Mean Residual Lifetime Function for Generalized Gamma Distribution (Prentice)

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The PDF of the generalized gamma (GG) distribution (Prentice) is:

$$f(x;\beta,\sigma,\lambda) = \frac{|\lambda|}{\sigma x\Gamma(\lambda^{-2})}[\lambda^{-2}(e^{-\beta}x)^{\frac{\lambda}{\sigma}}]^{\lambda^{-2}}\texttt{exp}[-\lambda^{-2}(e^{-\beta}x)^{\frac{\lambda}{\sigma}}] \tag{1}$$

where β,σ,λ are parameters of GG, $\Gamma(\lambda^{-2})$ is Gamma function.

We can represent Mean Residual Lifetime (MRL) function, $m(x)$, for a specific distribution using the survival function, $S(x)$, and its pdf:

$$m(x) = \frac{\int_x^\infty tf(t)\texttt{dt}}{S(x)} - x \tag{2}$$

The numerator of equation 2 can be written as:

$$\begin{aligned}\int_x^\infty tf(t) &= \int_x^\infty t\frac{|\lambda|}{\sigma t\Gamma(\lambda^{-2})}[\lambda^{-2}(e^{-\beta}t)^{\frac{\lambda}{\sigma}}]^{\lambda^{-2}}\texttt{exp}[-\lambda^{-2}(e^{-\beta}t)^{\frac{\lambda}{\sigma}}]\texttt{dt} \quad (3)\\ &= \frac{|\lambda|}{\sigma\Gamma(\lambda^{-2})}\int_x^\infty [\lambda^{-2}(e^{-\beta}t)^{\frac{\lambda}{\sigma}}]^{\lambda^{-2}}\texttt{exp}[-\lambda^{-2}(e^{-\beta}t)^{\frac{\lambda}{\sigma}}]\texttt{dt}\end{aligned}$$

Let $u = \lambda^{-2}(e^{-\beta}t)^{\frac{\lambda}{\sigma}}$, then, by differentiating both sides with respect to t, we can obtain by

$$\begin{aligned}\texttt{du} &= \lambda^{-2}\left(\frac{\lambda}{\sigma}\right)e^{-\beta}(te^{-\beta})^{\frac{\lambda}{\sigma}-1}\texttt{dt} \quad (4)\\ &= \left(\frac{1}{\sigma\lambda}\right)e^{-\beta}(te^{-\beta})^{\frac{\lambda}{\sigma}-1}\texttt{dt}\\ &= \left(\frac{1}{\sigma\lambda}\right)e^{-\frac{\lambda\beta}{\sigma}}(t)^{\frac{\lambda}{\sigma}-1}\texttt{dt}\end{aligned}$$

We can also express `dt` in terms of `du`:

$$\mathtt{dt} = (\lambda\sigma)e^{\frac{\lambda\beta}{\sigma}}t^{-\frac{\lambda}{\sigma}+1}\mathtt{du} \tag{5}$$

Now, we can rewrite the integral 3 as:

$$\begin{aligned}
\int_x^\infty tf(t) &= \frac{|\lambda|}{\sigma\Gamma(\lambda^{-2})}\int_x^\infty [\lambda^{-2}(e^{-\beta}t)^{\frac{\lambda}{\sigma}}]^{\lambda^{-2}}\mathtt{exp}[-\lambda^{-2}(e^{-\beta}t)^{\frac{\lambda}{\sigma}}]\mathtt{dt} \qquad (6)\\
&= \frac{|\lambda|}{\sigma\Gamma(\lambda^{-2})}\int_{u(x)}^\infty u^{\lambda^{-2}}e^{u}(\lambda\sigma)e^{\frac{\lambda\beta}{\sigma}}t^{(-\frac{\lambda}{\sigma}+1)}\mathtt{du}\\
&= \frac{|\lambda|}{\sigma\Gamma(\lambda^{-2})}\int_{u(x)}^\infty u^{\lambda^{-2}}u^{-\frac{\lambda+\sigma}{\lambda}}e^{u}e^{-\beta}\lambda^{(1-\frac{2(\lambda+\sigma)}{\lambda})}\sigma\mathtt{du}\\
&= C_1\int_{u(x)}^\infty u^{\frac{1-\lambda\sigma}{\lambda^2}-1}e^{u}\mathtt{du}\\
&=
\end{aligned}$$

where $C_1 = \frac{\lambda^{(1-\frac{2(\lambda+\sigma)}{\lambda})}|\lambda|}{\Gamma(\lambda^{-2})}e^{-\beta}$, $u(x) = \lambda^{-2}(e^{-\beta}x)^{\frac{\lambda}{\sigma}}$

The integral part of equation 6 is known as the upper incomplete gamma function,

$$\Gamma(x, a) = \int_x^\infty t^{a-1}\mathtt{exp}(-t)\mathtt{dt} \tag{7}$$

which is defined as $\Gamma((u(x), \frac{1-\lambda\sigma}{\lambda^2})$.

Since the survival function can be written as:

$$S(x) = \Gamma(u(x); \lambda^{-2}) \tag{8}$$

where $\Gamma(u(x); \lambda^{-2})$ is the cumulative distribution function (CDF) for the particular case of the gamma distribution with mean and variance equal to λ^{-2}

Thus,

We can express MRL for GG distribution (Prentice) as:

$$m(x) = C_1\frac{\Gamma((u(x), \frac{1-\lambda\sigma}{\lambda^2})}{\Gamma(u(x); \lambda^{-2})} - x \tag{9}$$