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Percentile Residual Life Functions

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The α-percentile ($0 < \alpha < 1$) residual life function at time t is defined as the α-percentile of the remaining life given survival up to time t. (Note that α-percentile is used in place of 100 α-percentile.) Two classes of life distributions defined by the α-percentile residual life function are the decreasing α-percentile residual life (DPRL-α) class and the new better than used with respect to the α-percentile (NBUP-α) class. These two classes are, respectively, analogous to the well-known decreasing mean residual life (DMRL) class and the new better than used in expectation (NBUE) class that involve the mean residual life function. We obtain properties of the α-percentile residual life function and of the DPRL-α, NBUP-α and their dual classes. Some results differ notably from corresponding results for the mean residual life function.

THE MEAN residual life function is of interest in biometry, actuarial studies and reliability. At time t, it is defined to be the expected remaining life given survival up to time t. Similarly, the α-percentile or quantile ($0 < \alpha < 1$) residual life function at time t is defined to be the α-percentile of the remaining life given survival up to time t. (Note that α-percentile is used throughout in place of 100 α-percentile.) In statistical practice, the median and other percentiles are used as well as the mean—for example, in situations where the underlying distribution is skewed. So, it should be of interest to study the median residual life function or more generally the α-percentile residual life function. The only previous work on this subject is by Haines and Singpurwalla [1974]. Some recent papers studying properties of the mean residual life function are by Hall and Wellner [1981] and Bhattacharjee [1982]. Other papers in the statistical literature consider estimation of the mean residual life function or consider testing situations involving the mean residual life function.

Statisticians find it useful to categorize life distributions according to different aging properties. These categories of distributions are useful for modeling situations where items deteriorate (or improve) with age. We define the decreasing α-percentile residual life (DPRL-α) class and the new better than used with respect to the α-percentile (NBUP-α) class. Haines and Singpurwalla have studied the DPRL-α classes, $0 < \alpha < 1$,

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and obtained some probabilistic properties of this class. The NBUP-α classes, $0 < \alpha < 1$, being considered here, are new.

In this paper, we study properties of α-percentile residual life functions and of the DPRL-α, NBUP-α and their dual classes. Section 1 gives definitions and Section 2 gives the main results. In Section 3, we discuss the preservation of the DPRL-α, NBUP-α, and their dual classes under some reliability operations.

1. DEFINITIONS AND PRELIMINARIES

Let F be the life distribution (a distribution such that $F(0^-) = 0$) of a system and let $\bar{F} \equiv 1 - F$ be the survival function. Let $T_F = F^{-1}(1) = \sup\{x: F(x) < 1\}$ be the right-hand endpoint of support of F. Given that the system has survived up to time t, its remaining life has survival function given by

$$1 - F_t(x) \equiv \bar{F}_t(x) = \bar{F}(t + x)/\bar{F}(t), \qquad x \geq 0, \qquad 0 \leq t < T_F.$$

For $0 < \alpha < 1$, the α-percentile residual life function $q_{\alpha,F}(t)$ of F is defined as

$$\begin{aligned} q_{\alpha,F}(t) &= F_t^{-1}(\alpha) = \inf\{x: F_t(x) \geq \alpha\} \\ &= \inf\{x: \bar{F}(t + x) \leq (1 - \alpha)\bar{F}(t)\} \\ &= \inf\{y: \bar{F}(y) \leq (1 - \alpha)\bar{F}(t)\} - t \\ &= F^{-1}(1 - \bar{\alpha}\bar{F}(t)) - t, \qquad 0 \leq t < T_F, \end{aligned}$$

where $\bar{\alpha} = 1 - \alpha$. Note that we are defining $q_{\alpha,F}(t)$ using the left-continuous version of the inverse of a distribution function. The mean residual life function $m_F(t)$ of F is defined as

$$m_F(t) = \int_0^\infty \bar{F}_t(x)dx = \int_t^\infty \bar{F}(x)dx/\bar{F}(t), \qquad 0 \leq t < T_F,$$

if F has a finite mean. Whenever possible, we will suppress the subscript F and use T, q_α, m in place of T_F, $q_{\alpha,F}$, m_F, respectively.

Definition. A life distribution F with finite mean is a *decreasing mean residual life* (DMRL) distribution if $F(0) = 0$ and $m(t)$ is decreasing in $t \in [0, T)$. It is an *increasing mean residual life* (IMRL) distribution if $m(t)$ is increasing in $t \in [0, T)$.

(Throughout, we use the word decreasing in place of nonincreasing and increasing in place of nondecreasing.)

Definition. A life distribution F with finite mean is a *new better than used in expectation* (NBUE) distribution if $F(0) = 0$ and $m(0) \geq m(t)$ for all $0 \leq t < T$. It is a *new worse than used in expectation* (NWUE) distribution if $m(0) \leq m(t)$ for all $0 \leq t < T$.

These classes of distributions are well-known to reliabilists. We can define analogous classes of distributions with the α-percentile residual life function $q_\alpha(t)$. Haines and Singpurwalla introduced the first group of classes; the second group of classes is new.

Definition. A life distribution F is a *decreasing α-percentile residual life* (DPRL-α) distribution if $F(0) = 0$ and $q_\alpha(t)$ is decreasing in $t \in [0, T)$. It is an *increasing α-percentile residual life* (IPRL-α) distribution if $q_\alpha(t)$ is increasing in $t \in [0, T)$. F is a *constant α-percentile residual life* (CPRL-α) distribution if it is both a DPRL-α and IPRL-α distribution.

Definition. A life distribution F is a *new better than used with respect to the α-percentile* (NBUP-α) distribution if $F(0) = 0$ and $q_\alpha(0) \geq q_\alpha(t)$ for all $0 \leq t \leq T$. It is a *new worse than used with respect to the α-percentile* (NWUP-α) distribution if $q_\alpha(0) \leq q_\alpha(t)$ for all $0 \leq t < T$.

Other well-known classes of life distributions consist of the increasing failure rate (IFR), decreasing failure rate (DFR), new better than used (NBU), and new worse than used (NWU) distributions. Barlow and Proschan [1981] have studied these classes and their properties.

Note that if F is DFR, IMRL, NWU, NWUE, IPRL-α or NWUP-α, then $T = \infty$. From the above definitions and the fact that IFR and NBU distributions have finite means, we can easily see the following relationships among the classes:

$$F \text{ is IFR} \Rightarrow F \text{ is DMRL} \Rightarrow F \text{ is NBUE,}$$

$$F \text{ is IFR} \Rightarrow F \text{ is NBU} \Leftrightarrow F \text{ is NBUP-}\alpha,$$

$$\text{for all } 0 < \alpha < 1 \Rightarrow F \text{ is NBUE,}$$

$$F \text{ is IFR} \Leftrightarrow F \text{ is DPRL-}\alpha \text{ for all } 0 < \alpha < 1,$$

$$F \text{ is DPRL-}\alpha \Rightarrow F \text{ is NBUP-}\alpha, \text{ for any } 0 < \alpha < 1.$$

Similar relationships exist for the dual classes, DFR, IMRL, NWU, NWUE, IPRL-α and NWUP-α, but we must assume that the mean of F is finite in some implications. It is well known that the exponential distributions are the only distributions that are both IFR and DFR, both DMRL and IMRL, both NBU and NWU, and both NBUE and NWUE. However, for any $0 < \alpha < 1$, there are distributions other than the exponential distributions that are both DPRL-α and IPRL-α, that is, CPRL-α. Note also that if a distribution is both NBUP-α and NWUP-α, then it is CPRL-α. It is true, though, that the exponential distributions are the only distributions that are CPRL-α for all $0 < \alpha < 1$.

2. PROPERTIES

In this section, we obtain some properties of the α-percentile residual life function and of the DPRL-α, IPRL-α, CPRL-α, NBUP-α and

NWUP-α classes. The first property of $q_\alpha(t)$ is that it has both right-hand and left-hand limits. These limits are easily deduced from the right continuity of F and the left continuity of F^{-1}.

THEOREM 1. *Let $q_\alpha(t^+)$ and $q_\alpha(t^-)$ denote the right-hand and left-hand limits of q_α at t, respectively. Then, for $0 < t < T$,*

(i) $q_\alpha(t^+) = F^{-1}((1 - \bar{\alpha}\bar{F}(t))^+) - t$ *if* $F(t + \epsilon) - F(t) > 0$ *for every* $\epsilon > 0$,

(ii) $q_\alpha(t^+) = F^{-1}(1 - \bar{\alpha}\bar{F}(t)) - t$ *if* $F(t + \epsilon) - F(t) = 0$ *for some* $\epsilon > 0$,

(iii) $q_\alpha(t^-) = F^{-1}(1 - \bar{\alpha}\bar{F}(t^-)) - t$.

Also, $\lim_{t\uparrow T} q_\alpha(t) = 0$ *if* $T < \infty$.

Remarks. (1) $q_\alpha(t)$ is continuous at t if (a) F is constant on an interval containing t, (b) F^{-1} is continuous at $1 - \bar{\alpha}\bar{F}(t)$, and (i) F is continuous at t, or (ii) F is discontinuous at both t and $F^{-1}(1 - \bar{\alpha}\bar{F}(t))$, and $\bar{F}(F^{-1}(1 - \bar{\alpha}\bar{F}(t))^-) \geq \bar{\alpha}\bar{F}(t^-)$ [since $\bar{F}[F^{-1}(1 - \bar{\alpha}\bar{F}(t))] \leq \bar{\alpha}\bar{F}(t)$ by the definition of $q_\alpha(t)$, this condition means that $F[F^{-1}(1 - \bar{\alpha}\bar{F}(t))] - F(F^{-1}(1 - \bar{\alpha}\bar{F}(t))^-) \geq \bar{\alpha}(F(t) - F(t^-))$].

(2) If F is constant on an interval, then q_α is decreasing linearly with slope -1 on this interval.

(3) $q_\alpha(t^+) \geq q_\alpha(t^-)$, that is, if q_α has a jump discontinuity at t, then the jump is positive.

(4) It is possible that q_α is neither right continuous nor left continuous at t—for example, when F is discontinuous at t and F^{-1} is discontinuous at $1 - \bar{\alpha}\bar{F}(t)$.

(5) If F is continuous at $F^{-1}(1 - \bar{\alpha}\bar{F}(t))$, then $\bar{F}(t + q_\alpha(t)) = \bar{\alpha}\bar{F}(t)$.

THEOREM 2. *If F is a DPRL-α distribution, then $q_\alpha(t)$ is continuous for all $t \in [0, T)$. If F is also continuous, then $F(t)$ is strictly increasing for all $t \in [F^{-1}(\alpha), T)$.*

Proof. Since q_α is decreasing, $q_\alpha(t^-) \geq q_\alpha(t^+)$ for all $0 < t < T$. By Remark (3) above, $q_\alpha(t^-) = q_\alpha(t^+)$ for all $0 < t < T$. Similarly, $q_\alpha(0) = q_\alpha(0^+)$. Thus, q_α is continuous for $t \in [0, T)$.

If F is continuous, then the continuity of q_α implies that $F^{-1}((1 - \bar{\alpha}\bar{F}(t))^+) = F^{-1}(1 - \bar{\alpha}\bar{F}(t))$ for all $0 \leq t < T$ such that $F(t + \epsilon) - F(t) > 0$ for every $\epsilon > 0$. Thus, $F^{-1}(u)$ is continuous for $u \in [\alpha, 1)$ and $F(t)$ is strictly increasing for $t \in [F^{-1}(\alpha), T)$.

THEOREM 3. *Suppose F has a density f and thus a failure rate function $r = f/\bar{F}$. Then $q_\alpha(t)$ satisfies the equation*

$$\int_t^{t+q_\alpha(t)} r(y)dy = -\log(1 - \alpha), \qquad 0 \leq t < T.$$

If r is continuous and q_α is differentiable, then $r(t + q_\alpha(t))(1 + q_\alpha'(t)) = r(t)$ or $q_\alpha'(t) = r(t)/r(t + q_\alpha(t)) - 1$ if $r(t + q_\alpha(t)) > 0$.

Proof. Let $0 \le t < T$; $-\log \bar{F}(x) = \int_0^x r(y)dy$, so that $-\log \bar{F}_t(x) = -\log \bar{F}(t+x) + \log \bar{F}(t) = \int_t^{t+x} r(y)dy$. Since F_t is continuous, $F_t(q_\alpha(t)) = \alpha$. Thus $-\log \bar{F}_t(q_\alpha(t)) = -\log(1-\alpha) = \int_t^{t+q_\alpha(t)} r(y)dy$. Upon differentiation, we obtain the last result of the theorem.

COROLLARY 4. *Suppose F has a continuous density f. Then F is DPRL-α if and only if $r(t) \le r(t + q_\alpha(t))$ for all $0 \le t < T$ and F is IPRL-α if and only if $r(t) \ge r(t + q_\alpha(t))$ for all $0 \le t < T$.*

Remark. Suppose that the failure rate function r of F exists, and that there exists a point t_0 such that $r(t) \le r(t_0)$ for all $0 \le t \le t_0$ and $r(t)$ is increasing for $t \ge t_0$ (see Figure 1). Then, as a consequence of Theorem 3, F is a DPRL-α distribution for all $\alpha_0 \le \alpha < 1$ for some $0 < \alpha_0 \le F(t_0)$. Thus the class of DPRL-α distributions includes not only IFR distributions, but also some distributions with failure rate functions that are eventually increasing (this condition may be more realistic in some situations). For example, a distribution with bathtub-shaped failure rate function is DPRL-α for all $\alpha_0 \le \alpha < 1$ for some $\alpha_0 > 0$, provided there exists a t_0 with $r(t_0) \ge r(0)$.

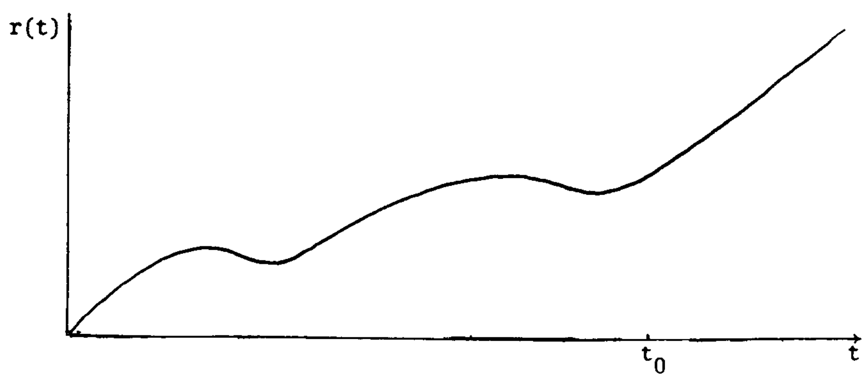

Figure 1. Failure rate function corresponding to F.

Haines and Singpurwalla have established a version of the next theorem for DPRL-α and IPRL-α distributions.

THEOREM 5. (a) *If F is an NBUP-α distribution, then $\bar{F}(t) \le (1-\alpha)^n$ for $nq_\alpha(0) \le t < (n+1)q_\alpha(0)$, $n = 0, 1, 2, \cdots$.* (b) *If F is a continuous NWUP-α distribution, then $\bar{F}(t) \ge (1-\alpha)^{n+1}$ for $nq_\alpha(0) \le t < (n+1)q_\alpha(0)$, $n = 0, 1, 2, \cdots$.*

Proof. (a) If $0 \le t < T$, then $F^{-1}(1 - \bar{\alpha}\bar{F}(t)) - t = q_\alpha(t) \le q_\alpha(0)$, and if $T < \infty$ and $t \ge T$, then $F^{-1}(1 - \bar{\alpha}\bar{F}(t)) - t \le 0 < q_\alpha(0)$. Therefore, for all

$t \geq 0$, $1 - \bar{\alpha}\bar{F}(t) \leq F[F^{-1}(1 - \bar{\alpha}\bar{F}(t))] \leq F(t + q_\alpha(0))$ or $\bar{F}(t + q_\alpha(0)) \leq \bar{\alpha}\bar{F}(t)$. By induction, $\bar{F}(t + nq_\alpha(0)) \leq \bar{\alpha}^n\bar{F}(t)$ for $n = 0, 1, 2, \cdots$. Thus, for $0 \leq t < q_\alpha(0)$, $\bar{F}(t + nq_\alpha(0)) \leq \bar{\alpha}^n = (1 - \alpha)^n$ for $n = 0, 1, 2, \cdots$. (b) Since $q_\alpha(t) \geq q_\alpha(0)$ for all $t \geq 0$ and F is continuous, then $1 - \bar{\alpha}\bar{F}(t) = F[F^{-1}(1 - \bar{\alpha}\bar{F}(t))] \geq F(t + q_\alpha(0))$. By induction, $\bar{F}(t + nq_\alpha(0)) \geq \bar{\alpha}^n\bar{F}(t)$ for $n = 0, 1, 2, \cdots$. Thus, for $0 \leq t < q_\alpha(0)$, $\bar{F}(t + nq_\alpha(0)) \geq \bar{\alpha}^n\bar{F}(q_\alpha(0)) = (1 - \alpha)^{n+1}$ for $n = 0, 1, 2, \cdots$.

COROLLARY 6. *An NBUP-α distribution F has a finite mean that is bounded above by $q_\alpha(0)/\alpha$. A continuous NWUP-α distribution F has a mean (possibly infinite) that is bounded from below by $(1 - \alpha)q_\alpha(0)/\alpha$.*

Proof. If F is an NBUP-α distribution, then by Theorem 5, $\int_0^\infty \bar{F}(t)dt = \sum_{n=0}^\infty \int_{nq_\alpha(0)}^{(n+1)q_\alpha(0)} \bar{F}(t)dt \leq \sum_{n=0}^\infty (1 - \alpha)^n q_\alpha(0) = q_\alpha(0)/\alpha$. If F is a continuous NWUP-α distribution, then by Theorem 5, $\int_0^\infty \bar{F}(t)dt \geq \sum_{n=0}^\infty (1 - \alpha)^{n+1}q_\alpha(0) = (1 - \alpha)q_\alpha(0)/\alpha$.

COROLLARY 7. *An NBUP-α distribution F has finite moment of order γ for every $\gamma \geq 1$ (and hence for every $\gamma > 0$).*

Proof. Let $\gamma \geq 1$. Then, by the bound of Theorem 5, $\int_0^\infty t^\gamma dF(t) = \gamma \int_0^\infty t^{\gamma-1}\bar{F}(t)dt = \gamma \sum_{n=0}^\infty \int_{nq_\alpha(0)}^{(n+1)q_\alpha(0)} t^{\gamma-1}\bar{F}(t)dt \leq \gamma \sum_{n=0}^\infty [(n + 1)q_\alpha(0)]^{\gamma-1} (1 - \alpha)^n q_\alpha(0) = \gamma[q_\alpha(0)]^\gamma \sum_{n=0}^\infty (n + 1)^{\gamma-1} (1 - \alpha)^n < \infty$.

The next theorem characterizes the class of CPRL-α distributions.

THEOREM 8. *F is a CPRL-α distribution if and only if $F(0) = 0$, F is not constant on any interval, and there exists a $t_0 > 0$ such that $\bar{F}(t + t_0) = \bar{\alpha}\bar{F}(t)$ for all $t \geq 0$. In this case, $q_\alpha(t) \equiv t_0$.*

Proof. Suppose first that F is a CPRL-α distribution and let $q_\alpha(t) \equiv t_0$. If F is continuous at $t + t_0$, then $\bar{F}(t + t_0) = \bar{\alpha}\bar{F}(t)$. If F is discontinuous at $t + t_0$, then $\bar{F}(t + t_0) \leq \bar{\alpha}\bar{F}(t)$ by the definition of $q_\alpha(t)$. If $\bar{F}(t + t_0) < \bar{\alpha}\bar{F}(t)$, then $\bar{F}(t + t_0) \leq \bar{\alpha}\bar{F}(t + \delta)$ for some $\delta > 0$ since F is right continuous. This result would imply $q_\alpha(t + \delta) \leq t_0 - \delta$, which is a contradiction.

Now, suppose that F is not constant on any interval and $\bar{F}(t + t_0) = \bar{\alpha}\bar{F}(t)$ for all $t \geq 0$, where $t_0 > 0$. Then, by the definition of $q_\alpha(t)$, $q_\alpha(t) \equiv t_0$.

The next theorem characterizes the class of CPRL-α distributions that have densities. Haines and Singpurwalla have given one direction of this characterization.

THEOREM 9. *Let F be a distribution with density f. Then F is CPRL-α if and only if $r = f/\bar{F}$ is not identically zero on any interval and there exists a $t_0 > 0$ such that r is an (essentially) periodic function with period t_0 and $-\log(1 - \alpha) = \int_0^{t_0} r(y)dy$. In this case, $q_\alpha(t) \equiv t_0$.*

Proof. Suppose first that F is CPRL-α and let $q_\alpha(t) \equiv t_0$. By Theorem 3, $\int_t^{t+t_0} r(y)dy = -\log(1-\alpha)$ for all $t \geq 0$. Upon differentiation, $r(t+t_0) = r(t)$ a.e. Thus, r is (essentially) periodic with period t_0 and $\int_0^{t_0} r(y)dy = -\log(1-\alpha)$. Also, r cannot be identically zero on any interval, because then q_α would be strictly decreasing on the interval.

Now suppose that r is periodic with period $t_0 > 0$, r is not identically zero on any interval, and $-\log(1-\alpha) = \int_0^{t_0} r(y)dy$. Then $-\log(1-\alpha) = \int_t^{t+t_0} r(y)dy$ for all $t \geq 0$, and by Theorem 3 and the condition that r is not identically zero on any interval, $q_\alpha(t) = t_0$ for all $t \geq 0$. Thus F is CPRL-α.

The above theorem clearly shows that the exponential distributions are not the only distributions that are CPRL-α. Also, from the theorem, we see that if F is both CPRL-α and CPRL-β ($\alpha \neq \beta$), F has a density and $q_\alpha(t)/q_\beta(t)$ is irrational, then F must be exponential. (This result is true since if r is periodic with periods t_1 and t_2 such that t_1/t_2 is irrational, then r is periodic with a dense set of periods and hence is constant a.e. (cf. Semadeni [1964]). Actually, this result can be proved without assuming that F has a density.

THEOREM 10. *Suppose that F is CPRL-α and CPRL-β ($\alpha \neq \beta$) and that $q_\alpha(t)/q_\beta(t)$ is irrational. Then F is exponential.*

Proof. Let $t_\alpha \equiv q_\alpha(t)$, $t_\beta \equiv q_\beta(t)$, $\bar{\alpha} = 1-\alpha$ and $\bar{\beta} = 1-\beta$. By Theorem 8 and the condition that $F(0) = 0$, $\bar{F}(t+t_\alpha) = \bar{\alpha}\bar{F}(t) = \bar{F}(t_\alpha)\bar{F}(t)$ and $\bar{F}(t+t_\beta) = \bar{\beta}\bar{F}(t) = \bar{F}(t_\beta)\bar{F}(t)$ for all $t \geq 0$. By Theorems 1 and 2 of Marsaglia and Tubilla [1975], F is exponential.

In general, an α-percentile residual life function $q_\alpha(t)$ does not uniquely determine a distribution function, which is in sharp contrast to the situation for the mean residual life function $m(t)$, since $m(t)$ uniquely determines a distribution function. In fact, the inversion formula is given by $\bar{F}(t) = \bar{F}(0)[m(0)/m(t)]\exp\{-\int_0^t [m(x)]^{-1}dx\}$ for all $0 \leq t < T$. However, under some conditions, a distribution function F is determined by its α-percentile residual life function $q_\alpha(t)$ together with its values on $[0, q_\alpha(0)]$.

THEOREM 11. *Let F be a continuous distribution with α-percentile residual life function $q_\alpha(t)$. Let $s_\alpha(t) = \inf\{x\colon x + q_\alpha(x) > t\}$ whenever $q_\alpha(0) \leq t < T$. Then $F(t) = 1 - \bar{\alpha}\bar{F}(s_\alpha(t))$ for all $q_\alpha(0) \leq t < T$. That is, given $q_\alpha(t)$ for all $0 \leq t < T$, and $F(t)$ for all $0 \leq t \leq q_\alpha(0)$, we can deduce $F(t)$ for $q_\alpha(0) \leq t < T$.*

Proof. Since F is continuous, $\bar{F}(t + q_\alpha(t)) = \bar{\alpha}\bar{F}(t)$ for all $0 \leq t < T$. Since $t + q_\alpha(t)$ is increasing in $t \in [0, T)$, it has a right continuous inverse, which is $s_\alpha(t)$ for all $q_\alpha(0) \leq t < T$. Thus, $\bar{F}(t) = \bar{\alpha}\bar{F}(s_\alpha(t))$ for all $q_\alpha(0) \leq t < T$.

Remark. The equation $F(t) = 1 - \bar{\alpha}\bar{F}(s_\alpha(t))$ for all $q_\alpha(0) \le t < T$ is valid for a distribution F with discontinuities if q_α is continuous at the discontinuity points of F. An example where the equation is not satisfied occurs when F is an empirical distribution function of a sample.

From Theorem 11, we can see how nonuniqueness arises for a possible α-percentile residual life function $q(t)$. Theorem 12 below shows the nonuniqueness for $q(t)$ satisfying the condition that $t + q(t)$ is strictly increasing and continuous on the domain of q. More care is needed to show nonuniqueness for a general $q(t)$. A necessary condition for $q(t)$ to be an α-percentile residual life function is that $q(t)$ has right-hand and left-hand limits and $t + q(t)$ is increasing on the domain of q. This condition is not sufficient; there is a restriction on the rate at which $t + q(t)$ can increase. (This situation is analogous to the result of Hall and Wellner that a mean residual life function $m(t)$ satisfies $\int_0^T [m(t)]^{-1}dt = \infty$.) For example, let $q^*(t) = 2$ for $0 \le t \le 1$ and let $q^*(t) = 102$ for $t > 1$. Since q^* has a jump of 100 at $t = 1$, if F is a distribution with α-percentile residual life function $q^*(t)$, then it must have an interval of length 100 where it is constant. But then $q^*(t)$ should be strictly decreasing on this interval. Thus, $q^*(t)$ cannot be an α-percentile residual life function.

THEOREM 12. *Let $q(t)$ be defined on $[0, T^*)$, where $0 < T^* \le \infty$. Suppose that $t + q(t)$ is strictly increasing and continuous on $[0, T^*)$ and that $\lim_{t \uparrow T^*} q(t) = 0$ if $T^* < \infty$. Then there are infinitely many distribution functions with α-percentile residual life function $q(t)$.*

Proof. Let $s(t)$ for all $q(0) \le t < T^*$ be the inverse of $t + q(t)$. Let $G(t)$ be defined on $[0, q(0)]$, and suppose that G is continuous and strictly increasing, $G(0) = 0$ and $G(q(0)) = \alpha$. Define $F(t) = G(t)$ for all $0 \le t \le q(0)$ and $F(t) = 1 - \bar{\alpha}\bar{F}(s(t))$ for all $q(0) < t < T^*$. Then $F(t)$ has α-percentile residual life function $q(t)$.

Remark. Schmittlein and Morrison [1981] stated the following characterization theorem involving the median residual life function: A distribution F has a linearly increasing median residual life function (i.e., $q_{0.5}(t) = a + bt$, where $a > 0$ and $b > 0$) if and only if F is a Pareto distribution of the second kind (gamma mixture of exponentials). As stated, this result contradicts Theorem 12. Schmittlein and Morrison, in fact, proved a slightly different theorem since in their proof they assume that F has a density and furthermore that the failure rate function is analytic or analytic except for a pole of finite order at $-a/b$ (when the domain is extended). Thus, they proved that the Pareto distribution of the second kind is the only "very smooth" distribution with linearly increasing median residual life function.

The next theorem shows that if F is continuous, then F is uniquely determined by the function q_β for all large enough β.

THEOREM 13. *Let $q_\beta(t)$, $\alpha \le \beta < 1$, be the β-percentile residual life function of a continuous life distribution. Then the continuous distribution with these percentile residual life functions is unique.*

Proof. Let F and G be two continuous distributions and let their β-percentile ($\alpha \le \beta < 1$) residual life functions be denoted by $q_{\beta,F}(t)$ and $q_{\beta,G}(t)$ respectively. Suppose $q_{\beta,F}(t) = q_{\beta,G}(t)$ for all $0 \le t < F^{-1}(1) = G^{-1}(1)$ and for all $\alpha \le \beta < 1$. Then $F^{-1}(1 - \bar{\beta}\bar{F}(t)) = G^{-1}(1 - \bar{\beta}\bar{G}(t))$ for all t and for all $\alpha \le \beta < 1$, where $\bar{\beta} = 1 - \beta$. Letting $t = 0$, we see that $F^{-1}(\beta) = G^{-1}(\beta)$ for all $\alpha \le \beta < 1$. Let $y \ge F^{-1}(\alpha)$. Then $G^{-1}(F(y)) = F^{-1}(F(y)) = x$, where $x = \{\inf z\colon F(z) = F(y)\}$. Since F and G are continuous, $F(y) = G(x) = F(x)$. If $x \ne y$, then F is constant on $[x, y]$ and $q_{\beta,F}$ is decreasing with slope -1 on $[x, y]$. Since $q_{\beta,F} = q_{\beta,G}$ for all $\alpha \le \beta < 1$ and G is continuous, G must be constant on $[x, y]$ and hence $F(y) = G(y)$. Thus $F = G$ on $[F^{-1}(\alpha), F^{-1}(1))$. The equations $\bar{F}(t + q_{\alpha,F}(t)) = \bar{\alpha}\bar{F}(t)$, $\bar{G}(t + q_{\alpha,G}(t)) = \bar{\alpha}\bar{G}(t)$ then show that $F = G$ on $[0, F^{-1}(\alpha))$.

Remark. The continuity assumption in Theorem 13 is necessary. For example, if $F_1(t) = 0$ for all $0 \le t < 1$, $F_1(t) = 1$ for all $t \ge 1$ and $F_2(t) = 0.5t$ for all $0 \le t < 1$, $F_2(t) = 1$ for all $t \ge 1$, then both have β-percentile residual life function $q_\beta(t) = 1 - t$, $0 \le t \le 1$, whenever $0.5 \le \beta \le 1$.

3. PRESERVATION UNDER RELIABILITY OPERATIONS

For each of the well-known classes of life distributions defined in Section 1, reliabilists have studied closure under the reliability operations of (a) formation of coherent systems, (b) addition of independent life lengths (i.e., convolution of distributions), and (c) mixture of distributions (see Barlow and Proschan or Haines and Singpurwalla). In this section, we consider closure under these operations for the DPRL-α, NBUP-α, IPRL-α, and NWUP-α classes.

Haines and Singpurwalla showed that the DPRL-α class is not closed under formation of coherent systems or under mixture of distributions, and that the IRPL-α class is not closed under any of the above three reliability operations. The same counterexamples work for the corresponding operations with the NBUP-α and NWUP-α classes. The only question remaining is whether or not the DPRL-α and NBUP-α classes are preserved under convolution. It is well known that the IFR, NBU, and NBUE classes are closed under convolution. However, Bondesson [1982] has recently shown by an example that the DMRL class is *not* closed under convolution. A similar example (given below) shows that the DPRL-α and NBUP-α classes also are *not* closed under convolution.

Let $F(t) = pU(t) + (1 - p)I(t - 2)$ for all $t \ge 0$, where

$$U(x) = \begin{cases} x & 0 \le x \le 1 \\ 1 & x \ge 1 \end{cases}, \qquad I(x) = \begin{cases} 0 & x < 0 \\ 1 & x \ge 0 \end{cases},$$

and $0 < p < 1$. That is, F is a mixture of a uniform distribution on (0, 1) and a mass at 2. Note that F is a DPRL-α distribution for all $p < \alpha < 1$. Let $*$ denote the convolution operator and let $H = F * F$. Then

$$H(t) = p^2 U{*}U(t) + 2p(1 - p)U(t - 2) + (1 - p)^2 I(t - 4).$$

$H(t)$ is strictly increasing on [0, 3), constant on [3, 4) and has a jump of size $(1 - p)^2$ at $t = 4$. Thus H is a DPRL-α distribution for all $p_2 < \alpha < 1$, where $p_2 = 1 - (1 - p)^2$. But for $0 < \alpha \le p_2$, H is not a DPRL-α distribution because there is a $t_\alpha \in [0, 3)$ such that $q_{\alpha,H}(t_\alpha^-) = 3 - t_\alpha$ and $q_{\alpha,H}(t_\alpha^+) = 4 - t_\alpha$. Also, there exists $\delta(p) > 0$ such that for $p_2 - \delta(p) < \alpha < p_2$, $q_{\alpha,H}(0) < 3$ and $q_{\alpha,H}(t_\alpha^+) > 3$, so that H is not an NBUP-α distribution for α in this range. For fixed $0 < \alpha < 1$, choosing p such that $p < \alpha < p_2$ shows that the DPRL-α class is not closed under convolution and choosing p such that $p < p_2 - \delta(p) < \alpha < p_2$ shows that the NBUP-α class is not closed under convolution.

Note that in the above example, F is not a continuous distribution function. If $G = F * U_\theta$ and $H = G * G$, where $U_\theta(x) = U(x/\theta)$, then G and H are continuous and the above conclusions are still valid for a small $\theta > 0$.

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