Extended Fourier amplitude sensitivity test (eFAST) algorithm

1. Sampling parameter values using sinusoidal search curves.
2. Generate sampling matrix.
3. Estimate the full set of first and total-order sensitivity indices.
4. Resampling scheme. Repeat this process NR number of times with respect
5. Calculate the partial variance contributed by each parameter to each parameter.

Detailed algorithm

\begin{enumerate}

\item {\it Sampling parameter values using sinusoidal search curves.} A unique feature of eFAST is to use Fourier transformation functions (namely, {\it search curves}) to generate sampling values for each parameter over their feasible ranges.

Saltelli and Lu et al. \cite{saltelli1999quantitative,lu2001sensitivity} suggested the following routine to yield the most uniform search throughout the parameter space,

\begin{align}

&\alpha\_i(s\_{}) = \frac{1}{2}+ \frac{1}{\pi} \arcsin(\sin(\omega\_i s\_{}+ \phi\_i))\label{eqn:searchcurvealpha} \hspace{0.5cm} \forall i = 1,2,\dots, K \hspace{0.5cm}\\

&X\_i(s\_{}) = F^{-1}\_i (\alpha\_i(s\_{}))\label{eqn:searchcurve}

\end{align}

where $s$ is a random variable uniformly distributioned between $(-\pi, \pi)$, and $F^{-1}\_i(\cdot)$ is the parameter's inverse cumulative distribution function, $\omega\_i$ and $\phi\_i$ are the function's frequency and phase shift, respectively.

Equation \eqref{eqn:searchcurvealpha} generates a uniform sample between $[0,1]$, which equation \eqref{eqn:searchcurve} uses as an input to generate a sampling value for $X\_i$ within the range of uncertainty.

In general, an inverse uniform distribution is chosen for $F^{-1}$ if we do not know specific information about the model parameters' distributions.

The sampling curves frequencies are the critical components of the scheme, which are used to estimate the partial variance contributed by the parameter $X\_i$ and the remaining parameters $X\_{\sim i}$. An automated algorithm to select these frequencies is given in \cite{saltelli1999quantitative}.

\item {\it Generate sampling matrix.} The input from step 1 can be used to generate a sampling matrix. Then, we simulate the model output for each sampling of parameter combination and estimate model outcome's total variance.

\item {\it Calculate the partial variance contributed by each parameter.} The partial variance contributed by $X\_i$ and the complementary set $X\_{\sim i}$ are estimated by decomposing the total variance from step 2 using the first

few terms of the Fourier series expansion at the frequency $\omega\_{ i}$ and $\omega\_{\sim i}$ respectively as indicated in Equations \ref{eqn:estimatepartialvar}-\ref{eqn:eFASTVarEsimi}. Cukier et al. \cite{cukier1978nonlinear} suggested that four terms was sufficient.

\begin{align}

&\widehat{Var}(E(Y|X\_i=x\_i^\*)) \approx 2 \sum\_{p =1}^{4} (A\_{p\omega\_i}^2 + B\_{p\omega\_i}^2) \hspace{0.5cm}\text{for $p = 1,2,\dots, $}

\label{eqn:estimatepartialvar}\\

&A\_{p\omega\_i} = \frac{1}{2\pi}\int\_{-\pi}^{\pi}f(s)\cos(p\omega\_is)\, ds \label{eqn:eFAST\_hatVarE2}\\

&B\_{p\omega\_i} = \frac{1}{2\pi} \int\_{-\pi}^{\pi}f(s)\sin(p\omega\_is)\, ds \label{eqn:eFAST\_hatVarE3}\\

&\widehat{Var} (E(Y|X\_{\sim i} = x\_{\sim i}^\*)) \approx 2 \sum\_{p =1}^{4} (A\_{p\omega\_{\sim i}}^2 + B\_{p\omega\_{\sim i}}^2), \hspace{0.5cm} \text{for $p = 1,2,\dots, $}\\

&A\_{p\omega\_{\sim i}} = \frac{1}{2\pi}\int\_{-\pi}^{\pi}f(s)\cos(p\omega\_{\sim i}s)\, ds\\

&B\_{p\omega\_i} = \frac{1}{2\pi} \int\_{-\pi}^{\pi}f(s)\sin(p\omega\_{\sim i}s)\, ds

\label{eqn:eFASTVarEsimi}

\end{align}

where $f(s)= f(X\_1(s)), X\_2(s), \dots, X\_K(s))$ is the simulated model outcome from step 2.

\item The full set of both first and total-order sensitivity indices are calculated by taking the ratio between the partial variance and the total variance of the model output as in equation \eqref{eqn:Si}-\eqref{eqn:STi}

\item {\it Resampling scheme.}

Due to the symmetry properties of the sampling functions, parameter samples will eventually repeat. Therefore, to ensure an adequate sampling, each parameter must be resampled by introducing a small adjustment to the phase shift $\phi\_i$ of the search curve function \eqref{eqn:searchcurvealpha}. The sensitivity indices should be re-computed. This procedure is called a {\it resampling scheme} \cite{saltelli1999quantitative}. For example, for a model with eight parameters and three resampling schemes are used, there will be three sets of $\{S\_i,S^{tot}\_i\}$ for each parameter.

The resampling size $N\_R$, sampling size $N\_S$ and the maximal frequency $\omega\_{\max}$ must be chosen to satisfy equation \ref{eqn:NS,NR,omega} to guarantee a fair and balanced sampling scheme. If $\omega\_{\max}$ is too low and $N\_R$ is too large, the sampling over each curve is too sparse. On the other hand, if $\omega\_{\max}$ is too large and $N\_R$ is too small, the sampling is too dense over a small number of closed paths \cite{saltelli1999quantitative}. In cross-referencing the existing implementation of eFAST, we noted that equation \ref{eqn:NS,NR,omega} was not accurately implemented in the existing algorithm and we addressed in our version.

\begin{equation}

\omega\_{\max}= \frac{N\_S-N\_R -1}{2M}

\label{eqn:NS,NR,omega}

\end{equation}

\item We repeat the above steps for all $K$ parameters.

\end{enumerate}