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David C. Schmittlein, Donald G. Morrison,

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DAVID C. SCHMITTLEIN
University of Pennsylvania, Philadelphia, Pennsylvania

DONALD G. MORRISON
Columbia University, New York, New York

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The concept of the median residual lifetime for a probability distribution is outlined and shown to have several advantages over its more common counterpart, the mean residual lifetime A characterization theorem shows that a Pareto distribution of the second kind (or equivalently, a gamma mixture of exponentials) is the unique distribution with a linearly increasing median residual lifetime. An empirical example from the literature on strike durations is presented.

THE MEAN residual lifetime function of a random variable T (at t) is $E[T - t \mid T > t]$, which we denote by $E[S \mid t]$. It has frequently been useful in analyzing duration times for random variables T having the range $(0, \infty)$. Examples from the reliability literature include Watson and Wells (1961), Bryson and Siddiqui (1969) and Muth (1977). In the social sciences, the common empirical phenomenon of an increasing mean residual lifetime has been referred to as "inertia," and its presence in the data on durations of jobs, strikes and wars has been considered by Bartholomew (1973), Singer and Spilerman (1974) and Morrison and Schmittlein (1980).

When considering a histogram of duration times, however, the function $E[S \mid t]$ has a number of theoretical and practical shortcomings. For instance, in an experiment it is often impossible or impractical to wait until all items have failed. Recording those remaining as having a lifetime simply "greater than t_{max}" may still enable estimation of the distribution of T, but the empirical mean residual lifetime cannot be calculated using these censored data. Furthermore, even if all failure times are recorded the estimated mean residual lifetime will tend to be unstable due to its strong dependence on the few very long durations. This is particularly evident with the "fat-tailed" distributions common in analyzing job and strike durations. Finally, in some instances the mean residual lifetime may not even exist. This is the case for a gamma mixture of exponentials

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where the shape parameter of the gamma mixing distribution is less than 1 (Johnson and Kotz [1970], pp. 169 and 233). In such a situation one might hope to say something more concrete about the anticipated future behavior of the phenomenon under study.

As an alternative we define the median residual lifetime function, $M[S|t]$ say, as the median additional time to failure given no failure by time t. (Thus, for a number of components operating at time t, we expect half to fail in an additional period of length $M[S|t]$.) Calculation of this statistic with censored data poses no difficulty as long as at least half of those remaining have recorded failure times. In addition, the median is less sensitive to the distribution's upper tail, and in particular to any outliers. Lastly, $M[S|t]$ has the advantages of always being finite, easily obtained from the distribution function, and available in closed-form for most commonly used distributions. Given these points, it may be more useful to consider inertia in terms of an increasing median, rather than mean residual lifetime.

Morrison (1978) has shown that a gamma mixture of exponentials, or equivalently a Pareto distribution of the second kind (a Pearson type VI distribution) is the unique distribution resulting in a linearly increasing mean residual lifetime. In Section 1 it is shown that this same distribution is unique in generating a linearly increasing median residual lifetime. In Section 2, implications for analyzing data are considered. In particular, some published data on the durations of strikes are reanalyzed using the empirical median residual lifetime function.

1. LINEARLY INCREASING $M[S|t]$

Letting $H(t)$ and $R(t) = 1 - H(t)$ denote the c.d.f. and reliability function of T, respectively, we have

$$\begin{aligned} M[S|t] &= H^{-1}[1 - (1/2)(1 - H(t))] - t \\ &= R^{-1}[(1/2)R(t)] - t, \end{aligned} \tag{1}$$

where H^{-1} and R^{-1} are the inverse functions. The notation H for the c.d.f. is used because in the probability mixture model literature it is common to use F for the individual c.d.f., G for the mixing distribution and H for the resulting mixture.

THEOREM. *The random variable T has median residual lifetime*

$$M[S|t] = a + bt, \qquad a, b > 0,$$

if and only if T has the Pareto distribution of the second kind.

Proof. We begin with the negative exponential density function

$$f(t; \lambda) = \lambda e^{-\lambda t}, \qquad t > 0, \lambda > 0,$$

and allow λ to be mixed according to a gamma distribution

$$g(\lambda) = (\alpha/\Gamma(r))(\alpha\lambda)^{r-1}e^{-\alpha\lambda}, \qquad \lambda > 0;\ r, \alpha > 0.$$

The resulting mixture has c.d.f. $H(t) = \int_0^t \int_0^\infty f(x;\ \lambda)g(\lambda)d\lambda dx = 1 - (\alpha/(\alpha + t))^r$, and reliability function

$$R(t) = (\alpha/(\alpha + t))^r. \tag{2}$$

Solving for t in (2) yields

$$R^{-1}(x) = (x^{-1/r} - 1)\alpha. \tag{3}$$

Substituting (2) and (3) in (1) gives $M[S\,|\,t] = (2^{1/r} - 1)(t + \alpha)$, which has the desired linear form.

To prove uniqueness assume linearity of $M[S\,|\,t]$, and rewrite (1) as

$$R(a + (b + 1)t) = (1/2)R(t), \qquad a, b > 0. \tag{4}$$

It will be easier to work with the hazard function

$$u(t) = h(t)/R(t), \tag{5}$$

where $h(t)$ is the p.d.f. of T. Substituting (4) into (5) and noting that $h(t) = -(d/dt)R(t)$ yields the functional equation

$$u(a + (b + 1)t) = (b + 1)^{-1}u(t), \tag{6}$$

which needs to be solved. Equation (6) is a special case of Schröder's equation

$$\psi[\phi(t)] = \delta\psi(t), \tag{7}$$

where δ and $\phi(t)$ are known (see Kuczma [1968], Chapter VI).

It will be convenient to transform the function $\phi(t)$ so as to have a fixed point (i.e. a point t_0 for which $\phi(t_0) = t_0$) at the origin. Consider the resulting new equation

$$u^*((b + 1)t) = (b + 1)^{-1}u^*(t). \tag{8}$$

It can be shown (Lemma 6.1 of Kuczma) that since $t_0 = -a/b$ is a fixed point for $\phi(t) = a + (b + 1)t$, the solutions to (8) and (6) are related by

$$u(t) = u^*(t + a/b). \tag{9}$$

Therefore the proof requires solving (8), which can be rewritten as

$$u^*((b + 1)^{-1}t) = (b + 1)u^*(t). \tag{10}$$

Only functions $u^*(t)$ which are analytic, or those having a pole of order K at $t = 0$ are admitted.

First, consider the more general functional equation

$$\psi[\phi(t)] = \gamma(t)\psi(t), \tag{11}$$

where ϕ and γ are analytic in a neighborhood of $t = 0$ and $\phi(0) = 0$. The following analog of Lemma 8.1 in Kuczma for real-valued functions is needed.

LEMMA. *Let $\psi(t)$, not identically equal to 0, be a solution to (11), analytic or having a pole at $t = 0$. Then $\gamma(0) = [\phi'(0)]^{-K}$ where*

$$K = \begin{cases} \text{the order of the pole} \\ 0, \text{ if } \psi \text{ is analytic.} \end{cases}$$

Proof. If at $t = 0$, $\psi(t)$ has a zero of order $(-K)$ or a pole of order K, then $\Psi(t) \equiv t^K\psi(t)$ is regular at zero, $\Psi(0) \neq 0$, yielding the relation

$$\Psi[\phi(t)] = \gamma_1(t)\Psi(t), \tag{12}$$

where

$$\gamma_1(t) = (\phi(t)/t)^K\gamma(t) \tag{13}$$

is regular at $t = 0$.

Setting $t = 0$ in (12) requires $\gamma_1(0) = 1$, since $\Psi(0) \neq 0$. Substituting into (13), with $\phi(0) = 0$, yields the desired result.

Equation (11) is a generalization of (10) with $\phi(t) = (b + 1)^{-1}t$ and $\gamma(t) = (b + 1)$; hence, $\phi'(0) = (b + 1)^{-1}$ and $\gamma(0) = [\phi'(0)]^{-1}$. Consequently, the only possible solutions $u^*(t)$ to (10) must have a pole of order 1 at $t = 0$. Defining

$$\hat{u}^*(t) = tu^*(t), \tag{14}$$

yields $\hat{u}^*(t)$ regular at $t = 0$, and $\hat{u}^*(0) \neq 0$. Using (10) and (14),

$$\begin{aligned} \hat{u}^*((b + 1)^{-1}t) &= (b + 1)^{-1}t(b + 1)u^*(t) \\ &= tu^*(t). \end{aligned}$$

Obviously, then

$$\hat{u}^*((b + 1)^{-m}t) = tu^*(t) \tag{15}$$

for all integers m. Since the function $\hat{u}^*(t)$ is defined and $\neq 0$ at $t = 0$, the limit of (15) as $m \to \infty$ yields

$$tu^*(t) = \hat{u}^*(0),$$

or equivalently,

$$u^*(t) = c/t, \tag{16}$$

where $c = \hat{u}^*(0)$ is an unknown constant.

Using (9) and (16) the desired hazard rate is therefore

$$u(t) = cb/(a + bt). \tag{17}$$

Since the hazard and reliability functions are related by

$$R(t) = \exp - \int_0^t u(x)dx,$$

the reliability function is

$$R(t) = (a/(a + bt)^c. \tag{18}$$

Substituting $t = 0$ in (4) and comparing with (18) we have

$$R(a) = 1/2 = (1/(1 + b))^c$$

which uniquely determines the constant c, namely

$$c = (\ln 2)/(\ln(1 + b)).$$

The constants a and b in $M[S|t] = a + bt$ completely determine the reliability function (and hence the c.d.f.). The theorem is proven since (18) is the reliability function of the Pareto distribution of the second kind.

This characterization theorem can be easily extended to a general percentile residual lifetime, as follows. Haines and Singpurwalla (1974) have defined a β-percentile residual lifetime as (in our notation)

$$M_\beta(S|t) = R^{-1}(\beta R(t)) - t, \qquad 0 < \beta < 1. \tag{19}$$

Clearly, the median residual life is a special case with $\beta = 1/2$. For the Pareto distribution, substituting Equations (2) and (3) into (19) gives the result

$$M_\beta(S|t) = (\beta^{-1/r} - 1)(t + \alpha)$$

which is again linear in t. Conversely, the assumption that $M_\beta(S|t)$ is linear leads to exactly the same functional equation for the hazard rate as that given in Equation (6). Thus a linear β-percentile residual lifetime also characterizes the Pareto distribution. Moreover, if $M_{\beta 1}(S|t)$ is linear for *some* β_1 then $M_\beta(S|t)$ is linear for *all* β in (0, 1).

2. AN APPLICATION OF THE MEDIAN RESIDUAL LIFETIME

Lancaster (1972) collected data on strikes in the United Kingdom that commenced in 1965, recording the durations for those lasting longer than one day. The Ministry of Labour attempts to record all strikes except those in companies with less than ten workers or those lasting only one day (often considered to be simply union pressure tactics and hence not real strikes). Six industries were examined (nonelectrical engineering, construction, shipbuilding, vehicles and cycles, metal manufacturing and the distributive trades sector) leading to a total of 840 cases, with the histogram included in Table I.

TABLE I

LANCASTER STRIKE DATA

Time (days)	No. of Strikes	Time (days)	No. of Strikes
1–2	203	13–14	11
2–3	149	14–15	15
3–4	100	15–16	6
4–5	71	16–17	7
5–6	49	17–18	6
6–7	33	18–19	4
7–8	29	19–20	4
8–9	26	20–25	17
9–10	23	25–30	16
10–11	14	30–35	8
11–12	12	35–40	8
12–13	9	40–50	12
		>50	8

The calculated values for the median residual lifetime are:

Time (days)	$M[S\|t]$
1	2.680
2	2.979
3	3.727
4	4.462
5	4.935
6	5.750
7	6.409

Given the results of the first two sections two insights are suggested by this short table. First, the near-linear increase in median residual lifetime is consistent with strike durations following a gamma mixture of exponentials. In fact, estimating that model by maximum likelihood (using a pattern search) yields the parameters $r = 1.83$, $\alpha = 4.94$. The χ^2 goodness of fit statistic (22 degrees of freedom) is $\chi^2 = 22.44$, which is far short of any critical values for rejection of the model.

TABLE II

MEDIAN RESIDUAL LIFETIME FUNCTIONS

Distribution	c.d.f.	$M[S\|t]$
Exponential	$1 - e^{-\lambda t}$	$\lambda^{-1}\ln 2$
Weibull	$1 - e^{-\lambda t^{\beta}}$	$[\lambda^{-1}\ln 2 + t^{\beta}]^{1/\beta} - t$
Pareto—1st Kind ($T > K$)	$1 - (K/t)^r$	$(2^{1/r} - 1)t$
Pareto—2nd Kind (gamma mixture of exponentials)	$1 - (\alpha/(\alpha + t))^r$	$(2^{1/r} - 1)(t + \alpha)$
Generalized Burr (gamma mixture of Weibulls)	$1 - (\alpha/(\alpha + t^{\beta}))^r$	$[2^{1/r}(\alpha + t^{\beta}) - \alpha]^{1/\beta} - t$
Half-Cauchy	$2\pi^{-1}\tan^{-1}(\lambda t)$	$\lambda^{-1}\tan[\pi/4 + \frac{1}{2}\tan^{-1}(\lambda t)] - t$

Second, these data clearly show the phenomenon of inertia—that is, strikes still in progress after (say) seven days have a longer median *additional* time to settlement than those which are in their second or third day. However, if heterogeneity in the mean strike durations is present, this inertia at the aggregate level can be due simply to the longer mean durations of strikes which have survived seven days, and may not be a characteristic of any individual strike (see Morrison and Schmittlein).

3. CONCLUSION

Silcock (1954) has argued that the median of the distribution of job durations ($M[S \mid t = 0]$) is a better measure of the employer's power to retain labor than that mean. This paper further studies the median residual lifetime and shows how this function characterizes the Pareto distribution when it is linear. Clearly, hypotheses about the median residual lifetime may not always include linearity, so using Equation (1) we list in Table II the distribution function and $M[S \mid t]$ for a number of common lifetime distributions. The pervasive nature of the study of durations in the social, biological and physical sciences will doubtless lead to additional distributional assumptions. In view of its desirable properties the median residual lifetime should become a valuable theoretical and empirical concept for analyzing a variety of duration times.

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