left(\int_0^{2\pi} du(e_{\alpha_1} e^{-2\phi} + e_{\alpha_0} e^{2\phi})\right), \end{aligned} \tag{26}$$

such that

$$\pi_s(\lambda)\mathrm{L} = \mathrm{L}_s(\lambda) \tag{27}$$

where h_{α_1}, e_{α_1}, e_{α_0} are the Chevalley generators of the affine algebra $A_1^{(1)}$:

$$[h_{\alpha_0}, h_{\alpha_1}] = 0, \quad [h_{\alpha_0}, e_{\pm\alpha_1}] = \mp 2 e_{\pm\alpha_1}, \tag{28}$$

$$
\begin{aligned}
&[h_{\alpha_1}, e_{\pm\alpha_0}] = \mp 2 e_{\pm\alpha_0},\\
&[h_{\alpha_i}, e_{\pm\alpha_i}] = \pm 2 e_{\pm\alpha_i} \quad (i = 0, 1),\\
&[e_{\alpha_i}, e_{-\alpha_j}] = \delta_{i,j} h_{\alpha_i} \quad (i, j = 0, 1),\\
&ad^3_{e_{\pm\alpha_1}} e_{\pm\alpha_0} = 0, \quad ad^3_{e_{\pm\alpha_0}} e_{\pm\alpha_1} = 0
\end{aligned}
$$

and $\pi_s(\lambda)$ is its evaluation representation. The quantum scalar field have the following mode expansion:

$$
\phi(u) = iQ + iPu + \sum_n \frac{a_{-n}}{n} e^{inu}, \tag{29}
$$

$$
[Q, P] = \frac{i}{2}\beta^2, \quad [a_n, a_m] = \frac{\beta^2}{2} n \delta_{n+m,0}.
$$

The quantum generalization of the auxiliary L-matrix is:

$$
\begin{aligned}
&\mathrm{L}^{(q)} = e^{-\pi i P h_{\alpha_1}} \mathrm{M}^{(q)} = \\
&e^{\pi i p h_{\alpha_1}} Pexp(\int_0^{2\pi} du(e_{\alpha_1} : e^{-2\phi} : + e_{\alpha_0} : e^{2\phi} :),
\end{aligned} \tag{30}
$$

where $h_{\alpha_1}, e_{\alpha_1}, e_{\alpha_0}$ are now the generators of the quantum affine algebra $A_1^{(1)}$:

$$
\begin{aligned}
&[h_{\alpha_0}, h_{\alpha_1}] = 0, \quad [h_{\alpha_0}, e_{\pm\alpha_1}] = \mp e_{\pm\alpha_1},\\
&[h_{\alpha_1}, e_{\pm\alpha_0}] = \mp e_{\pm\alpha_0},\\
&[h_{\alpha_i}, e_{\pm\alpha_i}] = \pm 2 e_{\pm\alpha_i} \quad (i = 0, 1),\\
&[e_{\alpha_i}, e_{-\alpha_j}] = \delta_{i,j} [h_{\alpha_i}] \quad (i, j = 0, 1),\\
&[e_{\pm\alpha_1}, [e_{\pm\alpha_1}, [e_{\pm\alpha_1}, e_{\pm\alpha_0}]_q]_q]_q = 0,\\
&[[[e_{\pm\alpha_1}, e_{\pm\alpha_0}]_q, e_{\pm\alpha_0}]_q, e_{\pm\alpha_0}]_q = 0,
\end{aligned} \tag{31}
$$

where $[h] = \frac{q^h - q^{-h}}{q - q^{-1}}$, q-commutator is defined in the following way:

$$
\begin{aligned}
&[e_\gamma, e_{\gamma'}]_q \equiv e_\gamma e_{\gamma'} - q^{(\gamma,\gamma')} e_{\gamma'} e_\gamma,\\
&q = e^{i\pi\beta^2}
\end{aligned} \tag{32}
$$

and the ordered exponentials are defined as usual:

$$
\begin{aligned}
: e^{c\phi(u)} := \quad & \exp\,(c\sum_{n=1}^{\infty}\frac{a_{-n}}{n}e^{inu})\exp\,(ci(Q+Pu))\,\cdot \\
& \exp\,(-c\sum_{n=1}^{\infty}\frac{a_{n}}{n}e^{-inu}).
\end{aligned} \tag{33}
$$

Let's prove that the RTT-relation holds for $\mathrm{L}^{(q)}$:

$$
\begin{aligned}
& (\mathrm{L}^{(q)}\otimes I)(I\otimes \mathrm{L}^{(q)}) = \\
& e^{i\pi P\Delta(h_{\alpha_1})} Pexp\int_0^{2\pi} du\tilde{K}_1(u) Pexp\int_0^{2\pi} du K_2(u), \\
& \tilde{K}_1(u) =: e^{-2\phi(u)} : e_{\alpha_1}\otimes q^{h_{\alpha_1}} + : e^{2\phi(u)} : e_{\alpha_0}\otimes q^{h_{\alpha_0}}, \\
& K_2(u) = 1\otimes : e^{-2\phi(u)} : e_{\alpha_1} + 1\otimes : e^{2\phi(u)} : e_{\alpha_0}
\end{aligned} \tag{34}
$$

Using that $: e^{\sigma 2\phi(u)} :: e^{\sigma' 2\phi(u')} := q^{\sigma\sigma'} : e^{\sigma' 2\phi(u')} :: e^{\sigma 2\phi(u)} :$ for $u > u'$ $(\sigma, \sigma' = \pm 1)$ one can see that

$$
[\tilde{K}_1(u), K_2(u')] = 0, \quad u < u'. \tag{35}
$$

and therefore we can write

$$
(\mathrm{L}^{(q)}\otimes I)(I\otimes \mathrm{L}^{(q)}) = \Delta(\mathrm{L}^{(q)}) \tag{36}
$$

where the coproduct Δ is defined in the following way:

$$
\begin{aligned}
& \Delta(h_{\alpha_j}) = h_{\alpha_j}\otimes 1 + 1\otimes h_{\alpha_j}, \\
& \Delta(e_{\alpha_j}) = e_{\alpha_j}\otimes q^{h_{\alpha_j}} + 1\otimes e_{\alpha_j}, \\
& \Delta(e_{-\alpha_j}) = e_{-\alpha_j}\otimes 1 + q^{-h_{\alpha_j}}\otimes e_{-\alpha_j}.
\end{aligned} \tag{37}
$$

In a similar way we obtain that

$$
(I\otimes \mathrm{L}^{(q)})(\mathrm{L}^{(q)}\otimes I) = \Delta^{op}(\mathrm{L}^{(q)}) \tag{38}
$$

where the opposite coproduct $\Delta^{op} = \tau\Delta$, and τ is a permutation map: $\tau(a\otimes b) = b\otimes a$. One of the fundamental properties of the universal R-matrix is that it relates two coproducts:

$$\mathrm{R}\Delta = \Delta^{op}\mathrm{R} \tag{39}$$

That is quantum L-operators satisfy RTT-relation:

$$\mathrm{R}(\mathrm{L}^{(q)} \otimes I)(I \otimes \mathrm{L}^{(q)}) = (I \otimes \mathrm{L}^{(q)})(\mathrm{L}^{(q)} \otimes I)\mathrm{R} \tag{40}$$

And making the spectral parameter dependence explicit we find:

$$\begin{gathered}\mathrm{R}(\lambda\mu^{-1})(\mathrm{L}^{(q)}(\lambda) \otimes I)(I \otimes \mathrm{L}^{(q)}(\mu)) = \\ (I \otimes \mathrm{L}^{(q)}(\mu))(\mathrm{L}^{(q)}(\lambda) \otimes I)\mathrm{R}(\lambda\mu^{-1})\end{gathered} \tag{41}$$

Recalling that the monodromy matrix is equal to $\mathrm{M}^{(q)}(\lambda) = e^{\pi i P h_{\alpha_1}}\mathrm{L}^{(q)}(\lambda)$ we find that the traces of the monodromy matrices commute:

$$[\mathrm{t}^{(q)}(\lambda), \mathrm{t}^{(q)}(\mu)] = 0 \tag{42}$$

leading to quantum integrability. The quantum counterpart of the Miura map gives the free field representation for the Virasoro algebra:

$$\begin{gathered}-\beta^2 T(u) =: \phi'^2(u) : + (1-\beta^2)\phi''(u) + \frac{\beta^2}{24}, \\ c = 13 - 6(\beta^{-2} + \beta^2), \quad T(u) = \sum_n L_n e^{-inu} - \frac{c}{24}\end{gathered} \tag{43}$$

The classical limit is given by $c \to -\infty, \quad T(u) \to -\frac{c}{6}U(u)$, $[,] \to \frac{6\pi}{ic}\{,\}$
With such a deformation of the Miura map we find that

$$[L_n, : e^{-2\phi(u)} :] \sim \partial_u(e^{inu} : e^{-2\phi(u)} :), \tag{44}$$

that is vertex operator is the screening one. The other one does not satisfy such a good property but we have the following one:

$$[I_n^{(q)}, : e^{\pm 2\phi(u)} :] = \partial_u \Theta_n^{\pm}(u) \tag{45}$$

because KdV IM have the symmetry $\phi \to -\phi$. That is $: e^{\pm 2\phi(u)} :$ can be treated as an integrable perturbation of some CFT of dimension $h_{1,3}$. This properties are not unexpected, because the $\mathcal{L}_{mKdV}$- operator is the L-operator for the sinh-Gordon model:

$$S = \beta^{-2} \int d^2u(\partial\phi\bar{\partial}\phi + g cosh(2\phi)) \tag{46}$$

2 Bosonic Toda-mKdV hierarchies

Each Toda or mKdV hierarchy related with affine Lie algebra is generated by the following L-operator:

$$\mathcal{L} = \partial_u - \partial_u\phi^i(u)H^i - (\sum_{i=0}^{r} e_{\alpha_i}), \tag{47}$$

where u lies on a cylinder of circumference 2π, ϕ^i are the scalar fields with the Poisson brackets:

$$\{\partial_u\phi^i(u), \partial_v\phi^j(v)\} = -\delta^{ij}\delta'(u-v) \tag{48}$$

with quasiperiodic boundary condition:

$$\phi^i(u+2\pi) = \phi^i(u) + 2\pi i p^i. \tag{49}$$

e_{α_i} are the Chevalley generators of the underlying affine Lie algebra and H^i $(i = 1, ..., r)$ form a basis in the Cartan subalgebra of the corresponding simple Lie algebra:

$$\begin{aligned} &[H^i, e_{\alpha_k}] = \alpha^i_k e_{\alpha_k}, \quad [e_{\alpha_k}, e_{-\alpha_l}] = \delta_{kl} h_{\alpha_k}, \\ &ad^{1-a_{kj}}_{e_{\pm\alpha_k}} e_{\pm\alpha_j} = 0, \end{aligned} \tag{50}$$

where a_{kj} is a Cartan matrix and $h_{\alpha_k} \equiv (\alpha_k, H) = \alpha^i_k H^i$. In our case this algebra is considered in the evaluation representation, when $e_{\alpha_0} = \lambda e_{-\theta}$ The classical monodromy matrix for the linear problem associated with the L-operator (47) can be expressed in the following way:

$$\pi_s(\lambda)(\mathrm{M}) \equiv \mathrm{M_s}(\lambda) = \tag{51}$$
$$e^{2\pi i p^k H^k} Pexp \int_0^{2\pi} du(\sum_{i=0}^{r} e^{-(\alpha_i,\phi)} e_{\alpha_i}),$$

where π_s is some evaluation representation of the corresponding affine Lie algebra. Defining the auxiliary L-matrix:

$$\mathbf{L} = e^{-\pi i p^k H^k}\mathrm{M} \tag{52}$$

one can find that they satisfy the quadratic Poisson bracket relation:

$$\{\mathbf{L}_s(\lambda)\otimes, \mathbf{L}_{s'}(\mu)\} = [\mathrm{r}_{ss'}(\lambda\mu^{-1}), \mathbf{L}_s(\lambda)\otimes \mathbf{L}_{s'}(\mu)], \tag{53}$$

where r is the trigonometric r-matrix related with the corresponding simple Lie algebra. The traces of the monodromy matrices in different evaluation representations π_s:

$$\mathrm{t_s}(\lambda) = tr\pi_s(\lambda)(\mathrm{M}) \tag{54}$$

are in involution under the Poisson brackets:

$$\{\mathrm{t_s}(\lambda), \mathrm{t_{s'}}(\mu)\} = 0. \tag{55}$$

The quantization means that we move from the quadratic Poisson bracket relation (53) to the RTT-relation with the underlying affine Lie algebra deformed to the quantum affine algebra.

First, let's quantize the scalar fields ϕ^i:

$$\phi^k(u) = iQ^k + iP^k u + \sum_n \frac{a^k_{-n}}{n} e^{inu}, \tag{56}$$

$$[Q^k, P^j] = \frac{i}{2}\beta^2\delta^{kj}, \quad [a^k_n, a^j_m] = \frac{\beta^2}{2} n\delta^{kj}\delta_{n+m,0},$$

and define the vertex operators

$$\tilde{V}_{\alpha_k}(u) :=: e^{(\alpha_k,\phi(u))} := \exp\,(\sum_{n=1}^{\infty}\frac{(\alpha_k, a_{-n})}{n}e^{inu})$$
$$\exp\,(i((\alpha_k, Q) + (\alpha_k, P)u))$$
$$\exp\,(\mp\sum_{n=1}^{\infty}\frac{(\alpha_k, a_n)}{n}e^{-inu}).$$

Then one can define the quantum generalizations of auxiliary L-operators:

$$\mathrm{L}^{(q)} = e^{\pi i P^k H^k} Pexp\int_0^{2\pi} du(\sum_{i=0}^{r} : e^{-(\alpha_i,\phi)} : e_{\alpha_i}), \qquad (57)$$

where now e_{α_i}, H^k are the generators of the corresponding quantum affine algebra:

$$[H^i, e_{\alpha_k}] = \alpha_k^i e_{\alpha_k}, \quad [e_{\alpha_k}, e_{-\alpha_l}] = \delta_{kl}[h_{\alpha_k}]_q,$$
$${ad^{(q)}_{e_{\pm\alpha_k}}}^{1-a_{kj}} e_{\pm\alpha_j} = 0, \qquad (58)$$

where $q = e^{i\pi\frac{\beta^2}{2}}$, $[a]_q = (q^a - q^{-a})/(q - q^{-1})$ and $ad^{(q)}_{e_\alpha} e_\beta = e_\alpha e_\beta - q^{(\alpha,\beta)} e_\beta e_\alpha$. The object (57) is defined in the interval $0 < \beta^2 < 2b^{-1}$, $b = max(|b_{ij}|)$ $(i \neq j)$, where b_{ij} is a symmetrized Cartan matrix, but can be analitically continuated to a wider region. The quantum monodromy matrix is:

$$\mathrm{M}^{(q)}(\lambda) = e^{\pi i P^k H^k}\mathrm{L}^{(q)}(\lambda). \qquad (59)$$

It can be shown that the $\mathrm{L}^{(q)}$ operator satisfies the mentioned RTT-relation in the two ways. The first way is to consider the following product:

$$(\mathrm{L}^{(q)} \otimes I)(I \otimes \mathrm{L}^{(q)}) \qquad (60)$$

Then, moving all the Cartan multipliers to left we find the following:

$$e^{i\pi P^i \Delta(H^i)} Pexp \int_0^{2\pi} du \tilde{K}_1(u) Pexp \int_0^{2\pi} du K_2(u),$$
$$\tilde{K}_1(u) = \sum_{j=0}^{r} : e^{-(\alpha_j,\phi(u))} : e_{\alpha_j} \otimes q^{h_{\alpha_j}}, \tag{61}$$
$$K_2(u) = \sum_{j=0}^{r} 1 \otimes : e^{-(\alpha_j,\phi(u))} : e_{\alpha_j},$$

where $\Delta(H^i) = H^i \otimes I + I \otimes H^i$. The commutation relations between vertex operators on a circle:

$$\tilde{V}_{\alpha_k}(u)\tilde{V}_{\alpha_j}(u') = q^{b_{kj}} \tilde{V}_{\alpha_j}(u')\tilde{V}_{\alpha_k}(u), \quad u > u', \tag{62}$$

lead to

$$[\tilde{K}_1(u), K_2(u')] = 0, \quad u < u'. \tag{63}$$

Due to this property one can unify two P-exponents into the single one which is equal to $\Delta(\mathrm{L}^{(q)})$, where coproduct Δ of the quantum affine (super)algebra is defined in the following way:

$$\Delta(H^i) = H^i \otimes I + I \otimes H^i, \tag{64}$$
$$\Delta(e_{\alpha_j}) = e_{\alpha_j} \otimes q^{h_{\alpha_j}} + 1 \otimes e_{\alpha_j},$$
$$\Delta(e_{-\alpha_j}) = e_{-\alpha_j} \otimes 1 + q^{-h_{\alpha_j}} \otimes e_{-\alpha_j}.$$

Next, considering opposite product of the L-operators similarly, one finds that it coincide with opposite coproduct of the L-operators:

$$(I \otimes \mathrm{L}^{(q)})(\mathrm{L}^{(q)} \otimes I) = \Delta^{op}(\mathrm{L}^{(q)}), \tag{65}$$

where $\Delta^{op} = \tau\Delta$ and the map τ is defined as follows: $\tau(a \otimes b) = b \otimes a$. Using the property of the universal

R-matrix, namely $\mathrm{R}\Delta = \Delta^{op}\mathrm{R}$, we arrive to the RTT-relation:

$$\begin{aligned} &\mathrm{R}(\mathrm{L}^{(q)} \otimes I)(I \otimes \mathrm{L}^{(q)}) = \\ &(I \otimes \mathrm{L}^{(q)})(\mathrm{L}^{(q)} \otimes I)\mathrm{R}. \end{aligned} \tag{66}$$

Remembering the expression for the monodromy matrix (59) one obtains that the RTT-relation is no longer valid for the monodromy matrices, however, it is easy to see that multiplying both RHS and LHS of (66) by $e^{i\pi\Delta H^k P^k}$ one obtains:

$$\begin{aligned} &\mathrm{R}_{12}(\lambda\mu^{-1})\tilde{\mathrm{M}}_1^{(q)}(\lambda)\mathrm{M}_2^{(q)}(\mu) = \\ &\tilde{\mathrm{M}}_2^{(q)}(\mu)\mathrm{M}_1^{(q)}(\lambda)\mathrm{R}_{12}(\lambda\mu^{-1}). \end{aligned} \tag{67}$$

where we have denoted $\tilde{\mathrm{M}}_1^{(q)}(\lambda)$ the monodromy matrix with $e_{\alpha_i} \otimes 1$ replaced by $e_{\alpha_i} \otimes q^{-h_{\alpha_i}}$ and $\tilde{\mathrm{M}}_2^{(q)}(\lambda)$ the monodromy matrix with $1 \otimes e_{\alpha_i}$ replaced by $q^{-h_{\alpha_i}} \otimes e_{\alpha_i}$. Taking the supertrace the above additional cartan factors in $\tilde{\mathrm{M}}^{(q)}$ cancel and we obtain the quantum integrability condition for their traces (transfer-matrices):

$$[\mathrm{t}_s(\lambda), \mathrm{t}_{s'}(\mu)] = 0. \tag{68}$$

Another more universal way to obtain the RTT-relation is the correspondence between the reduced universal R-matrix (see below) and the P-exponential form of the auxiliary $\mathrm{L}^{(q)}$ operator.

Universal R-matrix

The universal R-matrix is an object, satisfying three properties:

$$\begin{aligned} \mathrm{R}\Delta &= \Delta^{op}\mathrm{R} \\ (I \otimes \Delta)\mathrm{R} &= \mathrm{R}^{13}\mathrm{R}^{12} \\ (\Delta \otimes I)\mathrm{R} &= \mathrm{R}^{13}\mathrm{R}^{23} \end{aligned} \tag{69}$$

Easy to see that the Yang-Baxter equation is satisfied:

$$\mathrm{R}^{12}\mathrm{R}^{13}\mathrm{R}^{23}=\mathrm{R}^{23}\mathrm{R}^{13}\mathrm{R}^{12} \tag{70}$$

The universal R-matrix for the quantum affine algebra has the following structure:

$$\mathrm{R}=\boldsymbol{K}\bar{\boldsymbol{R}}=\boldsymbol{K}(\prod_{\alpha\in\Delta_+}^{\rightarrow}\boldsymbol{R}_\alpha), \tag{71}$$

where $\bar{R}$ is a reduced R-matrix and R_α are defined by the formulae:

$$R_\alpha=exp_{q_\alpha^{-1}}((q-q^{-1})(a(\alpha))^{-1}(e_\alpha\otimes e_{-\alpha}) \tag{72}$$

for real roots and

$$R_{n\delta}=exp((q-q^{-1})(\sum_{i,j}^{mult}c_{ij}(n)e_{n\delta}^{(i)}\otimes e_{-n\delta}^{(j)})) \tag{73}$$

for pure imaginary roots.Here Δ_+ is the positive root system. The q-exponentials are defined in the usual way:

$$\begin{aligned} exp_q(x) &= 1+x+\frac{x^2}{(2)_q!}+...+\frac{x^n}{(n)_q!}+...=\sum_{n\geq 0}\frac{x^n}{(n)_q!} \\ (a)_q &\equiv \frac{q^a-1}{q-1}, \quad q_\alpha\equiv q^{(\alpha,\alpha)}. \end{aligned} \tag{74}$$

The generators corresponding to the composite roots are defined according to the construction of the Cartan-Weyl basis. K is the Cartan factor and is given by the expression:

$$K=q^{\sum_{ij}d_{ij}h_{\alpha_i}\otimes h_{\alpha_j}} \tag{75}$$

where d_{ij} is the extended symmetrized Cartan matrix.

Let's now consider integrals of vertex operators:

$$V_{\alpha_k}(u_2,u_1)=\frac{1}{q-q^{-1}}\int_{u_1}^{u_2}du\tilde{V}_{\alpha_k}(u) \tag{76}$$

Via the contour technique one can show that these objects satisfy the quantum Serre relations of the lower Borel subalgebra of the associated affine superalgebra with simple roots α_k. Using the structure of the reduced universal R-matrix we can write $\bar{R} = K^{-1}\mathrm{R} = \bar{R}(\bar{e}_{\alpha_i}, \bar{e}_{-\alpha_i})$, where

$$\bar{e}_{\alpha_i} = e_{\alpha_i} \otimes 1, \quad \bar{e}_{-\alpha_i} = 1 \otimes e_{-\alpha_i}, \tag{77}$$

R is a universal R-matrix and K depends on the elements from Cartan subalgebra, because $\bar{R}$ is represented as a power series of these elements. Then, using the fundamental feature of the universal R-matrix:

$$(I \otimes \Delta)\mathrm{R} = \mathrm{R}^{13}\mathrm{R}^{12} \tag{78}$$

one can show that the reduced R-matrix has the following property:

$$\bar{R}(\bar{e}_{\alpha_i}, e'_{-\alpha_i} + e''_{-\alpha_i}) = \bar{R}(\bar{e}_{\alpha_i}, e'_{-\alpha_i})\bar{R}(\bar{e}_{\alpha_i}, e''_{-\alpha_i}), \tag{79}$$

where

$$\begin{aligned} e'_{-\alpha_i} \ + \ e''_{-\alpha_i} &= (I \otimes \Delta)(I \otimes e_{-\alpha_i}), \\ e'_{-\alpha_i} &= 1 \otimes q^{-h_{\alpha_i}} \otimes e_{-\alpha_i}, \quad e''_{-\alpha_i} = 1 \otimes e_{-\alpha_i} \otimes 1, \\ \bar{e}_{\alpha_i} &= e_{\alpha_i} \otimes 1 \otimes 1. \end{aligned} \tag{80}$$

The commutation relations between them are

$$\begin{aligned} e'_{-\alpha_i}\bar{e}_{\alpha_j} &= \bar{e}_{\alpha_j}e'_{-\alpha_i}, \quad e''_{-\alpha_i}\bar{e}_{\alpha_j} = \bar{e}_{\alpha_j}e''_{-\alpha_i}, \\ e'_{-\alpha_i}e''_{-\alpha_j} &= q^{b_{ij}}e''_{-\alpha_j}e'_{-\alpha_i}. \end{aligned} \tag{81}$$

Now, denoting $\bar{\mathrm{L}}^{(q)}(u_2, u_1)$ the reduced R-matrix with $e_{-\alpha_i}$ represented by $V_{\alpha_i}(u_2, u_1)$ and using the above property of $\bar{R}$ with $e'_{-\alpha_i}$ replaced by appropriate vertex operators we find:

$$\bar{\mathrm{L}}^{(q)}(u_3, u_1) = \bar{\mathrm{L}}^{(q)}(u_3, u_2)\bar{\mathrm{L}}^{(q)}(u_2, u_1), \ u_3 \geq u_2 \geq u_1. \tag{82}$$

So, $\bar{\mathrm{L}}^{(q)}$ has the property of P-exponent. When the interval $\delta = [u_2, u_1]$ is small enough we see that

$$\bar{\mathrm{L}}^{(q)}(u_2, u_1) = \\ 1 + \int_{u_1}^{u_2} du(\sum_{i=0}^{r} :e^{-(\alpha_i,\phi)}: e_{\alpha_i}) + O(\delta^2) \tag{83}$$

That is, we obtain that

$$\bar{\mathrm{L}}^{(q)}(u_2, u_1) = Pexp \int_{u_1}^{u_2} du(\sum_{i=0}^{r} :e^{-(\alpha_i,\phi)}: e_{\alpha_i}) \tag{84}$$

and $\mathrm{L}^{(q)} = e^{i\pi H^i P^i} \bar{\mathrm{L}}^{(q)}(2\pi, 0)$ satisfies RTT relation by construction.

3 Super-KdV: Classical Theory

In the previous section we have considered KdV hierarchies based on affine Lie algebras. Now we consider the super-KdV hierarchy related with affine superalgebra $\widehat{osp}(1|2)$.

$osp(1|2)$ superalgebra

$$\begin{aligned} [h, X_\pm] &= \pm 2X_\pm, \quad [h, v_\pm] = \pm v_\pm, \quad [X_+, X_-] = h, \\ [v_\pm, v_\pm] &= \pm 2X_\pm, \quad [v_+, v_-] = -h, \\ \{X_\pm, v_\mp\} &= v_\pm, \quad [X_\pm, v_\pm] = 0. \end{aligned} \tag{85}$$

Here [,] means supercommutator:
$[a, b] = ab - (-1)^{p(a)p(b)} ba$ and the parity p is defined as follows: $p(v_\pm) = 1$, $p(X_\pm) = 0$, $p(h) = 0$. The defining representation is:

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$v_- = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad v_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad X_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The supermatrix L-operator, corresponding to super-KdV theory is the following one:

$$\mathcal{L}_F = D_{u,\theta} - D_{u,\theta}\Psi h - (iv_+\sqrt{\lambda} - \theta\lambda X_-), \tag{86}$$

where $D_{u,\theta} = \partial_\theta + \theta\partial_u$ is a superderivative, variable u lies on a cylinder of circumference 2π, θ is a Grassmann variable, $\Psi(u,\theta) = \phi(u) - i\theta\xi(u)/\sqrt{2}$ is a bosonic superfield. The "fermionic" operator $\mathcal{L}_F$ considered together with a linear problem $\mathcal{L}_F\chi(u,\theta) = 0$ is equivalent to the "bosonic" one:

$$\mathcal{L}_B = \partial_u - \phi'(u)h + \sqrt{\lambda/2}\xi(u)v_+ - \lambda(X_+ + X_-). \tag{87}$$

The fields ϕ, ξ satisfy the following boundary conditions:

$$\begin{aligned} \phi(u+2\pi) &= \phi(u) + 2\pi ip, \\ \xi(u+2\pi) &= \pm\xi(u), \end{aligned} \tag{88}$$

where "+" corresponds to the so-called Ramond (R) sector of the model and "–" to the Neveu-Schwarz (NS) one. The Poisson brackets, given by the Drinfeld-Sokolov construction are the following:

$$\begin{aligned} \{\xi(u),\xi(v)\} &= -2\delta(u-v), \\ \{\phi(u),\phi(v)\} &= \frac{1}{2}\epsilon(u-v). \end{aligned} \tag{89}$$

The L-operators (86), (87) correspond to the super-mKdV, they are written in the Miura form. Making a gauge transformation to proceed to the super-KdV L-operator one obtains two fields:

$$\begin{aligned} U(u) &= -\phi''(u) - \phi'^2(u) - \frac{1}{2}\xi(u)\xi'(u), \\ \alpha(u) &= \xi'(u) + \xi(u)\phi'(u), \end{aligned} \tag{90}$$

which generate the superconformal algebra under the Poisson brackets:

$$\begin{aligned} \{U(u), U(v)\} &= \delta'''(u-v) + 2U'(u)\delta(u-v) \\ &+ 4U(u)\delta'(u-v), \\ \{U(u), \alpha(v)\} &= 3\alpha(u)\delta'(u-v) + \alpha'(u)\delta(u-v), \\ \{\alpha(u), \alpha(v)\} &= 2\delta''(u-v) + 2U(u)\delta(u-v). \end{aligned} \tag{91}$$

These brackets describe the second Hamiltonian structure of the super-KdV hierarchy. One can obtain evolution equation by taking one of the corresponding infinite set of local IM (they could be obtained by expanding $\log(\mathrm{t}_1(\lambda))$, where $\mathrm{t}_1(\lambda)$ is the supertrace of the monodromy matrix, see below):

$$\begin{aligned} I_1^{(cl)} &= \int \frac{du}{2\pi} U(u), \\ I_3^{(cl)} &= \int \frac{du}{2\pi} (U^2(u) - 2\alpha(u)\alpha'(u)), \\ I_5^{(cl)} &= \int \frac{du}{2\pi} (U^3(u) - \frac{(U')^2(u)}{2} - \\ &\quad 4\alpha'(u)\alpha''(u) - 6\alpha'(u)\alpha(u)U(u)). \\ &\quad . \quad . \quad . \end{aligned} \tag{92}$$

These IM form an involutive set under the Poisson brackets:

$$\{I_{2k-1}^{(cl)}, I_{2l-1}^{(cl)}\} = 0 \tag{93}$$

and the $I_3^{(cl)}$ leads to the super-KdV equation:

$$\begin{aligned} U_t &= -U_{uuu} - 6UU_u - 6\alpha\alpha_{uu}, \\ \alpha_t &= -4\alpha_{uuu} - 6U\alpha_u - 3U_u\alpha. \end{aligned} \tag{94}$$

Now let's consider the "bosonic" linear problem $\pi_s(\mathcal{L}_B)\chi(u) = 0$, where π_s means irreducible representation of $osp(1|2)$ labeled by an integer $s \geq 0$. The formal solution of this problem is:

$$\begin{aligned} \chi(u) &= \pi_s\Big(e^{\phi(u)h} P \exp \int_0^u du'(- \sqrt{\lambda/2}\xi(u')v_+ e^{-\phi(u')} \\ &+ \lambda(X_+ e^{-2\phi(u')} + X_- e^{2\phi(u')}))\Big)\chi_0, \end{aligned}$$

This could be rewritten in a more general way:

$$\begin{aligned} \chi(u) &= \pi_s(\lambda)\Big(e^{-\phi(u)h_{\alpha_0}} P \exp \int_0^u du'(i\xi(u')e^{-\phi(u')}e_\alpha \\ &- e^{-2\phi(u')}2e_\alpha^2 + e^{2\phi(u')}e_{\alpha_0})\Big)\chi_0, \end{aligned}$$

where e_α, e_{α_0}, h_{α_0} are part of the Chevalley generators of $\widehat{osp}(1|2)$, which coincide in the evaluation representations $\pi_s(\lambda)$ with $i\sqrt{\lambda/2}v_+$, λX_-, $-h$ correspondingly. The associated monodromy matrix then has the form:

$$\begin{aligned} \mathrm{M}_s(\lambda) &= \pi_s\Big(e^{-2\pi i p h_{\alpha_0}} P \exp \int_0^{2\pi} du'(i\xi(u')e^{-\phi(u')}e_\alpha \\ &- e^{-2\phi(u')}2e_\alpha^2 + e^{2\phi(u')}e_{\alpha_0})\Big). \end{aligned}$$

The auxiliary L-matrices are:

$$\pi_s(\lambda)(\mathrm{L}) = \mathrm{L}_s(\lambda) = \pi_s(\lambda)(e^{\pi i p h_{\alpha_0}})\mathrm{M}_s(\lambda). \tag{95}$$

They satisfy quadratic Poisson bracket algebra:

$$\{\mathrm{L}_s(\lambda) \otimes, \mathrm{L}_{s'}(\mu)\} = [\mathrm{r}_{ss'}(\lambda\mu^{-1}), \mathrm{L}_s(\lambda) \otimes \mathrm{L}_{s'}(\mu)], \tag{96}$$

where $\mathrm{r}_{ss'}(\lambda\mu^{-1}) = \pi_s(\lambda)\otimes\pi_{s'}(\mu)(\mathrm{r})$ is the classical trigonometric $\widehat{osp}(1|2)$ r-matrix taken in the corresponding representations:

$$\begin{aligned}\mathrm{r}(\lambda\mu^{-1}) &= \frac{1}{2}\frac{\lambda\mu^{-1}+\lambda^{-1}\mu}{\lambda\mu^{-1}-\lambda^{-1}\mu} h\otimes h \\ &+ \frac{2}{\lambda\mu^{-1}-\lambda^{-1}\mu}(X_+\otimes X_- + X_-\otimes X_+) \\ &+ \frac{1}{(\lambda\mu^{-1}-\lambda^{-1}\mu)}\Big(\sqrt{\frac{\mu}{\lambda}}v_+\otimes v_- - \sqrt{\frac{\lambda}{\mu}}v_-\otimes v_+\Big).\end{aligned}$$

From the Poisson brackets for $\mathrm{L}_s(\lambda)$ one obtains that the traces of monodromy matrices $\mathrm{t}_s(\lambda) = str\mathrm{M}_s(\lambda)$ commute under the Poisson bracket:

$$\{\mathrm{t}_s(\lambda), \mathrm{t}_{s'}(\mu)\} = 0. \tag{97}$$

If one expands $\log(\mathrm{t}_1(\lambda))$ in series of λ^{-1}, one can see that the coefficients in this expansion are local IM, as we mentioned earlier:

$$\frac{1}{2\pi}\log(\mathrm{t}_1(\lambda)) = \lambda - \sum_{n=1}^{\infty} c_n I^{(cl)}_{2n-1}\lambda^{-4n+2}, \quad \lambda\to\infty, \tag{98}$$

where $c_1 = \frac{1}{2}$, $c_n = \frac{(2n-3)!!}{2^n n!}$ for $n > 1$.

4 Free field representation of Superconformal algebra and Vertex operators

To quantize the introduced classical quantities, we start from a quantum version of the Miura transformation (90), the so-called free field representation of the superconformal algebra:

$$-\beta^2 T(u) \; = \; :\phi'^2(u): + (1-\beta^2/2)\phi''(u)$$

$$
\begin{aligned}
&+\ \frac{1}{2}:\xi\xi'(u):+\frac{\epsilon\beta^2}{16} \qquad (99)\\
\frac{i^{1/2}\beta^2}{\sqrt{2}}G(u) \ &= \ \phi'\xi(u)+(1-\beta^2/2)\xi'(u),
\end{aligned}
$$

$$
\begin{aligned}
&\phi(u)=iQ+iPu+\sum_n\frac{a_{-n}}{n}e^{inu}, \qquad (100)\\
&\xi(u)=i^{-1/2}\sum_n\xi_n e^{-inu},\\
&[Q,P]=\frac{i}{2}\beta^2, \quad [a_n,a_m]=\frac{\beta^2}{2}n\delta_{n+m,0},\\
&\{\xi_n,\xi_m\}=\beta^2\delta_{n+m,0}.
\end{aligned}
$$

Parameter β^2 plays a role of a semiclassical parameter (Planck's constant). Recall that there are two types of boundary conditions on ξ: $\xi(u+2\pi)=\pm\xi(u)$. The sign "+" corresponds to the R sector,the case when ξ is integer modded, the "–" sign corresponds to the NS sector and ξ is half-integer modded. The variable ϵ in (20) is equal to zero in the R case and equal to 1 in the NS case.
One can expand $T(u)$ and $G(u)$ by modes in such a way:

$$
T(u)=\sum_n L_{-n}e^{inu}-\frac{\hat{c}}{16}, \quad G(u)=\sum_n G_{-n}e^{inu}, \quad (101)
$$

where the central charge $\hat{c}=5-2(\frac{\beta^2}{2}+\frac{2}{\beta^2})$ and L_n, G_m generate the superconformal algebra:

$$
\begin{aligned}
[L_n,L_m] \ &= \ (n-m)L_{n+m}+\frac{\hat{c}}{8}(n^3-n)\delta_{n,-m} \qquad (102)\\
[L_n,G_m] \ &= \ (\frac{n}{2}-m)G_{m+n}\\
[G_n,G_m] \ &= \ 2L_{n+m}+\delta_{n,-m}\frac{\hat{c}}{2}(n^2-1/4).
\end{aligned}
$$

In the classical limit $c \to -\infty$ (the same is $\beta^2 \to 0$) the following substitution: $T(u) \to -\frac{\hat{c}}{4}U(u)$, $\quad G(u) \to -\frac{\hat{c}}{2\sqrt{2i}}\alpha(u)$, $\quad [,] \to \frac{4\pi}{i\hat{c}}\{,\}$ reduce the above algebra to the Poisson bracket algebra of super-KdV theory.

Let F_p be the Fock representation with the vacuum $|p\rangle$ (highest weight vector). Vector $|p\rangle$ is determined by the eigenvalue of P and nilpotency condition of the action of the positive modes:

$$P|p\rangle = p|p\rangle, \quad a_n|p\rangle = 0, \quad \xi_m|p\rangle = 0 \quad n, m > 0. \tag{103}$$

In the case of the R sector the highest weight becomes doubly degenerate due to the presence of zero mode ξ_0,i.e. there are two ground states $|p, +\rangle$ and $|p, -\rangle$:

$$|p, +\rangle = \xi_0|p, -\rangle. \tag{104}$$

Using the above free field representation of the superconformal algebra one can obtain that for generic $\hat{c}$ and p, F_p is isomorphic to the super-Virasoro module with the highest weight vector $|p\rangle$:

$$L_0|p\rangle \;=\; \Delta_{NS}|p\rangle, \quad \Delta_{NS} = \left(\frac{p}{\beta}\right)^2 + \frac{\hat{c}-1}{16} \tag{105}$$

in the NS sector and module with two highest weight vectors in the Ramond case:

$$\begin{aligned} L_0|p,\pm\rangle \;&=\; \Delta_R|p,\pm\rangle, \quad \Delta_R = \left(\frac{p}{\beta}\right)^2 + \frac{\hat{c}}{16}, \\ |p,+\rangle \;&=\; \frac{\beta^2}{\sqrt{2}p}G_0|p,-\rangle. \end{aligned} \tag{106}$$

The space F_p, considered as super-Virasoro module, splits in the sum of finite-dimensional subspaces, determined by the value of L_0:

$$F_p = \oplus_{k=0}^{\infty} F_p^{(k)}, \quad L_0 F_p^{(k)} \;=\; (\Delta + k)F_p^{(k)}. \tag{107}$$

The quantum versions of local integrals of motion should act invariantly on the subspaces $F_p^{(k)}$. Thus, the problem of the diagonalization of IM reduces (in a given subspace $F_p^{(k)}$) to the finite purely algebraic problem, which however rapidly become very complex for large k. It should be noted also that in the case of the Ramond sector G_0 does not commute with IM (even classically), so IM mix $|p,+\rangle$ and $|p,-\rangle$.

In the end of this section we introduce vertex operators, which we will use. We need two types of them:"bosonic" and "fermionic":

$$V_B^{(a)} = \int d\theta\theta : e^{a\Phi} :, \quad V_F^{(b)} = \int d\theta : e^{b\Phi} :, \tag{108}$$

where $\Phi(u,\theta) = \phi(u) - \theta\xi(u)$ is a superfield, so

$$V_B^{(a)} =: e^{a\phi} :, \quad V_F^{(b)} = -\frac{ib}{\sqrt{2}}\xi : e^{b\phi} : \tag{109}$$

5 Super-KdV: Quantum Monodromy Matrix and Fusion Relations

In this section we will construct the quantum versions of monodromy matrices, operators L_s and t_s.

The classical monodromy matrix is based on the $\widehat{osp}(1|2)$ affine Lie algebra. In the quantum case the underlying algebra is quantum $\widehat{osp}_q(1|2)$ with $q = e^{i\pi\beta^2}$ and generators, corresponding to even root α_0 and odd root α:

$$\begin{aligned}
&[h_\gamma, h_{\gamma'}] = 0 \quad (\gamma, \gamma' = \alpha, d, \alpha_0), \\
&[e_\beta, e_{\beta'}] = \delta_{\beta,-\beta'}[h_\beta] \quad (\beta = \alpha, \alpha_0), \\
&[h_{\alpha_0}, e_{\pm\alpha_0}] = \pm 2 e_{\pm\alpha_0}, \\
&[h_{\alpha_0}, e_{\pm\alpha}] = \mp e_{\pm\alpha}, \\
&[h_\alpha, e_{\pm\alpha}] = \pm\frac{1}{2} e_{\pm\alpha},
\end{aligned} \tag{110}$$

$$[h_\alpha, e_{\pm\alpha_0}] = \mp e_{\pm\alpha_0},$$
$$[[e_{\pm\alpha}, e_{\pm\alpha_0}]_q, e_{\pm\alpha_0}]_q = 0,$$
$$[e_{\pm\alpha}, [e_{\pm\alpha}, [e_{\pm\alpha}[e_{\pm\alpha}, [e_{\pm\alpha}, e_{\pm\alpha_0}]_q]_q]_q]_q]_q = 0.$$

Here $[,]_q$ is the super q-commutator: $[e_a, e_b] = e_a e_b - q^{(a,b)}(-1)^{p(a)p(b)} e_b e_a$ and parity p is defined as follows: $p(h_{\alpha_0}) = 0$, $p(h_\alpha) = 0$, $p(e_{\pm\alpha_0}) = 0$, $p(e_{\pm\alpha}) = 1$. Also, as usual, $[h_\beta] = \frac{q^{h_\beta} - q^{-h_\beta}}{q - q^{-1}}$. The finite dimensional representations $\pi_s^{(q)}(\lambda)$ of $\widehat{osp}_q(1|2)$ can be characterized by integer number s and have the following explicit form:

$$\begin{aligned}
h_{\alpha_0}|j,m\rangle &= 2m|j,m\rangle, \\
e_{\alpha_0}|j,m\rangle &= \lambda\sqrt{[j-m][j+m+1]}|j,m+1\rangle, \\
e_{-\alpha_0}|j,m\rangle &= \lambda^{-1}\sqrt{[j+m][j-m+1]}|j,m-1\rangle, \\
e_{\alpha}|j,m\rangle &= \sqrt{\lambda}((-1)^{-2j}\sqrt{\alpha(j)[j-m+1]}|j \\
&+ 1/2, m-1/2\rangle \\
&+ \sqrt{\alpha(j-1/2)[j+m]}|j-1/2, m-1/2\rangle), \\
e_{-\alpha}|j,m\rangle &= \sqrt{\lambda}^{-1}(-\sqrt{\alpha(j)[j+m+1]}|j \\
&+ 1/2, m+1/2\rangle \\
&- (-1)^{2j}\sqrt{\alpha(j-1/2)[j-m]}|j \\
&- 1/2, m+1/2\rangle), \\
h_{\alpha_0} &= -2h_\alpha,
\end{aligned} \tag{111}$$

where $j = 0, 1/2, ..., s/2$, $m = -j, -j+1..., j$.
The normalization coefficients

$$\begin{aligned}
\alpha(j) &= \frac{[j+1][j+1/2][1/4]}{[2j+2][2j+1][1/2]}\Big((-1)^{s-2j+1}\frac{[s+3/2]}{[s/2+3/4]} \\
&+ \frac{[j+3/2]}{[j/2+3/4]}\Big)
\end{aligned} \tag{112}$$

are defined by the recurrence relation:

$$\alpha(j)\frac{[2j+2]}{[j+1]} + \alpha(j-1/2)\frac{[2j]}{[j]} = 1, \;\; \alpha(s/2) = 0. \tag{113}$$

It is not hard to see that in the classical limit $q \to 1$ $\alpha(s/2-k)=0$, if $k<s/2$ is a nonnegative integer and $\alpha(s/2-k)=1/2$, if $k<s/2$ is nonnegative half-integer. Using this fact one can obtain that this representation in the classical limit appears to be a direct sum of finite dimensional irreducible representations of $\widehat{osp}(1|2)$:

$$\pi_s^{(1)}(\lambda) = \oplus_{k=0}^{[s/2]} \pi_{s-2k}(\lambda). \tag{114}$$

In this sum k runs through integer numbers. One can notice that the structure of irreducible finite dimensional representations of $\widehat{osp}_q(1|2)$ is similar to those of $(A_2^{(2)})_q$. This is the consequence of the coincidence of their Cartan matrices.

The quantum counterparts of L_s operators are:

$$\begin{aligned}
\mathrm{L}_s^{(q)} &= \pi_s^{(q)}(\lambda)(\mathrm{L}^{(q)}) \qquad (115)\\
&= \pi_s^{(q)}\Big(e^{-i\pi P h_{\alpha_0}} P \exp\Big(\int_0^{2\pi} du(: e^{2\phi(u)} : e_{\alpha_0} \\
&+ i\xi(u) : e^{-\phi(u)} : e_{\alpha}))\Big).
\end{aligned}$$

One can see that one term is missing in the P-exponent in comparison with the classical case. Analyzing the singularity properties of the integrands in P-exponent of $\mathrm{L}_s^{(q)}(\lambda)$ one can find that the integrals are convergent for values of $\hat{c}$ from the interval:

$$-\infty < \hat{c} < 0 \tag{116}$$

With the use of regularization P-exponent can be continuated on a wider region of $\hat{c}$.

Now let's prove that in the classical limit $\mathrm{L}^{(q)}$ will coincide with L.

First let's analyse the products of the operators we have in the P-exponent. The product of the two fermion operators can be written in such a way:

$$\xi(u)\xi(u') =: \xi(u)\xi(u') : -i\beta^2 \frac{e^{-\kappa\frac{i}{2}(u-u')}}{e^{\frac{i}{2}(u-u')} - e^{-\frac{i}{2}(u-u')}} \quad (117)$$

where κ is equal to zero in the NS sector and equal to 1 in the R sector. for vertex operators the corresponding operator product is:

$$\begin{aligned} &: e^{a\phi(u)} :: e^{b\phi(u')} := \quad (118) \\ &(e^{\frac{i}{2}(u-u')} - e^{-\frac{i}{2}(u-u')})^{\frac{ab\beta^2}{2}} : e^{a\phi(u)+b\phi(u')} :, \end{aligned}$$

where

$$\begin{aligned} &: e^{a\phi(u)+b\phi(u')} := \quad (119) \\ &\exp\,(a\sum_{n=1}^{\infty}\frac{a_{-n}}{n}e^{inu} + b\sum_{n=1}^{\infty}\frac{a_{-n}}{n}e^{inu'}) \\ &\exp\,(ai(Q+Pu) + bi(Q+Pu')) \\ &\exp\,(-a\sum_{n=1}^{\infty}\frac{a_n}{n}e^{-inu} - b\sum_{n=1}^{\infty}\frac{a_n}{n}e^{-inu'}). \end{aligned}$$

It would be more useful to rewrite these products picking out the singular part (when $u \to u'$):

$$\begin{aligned} &\xi(u)\xi(u') = \quad (120) \\ &-\frac{i\beta^2}{(iu-iu')} + \sum_{k=0}^{\infty} c_k(u)(iu-iu')^k, \end{aligned}$$

$$\begin{aligned} &: e^{a\phi(u)} :: e^{b\phi(u')} := (iu-iu')^{\frac{ab\beta^2}{2}}(: e^{(a+b)\phi(u)} : \\ &+ \sum_{k=1}^{\infty} d_k(u)(iu-iu')^k), \quad (121) \end{aligned}$$

where $c_k(u)$ and $d_k(u)$ are operator-valued functions of u. Now let's return to the $\mathrm{L}^{(q)}(\lambda)$ operator, which one can express it in the following way:

$$\begin{aligned}
\mathrm{L}^{(q)} &= e^{-i\pi P h_{\alpha_0}} \lim_{N\to\infty} \prod_{m=1}^{N} \tau_m^{(q)}, \\
\tau_m^{(q)} &= P \exp \int_{x_{m-1}}^{x_m} du K(u), \\
K(u) &\equiv \, : e^{2\phi(u)} : e_{\alpha_0} + i\xi(u) : e^{-\phi(u)} : e_{\alpha}.
\end{aligned} \tag{122}$$

Here we have divided the interval $[0, 2\pi]$ into small intervals $[x_m, x_{m+1}]$ with $x_{m+1} - x_m = \Delta = 2\pi/N$. Let's look on the behaviour of the first two iterations when $\beta^2 \to 0$:

$$\begin{aligned}
\tau_m^{(q)} &= 1 + \int_{x_{m-1}}^{x_m} du K(u) \\
&+ \int_{x_{m-1}}^{x_m} du K(u) \int_{x_{m-1}}^{u} du' K(u') + O(\Delta^2).
\end{aligned} \tag{123}$$

It appears, that in $\beta^2 \to 0$ limit terms from the second iteration can give contribution to the first one. To see this let's consider the expression that comes from the second iteration:

$$\int_{x_{m-1}}^{x_m} du \xi(u) \int_{x_{m-1}}^{u} du' \xi(u') : e^{-\phi(u)} :: e^{-\phi(u')} : e_{\alpha}^2. \tag{124}$$

Using the above operator products and seeking the terms of order $\Delta^{1+\beta^2}$ (only those can give us the first iteration terms in $\beta^2 \to 0$ limit) one obtains that their contribution is:

$$\begin{aligned}
&-i\beta^2 \int_{x_{m-1}}^{x_m} du \int_{x_{m-1}}^{u} du' (iu - iu')^{\frac{\beta^2}{2}-1} : e^{-2\phi(u)} : e_{\alpha}^2 \\
&= -2 \int_{x_{m-1}}^{x_m} du : e^{-2\phi(u)} : (iu - ix_{m-1})^{\frac{\beta^2}{2}} e_{\alpha}^2.
\end{aligned} \tag{125}$$

Considering this in the classical limit we recognize the familiar terms from L:

$$\tau_m^{(1)} = 1 + \int_{x_{m-1}}^{x_m} du(i\xi(u)e^{-\phi(u)}e_\alpha + e^{2\phi(u)}e_{\alpha_0} - e^{-2\phi(u)}2e_\alpha^2) + O(\Delta^2). \tag{126}$$

Collecting all $\tau_m^{(1)}$ one obtains the desired result:

$$\mathrm{L}^{(1)} = \mathrm{L}. \tag{127}$$

Recalling the structure of $\widehat{osp}_q(1|2)$ representations we get:

$$\mathrm{L}_s^{(1)}(\lambda) = \sum_{k=0}^{[s/2]} \mathrm{L}_{s-2k}(\lambda). \tag{128}$$

Using the properties of quantum R-matrix it follows that $\mathrm{R}\Delta(\mathrm{L}^{(q)}) = \Delta^{op}(\mathrm{L}^{(q)})\mathrm{R}$, where Δ and Δ^{op} are coproduct and opposite coproduct of $\widehat{osp}_q(1|2)$ correspondingly. Factorizing $\Delta(\mathrm{L}^{(q)})$ and $\Delta^{op}(\mathrm{L}^{(q)})$, according to the properties of vertex operators and P-exponent, we get the RTT-relation:

$$\begin{aligned} &\mathrm{R}_{ss'}(\lambda\mu^{-1})(\mathrm{L}_s^{(q)}(\lambda)\otimes \mathrm{I})(\mathrm{I}\otimes \mathrm{L}_{s'}^{(q)}(\mu)) \\ &= (\mathrm{I}\otimes \mathrm{L}_{s'}^{(q)}(\mu))(\mathrm{L}_s^{(q)}(\lambda)\otimes \mathrm{I})\mathrm{R}_{ss'}(\lambda\mu^{-1}), \end{aligned} \tag{129}$$

where $\mathrm{R}_{ss'}$ is the universal R-matrix for $\widehat{osp}_q(1|2)$ which acts in the space $\pi_s(\lambda)\otimes\pi_{s'}(\mu)$. Let's define now the "transfer matrices" which are the quantum counterparts of the traces of monodromy matrices:

$$\mathrm{t}_s^{(q)}(\lambda) = str\pi_s(\lambda)(e^{-i\pi p h_{\alpha_0}}\mathrm{L}^{(q)}). \tag{130}$$

According to the RTT-relation we obtain:

$$[\mathrm{t}_s^{(q)}(\lambda), \mathrm{t}_{s'}^{(q)}(\mu)] = 0. \tag{131}$$

First few orders of expansion in λ^2 of $\mathrm{t}_s^{(q)}(\lambda)$ operators the following fusion relation between transfer matrices can be obtained:

$$\begin{aligned}&\mathrm{t}_s^{(q)}(q^{1/4}\lambda)\mathrm{t}_s^{(q)}(q^{-1/4}\lambda) = \\ &\mathrm{t}_{s+1}^{(q)}(q^{\frac{1}{2\beta^2}}\lambda)\mathrm{t}_{s-1}^{(q)}(q^{\frac{1}{2\beta^2}}\lambda) + \mathrm{t}_s^{(q)}(\lambda).\end{aligned} \tag{132}$$

This result is very similar to that in the $(A_2^{(2)})_q$ case. Such correspondence should not seem to be extraordinary because of the coincidence of their Cartan matrices and similarities in the spectra of representations as we mentioned above.

6 SUSY N=1 KdV: Classical Theory

The SUSY N=1 KdV system can be constructed by means of the Drinfeld-Sokolov reduction applied to the $C(2)^{(2)} = sl(1|2)^{(2)}$ twisted affine superalgebra.

$C(2)$ and $C(2)^{(2)}$ superalgebras

The superalgebra $C(2)$ has six Chevalley generators: h_1, h_2, ${e_1}^{\pm}$, ${e_2}^{\pm}$ (${e_1}^{\pm}$, ${e_2}^{\pm}$ are odd) with the following commutation relations:

$$\begin{aligned}&[h_1, h_2] = 0, \quad [h_1, {e_2}^{\pm}] = \pm {e_2}^{\pm}, \\ &[h_2, {e_1}^{\pm}] = \pm {e_1}^{\pm}, \\ &ad^2_{{e_1}^{\pm}} {e_2}^{\pm} = 0, \quad ad^2_{{e_2}^{\pm}} {e_1}^{\pm} = 0, \\ &[h_\alpha, {e_\alpha}^{\pm}] = 0 \quad (\alpha = 1, 2), \\ &[{e_\beta}^{\pm}, {e_{\beta'}}^{\mp}] = \delta_{\beta,\beta'} h_\beta \quad (\beta, \beta' = 1, 2),\end{aligned} \tag{133}$$

The fundamental 3-dimensional representation is:

$$e_1^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_1^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_2^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

$$h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The following correspondence:

$$\begin{aligned} & e_{\alpha_1} \to \lambda(e_1^+ + e_2^+), \quad e_{-\alpha_1} \to \lambda(e_1^- + e_2^-), \\ & h_{\alpha_1} \to h_1 + h_2, \\ & e_{\alpha_0} \to \lambda(e_1^- - e_2^-), \quad e_{-\alpha_0} \to \lambda(-e_1^+ + e_2^+), \\ & h_{\alpha_0} \to -h_1 - h_2 \end{aligned} \tag{134}$$

give the evaluation representation (we will consider representations only of this type) for the $C(2)^{(2)}$ twisted Kac-Moody superalgebra with Chevalley generators $h_{\alpha_{0,1}}$, $e_{\pm\alpha_1}$, $e_{\pm\alpha_0}$:

$$\begin{aligned} & [h_{\alpha_1}, h_{\alpha_0}] = 0, \quad [h_{\alpha_0}, e_{\pm\alpha_1}] = \mp e_{\pm\alpha_1}, \\ & [h_{\alpha_1}, e_{\pm\alpha_0}] = \mp e_{\pm\alpha_0}, \quad [h_{\alpha_i}, e_{\pm\alpha_i}] = \pm e_{\pm\alpha_i}, \\ & [e_{\pm\alpha_i}, e_{\mp\alpha_j}] = \delta_{i,j} h_{\alpha_i}, \quad (i,j = 0,1), \\ & ad^3_{e_{\pm\alpha_0}} e_{\pm\alpha_1} = 0, \quad ad^3_{e_{\pm\alpha_1}} e_{\pm\alpha_0} = 0 \end{aligned} \tag{135}$$

So, the corresponding $\mathcal{L}$-operator has the following form:

$$\begin{aligned} \tilde{\mathcal{L}}_F \; = \; & D_{u,\theta} - D_{u,\theta}\Phi(h_1 + h_2) \\ - \; & \lambda({e_1}^+ + {e_2}^+ + {e_1}^- - {e_2}^-), \end{aligned} \tag{136}$$

where $D_{u,\theta} = \partial_\theta + \theta\partial_u$ is a superderivative, the variable u lies on a cylinder of circumference 2π, θ is a Grassmann variable, $\Phi(u,\theta) = \phi(u) - \frac{i}{\sqrt{2}}\theta\xi(u)$ is a bosonic superfield. The operator (1) can be considered as more general one, taken in the evaluation representation of $C(2)^{(2)}$:

$$\mathcal{L}_F = D_{u,\theta} - D_{u,\theta}\Phi h_\alpha - (e_{\delta-\alpha} + e_\alpha), \tag{137}$$

The Poisson brackets for the field Φ are:

$$\begin{aligned}&\{D_{u,\theta}\Phi(u,\theta), D_{u',\theta'}\Phi(u',\theta')\} = \\ &D_{u,\theta}(\delta(u-u')(\theta-\theta'))\end{aligned} \tag{138}$$

and the following boundary conditions are imposed on the components of Φ: $\phi(u+2\pi) = \phi(u) + 2\pi i p$, $\xi(u+2\pi) = \pm\xi(u)$. The $\mathcal{L}_F$-operator is written in the Miura form, making a gauge transformation one can obtain a new superfield $\mathcal{U}(u,\theta) \equiv D_{u,\theta}\Phi(u,\theta)\partial_u\Phi(u,\theta) + D^3_{u,\theta}\Phi(u,\theta) = -\theta U(u) - i\alpha(u)/\sqrt{2}$, where U and α generate the superconformal algebra under the Poisson brackets:

$$\begin{aligned}\{U(u), U(v)\} &= \delta'''(u-v) + 2U'(u)\delta(u-v) \\ &+ 4U(u)\delta'(u-v), \\ \{U(u), \alpha(v)\} &= 3\alpha(u)\delta'(u-v) + \alpha'(u)\delta(u-v), \\ \{\alpha(u), \alpha(v)\} &= 2\delta''(u-v) + 2U(u)\delta(u-v).\end{aligned} \tag{139}$$

The involutive family of IM:

$$\begin{aligned}I_1^{(cl)} &= \int \frac{du}{2\pi} U(u), \\ I_3^{(cl)} &= \int \frac{du}{2\pi}(U^2(u) + \alpha(u)\alpha'(u)/2), \\ I_5^{(cl)} &= \int \frac{du}{2\pi}(U^3(u) - (U')^2(u)/2 \\ &- \alpha'(u)\alpha''(u)/4 - \alpha'(u)\alpha(u)U(u)), \\ &\quad . \quad\quad . \quad\quad .\end{aligned} \tag{140}$$

One can obtain an evolution equations; for example, taking I_2 we get the SUSY N=1 KdV equation:

$$\mathcal{U}_t = -\mathcal{U}_{uuu} + 3(\mathcal{U} D_{u,\theta} \mathcal{U})_u \tag{141}$$

and in components:

$$\begin{aligned} U_t &= -U_{uuu} - 6UU_u - \frac{3}{2}\alpha\alpha_{uu}, \\ \alpha_t &= -4\alpha_{uuu} - 3(U\alpha)_u. \end{aligned} \tag{142}$$

As we have noted in the introduction one can show that the IM are invariant under supersymmetry transformation generated by $\int_0^{2\pi} du \alpha(u)$.

In order to construct the monodromy matrix we introduce the $\mathcal{L}_B$-operator, equivalent to the $\mathcal{L}_F$ one:

$$\mathcal{L}_B = \partial_u - \phi'(u) h_{\alpha_1} + (e_{\alpha_1} + e_{\alpha_0} - \frac{i}{\sqrt{2}}\xi h_{\alpha_1})^2 \tag{143}$$

The equivalence can be easily established if one considers the linear problem associated with the $\mathcal{L}_F$-operator: $\mathcal{L}_F \chi(u,\theta) = 0$ (we consider this operator acting in some representation of $C(2)^{(2)}$ and $\chi(u,\theta)$ is the vector in this representation). Then, expressing $\chi(u,\theta)$ in components: $\chi(u,\theta)=\chi_0(u) + \theta\chi_1(u)$, we find: $\mathcal{L}_B\chi_0 = 0$ and $\chi_1 = (e_{\alpha_1} + e_{\alpha_0} - \frac{i}{\sqrt{2}}\xi h_{\alpha_1})\chi_0$.

The formal solution to the equation $\mathcal{L}_B\chi_0 = 0$ can be written in the following way:

$$\begin{aligned} \chi_0(u) &= e^{\phi(u) h_{\alpha_1}} P \exp \int_0^u du' (\frac{i}{\sqrt{2}}\xi(u') e^{-\phi(u')} e_{\alpha_1} \\ &- \frac{i}{\sqrt{2}}\xi(u') e^{\phi(u')} e_{\alpha_0} - e_{\alpha_1}^2 e^{-2\phi(u')} \\ &- e_{\alpha_0}^2 e^{2\phi(u')} - [e_{\alpha_1}, e_{\alpha_0}])\eta, \end{aligned} \tag{144}$$

Therefore we can define the monodromy matrix in the following way:

$$
\begin{aligned}
\mathrm{M} &= e^{2\pi i p h_{\alpha_1}} P \exp \int_0^{2\pi} du \Big(\frac{i}{\sqrt{2}} \xi(u) e^{-\phi(u)} e_{\alpha_1} \\
&- \frac{i}{\sqrt{2}} \xi(u) e^{\phi(u)} e_{\alpha_0} - e_{\alpha_1}^2 e^{-2\phi(u)} \\
&- e_{\alpha_0}^2 e^{2\phi(u)} - [e_{\alpha_1}, e_{\alpha_0}] \Big).
\end{aligned} \tag{145}
$$

Introducing then the auxiliary L-operators:
$\mathrm{L} = e^{-\pi i p h_{\alpha_1}} \mathrm{M}$ we find that in the evaluation representation (when λ, the spectral parameter appears) the familiar Poisson bracket relation is satisfied:

$$
\{\mathrm{L}(\lambda) \otimes_, \mathrm{L}(\mu)\} = [\mathrm{r}(\lambda\mu^{-1}), \mathrm{L}(\lambda) \otimes \mathrm{L}(\mu)], \tag{146}
$$

where $\mathrm{r}(\lambda\mu^{-1})$ is a classical trigonometric $C(2)^{(2)}$ r-matrix. From this relation one obtains that the supertraces of monodromy matrices $\mathrm{t}(\lambda) = str\mathrm{M}(\lambda)$ commute under the Poisson bracket: $\{\mathrm{t}(\lambda), \mathrm{t}(\mu)\} = 0$. Expanding $\log(\mathrm{t}(\lambda))$ in λ in the evaluation representation corresponding to the defining 3-dimensional representation of $C(2)$ $\pi_{1/2}$ we find:

$$
\log(\mathrm{t}_{1/2}(\lambda)) = -\sum_{n=1}^{\infty} c_n I_{2n-1}^{(cl)} \lambda^{-4n+2}, \qquad \lambda \to \infty \tag{147}
$$

where $c_1 = \frac{1}{2}$, $c_n = \frac{(2n-3)!!}{2^n n!}$ for $n > 1$.

7 SUSY N=1 KdV: Quantum Monodromy Matrix and RTT-relation

In this part of the work we will consider the quantum $C_q(2)^{(2)}$ R-matrix and show that the vertex operator representation of the lower Borel subalgebra of $C_q(2)^{(2)}$ allows to represent this R-matrix in the P-exponent like form which in the classical limit coincide with the auxiliary L-operator.

Universal R-matrix for $C_q(2)^{(2)}$ twisted affine superalgebra

The $C_q(2)^{(2)}$ quantum affine superalgebra has the following commutation relations:

$$\begin{aligned}
&[h_{\alpha_0}, h_{\alpha_1}] = 0, \quad [h_{\alpha_0}, e_{\pm\alpha_1}] = \mp e_{\pm\alpha_1}, \\
&[h_{\alpha_1}, e_{\pm\alpha_0}] = \mp e_{\pm\alpha_0}, \quad [h_{\alpha_i}, e_{\pm\alpha_i}] = \pm e_{\pm\alpha_i} \quad (i = 0, 1), \\
&[e_{\pm\alpha_i}, e_{\mp\alpha_j}] = \delta_{i,j}[h_{\alpha_i}] \quad (i, j = 0, 1), \\
&[e_{\pm\alpha_1}, [e_{\pm\alpha_1}, [e_{\pm\alpha_1}, e_{\pm\alpha_0}]_q]_q]_q = 0, \\
&[[[e_{\pm\alpha_1}, e_{\pm\alpha_0}]_q, e_{\pm\alpha_0}]_q, e_{\pm\alpha_0}]_q = 0,
\end{aligned} \tag{148}$$

where $[h] = \frac{q^h - q^{-h}}{q - q^{-1}}$, $p(h_{\alpha_{0,1}}) = 0$, $p(e_{\pm\alpha_{0,1}}) = 1$ and q-supercommutator is defined in the following way: $[e_\gamma, e_{\gamma'}]_q \equiv e_\gamma e_{\gamma'} - (-1)^{p(e_\gamma)p(e_{\gamma'})} q^{(\gamma,\gamma')} e_{\gamma'} e_\gamma$, $q = e^{i\pi\frac{\beta^2}{2}}$. The corresponding coproducts are:

$$\begin{aligned}
\Delta(h_{\alpha_j}) &= h_{\alpha_j} \otimes 1 + 1 \otimes h_{\alpha_j}, \\
\Delta(e_{\alpha_j}) &= e_{\alpha_j} \otimes q^{h_{\alpha_j}} + 1 \otimes e_{\alpha_j}, \\
\Delta(e_{-\alpha_j}) &= e_{-\alpha_j} \otimes 1 + q^{-h_{\alpha_j}} \otimes e_{-\alpha_j}.
\end{aligned} \tag{149}$$

The associated universal R-matrix can be expressed in such a way:

$$\mathrm{R} = K\bar{R} = KR_+R_0R_-, \tag{150}$$

where

$$\begin{aligned}
&K = q^{h_\alpha \otimes h_\alpha}, \quad R_+ = \prod_{n\geq 0}^{\rightarrow} R_{n\delta+\alpha}, \quad R_- = \prod_{n\geq 1}^{\leftarrow} R_{n\delta-\alpha}, \\
&R_0 = \exp((q - q^{-1}) \sum_{n>0} d(n) e_{n\delta} \otimes e_{-n\delta}).
\end{aligned} \tag{151}$$

and $\bar{\mathrm{R}}$ is usually called "reduced" universal R-matrix. Here

$$\begin{aligned}
&R_\gamma = \exp_{(-q^{-1})}(A(\gamma)(q-q^{-1})(e_\gamma \otimes e_{-\gamma})), \qquad (152)\\
&d(n) = \frac{n(q-q^{-1})}{q^n - q^{-n}},\\
&A(\gamma) = \{(-1)^n \quad if \quad \gamma = n\delta + \alpha; (-1)^{n-1}\\
&\quad if \quad \gamma = n\delta - \alpha\}.
\end{aligned}$$

The generators $e_{n\delta}$, $e_{n\delta\pm\alpha}$ are defined via the q-commutators of Chevalley generators, for example: $e_\delta = [e_{\alpha_0}, e_{\alpha_1}]_{q^{-1}}$ and $e_{-\delta} = [e_{-\alpha_1}, e_{-\alpha_0}]_q$. The elements $e_{n\delta\pm\alpha}$ are expressed as multiple commutators of e_δ with corresponding Chevalley generators, $e_{n\delta}$ ones have more complicated form.

The reduced R-matrix $\bar{R} \equiv K^{-1}\mathrm{R}$ can be rewritten as $\bar{R}(\bar{e}_{\alpha_i}, \bar{e}_{-\alpha_i})$, where $\bar{e}_{\alpha_i} = e_{\alpha_i} \otimes 1$ and $\bar{e}_{-\alpha_i} = 1 \otimes e_{-\alpha_i}$, because it is represented as power series of these elements. Let's introduce then the vertex operators:

$$\begin{aligned}
V_\pm &= \frac{1}{q^{-1}-q}\int d\theta \int_{u_2}^{u_1} du : e^{\pm\Phi} \qquad (153)\\
&:= \frac{1}{q^{-1}-q}\int_{u_2}^{u_1} du W_\pm(u),
\end{aligned}$$

where $2\pi \geq u_1 \geq \ u_2 \geq 0$, $\Phi = \phi(u) - \frac{i}{\sqrt{2}}\theta\xi(u)$ is a superfield. One can show via the standard contour technique that these operators satisfy the same commutation relations as $e_{-\alpha_1}$, $e_{-\alpha_0}$ correspondingly.

Then, as in the bosonic case one can show, using the fundamental property of the universal R-matrix: $(I \otimes \Delta)\mathrm{R} = \mathrm{R}^{13}\mathrm{R}^{12}$ that the reduced R-matrix has the following property:

$$\bar{R}(\bar{e}_{\alpha_i}, e'_{-\alpha_i} + e''_{-\alpha_i}) = \bar{R}(\bar{e}_{\alpha_i}, e'_{-\alpha_i})\bar{R}(\bar{e}_{\alpha_i}, e''_{-\alpha_i}), \qquad (154)$$

where $e'_{-\alpha_i} = 1 \otimes q^{-h_{\alpha_i}} \otimes e_{-\alpha_i}$, $e''_{-\alpha_i} = 1 \otimes e_{-\alpha_i} \otimes 1$, $\bar{e}_{\alpha_i} = e_{\alpha_i} \otimes 1 \otimes 1$. The commutation relations between them are:

$$\begin{aligned} e'_{-\alpha_i}\bar{e}_{\alpha_j} &= -\bar{e}_{\alpha_j}e'_{-\alpha_i}, \quad e''_{-\alpha_i}\bar{e}_{\alpha_j} = -\bar{e}_{\alpha_j}e''_{-\alpha_i}, \\ e'_{-\alpha_i}e''_{-\alpha_j} &= -q^{b_{ij}}e''_{-\alpha_j}e'_{-\alpha_i}, \end{aligned} \tag{155}$$

where b_{ij} is the symmetric matrix with the following elements: $b_{00} = b_{11} = -b_{01} = 1$. Now, denoting $\bar{\mathrm{L}}^{(q)}(u_2, u_1)$ the reduced R-matrix with $e_{-\alpha_i}$ represented by $V_\pm$ and the commutation relations of $W_\pm$:

$$\begin{aligned} & W_{\sigma_1}(u_1)W_{\sigma_2}(u_2) = -q^{\sigma_1\sigma_2}W_{\sigma_2}(u_2)W_{\sigma_1}(u_1), \\ & q = e^{\frac{\pi i \beta^2}{2}}, \quad u_1 > u_2 \\ & PW_\pm(u) = W_\pm(u)(P \pm \frac{\beta^2}{2}) \end{aligned} \tag{156}$$

we find using the above property of $\bar{R}$ with $e'_{-\alpha_i}$ replaced by appropriate vertex operators:
$\bar{\mathrm{L}}^{(q)}(u_3, u_1) = \bar{\mathrm{L}}^{(q)}(u_2, u_1)\bar{\mathrm{L}}^{(q)}(u_3, u_2)$ with $u_1 \geq u_2 \geq u_3$. So, $\bar{\mathrm{L}}^{(q)}$ has the property of P-exponent. But because of singularities in the operator products of vertex operators it can not be written in the usual P-ordered form. Thus, we propose a new notion, the quantum P-exponent:

$$\begin{aligned} \bar{\mathrm{L}}^{(q)}(u_1, u_2) &= Pexp^{(q)} \int_{u_2}^{u_1} du(W_-(u)e_{\alpha_1} \\ &+ W_+(u)e_{\alpha_0}). \end{aligned} \tag{157}$$

Introducing new object: $\mathrm{L}^{(q)} \equiv e^{i\pi P h_{\alpha_1}}\bar{\mathrm{L}}^{(q)}(0, 2\pi)$, which coincides with R-matrix with $1 \otimes h_{\alpha_1}$ replaced by $2P/\beta^2$ and $1 \otimes e_{-\alpha_1}$, $1 \otimes e_{-\alpha_0}$ replaced by V_- and V_+ (with integration from 0 to 2π) correspondingly, we find that it satisfies the RTT-relation:

$$\begin{aligned} & \mathrm{R}(\lambda\mu^{-1})(\mathrm{L}^{(q)}(\lambda) \otimes \mathrm{I})(\mathrm{I} \otimes \mathrm{L}^{(q)}(\mu)) \\ & = (\mathrm{I} \otimes \mathrm{L}^{(q)}(\mu))(\mathrm{L}^{(q)}(\lambda) \otimes \mathrm{I})\mathrm{R}(\lambda\mu^{-1}), \end{aligned} \tag{158}$$

where the dependence on λ, μ means that we are considering $\mathrm{L}^{(q)}$-operators in the evaluation representation of $C_q(2)^{(2)}$. Thus the supertraces of "shifted" $\mathrm{L}^{(q)}$-operators, the transfer matrices
$\mathrm{t}^{(q)}(\lambda) \equiv str(e^{i\pi P h_{\alpha_1}}\mathrm{L}^{(q)}(\lambda))$ commute: $[\mathrm{t}^{(q)}(\lambda), \mathrm{t}^{(q)}(\mu)] = 0$, giving the quantum integrability.

Now we show that in the classical limit ($q \to 1$) the $\mathrm{L}^{(q)}$-operator will give the classical auxiliary L-matrix. We will use the P-exponent property of $\bar{\mathrm{L}}^{(q)}(0, 2\pi)$. Let's decompose $\bar{\mathrm{L}}^{(q)}(0, 2\pi)$ in the following way:
$\bar{\mathrm{L}}^{(q)}(0, 2\pi) = \lim_{N\to\infty} \prod_{m=1}^{N} \bar{\mathrm{L}}^{(q)}(x_{m-1}, x_m)$, where we divided the interval $[0, 2\pi]$ into infinitesimal intervals $[x_m, x_{m+1}]$ with $x_{m+1} - x_m = \epsilon = 2\pi/N$. Let's find the terms that can give contribution of the first order in ϵ in $\bar{\mathrm{L}}^{(q)}(x_{m-1}, x_m)$. In this analysis we will need again the operator product expansion of vertex operators:

$$\xi(u)\xi(u') = -\frac{i\beta^2}{(iu - iu')} + \sum_{k=1}^{\infty} c_k(u)(iu - iu')^k,$$
$$: e^{a\phi(u)} :: e^{b\phi(u')} := (iu - iu')^{\frac{ab\beta^2}{2}} \cdot \tag{159}$$
$$(: e^{(a+b)\phi(u)} : + \sum_{k=1}^{\infty} d_k(u)(iu - iu')^k),$$

where $c_k(u)$ and $d_k(u)$ are operator-valued functions of u. Now one can see that only two types of terms can give the contribution of the order ϵ in $\bar{\mathrm{L}}^{(q)}(x_{m-1}, x_m)$ when $q \to 1$. The first type consists of operators of the first order in $V_\pm$ and the second type is formed by the operators, quadratic in $V_\pm$, which give contribution of the order $\epsilon^{1\pm\beta^2}$ by virtue of operator product expansion. Let's look on the terms of the second type in detail. At first we consider the terms appearing from the R_0-part of R-matrix, represented by vertex operators:

$$\frac{e_\delta}{2(q-q^{-1})}\Big(\int_{x_{m-1}}^{x_m} du_1 :e^{-\phi}: \xi(u_1-i0)$$
$$\int_{x_{m-1}}^{x_m} du_2 :e^{\phi}: \xi(u_2+i0)+$$
$$q^{-1}\int_{x_{m-1}}^{x_m} du_2 :e^{\phi}: \xi(u_2-i0)$$
$$\int_{x_{m-1}}^{x_m} du_1 :e^{-\phi}: \xi(u_1+i0)\Big) \tag{160}$$

Neglecting the terms, which give rise to $O(\epsilon^2)$ contribution, we obtain, using the operator products of vertex operators:

$$\frac{e_\delta}{2(q-q^{-1})}\Big(\int_{x_{m-1}}^{x_m} du_1 \int_{x_{m-1}}^{x_m} du_2 \frac{-i\beta^2}{(i(u_1-u_2-i0))^{\frac{\beta^2}{2}+1}}+$$
$$q^{-1}\int_{x_{m-1}}^{x_m} du_2 \int_{x_{m-1}}^{x_m} du_1 \frac{-i\beta^2}{(i(u_2-u_1-i0))^{\frac{\beta^2}{2}+1}}\Big) \tag{161}$$

In the $\beta^2 \to 0$ limit we get:

$$\frac{[e_{\alpha_1},e_{\alpha_0}]}{2i\pi}\int_{x_{m-1}}^{x_m} du_1 \int_{x_{m-1}}^{x_m} du_2 \Big(\frac{1}{u_1-u_2+i0}-\frac{1}{u_1-u_2-i0}\Big)$$

which, by the well known formula: $\frac{1}{x+i0} = P\frac{1}{x} - i\pi\delta(x)$ gives:

$$-\int_{x_{m-1}}^{x_m} du [e_{\alpha_1},e_{\alpha_0}]. \tag{162}$$

Another terms arise from the R_+ and R_- parts of R-matrix and are very similar to each other:

$$\frac{e_{\alpha_0}^2}{2(2)_{(-q^{-1})}}\int_{x_{m-1}}^{x_m} du_1 :e^{\phi}: \xi(u_1-i0) \tag{163}$$

$$\int_{x_{m-1}}^{x_m} du_2 :e^{\phi}: \xi(u_2+i0),$$

$$\frac{e_{\alpha_1}^2}{2(2)_{(-q^{-1})}} \int_{x_{m-1}}^{x_m} du_1 :e^{-\phi}: \xi(u_1-i0)$$

$$\int_{x_{m-1}}^{x_m} du_2 :e^{-\phi}: \xi(u_2+i0).$$

The integrals can be reduced to the ordered ones:

$$\frac{e_{\alpha_0}^2}{2} \int_{x_{m-1}}^{x_m} du_1 :e^{\phi}: \xi(u_1) \int_{x_{m-1}}^{u_1} du_2 :e^{\phi}: \xi(u_2), \tag{164}$$

$$\frac{e_{\alpha_1}^2}{2} \int_{x_{m-1}}^{x_m} du_1 :e^{-\phi}: \xi(u_1) \int_{x_{m-1}}^{u_1} du_2 :e^{-\phi}: \xi(u_2).$$

From super-KdV case we know that their contribution (of order ϵ) in the classical limit is:

$$-e_{\alpha_0}^2 \int_{x_{m-1}}^{x_m} due^{2\phi(u)}, \quad -e_{\alpha_1}^2 \int_{x_{m-1}}^{x_m} due^{-2\phi(u)}. \tag{165}$$

Gathering now all the terms of order ϵ we find:

$$\begin{aligned} \bar{\mathrm{L}}^{(1)}(x_{m-1},x_m) = 1 + \int_{x_{m-1}}^{x_m} du\Big(&\frac{i}{\sqrt{2}}\xi(u)e^{-\phi(u)}e_{\alpha_1} \\ -&\frac{i}{\sqrt{2}}\xi(u)e^{\phi(u)}e_{\alpha_0} - e_{\alpha_1}^2 e^{-2\phi(u)} \\ -&e_{\alpha_0}^2 e^{2\phi(u)} - [e_{\alpha_1}, e_{\alpha_0}]\Big) + O(\epsilon^2) \end{aligned} \tag{166}$$

and collecting all $\bar{\mathrm{L}}^{(1)}(x_{m-1},x_m)$ we find that $\bar{\mathrm{L}}^{(1)}$=$e^{-i\pi p h_{\alpha_1}}\mathrm{L}$. Therefore $\mathrm{L}^{(1)}$=L.

8 SUSY N=1 KdV: The Construction of the Q-operator and the Fusion Relations

Evaluation representations of $C_q(2)^{(2)}$

We have found the evaluation representations ρ_s of $C_q(2)^{(2)}$ which we have called "$osp_q(1|2)$-induced" because each triple h_{α_i}, e_{α_i}, $e_{-\alpha_i}$ forms an $osp_q(1|2)$ subalgebra and their representations are irreducible in ρ_s. The representations are $4s+1$-dimensional, where s is an integer or half-integer. Labeling the vectors in the representation space as $|s,l\rangle$ where $l=-s,-s-1/2,..,s$ we can write an explicit formulae for the action of the Chevalley generators on these vectors:

$$
\begin{aligned}
&h_{\alpha_1}|s,l\rangle = -h_{\alpha_0}|s,l\rangle = 2l|s,l\rangle \\
&\lambda^{-1}e_{\alpha_1}|s,l\rangle = c[s-l][s+l+1/2]^{+}|s,l+1/2\rangle, \\
&\quad s-l\in\mathbb{Z} \\
&\lambda^{-1}e_{\alpha_1}|s,l\rangle = c[s-l]^{+}[s+l+1/2]|s,l+1/2\rangle, \\
&\quad s-l\in\mathbb{Z}+1/2 \\
&\lambda e_{-\alpha_1}|s,l\rangle = |s,l-1/2\rangle, \quad s-l\in\mathbb{Z} \\
&\lambda e_{-\alpha_1}|s,l\rangle = -|s,l-1/2\rangle, \quad s-l\in\mathbb{Z}+1/2 \\
&\lambda^{-1}e_{\alpha_0}|s,l\rangle = |s,l-1/2\rangle, \quad s-l\in\mathbb{Z} \\
&\lambda^{-1}e_{\alpha_0}|s,l\rangle = |s,l-1/2\rangle, \quad s-l\in\mathbb{Z}+1/2 \\
&\lambda e_{-\alpha_0}|s,l\rangle = c[s-l][s+l+1/2]^{+}|s,l+1/2\rangle, \\
&\quad s-l\in\mathbb{Z} \\
&\lambda e_{-\alpha_0}|s,l\rangle = -c[s-l]^{+}[s+l+1/2]|s,l+1/2\rangle, \\
&\quad s-l\in\mathbb{Z}+1/2,
\end{aligned}
\tag{167}
$$

where $[x]=\frac{q^x-q^{-x}}{q-q^{-1}}$, $[x]^{+}=\frac{q^x+q^{-x}}{q+q^{-1}}$, $c=([1/2]^{+})^{-1}$.

We also note (this will be important in taking the supertrace) that we choose the highest weight $|s,s\rangle$ to be even.

The representations ρ_s^+ of $C_q(2)^{(2)}$, which play also a crucial role in the construction of the Q-operators are infinite-dimensional and their explicit form is similar to that of (167) but with l going from s to $-\infty$. From a point of view of the $osp_q(1|2)$ subalgebra, generated by the triple $\{h_{\alpha_1}, e_{\alpha_1}, e_{-\alpha_1}\}$ these representation look like Verma module for $osp_q(1|2)$ with the highest weight $|s,s\rangle$. One can easily obtain that ρ_s representations arise from the factor of two infinite-dimensional representations of ρ^+-type: $\rho_s^+/\rho^+_{-s-\frac{1}{2}}$.

To simplify the notation we denote $\rho_s(\lambda)(\mathrm{L})$ as $\mathrm{L}_s(\lambda)$ and $\rho_s^+(\lambda)(\mathrm{L})$ as $\mathrm{L}_s^+(\lambda)$.

In order to construct the Q-operator we introduce the auxiliary object, the super q-oscillator algebra with the following commutation relations:

$$[H, \varepsilon_\pm] = \pm\varepsilon_\pm,\ q^{\frac{1}{2}}\varepsilon_+\varepsilon_- + q^{-\frac{1}{2}}\varepsilon_-\varepsilon_+ = \frac{1}{q-q^{-1}}. \tag{168}$$

Using these relations it is possible to build realization for the upper Borel subalgebra b_+ generated by $\{h_{\alpha_1}, h_{\alpha_0}, e_{\alpha_1}, e_{\alpha_0}\}$:

$$\begin{aligned} &\chi_\pm(\lambda): \quad e_{\alpha_1} \to \lambda\varepsilon_\pm, \quad e_{\alpha_0} \to \mp\lambda\varepsilon_\mp, \\ &h_{\alpha_1} = -h_{\alpha_0} \to \pm H \end{aligned} \tag{169}$$

and we denote $\chi_\pm(\lambda)(\mathrm{L})$ as $\mathrm{L}_\pm(\lambda)$. Next, following we define two operators which do not depend on representation of oscillator algebra but depend on (and completely determined by) the commutation relations and cyclic property of the supertrace:

$$\mathrm{A}_\pm(\lambda) = Z_\pm^{-1}(P) str_{\chi_\pm(\lambda)}(e^{\pm\pi i PH}\mathrm{L}_\pm(\lambda)), \tag{170}$$

where

$$Z_\pm^{-1}(P) = str_{\chi_\pm(\lambda)}(e^{\pm 2\pi i PH}) \tag{171}$$

Multiplying two A operators we arrive to such a result:

$$\mathrm{A}_+(q^{s+\frac{1}{4}}\lambda)\mathrm{A}_-(q^{-s-\frac{1}{4}}\lambda) = 2cos(\pi P)e^{-4\pi i P(s+\frac{1}{4})}\mathrm{t}_s^+(\lambda), \tag{172}$$

where $\mathrm{t}_s^+(\lambda)$ is the supertrace of the monodromy matrix in $\rho_s^+(\lambda)$ representation. Defining then the Q-operators:

$$\mathrm{Q}_\pm(\lambda) = \lambda^{\pm\frac{2P}{\beta^2}}\mathrm{A}_\pm(\lambda) \tag{173}$$

we find that the expression (172) can be written in such a form:

$$\mathrm{Q}_+(q^{s+\frac{1}{4}}\lambda)\mathrm{Q}_-(q^{-s-\frac{1}{4}}\lambda) = 2cos(\pi P)\mathrm{t}_s^+(\lambda). \tag{174}$$

Now let's recall that $\mathrm{t}_s^+(\lambda)$ corresponds to infinite-dimensional representation ρ_s^+. In order to obtain the transfer matrix corresponding to the finite-dimensional representation ρ_s one should consider factor-representation $\rho_s^+/\rho_{-s-\frac{1}{2}}^+ \cong \rho_s$. Thus one obtains:

$$\mathrm{t}_s^+(\lambda) + \mathrm{t}_{-s-\frac{1}{2}}^+(\lambda) = \mathrm{t}_s(\lambda). \tag{175}$$

Sign + in this formula appears because of taking a supertrace. Really, $\rho_{-s-\frac{1}{2}}^+$ subrepresentation in ρ_s^+ representation space has a highest weight vector which is odd (though our convention is that highest weight vector in ρ_j^+ representation is even. That's why we have sign + instead of the usual −.

Rewriting (175) in terms of the $\mathrm{Q}_\pm$ operators we find:

$$2cos(\pi P)\mathrm{t}_s(\lambda) = \mathrm{Q}_+(q^{s+\frac{1}{4}}\lambda)\mathrm{Q}_-(q^{-s-\frac{1}{4}}\lambda) + \mathrm{Q}_+(q^{-s-\frac{1}{4}}\lambda)\mathrm{Q}_-(q^{s+\frac{1}{4}}\lambda). \tag{176}$$

Considering the case of 1-dimensional representation of $C_q(2)^{(2)}$ ($s = 0$) we obtain a quantum super-Wronskian (qsW) relation:

$$2cos(\pi P) \;=\; Q_+(q^{\frac{1}{4}}\lambda)Q_-(q^{-\frac{1}{4}}\lambda) + Q_+(q^{-\frac{1}{4}}\lambda)Q_-(q^{\frac{1}{4}}\lambda). \tag{177}$$

In order to construct the Baxter type relation we introduce the auxiliary "quarter" operators:

$$2cos(\pi P)t_{\frac{k}{4}}(\lambda) = Q_+(q^{\frac{k}{4}+\frac{1}{4}}\lambda)Q_-(q^{-\frac{k}{4}-\frac{1}{4}}\lambda) - Q_+(q^{-\frac{k}{4}-\frac{1}{4}}\lambda)Q_-(q^{\frac{k}{4}+\frac{1}{4}}\lambda) \tag{178}$$

for odd integer k. Let's consider the $t_{\frac{1}{4}}(\lambda)$ operator and multiply it on $Q_\pm(\lambda)$ operators. Using (177) one can get the following relation of Baxter type, which is similar to that one obtained in $A_1^{(1)}$ (usual KdV) case:

$$t_{\frac{1}{4}}(\lambda)Q_\pm(\lambda) = \pm Q_\pm(q^{\frac{1}{2}}\lambda) \mp Q_\pm(q^{-\frac{1}{2}}\lambda). \tag{179}$$

The main difference is that unlike the $A_1^{(1)}$ case we are not seeking for the eigenvalues of $t_{\frac{1}{4}}(\lambda)$ operator but we are interested in the operator corresponding to the 3-dimensional representation, $t_{\frac{1}{2}}(\lambda)$, because we know that it is a generating function of the local IM. Thus we are looking for the relation of Baxter type which includes this operator.

In order to do this we need to proceed through some additional steps. First, let's unify the expressions (176), (178):

$$2cos(\pi P)t_s(\lambda) = Q_+(q^{s+\frac{1}{4}}\lambda)Q_-(q^{-s-\frac{1}{4}}\lambda) - (-1)^{4s+1}Q_+(q^{-s-\frac{1}{4}}\lambda)Q_-(q^{s+\frac{1}{4}}\lambda), \tag{180}$$

where $s \in \mathbb{Z}/4$, $s \geq 0$. Using the qsW relation it is not hard to obtain the expressions for $t_s(\lambda)$ operators only in terms of $Q_+(\lambda)$ or $Q_-(\lambda)$:

$$t_s(\lambda) = Q_\pm(q^{s+\frac{1}{4}}\lambda)Q_\pm(q^{-s-\frac{1}{4}}\lambda)\sum_{k=-s}^{s} \cdot \frac{(-1)^{2(k\pm s)}}{Q_\pm(q^{k+\frac{1}{4}}\lambda)Q_\pm(q^{k-\frac{1}{4}}\lambda)} \tag{181}$$

For example for $t_{\frac{1}{2}}(\lambda)$ we obtain the "triple relation":

$$t_{\frac{1}{2}}(\lambda) = \frac{Q_\pm(q^{-\frac{3}{4}}\lambda)}{Q_\pm(q^{\frac{1}{4}}\lambda)} - \frac{Q_\pm(q^{\frac{3}{4}}\lambda)Q_\pm(q^{-\frac{3}{4}}\lambda)}{Q_\pm(q^{\frac{1}{4}}\lambda)Q_\pm(q^{-\frac{1}{4}}\lambda)} + \frac{Q_\pm(q^{\frac{3}{4}}\lambda)}{Q_\pm(q^{-\frac{1}{4}}\lambda)} \tag{182}$$

Due to the unifying relation (180) and using again (177) we find the fusion relations:

$$t_j(q^{\frac{1}{4}}\lambda)t_j(q^{-\frac{1}{4}}\lambda) = t_{j+\frac{1}{4}}(\lambda)t_{j-\frac{1}{4}}(\lambda) + (-1)^{4j}, \quad j = 0, \frac{1}{4}, \frac{1}{2}, 1, \ldots \tag{183}$$

Let's write the fusion relations for $s = \frac{1}{4}$ with λ multiplied on $q^{\frac{1}{4}}$:

$$t_{\frac{1}{2}}(q^{\frac{1}{4}}\lambda) = t_{\frac{1}{4}}(q^{\frac{1}{2}}\lambda)t_{\frac{1}{4}}(\lambda) + 1. \tag{184}$$

Multiplying both sides on $Q_\pm(\lambda)$ we obtain second relation of Baxter type:

$$Q_\pm(q\lambda) \mp t_{\frac{1}{4}}(q^{\frac{1}{2}}\lambda)Q_\pm(q^{-\frac{1}{2}}\lambda) = t_{\frac{1}{2}}(q^{\frac{1}{4}}\lambda)Q_\pm(\lambda). \tag{185}$$

9 Truncation of the Fusion Relations

Though the Baxter Q-operator method is very powerful there exists another approach which is useful for rational values of the central charge, it is the SCFT counterpart of Baxter's commuting transfer matrix method. In this section we will show that when q is the root of unity the system of fusion relations becomes finite. So, let's put $q^N = \pm 1$, $N \in \mathbb{Z}$, $N > 0$. Then

$$2cos(\pi P)\mathrm{t}_{\frac{N}{2}-\frac{1}{2}-s}(\lambda q^{\frac{N}{2}}) = \\ \mathrm{Q}_+(q^{N-\frac{1}{2}-s}\lambda)\mathrm{Q}_-(q^{\frac{1}{4}+s}\lambda) - \\ (-1)^{4s+1}\mathrm{Q}_+(q^{\frac{1}{4}+s}\lambda)\mathrm{Q}_-(q^{N-\frac{1}{2}-s}\lambda). \tag{186}$$

Here $s \in \mathbb{Z}/4$, $s \leq \frac{N}{2} - \frac{1}{2}$. Note also that

$$\mathrm{Q}_\pm(q^N\lambda) = e^{\pm\pi i N P}\mathrm{Q}_\pm(\lambda). \tag{187}$$

With the use of such a transform one obtains:

$$2cos(\pi P)\mathrm{t}_{\frac{N}{2}-\frac{1}{2}-s}(\lambda q^{\frac{N}{2}}) = \\ e^{\pi i N P}\mathrm{Q}_+(q^{-\frac{1}{4}-s}\lambda)\mathrm{Q}_-(q^{\frac{1}{4}+s}\lambda) - \\ (-1)^{4s+1}e^{-\pi i N P}\mathrm{Q}_+(q^{\frac{1}{4}+s}\lambda)\mathrm{Q}_-(q^{-\frac{1}{4}-s}\lambda). \tag{188}$$

Recalling the formula (176) for t_s we arrive to the following expression:

$$\mathrm{t}_{\frac{N}{2}-\frac{1}{2}-s}(\lambda q^{\frac{N}{2}}) + (-1)^{4s+1}e^{\pi i N P}\mathrm{t}_s(\lambda) = \\ (-1)^{4s+1}i\frac{sin(\pi N P)}{cos(\pi P)}\mathrm{Q}_+(q^{\frac{1}{4}+s}\lambda)\mathrm{Q}_-(q^{-\frac{1}{4}-s}\lambda), \\ s = 0, \frac{1}{4}, \frac{1}{2}, ..., \frac{N}{2} - \frac{1}{2}. \tag{189}$$

For example for $s = 0$ we obtain:

$$\mathrm{t}_{\frac{N}{2}-\frac{1}{2}}(\lambda q^{\frac{N}{2}}) - e^{\pi i N P} = \\ -i\frac{sin(\pi N P)}{cos(\pi P)}\mathrm{Q}_+(q^{\frac{1}{4}}\lambda)\mathrm{Q}_-(q^{-\frac{1}{4}}\lambda). \tag{190}$$

Using the basic formula (176) and property (187) $t_{\frac{N}{2}-\frac{1}{4}}(\lambda)$ and for $t_{\frac{N}{2}}(\lambda)$ one arrives to the following relations:

$$t_{\frac{N}{2}-\frac{1}{4}}(\lambda) = i\frac{sin(\pi NP)}{cos(\pi P)}Q_+(q^{\frac{N}{2}}\lambda)Q_-(q^{\frac{N}{2}}\lambda), \tag{191}$$

$$\begin{gathered} t_{\frac{N}{2}}(\lambda q^{\frac{N}{2}}) - e^{-\pi iNP} = \\ i\frac{sin(\pi NP)}{cos(\pi P)}Q_+(q^{\frac{1}{4}}\lambda)Q_-(q^{-\frac{1}{4}}\lambda). \end{gathered} \tag{192}$$

Comparing (192) and (190) we obtain the following simple formula:

$$t_{\frac{N}{2}}(\lambda) + t_{\frac{N}{2}-\frac{1}{2}}(\lambda) = 2cos(\pi NP). \tag{193}$$

In the case when $p = \frac{l+1}{N}$, where $l \geq 0$, $l \in \mathbb{Z}$ there exist additional number of truncations:

$$\begin{gathered} t_{\frac{N}{2}-\frac{1}{4}}(\lambda) = 0, \quad t_{\frac{N}{2}}(\lambda) = t_{\frac{N}{2}-\frac{1}{2}}(\lambda) = (-1)^{l+1}, \\ t_{\frac{N}{2}-\frac{1}{2}-s}(\lambda q^{\frac{N}{2}}) = (-1)^{4s}t_s(\lambda)(-1)^{l+1}. \end{gathered} \tag{194}$$

The relation (193) allows us to to rewrite the fusion relation system in the Thermodynamic Bethe Ansatz Equations of $D_{N'}$ type as in $A_1^{(1)}$ case:

$$\begin{aligned} & Y_s(\theta + \frac{i\pi\xi}{2})Y_s(\theta - \frac{i\pi\xi}{2}) = \\ & (1 + Y_{s+\frac{1}{2}}(\theta))(1 + Y_{s-\frac{1}{2}}(\theta)), \\ & s = \frac{1}{2}, 1, ..., \frac{N'}{2} - \frac{3}{2}, \\ & Y_{\frac{N'}{2}-1}(\theta + \frac{i\pi\xi}{2})Y_{\frac{N'}{2}-1}(\theta - \frac{i\pi\xi}{2}) = (1 + Y_{\frac{N'}{2}-\frac{3}{2}}(\theta)) \\ & (1 + e^{\pi iP\frac{N'}{2}}\bar{Y}(\theta))(1 + e^{-\pi iP\frac{N'}{2}}\bar{Y}(\theta)), \\ & \bar{Y}(\theta + \frac{i\pi\xi}{2})\bar{Y}(\theta - \frac{i\pi\xi}{2}) = 1 + Y_{\frac{N'}{2}-1}(\theta). \end{aligned} \tag{195}$$

with the use of identification:

$$
\begin{aligned}
Y_{2s}(\theta) &= t_{s+\frac{1}{4}}(\lambda)t_{s-\frac{1}{4}}(\lambda)(-1)^{4s}, \\
\bar{Y}(\theta) &= -t_{\frac{N}{2}-\frac{1}{2}}(\lambda), \\
\frac{\beta^2}{2} &= \frac{2\xi}{1+2\xi}, \quad \lambda = e^{\frac{\theta}{1+2\xi}}, \quad N' = 2N,
\end{aligned} \tag{196}
$$

where we have denoted $t_s(\lambda)$ the eigenvalue of $\mathrm{t}_s(\lambda)$.

10 Quantum Toda-mKdV hierarchies based on superalgebras

Now let's generalize the above results to the case when the underlying algebraic structures are related to the general affine Lie superalgebra. In the previous part we have moved from classical theory to the quantum one, here we will go in opposite direction, moving from the quantum version of the monodromy matrix and related auxiliary L-operators, satisfying RTT-relation to their classical counterparts.

First, let's introduce two types of vertex operators, bosonic and fermionic ones:

$$
\begin{aligned}
W^F_{\alpha_i}(u) &\equiv \int d\theta : e^{-(\alpha_i, \Phi(u,\theta))} \\
&:= \frac{i}{\sqrt{2}}(\alpha_i, \xi(u)) : e^{-(\alpha_i, \phi(u))} : \\
W^B_{\alpha_i}(u) &\equiv \int d\theta\theta : e^{-(\alpha_i, \Phi(u,\theta))} :=: e^{-(\alpha_i, \phi(u))} :
\end{aligned} \tag{197}
$$

where Φ^k are the superfields: $\Phi^k(u,\theta) = \phi^k(u) - \frac{i}{\sqrt{2}}\theta\xi^k(u)$ and θ is a Grassmann variable. Their commutation relations on a circle are:

$$
\begin{aligned}
&W^s_{\alpha_i}(u)W^{s'}_{\alpha_k}(u') = \\
&(-1)^{p(s)p(s')}q^{b_{ik}}W^{s'}_{\alpha_k}(u')W^s_{\alpha_i}(u), u > u',
\end{aligned} \tag{198}
$$

where b_{kj} is the symmetrized Cartan matrix for the corresponding affine Lie superalgebra, s, s' are B, F and $p(F) = 1, p(B) = 0$.

The mode expansion for the bosonic fields is the same as in (56) and for fermionic fields $\xi^k(u)$ is the following:

$$\xi^l(u) = i^{-1/2} \sum_n \xi^l_n e^{-inu}, \qquad (199)$$
$$\{\xi^k_n, \xi^l_m\} = \beta^2 \delta^{kl} \delta_{n+m,0}.$$

These fermion fields may satisfy two boundary conditions periodic and antiperiodic $\xi^i(u+2\pi) = \pm\xi^i(u)$ corresponding to the two sectors of SCFT – Ramond (R) and Neveu-Schwarz (NS) (the supersymmetry operator appears only when all fermions are in the R sector).

It can be shown that the integrals of the introduced vertex operators as in the bosonic case satisfy the Serre and "non Serre relations" for the lower Borel subalgebra:

$$ad^{(q)}_{e_{-\alpha_k}}{}^{1-a_{kj}} e_{-\alpha_j} = 0, \qquad (200)$$
$$[[e_{-\alpha_r}, e_{-\alpha_s}]_q, [e_{-\alpha_r}, e_{-\alpha_p}]_q]_q = 0$$

(where the $ad^{(q)}$ operator is defined in the following way: $ad^{(q)}_{e_\alpha} e_\beta = e_\alpha e_\beta - (-1)^{p(\alpha)p(\beta)} q^{(\alpha,\beta)} e_\beta e_\alpha$, and $e_{\pm\alpha_r}$ is a so-called "grey" root) of the quantum affine superalgebra with the corresponding bosonic and fermionic roots α_i.

Substituting them with the appropriate multiplier $(q - q^{-1})^{-1}$ in the reduced R-matrix one can find that it satisfies the P-exponential multiplication property:

$$\begin{aligned} \bar{\mathrm{L}}^{(q)}(u_2, u_1) &= Pexp^{(q)} \int_{u_1}^{u_2} du (\sum_f W^F_{\alpha_f}(u) e_{\alpha_f} \\ &+ \sum_b W^B_{\alpha_b}(u) e_{\alpha_b}) \\ \bar{\mathrm{L}}^{(q)}(u_3, u_1) &= \bar{\mathrm{L}}^{(q)}(u_3, u_2) \bar{\mathrm{L}}^{(q)}(u_2, u_1), \quad u_3 \geq u_2 \geq u_1, \end{aligned} \qquad (201)$$

where indices f and b imply that we are summing over fermionic and bosonic simple roots. The letter q over the $Pexp$ means that the object introduced above in some cases (more precisely when a number of fermionic roots is more than one) cannot be written as P-exponential for any value of the deformation parameter due to the singularities in the operator products generated by the fermion fields ξ^i. Thus we call this object quantum P-exponential.

Defining then $\mathrm{L}^{(q)} \equiv e^{\pi i p^i H^i} \bar{\mathrm{L}}^{(q)}(2\pi, 0)$ we find (similarly to the purely bosonic case) that it satisfies the RTT relation (66) and defining the monodromy matrix $\mathrm{M}^{(q)} \equiv e^{\pi i p^i H^i} \mathrm{L}^{(q)}$ we again arrive to the property (67) and obtain again the quantum integrability condition (68) for $\mathrm{t}^{(q)} = str \mathrm{M}^{(q)}$. We mention here that the relation (67) can be rewritten in a more universal way, as a specialization of the reflection equation or braided RTT-relation:

$$\begin{aligned} &\mathrm{R}_{12}(\lambda\mu^{-1}) F_{12} \mathrm{M}_1^{(q)}(\lambda) F_{12}^{-1} \mathrm{M}_2^{(q)}(\mu) = \\ &F_{12} \mathrm{M}_2^{(q)}(\mu) F_{12}^{-1} \mathrm{M}_1^{(q)}(\lambda) \mathrm{R}_{12}(\lambda\mu^{-1}), \end{aligned} \tag{202}$$

where $F = K^{-1}$ the Cartan's factor from the universal R-matrix.

Universal R-matrix for the affine superalgebra

The universal R-matrix for the affine Lie superalgebra has the following structure:

$$R = K\bar{R} = K\Big(\overrightarrow{\prod_{\alpha\in\Delta_+}} R_\alpha\Big), \tag{203}$$

where $\bar{R}$ is a reduced R-matrix and R_α are defined by the formulae:

$$R_\alpha = exp_{q_\alpha^{-1}}((-1)^{p(\alpha)}(q - q^{-1})(a(\alpha))^{-1}(e_\alpha \otimes e_{-\alpha}) \tag{204}$$

for real roots and

$$R_{n\delta} = exp((q - q^{-1})(\sum_{i,j}^{mult} c_{ij}(n) e_{n\delta}^{(i)} \otimes e_{-n\delta}^{(j)})) \qquad (205)$$

for pure imaginary roots. Here Δ_+ is the reduced positive root system (the bosonic roots which are two times fermionic roots are excluded). K is the Cartan factor and is given by the expression:

$$K = q^{\sum_{ij} d_{ij} h_{\alpha_i} \otimes h_{\alpha_j}} \qquad (206)$$

where d_{ij} is the extended symmetrized Cartan matrix. The generators corresponding to the composite roots are defined according to the construction of the Cartan-Weyl basis. For example the generators of the type $e_{\pm\alpha_{f_1} \pm \alpha_{f_2}}$ are constructed by means of the following q-commutators:

$$\begin{aligned} e_{\alpha_{f_1}+\alpha_{f_2}} &= [e_{\alpha_{f_2}}, e_{\alpha_{f_1}}]_{q^{-1}}, \\ e_{-\alpha_{f_1}-\alpha_{f_2}} &= [e_{-\alpha_{f_1}}, e_{-\alpha_{f_2}}]_q \end{aligned} \qquad (207)$$

The $a(\alpha)$ coefficients are defined as follows:

$$[e_\gamma, e_{-\gamma}] = a(\gamma) \frac{k_\gamma - k_\gamma^{-1}}{q - q^{-1}} \qquad (208)$$

We will need the values of $a(\gamma)$ when γ is equal to $\alpha_{f_1} + \alpha_{f_2}$, where α_{f_1} and α_{f_2} are fermionic simple roots:

$$a(\alpha_{f_1} + \alpha_{f_2}) = \frac{q^{-b_{f_1 f_2}} - q^{b_{f_1 f_2}}}{q - q^{-1}}. \qquad (209)$$

The q-exponentials in (203) are defined in the usual way:

$$\begin{aligned} exp_q(x) &= 1 + x + \frac{x^2}{(2)_q!} + ... + \frac{x^n}{(n)_q!} + ... \\ &= \sum_{n \geq 0} \frac{x^n}{(n)_q!} \\ (a)_q &\equiv \frac{q^a - 1}{q - 1}, \quad q_\alpha \equiv (-1)^{p(\alpha)} q^{(\alpha,\alpha)}. \end{aligned} \qquad (210)$$

Now let's analyse the classical limit of the defined objects. We will use the P-exponential property of $\bar{\mathrm{L}}^{(q)}(2\pi, 0)$. Let's decompose $\bar{\mathrm{L}}^{(q)}(2\pi, 0)$ in the following way:

$$\bar{\mathrm{L}}^{(q)}(2\pi, 0) = \lim_{N\to\infty} \prod_{m=1}^{N} \bar{\mathrm{L}}^{(q)}(x_m, x_{m-1}), \qquad (211)$$

where we divided the interval $[0, 2\pi]$ into infinitesimal intervals $[x_m, x_{m+1}]$ with $x_{m+1} - x_m = \epsilon = 2\pi/N$. Let's find the terms that can give contribution of the first order in ϵ in $\bar{\mathrm{L}}^{(q)}(x_m, x_{m-1})$. In this analysis one needs the operator product expansion of fermion fields and vertex operators:

$$\xi^k(u)\xi^l(u') = -\frac{i\beta^2\delta^{kl}}{(iu - iu')} + \sum_{p=0}^{\infty} c_p^{kl}(u)(iu - iu')^p, \qquad (212)$$

$$: e^{-(\alpha_k, \phi(u))} :: e^{-(\alpha_l, \phi(u'))} := (iu - iu')^{\frac{(\alpha_k,\alpha_l)\beta^2}{2}}(: e^{(\alpha_k+\alpha_l, \phi(u))} : + \sum_{p=1}^{\infty} d_p^{kl}(u)(iu - iu')^p),$$

where $c_p^{kl}(u)$ and $d_p^{kl}(u)$ are operator-valued functions of u. Now one can see that only two types of terms can give the contribution of the order ϵ in $\bar{\mathrm{L}}^{(q)}(x_{m-1}, x_m)$ when $q \to 1$. The first type consists of operators of the first order in W_{α_i} and the second type is formed by the operators, quadratic in W_{α_i}, which give contribution of the order $\epsilon^{1\pm\beta^2}$ by virtue of operator product expansion. Let's look on the terms of the second type in detail.

The terms of the second type appear from the quadratic products of vertex operators arising from:

i)the composite roots (more precisely q-commutators of two fermionic roots),

ii)the quadratic terms of the q-exponentials which are present in the universal R-matrix.

At first we consider terms emerging from composite roots, which have the following form:

$$\frac{1}{a(\alpha_i+\alpha_j)(q-q^{-1})}[e_{\alpha_j},e_{\alpha_i}]_{q^{-1}} \tag{213}$$
$$(\int_{x_{m-1}}^{x_m} du_1 W_{\alpha_i}(u_1-i0)$$
$$\int_{x_{m-1}}^{x_m} du_2 W_{\alpha_j}(u_2+i0)+q^{b_{ij}}\int_0^{2\pi} du_2 W_{\alpha_j}(u_2-i0)$$
$$\int_0^{2\pi} du_1 W_{\alpha_i}(u_1+i0)$$

Using the fact that $q=e^{i\pi\frac{\beta^2}{2}}$ and that in the limit $\beta^2\to 0$, $a(\alpha_i+\alpha_j)\to -b_{ij}$, one can rewrite this as follows (leaving only terms that can give contribution to the first order in ϵ:

$$(2\pi i)^{-1}[e_{\alpha_j},e_{\alpha_i}] \tag{214}$$
$$\int_{x_{m-1}}^{x_m} du_1 \int_{x_{m-1}}^{x_m} du_2 (\frac{1}{u_2-u_1+i0}-\frac{1}{u_2-u_1-i0})$$
$$: e^{-(\alpha_i+\alpha_j,\phi(u_2))} :$$

Now using the well known formula

$$\frac{1}{x+i0}-\frac{1}{x-i0}=-2i\pi\delta(x) \tag{215}$$

we obtain that (213) in the classical limit gives

$$-[e_{\alpha_j},e_{\alpha_i}]\int_{x_{m-1}}^{x_m} du : e^{-(\alpha_i+\alpha_j,\phi(u))} : \tag{216}$$

Next let's consider the quadratic products arising from quadratic parts of q-exponentials of fermionic roots. They look as follows:

$$\frac{-1}{(2)_{q_{\alpha_i}^{-1}}}\int_{x_{m-1}}^{x_m} du_1 W_{\alpha_i}(u_1 - i0)\int_{x_{m-1}}^{x_m} du_2 W_{\alpha_i}(u_2 + i0) e_{\alpha_i}^2 \tag{217}$$

One can rewrite this product via the ordered integrals:

$$\frac{q^{b_{ii}} - 1}{(2)_{q_{\alpha_i}^{-1}}}\int_{x_{m-1}}^{x_m} du_1 W_{\alpha_i}(u_1)\int_{x_{m-1}}^{u_1} du_2 W_{\alpha_i}(u_2) e_{\alpha_i}^2 \tag{218}$$

In the limit $\beta^2 \to 0$ we obtain (forgetting about the terms that could give contribution of the order ϵ^2):

$$-\frac{ib_{ii}\beta^2}{2}\int_{x_{m-1}}^{x_m} du_1 \tag{219}$$

$$\int_{x_{m-1}}^{u_1} du_2 (iu_1 - iu_2)^{\frac{b_{ii}\beta^2}{2}-1} e^{-2(\alpha_i, \phi(u_2))} e_{\alpha_i}^2$$

Therefore the final contribution is:

$$-\int_{x_{m-1}}^{x_m} du e^{-2(\alpha_i, \phi(u))} e_{\alpha_i}^2 \tag{220}$$

Collecting now all the terms of order ϵ we find:

$$\bar{\mathrm{L}}^{(q)}(x_m, x_{m-1}) = 1 + \int_{x_{m-1}}^{x_m} du (\sum_f W_{\alpha_f}^F(u) e_{\alpha_f} + \tag{221}$$

$$\sum_b W_{\alpha_b}^B(u) e_{\alpha_b} + \sum_{f_1 \geq f_2} [e_{\alpha_{f_1}}, e_{\alpha_{f_2}}] W_{\alpha_{f_1}+\alpha_{f_2}}^B(u)) + O(\epsilon^2)$$

Gathering the $\bar{\mathrm{L}}^{(q)}(x_m, x_{m-1})$ it is easy to see that in the $q \to 1$ limit $\bar{\mathrm{L}}^{(q)}(x_m, x_{m-1})$ is equal to:

$$\bar{\mathrm{L}}^{(cl)}(2\pi, 0) = Pexp\int_0^{2\pi} du (\sum_f W_{\alpha_f}^F(u) e_{\alpha_f} + \tag{222}$$

$$\sum_b W_{\alpha_b}^B(u) e_{\alpha_b} - \sum_{f_1 \geq f_2} [e_{\alpha_{f_1}}, e_{\alpha_{f_2}}] W_{\alpha_{f_1}+\alpha_{f_2}}^B(u))$$

Defining then $\mathrm{L}^{(cl)} \equiv e^{\pi i p^i H^i} \bar{\mathrm{L}}^{(cl)}(2\pi, 0)$ we find that it satisfies the quadratic Poisson bracket relation (53) and defining the monodromy matrix $\mathrm{M}^{(cl)} \equiv e^{\pi i p^i H^i} \mathrm{L}^{(cl)}$ we again obtain the classical integrability condition (55).

Now let's find the L-operator which corresponds to the monodromy matrix defined above. Let's consider the following one:

$$\mathcal{L}_F = D_{u,\theta} - D_{u,\theta}\Phi^i(u,\theta)H^i - (\sum_f e_{\alpha_f} + \sum_b \theta e_{\alpha_b}), \tag{223}$$

where $D_{u,\theta} = \partial_\theta + \theta\partial_u$ is a superderivative and Φ^i are the classical superfields with the following Poisson brackets:

$$\begin{aligned}\{D_{u,\theta}\Phi^i(u,\theta), D_{u',\theta'}\Phi^j(u',\theta')\} = \\ \delta^{ij} D_{u,\theta}(\delta(u-u')(\theta-\theta'))\end{aligned} \tag{224}$$

The associated "fermionic" linear problem can be reduced to the "bosonic" one. The linear problem

$$\mathcal{L}_F \Psi(u,\theta) = (D_{u,\theta} + N_1 + \theta N_0)(\chi + \theta\eta), \tag{225}$$

where $\Psi(u,\theta) = \chi + \theta\eta$, $N_1 = \frac{i}{\sqrt{2}}\xi^i H^i - \sum_f e_{\alpha_f}$, $N_0 = -\partial_u\phi^i H^i - \sum_b e_{\alpha_b}$, can be reduced to the linear problem on χ:

$$\mathcal{L}_B \chi(u) = (\partial_u + N_1^2 + N_0)\chi(u) \tag{226}$$

That is:

$$\mathcal{L}_B = \partial_u - \partial_u\phi^i H^i + (\frac{i}{\sqrt{2}}\xi^i H^i - \sum_f e_{\alpha_f})^2 - \sum_b e_{\alpha_b}. \tag{227}$$

One can easily see that the monodromy matrix for the corresponding linear problem is that described above.

11 Integrals of Motion and Supersymmetry Invariance

It is well known that (both classical and quantum) integrability conditions lead to the involutive family of (both local and nonlocal) integrals of motion (IM). For superversions of these systems it is also known that sometimes it is possible to include supersymmetry generator

$$G_0 = \beta^{-2}\sqrt{2}i^{-1/2}\int_0^{2\pi} du\phi'^l(u)\xi^l(u) \tag{228}$$
$$= \sum_{l=0}^{r}\sum_{n\in Z}\beta^{-2}\xi_n^l a_{-n}^l$$

in these series. Here we will show that the transfer matrix $\mathrm{t}^{(q)}(\lambda) = str\mathrm{M}^{(q)}(\lambda)$ commute with G_0 if the simple root system is purely fermionic, that is:

$$\mathrm{t}^{(q)}(\lambda) = \tag{229}$$
$$str(\pi(\lambda)(e^{2i\pi P^k H^k} Pexp\int_0^{2\pi} du(\sum_{f=0}^{r} W_{\alpha_f}^F(u)e_{\alpha_f}))),$$

where π denote some representation of the corresponding superalgebra in which the supertrace is taken.

We note the crucial property:

$$[G_0, W_\alpha^F(u)] = -\partial_u W_\alpha^B(u) \tag{230}$$

Integrating over u and multiplying by the appropriate coefficient one obtains:

$$[G_0, e_{-\alpha_i}] = \frac{W_{\alpha_i}^F(0) - W_{\alpha_i}^F(2\pi)}{q - q^{-1}}, \tag{231}$$

where $e_{-\alpha_i}$ is represented by the vertex operator (see previous Section). Next, we will use the important Proposition.

For the objects A_i, B_i, I, satisfying the commutation relations

$$[I, e_{-\alpha_i}] = \frac{A_i - B_i}{q - q^{-1}}, \quad A_i e_{-\alpha_j} = q^{-b_{ij}} e_{-\alpha_j} A_i, \tag{232}$$
$$B_i e_{-\alpha_j} = q^{b_{ij}} e_{-\alpha_j} B_i,$$

the following relation holds:

$$[(1 \otimes I), \bar{R}] = \bar{R}(\sum_i e_{\alpha_i} \otimes A_i) - (\sum_i e_{\alpha_i} \otimes B_i)\bar{R}. \quad (233)$$

Applying this relations to our case we find (identifying A_i with $W^F_{\alpha_i}(0)$, B_i with $W^F_{\alpha_i}(2\pi)$ and I with G_0):

$$\begin{aligned} &[G_0, \bar{\mathrm{L}}^{(q)}(2\pi, 0)] = \\ &\bar{\mathrm{L}}^{(q)}(2\pi, 0) W^B(0) - W^B(2\pi)\bar{\mathrm{L}}^{(q)}(2\pi, 0), \end{aligned} \quad (234)$$

where

$$W^B(u) = \sum_{i=0}^{r} W^B_{\alpha_i}(u) e_{\alpha_i}. \quad (235)$$

Now using the property (198), periodicity properties of vertex operators:

$$W^s_\alpha(u + 2\pi) = q^{-(\alpha,\alpha)} e^{-2i\pi(\alpha,P)} W^s_\alpha(u + 2\pi) \quad (236)$$

(here $s = B, F$) and cyclic property of the supertrace one obtains, multiplying both sides of (234) by $e^{2i\pi P^k H^k}$ and taking the supertrace:

$$[G_0, \mathrm{t}^{(q)}] = 0. \quad (237)$$

We note here that if there were bosonic simple roots in the construction of the transfer-matrix the above reasonings are no longer valid, because the corresponding bosonic vertex operators commuting with G_0 give the fermionic vertex operators associated with the same root vector, but not the total derivative as in (230). It was already shown explicitly on the concrete examples that the hierarchies, based on the partly bosonic simple root systems are not invariant under the supersymmetry transformation. The affine superalgebras which allow such root systems are of the following type:

$A(m,m)^{(1)} = sl(m+1,m+1)^{(1)}$,
$A(2m,2m)^{(4)} = sl(2m+1,2m+1)^{(4)}$,
$A(2m+1,2m+1)^{(2)} = sl(2m+2,2m+2)^{(2)}$,
$A(2m+1,2m)^{(2)} = sl(2m+2,2m+1)^{(2)}$,
$B(m,m)^{(1)} = osp(2m+1,2m)^{(1)}$,
$D(m+1,m)^{(1)} = osp(2m+2,2m)^{(1)}$,
$D(m,m)^{(2)} = osp(2m,2m)^{(2)}$,
$D(2,1,\alpha)^{(1)}$.

The involutive family of the (both classical and quantum) IM in the Toda field theories have the property, that the commutators of IM with the corresponding vertex operators reduce to the total derivatives:

$$[I_l, W_{\alpha_k}(u)] = \partial_u(: O^{(l)}_{\alpha_k}(u) W_{\alpha_k}(u) :) = \partial_u \Theta^{(l)}_k(u), \quad (238)$$

where $O^{(k)}_{\pm}(u)$ is the polynomial of $\partial_u \phi^i(u)$, $\xi^i(u)$ and their derivatives. That is using the arguments above one can show that I_l commute with the transfer matrix.

12 Main Results

1) The quantization of the generalized KdV hierarchies related with superalgebras is given. The quantum monodromy matrix is shown to be related with the R-matrix represented by the integrated vertex operators.

2) It was shown that the KdV hierarchies corresponding to superalgebras with purely fermionic system of simple roots are supersymmetric.

3)The explicit form of the evaluation representations of the quantum affine $osp_q(1|2)^{(1)}$ and twisted affine $sl_q(1|2)^{(2)}$ superalgebras was obtained.

4) The SCFT analogues of the Baxter Q-operator, satisfying the quantum super-Wronskian relation were built.

5) The fusion relations for the transfer matrices were constructed in two particular cases of super-KdV and SUSY N=1 KdV systems; and for the last one their truncations were shown to coincide with TBA of D_N type.