Mean Residual Lifetime Function for Generalized Gamma Distribution

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The PDF of the generalized gamma (GG) distribution (Original) is:

$$f(x; a, b, k) = \frac{bx^{bk-1}}{\Gamma(k)a^{bk}} \texttt{exp}((-\frac{x}{a})^b) \tag{1}$$

where a,b,k are parameters of GG, $\Gamma(k)$ is Gamma function.

We can represent Mean Residual Lifetime (MRL) function, $m(x)$, for a specific distribution using the survival function, $S(x)$, and its pdf:

$$m(x) = \frac{\int_x^\infty tf(t)\texttt{dt}}{S(x)} - x \tag{2}$$

The numerator of equation 2 can be written as:

$$\begin{aligned} \int_x^\infty tf(t) &= \int_x^\infty t\frac{bt^{bk-1}}{\Gamma(k)a^{bk}} \texttt{exp}((-\frac{t}{a})^b)\texttt{dt} \\ &= \frac{b}{\Gamma(k)} \int_x^\infty (\frac{t}{a})^{bk} \texttt{exp}((-\frac{t}{a})^b)\texttt{dt} \end{aligned} \tag{3}$$

Let $u = (\frac{t}{a})^b$, then we can obtain $\texttt{du} = \frac{b}{a}(\frac{t}{a})^{b-1}\texttt{dt}$, and $\texttt{dt} = \frac{a}{b}(u)^{\frac{1-b}{b}}\texttt{du}$

Thus, the new limit for u become $u = (\frac{x}{a})^b$ when $t = x$ and $u \to \infty$, when $t \to \infty$.

Make a substitution to simplify the integral 3:

$$\begin{aligned} \int_x^\infty tf(t)\texttt{dt} &= \frac{b}{\Gamma(k)} \int_x^\infty (\frac{t}{a})^{bk} \texttt{exp}((-\frac{t}{a})^b)\texttt{dt} \\ &= \frac{b}{\Gamma(k)} \int_{(\frac{x}{a})^b}^\infty u^k \texttt{exp}(-u) \frac{a}{b} u^{\frac{1-b}{b}} \texttt{du} \\ &= \frac{a}{\Gamma(k)} \int_{(\frac{x}{a})^b}^\infty u^{k+\frac{1}{b}-1} \texttt{exp}(-u)\texttt{du} \end{aligned} \tag{4}$$

The integral part of equation 5 is known as the upper incomplete gamma function,

$$\Gamma(x,a) = \int_x^\infty t^{a-1}\mathtt{exp}(-t)\mathtt{dt} \tag{5}$$

which is defined as $\Gamma((\frac{x}{a})^b, k+\frac{1}{b})$.
Thus,

$$\int_x^\infty tf(t)\mathtt{dt} \quad = \quad a\frac{\Gamma((\frac{x}{a})^b, k+\frac{1}{b})}{\Gamma(k)} \tag{6}$$

Since the survival function of GG can be represented as:

$$S(x) = \frac{\Gamma((\frac{x}{a})^b, k)}{\Gamma(k)} \tag{7}$$

We can express MRL for GG distribution as:

$$\begin{aligned} m(x) &= \frac{\int_x^\infty tf(t)\mathtt{dt}}{S(x)} - x \qquad (8) \\ &= a\frac{\Gamma((\frac{x}{a})^b, k+\frac{1}{b})}{\Gamma((\frac{x}{a})^b, k)} - x \end{aligned}$$