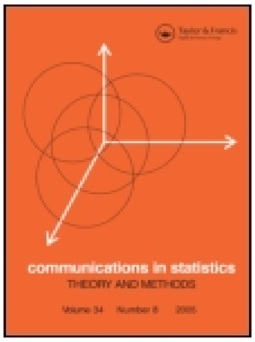


Communications in Statistics - Theory and Methods

ISSN: 0361-0926 (Print) 1532-415X (Online) Journal homepage: https://www.tandfonline.com/loi/lsta20

On a recent generalization of gamma distribution

M.E. Ghitany

COMMUN. STATIST.—THEORY METH., 27(1), 223-233 (1998)

ON A RECENT GENERALIZATION OF GAMMA DISTRIBUTION

M. E. Ghitany

Department of Statistics & Operations Research
Faculty of Science, Kuwait University
P.O.Box 5969, Safat 13060, Kuwait

Key Words: generalized gamma function; generalized incomplete gamma function; generalized gamma distribution; hazard rate; mean residual life.

ABSTRACT

In this paper, we investigate a generalized gamma distribution recently developed by Agarwal and Kalla (1996). Also, we show that such generalized distribution, like the ordinary gamma distribution, has a unique mode and, unlike the ordinary gamma distribution, may have a hazard rate (mean residual life) function which is upside-down bathtub (bathtub) shaped.

1. INTRODUCTION

Recently, Agarwal and Kalla (1996) introduced a new generalization of the ordinary gamma distribution. Their generalization is mathematically motivated by a generalized gamma function commonly used in wave scattering and diffraction theory. Analytical properties of the generalized gamma function as well as the generalized incomplete gamma function have been studied by Kobayashi (1991).

In this paper, we show that some of the results given by Agarwal and Kalla (1996) are incorrect. These include, among others, a special case of the distribution function and the moment generating function of the generalized gamma distribution. The corrections are given, respectively, in Sections 2 and 3. Also, in Section 2, we show that the unimodality of the generalized distribution depends only on one of the four parameters of the distribution. An explicit expression for the higher order derivatives of the moment generating function is also given in Section 3. In Sections 4 and 5, we show that, unlike the ordinary gamma distribution, the generalized gamma distribution may have a hazard rate (mean residual life) function that initially increases (decreases) and then decreases (increases), depending only on two of the four parameters of the distribution.

2. THE DENSITY AND DISTRIBUTION FUNCTIONS

A random variable X is said to have a generalized gamma distribution with parameters λ, m, α, n, if its p.d.f. is given by

$$f(x) = \frac{\alpha^{m-\lambda}}{\Gamma_\lambda(m,\alpha n)} \frac{x^{m-1}\, e^{-\alpha x}}{(x+n)^\lambda}, \qquad x \geq 0, \quad \lambda \geq 0, m, \alpha, n > 0, \tag{1}$$

where

$$\Gamma_\lambda(m,\alpha n) = \int_0^\infty \frac{y^{m-1}\, e^{-y}}{(y+\alpha n)^\lambda}\, dy, \tag{2}$$

is the generalized gamma function.

Clearly

$$\Gamma_0(m,\alpha n) = \int_0^\infty y^{m-1}\, e^{-y} dy = \Gamma(m), \tag{3}$$

is termed the ordinary gamma function. Hence, equation (2.2) of Agarwal and Kalla (1996) should read $\Gamma_0(m,\alpha n) = \Gamma(m)$. Also the left hand side of their equation (2.4) should read $\Gamma_1(\frac{1}{2},\alpha n)$ but not $\Gamma_1(\frac{1}{2},n)$.

Note that

$$f(0) = \begin{cases} \infty & \text{if } m < 1 \\ \frac{\alpha^{1-\lambda}}{\Gamma_\lambda(1,\alpha n)\; n^\lambda} & \text{if } m = 1 \\ 0 & \text{if } m > 1, \end{cases} \tag{4}$$

and $f(\infty) = 0$, for all $m > 0$.

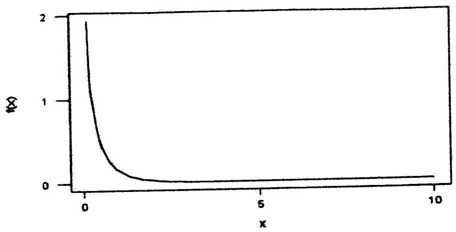
Throughout, primes will indicate differentiation w.r.t. x. We have

$$f'(x) = -\left(\alpha + \frac{\lambda}{x+n} - \frac{m-1}{x} \right) f(x). \tag{5}$$

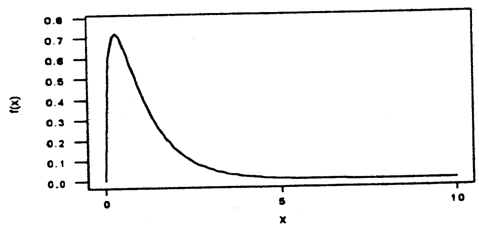
If $m \leq 1$, then $f'(x) < 0$ and hence $f(x)$ is decreasing with mode 0. On the other hand, if $m > 1$, $f(x)$ has a unique maximum at the point

$$x_{mode} = \frac{1}{2\alpha}\{m - 1 - (\lambda + \alpha n) + \sqrt{[m-1-(\lambda+\alpha n)]^2 + 4\alpha n(m-1)}\}.$$

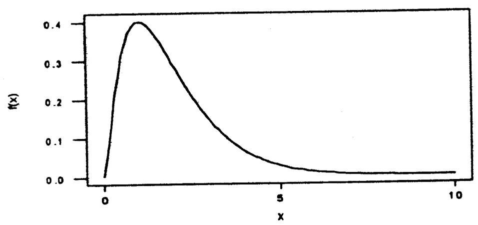
Figure 1 shows the graphs of $f(x)$ for some values of λ, m, α, n.



(a) λ=1, m=0.5, α=1, n=1



(b) λ=1, m=1.5, α=1, n=1



(c) λ=1, m=2.5, α=1, n=1

Figure 1. Graphs of some generalized gamma probability density functions.

If $\lambda = 0$, we obtain

$$f(0) = \begin{cases} \infty & \text{if } m < 1 \\ \alpha & \text{if } m = 1 \\ 0 & \text{if } m > 1, \end{cases}$$

and $f(\infty) = 0$, for all $m > 0$. Moreover, in this case, $f(x)$ decreases with mode 0 if $m \leq 1$ and has a unique maximum at the point $x_{mode} = (m-1)/\alpha$ if $m > 1$.

These, of course, are well known results for the ordinary gamma distribution (see e.g., Milton and Arnold (1990), page 100).

The c.d.f. of generalized gamma distribution with p.d.f. (1) is

$$F(x) = \int_0^x f(y)dy = \frac{\Gamma_\lambda(m, \alpha n) - \Gamma_\lambda(m, \alpha n; \alpha x)}{\Gamma_\lambda(m, \alpha n)}, \tag{6}$$

where

$$\Gamma_\lambda(m, \alpha n; \alpha x) = \int_{\alpha x}^{\infty} \frac{y^{m-1}\ e^{-y}}{(y + \alpha n)^\lambda}\ dy, \tag{7}$$

is termed the generalized incomplete gamma function.

Clearly

$$\Gamma_\lambda(m, \alpha n; 0) = \Gamma_\lambda(m, \alpha n). \tag{8}$$

Remarks:

(i) If $\lambda = 0$, then

$$\Gamma_0(m, \alpha n) - \Gamma_0(m, \alpha n; \alpha x) = \int_0^{\alpha x} y^{m-1}\ e^{-y} dy = \gamma(m, \alpha x), \tag{9}$$

which is the ordinary incomplete gamma function. Hence, using (3) and (9), the c.d.f. (6), in this case, reduces to

$$F(x) = \frac{1}{\Gamma(m)}\ \gamma(m, \alpha x),$$

which is the c.d.f. of the ordinary gamma distribution.

(ii) If $\lambda = 1$ and $m = 1$, then, using the substitution: $t = -\alpha(y + n)$, we have

$$\Gamma_1(1, \alpha n; \alpha x) = e^{\alpha n} \int_{-\infty}^{-\alpha(x+n)} \frac{e^t}{t}\ dt = e^{\alpha n}\ E_i(-\alpha(x + n)), \tag{10}$$

where

$$E_i(z) = \int_{-\infty}^{z} \frac{e^t}{t}\ dt, \qquad z < 0,$$

is the exponential integral function.

Hence, using (8) and (10), the c.d.f. (6), in this case, reduces to

$$F(x) = 1 - \frac{\Gamma_1(1, \alpha n; \alpha x)}{\Gamma_1(1, \alpha n; 0)} = 1 - \frac{E_i(-\alpha(x + n))}{E_i(-\alpha n)}.$$

3. THE MOMEMT GENERATING FUNCTION

The following theorem corrects the expression and proof of the m.g.f. of the generalized gamma distribution given in Theorem 2 of Agarwal and Kalla (1996), page 205.

Theorem 1. The m.g.f. of the generalized gamma distribution with p.d.f. (1) is

$$M(t) = (1-t/\alpha)^{\lambda-m} \frac{\Gamma_\lambda(m, \alpha n(1-t/\alpha))}{\Gamma_\lambda(m, \alpha n)}, \qquad t < \alpha.$$

Proof. By definition, we have

$$M(t) = E(e^{tX}) = \frac{\alpha^{m-\lambda}}{\Gamma_\lambda(m, \alpha n)} \int_0^\infty \frac{x^{m-1}\, e^{-\alpha(1-t/\alpha)x}}{(x+n)^\lambda}\, dx.$$

Using the substitution: $y = \alpha(1-t/\alpha)x$, we obtain

$$M(t) = \frac{(1-t/\alpha)^{\lambda-m}}{\Gamma_\lambda(m, \alpha n)} \int_0^\infty \frac{y^{m-1}\, e^{-y}}{[y+\alpha n(1-t/\alpha)]^\lambda}\, dy,$$

proving the theorem. □

The following theorem gives the higher order derivatives of $M(t)$.

Theorem 2. The rth derivative, w.r.t. t, of $M(t)$ is:

$$M^{(r)}(t) = \alpha^{-r}(1-t/\alpha)^{\lambda-m-r} \frac{\Gamma_\lambda(m+r, \alpha n(1-t/\alpha))}{\Gamma_\lambda(m, \alpha n)}, \qquad t < \alpha.$$

Proof. The proof of the theorem follows by mathematical induction. □

Some special cases of Theorems 1 and 2 above are given below.

(i) If $\lambda = 0$, then $M(t) = (1-t/\alpha)^{-m}$, $t < \alpha$, which is the m.g.f. of gamma distribution with shape parameter m and scale parameter $1/\alpha$.

(ii) Since

$$M(ct) = (1-ct/\alpha)^{\lambda-m} \frac{\Gamma_\lambda(m, \alpha n(1-ct/\alpha))}{\Gamma_\lambda(m, \alpha n)}, \qquad t < \frac{\alpha}{c},\ c > 0,$$

it follows that the random variable $Y = cX$, $c > 0$, has also a generalized gamma distribution with parameters $\lambda, m, \frac{\alpha}{c}, cn$.

(iii) By Theorem 2, the rth moment about the origin of X is

$$M^{(r)}(0) = \alpha^{-r} \frac{\Gamma_\lambda(m+r, \alpha n)}{\Gamma_\lambda(m, \alpha n)}.$$

4. THE HAZARD RATE FUNCTION

Let X be a non-negative continuous random variable which represents the survival time of an item. Let f and F be the p.d.f. and c.d.f. of X. The hazard rate function $h(x)$ (also called failure rate, force of mortality or age-specific mortality rate) is defined as

$$h(x) = \frac{f(x)}{S(x)}, \qquad S(x) = 1 - F(x) > 0, \qquad x \geq 0.$$

The function $S(x)$ is known as the survivor (or reliability) function of X.

Increasing (decreasing) hazard rate functions represent positive (negative) aging. Constant hazard rate functions represent no aging. Hazard rate functions that first decrease (increase) and then increase (decrease) are usually termed bathtub (upside-down bathtub) shaped.

For the generalized gamma distribution with p.d.f. (1), we have

$$h(x) = \frac{\alpha^{m-\lambda}}{\Gamma_\lambda(m, \alpha n; \alpha x)} \; \frac{x^{m-1}\, e^{-\alpha x}}{(x+n)^\lambda}, \qquad x \geq 0. \tag{11}$$

Note that $h(0) = f(0)$ and, by L'Hôpital rule and (5), we have

$$h(\infty) = \lim_{x \to \infty} - \frac{f'(x)}{f(x)} = \alpha. \tag{12}$$

It can be seen that (11) does not allow an easy study of the shape behaviour of $h(x)$. Fourtunately, the following theorem by Glaser (1980) provides us with simple devices that facilitate such study. Throughout, monotone properties of functions are w.r.t. x.

Theorem 3. Let X be a continuous random variable with twice differentiable p.d.f. $f(x)$ and hazard rate function $h(x), x \geq 0$. Define $\eta(x) = -f'(x)/f(x)$.

(I) If $\eta'(x) < 0$ for all $x > 0$, then $h(x)$ is decreasing.

(II) If $\eta'(x) > 0$ for all $x > 0$, then $h(x)$ is increasing.

(III) Suppose $\eta'(x) > 0$ for all $0 < x < x_0$, $\eta'(x_0) = 0$, and $\eta'(x) < 0$ for all $x > x_0$. If, in addition, $f(0) = 0$, then $h(x)$ is upside-down bathtub shaped with a unique change point $x_h \in (0, x_0)$.

In the following theorem, we show that the generalized gamma distribution possesses four types of hazard rate functions, *viz.* constant, decresing, increasing, or upside-down bathtub shaped, depending only on the values of λ and m.

Theorem 4. Let X be a generalized gamma random variable with hazard rate function $h(x)$ given by (11).

(a) $h(x) = \alpha$ if and only if $\lambda = 0$ & $m = 1$.

(b) $h(x)$ is decreasing if $\lambda = 0$ & $m < 1$, or $\lambda > 0$ & $m \leq 1$.

(c) $h(x)$ is increasing if $\lambda = 0$ & $m > 1$, or $0 < \lambda \leq m - 1$.

(d) $h(x)$ is upside-down bathtub shaped with a unique change point $x_h \in (0, x_0)$, where

$$x_0 = \frac{m - 1 + \sqrt{(m-1)\lambda}}{\lambda - (m-1)} n, \tag{13}$$

if $\lambda > m - 1 > 0$.

Proof. (a)This is a well known property for the exponential distribution. For the generalized gamma distribution, we have, we have

$$\eta'(x) = \frac{m-1}{x^2} - \frac{\lambda}{(x+n)^2}.$$

Hence, (b)-(d) follow by Theorem 3 (I)-(II), respectively. □

Figure 2 shows the graphs of $h(x)$ for some values of λ, m, α, n.

Figure 2 (b) shows that $x_h = 1.915 < 2.414 = x_0$.

Remarks:

(i) If $\lambda = 0$, then Theorem 4 states that

$$h(x) \begin{cases} \text{decreasing} & \text{if } m < 1 \\ = \alpha & \text{if } m = 1 \\ \text{increasing} & \text{if } m > 1. \end{cases}$$

These are well known properties about gamma distribution with shape parameter m and scale parameter $1/\alpha$ (see e.g. Barlow and Proschan (1975), page 75).

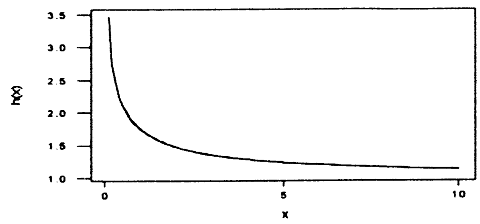
(ii) Consider the case $\lambda > m - 1 > 0$ and claim that $h(x_h) \leq \alpha$. Since, in this case, $h(x)$ decreases on (x_h, ∞), then $h(\infty) = 0$, contradicting (12). Hence, we must have $h(x_h) > \alpha$. As an example, see Figure 2 (b), if $\lambda = 1, m = 1.5, \alpha = 1, n = 1$, then $h(x_h) = h(1.915) = 1.08193$.

5. THE MEAN RESIDUAL LIFE

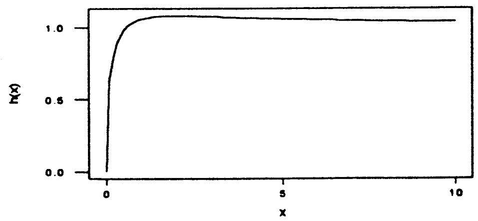
Another important aging property is the mean residual life function:

$$k(x) = E\{X - x | X \geq x\} = \frac{1}{S(x)} \int_x^\infty (y - x) f(y) dy, \qquad x \geq 0.$$

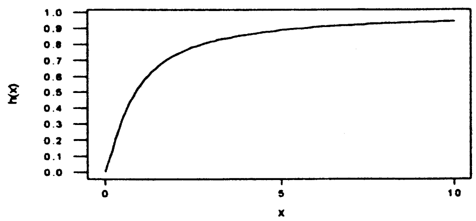
Increasing (decreasing) mean residual life functions represent beneficial (adverse) aging. Mean residual life functions that first decrease (increase) and then increase (decrease) are usually termed bathtub (upside-down bathtub) shaped.



(a) $\lambda=1,\ m=0.5,\ \alpha=1,\ n=1$



(b) $\lambda=1,\ m=1.5,\ \alpha=1,\ n=1$



(c) $\lambda=1,\ m=2.5,\ \alpha=1,\ n=1$

Figure 2. Graphs of some generalized gamma hazard rate functions.

For the generalized gamma distribution with p.d.f. (1), we have

$$k(x) = \alpha^{-1}\ \frac{\Gamma_\lambda(m+1, \alpha n; \alpha x)}{\Gamma_\lambda(m, \alpha n; \alpha x)}\ -\ x, \qquad x \geq 0. \tag{14}$$

The following theorem gives the connection between the shape of the mean residual life and the shape of the corresponding hazard rate function.

Theorem 5. Let X be a non-negative continuous random variable with p.d.f. $f(x)$, survival function $S(x)$, and differentiable hazard rate function $h(x)$. Assume that X has a finite mean μ.

(i) $k(x) = \mu$ if and only if X has an exponential distribution with mean μ.

(ii) $k(x)$ is decreasing (increasing) if $h(x)$ is increasing (decreasing).

(iii) $k(x)$ is bathtub shaped with a unique change point x_k if $h(x)$ is upside-down bathtub shaped with a unique change point x_h and $f(0)\mu < 1$.

Proof. (i) This is a well known characterization , see e.g. Shanbhag (1970).
(ii) This is a well known implication, see e.g. Klefsjö (1989), page 560.
(iii) By definition, we have

$$k(x) = \frac{1}{S(x)} \int_x^\infty S(y)\, dy.$$

Hence, it can be shown that

$$k'(x) = \frac{g(x)}{S(x)}, \tag{15}$$

where

$$g(x) = h(x) \int_x^\infty S(y)\, dy \ - \ S(x).$$

Since $h(x)$ is (positive) upside-down shaped with a unique change point x_h and

$$g'(x) = h'(x) \int_x^\infty S(y)\, dy,$$

then $g(x)$ is also upside-down shaped with the same unique change point x_h.

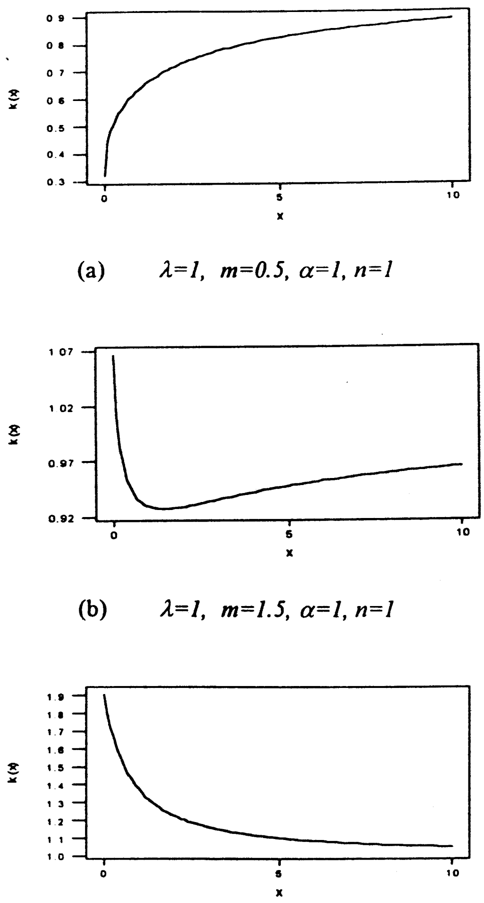
Since $g(x)$ decreases on $[x_h, \infty)$, we have

$$h(x) \int_x^\infty S(y)\, dy \ > \int_x^\infty h(y)\, S(y)\, dy = S(x).$$

Hence, $g(x) > 0$, for all $x \in [x_h, \infty)$. Also, since $g(x)$ increases on $[0, x_h)$ and $g(0) = f(0)\ \mu \ - \ 1 \ < 0$, then there exists a unique point $x_k \in (0, x_h)$ s.t. $g(x)$ is negative on $[0, x_k)$, is 0 at x_k, and is positive on (x_k, ∞). Finally, using (15), it can be seen that $k(x)$ is bathtub shaped with a unique change point $x_k \in (0, x_h)$. □

The following theorem gives conditions for the properties of the mean residual life function of the generalized gamma distribution.

Theorem 6. Let X be a generalized gamma random variable with mean residual life function $k(x)$ given by (14).



(a) $\lambda=1,\ m=0.5,\ \alpha=1,\ n=1$

(b) $\lambda=1,\ m=1.5,\ \alpha=1,\ n=1$

(c) $\lambda=1,\ m=2.5,\ \alpha=1,\ n=1$

Figure 3. Graphs of some generalized gamma mean residual life functions.

(a) $k(x) = 1/\alpha$ if and only if $\lambda = 0$ & $m = 1$.

(b) $k(x)$ is increasing if $\lambda = 0$ & $m < 1$, or $\lambda > 0$ & $m \leq 1$.

(c) $k(x)$ is decreasing if $\lambda = 0$ & $m > 1$, or $0 < \lambda \leq m - 1$.

(d) $k(x)$ is bathtub shaped with a unique change point, x_k, if $\lambda > m - 1 > 0$.

Proof. By Theorem 4, the conditions for (a)-(d) imply, respectively, that $h(x)$

is constant, decreasing, increasing, and upside-down bathtub shaped with a unique change point x_h. Hence, (a)– (d) follow by Theorem 5. □

Figure 3 shows the graphs of $k(x)$ for some values of λ, m, α, n.

Figure 3 (b) shows that $x_k = 1.425 < 1.915 = x_h$.

Remark.

If $\lambda = 0$, then Theorem 6 states that

$$k(x) \begin{cases} \text{increasing} & \text{if } m < 1 \\ = 1/\alpha & \text{if } m = 1 \\ \text{decreasing} & \text{if } m > 1. \end{cases}$$

which proves a conjecture by Osaki and Li (1988), page 381, regarding the mean residual life of a gamma distribution with shape parameter m and scale parameter $1/\alpha$.

ACKNOWLEDGEMENT

The research reported in this paper is partially supported by the Research Administration of Kuwait University (Research Project SS020).

BIBLIOGRAPHY

Agarwal, S.K. and Kalla, S.L. (1996). "A generalized gamma distribution and its application in reliability." *Commun. Statist.-Theory Meth.*, **25** (1), 201–210.

Barlow, R.E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing – Probability Models*, New York: Holt, Rinehart & Winston.

Glaser, R. E. (1980). "Bathtub and related failure rate characterization." *J. Amer. Statist. Assoc.*, **75**, 667–672.

Klefsjö, B. (1989). "Testing against a change in the NBUE property." *Microelectron. Reliab.*, **29** (4), 559–570.

Kobayashi, K. (1991). "On generalized gamma functions occurring in diffraction theory." *J. Phys. Soc. Japan*, **60** (5), 1501–1512.

Milton, J.S., and Arnold, J.C. (1990). *Introduction to Probability and Statistics – Principles and Applications for Engineering and the Computing Sciences*, Second edition, New York: McGraw-Hill.

Osaki, S. and Li, X. (1988). "Characterization of gamma and negative binomial distributions." *IEEE Trans. Reliability*, **37** (4), 379–382.

Shanbhag, D.N. (1970). "The characterization of exponential and geometric distributions." *J. Amer. Statist. Assoc.*, **65**, 1256–1259.

Received July, 1996; Revised July, 1997.