Mean Residual Lifetime Function for Generalized Gamma Distribution

Zekai Wang

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The PDF of the generalized gamma (GG) distribution is:

$$f(x;a,b,k) = \frac{bx^{bk-1}}{\Gamma(k)a^{bk}}\mathtt{exp}((-\frac{x}{a})^b) \tag{1}$$

where a,b,k are parameters of GG, $\Gamma(k)$ is Gamma function.

We can represent Mean Residual Lifetime (MRL) function, $m(x)$, for a specific distribution using the survival function, $S(x)$, and its pdf:

$$m(x) = \frac{\int_x^\infty tf(t)\mathtt{dt}}{S(x)} - x \tag{2}$$

The numerator of equation 2 can be written as:

$$\begin{aligned} \int_x^\infty tf(t) &= \int_x^\infty t\frac{bt^{bk-1}}{\Gamma(k)a^{bk}}\mathtt{exp}((-\frac{t}{a})^b)\mathtt{dt} \\ &= \frac{b}{\Gamma(k)}\int_x^\infty (\frac{t}{a})^{bk}\mathtt{exp}((-\frac{t}{a})^b)\mathtt{dt} \end{aligned} \tag{3}$$

Let $u = (\frac{t}{a})^b$, then we can obtain $\mathtt{du} = \frac{b}{a}(\frac{t}{a})^{b-1}\mathtt{dt}$

Thus, the new limit for u become $u = (\frac{x}{a})^b$ when $t = x$ and $u \to \infty$, when $t \to \infty$.

Make a substitution to simplify the integral 3:

$$\begin{aligned} \int_x^\infty tf(t)\mathtt{dt} &= \frac{b}{\Gamma(k)}\int_x^\infty (\frac{t}{a})^{bk}\mathtt{exp}((-\frac{t}{a})^b)\mathtt{dt} \\ &= \frac{b}{\Gamma(k)}\int_{(\frac{x}{a})^b}^\infty u^k\mathtt{exp}(-u)\mathtt{du} \end{aligned} \tag{4}$$

The integral part of equation 4 is known as the upper incomplete gamma function, which is defined as $\Gamma(k+1,(\frac{x}{a})^b)$.

Thus,

$$\int_x^\infty tf(t)\mathtt{dt} \quad = \quad b\frac{\Gamma(k+1,(\frac{x}{a})^b)}{\Gamma(k)} \tag{5}$$

Since the survival function of GG can be represented as:

$$S(x) = \frac{\Gamma(k,(b^{\frac{1}{bk}}a{-}1t)^b)}{\Gamma(k)} \tag{6}$$

We can express MRL for GG distribution as:

$$\begin{aligned} m(x) \quad &= \quad \frac{\int_x^\infty tf(t)\mathtt{dt}}{S(x)} - x \\ &= \quad b\frac{\Gamma(k+1,(\frac{x}{a})^b)}{\Gamma(k,(b^{\frac{1}{bk}}a{-}1t)^b} - x \end{aligned} \tag{7}$$