STATISTICAL RETHINKING

SECOND EDITION

PRACTICE PROBLEM SOLUTIONS

Contents

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2. Chapter 2 Solutions

2E1. Both (2) and (4) are correct. (2) is a direct interpretation, and (4) is equivalent.

2E2. Only (3) is correct.

2E3. Both (1) and (4) are correct. For (4), the product $\Pr(\text{rain}|\text{Monday})\Pr(\text{Monday})$ is just the joint probability of rain and Monday, $\Pr(\text{rain}, \text{Monday})$. Then dividing by the probability of rain provides the conditional probability.

2E4. This problem is merely a prompt for readers to explore intuitions about probability. The goal is to help understand statements like "the probability of water is 0.7" as statements about partial knowledge, not as statements about physical processes. The physics of the globe toss are deterministic, not "random." But we are substantially ignorant of those physics when we toss the globe. So when someone states that a process is "random," this can mean nothing more than ignorance of the details that would permit predicting the outcome.

As a consequence, probabilities change when our information (or a model's information) changes. Frequencies, in contrast, are facts about particular empirical contexts. They do not depend upon our information (although our beliefs about frequencies do).

This gives a new meaning to words like "randomization," because it makes clear that when we shuffle a deck of playing cards, what we have done is merely remove our knowledge of the card order. A card is "random" because we cannot guess it.

2M1. Since the prior is uniform, it can be omitted from the calculations. But I'll show it here, for conceptual completeness. To compute the grid approximate posterior distribution for (1):

R code 2.1
```
p_grid <- seq( from=0 , to=1 , length.out=100 )
# likelihood of 3 water in 3 tosses
likelihood <- dbinom( 3 , size=3 , prob=p_grid )
prior <- rep(1,100) # uniform prior
posterior <- likelihood * prior
posterior <- posterior / sum(posterior) # standardize
```

And `plot(posterior)` will produce a simple and ugly plot. This will produce something with nicer labels and a line instead of individual points:

R code 2.2
```
plot( posterior ~ p_grid , type="l" )
```

The other two data vectors are completed the same way, but with different likelihood calculations. For (2):

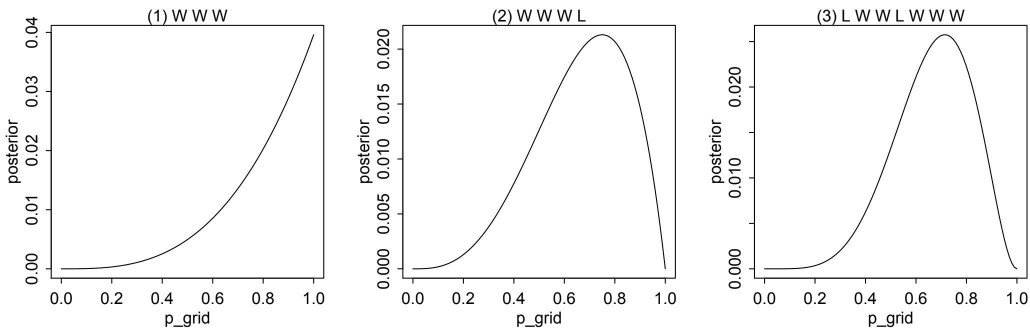
R code 2.3
```
# likelihood of 3 water in 4 tosses
likelihood <- dbinom( 3 , size=4 , prob=p_grid )
```

And for (3):

R code 2.4

```
# likelihood of 5 water in 7 tosses
likelihood <- dbinom( 5 , size=7 , prob=p_grid )
```

And this is what each plot should look like:

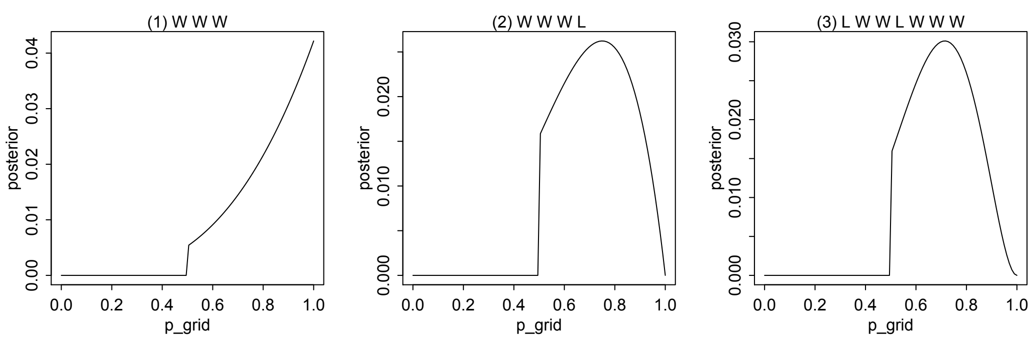


2M2. Only the prior has to be changed. For the first set of observations, W W W, this will complete the calculation and plot the result:

R code 2.5

```
p_grid <- seq( from=0 , to=1 , length.out=100 )
likelihood <- dbinom( 3 , size=3 , prob=p_grid )
prior <- ifelse( p_grid < 0.5 , 0 , 1 ) # new prior
posterior <- likelihood * prior
posterior <- posterior / sum(posterior) # standardize
plot( posterior ~ p_grid , type="l" )
```

The other two plots can be completed by changing the likelihood, just as in the previous problem. Here are the new plots, demonstrating that the prior merely truncates the posterior distribution below 0.5:



2M3. Here's what we know from the problem definition, restated as probabilities:

$$\Pr(\text{land}|\text{Earth}) = 1 - 0.7 = 0.3$$
$$\Pr(\text{land}|\text{Mars}) = 1$$

We also have, as stated in the problem, equal prior expectation of each globe. This means:

$$\Pr(\text{Earth}) = 0.5$$
$$\Pr(\text{Mars}) = 0.5$$

We want to calculate $\Pr(\text{Earth}|\text{land})$. By definition:

$$\Pr(\text{Earth}|\text{land}) = \frac{\Pr(\text{land}|\text{Earth}) \Pr(\text{Earth})}{\Pr(\text{land})}$$

The $\Pr(\text{land})$ in the denominator is just the average probability of land, averaging over the two globes. So the above expands to:

$$\Pr(\text{Earth}|\text{land}) = \frac{\Pr(\text{land}|\text{Earth}) \Pr(\text{Earth})}{\Pr(\text{land}|\text{Earth}) \Pr(\text{Earth}) + \Pr(\text{land}|\text{Mars}) \Pr(\text{Mars})}$$

Plugging in the numerical values:

$$\Pr(\text{Earth}|\text{land}) = \frac{(0.3)(0.5)}{(0.3)(0.5) + (1)(0.5)}$$

Let's compute the result using R:

R code 2.6

```
0.3*0.5 / ( 0.3*0.5 + 1*0.5 )
```

```
[1] 0.2307692
```

And there's the answer, $\Pr(\text{Earth}|\text{land}) \approx 0.23$. You can think of this posterior probability as an updated prior, of course. The prior probability was 0.5. Since there is more land coverage on Mars than on Earth, the posterior probability after observing land is smaller than the prior.

2M4. Label the three cards as (1) B/B, (2) B/W, and (3) W/W. Having observed a black (B) side face up on the table, the question is: How many ways could the other side also be black?

First, count up all the ways each card could produce the observed black side. The first card is B/B, and so there are 2 ways is could produce a black side face up on the table. The second card is B/W, so there is only 1 way it could show a black side up. The final card is W/W, so it has zero ways to produce a black side up.

Now in total, there are 3 ways to see a black side up. 2 of those ways come from the B/B card. The other comes from the B/W card. So 2 out of 3 ways are consistent with the other side of the card being black. The answer is 2/3.

2M5. With the extra B/B card, there are now 5 ways to see a black card face up: 2 from the first B/B card, 1 from the B/W card, and 2 more from the other B/B card. 4 of these ways are consistent with a B/B card, so the probability is now 4/5 that the other side of the card is also black.

2M6. This problem introduces uneven numbers of ways to draw each card from the bag. So while in the two previous problems we could treat each card as equally likely, prior to the observation, now we need to employ the prior odds explicitly.

There are still 2 ways for B/B to produce a black side up, 1 way for B/W, and zero ways for W/W. But now there is 1 way to get the B/B card, 2 ways to get the B/W card, and 3 ways to get the W/W card. So there are, in total, $1 \times 2 = 2$ ways for the B/B card to produce a black side up and $2 \times 1 = 2$ ways for the B/W card to produce a black side up. This means there are 4 ways total to see a black side up, and 2 of these are from the B/B card. 2/4 ways means probability 0.5.

2M7. The observation is now the sequence: black side up then white side up. We're still interested in the probability the other side of the first card is black. Let's take each possible card in turn.

First the B/B card. There are 2 ways for it to produce the first observation, the black side up. This leaves the B/W card and W/W card to produce the next observation. Each card is equally likely (has same number of ways to get drawn from the bag). But the B/W card has only 1 way to produce a white side up, while the W/W card has 2 ways. So 3 ways in total to get the second card to show white side up. All together, assuming the first card is B/B, there are $2 \times 3 = 6$ ways to see the BW sequence of sides up.

Now consider the B/W card being drawn first. There is 1 way for it to show black side up. This leaves the B/B and W/W cards to produce the second side up. B/B cannot show white up, so zero ways there. W/W has 2 ways to show white up. All together, that's $1 \times 2 = 2$ ways to see the sequence BW, when the first card is B/W.

The final card, W/W, cannot produce the sequence when drawn first. So zero ways.

Now let's bring it all together. Among all three cards, there are $6 + 2 = 8$ ways to produce the sequence BW. 6 of these are from the B/B being drawn first. So that's $6/8 = 0.75$ probability that the first card is B/B.

2H1. To solve this problem, realize first that it is asking for a conditional probability:

$$\Pr(\text{twins}_2|\text{twins}_1)$$

the probability the second birth is twins, conditional on the first birth being twins. Remember the definition of conditional probability:

$$\Pr(\text{twins}_2|\text{twins}_1) = \frac{\Pr(\text{twins}_1, \text{twins}_2)}{\Pr(\text{twins})}$$

So our job is to define $\Pr(\text{twins}_1, \text{twins}_2)$, the joint probability that both births are twins, and $\Pr(\text{twins})$, the unconditioned probability of twins.

$\Pr(\text{twins})$ is easier, so let's do that one first. The "unconditioned" probability just means that we have to average over the possibilities. In this case, that means the species have to averaged over. The problem implies that both species are equally common, so there's a half chance that any given panda is of either species. This gives us:

$$\Pr(\text{twins}) = \underbrace{\tfrac{1}{2}(0.1)}_{\text{Species A}} + \underbrace{\tfrac{1}{2}(0.2)}_{\text{Species B}}.$$

A little arithmetic tells us that $\Pr(\text{twins}) = 0.15$.

Now for $\Pr(\text{twins}_1, \text{twins}_2)$. The probability that a female from species A has two sets of twins is $0.1 \times 0.1 = 0.01$. The corresponding probability for species B is $0.2 \times 0.2 = 0.04$. Averaging over species identity:

$$\Pr(\text{twins}_1, \text{twins}_2) = \tfrac{1}{2}(0.01) + \tfrac{1}{2}(0.04) = 0.025.$$

Finally, we combine these probabilities to get the answer:

$$\Pr(\text{twins}_2|\text{twins}_1) = \frac{0.025}{0.15} = \frac{25}{150} = \frac{1}{6} \approx 0.17.$$

Note that this is higher than $\Pr(\text{twins})$. This is because the first set of twins provides some information about which species we have, and this information was automatically used in the calculation.

2H2. Our target now is $\Pr(A|\text{twins}_1)$, the posterior probability that the panda is species A, given that we observed twins. Bayes' theorem tells us:

$$\Pr(A|\text{twins}_1) = \frac{\Pr(\text{twins}_1|A)\Pr(A)}{\Pr(\text{twins})}.$$

We calculated $\Pr(\text{twins}) = 0.15$ in the previous problem, and we were given $\Pr(\text{twins}|A) = 0.1$ as well. The only stumbling block may be $\Pr(A)$, the prior probability of species A. This was also given, implied by the equal abundance of both species. So prior to observing the birth, we have $\Pr(A) = 0.5$. So:

$$\Pr(A|\text{twins}_1) = \frac{(0.1)(0.5)}{0.15} = \frac{5}{15} = \frac{1}{3}.$$

So the posterior probability of species A, after observing twins, falls to 1/3, from a prior probability of 1/2. This also implies a posterior probability of 2/3 that our panda is species B, since we are assuming only two possible species. These are *small world* probabilities that trust the assumptions.

2H3. There are a few ways to arrive at the answer. The easiest is perhaps to recall that Bayes' theorem accumulates evidence, using Bayesian updating. So we can take the posterior probabilities from the previous problem and use them as prior probabilities in this problem. This implies $\Pr(A) = 1/3$. Now we can ignore the first observation, the twins, and concern ourselves with only the latest observation, the singleton birth. The previous observation is embodied in the prior, so there's no need to account for it again.

The formula is:

$$\Pr(A|\text{singleton}) = \frac{\Pr(\text{singleton}|A)\Pr(A)}{\Pr(\text{singleton})}$$

We already have the prior, $\Pr(A)$. The other pieces are straightforward:

$$\begin{aligned}\Pr(\text{singleton}|A) &= 1 - 0.1 = 0.9\\ \Pr(\text{singleton}) &= \Pr(\text{singleton}|A)\Pr(A) + \Pr(\text{singleton}|B)\Pr(B)\\ &= (0.9)\tfrac{1}{3} + (0.8)\tfrac{2}{3} = \frac{5}{6}\end{aligned}$$

Combining everything, we get:

$$\Pr(A|\text{singleton}) = \frac{(0.9)\frac{1}{3}}{\frac{5}{6}} = \frac{9}{25} = 0.36$$

This is a modest increase in posterior probability of species A, an increase from about 0.33 to 0.36.

The other way to proceed is to go back to the original prior, $\Pr(A) = 0.5$, before observed any births. Then you can treat both observations (twins, singleton) as data and update the original prior. I'm going to start abbreviating "twins" as T and "singleton" as S. The formula:

$$\Pr(A|T,S) = \frac{\Pr(T,S|A)\Pr(A)}{\Pr(T,S)}$$

Let's start with the average likelihood, $\Pr(T,S)$, because it will force us to define the likelihoods anyway.

$$\Pr(T,S) = \Pr(T,S|A)\Pr(A) + \Pr(T,S|B)\Pr(B)$$

I'll go slowly through this, so I don't lose anyone along the way. The first likelihood is just the probability a species A mother has twins and then a singleton:

$$\Pr(T,S|A) = (0.1)(0.9) = 0.09$$