

# *Rarefied Gas Dynamics*

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**Mikhail N. Kogan**

*Computer Center  
Academy of Sciences of the USSR  
Moscow*

Translated from Russian

Translation Editor  
**Leon Trilling**

*Department of Aeronautics and Astronautics  
Massachusetts Institute of Technology  
Cambridge, Massachusetts*



Mikhail Naumovich Kogan holds a professorship at the Moscow Institute of Physics and Engineering. His scientific interests include aerodynamics at supersonic and hypersonic velocities, in particular the study of optimum airfoils and bodies; magnetohydrodynamics and plasma physics; the investigation of shock waves in a plasma and of flow of plasma over bodies, particularly the detection of shock waves traveling upstream; and dynamics of rarefied gases and the kinetic theory of gases, in particular the investigation of hypersonic flow of a highly rarefied gas, the derivation of the equations of nonequilibrium flows, and the development of Monte-Carlo methods for the calculation of flows with arbitrary Knudsen number.

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*Михаил Наумович Коган*  
ДИНАМИКА РАЗРЕЖЕННОГО ГАЗА  
DINAMIKA RAZREZHENNOGO GAZA

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## FOREWORD

Rarefied gas dynamics, or superaerodynamics as it is sometimes called, studies phenomena taking place at an arbitrary ratio of the mean free path (time between collisions) of molecules to the characteristic dimension (time) of the phenomena. The phenomena studied can be as remote as desired from equilibrium. In the general case, an investigation of such phenomena requires consideration of the molecular structure of the gas, a kinetic description, and use of Boltzmann's equation. The range of problems of rarefied gas dynamics includes, for example, problems of flow past aircraft flying at high altitudes, motion of gases in vacuum apparatus, ultrasonic vibrations in gases, structure of shock waves, nonequilibrium flows, etc.

In the limiting case of small mean free paths we arrive at problems which can be solved within the continuum theory or, more exactly, with the use of Navier-Stokes equations. Essentially, these are problems of ordinary gas dynamics. However, according to established tradition, some of them are studied by rarefied gas dynamics. Among such problems are, for example, certain problems of viscous flows at small Reynolds numbers, flows with interaction of the boundary layer with a nonviscous flow, nonequilibrium flows with relaxation of excitation of internal degrees of freedom, flows with slip and a temperature jump near the wall, etc. Gasdynamic methods can be used for the solution of these problems. At the same time, these problems, solvable within the continuum theory, are intimately related with kinetic theory, since, by means of kinetic theory we can derive, from an analysis of Boltzmann's equation, the Euler and Navier-Stokes equations and their analogs for relaxing media, establish their area of applicability, and supply them with correct initial and boundary conditions and transfer coefficients.

This monograph examines mainly problems requiring a kinetic description, for the solution of which the gasdynamic methods are inapplicable, and new methods, approaches, and models are needed. Prime attention is given to the Boltzmann kinetic equation, a study of its properties and methods of solution. At the same time, considerable attention is devoted to the derivation from the Boltzmann kinetic equation of equations of gasdynamics and corresponding boundary conditions (slip conditions) and to an establishment of the areas of their applicability.

The first chapter deals with basic concepts of the kinetic theory of gases. The second and third chapters are devoted to derivation of kinetic equations and general methods of their solution. Despite the fact that only gases consisting of neutral molecules are examined, some of the methods presented find use also in plasma theory, and we hope that the interpretation of the general methods given here will promote a more critical approach to their use. In the fourth chapter the use of these methods is illustrated on simple, mainly one-dimensional, problems. The fifth and sixth chapters examine limiting cases of flows at small and large Knudsen numbers.

The author's interests are, of course, reflected in the selection of the material. Some room has been given to an account of the results of works of the author and his colleagues. Certain problems which are not taken up in the book can be found by the interested reader in the literature.\* The literature references do

\*See, for example, S. Chapman and T. G. Cowling, "Mathematical Theory of Nonuniform Gases," Cambridge University Press, New York, 1952; J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, "Molecular Theory of Gases and Liquids," Wiley, New York, 1964; G. N. Patterson, "Molecular Flow of Gas," Wiley, New York, 1956; M. Devienne, "Flows and Heat Exchange of Rarefied Gases," [Russian translation], IL, 1962; V. P. Shidlovskii, "Introduction to Dynamics of Rarefied Gases [in Russian]," Nauka Press, 1965. Numerous works on the dynamics of rarefied gases are in the proceedings of the international symposia: "Rarefied Gas Dynamics," First Symposium, Pergamon Press, New York, 1960; "Rarefied Gas Dynamics," Second Symposium, Academic Press, New York, 1961; "Rarefied Gas Dynamics," Third Symposium, Academic Press, New York, 1963; "Rarefied Gas Dynamics," Fourth Symposium, Academic Press, New York, 1966. See also: S. V. Vallander, editor, "Aerodynamics of Rarefied Gases," Leningrad State University, Vol. 1 (1963) and Vol. 2 (1965). See also: "Certain Problems in the Kinetic Theory of Gases" [Russian translation], Mir, 1965; "Interaction of Gases with Surfaces" [Russian translation], Mir, 1965.

not claim to be complete. Only works used directly in the writing of this book are cited.

Reading of the book does not require previous familiarity with the kinetic theory of gases and statistical physics. Knowledge of gasdynamics is presumed in certain places.

A course of lectures given by the author at the Moscow Physicotechnical Institute served as the basis of the book.

The author is sincerely grateful for the useful comments of Yu. P. Raizer, who carefully read the entire manuscript, A. A. Dorodnitsyn, who read its first chapters, and V. S. Galkin, who examined certain sections of the manuscript. The author is also grateful to E. M. Shakhov, who edited the book. The author would like to express special thanks to I. N. Sokolova for help in preparing the manuscript.

M. N. Kogan

#### NOTE TO THE AMERICAN EDITION

I am very happy to learn that my book is being translated into English, since this gives evidence of the interest of my American colleagues in the rapidly expanding field of gas dynamics. The American edition mirrors the Russian edition exactly, except that some minor errors and misprints have been corrected.

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## INTRODUCTION

### §1.1. Molecular Structure of a Gas

The kinetic theory is based on the hypothesis that all substances, including gases, are composed of molecules. Although the motion of the individual molecules cannot be followed even with the aid of the most powerful microscopes, the molecular hypothesis is nevertheless well supported. Complex polyatomic molecules, as well as individual atoms, ions, protons, electrons, etc., may all be treated as molecules.

A gas is a collection of molecules separated by distances so large that most of the time the molecules interact only weakly with each other. The brief periods during which the molecules interact strongly are considered as collisions.

If the time average of the potential energy of interaction of the molecules may be neglected in comparison with their kinetic energy, then the gas is called perfect. When this condition is not satisfied, the gas is called a van der Waals gas.

Only perfect gases are considered below. If the molecules possess a weak attractive potential when at large distances from each other, and a rapidly decreasing repulsive potential at small distances, then as the gas density decreases (i.e., as the mean distance between the molecules increases) the potential energy of interaction drops off rapidly. In practice, gases of neutral molecules for pressures up to hundreds of atmospheres may be considered as perfect. Up to such pressures the probability of ternary (or three-body) collisions (i.e., collisions in which three mole-

cules take part) is small in comparison with that of binary (or two-body) collisions.

It is further assumed in what follows that the motion of the molecules may be described by classical Newtonian mechanics. Quantum effects are important only at very low temperatures and for light molecules (hydrogen, helium, and electrons). For hydrogen and helium quantum corrections are important even under normal conditions. The majority of gases liquefy at temperatures where it is still not necessary to apply the quantum theory of molecular collisions.

Quantum effects must be taken into account in inelastic collisions of atoms and molecules (excitation of internal degrees of freedom of molecules, excitation of electronic levels, etc.). The potentials of elastic molecular interactions also may be calculated only by means of quantum mechanics. At a certain interaction potential, however, elastic collisions may be treated classically.

Relativistic effects are important only at very high temperatures (large molecular velocities). In practice, these effects may be neglected at temperatures of the order of tens and hundreds of thousands of degrees. For hydrogen, for example, the mean molecular speed at a temperature of  $10^5$  °K is 0.0001 of the speed of light. Even the speed of an electron at this temperature is a few thousandths of the speed of light.

Thus, the theory given below for a perfect gas, taking into account two-body collisions within the framework of classical mechanics, gives a satisfactory description of the motion of a gas over a wide range of temperatures and pressures (for temperatures from tens of degrees Kelvin to hundreds of thousands of degrees, and for pressures up to hundreds of atmospheres).

### §1.2. Laws of Molecular Interaction

The state of a gas is determined by the interaction of its molecules with each other and with the solid or liquid boundaries of the gas. In this section we will present only certain information on the interaction of the molecules with each other.\* The more

\*More detailed information about the interaction of molecules may be found, for example, in the following monographs: J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, "Molecular Theory of Gases and Liquids," Wiley, New York, 1954; J. B. Hasted, "Physics of Atomic Collisions," Butterworth, London, 1964.

complicated problem of the interaction of molecules with solid surfaces will be touched upon briefly in Chapter II.

Molecules or atoms consist of positive and negative charges (nuclei and electrons). Strictly speaking, when considering the interaction of two molecules, we may speak only of the interaction of the sum total of the charges of the two molecules. The interaction potential of this system of charges depends on the location of all the nuclei and all the electrons. Let us denote by  $\mathbf{r}_A^a$  and  $\mathbf{r}_A^e$  the radius vectors of the nuclei and electrons of molecule A, and by  $\mathbf{r}_B^a$  and  $\mathbf{r}_B^e$  the corresponding quantities for molecule B. Then the interaction potential of all the charges has the form  $U(\mathbf{r}_A^a, \mathbf{r}_A^e, \mathbf{r}_B^a, \mathbf{r}_B^e)$ .

The interaction potential of the molecules is usually given by the quantity

$$U_{AB} = U(\mathbf{r}_A^a, \mathbf{r}_A^e, \mathbf{r}_B^a, \mathbf{r}_B^e) - U_A(\mathbf{r}_A^a, \mathbf{r}_A^e) - U_B(\mathbf{r}_B^a, \mathbf{r}_B^e), \quad (2.1)$$

where  $U_A(\mathbf{r}_A^a, \mathbf{r}_A^e)$  is the potential dependent only on the mutual positions of the charges of molecule A, and  $U_B(\mathbf{r}_B^a, \mathbf{r}_B^e)$  is the potential dependent on the mutual positions of the charges of molecule B. The functions  $U_A$  and  $U_B$  define the potential energy of the molecules only when the distance between the molecules is infinite. The interaction potential of the molecules depends on the relative position of the charges of both molecules.

The forces acting between molecules may be roughly classified as chemical or valence repulsive forces which arise at small distances, and van der Waals attractive forces, which act at large separation distances between the molecules. This division is somewhat arbitrary.

The chemical or valence forces are dominant when the electron clouds of the molecules overlap. These forces are of quantum-mechanical origin, and are connected mainly with exchange interactions. Usually the potential of this interaction drops off exponentially with distance:

$$U_{AB} = C_1 e^{-C_2 r_{AB}}, \quad (2.2)$$

where  $C_1$  and  $C_2$  are constants.

The van der Waals forces have both an electrostatic component and a quantum-mechanical dispersion component.

Neutral molecules may possess dipole moments, quadrupole moments, etc. Therefore, interaction forces arise between the molecules of dipole-dipole type, dipole-quadrupole type, etc. If the colliding molecules have a dipole moment, the interaction force falls off as the inverse fourth power of the distance between the molecules. This force depends in an important way on the orientation of the dipoles, and can be a force of attraction as well as of repulsion.

Frequently, the interaction energy of the molecules is averaged over all orientations. If the gas is in equilibrium, the probability of finding two molecules which interact with a potential  $U_{AB}$  is proportional to the Boltzmann factor  $e^{-U_{AB}/kT}$ , where  $T$  is the temperature of the gas, and  $k$  is the Boltzmann constant. In this case the average potential is

$$\bar{U}_{AB} = \frac{\int U_{AB} e^{-U_{AB}/kT} \sin \theta_A \sin \theta_B d\theta_A d\varphi_A d\theta_B d\varphi_B}{\int e^{-U_{AB}/kT} \sin \theta_A \sin \theta_B d\theta_A d\varphi_A d\theta_B d\varphi_B},$$

where  $\theta$  and  $\varphi$  are the polar and azimuthal angles, respectively, which determine the direction of the dipole axis relative to the line joining the centers of the molecules. For weak interactions (i.e., at large distances), when the quantity  $|U_{AB} - \bar{U}_{AB}|$  is small in comparison with  $kT$  for all orientations, the exponential term may be expanded in a series, from which we obtain

$$\bar{U}_{AB} = \frac{1}{64\pi^6} \int U_{AB} d\omega - \frac{1}{kT} \left\{ \frac{1}{64\pi^4} \int U_{AB}^2 d\omega - \left( \frac{1}{64\pi^6} \int U_{AB} d\omega \right)^2 \right\} + \dots,$$

where

$$d\omega = \sin \theta_A \sin \theta_B d\theta_A d\varphi_A d\theta_B d\varphi_B.$$

Since, for a dipole-dipole interaction, the probability of attractive and repulsive orientations is the same, the first term is equal to zero, and the average interaction potential is seen to fall off as the inverse sixth power of distance ( $U_{AB} \sim 1/r_{AB}^6$ ).

It should be noted that for typical states far from equilibrium in the theory of rarefied gases, the above method of averaging

cannot be carried out and dipole-dipole interactions may fall off relatively slowly.

In the interaction of two molecules, the charge of one molecule polarizes the other molecule, giving rise to an induced dipole. Simple electrostatic considerations \* show the induced interaction potential falls off as the inverse sixth power of the distance. The dispersion interaction, which is of quantum-mechanical origin, also falls off as the inverse sixth power of the distance.

For the majority of molecules, the main contribution to the interaction potential comes from the dispersion component. For some molecules, however, the dipole-dipole interaction is dominant. The induced interaction is almost always small. Although a quantum-mechanical calculation of the interaction potential is possible in principle for all molecules, in practice more or less exact interaction calculations have been carried out only for the simplest molecules.

In practice it is customary to use empirical and semi-empirical interaction laws, and we shall present some of the more widely used of these.†

1. Elastic Spheres (Fig. 1a). If  $d$  is the sphere diameter, the interaction potential of two spheres takes the form

$$\begin{cases} U(r) = \infty & \text{for } r < d, \\ U(r) = 0 & \text{for } r > d. \end{cases} \quad (2.3)$$

Although this model gives a crude representation only of the short-range repulsive forces, it is frequently used in calculations because of its simplicity. It is also widely used in qualitative considerations of molecular collisions since the collision process is clearest for hard spheres. Moreover, when applying other, more complicated interaction laws, it is usual to introduce so-called effective collision cross sections (see §1.3), by means of which the collision of the actual molecules is replaced by the collision of elastic spheres which are in some sense equivalent. For each type

\* See, for example, the previously cited monograph of Hirschfelder et al.

† In what follows, we shall designate the distance between molecules and the interaction potential between them by  $r$  and  $U$ , respectively.

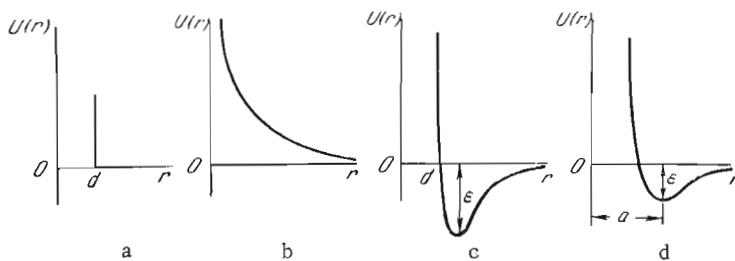


Fig. 1

of collision (collisions for different relative velocities, inelastic collisions with various transitions, etc.) we may in some manner choose a sphere that is equivalent. Even for elastic collisions with a given relative velocity, the spheres that are equivalent to the actual molecules, for example, with respect to change of momentum in a particular direction, may differ from the spheres that are equivalent with respect to change of momentum in another direction.

Nevertheless, in the qualitative arguments which follow we shall often consider the molecules as elastic spheres of fixed diameter. This is permissible in many cases, since the diameters of the equivalent spheres for various types of interactions are of the same order. In some cases, however, the change in the effective diameter of the molecules is the main effect. For example, if the relative velocities of the molecules vary over a wide range, the neglect of the variation in the effective sphere diameters as a function of velocity may lead to a qualitative distortion of the actual phenomena (see, for example, §§4.2, 4.4, or 6.6).

The diameters of the molecules are of the order of  $10^{-8}$  cm.

#### 2. Centers of Repulsion (Fig. 1b) with the potential

$$U = \frac{K}{r^{s-1}}. \quad (2.4)$$

The quantity  $s - 1$  (and sometimes also  $s$ ) is called the repulsion index. For purely mathematical convenience (see §§2.2, 3.3, 3.11, etc.) hypothetical molecules with the repulsion index  $s = 5$ , called Maxwellian molecules, have been widely studied. A gas

composed of such molecules is called a Maxwellian gas. Values of  $s$  larger than 5, for example 9 or 12, are closer to the actual values. Maxwellian molecules are too "soft," while elastic spheres are too "hard." The actual interaction potentials lie between these two most widely employed model potentials. It should be noted that the use of the mathematically convenient Maxwellian molecules sometimes leads to incorrect results. For example, there is no thermal diffusion in a Maxwellian gas.

The so-called pseudo-Maxwellian molecules, which will be defined in § 2.2, are also advantageous from a mathematical point of view.

#### 3. Lennard-Jones Potential (Fig. 1c)

$$U(r) = \frac{d}{r^n} - \frac{e}{r^m}. \quad (2.5)$$

The first term with the exponent  $n$  characterizes repulsion, and the second, attraction;  $d$  and  $e$  are constants.

The most often used relation of this form is the so-called Lennard-Jones (6-12) potential

$$U(r) = 4\epsilon \left[ \left( \frac{d}{r} \right)^{12} - \left( \frac{d}{r} \right)^6 \right]. \quad (2.6)$$

As was pointed out above, the sixth power dropoff in potential simulates electrostatic dipole-dipole and dispersion attraction. The twelfth power dropoff in collision potential was chosen from considerations of mathematical convenience. At the same time it simulates a rather hard collision. For  $r = d$  the potential is equal to zero. The quantity  $\epsilon$  characterizes the depth of the potential well.

#### 4. Morse Potential (Fig. 1d)

$$U(r) = \epsilon [e^{-2\beta(r-a)} - 2e^{-\beta(r-a)}]. \quad (2.7)$$

Here the quantity  $a$  determines the position of the minimum of the potential well, and  $\epsilon$  the depth of the potential well. The quantity  $\beta$  determines the curvature of the potential curve near the minimum. By means of these parameters the potential curve can be well approximated near the minimum. However, the potential (2.7) falls off too rapidly as  $r \rightarrow \infty$ . This potential is convenient for calculations in many cases.

Other, more complex potentials with a larger number of free parameters are also used.

The parameters which appear in the formulas discussed here and in similar formulas for the interaction potential are most often determined experimentally. This may be done, for example, by studying the scattering of a beam of molecules by molecules of the same or another gas. More often, however, indirect methods are used. One or more macroscopic quantities are calculated for a given interaction law. From a measurement of these quantities in a macroscopic experiment, the unknown potential interaction parameters appearing in them can be found. Obviously, the simplest procedure is an experiment to determine the second virial coefficient. As is known, the equation of state for a van der Waals gas has the form\*

$$\frac{pV}{RT} = 1 + \frac{B_2(T)}{V} + \frac{B_3(T)}{V^2} + \dots,$$

where  $p$ ,  $V$ , and  $T$  are the pressure, specific volume, and temperature of the gas,  $R$  is the universal gas constant, and  $B_2$ ,  $B_3$ , etc., are the second, third, etc., virial coefficients. By varying the temperature of the gas in a container, and measuring the pressure, the dependence of the virial coefficient on temperature can be found. On the other hand, the dependence of the virial coefficient on the interaction potential is known from statistical mechanics. From the experimental dependence of the virial coefficient on temperature, the interaction parameters of the molecule can be found. Usually for any given interaction law, these parameters lead to a satisfactory approximation of the experimental dependence only over a limited temperature range.

Knowing the molecular interaction law, we can determine, as will be shown in Chapter III, the transport properties of gases (viscosity, thermal conductivity, diffusion coefficient, etc.). By measuring these quantities we can also determine the parameters which enter in the interaction potential formulas.

As a rule, the parameters entering in the interaction law, as determined from measurements of the virial coefficient and transport properties, differ somewhat. This is due to the fact that the

\*See, for example, L. D. Landau and E. M. Lifshits, "Statistical Physics," Nauka, Moscow, 1964. [English translation of an earlier edition: Addison-Wesley, Reading, Mass., 1958.]

virial coefficient is determined largely by the long-range attractive forces, while the transport properties are determined in large measure by the molecular collisions.

The molecular interaction laws are still further complicated if a molecule undergoes a transition from one quantum state to another during a collision. In general, there must be a potential curve for each state of the colliding molecules. If a molecule goes from one state to another in the collision process (for example, from the ground state to some excited state), then it is necessary, at the point of transition, to go from one potential curve to another.

Although the study of molecular interaction laws is far from complete, we shall regard the interaction laws as given in what follows, and for estimates and specific calculations we shall use only the simplest of the empirical laws given above.

Only spherically symmetric potentials have been discussed. In general, the interaction potential depends on the relative angular positions of the molecules. We may represent the simplest model of a nonspherical molecule in the form of an ellipsoid of revolution. The absence of spherical symmetry in the interaction potential significantly complicates any study. Sometimes the interaction of such molecules can be averaged with respect to the angle, thereby reducing the analysis to some appropriately defined equivalent spherical molecules. Since we shall only be concerned with spherical molecules, we shall not present the various potential models which depend on angle, but refer the interested reader to the appropriate literature.\*

### §1.3. Particle Collisions

The motion of a rarefied perfect gas, which is an assembly of molecules, is determined in the final analysis by the collisions of the particles. In what follows we shall be interested only in

\*See, for example, J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, "Molecular Theory of Gases and Liquids," Wiley, New York, 1954; S. Chapman and T. Cowling, "Mathematical Theory of Nonuniform Gases," Cambridge University Press, London, 1952.

binary interactions of particles. We shall therefore recall certain facts from the theory of two-body collisions.\*

Consider a collision between two particles of masses  $m_1$  and  $m_2$  and velocities  $\xi_1 \equiv (\xi_{11}, \xi_{12}, \xi_{13})$  and  $\xi_2 \equiv (\xi_{21}, \xi_{22}, \xi_{23})$  before collision, respectively. We let the interaction potential  $U(r)$  depend only on the separation distance  $r$  of the particles, so that the interaction force  $F$  is directed along the line of centers of the colliding particles.

The collision process satisfies the laws of conservation of mass, momentum, and energy, which may be written in the form

$$\psi_i(\xi_1) + \psi_i(\xi_2) = \psi_i(\xi'_1) + \psi_i(\xi'_2) \quad (i = 0, 1, 2, 3, 4), \quad (3.1)$$

where

$$\begin{aligned} \psi_0(\xi_k) &= m_k, & \psi_j(\xi_k) &= m_k \xi_{kj}, & \psi_4(\xi_k) &= m_k \xi_k^2; \\ \xi_k^2 &= \xi_{k1}^2 + \xi_{k2}^2 + \xi_{k3}^2, & k &= 1, 2; & j &= 1, 2, 3, \end{aligned}$$

and where the primes denote the corresponding quantities after collision.

The quantities  $\psi_i$  are called summational invariants. Any additive function of the velocities is a linear combination of the summational invariants. We shall assume that the particle masses do not change in the collision process. Then Eq. (3.1) imposes four conditions on the six velocity components of the molecules after collision. To determine the velocities of the molecules after collision completely, two more parameters are required.

The motion of the molecules satisfies Newton's equations:

$$\begin{aligned} m_1 \ddot{\mathbf{r}}_1 &= \mathbf{F}_1(r), & m_2 \ddot{\mathbf{r}}_2 &= \mathbf{F}_2(r), \\ \mathbf{F}_1 &= -\mathbf{F}_2, & \mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1, & r &= |\mathbf{r}|; \end{aligned} \quad (3.2)$$

the dot denotes differentiation with respect to time. Subtracting the first equation (3.2), multiplied by  $m_2$ , from the second equation, multiplied by  $m_1$ , we obtain

$$\mu \ddot{\mathbf{r}} = \mathbf{F}_2 = -\frac{dU}{dr} \frac{\mathbf{r}}{r}, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}, \quad (3.3)$$

\*See, for example, L. D. Landau and E. M. Lifshits, "Mechanics," Addison-Wesley, Reading, Mass., 1960.

i.e., the problem of the motion of the two particles has been reduced to the problem of the motion of a single particle of reduced mass  $\mu$  in a central force field with a potential  $U(r)$ . From the solution  $\mathbf{r}(t)$  of Eq. (3.3), we can easily find the law of motion  $\mathbf{r}_k(t)$  of each of the particles, if we take account of the fact that the velocity of the center of mass of the system does not change during collision:

$$m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2 = \text{const.} \quad (3.4)$$

In particular, in the coordinates of the center of mass,

$$m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 = 0$$

and

$$\mathbf{r}_1 = -\frac{m_2}{m_1 + m_2} \mathbf{r}, \quad \mathbf{r}_2 = \frac{m_1}{m_1 + m_2} \mathbf{r}. \quad (3.5)$$

For the motion of a particle of mass  $\mu$  in a central force field the angular momentum of the system  $\mathbf{M} = \mu(\mathbf{r} \times \dot{\mathbf{r}})$  is conserved relative to the center of the field; in fact, using (3.3), we have

$$\dot{\mathbf{M}} = \mu \frac{d}{dt} (\mathbf{r} \times \dot{\mathbf{r}}) = \mu (\dot{\mathbf{r}} \times \dot{\mathbf{r}}) - \left( \mathbf{r} \times \frac{dU}{dr} \frac{\mathbf{r}}{r} \right) = 0.$$

Therefore, the particle motion is confined to a single plane. We shall introduce a polar system of coordinates (Fig. 2) in that plane. The equations of conservation of angular momentum and of energy take the form

$$M = \mu r^2 \dot{\theta} = \text{const.} \quad (3.6)$$

$$E = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) + U(r) = \frac{1}{2} \mu \dot{r}^2 + \frac{M^2}{2\mu r^2} + U(r) = \text{const.} \quad (3.7)$$

Eliminating time from (3.6) by means of (3.7) and integrating, we obtain

$$\dot{\theta} = M \int \frac{dr}{r \sqrt{2\mu r^2 [E - U(r)] - M^2}} + \text{const.} \quad (3.8)$$

It is easy to show that the particle trajectory is symmetrical with respect to the line OA. It follows from the law of conservation of energy that the relative velocity of the molecules does not change:

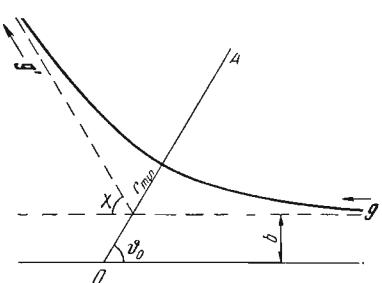


Fig. 2

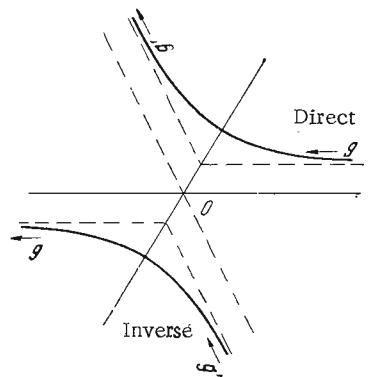


Fig. 3

$$g = g',$$

and it follows from the conservation of angular momentum that

$$gb = g'b' \quad \text{or} \quad b = b',$$

i.e., that the impact parameter is also conserved during the collision. We are usually interested in the deflection angle  $\chi = \pi - 2\vartheta_0$ , where

$$\vartheta_0(b) = M \int_{r_{\min}}^{\infty} \frac{dr}{r \sqrt{2\mu r^2 [E - U(r)] - M^2}}, \quad (3.9)$$

$$M = gb\mu, \quad E = \frac{1}{2}\mu g^2.$$

For molecules with a power-law interaction  $U = K/r^{s-1}$  we obtain

$$\chi = \pi - 2 \int_0^{\beta_0} \frac{d\beta}{\sqrt{1 - \beta^2 - \frac{1}{s-1} \left(\frac{\beta}{\rho}\right)^{s-1}}}, \quad (3.10)$$

where

$$\rho = b \left( \frac{\mu g^2}{2(s-1)K} \right)^{\frac{1}{s-1}}, \quad \beta = \frac{b}{r},$$

and  $\beta_0 = b/r_{\min}$  is determined from the equation

$$1 - \beta_0^2 - \frac{1}{s-1} \left( \frac{\beta_0}{\rho} \right)^{s-1} = 0. \quad (3.11)$$

It is interesting to note that for a power-law interaction the collision is determined by the single parameter  $\rho$ .

For hard spheres,

$$\left. \begin{array}{ll} \chi = 2 \arccos \left( \frac{b}{d} \right) & \text{for } b \leq d, \\ \chi = 0 & \text{for } b > d. \end{array} \right\} \quad (3.12)$$

We shall call a collision of molecules the "inverse" of that considered above, if molecules with velocities  $\xi'$  and  $\xi_1'$  are colliding with the same impact as parameter  $b$ . It is easy to see that the molecules acquire velocities  $\xi$  and  $\xi_1$  as the result of such a collision. In fact, if we reverse the signs of the coordinates and the time, Eq. (3.3), which describes the collision process, remains unchanged (it is assumed that the potential is spherically symmetric). The velocity also does not change under this transformation. But, since time goes in the reverse direction, the solution found above now describes the inverse collision (Fig. 3), in which the velocities are  $\xi'$  and  $\xi_1'$  before collision, and  $\xi$  and  $\xi_1$  afterwards.

The scattering process is often characterized by effective scattering cross sections. Consider the molecules passing through the annulus lying between the circles  $b$  and  $b + db$  in the plane perpendicular to the relative velocity  $\mathbf{g}$ . All these

molecules are scattered in the angular range between  $\chi$  and  $\chi + d\chi$ , or into the solid angle

$$d\Omega = 2\pi \sin \chi \, d\chi.$$

Let us assume that  $\chi$  and  $b$  are uniquely connected by the relations (3.9) or (3.10). The differential effective cross section  $\sigma(\chi, g)$  is the area of the annulus taken over a unit solid angle  $\Omega$ , i.e.,

$$\sigma(\chi, g) d\Omega = 2\pi b(\chi) db = 2\pi b(\chi) \left| \frac{db}{d\chi} \right| d\chi = \frac{b}{\sin \chi} \left| \frac{db}{d\chi} \right| d\Omega.$$

Therefore,

$$\sigma(\chi, g) = \frac{b}{\sin \chi} \left| \frac{db}{d\chi} \right|. \quad (3.13)$$

Sometimes the differential effective cross section refers not to a unit solid angle, but to a unit angle  $\chi$ , and then

$$\sigma_\chi(\chi, g) = 2\pi b(\chi) \left| \frac{db}{d\chi} \right|, \quad (3.14)$$

where the subscript  $\chi$  on  $\sigma$  indicates that the cross section is referred to a unit angle  $\chi$ .

The total effective cross section is

$$\sigma(g) = \int_{\chi=0}^{\pi} \sigma(\chi, g) d\Omega = 2\pi \int_0^{\pi} \sigma(\chi, g) \sin \chi \, d\chi = 2\pi \int_0^{b_{\max}} b \, db, \quad (3.15)$$

where  $b_{\max}$  is the impact parameter, such that when  $b \geq b_{\max}$  the potential  $U \equiv 0$ . Molecules for which  $b_{\max}$  is finite are called molecules with a finite radius of action. Among the potentials given in the previous section, only the hard spheres satisfy this condition. For the other interaction laws both  $b_{\max}$  and the total effective interaction cross section are infinite. If instead of a single molecule we consider a uniform stream of molecules with a density of  $n$  particles per unit volume, it is evident that  $ng\sigma(\chi, g)d\Omega$  is the number of molecules undergoing a deflection through angles between  $\chi$  and  $\chi + d\chi$ . It is thus evident that for molecules with an infinite radius of action the number of molecules deflected through all possible angles is infinite.

Table 1

$s$	$A^{(1)}(s)$	$A^{(2)}(s)$
5	0.298	0.308
7	0.306	0.289
13	0.321	0.279

At the same time, the change of momentum, energy, and other properties often remains finite as a result of collisions. For example, the change in the longitudinal momentum of a molecule in the center-of-mass system is equal to  $mg(1 - \cos \chi)$ . As an average characteristic of the change of momentum we introduce the effective cross section

$$\sigma^{(1)}(g) = \int \sigma(\chi, g)(1 - \cos \chi) d\Omega = 2\pi \int_0^\infty (1 - \cos \chi) b \, db,$$

often called the transport cross section.

The expression

$$\sigma^{(v)}(g) = 2\pi \int_0^\infty (1 - \cos^v \chi) b \, db \quad (3.16)$$

is called the effective cross section of order  $v$ . Other definitions of the effective cross section are also used.

It is easy to verify that for hard spheres,

$$\sigma^{(1)} = \pi d^2, \quad \sigma^{(v)} = \pi d^2 \left( 1 - \frac{1}{2} \frac{1 + (-1)^v}{1 + v} \right). \quad (3.17)$$

Substituting (3.10) into (3.16), we obtain, for molecules with a power-law interaction

$$\sigma^{(v)}(g) = 2\pi \left( \frac{(s-1)K}{1/2 \mu g^2} \right)^{\frac{2}{s-1}} A^{(v)}(s). \quad (3.18)$$

The values of  $A^{(v)}(s)$  obtained by numerical integration are given in Table 1.\*

It is important to note that molecules cannot be characterized by any single effective cross section, since these differ for various kinds of interaction (e.g., change of momentum, energy, etc.). At the same time, we see in Table 1, that the cross sections for different  $v$  are of the same order. In qualitative discuss-

\* A more detailed table may be found in the previously cited monographs by J. O. Hirschfelder et al. and S. Chapman and T. Cowling.

sions, therefore, one may often speak simply of the effective cross section for a given molecule.

In general, as the relative velocity of the molecules increases, the effective collision cross sections decrease. Thus, for molecules with the power-law interaction  $\sigma \sim g^{-4/s-1}$ . In particular, for Maxwellian molecules ( $s = 5$ ) the collision cross section is inversely proportional to the relative velocity:  $\sigma(v) \sim g^{-1}$ .

The time during which the molecules interact is called the collision time  $\tau_c$ . Strictly speaking, molecules may interact at any distance, and the collision time of such molecules is infinite. However, the main contribution to the effect of a collision comes from interaction at distances of the order of the effective interaction diameters, i.e.,

$$\tau_c \sim \frac{V\bar{\sigma}}{g}. \quad (3.19)$$

#### §1.4. Mean Free Path

The concept of effective cross section introduced in the previous paragraph is intimately connected with other concepts which play a fundamental role in kinetic theory. We consider first a gas composed of hard-sphere molecules of diameter  $d$ , uniformly distributed with density  $n$ , and moving at the same velocity  $\xi_1$ . We call these field molecules. We consider a test molecule moving in that gas with velocity  $\xi$ , so that relative to the gas it has a velocity  $\mathbf{g} = \xi - \xi_1$ . Clearly, in unit time, the probe molecule collides with all the molecules whose centers lie within the cylinder of base  $\pi d^2 = \sigma$  and height  $g$ . The number of such molecules is

$$v = \sigma g n. \quad (4.1)$$

This quantity is called the collision frequency. The mean time between collisions is equal to

$$\tau = 1/v.$$

In that time the test molecule traverses a distance  $\xi / v$ , which is called the mean free path.

According to (4.1), the mean free path is

$$\lambda = \frac{\xi}{\sigma g n}. \quad (4.2)$$

Clearly, the concepts introduced here should not be applied directly to molecules with an infinite radius of action, since  $\sigma = \infty$  for them. For those molecules the concepts of mean free path and collision frequency may be introduced formally, if in the definitions (4.1) and (4.2) effective cross sections of type (3.16) defined in the preceding section are substituted. The meaning of these concepts may be illustrated by the following argument. Consider a group of hard-sphere molecules of density  $n$ , moving with velocity  $\xi$  in a gas of identical but stationary molecules of density  $n_1$ . It is easy to see that in a unit area layer of thickness  $dx$  perpendicular to  $\xi$ , the number of collisions per unit time is

$$\sigma n n_1 \xi dx = n \xi \lambda^{-1} dx.$$

In other words, if a flux  $N = n \xi$  of molecules of velocity  $\xi$  passes through section  $x$ , then the flux through section  $x + dx$  is  $N(1 - \lambda^{-1} dx)$ , i.e.,  $N \lambda^{-1} dx$  molecules are lost from the original beam as a result of collisions.

We now consider the same two groups of molecules, with an arbitrary interaction potential. The molecules passing through a unit area at section  $x$  carry a momentum  $mN\xi$ . It follows from the previous paragraph that a molecule which has been deflected by an angle  $\chi$  as a result of a collision loses an amount of momentum in the  $x$  direction equal to  $\frac{1}{2}\xi(1 - \cos \chi)$ . As a result of colliding with one molecule at rest, the  $x$  momentum lost by the passing molecules is

$$\pi n \xi^2 m \int_0^\infty (1 - \cos \chi) b db.$$

Since there are  $n_1 dx$  stationary molecules in a layer of thickness  $dx$ , the total momentum lost in the layer is

$$\pi m n n_1 \xi^2 dx \int_0^\infty (1 - \cos \chi) b db = \frac{1}{2} m \xi N n_1 \sigma^{(1)} dx = \frac{1}{2} m \xi N \frac{dx}{\lambda}.$$

The effective cross section  $\sigma^{(1)}$  introduced here has an obvious interpretation. When molecules with an infinite interaction range pass through a layer of thickness  $dx$ , each molecule undergoes a collision and loses some momentum. If the gas is replaced by a gas of hard spheres of cross section  $\sigma^{(1)}$ , then, in the layer of thickness  $dx$ , its molecules undergo  $Nn_1\sigma^{(1)}dx = N\lambda^{-1}dx$  collisions, in each of which an average momentum of  $\frac{1}{2}m\xi$  is lost. The total momentum lost in the layer by the real gas and the hard-sphere gas is identical. Thus, in that example, the hard-sphere gas is equivalent to the real gas with respect to total momentum loss. But, if we are interested in the change of transport of some other molecular property, the effective cross section of the hard-sphere molecules of the equivalent gas for that property may be different from  $\sigma^{(1)}$ .

When the concept of mean free path is applied to nonspherical molecules of finite or infinite range, we imply the mean free path corresponding to some effective cross section. Since, in many cases, the mean free paths for transport of various properties are of the same order, we shall talk, in qualitative discussions, of mean free path without indicating the property to which it refers.

It was pointed out above that the number  $dN$  of molecules knocked out of a beam of intensity  $N(x)$  in a layer of thickness  $dx$  as a result of collisions is

$$dN = -N \frac{dx}{\lambda}. \quad (4.3)$$

If  $N_0$  molecules pass through the section  $x = 0$ , the intensity of the flux at section  $x$  is evidently

$$N = N_0 e^{-x/\lambda}. \quad (4.4)$$

The number of particles undergoing collisions between  $x$  and  $x + dx$ , according to (4.3) and (4.4), is

$$dN_{x \rightarrow x+dx} = N_0 e^{-x/\lambda} \frac{dx}{\lambda}, \quad (4.5)$$

i.e., in other words, the probability that a molecule travels a distance  $x$  without a collision, and collides in the element  $dx$ , is

Our examination above has involved only two groups of molecules, of densities  $n$  and  $n_1$ , respectively, undergoing collisions. If there are  $k$  groups of molecules with densities  $n_i$  and velocities  $\xi_i$  ( $i = 1, \dots, k$ ) we may introduce the mean free path of each of the groups with respect to any of the other groups.

The mean free path of molecules of group  $i$  relative to molecules of group  $j$  is

$$\lambda_{ij} = \frac{\xi_i}{\sigma_{ij} n_j g_{ij}}, \quad g_{ij} = |\xi_i - \xi_j|, \quad (4.7)$$

where  $\sigma_{ij}$  is the cross section for collision of molecules of the  $i$ -th kind with those of the  $j$ -th kind;  $\sigma_{ij}$  may depend on the relative velocity  $g_{ij}$ . The mean free path of  $j$ -molecules relative to  $i$ -molecules,

$$\lambda_{ji} = \frac{\xi_j}{\sigma_{ji} n_i g_{ji}} \quad (4.8)$$

is generally not equal to  $\lambda_{ij}$ .

We may introduce the mean free path of molecules of a given kind, for example of the  $i$ -th kind, relative to collection of groups of molecules

$$\lambda_{i\Sigma} = \frac{\xi_i}{\sum_j \sigma_{ij} n_j g_{ij}}, \quad (4.9)$$

where the sum  $\sum_j$  is taken over all the groups of molecules relative to which the mean free path is being considered.

Finally, for a criterion of the state of a gas as a whole, it is often convenient to introduce the mean free path with regard to all possible collisions, which may be determined, for example, by the relation

$$\bar{\lambda} = \frac{\sum_i^k \xi_i n_i}{\frac{1}{2} \sum_{i,j=1}^k \sigma_{ij} n_i n_j g_{ij}}. \quad (4.10)$$

Table 2

	$n, 1/cm^3$	$\xi, cm/sec$	$\lambda, cm$	$\tau, sec$	$\tau_c, sec$
Nitrogen at pressure of 100 atm and 300°K .....	$10^{21}$	$5 \cdot 10^4$	$10^{-7}$	$10^{-11}$	$10^{-13}$
Air at normal conditions .....	$10^{19}$	$5 \cdot 10^4$	$10^{-5}$	$10^{-9}$	$10^{-13}$
Earth's atmosphere at a height of 100 km .....	$10^{13}$	$10^4$	$10^{-1}$	$10^{-5}$	$10^{-13}$
Earth's atmosphere at a height of 300 km .....	$10^9$	$10^5$	$10^5$	1	$10^{-13}$
Earth's atmosphere at a height of 3000 km .....	$10^3$				

If all the colliding molecules are identical and  $\sigma_{ij} = \sigma$  does not depend on  $g_{ij}$ , the average mean free path is

$$\bar{\lambda} = \frac{\bar{\xi}}{ogn}, \quad (4.11)$$

where

$$\bar{\xi} = \frac{1}{n} \sum_{l=1}^k \xi_l n_l \quad \left( n = \sum_l n_l \right) \quad (4.12)$$

is the mean velocity of the molecules, and

$$\bar{g} = \frac{1}{2n^2} \sum_{i,j=1}^k n_i n_j g_{ij} \quad (4.13)$$

is the mean relative velocity.

Note that the mean free path is different in different inertial coordinate systems.

To conclude this section we present some typical orders of magnitude of the values of the properties introduced above (Table 2).

It is seen in the table that the time between collisions  $\tau$  is much larger than the time of collision  $\tau_c$ , even at pressures of the order of hundreds of atmospheres. We may therefore consider that most of the time molecules move in the field of external forces (in the absence of which their motion is rectilinear), and sharply

### § 1.5] ELEMENTARY KINETIC THEORY

change their direction and velocity during short-duration collisions.

#### §1.5. Elementary Kinetic Theory

In this section, the fundamental concepts of the theory of gases are examined from the kinetic point of view. This is, in essence, an order of magnitude estimate. However, from elementary considerations we can obtain qualitative basic relationship laws. Those elementary procedures prove useful also in the analysis of more complex phenomena (see, for example, §6.6).

From the molecular viewpoint, the pressure of a gas on the wall of a container is the result of the transfer of the momentum of the molecules striking the wall. Consider a gas in a container of characteristic dimension  $L$ ; let the number density of the gas molecules be  $n$ , and their mean velocity  $c$ . If the gas is in equilibrium with the walls, the mean velocity of the molecules rebounding from the wall is equal to the mean velocity of those striking the wall. On the average, each molecule undergoes  $c/L$  collisions with the wall per unit time, transferring a momentum  $2mc$  to the wall at each collision. The momentum transferred per unit area in unit time (the pressure  $p$ ) is therefore

$$p \sim mnc^2. \quad (5.1)$$

Let us define the gas temperature as a measure of the mean kinetic energy of the molecules by the relation

$$\frac{3}{2} kT = \frac{mc^2}{2}, \quad (5.2)$$

where  $k$  is the Boltzmann constant. Then relation (5.1) may be re-written in the form

$$p = kTn, \quad (5.3)$$

i.e., we obtain the equation of state of a perfect gas.

It follows from (5.2) that the thermal velocity of the molecules is of the order of the speed of sound  $a$ :

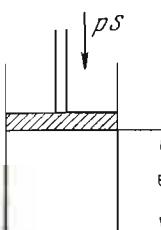


Fig. 4

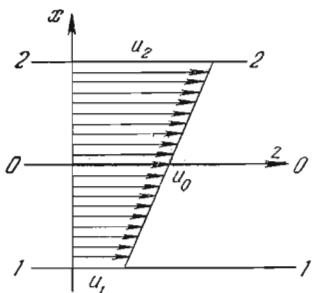


Fig. 5

$$c = \sqrt{\frac{3kT}{m}} \sim \sqrt{\frac{\gamma k}{m} T} = a$$

( $\gamma$  is the specific heat ratio). We shall use this estimate repeatedly below.

If there is a mixture of  $N$  gases at the same temperature  $T$ , then it is obvious that

$$p \sim \sum_{v=1}^N m^v n^v c^{v2} = \sum_{v=1}^N k T n^v = \sum_{v=1}^N p^v, \quad (5.4)$$

where  $p^v$  is the partial pressure of the  $v$ -th component of the mixture. Formula (5.4) expresses Dalton's law.

Consider the process of adiabatic equilibrium compression. Let a gas in a container be compressed by the action of a piston (Fig. 4) whose velocity is  $v$ . For the process to be reversible, we consider the velocity  $v$  to be small, equal to zero in the limit. During the time interval  $\Delta t$  the piston moves a distance  $\Delta x = v\Delta t$ . In the same time a molecule moving in the direction  $x$  collides with the piston  $c$ ,  $\Delta t/2x$  times, since it travels a distance  $2x$  between successive collision with the piston. At each collision with the piston the molecule acquires a velocity  $2v$ . Therefore, in time  $\Delta t$  the molecule acquires an energy

$$\frac{m}{2} \left( c + \frac{cv\Delta t}{x} \right)^2 - \frac{mc^2}{2} = mc^2 \frac{\Delta x}{x} = mc^2 \frac{\Delta n}{n}.$$

This change of mean molecular energy is proportional to the change in temperature:

$$\frac{3}{2} k \Delta T \sim mc^2 \frac{\Delta n}{n} \sim p \frac{\Delta n}{n^2}.$$

Introducing the specific volume  $V = 1/mn$ , we obtain the relation, well known in thermodynamics:

$$p dV \sim -c_V dT, \quad c_V = \frac{3}{2} \frac{k}{m}, \quad (5.5)$$

where  $c_V$  is the specific heat per molecule.

The phenomena examined above for a perfect gas (not a van der Waals gas) do not depend on the laws of molecular interaction, nor, therefore, on the mean free path. The pressure on the wall, for example, does not depend on whether one and the same molecule moves from wall to wall, or whether it acquires the appropriate momentum as it approaches the wall from another molecule as a result of a collision.

We now examine the simplest processes of transfer of molecular quantities which depend appreciably on the mean free path. We assume that at every point of the flow the gas is in a near-equilibrium state. Then the thermal velocities of the molecules, i.e., the velocities of the molecules in a coordinate system moving with the gas, are of the same order in all directions. The mean velocity, the root-mean-square velocity, and the mean relative velocity of thermal motion of the molecules are also of the same order, and we shall therefore not distinguish between them below. Within that accuracy, the mean free path is  $\lambda \sim 1/n\sigma$ , in a coordinate system moving with the gas.

For simplicity we consider uniform motion of a gas. Let the velocity component of the gas  $u_z = u$  vary as a function of  $x$ . We consider the other two velocity components to be zero (Fig. 5), and the density, temperature, and pressure of the gas to be constant. As a result of the thermal motion, molecules cross the surface element  $00$ . The number of those molecules is proportional to the density  $n$  and the mean thermal velocity of the molecules,  $c$ . Molecules which cross the surface element  $00$  have experienced their last collision on the average at a distance of the order of the mean free path  $\lambda$  from  $00$ . Therefore, molecules crossing the surface  $00$  from below carry a momentum component of order

$$m \left( u_0 - \frac{\partial u}{\partial x} \lambda \right),$$

while molecules going downwards carry a momentum

$$m \left( u_0 + \frac{\partial u}{\partial x} \lambda \right),$$

where  $u_0$  is the velocity of the gas at  $x = 0$ . Therefore, the momentum per unit area per unit time transferred upward by the thermal motion of the molecules is of order

$$mnc\lambda \frac{\partial u}{\partial x}. \quad (5.6)$$

From the macroscopic point of view the transfer of tangential momentum across the area 00 is equal in magnitude to a friction stress

$$\mu \frac{\partial u}{\partial x}, \quad (5.6a)$$

where  $\mu$  is the viscosity. Comparing (5.6) and (5.6a), we have

$$\mu \sim nmc\lambda \sim \frac{mc}{\sigma} \sim \frac{V kmT}{\sigma}. \quad (5.7)$$

Therefore, the viscosity does not depend on the pressure, and is a function of the temperature. For a gas of hard-sphere molecules ( $\sigma = \text{const}$ ), the viscosity  $\mu \sim \sqrt{T}$ . For Maxwellian molecules the collision cross section (see §1.3) is inversely proportional to the relative velocity of the molecules, and, therefore, also to the square root of the temperature; in that case,  $\mu \sim T$ .

If the gas is at rest, while the temperature varies along the  $x$  axis, the energy transferred by thermal motion of the molecules is of order

$$nc\lambda \frac{\partial}{\partial x} \left( \frac{mc^2}{2} \right) \sim nc\lambda k \frac{\partial T}{\partial x}.$$

On the other hand, the heat flux is

$$\lambda_T \frac{\partial T}{\partial x},$$

where  $\lambda_T$  is the thermal conductivity. Therefore,

$$\lambda_T \sim nck\lambda \sim \frac{k}{m} \mu \sim c_V \mu. \quad (5.8)$$

This relation is valid for any law of molecular interaction.

We now consider a mixture of two gases of densities  $n^{(1)}$  and  $n^{(2)}$ . At every point both gases have the same temperature  $T$ . We assume the pressure of the gas to be uniform. Let the densities  $n^{(1)}$  and  $n^{(2)}$  and the temperature vary only along the  $x$  axis. As a result of the temperature variation, the mean thermal velocity of the molecules

$$c^{(v)} = \sqrt{\frac{3kT}{m^{(v)}}} \quad (v = 1, 2) \quad (5.9)$$

also varies.

Let the area 00 move with the gas so that molecules cross it only because of thermal motion. If the subscript 0 designates quantities at the section 00, as above, the number of molecules of the  $v$ -th kind which cross 00 from below is of order

$$\left( n_0^{(v)} - \frac{\partial n^{(v)}}{\partial x} \lambda^{(v)} \right) \left( c_0^{(v)} - \frac{\partial c_0^{(v)}}{\partial x} \lambda^{(v)} \right). \quad (5.10)$$

Similarly,

$$\left( n_0^{(v)} + \frac{\partial n^{(v)}}{\partial x} \lambda^{(v)} \right) \left( c_0^{(v)} + \frac{\partial c_0^{(v)}}{\partial x} \lambda^{(v)} \right) \quad (5.10a)$$

molecules are transferred from above.

Therefore, the total number of molecules of the  $v$ -th kind transferred upward per unit area and unit time is of order

$$N^{(v)} = - \left( \frac{\partial n^{(v)}}{\partial x} c_0^{(v)} + \frac{\partial c_0^{(v)}}{\partial x} n^{(v)} \right) \lambda^{(v)}. \quad (5.11)$$

The velocity of the  $v$ -th component of the gas is

$$u^{(v)} = \frac{N^{(v)}}{n^{(v)}}.$$

For the relative velocity of components of the gas or for the velocity of diffusion, we have, according to (5.11),

$$\begin{aligned} u^{(1)} - u^{(2)} &= \frac{N^{(1)}}{n^{(1)}} - \frac{N^{(2)}}{n^{(2)}} \\ &= - \left[ \frac{\lambda^{(1)}}{n^{(1)}} \frac{\partial n^{(1)}}{\partial x} c_0^{(1)} - \frac{\lambda^{(2)}}{n^{(2)}} \frac{\partial n^{(2)}}{\partial x} c_0^{(2)} + \frac{\partial c^{(1)}}{\partial x} \lambda^{(1)} - \frac{\partial c^{(2)}}{\partial x} \lambda^{(2)} \right]. \end{aligned} \quad (5.12)$$

We denote the concentrations of the components of the mixture by  $n_0^{(v)} = n^{(v)}/n$ . Clearly, the partial pressure  $p^{(v)}$  is

$$p^{(v)} = n^{(v)} kT = n_0^{(v)} p.$$

Therefore,

$$\frac{\partial n_0^{(v)}}{\partial x} = \frac{p}{k} \left( \frac{1}{T} \frac{\partial n_0^{(v)}}{\partial x} - \frac{n_0^{(v)}}{T^2} \frac{\partial T}{\partial x} \right). \quad (5.13)$$

Substituting this expression into (5.12) and eliminating  $c^{(v)}$  in terms of  $T$  by (5.9), we obtain the following expression for the diffusion velocity:

$$u^{(1)} - u^{(2)} = - \frac{1}{n_0^{(1)} n_0^{(2)}} \left( D_{12} \frac{\partial n_0^{(1)}}{\partial x} + D_T \frac{1}{T} \frac{\partial T}{\partial x} \right), \quad (5.14)$$

where

$$D_{12} \sim \sqrt{\frac{3kT}{m^{(1)} m^{(2)}}} (\lambda^{(1)} n_0^{(2)} \sqrt{m^{(2)}} + \lambda^{(2)} n_0^{(1)} \sqrt{m^{(1)}})$$

is the diffusion coefficient, and

$$D_T \sim \sqrt{\frac{3kT}{m^{(1)} m^{(2)}}} (\lambda^{(2)} \sqrt{m^{(1)}} - \lambda^{(1)} \sqrt{m^{(2)}}) n_0^{(1)} n_0^{(2)}$$

is the thermal diffusion coefficient.

Here we take into consideration the fact that  $n_0^{(1)} + n_0^{(2)} = 1$ , and, therefore,

$$\frac{\partial n_0^{(1)}}{\partial x} = - \frac{\partial n_0^{(2)}}{\partial x}.$$

The order of magnitude of the mean free path  $\lambda^{(v)}$  is

$$\lambda^{(v)} \sim \frac{c^{(v)}}{n^{(v)} \sigma_{vv} c^{(v)} + n^{(\mu)} (c^{(v)} + c^{(\mu)}) \sigma_{v\mu}} = \frac{1}{n^{(v)} \sigma_{vv} + n^{(\mu)} \sigma_{v\mu} \left( 1 + \sqrt{\frac{m^{(v)}}{m^{(\mu)}}} \right)}, \quad (5.15)$$

(v,  $\mu = 1, 2$ ),

where  $\sigma_{vv}$  is the collision cross section of the  $v$ -molecules among themselves, and

$$\sigma_{12} = \sigma_{21} = \frac{\pi}{4} \left( \sqrt{\frac{\sigma_{11}}{\pi}} + \sqrt{\frac{\sigma_{22}}{\pi}} \right)^2$$

is the cross section for collision of molecules of different kinds. It may be seen from (5.15) that in general the mean free paths  $\lambda^{(1)}$  and  $\lambda^{(2)}$  are different.

If  $\lambda^{(1)} = \lambda^{(2)}$ , the heavier molecules gravitate to the colder region. If the masses of the molecules are equal, then molecules with the larger collision cross sections gravitate to the cold region. If the masses of the molecules are equal, and  $\lambda^{(1)} = \lambda^{(2)}$ , there is no thermal diffusion. The diffusion coefficients are inversely proportional to the pressure.

The qualitative laws that we have established are confirmed by the exact theory.

A theory built along the lines discussed above is sometimes called mean free path theory. It may be improved by the introduction of several mean free paths, and by more-accurate account of the velocity distribution of the molecules. Then improved results are naturally attained, at the expense of complication in the analysis.

## THE EQUATIONS OF THE KINETIC THEORY OF GASES

### §2.1. Description of the Motion of a Many-Particle System

Consider a gas consisting of  $N$  monatomic molecules. We shall assume that the state of a molecule is fully determined by its coordinates  $x_i$  (by the vector  $\mathbf{x}$ ) and by the components of its translation velocity  $\xi_i$  (by the vector  $\boldsymbol{\xi}$ ), where  $i = 1, 2, 3$ , i.e., we ignore the possible excitation of internal degrees of freedom (rotational, vibrational, and electronic). We consider all the molecules identical. If the positions and velocities of all the molecules are given at time  $t = 0$ , then the subsequent behavior of the system is described completely by the system of  $N$  equations (Newton's law)

$$m \frac{d^2 \mathbf{x}_k}{dt^2} = \mathbf{X}_k \quad (k = 1, 2, \dots, N), \quad (1.1)$$

where  $\mathbf{X}_k$  is the force acting on the  $k$ -th molecule, and  $m$  is the mass of a molecule.

However, even if we assume that computing facilities are available for the solution of an enormous number\* of equations (1.1), it is evident here that this kind of description of the motion is unnecessarily complete, since it is impossible to grasp all the information resulting from a solution of this kind. Besides, we do not have the initial data necessary for the solution of the system

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\*We recall that in one cubic centimeter at atmospheric pressure there are  $10^{19}$  molecules, while at an altitude of 300 km there are  $10^9$  molecules. In the most elaborate vacuum chambers there are  $10^7$ - $10^8$  molecules per cubic centimeter.

(1.1). We therefore resort to a less complete statistical description of the behavior of the system.

The 6-dimensional space in which the state of a molecule is determined by its three space coordinates and its three velocity components is designated as the  $\gamma$ -phase space, or simply the  $\gamma$ -space. The system of  $N$  molecules is represented by  $N$  points with coordinates  $(\mathbf{x}_i, \xi_i)$ , where  $i = 1, \dots, N$ .

It is often convenient to use a  $6N$ -dimensional phase space or  $\Gamma$ -space, whose coordinates are the Cartesian position coordinates and the velocity components of all the  $N$  molecules. In that space the entire system of  $N$  molecules is represented by a single point.

To define the desired state of a system of many particles, we determine the probability of finding the system in a particular state. Let

$$\begin{aligned} F_N(t, \mathbf{r}_N, \mathbf{v}_N) d\mathbf{r}_N d\mathbf{v}_N &= F_N(t, z_1, \dots, z_N) dz_1 \dots dz_N \\ &= F_N(t, \mathbf{x}_1, \dots, \mathbf{x}_N, \xi_1, \dots, \xi_N) d\mathbf{x}_1 \dots d\mathbf{x}_N d\xi_1 \dots d\xi_N \\ (\mathbf{dx} &= dx_1 dx_2 dx_3, d\xi = d\xi_1 d\xi_2 d\xi_3, z \equiv (\mathbf{x}, \xi), \\ dz &= dx d\xi, \mathbf{r}_N \equiv (\mathbf{x}_1, \dots, \mathbf{x}_N), \mathbf{v}_N \equiv (\xi_1, \dots, \xi_N)) \end{aligned}$$

be the probability of finding the system in a state defined by  $\mathbf{r}_N$ ,  $\mathbf{v}_N$ , in the interval  $d\mathbf{r}_N d\mathbf{v}_N$  in the neighborhood of the point  $(\mathbf{r}_N, \mathbf{v}_N)$  in the  $\Gamma$ -space;  $F_N$  is the probability density function.

If all the molecules are identical, then the state of the system is invariant under permutation (interchange of position) of molecules in the  $\gamma$ -space. However, to each permutation (renumbering of the points) in the  $\gamma$ -space, there corresponds a new point in the  $\Gamma$ -space. Therefore, to a single cloud of  $N$  identical molecules in the  $\gamma$ -space there correspond  $N!$  points in the  $\Gamma$ -space. These  $N!$  points are so distributed that the function  $F_N(t, z_1, \dots, z_N)$  is symmetrical with respect to the variables  $z_i$ . If

$$f_N(t, \mathbf{x}_1, \dots, \mathbf{x}_N, \xi_1, \dots, \xi_N) dx_1 \dots dx_N d\xi_1 \dots d\xi_N$$

is the probability of finding the system at the points  $(\mathbf{x}_i, \xi_i)$  of the  $\gamma$ -space in the intervals  $dx_i d\xi_i$ , then, evidently, allowing for permutations,

### § 2.1] MOTION OF A MANY-PARTICLE SYSTEM

$$f_N dx_1 \dots dx_N d\xi_1 \dots d\xi_N = N! F_N d\mathbf{r}_N d\mathbf{v}_N.$$

For a less detailed description of the system we shall introduce the functions

$$\begin{aligned} F_s(t, z_1, \dots, z_s) &\equiv F_s(t, \mathbf{x}_1, \dots, \mathbf{x}_s, \xi_1, \dots, \xi_s) \\ &= \int F_N(t, z_1, \dots, z_N) dz_{s+1} \dots dz_N. \end{aligned} \quad (1.2)$$

The function  $F_s$  determines the probability of simultaneously observing  $s$  molecules in the state  $(z_1, \dots, z_s)$  independently of the states of the remaining  $N - s$  molecules.

Correspondingly,

$$f_s(t, z_1, \dots, z_s) = \frac{N!}{(N-s)!} F_s(t, z_1, \dots, z_s).$$

All the functions  $F_s \geq 0$  and are normalized so that

$$\int F_N dz_1 \dots dz_N = 1 \quad \text{and} \quad \int F_s dz_1 \dots dz_s = 1.$$

In what follows, we are interested mainly in the one-particle distribution function  $F_1(t, \mathbf{x}, \xi)$ , which determines the probability of finding a single particle in the element  $d\mathbf{x}_1 d\xi_1$  of phase space in the vicinity of the point  $(\mathbf{x}_1, \xi_1)$  at time  $t$ . As will be seen below, the functions  $F_1$  suffice to describe the motion of gases of moderate density which are the subject of this book. The state of denser gases depends on the joint configuration of two molecules (is determined by a binary distribution function), and so on.

For identical molecules, the function

$$f(t, \mathbf{x}, \xi) dx d\xi = N F_1(t, \mathbf{x}_1, \xi_1) dx d\xi \quad (1.3)$$

is the probable number of molecules in the element of volume of physical space  $d\mathbf{x}$  near the point  $\mathbf{x}$ , possessing velocities in the element of velocity space  $d\xi$  near the point  $\xi$ .

The distribution function  $f(t, \mathbf{x}, \xi)$  is fundamental to the kinetic theory of gases. But, even this function gives an unnecessarily detailed description of the gas. From an experiment we obtain only certain average quantities, such as the density, velocity, stress tensor, or energy flux of the gas. Therefore, in most prob-

lens we are interested in just these average properties. But, as we show below, a hydrodynamic description of a gas is possible only when the mean free path of the molecules is sufficiently small. In general we must solve the problem on the molecular scale, i.e., find the distribution function  $f(t, \mathbf{x}, \xi)$ , and then average over it to obtain the macroscopic quantities which interest us.

From the definition of the function  $f$ , it is easy to see that the number of molecules  $n$  in a unit volume of a gas is equal to

$$n(t, \mathbf{x}) = \int f(t, \mathbf{x}, \xi) d\xi, \quad (1.4)$$

where the integration is performed over all possible molecular velocities. Similarly, the mean velocity of the molecules, the stress tensor, and the energy flux vector are defined by the relations:

$$\mathbf{u}(t, \mathbf{x}) = \frac{1}{n} \int \xi f(t, \mathbf{x}, \xi) d\xi, \quad (1.5)$$

$$P_{ij} = m \int c_j c_i f(t, \mathbf{x}, \xi) d\xi, \quad (1.6)$$

$$q_i = \frac{m}{2} \int c^2 c_i f(t, \mathbf{x}, \xi) d\xi. \quad (1.7)$$

Here,  $\mathbf{c} = \xi - \mathbf{u}$  is the thermal, or peculiar, velocity of the molecules.

The mean energy of thermal motion of the molecules is usually characterized by the temperature, defined in the kinetic theory of gases by the relation

$$\frac{3}{2} kT = \frac{1}{n} \int \frac{mc^2}{2} f(t, \mathbf{x}, \xi) d\xi, \quad (1.8)$$

where  $k$  is the Boltzmann constant. The coefficient of  $T$  in relation (1.8) is so chosen that in the equilibrium state the fraction of energy in each translational degree of freedom is equal to  $\frac{1}{2}kT$ .

We also introduce the quantity

$$p = \frac{1}{3} (P_{11} + P_{22} + P_{33}), \quad (1.9)$$

which coincides with the conventional pressure in the hydrodynamic description of a gas. With the aid of (1.6) and (1.8), relation (1.9) may be rewritten in the form

$$p = nkT. \quad (1.10)$$

We note that the tensor  $P_{ij}$  is symmetric, and that the temperature  $T$  and the pressure  $p$  are not independent parameters, since they are represented by components of the tensor  $P_{ij}$ . Therefore, thirteen independent quantities in all appear in the hydrodynamic description of a gas. Instead of the stress tensor (1.6) it is often convenient to use the tensor

$$p_{ij} = P_{ij} - \delta_{ij} p. \quad (1.11)$$

We note, finally, that the relation between distribution functions of different orders is not unique. In fact, let the function  $F_s$  be specified at time  $t = 0$  with  $1 \leq s \leq N$ . Since  $F_s$  is the integral of  $F_N$ , a whole class of functions  $\{F_N(0)\}$  corresponding to the single function  $F_s(0)$  must exist. At an arbitrary time  $t > 0$ , to the family of initial states  $\{F_N(0)\}$  of the system corresponds a family of states  $\{F_N(t)\}$ , to each of which corresponds its own function  $F_s(t)$ , i.e., to a single function  $F_s(0)$  corresponds, in general, a whole family of functions  $\{F_s(t)\}$ , i.e., the description of the motion of a gas is not unique on the scales  $s < N$ . In order to make the description unique, we must restrict the class of functions  $F_N$ , i.e., restrict the class of phenomena to be examined.

If, for example,

$$\begin{aligned} F_N(t, \mathbf{x}_1, \dots, \mathbf{x}_N, \xi_1, \dots, \xi_N) \\ = F_1(t, \mathbf{x}_1, \xi_1) F_1(t, \mathbf{x}_2, \xi_2) \dots F_1(t, \mathbf{x}_N, \xi_N) \end{aligned} \quad (1.12)$$

then the single function  $F_N(0)$  corresponds to the initial function  $F_1(0)$ , and the description of the motion of the gas by means of the function  $F_1$  is unique. The condition (1.12) is satisfied if the probabilities of finding each of the molecules of the system in a given state are independent. This condition is called the condition of molecular chaos. Clearly, real gases always satisfy that condition.\* We only consider systems which satisfy the molecular chaos hypothesis.

\*See §§ 2.2 and 2.3.

### §2.2. The Boltzmann Equation\*

Let us consider a gas in which the time between collisions  $\tau$  is much greater than the collision time  $\tau_c$ . As was noted in §1.4, this condition is satisfied both by rarefied gases, which constitute the main subject of this book, and by gases rather dense in the ordinary sense of the word, at pressures of the order of tens and hundreds of atmospheres.

As we mentioned in §1.1, we are interested only in ideal gases, i.e., gases in which the volume occupied by the molecules is small compared to the volume available to the gas. If  $d$  is the effective diameter of a molecule, then in an ideal gas  $nd^3 = \epsilon \rightarrow 0$  asymptotically, where  $n$  is the number of molecules per unit volume. We call an ideal gas a Boltzmann gas, if the ratio of the mean free path of the molecules in the gas to a characteristic flow dimension  $L$  is finite, i.e., if

$$Lnd^2 = \text{const} \quad \text{for } nd^3 \rightarrow 0,$$

since the mean free path is inversely proportional to  $nd^2$  (see §1.4). If, at the same time,

$$Lnd^2 \rightarrow 0 \quad \text{and } nd^3 \rightarrow 0,$$

we call the gas a Knudsen gas. Molecular collisions are negligible in such a gas. We examine a Boltzmann gas below.

1. The motion of a molecule between collisions is completely determined by external forces and does not depend on the position or the velocity of other molecules. Since the molecules do not interact with each other during most of the time, the state of the gas may be described by the one-particle distribution function  $f(t, \mathbf{x}, \xi)$  defined in the preceding paragraph. The relative position of the molecules is relevant only at the moment of collision. When  $\tau_c \ll \tau$ , it is clear that triple collisions are much less probable than binary collisions, and we neglect them.

According to the definition of the function  $f$ , the probable number of molecules in the element of physical volume  $d\mathbf{x}$  near the point  $\mathbf{x}$  at time  $t$ , with velocities in the element  $d\xi$  near  $\xi$  is

\*An excellent account of the foundations of the kinetic theory has been given by L. Boltzmann in "Lectures on Gas Theory," University of California Press, 1964.

### §2.2] THE BOLTZMANN EQUATION

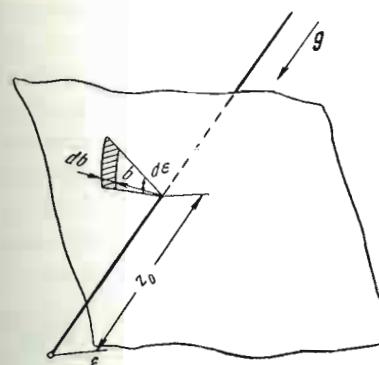


Fig. 6

If the molecules in  $d\mathbf{x}$  do not undergo any collision, then at time  $t + dt$  all these molecules are in the volume element  $d\mathbf{x}$  near the point  $\mathbf{x} + \xi dt$  and possess velocities in the element  $d\xi$  near velocity  $\xi + (\mathbf{X}_0/m)dt$ , where  $\mathbf{X}_0(t, \mathbf{x})$  is the external force acting on a molecule. [It is clear that the force  $\mathbf{X}_0(t, \mathbf{x})$  must vary little within the element  $d\mathbf{x}$  during the time  $dt$ .]

Therefore, in that case,

$$f(t, \mathbf{x}, \xi) d\mathbf{x} d\xi = f\left(t + dt, \mathbf{x} + \xi dt, \xi + \frac{\mathbf{X}_0}{m} dt\right) d\mathbf{x} d\xi. \quad (2.1)$$

But, in fact, these two quantities differ, since the fraction  $\Delta^- d\mathbf{x} d\xi dt$  of the group of molecules that we are considering changes its velocity as a result of collisions and does not appear in the element  $d\mathbf{x} d\xi$  of phase space near the point  $(\mathbf{x} + \xi dt, \xi + (\mathbf{X}_0/m)dt)$  at the time  $t + dt$ . On the other hand as a result of collisions  $\Delta^+ d\mathbf{x} d\xi dt$  molecules which were not located in the element  $d\mathbf{x} d\xi$  near the point  $(\mathbf{x}, \xi)$  at time  $t$  may appear in this element at time  $t + dt$ .

Thus,

$$\begin{aligned} & f\left(t + dt, \mathbf{x} + \xi dt, \xi + \frac{\mathbf{X}_0}{m} dt\right) d\mathbf{x} d\xi \\ &= f(t, \mathbf{x}, \xi) d\mathbf{x} d\xi + (\Delta^+ - \Delta^-) d\mathbf{x} d\xi dt. \end{aligned} \quad (2.2)$$

Let us determine the number of molecules  $\Delta^- d\mathbf{x} d\xi dt$ , lost as a result of collisions from among the molecules of velocity  $\xi$ . Con-

sider the collision of one of the molecules of our group ( $\xi$ -molecule) with a molecule of velocity  $\xi_1$  (a  $\xi_1$ -molecule). We draw a straight line (Fig. 6) through the center of the  $\xi$ -molecule, parallel to the relative velocity vector  $\mathbf{g} = \xi_1 - \xi$ . We define it as the axis of a cylindrical system of coordinates  $z, b, \varepsilon$ . The assumption made at the beginning of this paragraph, that the collision time  $\tau_c$  is small compared to the time between collisions, is equivalent to assuming that the range of the intermolecular forces is small compared to the mean free path. We may therefore assume that at some distance  $z_0$  from a  $\xi$ -molecule the trajectories of the  $\xi_1$ -molecules are effectively undisturbed. The element  $bdbd\varepsilon$  of the plane  $z_0 = \text{const}$  is crossed by

$$f(t, \mathbf{x}, \xi_1) gb db d\varepsilon dt \quad (2.3)$$

$\xi_1$ -molecules during the time interval  $dt$ . To obtain the total number of collisions experienced by the  $\xi$ -molecules, the expression (2.3) must be integrated over all angles  $\varepsilon$ , over all impact parameters  $b$ , and over all velocities  $\xi_1$ ; we then obtain

$$dt \int f(t, \mathbf{x}, \xi_1) gb db d\varepsilon d\xi_1. \quad (2.4)$$

Hence the total number of collisions experienced by all the  $\xi$ -molecules in the element  $d\mathbf{x}d\xi$  is equal to

$$\Delta^- dt d\mathbf{x} d\xi = dt d\mathbf{x} d\xi f(t, \mathbf{x}, \xi) \int f(t, \mathbf{x}, \xi_1) gb db d\varepsilon d\xi_1. \quad (2.5)$$

From a collision denoted by the angle  $\varepsilon$  and the impact parameter  $b$ , the  $\xi$ - and  $\xi_1$ -molecules acquire the velocities  $\xi'$  and  $\xi'_1$ , respectively. Let the interaction potential of the molecules be spherically symmetric. Then (see §1.3),

$$g = g', \quad b = b', \quad \text{and} \quad \varepsilon = \varepsilon', \quad (2.6)$$

where the primes denote the relative velocity and the target distance of the molecules after a collision, respectively. If the  $\xi'$ - and  $\xi'_1$ -molecules collide at a target distance  $b' = b$ , then, as was shown in §1.3, they acquire the velocities  $\xi$  and  $\xi_1$ , respectively, as a result of a collision. By analogy with relation (2.5), we have

$$\Delta^+ dt d\mathbf{x} d\xi = dt d\mathbf{x} d\xi' \int f(t, \mathbf{x}, \xi') f(t, \mathbf{x}, \xi'_1) g' b' db' d\varepsilon' d\xi'_1. \quad (2.7)$$

## §2.2] THE BOLTZMANN EQUATION

In contrast with expression (2.5), the function  $f(t, \mathbf{x}, \xi')$  here cannot be brought out from under the integral sign, since to each value of the velocity  $\xi'_1$  there corresponds a velocity  $\xi'$ , such that the  $\xi$ -molecule acquires the velocity  $\xi$  as a result of a collision. Furthermore, the Liouville's theorem of conservation of phase volume gives

$$d\xi d\xi_1 = d\xi' d\xi'_1. \quad (2.8)$$

We substitute expressions (2.5) and (2.7) into Eq. (2.2), replace  $\varepsilon'$ ,  $b'$ ,  $g'$ , and  $d\xi' d\xi'_1$  with the aid of relations (2.6) and (2.8), and expand the left side of (2.2) in a series in  $dt$ ; dividing by  $dtd\mathbf{x}d\xi$ , we obtain

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{X_0}{m} \cdot \frac{\partial f}{\partial \xi} = \int (f' f'_1 - f f_1) gb db d\varepsilon d\xi_1, \quad (2.9)$$

where†

$$\begin{aligned} f &= f(t, \mathbf{x}, \xi), & f_1 &= f(t, \mathbf{x}, \xi_1), & f' &= f(t, \mathbf{x}, \xi'), \\ f'_1 &= f(t, \mathbf{x}, \xi'_1). \end{aligned}$$

That equation is the fundamental equation of the kinetic theory of gases. It is usually called the Boltzmann equation, or the Maxwell - Boltzmann equation. The integral on the right side is called the collision integral.

It should be noted that, in deriving the Boltzmann equation we follow molecules along their trajectories in phase space. Therefore, to derive Eq. (2.9) it is necessary only to differentiate the distribution function along the phase trajectory, i.e., the derivative  $df/dt$  must exist while the partial derivatives  $\partial f/\partial \mathbf{x}$ ,  $\partial f/\partial \xi$ , and  $\partial f/\partial t$  need not exist.

\*See, for example, A. Sommerfeld, "Thermodynamics and Statistical Mechanics," Vol. 5 of "Lectures on Theoretical Physics," Academic Press, New York, 1956. This fact is also easily established by direct calculation of the transformation Jacobian.

†For any vector  $\mathbf{a} = (a_1, a_2, a_3)$ ,

$$\frac{\partial}{\partial \mathbf{a}} = i_1 \frac{\partial}{\partial a_1} + i_2 \frac{\partial}{\partial a_2} + i_3 \frac{\partial}{\partial a_3},$$

where  $i_1, i_2, i_3$  are unit vectors in a Cartesian coordinate system.

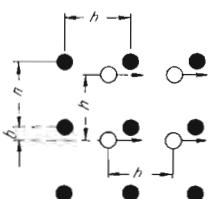


Fig. 7

( $\mathbf{x}_1, \xi_1$ ) and molecule two at the point ( $\mathbf{x}_2, \xi_2$ ) to be independent, i.e., we assumed that

$$F_2(t, \mathbf{x}_1, \xi_1, \mathbf{x}_2, \xi_2) = F_1(t, \mathbf{x}_1, \xi_1) F_1(t, \mathbf{x}_2, \xi_2).$$

This assumption is called the postulate of molecular chaos.

Finally, the third assumption was to regard as equally probable the collision pairs of molecules of arbitrary impact parameter  $b$ , i.e., we assumed that the function  $f(t, \mathbf{x}_1, \xi_1)$  does not change over distances of the order of the collision cross section, and that  $f(t, \mathbf{x}_1, \xi_1) = f(t, \mathbf{x}, \xi_1)$ .

Consider a simple example of a mechanical system. Let there be two ordered groups of hard-sphere molecules (Fig. 7). The sphere diameter  $d$  is small compared to the pitch  $h$ . Clearly, this system does not satisfy the third assumption. In fact, if the target distance  $d < b < h - d$ , the molecules generally do not collide. When  $b < d$ , all the molecules collide with the same impact parameter, and, therefore, the molecules of each group again possess the same velocity after the collision. If we examine these same groups of molecules within the framework of the assumption used in deriving the Boltzmann equation, then each molecule of one group could collide with a molecule of the other group with any impact parameter. The result would be that the molecules would possess a whole range of velocities after collision. Clearly, the behavior of the system is quite different in these two cases. Therefore, the Boltzmann equation is applicable only to disordered systems. In principle, an ordered mechanical system may remain ordered as long as we please (this is particularly easy to see in

\*For an analysis of the assumptions, see also the following paragraph.

the example of the system shown in Fig. 7, with  $b = 0$ ). In fact, however, there is always some scatter in the initial data. It follows from the Heisenberg uncertainty principle that there must be scatter in molecular systems. In fact, we shall examine, as an example, a beam of molecules approaching a molecule. If  $\Delta b$  is the scatter in target distance, and  $\Delta \xi_{\perp}$  is the scatter in transverse velocity ( $\Delta \xi_{\perp} / \xi$  is the scatter of the beam), then, according to the uncertainty principle,

$$m \Delta b \Delta \xi_{\perp} \geq \hbar,$$

where  $\hbar = 1.054 \cdot 10^{-27}$  erg/sec is the Planck constant. For estimating purposes let us take  $\Delta \xi_{\perp} = 10^4$  cm/sec (with  $\xi \sim 10^5$  cm per sec) and  $m = 10^{-23}$  g (the mass of the proton is  $1.67 \cdot 10^{-24}$  g). Then, for the scatter of impact parameters we obtain the estimate  $\Delta b \sim 10^{-8}$  cm, i.e., which is of the order of the diameter of the molecules. The greater the dispersion in the initial data, the smaller the number of collisions required to destroy the ordered state. The coarse estimate that we have made indicates that there is always considerable dispersion in molecular systems, and we may therefore consider molecular systems to be disordered. In addition, it is clear that the presence of dispersion (uncertainty in the impact parameters) must lead, as a result of collisions, to disorder in the sense indicated above (to independence), even if the molecules are correlated at time zero. Therefore, we may expect that the time required to establish molecular disorder is of the order of the time between collisions.

The Boltzmann equation may be used to describe processes which occur in intervals comparable to or smaller than the time between collisions, but much longer than the collision time. Then it is necessary that the molecular chaos condition be satisfied at time zero. In an overwhelming majority of cases actual molecular systems satisfy this requirement. In the exceptional cases when the condition of molecular disorder is not satisfied at the time zero or on the boundaries, we may expect that molecular disorder is established in a time of the order of the time between collisions.

As we shall see later, the Boltzmann equation, which is based on the reversible laws of mechanics, describes irreversible processes. That is, by making assumptions regarding molecular chaos and a lack of order (scatter, and uncertainty) in the target

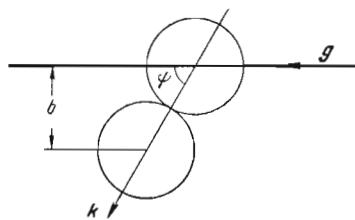


Fig. 8

any instant, then the velocities of the molecules also change sign, and the system passes in the reverse direction through the same sequence of states which it went through in the forward direction (there will be a change only in the sign of the velocities). If we consider the system to be disordered both in the forward and in the reverse motion (we allow scatter), then it is clear that the forward and reverse paths will be different, i.e., the process will be irreversible. In actual molecular systems we have in fact the second case, for which the Boltzmann equation has meaning.

**2.** The Boltzmann equation may be transformed somewhat for hard-sphere molecules of diameter  $d$ .

Let  $\psi$  be the angle between the relative velocity vector  $\mathbf{g}$  and the line of the molecules at the instant of collision (Fig. 8). It is easy to see that

$$b = d \sin \psi \text{ and } db = d \cos \psi d\psi.$$

Then Eq. (2.9) may be written in the form

$$\frac{df}{dt} = \frac{d^2}{2} \int (f' f'_1 - f f'_1) |\mathbf{g} \cdot \mathbf{k}| d\Omega d\xi_1, \quad (2.10)$$

where  $\mathbf{k}$  is the unit vector along the line of centers and  $d\Omega = \sin \psi d\psi d\varepsilon$  is a surface element on the unit sphere. The integration is carried out over the whole sphere. Therefore, the integral must be divided by two to avoid counting the same collisions twice.

In §1.3 it was shown [Eqs. (3.10), (3.11)], that for molecules with a power law of interaction, the deflection angle  $\chi$  is a function of the single parameter

$$\rho = b \left( \frac{mg^2}{4(s-1)K} \right)^{\frac{1}{s-1}}.$$

It is therefore convenient for such molecules to replace integration over  $b$  by integration over  $\rho$  in the Boltzmann equation (2.9). Carrying out the substitution of variables, we obtain

$$\frac{df}{dt} = \left( \frac{4(s-1)K}{m} \right)^{\frac{2}{s-1}} \int (f' f'_1 - f f'_1) g^{\frac{s-5}{s-1}} \rho d\rho d\varepsilon d\xi_1. \quad (2.11)$$

For Maxwellian molecules,  $s = 5$ , and the equation takes a simpler form, which does not contain  $\mathbf{g}$  explicitly:

$$\frac{df}{dt} = \left( \frac{16K}{m} \right)^{1/2} \int (f' f'_1 - f f'_1) \rho d\rho d\varepsilon d\xi_1. \quad (2.12)$$

Because of this simplification, Maxwellian molecules are widely used in theoretical investigations (see, for example, §3.3).

The so-called pseudo-Maxwellian molecules \* also enjoy great mathematical advantages. These are somewhat hypothetical molecules which, strictly speaking, do not have an interaction potential. By definition, pseudo-Maxwellian molecules are molecules for which the Boltzmann equation may be written in the form

$$\frac{df}{dt} = \left( \frac{16K}{m} \right)^{1/2} \int f' f'_1 \rho d\rho d\varepsilon d\xi_1 - Af \int f_1 d\xi_1, \quad (2.13)$$

where

$$A = \left( \frac{16K}{m} \right)^{1/2} \int \rho d\rho d\varepsilon$$

is some constant. (For Maxwellian molecules,  $A$  is infinite.) Since  $A = \text{const}$ , the integration with respect to  $\rho$  is carried out up to some finite  $\rho = \rho_{\max}$ , and the first integral in (2.13) is finite.

A distinction should be made between pseudo-Maxwellian molecules and Maxwellian molecules with a potential cutoff, i.e., molecules with a potential of the form

\*H. Grad, "Handbuch der Physik," Vol. 12, 1958.

$$U = \begin{cases} \frac{K}{r^{s-1}} & \text{for } b < b_0, \\ 0 & \text{for } b > b_0. \end{cases}$$

For those molecules the limits of integration with respect to  $\rho$  depend on  $g$  and the passage from the variable  $b$  to the variable  $\rho$  does not lead to the simplifications characteristic of Maxwellian molecules. The pseudo-Maxwellian molecules, which possess a finite collision cross section, retain all the conveniences of Maxwellian molecules; as a result, many proofs and calculations are appreciably simplified for them.

3. The Boltzmann equation may be put in a somewhat different form. Let us denote by  $W(\xi, \xi_1 | \xi', \xi'_1) \geq 0$  the probability that, as a result of a collision of molecules of velocities  $\xi$  and  $\xi_1$ , the  $\xi$ -molecule acquires a velocity  $\xi'$ , and the  $\xi_1$ -molecule—a velocity  $\xi'_1$ . Then the number of molecules per unit volume leaving the  $\xi$ -molecule group per unit time is

$$\Delta^- d\xi = d\xi f \int f_1 W(\xi, \xi_1 | \xi', \xi'_1) d\xi_1 d\xi' d\xi'_1.$$

Similarly,

$$\Delta^+ d\xi = d\xi \int f' f'_1 W(\xi', \xi'_1 | \xi, \xi_1) d\xi_1 d\xi' d\xi'_1;$$

therefore, the Boltzmann equation takes the form

$$\frac{df}{dt} = \int [f' f'_1 W(\xi', \xi'_1 | \xi, \xi_1) - f f_1 W(\xi, \xi_1 | \xi', \xi'_1)] d\xi_1 d\xi' d\xi'_1. \quad (2.14)$$

The specific form of the function  $W$  is determined by the properties of the molecular interaction, and does not depend on the form of the distribution function  $f$ .

It was shown in §1.3 that the result of a collision is determined by the relative velocity of the molecules and by the impact parameter. Since the relative velocity of the molecules is the same before and after the collision, then the probability of a given outcome of collisions is determined by the impact parameter. If the molecules have a spherical interaction potential, then, in collisions with the same impact parameter, the molecules with velocities  $\xi$  and  $\xi_1$  acquire velocities  $\xi'$  and  $\xi'_1$ , respectively and, con-

versely, the molecules with velocities  $\xi'$  and  $\xi'_1$  acquire velocities  $\xi$  and  $\xi_1$ . Therefore, the probability of the forward and the reverse collisions is the same, i.e.,

$$W(\xi, \xi_1 | \xi', \xi'_1) = W(\xi', \xi'_1 | \xi, \xi_1). \quad (2.15)$$

For molecules which satisfy the condition (2.15), the Boltzmann equation (2.14) takes the form

$$\frac{df}{dt} = \int (f' f'_1 - f f_1) W(\xi, \xi_1 | \xi', \xi'_1) d\xi_1 d\xi' d\xi'_1. \quad (2.16)$$

As noted above, the probability  $W$  is determined by the relative velocity and the impact parameter. To each impact parameter (for a given relative velocity) there corresponds a definite deflection angle  $\chi$ . Therefore, the probability  $W(\xi, \xi_1 | \xi', \xi'_1)$  may be replaced by the probability of scatter in any solid angle  $W_1(g, \chi)$ . Then the Boltzmann equation may be rewritten in the form

$$\frac{df}{dt} = \int (f' f'_1 - f f_1) W_1(g, \chi) \sin \chi d\chi d\xi_1 d\xi' d\xi'_1. \quad (2.17)$$

The relation between the transition probability, the impact parameter, and the differential collision cross section  $\sigma$ , is:

$$\begin{aligned} g b d\theta d\varepsilon &= g \sigma(\chi, g) \sin \chi d\chi d\varepsilon \\ &= W_1(g, \chi) \sin \chi d\chi d\varepsilon = W(\xi, \xi_1 | \xi', \xi'_1) d\xi' d\xi'_1. \end{aligned} \quad (2.18)$$

In essence, the relations (2.18) define the functions  $W$  and  $W_1$  introduced above.

Using differential cross sections, the Boltzmann equation takes the form

$$\frac{df}{dt} = \int (f' f'_1 - f f_1) g \sigma(\chi, g) \sin \chi d\chi d\varepsilon d\xi_1. \quad (2.19)$$

The Boltzmann equation is often written in the form

$$\frac{df}{dt} = \int B(\vartheta, g) (f' f'_1 - f f_1) d\vartheta d\varepsilon d\xi_1, \quad (2.20)$$

where

$$B(\vartheta, g) = gb \left| \frac{db}{d\vartheta} \right| \quad (2.21)$$

and  $2\vartheta = \pi - \chi$  (see §1.3).

For molecules with a power interaction law [cf. (2.11)],

$$B(\vartheta, g) = \left( \frac{4(s-1)K}{m} \right)^{\frac{2}{s-1}} g^{\frac{s-5}{s-1}} \rho \frac{dp}{d\vartheta}.$$

In particular, for Maxwellian molecules ( $s = 5$ ), the function  $B$  does not depend on  $g$ , i.e.,  $B = B(\vartheta)$ .

Note that Eq. (2.14) cannot be reduced to equations of the form (2.16), (2.17), or (2.19), for any molecules. Relation (2.15) is valid only for molecules with sufficient symmetry. Equation (2.15) is based on the fact that, for equal values of impact parameters, the transitions  $\xi, \xi_1 \rightarrow \xi', \xi'_1$  and  $\xi', \xi'_1 \rightarrow \xi, \xi_1$  both occur as a result of collisions. To prove this, in §1.3, we reversed the directions of coordinates in space (replaced  $x, y$ , and  $z$  by  $-x, -y$ , and  $-z$ ) and time (replaced  $t$  by  $-t$ ). It is easy to see that, for molecules without sufficient symmetry, that derivation is not valid. Moreover, in reversing the motion of molecules which do not possess sufficient symmetry we transform them to molecules which cannot be congruent with the original set, i.e., we transform them to molecules which possess other properties or, in other words, to molecules of another gas.

### §2.3. Derivation of the Boltzmann Equation from the Liouville Equation\*

In the preceding section we gave a direct derivation of the Boltzmann equation for a one-particle distribution function  $f(t, \mathbf{x}, \xi)$ . We now establish the relation between the Boltzmann equation and the general theorems of statistical mechanics.

1. We examine a Boltzmann gas consisting of a sufficiently large number  $N$  of particles. We describe the motion of the gas in  $6N$ -dimensional  $\Gamma$ -phase space, the coordinates of which are the

\*The reader who is interested only in applications of the Boltzmann equation may omit this section. However, for a deeper understanding both of the properties of the Boltzmann equation and the range of its application, and of its connection with the general theorems of statistical mechanics, this paragraph should be studied. The procedures used in this paragraph will be applied repeatedly in what follows.

$3N$  rectangular coordinates of the particles, and the  $3N$  components of their velocities. In that space, the system of  $N$  particles is represented by a point. The motion of the system in time is represented by a line called the phase trajectory of the system. Following the fundamental idea introduced into statistical mechanics by Gibbs, we examine not a single system, but a whole ensemble of identical systems, distributed over phase space according to an  $N$ -particle distribution function  $F_N(t, \mathbf{x}_1, \dots, \mathbf{x}_N, \xi_1, \dots, \xi_N)$ .

In place of the assertion (usual in mechanics) that at some instant of time  $t$  the system is located at a point in phase space, we speak of the probability of finding the system at time  $t$  in some element of phase space, given by

$$\begin{aligned} dW &= F_N(t, x_1, \dots, x_N, \xi_1, \dots, \xi_N) dx_1 \dots dx_N d\xi_1 \dots d\xi_N \\ &= F_N dz_1 \dots dz_N = F_N dz, \end{aligned} \quad (3.1)$$

where

$$dz_i = dx_i d\xi_i, \quad dz = dz_1 \dots dz_N.$$

From now on, in the interests of brevity, we use the following notation:

$$F_N(t, x_1, \dots, x_N, \xi_1, \dots, \xi_N) \equiv F_N(t, z_1, \dots, z_N) \equiv F_N(t, z).$$

Since all molecules are identical, the function  $F_N(t_1, z_1, \dots, z_N)$  is symmetrical with respect to the  $N$  variables  $z_i$  (see §2.1).

If we represent the ensemble in the form of a sufficiently large (in the limit infinite) number of systems  $\mathcal{N}$ , then the probability density  $F_N$  is equal to the number of density phase points of the ensemble,  $n(z)$  per unit phase volume.

Let the phase points of the ensemble, at some time  $t_0$ , be distributed in accordance with the assigned probability density at that time. Each phase point of the ensemble represents a system of  $N$  particles, moving in accordance with the laws of mechanics, i.e., its motion is described by the  $N$  Newtonian equations

$$m_i \frac{d^2 \mathbf{x}_i}{dt^2} = \mathbf{X}_i(t, z_1, \dots, z_N), \quad (i = 1, \dots, N), \quad (3.2)$$

where  $\mathbf{X}_i$  is the force acting on the  $i$ -th molecule.

Clearly, the phase trajectories of different systems of the ensemble cannot intersect; if they did, a single mechanical system (the systems of the ensemble are identical) could move in a different manner under the same initial conditions, which is impossible.

The motion of the ensemble is similar to the motion of a gas of density  $F_N$ . The systems which constitute the ensemble are neither created nor destroyed during the motion. Therefore, the total number of systems in the ensemble does not change, i.e., there are no sources or sinks, in the language of hydrodynamics. Therefore, the continuity equation must be satisfied, and for an  $N$ -dimensional space this takes the form

$$\frac{\partial F_N}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial z_i} (\dot{z}_i F_N) = 0, \quad (3.3)$$

where

$$\frac{\partial(\dot{z}_i F_N)}{\partial z_i} = \frac{\partial}{\partial x_i} \cdot \left( \frac{dx_i}{dt} F_N \right) + \frac{\partial}{\partial \xi_i} \cdot \left( \frac{d\xi_i}{dt} F_N \right)$$

and  $x_i$  and  $\xi_i$  are the radius vector and the velocity of the  $i$ -th particle of the system. It is evident that  $\dot{x}_i = \xi_i$ , and, in view of (3.2),

$$m \dot{\xi}_i = X_i(t, z).$$

Therefore, Eq. (3.3) takes the form

$$\frac{\partial F_N}{\partial t} + \sum_{i=1}^N \xi_i \cdot \frac{\partial F_N}{\partial x_i} + \frac{1}{m} \sum_{i=1}^N \frac{\partial}{\partial \xi_i} \cdot (X_i F_N) = 0. \quad (3.4)$$

This is the well-known Liouville equation.

If the force acting on the particles depends only on the coordinates of the particles and does not depend on their velocities, then, in the last term we may take  $X_i$  outside the differentiation sign\* so that

\*If the force is perpendicular to the velocity of motion of the molecule, it may also be taken outside the differentiation sign. An example of such a force is the ponderomotive force  $\mathbf{X} = e/c(\xi \times \mathbf{H})$ , acting on a charge moving in a magnetic field  $\mathbf{H}$ .

$$\frac{\partial F_N}{\partial t} + \sum_{i=1}^N \xi_i \cdot \frac{\partial F_N}{\partial x_i} + \frac{1}{m} \sum_{i=1}^N X_i \cdot \frac{\partial F_N}{\partial \xi_i} = 0, \quad (i = 1, \dots, N). \quad (3.5)$$

The Liouville equation (3.5) states that the probability density or the density of the phase points of the ensemble is constant along the phase trajectory of the system.

2. Bogolyubov, Kirkwood, and other authors\* have proposed general methods which allow us to obtain from the Liouville equation not only the Boltzmann equation, which takes account of binary collisions alone, but also general equations which take account of triple and multiple collisions of the molecules. These methods serve as a basis for the construction of equations which describe dense gases and liquids. The treatment which follows is based on the ideas contained in Bogolyubov's work. We also use the formalism of the multiple scaling method.†

We noted in § 2.1 that the  $N$ -particle function  $F_N$  gives an unnecessarily detailed description of the evolution of a gas. To obtain equations which will satisfy distribution functions of lower order ( $s$ -particle distribution functions  $F_s$  with  $s < N$ ), we integrate the Liouville equation (3.5) over  $z_{s+1}, \dots, z_N$ :

$$\int \left( \frac{\partial F_N}{\partial t} + \sum_{i=1}^N \xi_i \cdot \frac{\partial F_N}{\partial x_i} + \sum_{i=1}^N \frac{X_i}{m} \cdot \frac{\partial F_N}{\partial \xi_i} \right) dz_{s+1} \dots dz_N = 0. \quad (3.6)$$

\*N. N. Bogolyubov, Zh. Eksp. Teor. Fiz., 16(8): 681 (1946); "Problems of the Theory of Dynamics and Statistical Physics" [in Russian], Gostekhizdat, 1946; J. G. Kirkwood, J. Chem. Phys., 14: 180 (1946); 15: 72 (1947); M. Born and H. S. Green, Proc. Roy. Soc. A 188: 10 (1946); H. S. Green, Proc. Roy. Soc. A 189: 103 (1947); J. Yvon, "La Theorie Statistique des Fenids et l'Equation d'Etat," Paris, 1935; N. Prigogine, "Non-equilibrium Statistical Mechanics," Interscience, New York, 1963; see also G. Uhlenbeck and J. Ford, "Lectures in Statistical Mechanics," American Mathematical Society, 1963; K. P. Gurov, "Fundamentals of the Kinetic Theory of Gases" [in Russian], Nauka Press, Moscow, 1966.

†See, for example, the statement of this method in M. Van Dyke, "Perturbation Methods in Fluid Mechanics," Academic Press, New York, 1964. The method is applied to the problem being examined in J. E. McCune, G. Sandri, and E. A. Frieman, in: "Rarefied Gas Dynamics," Third Symposium, Academic Press, New York, 1963, and C. H. Su, Phys. Fluids, Vol. 7, No. 8 (1964). See also V. N. Zhigulev, Doklady Akad. Nauk SSSR, Vol. 161, No. 5 (1965); M. N. Kogan, Mekhanika Zhidkosti i Gaza, No. 6 (1966).

Let the gas be located in a vessel of volume  $V$ . Since the limits of integration do not depend on the time, we may interchange the order of integration and differentiation in the first term, and we therefore have

$$\int \frac{\partial F_N}{\partial t} dz_{s+1} \dots dz_N = \frac{\partial F_s}{\partial t}. \quad (3.7)$$

In an exactly similar manner, we may interchange the order of the operations also in the second term with  $i \leq s$ :

$$\int \sum_{i=1}^s \xi_i \cdot \frac{\partial F_N}{\partial x_i} dz_{s+1} \dots dz_N = \sum_{i=1}^s \xi_i \cdot \frac{\partial F_s}{\partial x_i}; \quad (3.8)$$

when  $i \geq s + 1$ , we have

$$\begin{aligned} \int \xi_i \cdot \frac{\partial F_N}{\partial x_i} dz_{s+1} \dots dz_N &= \int \xi_i \cdot \frac{\partial F_{s+1}(t, z_1, \dots, z_s, z_i)}{\partial x_i} dx_i d\xi_i \\ &= - \int_S F_{s+1}(t, z_1, \dots, z_s, z_i) (\xi_i \cdot n) dS d\xi_i, \end{aligned} \quad (3.9)$$

where  $S$  is the surface of the vessel, and  $n$  is the normal to it directed toward the gas. It is easy to see that the last integral determines the probability for the  $i$ -th particle to pass through the vessel wall while  $s$  particles are located in the states  $z_1, \dots, z_s$ , respectively. Since the walls are impermeable, this probability, and therefore the integral, vanish.

We now examine the last term in Eq. (3.6). Since the energy of the gas is finite, the probability density  $F_N$  must tend to zero when  $\xi \rightarrow \infty$ . Therefore, the integrals

$$\int \frac{X_t}{m} \cdot \frac{\partial F_N}{\partial \xi_i} dz_{s+1} \dots dz_N = \int \frac{\partial}{\partial \xi_i} \left( \frac{X_t}{m} F_N \right) dz_{s+1} \dots dz_N = 0 \quad (3.10)$$

vanish when  $i \geq s + 1$ .

For simplicity we assume that there are no external forces, and that

$$X_t = \sum_{j=1}^N X_{tj}, \quad (X_{tt} = 0), \quad (3.11)$$

where  $X_{ij}$  ( $x_i, x_j$ ) is the force on the  $i$ -th molecule due to the  $j$ -th molecule.

Taking account of (3.10) and (3.11), we may write the third term of Eq. (3.6) in the form

$$\begin{aligned} \int \sum_{t=1}^N \frac{X_t}{m} \cdot \frac{\partial F_N}{\partial \xi_t} dz_{s+1} \dots dz_N &= \sum_{i=1}^s \left( \sum_{j=1}^s \frac{X_{ij}}{m} \cdot \frac{\partial F_s}{\partial \xi_i} \right. \\ &\quad \left. + \sum_{j=s+1}^N \frac{\partial}{\partial \xi_i} \cdot \int \frac{X_{ij}}{m} F_{s+1}(t, z_1, \dots, z_s, z_j) dz_s \right). \end{aligned}$$

Because the particles are identical, this relation may be written in the form

$$\begin{aligned} \int \sum_{i=1}^N \frac{X_i}{m} \cdot \frac{\partial F_N}{\partial \xi_i} dz_{s+1} \dots dz_N &= \sum_{i,j}^s \frac{X_{ij}}{m} \cdot \frac{\partial F_s}{\partial \xi_i} + \sum_{i=1}^s (N-s) \frac{\partial}{\partial \xi_i} \cdot \int \frac{X_{i,s+1}}{m} F_{s+1} dz_{s+1}. \end{aligned} \quad (3.12)$$

Gathering the results obtained, we finally have for the  $s$ -particle distribution function

$$\begin{aligned} \frac{\partial F_s}{\partial t} + \sum_{i=1}^s \xi_i \cdot \frac{\partial F_s}{\partial x_i} + \sum_{i,j}^s \frac{X_{ij}}{m} \cdot \frac{\partial F_s}{\partial \xi_i} &= - \sum_{i=1}^s (N-s) \frac{\partial}{\partial \xi_i} \cdot \int \frac{X_{i,s+1}}{m} F_{s+1} dz_{s+1}. \end{aligned} \quad (3.13)$$

Putting  $s = 1, 2, \dots$ , in succession, we obtain the sequence of Bogolyubov equations.\* It is important to

\* In the non-Russian literature this system of equations is sometimes called the BBGKY-hierarchy (Bogolyubov–Born–Green–Kirkwood–Yvon; see the footnote on page 47).

note that the function  $F_2$  appears in the equation for the function  $F_1$ , the function  $F_3$  appears in the equation for  $F_2$  and, in general, the function  $F_{s+1}$  appears in the equation for  $F_s$ , i.e., we have a meshed system of equations. The forces  $\mathbf{X}_{ij}$  are assumed to fall off rapidly with increasing distance between the molecules  $i$  and  $j$ . Therefore, the terms of type

$$\frac{\mathbf{X}_{ij}}{m} \cdot \frac{\partial F_s}{\partial \xi_i}$$

on the left side of the equations (3.13) differ from zero only in small regions of the order of the collision cross section of the molecules  $d$ . Within these small regions, these terms are dominant, since we can make the obvious estimate

$$\frac{\mathbf{X}_{ij}}{m} \sim \frac{\xi_m^2}{d}, \quad (3.14)$$

where  $\xi_m$  is the mean velocity of the molecules. These sharply localized terms enter into all the equations, except the equation for  $F_1$ . We may therefore expect that the functions  $F_s$  with  $s \neq 1$  may vary sharply in regions of the order of the molecular diameter, whereas the function  $F_1$  will vary little on that scale.

We reduce equations (3.13) to dimensionless form, by introducing the following dimensionless quantities:

$$\begin{aligned} \bar{t} &= \frac{t \xi_m}{d} = \frac{t}{\tau_c}, & \bar{x}_i &= \frac{x_i}{d}, & \bar{\xi} &= \frac{\xi}{\xi_m}, \\ \bar{\mathbf{X}}_{ij} &= \frac{\mathbf{X}_{ij} d}{\xi_m^2 m}, & \bar{F}_s &= V^s \xi_m^{3s} F_s. \end{aligned} \quad (3.15)$$

The dimensionless quantities  $\bar{\xi}$ ,  $\bar{\mathbf{X}}_{ij}/m$ , and  $\bar{F}_s$  are of order unity.\*

In terms of these dimensionless quantities, equations (3.13) become (omitting the bars over the dimensionless quantities)

\*For  $F_s$  we have, by definition,

$$\int F_s dz_1 \dots dz_s = 1.$$

Therefore,

$$F_s = O(V^{-s} \xi_m^{-3s}).$$

$$\begin{aligned} & \frac{\partial F_s}{\partial t} + \sum_{i=1}^s \xi_i \cdot \frac{\partial F_s}{\partial x_i} + \sum_{i,j=1}^s \frac{\mathbf{X}_{ij}}{m} \cdot \frac{\partial F_s}{\partial \xi_i} \\ &= -\epsilon \sum_{i=1}^s \frac{\partial}{\partial \xi_i} \cdot \int \frac{\mathbf{X}_{i,s+1}}{m} F_{s+1}(t, z_1, \dots, z_s, z_{s+1}) dz_{s+1} \quad (3.16) \\ & \left( \epsilon = \frac{N}{V} d^3 = n d^3 \ll 1 \right). \end{aligned}$$

Here we have neglected  $s$  in comparison with  $N$ . The fact that first we considered the gas enclosed in the finite volume  $V$  is not important. We may pass to the limit  $V \rightarrow \infty$  when  $N \rightarrow \infty$  and  $n = N/V = \text{const}$ . Then the system of equations (3.16) does not change.

3. It is natural to seek a solution of equations (3.16) in the form of a power series in the parameter  $\epsilon$ , which is small for a Boltzmann gas:

$$F_s(t, x_i, \xi_i, \epsilon) = F_s^0(t, x_i, \xi_i) + \epsilon F_s^1(t, x_i, \xi_i) + \dots \quad (3.17)$$

Equations (3.16) may be rewritten in the form

$$\frac{D_s F_s}{Dt} = \epsilon J(F_{s+1}) = \epsilon J(t, z_1, \dots, z_s), \quad (3.18)$$

where  $D/Dt$  denotes the total derivative with respect to time of the function  $F_s$  along the phase trajectories of the  $s$  molecules which are interacting only with each other. Let our  $s$  molecules be located at the phase points  $z_i(t_0)$  at time  $t = t_0$ . Let them be at the phase points  $z_i(t)$ , at time  $t$ , after interacting only with each other. Then, from Eq. (3.18), we have

$$\begin{aligned} t, z_1(t), \dots, z_s(t) &-> F_s(t_0, z_1(t_0), \dots, z_s(t_0)) \\ &= \epsilon \int_{t_0}^t J(\tau, z_1(\tau), \dots, z_s(\tau)) d\tau. \end{aligned} \quad (3.19)$$

This equation confirms the obvious fact that the  $s$ -particle distribution function varies along the phase trajectories of the  $s$  molecules only as a result of interaction (collisions) with other molecules.

By substituting the series (3.17) into Eq. (3.19), and equating coefficients of equal powers of  $\epsilon$ , we obtain

$$\begin{aligned} F_s^0(t) &= F_s^0(t_0), \\ F_s^v(t) &= F_s^v(t_0) + \int_{t_0}^t J(F_{s+1}^{v-1}(\tau)) d\tau, \quad (v = 1, 2, \dots). \end{aligned} \quad (3.20)$$

It follows from these equations that the functions  $F_s^0$  are constant along phase trajectories, and that they are completely determined by their initial values. Therefore, in determining  $F_s^1$ , the integrals  $J(F_{s+1}^0)$  are assigned functions, in general different from zero. The dimensionless difference  $t - t_0 = O(1)$  throughout a time of the order of the collision time. Therefore, if the time during which we follow the molecules along their phase trajectories becomes much greater than the collision time, the functions  $F_s^1$  become infinite. Therefore, the solution in the form of the series (3.17) is valid only to describe processes which occur in a time of the order of the collision time. And this is not surprising. In essence, this result was already predetermined by the assumed structure of the functions  $F_s^v$ , when we postulated that these functions depend only on the variables  $\mathbf{x}/d$  and  $t/\tau_c$ , and indeed we postulated that  $d$  and  $\tau_c$  are unique characteristic scales of the phenomenon. In actual fact, the functions  $F_s$  may vary in magnitude with length and time of various scales.

It is clear from physical considerations that, beside the collision cross-section  $d$  and the collision time  $\tau_c$ , the mean free path  $\lambda$  and the time between collisions  $\tau$  must play an important part, as well as the characteristic flow dimension  $L$ , and the characteristic flow time  $\Theta$ . In the interests of brevity, in what follows, we call these scales d-,  $\lambda$ -, and L-scales, respectively.

The dimensionless equations (3.16) contain only characteristic quantities on the d-scale and the parameter  $\epsilon$ . Therefore, the solution of these equations depends on the variables  $t\epsilon$ ,  $t\epsilon^2$ ,  $\dots$ ,  $\mathbf{x}_i\epsilon$ ,  $\mathbf{x}_i\epsilon^2$ , etc., in addition to the variables  $t$  and  $\mathbf{x}_i$ . Moreover, the characteristic flow length  $L$  and the characteristic time  $\Theta$  must appear in the solution through the initial and boundary conditions. Therefore, the solution of equations (3.16) for a specific problem should have the structure

$$F_s = F_s(t_0, \mathbf{x}_{i0}, \epsilon t_0, \epsilon \mathbf{x}_{i0}, \epsilon^2 t_0, \dots, \frac{\tau_c}{\Theta} t_0, \frac{d}{L} \mathbf{x}_{i0}, \epsilon \frac{\tau_c}{\Theta} t_0, \epsilon \frac{d}{L} \mathbf{x}_{i0}, \dots, \xi_i, \epsilon) \equiv$$

$$\equiv F_s(t_0, \mathbf{x}_{i0}, t_1, \mathbf{x}_{i1}, \dots, t_L, \mathbf{x}_{iL}, \dots, \xi_i, \epsilon), \quad (3.21)$$

$$(i = 1, \dots, s),$$

where

$$t_1 = \epsilon t_0, \quad t_v = \epsilon^v t_0, \quad t_L = \frac{\tau_c}{\Theta} t_0, \quad \mathbf{x}_{iL} = \frac{d}{L} \mathbf{x}_{i0}.$$

The variables  $t_0$  and  $\mathbf{x}_{i0}$  are on the d-scale. By definition,  $\epsilon = nd^3$  and  $\lambda = 1/nd^2 = d/\epsilon$ . Therefore,

$$\mathbf{x}_{i1} = \epsilon \mathbf{x}_{i0} = \epsilon \frac{\mathbf{x}_i}{d} = \frac{\mathbf{x}_i}{\lambda}, \quad t_1 = \epsilon t_0 = \epsilon \frac{t}{\tau_c} = \frac{t}{\tau}.$$

Therefore, the variables  $t_1$  and  $\mathbf{x}_{i1}$  are variables on the  $\lambda$ -scale. The L-scale variables  $t_L$  and  $\mathbf{x}_{iL}$  may be written in the form

$$t_L = \frac{\tau_c}{\Theta} t_0 = \frac{\tau_c}{\tau} \frac{\tau}{\Theta} t_0 = \epsilon \epsilon_1 t_0,$$

$$\mathbf{x}_{iL} = \frac{d}{L} \mathbf{x}_{i0} = \frac{d}{\lambda} \frac{\lambda}{L} \mathbf{x}_{i0} = \epsilon \epsilon_1 \mathbf{x}_{i0},$$

where  $\epsilon_1 \sim \tau/\Theta \sim \lambda/L = Kn$  is a parameter which characterizes the ratio of the mean free path (the time between collisions) to the characteristic length (characteristic time) of the flow. This parameter is called the Knudsen number. The Knudsen number may, in general, take any value. Here we are interested in an asymptotic theory with  $\epsilon \rightarrow 0$  and with arbitrary, but fixed  $\lambda$  and  $L$ . Therefore, in the expansion with respect to  $\epsilon$ , we must take the parameter  $\epsilon_1$  as fixed.

We expand the functions  $F_s$  in a series with respect to  $\epsilon$ :

$$F_s = \sum_{v=0}^{\infty} F_s^v(t_0, \mathbf{x}_{i0}, t_1, \mathbf{x}_{i1}, \dots, t_L, \mathbf{x}_{iL}, \dots, \xi_i) \epsilon^v. \quad (3.22)$$

Then, for the derivatives with respect to  $t$  and  $\mathbf{x}_i$ , we have

$$\frac{\partial F_s}{\partial t} = \frac{\partial F_s^0}{\partial t_0} + \epsilon \left( \frac{\partial F_s^1}{\partial t_0} + \frac{\partial F_s^0}{\partial t_1} + \epsilon_1 \frac{\partial F_s^0}{\partial t_L} \right) + \dots \quad (3.23a)$$

$$\frac{\partial F_s}{\partial \mathbf{x}_l} = \frac{\partial F_s^0}{\partial \mathbf{x}_{l0}} + \epsilon \left( \frac{\partial F_s^1}{\partial \mathbf{x}_{l0}} + \frac{\partial F_s^0}{\partial \mathbf{x}_{l1}} + \epsilon_1 \frac{\partial F_s^0}{\partial \mathbf{x}_{lL}} \right) + \dots, \quad (3.23b)$$

Substituting the series (3.22) and the derivatives (3.23a) and (3.23b) into equations (3.16), and equating coefficients of the various powers of  $\epsilon$ , we obtain

$$\frac{D_s F_s^0}{Dt_0} = \frac{\partial F_s^0}{\partial t_0} + \sum_{l=1}^s \xi_l \cdot \frac{\partial F_s^0}{\partial \mathbf{x}_{l0}} + \sum_{l,j=1}^s \frac{X_{lj}}{m} \cdot \frac{\partial F_s^0}{\partial \xi_l} = 0, \quad (3.24)$$

$$\frac{D_s F_s^1}{Dt_0} + \frac{d_s F_s^0}{dt_1} = - \sum_{l=1}^s \frac{\partial}{\partial \xi_l} \cdot \int \frac{X_{l,s+1}}{m} F_{s+1}^0 d\zeta_{(s+1)0}, \quad (3.25)$$

$$\frac{D_s F_s^v}{Dt_0} + \frac{d_s F_s^{v-1}}{dt_1} + \dots + \frac{d_s F_s^0}{dt_v} = - \sum_{l=1}^s \frac{\partial}{\partial \xi_l} \cdot \int \frac{X_{l,s+1}}{m} F_{s+1}^{v-1} d\zeta_{(s+1)0}, \quad (v=1, \dots), \quad (3.25a)$$

where

$$\frac{d_s}{dt_v} = \frac{\partial}{\partial t_v} + \sum_{i=1}^s \xi_i \cdot \frac{\partial}{\partial \mathbf{x}_{iv}} + \epsilon_1 \left( \frac{\partial}{\partial (\epsilon^{v-1} t_L)} + \epsilon_1 \sum_{i=1}^s \frac{\partial}{\partial (\epsilon^{v-1} \mathbf{x}_{iL})} \right) \quad (3.26)$$

is the derivative along the trajectories of the  $s$  noninteracting molecules.

The variables  $(t_\nu, \mathbf{x}_{iv})$  and  $(\epsilon^{v-1} t_L, \epsilon^{v-1} \mathbf{x}_{iL})$  appear in the functions  $F_s$  in quite identical fashion. Therefore, to simplify the notation, we write  $(\tilde{t}_\nu, \tilde{\mathbf{x}}_{iv})$  instead of the combination of variables  $(t_\nu, \mathbf{x}_{iv}, \epsilon^{v-1} t_L, \epsilon^{v-1} \mathbf{x}_{iL})$ . We call the scale corresponding to these variables the  $v$ -scale ( $v = 1, 2, \dots$ ).

When we examine the evolution of the group of  $s$  molecules on the d-scale, it is evident that their relative position plays a decisive role. It is therefore convenient to introduce the variables

$$\mathbf{x}_{10} = \mathbf{x}_{10}, \quad \mathbf{r}_{1l} = \mathbf{x}_{10} - \mathbf{x}_{l0}. \quad (3.27)$$

Then

$$F_s^v = F_s^v(t_0, \mathbf{x}_{10}, \mathbf{r}_{12}, \dots, \mathbf{r}_{1s}, \tilde{t}_1, \tilde{\mathbf{x}}_{11}, \dots, \tilde{\mathbf{x}}_{s1}, \dots, \xi_1, \dots, \xi_s),$$

and

$$\frac{D_s F_s^v}{Dt_0} = \frac{\partial F_s^v}{\partial t_0} + \xi_1 \cdot \frac{\partial F_s^v}{\partial \mathbf{x}_{10}} + \sum_{i=2}^s (\xi_i - \xi_1) \cdot \frac{\partial F_s^v}{\partial \mathbf{r}_{1i}} + \sum_{i,j=1}^s \frac{X_{ij}}{m} \cdot \frac{\partial F_s^v}{\partial \xi_i}. \quad (3.28)$$

The Boltzmann equation is an equation for the function  $F_1$ . From Eq. (3.24) with  $s = 1$ , we have

$$\frac{d_1 F_1^{(0)}}{dt_0} = \frac{\partial F_1^0}{\partial t_0} + \xi_1 \cdot \frac{\partial F_1^0}{\partial \mathbf{x}_{10}} = 0, \quad (3.29)$$

i.e., the function  $F_1^0$  is constant along the rectilinear trajectories  $\mathbf{x}_{10} + \xi_1 t_0 = \text{const}$  on the d-scale. The equation (3.25) with  $s = 1$  has the form

$$\begin{aligned} & \frac{d_1 F_1^1(t_0, \mathbf{x}_{10}, \tilde{t}_1, \tilde{\mathbf{x}}_{11}, \xi_1)}{dt_0} + \frac{d_1 F_1^0(t_0, \mathbf{x}_{10}, \tilde{t}_1, \tilde{\mathbf{x}}_{11}, \xi_1)}{dt_1} \\ &= - \frac{\partial}{\partial \xi_1} \cdot \int \frac{X_{12}}{m} F_2^0(t_0, \mathbf{x}_{10}, \mathbf{x}_{20}, \tilde{t}_1, \tilde{\mathbf{x}}_{11}), \\ & \tilde{\mathbf{x}}_{21} = \tilde{\mathbf{x}}_{11}, \xi_1, \xi_2) d\mathbf{x}_{20} d\xi_2 = J_1. \end{aligned} \quad (3.30)$$

We transform the integral in (3.30). The function  $F_2^0$  on the d-scale satisfies the equation (3.24); using the solution of (3.24) in the form (3.28), we obtain

$$\begin{aligned} J_1 &= - \int \frac{X_{12}}{m} \cdot \frac{\partial F_2^0}{\partial \xi_1} d\mathbf{x}_{20} d\xi_2 = \int (\xi_1 - \xi_2) \cdot \frac{\partial}{\partial \mathbf{r}_{12}} F_2^0(t_0, \mathbf{x}_{10}, \mathbf{r}_{12}, \\ & \tilde{t}_1, \tilde{\mathbf{x}}_{11}, \tilde{\mathbf{x}}_{21} = \tilde{\mathbf{x}}_{11}, \xi_1, \xi_2) d\mathbf{x}_{20} d\xi_2 + \int \left( \frac{\partial F_2^0}{\partial t_0} + \xi_1 \cdot \frac{\partial F_2^0}{\partial \mathbf{x}_{10}} \right) d\mathbf{x}_{20} d\xi_2 \\ &+ \int \frac{X_{21}}{m} \cdot \frac{\partial F_2^0}{\partial \xi_2} d\mathbf{x}_{20} d\xi_2. \end{aligned} \quad (3.31)$$

The last integral vanishes for the same reasons as the integrals (3.10). As we shall see below, in the case of molecular chaos the first integral takes the form of a Boltzmann collision term. The second integral is in general different from zero, and, therefore, the equation does not reduce to the Boltzmann equation.

The integrand in the second integral is determined by the change in the distribution function  $F_2^0$  during the time of collision and at distances of the order of the diameter of the molecules, while the distance between the molecules remains fixed (i.e., with  $\mathbf{r}_{12} = \text{const}$ ). In what follows we restrict our examination to the

"homogeneous" case. We call a process homogeneous when the functions  $F_s$  on the d-scale are functions of  $\mathbf{r}_{1i}$ , but do not depend on  $t_0$  and  $\mathbf{x}_{10}$ . In other words, on the d-scale in the homogeneous case, the functions  $F_s$  change only with the relative position of the s molecules, but do not change with the displacement of a fixed group of s molecules at distances of order  $d$ ,\* or in times of order  $\tau_c$ . Two groups of s molecules are called similar when they are obtained from each other by displacement on the d-scale by a distance of order  $d$  and by a time of order  $\tau_c$ .

It is easy to see that the property of homogeneity is conserved if it obtains at time zero everywhere, including the boundaries. In fact, we examine at time  $t$  a group 1 of s molecules with phase coordinates  $(\tilde{\mathbf{x}}_1^1, \xi_1), \dots, (\tilde{\mathbf{x}}_s^1, \xi_s)$ . Let there be a similar group 2, which differ from group 1 only in that all the molecules are displaced by a distance  $\delta = O(d)$ , i.e.,  $\mathbf{x}_i^2 = \mathbf{x}_i^1 + \delta$ . If the molecules of each group interact only among themselves, it is evident that these groups will be similar at any instant of time.

Consider the difference

$$\Delta_s(t) = F_s(t, z_1^1(t), \dots, z_s^1(t)) - F_s(t, z_1^2(t), \dots, z_s^2(t)).$$

According to (3.19),

$$\Delta_s(t) = \Delta_s(0) - \epsilon \int_0^t \sum_{s=1}^s \frac{\partial}{\partial \xi_i} \cdot \int \frac{\mathbf{x}_{i,s+1}}{m} \Delta_{s+1}(\tau) dz_{s+1} d\tau. \quad (3.32)$$

In view of the homogeneity of the system, if  $\Delta_s(0) = 0$  for all s and for any of the groups, then  $\Delta_s(t) = 0$  and, therefore, the functions  $F_s^\nu$  do not depend on  $t_0$  and  $\mathbf{x}_{10}$ .

In the homogeneous case, Eq. (3.24) with  $s \geq 2$  takes the form

$$\sum_{i=2}^s (\xi_i - \xi_i) \frac{\partial F_s^0}{\partial r_{1i}} + \sum_{i,j=1}^s \frac{\mathbf{x}_{ij}}{m} \cdot \frac{\partial F_s^0}{\partial \xi_i} = 0. \quad (3.24a)$$

\*In particular, the function  $f(r, x, \xi)$  does not vary over distances of order  $d$ , i.e., the condition of independence of probability of collisions from impact parameter used in deriving the Boltzmann equation in the preceding paragraph is satisfied.

Therefore we may write the integral (3.31) in the form

$$\begin{aligned} J_1 &= \int (\xi_1 - \xi_2) \cdot \frac{\partial}{\partial r_{12}} F_2^0 d\mathbf{x}_{20} d\xi_2 \\ &= \int (\xi_2 - \xi_1) \cdot \frac{\partial}{\partial r_{21}} F_2^0(\mathbf{r}_{21}, \dots) dr_{21} d\xi_2. \end{aligned} \quad (3.33)$$

We introduce a cylindrical system of coordinates  $l, \rho, \varepsilon$  with the origin at the point  $\mathbf{x}_{10}$  and with the  $l$  axis parallel to the relative velocity vector  $\mathbf{g} = \xi_2 - \xi_1$ . Then,

$$\begin{aligned} J_1 &= \int g \rho d\rho de d\xi_2 \int_{-\infty}^{+\infty} \frac{\partial F_2^0}{\partial l} dl \\ &= \int g [F_2^0(+\infty, \tilde{t}_1, \tilde{\mathbf{x}}_{11}, \tilde{\mathbf{x}}_{21} = \tilde{\mathbf{x}}_{11}, \xi_1, \xi_2) - \\ &\quad - F_2(-\infty, \tilde{t}_1, \tilde{\mathbf{x}}_{11}, \tilde{\mathbf{x}}_{21} = \tilde{\mathbf{x}}_{11}, \xi_1, \xi_2)] \rho d\rho de d\xi_2. \end{aligned} \quad (3.34)$$

Clearly the function  $F_2^0(+\infty, \dots, \xi_1, \xi_2)$  is the probability of finding molecules with velocities  $\xi_1$  and  $\xi_2$ , respectively, before collision, while the function  $F_2^0(-\infty, \dots, \xi_1, \xi_2)$  is the probability of finding molecules with the same velocities after collision. If during a collision with impact parameter  $\rho$ , the molecules with velocities  $\xi_1$  and  $\xi_2$  acquire velocities  $\xi'_1$  and  $\xi'_2$ , respectively, then, as was shown in § 1.3, the molecules with velocities  $\xi'_1$  and  $\xi'_2$ , colliding with the same impact parameter, acquire velocities  $\xi_1$  and  $\xi_2$ . Therefore,

$$F_2^0(+\infty, \tilde{t}_1, \dots, \xi_1, \xi_2) = F_2^0(\tilde{t}_1, \dots, \xi'_1, \xi'_2).$$

Therefore, finally,

$$\begin{aligned} J_1(\tilde{t}_1, \tilde{\mathbf{x}}_{11}, \xi_1) &= \int g [F_2^0(\tilde{t}_1, \tilde{\mathbf{x}}_{11}, \tilde{\mathbf{x}}_{21} = \tilde{\mathbf{x}}_{11}, \xi'_1, \xi'_2) \\ &\quad - F_2^0(\tilde{t}_1, \tilde{\mathbf{x}}_{11}, \tilde{\mathbf{x}}_{21} = \tilde{\mathbf{x}}_{11}, \xi_1, \xi_2)] \rho d\rho de d\xi_2. \end{aligned} \quad (3.35)$$

We now turn to Eq. (3.30), and substitute  $J_1$  from (3.35) into it. In the homogeneous case  $dF_1^1/dt_0 = 0$ , and the equation takes the form

$$\begin{aligned} \frac{d_1 F_1^0(\tilde{t}_1, \tilde{\mathbf{x}}_{11}, \xi_1)}{dt_1} &= \int g [F_2^0(\tilde{t}_1, \tilde{\mathbf{x}}_{11}, \tilde{\mathbf{x}}_{21} = \tilde{\mathbf{x}}_{11}, \xi'_1, \xi'_2) \\ &\quad - F_2^0(\tilde{t}_1, \tilde{\mathbf{x}}_{11}, \tilde{\mathbf{x}}_{21} = \tilde{\mathbf{x}}_{11}, \xi_1, \xi_2)] \rho d\rho de d\xi_2. \end{aligned} \quad (3.36)$$

Thus, we have obtained an equation for  $F_1^0$  on the 1-scale. However, this equation, like the original equation (3.13) with  $s = 1$ , contains not only the function  $F_1^0$ , but also the function  $F_2^0$ . But, Eq. (3.36) differs from Eq. (3.13) because the function  $F_2^0$  which appears in it is also defined on the 1-scale, i.e., when the molecules 1 and 2 are located outside their interaction region before collision. To close Eq. (3.36) we require additional assumptions concerning the relation between the functions  $F_2^0$  and  $F_1^0$ .

If we assume that the molecular chaos condition

$$F_2^0(\tilde{t}_1, \tilde{x}_{11}, \tilde{x}_{21} = \tilde{x}_{11}, \xi_1, \xi_2) = F_1^0(\tilde{t}_1, \tilde{x}_{11}, \xi_1) F_1^0(\tilde{t}_1, \tilde{x}_{11}, \xi_2) \quad (3.37)$$

is satisfied before the collision, then Eq. (3.36) becomes identical with the Boltzmann equation

$$\begin{aligned} & \frac{\partial F_1^0(\tilde{t}_1, \tilde{x}_{11}, \xi_1)}{\partial \tilde{t}_1} + \xi_1 \cdot \frac{\partial F_1^0(\tilde{t}_1, \tilde{x}_{11}, \xi_1)}{\partial \tilde{x}_{11}} \\ &= \int [F_1^0(\tilde{t}_1, \tilde{x}_{11}, \xi'_1) F_1^0(\tilde{t}_1, \tilde{x}_{11}, \xi'_2) \\ &\quad - F_1^0(\tilde{t}_1, \tilde{x}_{11}, \xi_1) F_1^0(\tilde{t}_1, \tilde{x}_{11}, \xi_2)] g \rho d\rho d\xi_2, \end{aligned} \quad (3.38)$$

which differs from Eq. (2.9) of the previous paragraph only in notation.

Thus, to obtain the Boltzmann equation, we require only two conditions: the condition of molecular chaos, and the condition of homogeneity as defined above.

4. In general, it is difficult to represent the realization of an inhomogeneous process in the sense defined above, since, in such processes, one of the characteristic scales of variation of macroscopic (measurable) quantities would be the molecular diameter. If the characteristic scales of the boundary and initial conditions satisfy the inequalities

$$L \gg d \text{ and } \Theta \gg \tau_c,$$

which hold for most practical problems, then the condition of homogeneity is satisfied.

Let us examine the postulate of molecular chaos in greater detail. Molecular collisions constitute the only mechanism leading

to the establishment or the perturbation of chaos. It is therefore natural that chaos cannot be established in a time shorter than the time between collisions. As we have seen, the Boltzmann equation is applicable even for the description of processes with characteristic time  $\Theta \ll \tau$ . Therefore, in those cases, it is necessary that molecular chaos prevail at time zero.

On the other hand, collisions obviously disturb the state of chaos. In fact, the position and velocity of molecules which have just collided are correlated. However, the probability of a second collision of these same molecules tends to zero as  $N \rightarrow \infty$ . Before the molecules collide a second time, each of them experiences an enormous number of collisions with other molecules. Therefore, we may expect that chaos is preserved, if it prevails at time zero.

As we saw above, the derivation of the Boltzmann equation requires that molecular chaos prevail only before a molecular collision. Let us show\* that this "one-sided" molecular chaos is conserved, if there are no correlations at time zero. Let us introduce the correlation functions

$$\begin{aligned} g_{1,2} &= F_2(z_1, z_2) - F_1(z_1) F_1(z_2), \\ g_{1,23} &= F_3(z_1, z_2, z_3) - F_1(z_1) F_2(z_2, z_3), \\ g_{2,13} &= F_3(z_1, z_2, z_3) - F_1(z_2) F_2(z_1, z_3). \end{aligned} \quad (3.39)$$

Substituting  $s = 1, 2, \dots$  in succession in the Bogolyubov sequence (3.16), and replacing  $F_s$  by the functions  $F_{s-1}$ , and the correlation functions defined by (3.39), we obtain a sequence of equations for the correlation functions. Let us write one of them:

$$\begin{aligned} & \frac{\partial g_{1,2}}{\partial t} + \xi_1 \cdot \frac{\partial g_{1,2}}{\partial x_1} + \xi_2 \cdot \frac{\partial g_{1,2}}{\partial x_2} + \frac{1}{m} X_{1,2} \cdot \frac{\partial F_2(z_1, z_2)}{\partial \xi_1} \\ &+ \frac{1}{m} X_{21} \cdot \frac{\partial F(z_1, z_2)}{\partial \xi_2} + \epsilon \frac{\partial}{\partial \xi_1} \cdot \int \frac{X_{13}}{m} g_{2,13} dz_3 \\ &+ \epsilon \frac{\partial}{\partial \xi_2} \cdot \int \frac{X_{23}}{m} g_{1,23} dz_3 = 0. \end{aligned} \quad (3.40)$$

\*H. Grad, "Handbuch der Physik," Vol. 12, 1958. See also M. Kac, "Probability and Related Topics in Physical Sciences," Wiley, New York, 1959; I. Prigogine, "Non-equilibrium Statistical Mechanics," Interscience, New York, 1963.

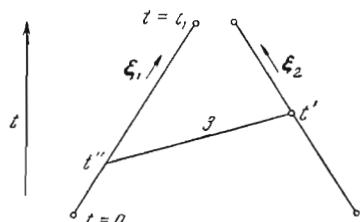


Fig. 9

$g_{1,2}$  is determined by integrals relating to collisions with third molecules; those terms are multiplied by the parameter  $\epsilon$  which is small for a Boltzmann gas.

A similar equation may be written for  $g_{1,23}$ ,  $g_{2,13}$ , etc. The equation for three-particle correlation functions involves integrals of the four-particle correlation function, etc.

Let all the correlation functions vanish at time zero. If no molecule had suffered a collision before the instant  $t = t_1 \neq 0$ , then the leading terms (of order unity) which characterize molecular interaction would be negligible in the whole sequence of equations, and the equations would be homogeneous. Clearly, all the functions  $g$  would vanish for  $t \leq t_1$ . However, when  $N \rightarrow \infty$ , collisions occur within an arbitrarily short time, and large inhomogeneous terms appear in the sequence of equations for the correlation functions.

We are interested in the conservation of molecular chaos up to a molecular collision within a finite time interval (in the 1-scale, i.e., with  $t \sim 1/\epsilon \gg 1$  in our dimensionless variables). Let the molecules 1 and 2 collide at time  $t = t_1$  with velocities  $\xi_1$  and  $\xi_2$ , respectively. The trajectories of these particles are shown in Fig. 9. The function  $g_{1,2}$  at time  $t = t_1$  is determined by integrals of the three-particle correlation function, which in turn is determined by similar integrals of the four-particle correlation, etc. For example, in calculating the integral of the three-particle function  $g_{1,23}$  at any time  $t'$ , one always finds the trajectory of a third molecule (Fig. 9), which intersected the trajectory of molecule 1 at some previous time  $t'' < t'$ . For such trajectories the leading inhomogeneous terms associated with collisions of molecules 3 and 1 play a major part, and the function  $g_{1,23}$  is of order unity, even if it was initially zero. However it is easy to see that, in the

This equation is written in terms of the dimensionless variable (3.15). When the molecules 1 and 2 approach each other to within a distance  $|\mathbf{x}_2 - \mathbf{x}_1| \leq 1$ , the terms which determine their interaction become of order unity in terms of those variables. Up to the time when that distance is reached, the behavior of the correlation function

### § 2.3] DERIVATION OF THE BOLTZMANN EQUATION

integration over  $\xi_3$ , the range of values of  $\xi_3$ , for which  $g_{1,23}$  is of order unity, is of order  $d$  or  $\epsilon$ . Therefore, in a time of order  $\epsilon^{-1}$ , the function  $g_{1,2}$  may become of order  $\epsilon$  [when  $g_{1,2}(0) = 0$ ]. To obtain the Boltzmann equation on the 1-scale, we required only the function  $F_2^0$ , and we did not calculate the function  $\epsilon F_2^{(1)}$ . To the same accuracy we may also neglect the correlation function  $g_{1,2} = O(\epsilon)$ , and assume that molecular chaos prevails up to the collision of molecules 1 and 2.

5. We did not distinguish above between the  $\lambda$ -scale and the L-scale, but brought them together under the general designation of 1-scale.

We recall that, according to (3.26),

$$\frac{dF_1}{dt_1} = \frac{\partial F_1}{\partial t_1} + \xi_1 \cdot \frac{\partial F_1}{\partial \mathbf{x}_{11}} + \epsilon_1 \left( \frac{\partial F_1}{\partial t_L} + \xi_1 \cdot \frac{\partial F_1}{\partial \mathbf{x}_{1L}} \right) = \frac{dF_1}{dt_1} + \epsilon_1 \frac{dF_1}{dt_L},$$

the Boltzmann equation may be rewritten in the form

$$\begin{aligned} & \frac{dF_1(t_1, \mathbf{x}_{11}, t_L, \mathbf{x}_{1L}, \xi_1)}{dt_1} + \epsilon_1 \frac{dF_1(t_1, \mathbf{x}_{11}, t_L, \mathbf{x}_{1L}, \xi_1)}{dt_L} \\ &= \int [F_1(t_1, \mathbf{x}_{11}, t_L, \mathbf{x}_{1L}, \xi'_1) F_1(t_1, \mathbf{x}_{11}, t_L, \mathbf{x}_{1L}, \xi'_2) \\ &\quad - F_1(t_1, \mathbf{x}_{11}, t_L, \mathbf{x}_{1L}, \xi_2) F_1(t_1, \mathbf{x}_{11}, t_L, \mathbf{x}_{1L}, \xi_1)] g_0 d\rho d\theta d\xi_2. \end{aligned} \quad (3.41)$$

When  $\epsilon_1 = Kn \ll 1$  we now have two substantially different scales: a small  $\lambda$ -scale and a large L-scale.

It is important to note that, for the Boltzmann equation to be valid, molecular chaos need only prevail on the  $\lambda$ -scale, when the molecules are located at a single point in the L-scale.

But if the particles are located at different points  $(t_L, \mathbf{x}_{1L})$  on the L-scale, then molecular chaos may not prevail. Chaos on the  $\lambda$ -scale is in fact molecular chaos.

A classical example of flow, in which there is chaos on the  $\lambda$ -scale (at the molecular level) while there is no chaos on the L-scale (at the macroscopic level), is hydrodynamic turbulence, since the scale of hydrodynamic turbulence  $L \gg \lambda$  and  $\Theta \gg \tau$ .

It is precisely because of the presence of an L-scale, much larger than the  $\lambda$ -scale, that ordered (turbulent) hydrodynamic

motions are possible, in spite of the fact that the initial and boundary conditions practically always satisfy the condition of molecular chaos.\*

If  $L \leq \lambda$  and  $\Theta \leq \tau$ , the flow may be turbulent only when the initial and boundary conditions do not satisfy the condition of molecular chaos. However, in practice, it is difficult to fulfill these conditions in a gas at low density (when  $\epsilon \rightarrow 0$ ) and, therefore, it is also difficult to realize turbulent flows with characteristic scales  $L \leq \lambda$  and  $\Theta \leq \tau$ .

#### §2.4. Certain Properties of the Collision Integral

Let us examine the symmetry properties of the collision integral, since these will be used repeatedly below.

The collision integral which appears on the right of the Boltzmann equation (2.9)

$$J(t, \mathbf{x}, \xi) = \int (f' f'_1 - f f_1) g b \, db \, de \, d\xi_1 \quad (4.1)$$

is a function of  $t$ ,  $\mathbf{x}$ , and  $\xi$ . In what follows we frequently have to integrate  $J$  with some weighting function of the velocities  $\varphi(\xi)$ :

$$I_\varphi = \int \varphi(\xi) J(t, \mathbf{x}, \xi) \, d\xi = \int \varphi(\xi) (f' f'_1 - f f_1) g b \, db \, de \, d\xi_1 \, d\xi_1. \quad (4.2)$$

This integral is evidently equal to

$$I_{\varphi'} = \int \varphi(\xi') (f f_1 - f' f'_1) g' b' \, db' \, de' \, d\xi' \, d\xi'_1. \quad (4.3)$$

Using (2.6) and (2.8), the last integral may be written in the form

\*Grad has shown [H. J. Grad, Chem. Phys., Vol. 33, No. 5 (1960)] that the uncorrelated initial conditions are in a certain sense the most probable. Generally speaking, molecules may be correlated at the surfaces bounding the gas. However, the probability of correlation of molecules reflected from the wall increases as the gas density and the time of interaction of the molecules with the surface increase (see §2.10). For the velocities of molecules which have undergone two or more reflections to be correlated, they must interact with each other at the moment of interaction with the wall. Under standard conditions (gas density  $n \approx 10^{19} \text{ cm}^{-3}$ ,  $\xi_m \approx 10^5 \text{ cm/sec}$ , adsorption time  $\tau_a \approx 10^{-13} \text{ sec}$ ), the probability of an encounter between molecules of a gas at the surface is of order  $10^{-5}$ .

$$I_{\varphi'} = - \int \varphi(\xi') (f' f'_1 - f f_1) g b \, db \, de \, d\xi \, d\xi'_1. \quad (4.4)$$

Adding (4.2) and (4.4) term by term, we obtain

$$I_\varphi = \frac{1}{2} \int (\varphi - \varphi') (f' f'_1 - f f_1) g b \, db \, de \, d\xi \, d\xi'_1. \quad (4.5)$$

The integral (4.2) is symmetrical with respect to  $\xi$  and  $\xi_1$ . Therefore,

$$\begin{aligned} I_{\varphi_1} &= \int \varphi(\xi_1) J(t, \mathbf{x}, \xi_1) \, d\xi_1 = I_\varphi \\ &= \frac{1}{2} \int (\varphi_1 - \varphi'_1) (f' f'_1 - f f_1) g b \, db \, de \, d\xi \, d\xi'_1. \end{aligned} \quad (4.6)$$

Adding (4.5) and (4.6) we obtain

$$I_\varphi = \frac{1}{4} \int (\varphi + \varphi_1 - \varphi' - \varphi'_1) (f' f'_1 - f f_1) g b \, db \, de \, d\xi \, d\xi'_1. \quad (4.7)$$

If we take any of the summational invariants  $\psi_i$  (see §1.3) as the weighting function  $\varphi$ , it is evident that

$$I_{\psi_i} = 0, \quad (i = 0, \dots, 4). \quad (4.8)$$

Equations (4.8) express physically the familiar properties of the collisions: the total mass, momentum, and energy of the colliding molecules does not change during the collision. This result plays an important role in the derivation of the equations for macroscopic quantities from the Boltzmann equation (see §3.1).

From symmetry we also obtain the following relations:

$$\begin{aligned} &\int (\varphi - \varphi') f f_1 g b \, db \, de \, d\xi \, d\xi'_1 \\ &= \int (\varphi' - \varphi) f' f'_1 g' b' \, db' \, de' \, d\xi' \, d\xi'_1, \end{aligned} \quad (4.9)$$

$$\begin{aligned} &\int (\varphi + \varphi_1 - \varphi' - \varphi'_1) f f_1 g b \, db \, de \, d\xi \, d\xi'_1 \\ &= \int (\varphi' + \varphi'_1 - \varphi - \varphi_1) f' f'_1 g' b' \, db' \, de' \, d\xi' \, d\xi'_1. \end{aligned} \quad (4.10)$$

Substituting these relations into (4.5) and (4.7), respectively, and using Eqs. (2.6) and (2.8), we obtain

$$I_\varphi = \int (\varphi' - \varphi) f f_1 g b \, db \, de \, d\xi \, d\xi_1, \quad (4.11)$$

$$I_\varphi = \frac{1}{2} \int (\varphi' + \varphi'_1 - \varphi - \varphi_1) f f_1 g b \, db \, de \, d\xi \, d\xi_1. \quad (4.12)$$

Because of the symmetry properties noted for the integrals  $I_\varphi$ , their calculation in practice for some specific interaction potential proves to be simply a calculation of the original collision integral  $J$ .

### §2.5. The Boltzmann H-Theorem

Let us divide the six-dimensional phase  $\gamma$ -space (see §2.1) into elementary cells of volume  $\delta v$ , equal to  $\delta \mathbf{x}_i \delta \xi_i$ . Let there be  $N$  molecules distributed in the  $\gamma$ -space in accordance with the distribution function  $f(\mathbf{x}, \xi)$ . Then, it follows from the definition of the distribution function that the number of molecules  $N_i$  in the  $i$ -th cell is equal to  $f(\mathbf{x}_i, \xi_i) \delta v$ , where  $\mathbf{x}_i$  and  $\xi_i$  are the coordinates of the center of the  $i$ -th cell. In the  $6N$ -dimensional  $\Gamma$ -space, the whole system is represented by a single point. If each of the  $N$  molecules in the  $\gamma$ -space is displaced within its elementary cell of volume  $\delta v$ , then the point representing the system in the  $\Gamma$ -space will be displaced within a volume  $(\delta v)^N$ . If we interchange the positions of two molecules without changing the distribution function, then the system is represented by a new point in the  $\Gamma$ -space and a new volume  $(\delta v)^N$  will correspond to it. The  $N$  molecules can be interchanged in  $N!$  ways. But since a permutation of the  $N_i$  molecules within each of the cells does not change the volume corresponding to the system in the  $\Gamma$ -space, the number of distinguishable permutations is  $N! / \prod_i N_i$ . Therefore, the volume of  $\Gamma$ -space corresponding to an assigned distribution function or to given occupation numbers  $N_i$ , is

$$\frac{N!}{\prod_i N_i!} (\delta v)^N.$$

The probability of finding the system in some state is proportional to the volume of  $\Gamma$ -space which corresponds to this state (to the given occupation numbers  $N_i$ ). Therefore, the probability that there are  $N_1$  molecules in the first cell of the  $\gamma$ -space,  $N_2$  in the second, etc., is

$$W = \frac{N!}{\prod_i N_i!} (\delta v)^N. \quad (5.1)$$

Taking the logarithm of the probability (5.1), we have

$$\ln W = \ln N! + N \ln (\delta v) - \sum_i \ln N_i!.$$

Using the well-known Stirling formula  $N! \sim N^N e^{-N}$ , we obtain

$$\ln W = N \ln N - N + N \ln (\delta v) - \sum_i N_i \ln N_i + \sum_i N_i.$$

Taking into account that  $\sum_i N_i = N$ , we have

$$\ln W = - \sum_i N_i \ln \frac{N_i}{N \delta v}. \quad (5.2)$$

Making the substitution

$$N_i = f \delta v = f \delta \mathbf{x}_i \delta \xi_i$$

and letting  $\delta v$  go to zero, we obtain for the probability of the given distribution (omitting an unimportant constant which determines only the zero reference probability)

$$\ln W = - \int f \ln f \, d\mathbf{x} \, d\xi. \quad (5.3)$$

The quantity

$$H_D(t) = \int_D f \ln f \, d\mathbf{x} \, d\xi \quad (5.4)$$

is called the H-function for the region of space  $D$ , over which the integration in (5.4) is carried out. Similarly, the function

$$H(t, \mathbf{x}) = \int f \ln f \, d\xi \quad (5.5)$$

is the H-function per unit volume, or simply the Boltzmann H-function. From the definitions (5.4) and (5.5), and the expression (5.3), it is clear that the negative of the H-function is equal to the logarithm of the probability of the given distribution  $f(t, \mathbf{x}, \xi)$ , up to a trivial constant.

Multiplying the Boltzmann equation (2.9) by  $(1 + \ln f)$ , and integrating with respect to  $\xi$  :\*

$$\int (1 + \ln f) \left( \frac{\partial f}{\partial t} + \xi_l \frac{\partial f}{\partial x_l} + \frac{X_l}{m} \frac{\partial f}{\partial \xi_l} \right) d\xi = \int (1 + \ln f) J d\xi \quad (5.6)$$

$(l = 1, 2, 3).$

It is easy to see that

$$(1 + \ln f) \frac{\partial f}{\partial t} = \frac{\partial}{\partial t} (f \ln f), \quad (1 + \ln f) \xi_l \frac{\partial f}{\partial x_l} = \frac{\partial}{\partial x_l} (\xi_l f \ln f), \quad (5.7)$$

$$(1 + \ln f) \frac{\partial f}{\partial \xi_l} = \frac{\partial}{\partial \xi_l} (f \ln f)$$

and that, on the basis of the symmetry property (4.7),

$$I_{(1+\ln f)} = \int (1 + \ln f) J d\xi = -\frac{1}{4} \int \ln \frac{f' f'_1}{f f_1} (f' f'_1 - f f_1) g b db d\varepsilon d\xi d\xi_1. \quad (5.8)$$

Substituting (5.7) and (5.8) into (5.6) and integrating, we obtain

$$\frac{\partial H}{\partial t} + \frac{\partial H_l}{\partial x_l} = I_{(1+\ln f)} = -\frac{1}{4} \int \ln \frac{f' f'_1}{f f_1} (f' f'_1 - f f_1) g b db d\varepsilon d\xi d\xi_1. \quad (5.9)$$

where

$$H_l = \int \xi_l f \ln f d\xi.$$

The integral of the third term on the left of (5.6) is equal to zero, since it is assumed that  $f$  tends to zero sufficiently rapidly as  $\xi$  tends to infinity.

Integrating (5.9) over the region D of the physical x-space, we obtain

$$\frac{dH_D}{dt} + \oint H_n dS = \int I_{(1+\ln f)} dx = G, \quad (5.10)$$

where the integration of the second term on the left is carried out over the surface bounding the region D, and  $H_n$  is the projection on the outward normal to region D of the vector  $\mathbf{H} \equiv (H_1, H_2, H_3)$ .

\* In all that follows it will be assumed that there is summation over repeated subscripts.

If that surface integral vanishes, then

$$\frac{dH_D}{dt} = G. \quad (5.11)$$

But, according to (5.8), the integral  $I_{(1+\ln f)} \leq 0$ , since  $f' f'_1 - f f_1$  and  $\ln (f' f'_1 / f f_1)$  always have the same sign; therefore,

$$\frac{dH_D}{dt} \leq 0. \quad (5.12)$$

Therefore, the function  $H_D$  decreases (the probability of the state increases), if  $G \neq 0$ . This result is the essence of the so-called Boltzmann H-theorem. It is easy to see that  $G = 0$ , i.e., the system is in equilibrium only when  $f f_1 = f' f'_1$  or

$$\ln f + \ln f_1 = \ln f'_1 + \ln f'_1.$$

Thus, in order for G to vanish, it is necessary that  $\ln f$  be a summational invariant of molecular collisions.

As is known from mechanics, any summational invariant may be represented in the form of a linear combination of the summational invariants given in § 1.3, i.e.,

$$\ln f = a m + b_l m \xi_l + d m \xi_l^2, \quad (5.13)$$

where  $a$ ,  $b_l$ , and  $d$  are functions of  $t$  and  $\mathbf{x}$ .

By substituting this form of the function  $f$  into expressions (1.4), (1.5), and (1.8), we can express the five quantities  $a$ ,  $b_l$ , and  $d$  in terms of the five physical quantities  $n$ ,  $u_i$ , and  $T$ . After performing some simple integrations,\* we finally obtain

$$f(t, \mathbf{x}, \xi) \equiv f_0(t, \mathbf{x}, \xi) = n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left\{ -\frac{m}{2kT} c^2 \right\} \quad (5.14)$$

$$(c^2 = c_1^2 + c_2^2 + c_3^2) = (\xi - \mathbf{u})^2.$$

\* For easy integration the left side of (5.13) may be written in the form

$$\begin{aligned} \frac{1}{m} \ln f &= d \left( \xi_1 + \frac{b_1}{2d} \right)^2 + d \left( \xi_2 + \frac{b_2}{2d} \right)^2 \\ &+ d \left( \xi_3 + \frac{b_3}{2d} \right)^2 + \left( a - \frac{1}{4} \frac{b_1^2 + b_2^2 + b_3^2}{d} \right). \end{aligned}$$

The integrals encountered here are given in Appendix II.

The distribution (5.14) is called the Maxwell distribution \* or, more accurately, the local Maxwell distribution, since the macroscopic quantities which appear in (5.14) depend in general on  $t$  and  $\mathbf{x}$ .

Thus, in order that the gas be in equilibrium in region D, it is necessary that the distribution function be locally Maxwellian. On the other hand, the equilibrium condition may be described by the Boltzmann equation

$$\xi_l \frac{\partial f}{\partial x_l} + \frac{X_l}{m} \frac{\partial f}{\partial \xi_l} = J. \quad (5.15)$$

It is evident that the collision integral  $J$  is equal to zero, since  $f'f \{ = ff_1$  for a Maxwellian distribution. However, it is easy to see that the Boltzmann equation imposes specific conditions on the dependence of the functions  $n$ ,  $u_i$ , and  $T$  on  $t$  and  $x_i$ . It is clear that the Maxwellian distribution satisfies Eq. (5.15), when the macroscopic parameters  $n$ ,  $u_i$ , and  $T$  are constant. There are also locally Maxwellian solutions of the Boltzmann equation, in which the hydrodynamic variables  $n$ ,  $u_i$ , and  $T$  depend on the coordinates (see §4.1). However, those solutions do not satisfy the condition assumed above as regards absence of flux of the H-function through the boundary of the region D.

Therefore, the gas is in equilibrium only for an absolute Maxwellian distribution, i.e., a Maxwellian distribution which is independent of the coordinates. It is necessary here that at the boundary of the region the molecules leaving the boundary also have the same Maxwellian distribution. Clearly, in this case, the surface integral appearing in (5.10) is equal to zero. If the gas is contained in a vessel, that boundary condition is satisfied both when the molecules are reflected specularly from the wall, and when the gas is in equilibrium with the wall.† If there is a flux of particles, momentum, or energy through the boundary, the surface integral is generally not zero, and the gas in region D is not in Maxwellian equilibrium.

We can verify directly that the Maxwellian is the most probable distribution. To that end we must seek the extremum of the functional

\*J. C. Maxwell, Phil. Mag. (4), 19(1):22 (1860).

†For greater detail concerning the boundary conditions, see §§2.9 and 2.10.

$$H = \int f \ln f d\xi \quad (5.16)$$

under conditions of conservation of number of particles, momentum, and energy:

$$\begin{aligned} \int f d\xi &= n, \quad m \int f \xi d\xi = mu, \\ \frac{m}{2} \int \xi^2 f d\xi &= \frac{3}{2} kT + \frac{mu^2}{2}. \end{aligned} \quad (5.17)$$

By introducing Lagrange multipliers, we reduce the problem to finding the extremum of the functional

$$F = \int (f \ln f + \lambda_1 f + \lambda_{21} \xi_1 f + \lambda_{22} \xi_2 f + \lambda_{23} \xi_3 f + \lambda_3 \xi^2 f) d\xi. \quad (5.18)$$

The Euler equation for this functional is

$$\ln f + 1 + \lambda_1 + \lambda_{2i} \xi_i + \lambda_3 \xi^2 = 0. \quad (5.19)$$

Substituting  $f$  as given by (5.19) into condition (5.17), we express  $\lambda_j$  in terms of  $n$ ,  $u_i$ , and  $T$ , and we again arrive at the Maxwellian distribution (5.14).

It will be shown in Chapter III that the Euler, Navier-Stokes, and Burnett equations of hydrodynamics for Maxwellian molecules are obtained from the Boltzmann equation, if the distribution function is represented in the approximate form

$$f = f_0(t, x, \xi) \quad (5.20)$$

for the Euler equation, and

$$f = f_0 \left[ 1 + \frac{p_{ij}m}{2pkT} c_i c_j - \frac{q_i m}{pkT} c_i \left( 1 - \frac{mc^2}{5kT} \right) \right] \quad (5.21)$$

for the Navier-Stokes and Burnett equations. Here,  $p_{ij}$  and  $q_i$  are the stress tensor and the heat flux vector, respectively.

We calculate  $H$  and  $H_i$  for the distribution functions (5.20) and (5.21); we have for the Maxwell distribution (5.20),

$$H = n \left[ \ln n \left( \frac{m}{2\pi kT} \right)^{\frac{3}{2}} - \frac{3}{2} \right] = -\frac{mn}{k} \mathcal{S}, \quad (5.22)$$

$$H_t = -\frac{mn}{k} u_t \mathcal{S} \quad (5.23)$$

and for the distribution (5.21),

$$H = -\frac{mn}{k} \mathcal{S} + \frac{m}{2\rho kT} p_{rs} p_{rs} + \frac{2m^2}{5\rho k^2 T^2} q_r q_r, \quad (5.24)$$

$$H_t = u_t H - \frac{m}{kT} q_t + \frac{4}{5} \frac{m}{\rho kT} p_{tr} q_r, \quad (5.25)$$

or, retaining only the leading terms, which correspond to the Navier-Stokes approximation (see Chapter III),

$$H = -\frac{mn}{k} \mathcal{S}, \quad (5.24a)$$

$$H_t = -\frac{mn}{k} u_t \mathcal{S} - \frac{m}{kT} q_t. \quad (5.25a)$$

Here

$$\mathcal{S} = \frac{3}{2} \frac{k}{m} \ln \frac{T}{(mn)^{2/3}} + \mathcal{S}_0 \quad (5.26)$$

is the specific thermodynamic entropy of a macroscopic particle of gas, and  $\mathcal{S}_0$  is a constant which determines the entropy reference.

These formulas show that for a gas in local equilibrium, the Boltzmann H-function is proportional to the negative entropy. Therefore, in the terminology proposed by Brillouin,\* the H-function may be considered a measure of the negative entropy. On the other hand, for a nonequilibrium gas, according to (5.24), the entropy is not proportional to the H-function and, therefore, does not determine the probability of the state of the system. The H-function provides a generalization of the concept of entropy and negative entropy in the case of a nonequilibrium gas.

We note that, in the Navier-Stokes approximation, i.e., for states close to equilibrium, the entropy is proportional to the H-

function and, therefore, retains its interpretation as the probability of a state. This fact is used in an important way in the thermodynamics of nonequilibrium processes (see § 3.17).

For gases in local Maxwellian equilibrium whose motion is described by the Euler equation, the entropy, according to (5.23), is carried along with the gas, i.e., the entropy of the macroscopic particles of the gas remains constant. In the flow of a gas which is not in equilibrium, the transport of the H-function (negative entropy) is also due to heat transfer, to the stress tensor, and in the case of distribution functions more general than (5.21), to other factors.

Let us examine again a region D of the flow. As may be seen from (5.10), molecular collisions lead to a continuous increase in entropy, or decrease in negative entropy, in the volume under examination. The collisions tend to disturb the ordered motion of the molecules. For the flow to be stationary, i.e.,  $dH_D/dt = 0$ , it is necessary that the increase in entropy (disorder) in the volume be compensated by an influx of negative entropy (order) through the boundary of the region.

From (5.10) the well-known result that entropy increases across a wave may easily be deduced. In fact, we examine two control surfaces parallel to the plane of the shock, one of them in the region of equilibrium flow ahead of the shock (plane 1), and the other in a region of equilibrium flow behind the shock (plane 2). We take the x axis perpendicular to the plane of the wave. Then, from (5.10), it follows that

$$H_{x2} - H_{x1} < 0. \quad (5.27)$$

In the region of equilibrium flow outside the shock wave, relation (5.23) is valid for the flux of the H-function. Therefore,

$$\rho_2 u_{x2} \mathcal{S}_2 - \rho_1 u_{x1} \mathcal{S}_1 > 0. \quad (5.28)$$

But, since mass flux is conserved in the wave, i.e.,  $\rho_2 u_{x2} = \rho_1 u_{x1}$ , then

$$\mathcal{S}_2 > \mathcal{S}_1. \quad (5.29)$$

\* L. Brillouin, "Science and Information Theory," Academic Press, New York, 1962.

### §2.6. The Kinetic Theory Equation for a Mixture of Gases, and for a Gas Consisting of Molecules with Internal Degrees of Freedom

In deriving the Boltzmann equation in §2.2, it was assumed that the gas consists of one kind of molecules, and that the molecules possess only translational degrees of freedom (monatomic gas). To describe the behavior of a mixture of gases, we must introduce separate distribution functions for the components of the gas —  $f_i(t, \mathbf{x}, \xi)$ , where  $i = 1, \dots, N$ , and  $N$  is the number of kinds of molecules.

Any change in the distribution function of molecules of the  $i$ -th kind along the trajectory of a particle in phase space is due to collisions both with molecules of the same kind, and with molecules of all other kinds. Therefore, the collision integral in (2.9) should be replaced by a sum of similar integrals, each of which takes account of collisions of molecules of the  $i$ -th kind with molecules of the  $j$ -th kind:

$$\frac{df_i}{dt} = \frac{\partial f_i}{\partial t} + \xi^i \cdot \frac{\partial f_i}{\partial \mathbf{x}} + \frac{\mathbf{X}^i}{m^i} \cdot \frac{\partial f_i}{\partial \xi^i} = \sum_{j=1}^N \int (f'_j f'_i - f_j f_i) g_{ij} b \, db \, de \, d\xi^j, \quad (6.1)$$

where  $\mathbf{X}^i$  is the external force acting on molecules of the  $i$ -th kind. Thus, the change in the state of an  $N$ -component mixture is described by a system of  $N$  equations for the  $N$  distribution functions  $f_i$ , all the functions  $f_i$  appearing in each of the equations (6.1).

The description of the behavior of a gas consisting of molecules with internal degrees of freedom,\* may also be reduced to the system of equations for a mixture of gases. Let us examine a gas consisting of molecules which possess internal degrees of freedom. Let the internal degrees of freedom be rotational and vibrational degrees for polyatomic gases, and excited electronic levels. Let us examine the translational degrees of freedom classically, and the internal degrees quantum-mechanically. Then the state of a molecule may be described by defining its velocity  $\xi$  and its quantum number  $i = 1, 2, \dots$ , which characterizes the excitation of its internal degrees of freedom. All the molecules in some  $i$ -th

\*See J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, "Molecular Theory of Gases and Liquids," Wiley, New York, 1954.

quantum state constitute a gas of the  $i$ -th kind. Thus, the initial gas with internal degrees of freedom is replaced by a mixture of reacting gases, since the collision of a molecule in the state  $i$  with a molecule in the state  $j$ , may cause the molecules to pass to states  $k$  and  $l$ , respectively. We designate by  $\sigma_{ij}^{kl}(\xi^i, \xi^j; \xi^k, \xi^l)$  the probability (effective cross section) that, as a result of a collision of a molecule in state  $i$ , moving with velocity  $\xi^i$ , with a molecule in state  $j$  and with velocity  $\xi^j$ , the first molecule transfers to state  $k$  and acquires velocity  $\xi^k$ , while the second passes to state  $l$  with velocity  $\xi^l$ . Then, assuming that the probabilities of direct and inverse transitions are equal, and repeating the reasoning of §2.2, we obtain

$$\begin{aligned} \frac{df_i}{dt} &= \frac{\partial f_i}{\partial t} + \xi^i \cdot \frac{\partial f_i}{\partial \mathbf{x}} + \frac{\mathbf{X}^i}{m^i} \cdot \frac{\partial f_i}{\partial \xi^i} \\ &= \sum_{j, k, l} \int (f_k f_l - f_l f_j) g_{ij} \sigma_{ij}^{kl}(\xi^i, \xi^j; \xi^k, \xi^l) d\xi^j d\xi^k d\xi^l. \end{aligned} \quad (6.2)$$

Similar equations may be written for a mixture in which chemical reactions occur, if we understand  $\sigma_{ij}^{kl}$  to be the respective probabilities (effective cross sections) for the reactions.

### §2.7. Integral Forms of the Boltzmann Equation

From a comparison of Eqs. (2.9), (6.1), and (6.2), it is easy to note that in all cases (for a one-component gas or a mixture, for a steady or an unsteady state, for a monatomic gas or a gas with internal degrees of freedom and chemical reactions), the Boltzmann equation has the same structure:

$$\frac{df(t, \mathbf{x}, \xi)}{dt} = J(t, \mathbf{x}, \xi) = J_1(t, \mathbf{x}, \xi) - f(t, \mathbf{x}, \xi) J_2(t, \mathbf{x}, \xi), \quad (7.1)$$

where the differentiation is carried out along the trajectory of a given molecule in the phase space, and

$$\begin{aligned} J_1(t, \mathbf{x}, \xi) &= \int f' f'_i g b \, db \, de \, d\xi_1, \\ J_2(t, \mathbf{x}, \xi) &= \int f_1 g b \, db \, de \, d\xi_1 \end{aligned} \quad (7.2)$$

for a monatomic gas or a sum of similar integrals for a reacting mixture and for polyatomic gases:

$$J_1(t, \mathbf{x}, \xi) = \sum_{j, k, l} \int f_k f_l \sigma_{ij}^{kl} (\xi^i, \xi^j; \xi^k, \xi^l) g_{ij} d\xi^j d\xi^k d\xi^l, \quad (7.2a)$$

$$J_2(t, \mathbf{x}, \xi) = \sum_{j, k, l} \int f_j f_l \sigma_{ij}^{kl} (\xi^i, \xi^j; \xi^k, \xi^l) g_{lj} d\xi^j d\xi^k d\xi^l.$$

For molecules of infinite range of interaction, the integrals (7.2) diverge, since they include among the colliding molecules some which interact at arbitrarily large distances with arbitrarily small resultant changes of state. In what follows, when the integrals  $J_1$  and  $J_2$  have to be calculated separately, we always assume that the range of interaction is finite. When the interaction potential falls off rapidly enough, distant collisions do not make an appreciable contribution; therefore, up to a given accuracy the actual potential of such molecules may be replaced by a "truncated" potential of finite interaction range. In general, the choice of the correct effective interaction range is far from trivial.

The collision integral  $J$  converges even when the integrals  $J_1$  and  $J_2$  diverge separately, since, for large target distances,  $\xi' \rightarrow \xi, \xi'_1 \rightarrow \xi_1$  and  $(f' f'_1 - f f_1) \rightarrow 0$ .

The integrodifferential equation (7.1) may be regarded as an ordinary differential equation for  $f$  of the type

$$\frac{df}{dt} = J(t, f) \quad (7.3)$$

or

$$\frac{df}{dt} + f J_2(t, f) = J_1(t, f). \quad (7.3a)$$

Assuming for the time being that the function  $f$  which appears in  $J$ ,  $J_1$ , and  $J_2$  is known, it is easy to write the solution of the linear equations (7.3) and (7.3a); we have

$$f(t, \mathbf{x}, \xi) = f(t_0, \mathbf{x} - \xi(t-t_0), \xi) + \int_{t_0}^t J(s, \mathbf{x} - \xi(t-s), \xi) ds \quad (7.4)$$

or\*

$$f(t, \mathbf{x}, \xi) = f(t_0, \mathbf{x} - \xi(t-t_0), \xi) \exp \left\{ - \int_{t_0}^t J_2(s, \mathbf{x} - \xi(t-s), \xi) ds \right\}$$

$$+ \int_{t_0}^t J_1(\tau, \mathbf{x} - \xi(t-\tau), \xi) \exp \left\{ - \int_{\tau}^t J_2(s, \mathbf{x} - \xi(t-s), \xi) ds \right\} d\tau. \quad (7.5)$$

Since the right sides of these equations contain the desired function  $f$ , these equations may be regarded as integral equations for  $f$ .

Equations (7.4) and (7.5) have been written for the special case of motion without external forces,  $\mathbf{X}(t, \mathbf{x}) \equiv 0$ . In that case,  $\xi = \text{const}$  along the trajectory of the particle, and the trajectory of a particle proceeding with velocity  $\xi$  to the point  $\mathbf{x}$  at time  $t$ , is given by the equation of the straight line  $\mathbf{x}(s) - \xi(t-s) = \mathbf{x}(t)$ . If there were no collisions, then, the number of particles of velocity  $\xi$  at point  $\mathbf{x}$  at time  $t$  would be equal to the number of particles at the point  $\mathbf{x} - \xi(t-t_0)$  at time  $t_0$ , i.e.,  $f(t, \mathbf{x}, \xi) = f(t_0, \mathbf{x} - \xi(t-t_0), \xi)$ . The integrals appearing in (7.4) and (7.5) determine the change in the number of particles as a result of collisions along the path of the molecules from the point  $\mathbf{x} - \xi(t-t_0)$  to the point  $\mathbf{x}$ .

If  $\mathbf{X}(t, \mathbf{x}) \neq 0$ , the trajectory of a particle is more complex, since the velocity  $\xi$  varies along the trajectory. In that case, the expressions  $\mathbf{x} - \xi(t-s)$  and  $\xi = \text{const}$  in Eqs. (7.4) and (7.5) should be replaced by  $\mathbf{x}(s)$  and  $\xi(s)$ , respectively, as determined by the equation of motion of the particle

$$m \frac{d^2 \mathbf{x}}{ds^2} = \mathbf{X}(s, \mathbf{x})$$

under the condition that when  $s = t$ , the particle is located at the assigned point  $\mathbf{x}(t)$  and has the assigned velocity  $\xi(t)$ .

In general, by analogy with (7.4) and (7.5), we can write an infinite number of different forms of the integral equations. In

\*D. Enskog, Ark. Mat. Astron. Phys. Ser. A21, Vol. 1 (1928); H. Grad, "Handbuch der Physik," Vol. 12, 1958; M. N. Kogan, Prikladnaya Matematika i Mekhanika, No. 4 (1958); S. V. Vallander, Doklady Akad. Nauk SSSR, Vol. 131, No. 1 (1960). In the last reference a direct derivation of Eq. (7.5) is given, bypassing the Boltzmann equation.

fact, an arbitrary function of  $f$  may be added to and subtracted from the right side of Eq. (7.1). Then, by dividing the right side of (7.1) in a different manner into a known and an unknown part, we obtain different differential equations, the solution of which leads to different integral equations. The choice of a particular integral form of the Boltzmann equation is determined by the convenience of its use for a particular problem.

The distribution function for each component of a mixture of polyatomic reacting gases (see §2.6) clearly satisfies the same integral equations (7.4) or (7.5), in which the integrals  $J_1$  and  $J_2$  are given by formulas (7.2).

### §2.8. Linearized and Model Boltzmann Equations

Because of the complex nonlinear structure of the collision integral, the Boltzmann equation is very difficult to solve and to analyze. It is of interest to make a qualitative study of the properties of its solutions by use of simplified model equations. Two approximate Boltzmann equations are examined below. The first is the linearized equation, obtained naturally from the Boltzmann equation for weakly perturbed flows; the second is a model equation, possessing many properties of the complete nonlinear Boltzmann equation, but not following strictly from it.

1. The Linearized Equation. If a particular solution of any nonlinear equation is known, the problem may be linearized for the investigation of solutions close to the special solution. For the Boltzmann equation only a small number of very special particular solutions is known (see §4.1). Therefore, the most universal linearization is that of the absolute Maxwellian distribution, which is the solution of the Boltzmann equation for a gas in equilibrium in the absence of body forces ( $\mathbf{X} \equiv 0$ ) (see §2.5).

For a weakly perturbed motion, the distribution function may be represented in the form

$$f(t, \mathbf{x}, \xi) = f_0(1 + \varphi). \quad (8.1)$$

where

$$f_0 = n_0 \left( \frac{m}{2\pi k T_0} \right)^{3/2} \exp \left\{ - \frac{m}{2kT_0} \xi^2 \right\}, \quad \varphi = \varphi(t, \mathbf{x}, \xi) \ll 1.$$

### §2.8] LINEARIZED AND MODEL BOLTZMANN EQUATIONS

and the subscript 0 denotes quantities relating to the undisturbed state. Since  $n_0$  and  $T_0$  do not depend on  $t$  and  $\mathbf{x}$ , we substitute (8.1) into the Boltzmann equation (2.9) in the absence of body forces, and neglecting second and higher powers of  $\varphi$ , we obtain

$$\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial t} + \xi_i \frac{\partial \varphi}{\partial x_i} = \int f_{01} (\varphi' + \varphi'_i - \varphi - \varphi_i) g b db de d\xi_1. \quad (8.2)$$

In deriving Eq. (8.2), we used the fact that  $f_0 f'_{01} = f_0 f_{01}$ , and that the function  $f_0(\xi)$  does not depend on the variable of integration  $\xi_1$ , and may be taken outside the integral sign.

Equation (8.2) is a very complex integrodifferential equation, but, because of its linearity, it is much simpler than the full Boltzmann equation. In particular, it describes weak sonic and ultrasonic oscillations (see §4.5), as well as very slow gas flows. However, in the majority of cases of practical interest, flows are not weakly perturbed. For example, flow over slender bodies or even over a flat plate aligned parallel to the stream, is accompanied by a large perturbation of the distribution function. Nevertheless, investigation of solutions of the linear equation allows us to clarify a number of properties also shared by the full Boltzmann equation (see §4.2).

We write the linear Boltzmann equation (8.2) in the form \* (see §2.2)

$$\frac{d\varphi}{dt} = \int f_{01} (\varphi' + \varphi'_i - \varphi - \varphi_i) W(\xi, \xi_i | \xi', \xi'_i) d\xi_1 d\xi' d\xi'_i. \quad (8.3)$$

Equation (8.3) may be written in the more symmetric form

$$\begin{aligned} f_0 \frac{d\varphi}{dt} &= L(\varphi) \\ &= \int (f_0 f_{01} f'_0 f'_{01})^{1/2} (\varphi' + \varphi'_i - \varphi - \varphi_i) W(\xi, \xi_i | \xi', \xi'_i) d\xi_1 d\xi' d\xi'_i \\ &= \int w(\xi, \xi_i | \xi', \xi'_i) (\varphi' + \varphi'_i - \varphi - \varphi_i) d\xi_1 d\xi' d\xi'_i, \end{aligned} \quad (8.4)$$

where

$$w(\xi, \xi_i | \xi', \xi'_i) = (f_0 f_{01} f'_0 f'_{01})^{1/2} W(\xi, \xi_i | \xi', \xi'_i).$$

\*L. Waldman, "Handbuch der Physik," Vol. 12, 1958.

It is easy to verify the relations

$$\int w(\xi, \xi_1 | \xi', \xi'_1) \varphi' d\xi_1 d\xi' d\xi'_1 = \int w(\xi, \xi' | \xi_1, \xi'_1) \varphi_1 d\xi_1 d\xi' d\xi'_1. \quad (8.5)$$

$$\begin{aligned} \int w(\xi, \xi_1 | \xi', \xi'_1) \varphi'_1 d\xi_1 d\xi' d\xi'_1 &= \int w(\xi, \xi' | \xi_1, \xi'_1) \varphi_1 d\xi_1 d\xi'_1 d\xi' \\ &= \int w(\xi, \xi' | \xi'_1, \xi_1) \varphi_1 d\xi_1 d\xi'_1 d\xi'. \end{aligned} \quad (8.6)$$

obtained by a change of signs.

Using those relations, we put the collision integral in the form

$$L(\varphi) = -\varphi \int w(\xi, \xi_1 | \xi', \xi'_1) d\xi_1 d\xi' d\xi'_1 + \int K(\xi, \xi_1) \varphi_1 d\xi_1, \quad (8.7)$$

where

$$K(\xi, \xi_1) = \int [-w(\xi, \xi_1 | \xi', \xi'_1) + w(\xi, \xi' | \xi_1, \xi'_1) + w(\xi, \xi' | \xi'_1, \xi_1)] d\xi' d\xi'_1.$$

Since the function  $w$  possesses properties of symmetry [see formula (2.15)]:

$$w(\xi', \xi'_1 | \xi, \xi_1) = w(\xi, \xi_1 | \xi', \xi'_1), \quad (8.8)$$

$$w(\xi, \xi_1 | \xi', \xi'_1) = w(\xi_1, \xi | \xi'_1, \xi'),$$

then

$$\begin{aligned} K(\xi, \xi_1) &= \int [-w(\xi_1, \xi | \xi'_1, \xi') + w(\xi_1, \xi' | \xi, \xi')] \\ &\quad + w(\xi_1, \xi'_1 | \xi', \xi)] d\xi' d\xi'_1. \end{aligned} \quad (8.9)$$

Interchanging  $\xi'$  and  $\xi'_1$ , we obtain

$$\begin{aligned} K(\xi, \xi_1) &= \int [-w(\xi_1, \xi | \xi', \xi'_1) + w(\xi_1, \xi' | \xi, \xi')] \\ &\quad + w(\xi_1, \xi'_1 | \xi', \xi)] d\xi' d\xi'_1 = K(\xi_1, \xi). \end{aligned} \quad (8.10)$$

i.e., the kernel  $K(\xi, \xi_1)$  is symmetrical.

By interchanging the variables of integration we can bring the integral of  $K(\xi, \xi_1)$  with respect to  $\xi_1$  to the form

$$\begin{aligned} \int K(\xi, \xi_1) d\xi_1 &= \int [-w(\xi, \xi_1 | \xi', \xi'_1) + w(\xi, \xi' | \xi_1, \xi'_1) \\ &\quad + w(\xi, \xi' | \xi'_1, \xi_1)] d\xi' d\xi'_1 d\xi_1 \\ &= \int [-w(\xi, \xi_1 | \xi', \xi'_1) + w(\xi, \xi_1 | \xi', \xi'_1) \\ &\quad + w(\xi, \xi_1 | \xi', \xi'_1)] d\xi_1 d\xi' d\xi'_1 = \int w(\xi, \xi_1 | \xi', \xi'_1) d\xi_1 d\xi' d\xi'_1. \end{aligned} \quad (8.11)$$

Therefore, the collision integral (8.7) is equal to

$$L(\varphi) = \int K(\xi, \xi_1) (\varphi_1 - \varphi) d\xi_1. \quad (8.12)$$

Thus, the integral operator of the linearized Boltzmann equation is an operator of the Fredholm type with a symmetrical kernel. This operator is often written with the term  $\varphi$  separated:

$$L(\varphi) = -\bar{K}(\xi) \varphi + \int K(\xi, \xi_1) \varphi_1 d\xi_1, \quad (8.13)$$

where

$$\bar{K}(\xi) = \int K(\xi, \xi_1) d\xi_1. \quad (8.14)$$

The standard technique for seeking solutions to integral equations with symmetrical kernels is to expand the solution in eigenfunctions of the integral operator  $L(\varphi)$ . It is more convenient to return to this question somewhat later (see § 3.11).

2. The Model Equation. In addition to this approximation, which is mathematically correct for small perturbations, wide use is also made of a model equation,\* which does not have a strict mathematical basis, but, as will be seen later, gives qualitatively correct results for a wide range of problems with arbitrary mean free paths and with perturbations of any intensity.

We write the Boltzmann equation in the form (7.1):

$$\frac{df}{dt} = J_1 - f J_2, \quad (8.15)$$

\*P. L. Bhatnagar, E. P. Gross, and M. Krook, Phys. Rev., 94: 511 (1954); P. Welander, Ark. Fysik, 7(6): 507 (1954); M. N. Kogan, Prikl. Matem. i Mekh., No. 4 (1958).

where

$$J_1 = \int f' f_1 g b \, db \, de \, d\xi_1, \quad J_2 = \int f_1 g b \, db \, de \, d\xi_1.$$

It was shown in § 2.2, that for Maxwellian molecules the Boltzmann equation may be written in a form which does not contain the relative velocity explicitly in the collision integral:

$$J_1 = \left(\frac{16K}{m}\right)^{1/2} \int f' f_1 \rho \, d\rho \, de \, d\xi_1, \quad J_2 = \left(\frac{16K}{m}\right)^{1/2} \int f_1 \rho \, d\rho \, de \, d\xi_1. \quad (8.16)$$

As we noted above when molecules interact at large distances, the momentum and the energy of the colliding particles does not change appreciably. Therefore, in the Boltzmann equation, we may integrate approximately with respect to the interaction parameter, not to infinity, but to a finite value, equal to the effective collision cross-section diameters. The larger the relative velocity of the molecules, the smaller the effective collision cross section. For pseudo-Maxwellian molecules [see formula (2.13)]

$$J_2 = \left(\frac{16K}{m}\right)^{1/2} \int f_1 \, d\xi_1 \int \rho \, d\rho \, de = A \int f_1 \, d\xi_1 = An, \quad (8.17)$$

where  $n$  is the number of molecules per unit volume and

$$A = \left(\frac{16K}{m}\right)^{1/2} \int \rho \, d\rho \, de.$$

The integration here is carried out up to a finite  $\rho$ , i.e., to a value of the interaction parameter  $b$ , which is inversely proportional to the square root of the relative velocity  $g$ , since

$$b = \rho \left(\frac{16K}{m}\right)^{1/4} \frac{1}{\sqrt{g}}.$$

Therefore, the effective collision cross section is inversely proportional to  $g$ .

Thus, a pseudo-Maxwellian molecule of velocity  $\xi$  collides with  $An$  molecules of all possible velocities  $\xi_1$  in unit time and unit volume. Therefore, a total of

$$N = \int Anf(\xi) \, d\xi = An^2 \quad (8.18)$$

particles collide per unit volume and unit time.

The integral  $J_2(t, \mathbf{x}, \xi)$  determines the number of particles of velocity  $\xi$  after the collision, i.e.,  $J_2$  is the molecular distribution function after the collision.

It follows from the H-theorem that the distribution function tends to its most probable value as a result of collisions. We assume that the distribution function of the molecules after collision,  $J_2$ , is the most probable for a specified number, momentum, and energy of molecules. That assumption is confirmed to some extent by the fact that a free system of particles tends to equilibrium exponentially (see §§ 3.3, 4.1) and the characteristic decay time is the time between collisions. Even after a single collision, a system with an arbitrary initial distribution function goes to a state close to equilibrium. A particle of velocity  $\xi$  takes part in  $An$  collisions, bringing with it a momentum  $mAn\xi$ . Therefore, the momentum  $\mathbf{P}$  of all the colliding molecules is equal to

$$\mathbf{P} = \int mAn\xi f \, d\xi = An^2 m \mathbf{u}, \quad (8.19)$$

where  $\mathbf{u}(t, \mathbf{x})$  is the mean velocity of the gas (see § 2.1).

Similarly, the energy  $E$  of the colliding molecules is

$$E = \int \frac{1}{2} m\xi^2 Anf \, d\xi = An^2 \left( \frac{3}{2} kT + \frac{mu^2}{2} \right), \quad (8.20)$$

where  $T(t, \mathbf{x})$  is the temperature of the gas.

It is easy to see that expressions (8.18)–(8.20) differ from the corresponding expressions (5.17) only in that the number of particles is equal not to  $n$ , but to  $An^2$ . Therefore, the most probable distribution function when  $N$ ,  $\mathbf{P}$ ,  $E$  are determined by the formulas (8.18)–(8.20), is

$$J_1 = An^2 \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left\{ -\frac{m}{2kT} c^2 \right\} = Anf_0. \quad (8.21)$$

Thus, under our assumptions, the Boltzmann equation takes the form

$$\frac{df}{dt} = An(f_0 - f). \quad (8.22)$$

Equation (8.22) is a nonlinear integrodifferential equation, since the distribution function  $f$  appears in a nonlinear fashion in  $f_0$  ( $n$ ,  $\mathbf{u}$ , and  $T$  are integrals of the function  $f$ ). The quantity  $1/An$  has the dimension of time and is equal to the time between collisions  $\tau = (n\sigma_m g)^{-1} = 1/An$ . When written in the form

$$\frac{df}{dt} = \frac{f_0 - f}{\tau}, \quad (8.23)$$

the Boltzmann equation coincides in form with equations in the theory of relaxation processes and is therefore sometimes called the **relaxation equation**.

Equation (8.22) may be represented in integral form.\* To achieve this, it is sufficient to use special values for the integrals  $J_1$  and  $J_2$  in Eqs. (7.4) and (7.5), namely,

$$J_1 = Anf_0 \text{ and } J_2 = An.$$

We have

$$\begin{aligned} f(t, \mathbf{x}, \xi) &= f(t_0, \mathbf{x} - \xi(t - t_0), \xi) \\ &+ A \int_{t_0}^t n[s, \mathbf{x} - \xi(t - s)] [f_0(s, \mathbf{x} - \xi(t - s), s) \\ &\quad - f(s, \mathbf{x} - \xi(t - s), \xi)] ds, \quad (8.24) \\ f(t, \mathbf{x}, \xi) &= f(t_0, \mathbf{x} - \xi(t - t_0), \xi) \\ &\times \exp \left\{ -A \int_{t_0}^t n(s, \mathbf{x} - \xi(t - s)) ds \right\} + A \int_{t_0}^t n(\tau, \mathbf{x} - \xi(t - \tau)) \\ &\times f_0(\tau, \mathbf{x} - \xi(t - \tau), \xi) \exp \left\{ -A \int_{\tau}^t n(s, \mathbf{x} - \xi(t - s)) ds \right\} d\tau. \quad (8.25) \end{aligned}$$

We examine a gas whose state at time  $t_0 = 0$  is described by a distribution function  $f(0, \xi)$ , which does not depend on the space coordinates. It follows, from the conservation of mass, momentum, and energy, that  $n$ ,  $\mathbf{u}$ , and  $T$  are constant in space and time and, therefore, that the function  $f_0$  is also constant. Then, from Eq. (8.25), we have

\* M. N. Kogan, Prikl. Matem. i Mekh., No. 4 (1958).

$$f(t, \xi) = f_0 + e^{-\frac{t}{\tau}} [f(0, \xi) - f_0], \quad (8.26)$$

where  $\tau = 1/An$  is the time between collisions.

Thus, the distribution function tends to the equilibrium function  $f_0$  exponentially, with a characteristic relaxation time  $\tau$ , equal to the time between collisions, i.e., the more dense the gas, the faster equilibrium is attained.\*

It may be seen from (8.23) that, at a given time and a given point in space, the relaxation times for molecules with all velocities are the same. This is a consequence of the crude assumptions made in deriving the equation.

For states close to equilibrium, Eqs. (8.22) or (8.23) confirm the obvious fact that the rate of approach to equilibrium is proportional to the deviation from equilibrium. The preceding considerations allow us, under definite assumptions, to apply this equation also for a qualitative description of flows which are far from equilibrium. The validity of this assertion is confirmed below (see §§4.1, 4.2, 4.4) by comparison with solutions of the full Boltzmann equation.

In §§3.12 and 6.8, more complex model equations are presented.

### §2.9. Formulation of Problems for the Boltzmann Equation

Let the motion of a gas be described by the Boltzmann equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} + \frac{X_i}{m} \frac{\partial f}{\partial \xi_i} = J(t, \mathbf{x}, \xi).$$

In order to construct a specific solution to this equation, we must prescribe initial and boundary conditions.

The distribution function  $f(t_0, \mathbf{x}, \xi) \geq 0$  at the initial time  $t = t_0$  may be assigned in an arbitrary manner.

On surfaces  $S$  which bound a flow region, it is necessary to assign a distribution function to the molecules moving from the

\* This same conclusion follows from the exact Boltzmann equation (see § 4.1).

boundary into the flow field. If  $\mathbf{x}$  is a point of the boundary, and  $\mathbf{n}(\mathbf{x})$  is the normal to the boundary surface directed into the flow at the point  $\mathbf{x}$ , we must prescribe a function  $f(t, \mathbf{x}, \xi) \geq 0$  for velocities  $\xi$ , which satisfy the condition  $\xi \cdot \mathbf{n} \geq 0$ . Then the distribution function at all the internal points of flow and at all points on the boundary surface for  $\xi \cdot \mathbf{n} \leq 0$  is determined by solution of the Boltzmann equation.

It is also possible to have a mixed problem, when a distribution function  $f(t, \mathbf{x}, \xi) \geq 0$  is assigned for all  $\xi$  on part of the boundary surface  $S_1$ . Then on the remaining part of the surface,  $S_2 = S - S_1$ , we may assign the function  $f(t, \mathbf{x}, \xi) \geq 0$  for  $\xi \cdot \mathbf{n} > 0$  and for those directions of the velocity vector  $\xi$  which do not intersect  $S_1$  on the way from  $\mathbf{x}$  into  $S_2$ .

The surface  $S$  may be both simply connected and multiply connected.

In the problem of the flow of a stream unbounded at infinity past finite bodies, the distribution function must evidently tend toward the solution of the Boltzmann equation describing the state of the gas unperturbed by the bodies. The most frequent problem is the study of the motion of bodies in a gas in equilibrium. Then, examining the flow in coordinates fixed with respect to the bodies, we have

$$f(\mathbf{x} \rightarrow \infty, \xi) = n_{\infty} \left( \frac{m}{2\pi k T_{\infty}} \right)^{3/2} \exp \left\{ -\frac{m}{2kT_{\infty}} [(\xi_1 - U_1)^2 + (\xi_2 - U_2)^2 + (\xi_3 - U_3)^2] \right\}, \quad (9.1)$$

where  $\mathbf{U} \equiv (U_1, U_2, U_3)$  is the velocity of the oncoming stream,  $n_{\infty}$  and  $T_{\infty}$  are the particle number density and the gas temperature at infinity, respectively.

We examine in somewhat greater detail the conditions which must be satisfied by the distribution function on solid surfaces which bound the flow. We examine a gas whose atoms do not undergo any chemical reaction on the surface. A molecule which strikes the surface at a velocity  $\xi_i$ , reflects from it at a velocity  $\xi_r$ . We designate by  $W(\xi_i, \xi_r) d\xi_i d\xi_r$  the probability that a molecule striking a surface at a velocity between  $\xi_i$  and  $\xi_i + d\xi_i$  is re-emitted from it at a velocity between  $\xi_r$  and  $\xi_r + d\xi_r$ . The number of mole-

cules which strike a unit area per unit time with a velocity between  $\xi_i$  and  $\xi_i + d\xi_i$  at point  $\mathbf{x}$  is equal to

$$-(\xi_i \cdot \mathbf{n}) f(\mathbf{x}, \xi_i) d\xi_i,$$

where  $\mathbf{n}$  is the normal to the surface directed into the gas. A fraction of these molecules, equal to

$$-(\xi_i \cdot \mathbf{n}) f(\mathbf{x}, \xi_i) W(\xi_i, \xi_r) d\xi_i d\xi_r,$$

is reflected with velocities in the range  $(\xi_r, \xi_r + d\xi_r)$ . The total number of molecules reflected per unit area at a velocity in the range  $(\xi_r, \xi_r + d\xi_r)$  is equal to

$$f(\mathbf{x}, \xi_r) (\xi_r \cdot \mathbf{n}) d\xi_r = -d\xi_r \int_{\xi \cdot \mathbf{n} < 0} (\xi_i \cdot \mathbf{n}) f(\mathbf{x}, \xi_i) W(\xi_i, \xi_r) d\xi_i, \quad (9.2)$$

where the integration is performed over all velocities of the molecules striking the wall:  $\xi \cdot \mathbf{n} < 0$ . Since the number of incoming molecules must be equal to the number of outgoing molecules, i.e.,

$$\int_{\xi \cdot \mathbf{n} > 0} f(\mathbf{x}, \xi_r) (\xi_r \cdot \mathbf{n}) d\xi_r = - \int_{\xi \cdot \mathbf{n} < 0} (\xi_i \cdot \mathbf{n}) f(\mathbf{x}, \xi_i) d\xi_i, \quad (9.3)$$

the function  $W(\xi_i, \xi_r)$  must satisfy the normalization condition

$$\int_{\xi \cdot \mathbf{n} > 0} W(\xi_i, \xi_r) d\xi_r = 1. \quad (9.4)$$

After division by  $d\xi_r$ , condition (9.2) may be written in the form

$$f_r(\mathbf{x})(\xi_r \cdot \mathbf{n}) = - \int_{\xi \cdot \mathbf{n} < 0} (\xi_i \cdot \mathbf{n}) f_l(\mathbf{x}) W(\xi_i, \xi_r) d\xi_i \quad (9.5)$$

or

$$f_r(\mathbf{x}) = \int_{\xi \cdot \mathbf{n} < 0} K(\xi_i, \xi_r) f_l(\mathbf{x}) d\xi_i, \quad (9.6)$$

where

$$f_l(\mathbf{x}) = f(\mathbf{x}, \xi_i), \quad f_r(\mathbf{x}) = f(\mathbf{x}, \xi_r),$$

and

$$K(\xi_i, \xi_r) = -W(\xi_i, \xi_r) \frac{(\xi_i \cdot n)}{(\xi_r \cdot n)}.$$

It is seen from (9.6) that the distribution function for the reflected molecules  $f_r$  may be quite different in nature from the distribution function for the incident molecules  $f_i$ , i.e., the distribution function is in general discontinuous with respect to  $\xi$  near the surface (when  $\xi \cdot n = 0$ ).

The function  $W$  (or  $K$ ) depends on the physical and chemical properties of the surface and of the incident particle, the treatment of the surface and its temperature  $T_w$ . In general, for example, when adsorbed layers are present, the function  $W$  may depend also on the total number of particles incident on the surface element, and on their total momentum and energy (see the following paragraph).

If the surface moves at a velocity  $\mathbf{U}_w(t)$ , then clearly the boundary condition at any time in the system of coordinates fixed with respect to the wall will also have the form (9.6), i.e., we have

$$\begin{aligned} f(t, \mathbf{x}, \xi_r)((\xi_r - \mathbf{U}_w) \cdot n) d\xi_r dt &= d\xi_r dt \int_{(\xi_i - \mathbf{U}_w) \cdot n < 0} ((\mathbf{U}_w - \xi_i) \cdot n) f(t, \mathbf{x}, \xi_i) \\ &\times W(t, \xi_i - \mathbf{U}_w, \xi_r - \mathbf{U}_w) d\xi_i \end{aligned} \quad (9.7)$$

and

$$K(t, \mathbf{U}_w, \xi_i, \xi_r) = W(t, \xi_i - \mathbf{U}_w, \xi_r - \mathbf{U}_w) \frac{(\mathbf{U}_w - \xi_i) \cdot n}{(\xi_r - \mathbf{U}_w) \cdot n}.$$

In deriving this relation, it has tacitly been assumed that the residence time of a molecule on the surface,  $\theta$  (time of adsorption), is much less than the characteristic time of the problem. If that condition is not satisfied, then we must know the probability

$$W(t, \xi_r - \mathbf{U}_w, \xi_i - \mathbf{U}_w, t - \theta) dt d\theta d\xi_i d\xi_r$$

that a molecule, incident on the surface with velocity  $\xi_i - \mathbf{U}_w$  relative to the surface at time  $t - \theta$  in the range  $d\theta$ , will leave the surface with velocity  $\xi_r - \mathbf{U}_w$  at time  $t$  in the range  $dt$ ; then,

$$f(t, \mathbf{x}(t), \xi_r)[(\xi_r - \mathbf{U}_w(t)) \cdot n(t)] d\xi_r dt =$$

$$= dt d\xi_r \int_0^\infty d\theta \int_{(\xi_i - \mathbf{U}_w) \cdot n < 0} [(U_w(t - \theta) - \xi_i) \cdot n] \\ \times f[t - \theta, \mathbf{x}(t - \theta), \xi_i] W(t, \xi_i - \mathbf{U}_w, t - \theta, \xi_i - \mathbf{U}_w) d\xi_i. \quad (9.8)$$

The probability  $W$  and the adsorption time  $\theta$  depend both on the physical and chemical properties of the surface and of the incident molecules, and on the velocity and number of molecules incident on the surface, since the state of the adsorbed layers\* depends on the latter.

With the indicated initial and boundary conditions, we can, in principle, construct a solution of the Boltzmann equation by any of a number of methods. However, the question of the existence and uniqueness of a solution cannot at present be resolved in such a general formulation.

The existence of solutions of the Boltzmann equation has been studied only for initial value problems in an unbounded region, and for molecules with finite interaction radius (with truncated interaction potential). For the spatially homogeneous case, an existence theorem has been proved both for hard-sphere molecules,<sup>†</sup> and also for pseudo-Maxwellian molecules<sup>‡</sup> for the full nonlinear Boltzmann equation. For the linear equation an existence theorem has been proved, and the asymptotic behavior of solutions has been investigated for a problem with initial conditions which depend on the space coordinates.<sup>\*\*</sup> The spatially inhomogeneous problem for the nonlinear Boltzmann equation has been examined by Grad.<sup>††</sup> However, the existence of solutions has been

\* J. H. de Boer, "The Dynamical Character of Adsorption," Oxford University Press, 1953; B. V. Filippov, in: "The Aerodynamics of Rarefied Gases" [in Russian], LGU, I (1963) and II (1965).

<sup>†</sup> T. Carleman, Acta Math., Stockholm, 60: 91 (1933).

<sup>‡</sup> D. Morgenstern, Proc. Nat. Acad. Sci., 40: 719 (1954).

<sup>\*\*</sup> T. Carleman, "Problemes mathematiques dans la Theorie Cinetique des Gaz," Almqvist and Wiksell, Uppsala, 1957; H. Grad, Phys. Fluids, 6: 147 (1963); H. Grad, "Rarefied Gas Dynamics," Third Symposium, Academic Press, 1963. H. Grad, Comm. Pure and Appl. Math., Vol. 18, No. 1/2 (1965); A. A. Arsen'ev, Zh. Vychislitel'noi Matematiki i Matematicheskoi Fiziki, Vol. 5, No. 5 (1965). See also § 3.7.

<sup>††</sup> H. Grad, Proc. Am. Math. Soc., Symposium on Application of Partial Differential Equations, 1964.

demonstrated for times which decrease, as the initial distribution function differs more from equilibrium. Thus, for times on the macro-scale, the existence has been proved only for small initial perturbations.

#### §2.10. Interaction of Molecules with Solid Surfaces. Accommodation Coefficients

From the discussion presented in the preceding paragraph, it is clear that, to calculate flows with boundaries which include solid surfaces, it is necessary to know the probability of reflection of molecules at a given velocity from the surface of the body. At the present time the theory of the interaction of molecules with solid or liquid surfaces is in the early stages of its development. The experimental data also are quite insufficient. The difficulties of the theoretical investigation of the interaction are due primarily to our lack of knowledge of the structure of the surface layers of solid bodies and, therefore, of the interaction potentials of the incident molecules with molecules of the solid. When a molecule arrives on a surface, it is adsorbed, and forms physical and chemical bonds with the molecules of the surface. On the surface a molecule may dissociate, and lose or acquire an electron. Molecules which arrive with large energies may displace surface molecules or molecules adsorbed on the surface. Depending on these interactions, a molecule may possess different momentum and internal energy during its adsorption time.

The state of a surface depends on the surface finish (roughness), the cleanliness of the surface, its temperature, etc. The nature of the interaction of molecules with the surface of a single specimen located in a vacuum may vary with time because of out-gassing from the surface. Preliminary heating of a surface also promotes purification of the surface through emission of adsorbed molecules.

Generally speaking, the nature of interaction of a given molecule with a surface also depends on the number and velocity of the other molecules incident on the same surface element.

Let us imagine a surface free of extraneous molecules. Depending on the potential of the interaction of the surface molecules with a gas molecule, the interaction time may vary over very wide

limits. If  $\tau_a$  is the adsorption time, then the fraction of the surface occupied by adsorbed molecules is equal to  $\gamma = N_i \tau_a \sigma$ , where  $N_i$  is the number of incident molecules per unit time on unit surface, and  $\sigma$  is the cross section of a gas molecule. The adsorption time  $\tau_a$  depends on the energy of the incident molecules. If  $\gamma$  is not close to zero, the nature of the interaction of each incident molecule depends on the total number and energy of the incident molecules. Under conditions of high vacuum (for example, for a body in orbit),  $\gamma \ll 1$ , even though the time of physical adsorption is of the order of  $10^{-6}$  sec. In that case, each incident molecule interacts with the surface independently of the others. In the other limiting case, when  $\gamma = 1$  (for example, in chemical adsorption, when the time  $\tau_a$  may be very large, or in a dense gas), the reflection may also be independent, if a second adsorbed layer is not formed. In that case, the molecule interacts with the adsorbed monolayer with a shorter adsorption time.

Molecular beams are widely used for the experimental investigation of the interaction of molecules with a surface. The technique of obtaining beams of molecules with thermal velocities (of the order  $10^4$ - $10^5$  cm/sec) has been rather well developed, but the problem of obtaining monochromatic, well-controlled beams sufficiently intense to perform measurements, at velocities of the order of  $10^6$ - $10^7$  cm/sec (for nitrogen and oxygen for energies of 10-100 eV) is very complex at the present time. A second difficulty is the control of conditions on the solid surface, a particular difficulty being the analysis of adsorbed films. Evidently, the latter causes the large scatter of experimental data, even in the thermal range of velocities.\*

The lack of reliable experimental data has led to the fact that at the present time we use more or less plausible assumptions about the form of the distribution function for reflected molecules, or the simplest theoretical models of interaction of molecules with a surface.†

\*F. M. Devienne, ed., "Rarefied Gas Dynamics," First Symposium, Pergamon Press, New York, 1960; H. Y. Wachman, ARS Journal, Vol. 32, No. 1 (1962); J. P. Hartnett, "Rarefied Gas Dynamics," Second Symposium, Academic Press, New York, 1961; F. C. Hurlbut, in: "Dynamics of Manned Lifting Planetary Entry," Wiley, New York, 1963.

†See, for example, B. Baule, Ann. Phys., 44: 145 (1914); A. I. Frenkel, Uspekhi Fiz.

From these or other considerations we postulate a function approximating the true distribution function for reflection of molecules, and containing a certain number of free parameters, i.e., we assume

$$f(t, \mathbf{x}, \xi_r) \equiv f_r(t, \mathbf{x}) = F(\xi_r, A_1, \dots, A_N), \quad (10.1)$$

where the  $A_i(t, \mathbf{x})$  depend on the properties of the surface and of the incident molecules, the surface temperature, etc. The function  $F$  must evidently satisfy the condition that the number of incident and reflected molecules is the same:

$$\int_{\xi \cdot \mathbf{n} \geq 0} F(t, \mathbf{x}, \xi_r) (\xi_r \cdot \mathbf{n}) d\xi_r = - \int_{\xi \cdot \mathbf{n} < 0} f(t, \mathbf{x}, \xi_l) (\xi_l \cdot \mathbf{n}) d\xi_l. \quad (10.2)$$

This condition determines one of the parameters  $A_i$ . The remaining parameters  $A_i$  may be chosen, for example, so that the function  $F$  correctly determines the total momentum, energy, and any further macroscopic characteristics of the reflected molecules. Their connection with the corresponding characteristics of the incident molecules is usually determined with the aid of an appropriate number of parameters, the so-called accommodation coefficients. The accommodation coefficients in turn are determined experimentally or with the help of the above-mentioned theoretical models of the interaction of the molecules with the surface.

One of the oldest and most widely used approximations to the distribution function for the reflected molecules is due to Maxwell.\*

Nauk, Vol. 20, No. 1 (1938); F. O. Goodman, J. Phys. Chem. Solids, 23: 1269 (1962); 24: 1451 (1963); 26: 85 (1965); F. O. Goodman, "Rarefied Gas Dynamics," Fourth Symposium, Academic Press, New York, 1966; L. Trilling, H. Y. Wachman, and P. B. Scott, Proceedings of the 6th Symposium on Fluid Mechanics, Polish Academy of Sciences, Zakopane, Poland, 1963; L. Trilling, J. Mecanique, Vol. 3, No. 2 (1964); R. A. Oman, A. Bogan, G. H. Weiser, and C. H. Li, AIAA Journal, Vol. 2, No. 10 (1964); R. A. Oman, A. Bogan, and C. H. Li, "Rarefied Gas Dynamics," Fourth Symposium, Academic Press, New York, 1966; R. G. Barantsev, in: "Aerodynamics of Rarefied Gases," Vol. 2, LGU, 1965; A. I. Erofeev, Inzhenernyi Zh., Vol. 4, No. 1 (1964); Vol. 5, No. 5 (1965); V. B. Leonas, Prikl. Mekh. i Tekhn. Fiz., No. 3, p. 71 (1965). A number of theoretical and experimental papers of recent years may be found in the collection "Interaction of Gases with Surfaces," [in Russian], Mir Press, 1965.

\*J. C. Maxwell, On the Stresses in Rarefied Gases, in "The Scientific Papers," Vol. 2, Paris, 1890.

It is constructed on the assumption that a fraction  $1 - \alpha_\tau$  of the molecules is reflected in a specular fashion from the surface, while the fraction  $\alpha_\tau$  is re-emitted diffusely, with the Maxwellian distribution

$$f_r(t, \mathbf{x}, \xi_r) = (1 - \alpha_\tau) f_i(t, \mathbf{x}, \xi_r - 2(\xi_r \cdot \mathbf{n}) \mathbf{n}) + \alpha_\tau n_r \left( \frac{h_r}{\pi} \right)^{3/2} e^{-h_r \xi_r^2}, \\ h_r = \frac{m}{2kT_r}, \quad (10.3)$$

i.e., the kernel in the integral (9.6) of the preceding paragraph has the form

$$K(\xi_i, \xi_r) = (1 - \alpha_\tau) \delta[\xi_i - (\xi_r - 2(\mathbf{n} \cdot \xi_r) \mathbf{n})] - \alpha_\tau \frac{2h_r^2}{\pi} e^{-h_r \xi_r^2} (\xi_i \mathbf{n}) \quad (10.3a)$$

Here  $\delta(x)$  is the Dirac delta-function,  $f_i$  is the distribution function of the molecules incident on the surface, and  $\alpha_\tau$ ,  $n_r$ , and  $T_r$  are free parameters. In the second term of (10.3a), condition (10.2) has been satisfied.

It is easy to see that the parameter  $\alpha_\tau$  gives the fraction of tangential momentum transferred to the wall by the incident molecules. In fact, the tangential momentum brought to the wall by the incident molecules is equal to

$$P_{it} = -m \int_{\xi \cdot \mathbf{n} < 0} f_i \xi_{it} (\xi_i \cdot \mathbf{n}) d\xi_i, \quad (10.4)$$

where  $\xi_{it}$  is the tangential component of the velocity of an incident molecule. The tangential momentum carried away by the reflected molecules, according to (10.3), is equal to

$$P_{rt} = m \int_{\xi \cdot \mathbf{n} > 0} f_r \xi_{rt} (\xi_r \cdot \mathbf{n}) d\xi_r = (1 - \alpha_\tau) P_{it}, \quad (10.5)$$

i.e.,  $\alpha_\tau P_{it}$  is the tangential momentum transmitted to the wall by the molecules. The parameter  $\alpha_\tau$  may be called the tangential momentum accommodation coefficient

$$\alpha_\tau = \frac{P_{it} - P_{rt}}{P_{it}}. \quad (10.6)$$

Evidently, for completely specular reflection,  $\alpha_\tau = 0$ , while for completely diffuse reflection,  $\alpha_\tau = 1$ .

Similarly, we may introduce an energy accommodation coefficient:

$$\alpha_e = \frac{E_i - E_r}{E_i - E_w}, \quad (10.7)$$

where  $E_i$  is the energy brought in by the incident molecules,  $E_r$  is the energy carried away by the reflected molecules, and  $E_w$  is the energy which would be carried away by the reflected molecules, if the gas were in equilibrium with the wall, i.e., when the parameter  $T_r$  in (10.3) is equal to the wall temperature  $T_w$ .

The accommodation coefficient  $\alpha_e = 0$  if the molecules do not exchange energy with the wall, and  $\alpha_e = 1$  if the incident molecules reach thermal equilibrium with the wall (they accommodate completely). In accordance with the general laws of mechanics, the energy exchanged with the surface decreases ( $\alpha_e$  is smaller), when the ratio of the mass of the incident molecules to that of molecules of the surface decreases.

The energy of the reflected molecules is equal to

$$E_r = \int_{\xi \cdot n > 0} \frac{m\xi_r^2}{2} (\xi_r \cdot n) f_r d\xi_r = (1 - \alpha_r) E_i + \alpha_r \frac{mn_r}{2\sqrt{\pi} h_r^{3/2}}, \quad (10.8)$$

where  $E_i$  is the energy of the incident molecules, equal to

$$E_i = \int_{\xi \cdot n < 0} \frac{m\xi_i^2}{2} (\xi_i \cdot n) f_i d\xi_i. \quad (10.9)$$

Similarly,

$$E_w = (1 - \alpha_r) E_i + \alpha_r \frac{mn_w}{2\sqrt{\pi} h_w^{3/2}}. \quad (10.10)$$

The number of molecules reflected per unit time from a unit area is equal to

$$N_r = \int_{\xi \cdot n > 0} (\xi_r \cdot n) f_r d\xi_r = (1 - \alpha_r) N_i + \alpha_r \frac{n_r}{2\sqrt{\pi} h_r}, \quad (10.11a)$$

where

$$N_i = \int_{\xi \cdot n < 0} (\xi_i \cdot n) f_i d\xi_i$$

is the number of molecules incident on this same surface per unit time.

Similarly, if the molecules are re-emitted after reaching equilibrium with the wall, i.e., when  $T_r = T_w$ , then

$$N_w = (1 - \alpha_r) N_i + \alpha_r \frac{n_w}{2\sqrt{\pi} h_w}. \quad (10.11b)$$

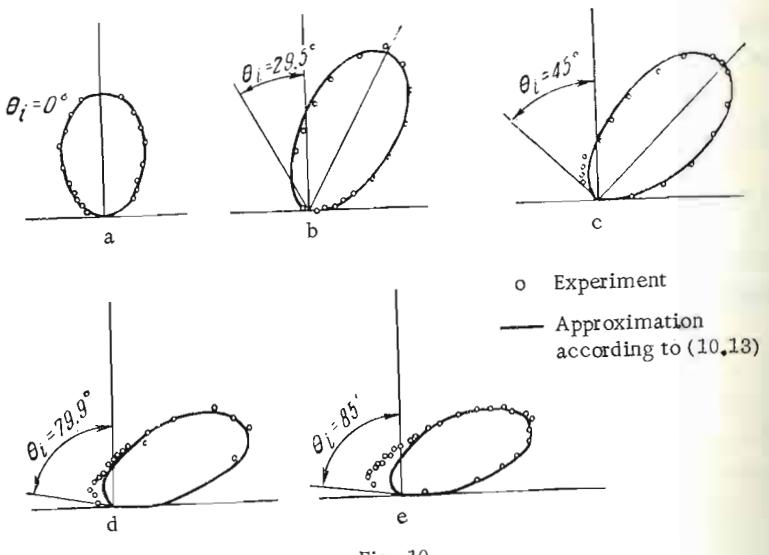
From condition (10.2) it follows that

$$N_i = N_r, \quad \text{or} \quad N_i = \frac{n_r}{2\sqrt{\pi} h_r}, \quad (10.12)$$

$$N_i = N_w, \quad \text{or} \quad N_i = \frac{n_w}{2\sqrt{\pi} h_w}.$$

If the wall temperature and the accommodation coefficients are given, then, from Eqs. (10.7) and (10.12), using relations (10.8)-(10.11), we can determine the parameters  $n_r$ ,  $n_w$ , and  $T_r$ , which appear in Eq. (10.3) for the distribution function for the reflected molecules.

It is clear that the approximation (10.3) may prove satisfactory only for a limited class of problems, not only because of the particular form of the approximation function, but also because the accommodation coefficients are assumed to be independent of the distribution function for the incident molecules. In addition, the approximation function (10.3) is inconsistent with the accommodation coefficients in the form (10.6) and (10.7). For example, let  $\alpha_r = 1$  and  $\alpha_e \neq 1$ . We examine two beams of molecules, incident on a surface with velocities  $\xi_1$  and  $\xi_2$ , respectively. If the density of the incident beams of molecules is not very large, the distribution function of the reflected molecules for each of the beams must not depend on the presence of the molecules of the other beam (we are concerned here with beams of molecules which have reached the wall, and with reflected molecules directly at the wall, so that collisions among the molecules cannot change our arguments). The distribution function of the reflected particles when the two beams strike the wall simultaneously must be equal to the sum of the distribution functions of the reflected molecules of each of the beams. However, it is easy to see that the function (10.3) with  $\alpha_e \neq 1$  does not satisfy this additive condition. In all three cases, the distribu-



tion function of the reflected molecules has the form

$$f_{rk} = \frac{2}{\pi} N_{ik} h_{rk} \exp \left\{ -h_{rk} \xi_r^2 \right\} \quad (k = 1, 2, 3),$$

where  $k = 1$  and  $k = 2$  refer to the beams with velocities  $\xi_1$  and  $\xi_2$ , while the subscript  $k = 3$  denotes the corresponding values when both beams are incident at once, and  $h_{rk}$  relates to the corresponding temperatures of the reflected molecules. Evidently,  $N_{i3} = N_{i1} + N_{i2}$ . From (10.7), we have

$$h_{rk} = \frac{\xi_{rk}^2}{2} (1 - \alpha_e) + \frac{\alpha_e}{h_w} \quad (k = 1, 2),$$

$$h_3 = \frac{N_{i1}\xi_1^2 + N_{i2}\xi_2^2}{2N_{i3}} (1 - \alpha_e) + \frac{\alpha_e}{h_w}.$$

It may be verified by direct substitution that

$$f_{r3} \neq f_{r1} + f_{r2}.$$

The approximation (10.3), with conditions (10.6) and (10.7), leads to a correct distribution function when the gas comes to equilibrium with the wall, which is a partial justification for it.

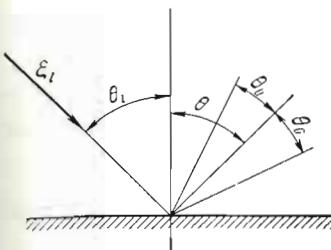


Fig. 11

From the physics of the phenomenon, it is easy to see – and this is confirmed experimentally\* – that the probability of re-emission of a molecule with a given velocity in a given direction depends on the velocity and angle of incidence of the incident molecule. Formula (10.3) may be improved by assuming, for example, that  $\alpha_\tau$  and  $\alpha_e$  depend on the angle of incidence and velocity of the molecules. The available experimental data show that the larger the angle of incidence  $\theta_i$  (the angle between the velocity vector and the normal to the surface), the more nearly the tangential momentum of the reflected molecules is conserved, i.e., the smaller  $\alpha_\tau$ . When  $\alpha_\tau = 1$ , the function of the reflected molecule is symmetrical with respect to the normal. If the number of molecules reflected in some direction is represented by a vector, then, according to (10.3), the length of the vector in this case describes a sphere (the cosine law). As the angle of incidence of the incident molecules increases, the distribution diagram for the reflected molecules begins more and more to resemble an ellipsoid as the re-emission angle increases. This may be allowed for in formula (10.3) by reducing  $\alpha_\tau$ . However, when the incident beam is monochromatic, for example, the distribution of reflected molecules will consist of two parts: molecules reflected according to the cosine law, and a parallel beam (a  $\delta$ -function on the diagram). The smaller  $\alpha_\tau$ , the larger the spike due to the specularly reflected molecules.

It is clear that we may approximate the experimental data better if, instead of assuming specular reflection of the molecules, we assume that the distribution function is Maxwellian, but with a certain macroscopic velocity†

$$f_r = n_r h_r^{3/2} \exp \left\{ -h_r (\xi - U_r)^2 \right\}. \quad (10.13)$$

\*See the references cited on page 89.

†H. Grad, "Handbuch der Physik," Vol. 12, 1958; S. Nocilla, "Rarefied Gas Dynamics," Third Symposium, Academic Press, New York, 1963.

Figure 10 shows an example copied from Nocilla's paper, where (by choosing the parameters  $h_r$  and  $T_r$ ) this formula has been used to correlate the results of an experiment on the reflection of thermal velocity nitrogen molecules from a clean LiF crystal.

The available experimental data may also be approximated by the distribution function\*

$$f_r = \int_{\xi \cdot n < 0} W(\xi_i, \xi_r) \frac{\xi_i \cos \theta_i}{\xi_r \cos \theta_r} f_i d\xi_i \\ = \begin{cases} \int_{\xi \cdot n < 0} \bar{n}_r \left( \frac{h_r}{\pi} \right)^{3/2} e^{-h_r \xi_r^2} \cos \left( \frac{\pi}{2} \frac{\theta_r - \theta}{\theta_0} \right) \frac{\xi_i \cos \theta_i}{\cos \theta_r} f_i d\xi_i & \text{for } \theta_r - \theta < \theta_0, \\ 0 & \text{for } \theta_r - \theta > \theta_0, \end{cases} \\ \sin \theta = (\sin \theta_i)^v, \quad (10.14)$$

where  $\theta_i$  and  $\theta_r$  are the trajectory angles of the incident and reflected molecules,  $\theta$  is the inclination of the axis of the cone of scatter of the reflected molecules,  $\theta_0$  is the limiting angle of the cone of scatter (Fig. 11). For a given surface and given incident molecules, the three parameters  $\theta_0$ ,  $v$ , and  $T_r$  are functions of  $\xi_r$  and  $\theta_i$ . The fourth parameter  $\bar{n}_r$  is determined by condition (9.4) of the preceding section:

$$\int_{\xi \cdot n > 0} W(\xi_i, \xi_r) d\xi_r = \int_{\xi \cdot n > 0} \bar{n}_r \left( \frac{h_r}{\pi} \right)^{3/2} e^{-h_r \xi_r^2} \xi_r \cos \left( \frac{\pi}{2} \frac{\theta_r - \theta}{\theta_0} \right) d\xi_r = 1. \quad (10.15)$$

If  $\theta_0 = \pi/2$ ,  $\theta = 0$  (i.e.,  $v = \infty$ ), and if  $\bar{n}_r$  and  $T_r$  do not depend on  $\xi_i$ , the function (10.14) becomes the Maxwellian distribution

$$f_r = n_r \left( \frac{h_r}{\pi} \right)^{3/2} e^{-h_r \xi_r^2},$$

where, according to (10.14) and (10.15),

$$n_r = \bar{n}_r \int_{\xi \cdot n < 0} \xi_i \cos \theta_i f_i d\xi_i = 2 \sqrt{\pi h_r} \int_{\xi \cdot n < 0} \xi_i \cos \theta_i f_i d\xi_i.$$

The parameters  $\theta_0(\xi_i, \theta_i)$ ,  $v(\xi_i, \theta_i)$ , and  $T_r(\xi_i, \theta_i)$  in formula (10.14) may be determined if any three macroscopic charac-

teristics of the reflected molecules, are specified, for example, the fraction of normal and tangential momentum and energy given to the surface by molecules incident with velocity  $\xi_i$  at angle  $\theta_i$ .

The fraction of momentum and energy imparted is described conveniently by accommodation coefficients, which take the form

$$\alpha_t(\xi_i, \theta_i) = \frac{P_{tr} - P_{rr}}{P_{tr}}, \quad \alpha_n(\xi_i, \theta_i) = \frac{P_{in} - P_{rn}}{P_{in}}, \\ \alpha_e(\xi_i, \theta_i) = \frac{E_l - E_r}{E_l}, \quad (10.16)$$

where

$$P_{tr} = N_t m \xi_i \sin \theta_i, \quad P_{in} = N_t m \xi_i \cos \theta_i, \\ E_l = \frac{1}{2} m \xi_i^2 N_i$$

are, respectively, the tangential and normal momentum and energy brought in by molecules incident with velocity  $\xi_i$  at an angle  $\theta_i$  with the normal to the surface;  $N_i$  is the number of molecules incident on unit area at that velocity:

$$P_{rr} = m \int_{\xi \cdot n > 0} \xi_r^2 \cos \theta_r \sin \theta_r f_r d\xi_r, \quad P_{rn} = m \int_{\xi \cdot n > 0} \xi_r^2 \cos^2 \theta_r f_r d\xi_r, \\ E_r = \frac{m}{2} \int_{\xi \cdot n > 0} \xi_r^3 \cos \theta_r f_r d\xi_r, \quad (10.17)$$

are the momentum and energy carried away from the surface by the molecules. If  $\alpha_t(\xi_i, \theta_i)$ ,  $\alpha_n(\xi_i, \theta_i)$ , and  $\alpha_e(\xi_i, \theta_i)$  are known then, substituting the distribution function (10.14) for an incident beam of molecules with the same velocity into (10.16), we obtain three equations to determine the three parameters  $v(\xi_i, \theta_i)$ ,  $\theta_0(\xi_i, \theta_i)$ , and  $T_r(\xi_i, \theta_i)$ .

The accommodation coefficients, given in the form (10.16), are also convenient for calculating free-molecular flow over blunt bodies (see § 6.1). In that calculation we are not interested in the distribution function of the reflected molecules. We only require the momentum and energy transmitted to the surface by the incident molecules. In that case a knowledge of the accommodation coefficients (10.16) solves the problem completely, since  $\alpha_t P_{tr}$ ,

\*R. Schamberg, Preprint, Heat Transfer and Fluid Mechanics Inst., Stanford, 1959.

$\alpha_n P_{in}$ , and  $\alpha_e E_i$  give precisely the momentum and energy transmitted to the wall by molecules with velocities  $\xi_i$ . Then the total momentum and energy transmitted to the wall by incident molecules of all velocities and directions, are equal to

$$\begin{aligned} P_\tau &= m \int_{\xi \cdot n < 0} \alpha_\tau(\xi_i, \theta_i) \xi_i \xi_i \cos \theta_i f_i d\xi_i, \\ P_n &= m \int_{\xi \cdot n < 0} \alpha_n(\xi_i, \theta_i) \xi_i^2 \cos^2 \theta_i f_i d\xi_i, \\ E &= \frac{m}{2} \int_{\xi \cdot n < 0} \alpha_e(\xi_i, \theta_i) \xi_i^3 \cos \theta_i f_i d\xi_i. \end{aligned} \quad (10.18)$$

We have been concerned above only with the energy of translational motion of the molecules. When a monatomic molecule collides with a wall, its internal energy may be both decreased or increased. The internal energy of the k-th species of reflected molecule may be represented, in the general case, by

$$E_r^k = \sum_j \alpha_{jk} E_i^j + \beta_k \xi_i^2 + \gamma_k T_w. \quad (10.19)$$

The coefficient  $\alpha_{jk}$  here defines the fraction of internal energy of class j which is transformed upon collision with the wall into energy of class k. The coefficient  $\beta_k$  determines the fraction of the translational energy of a molecule transformed into internal energy of type k. Finally, the coefficient  $\gamma_k$  determines the fraction of the wall energy going into the excitation of internal energy of class k. In a single collision a molecule is most likely to give up or acquire rotational energy.

In contrast with the accommodation coefficients examined above, the coefficients just introduced may be both positive and negative.

In analogy with (10.7), two more accommodation coefficients may be introduced as crude average characteristics:

$$E_r^i = E_i^i - \alpha_e^i (E_i^i - E_w^i) + \alpha_e^T (E_i - E_w). \quad (10.20)$$

The superscript i denotes the value of the internal energy;  $\alpha_e^i$  and  $\alpha_e^T$  are the new accommodation coefficients.

Although the accommodation coefficients  $\alpha_\tau$ ,  $\alpha_n$ ,  $\alpha_e$ , and especially  $\alpha_e^i$ ,  $\alpha_e^T$ , and the probability of reflection  $W(\xi_i, \xi_r)$  have not yet been sufficiently studied, we assume everywhere below that the nature of the interaction of molecules with a surface is known, and we use one of the above approximations for specific calculations.

### §2.11. Similarity Criteria

We bring the Boltzmann equation

$$\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} + \frac{X_i}{m} \frac{\partial f}{\partial \xi_i} = \int (f' f'_i - f f'_i) g b db de d\xi_i \quad (11.1)$$

to dimensionless form, by referring the variables appearing in Eq. (11.1) to their respective characteristic quantities; specifically,

$$\left. \begin{aligned} t &= \bar{t}\Theta, & x_i &= \bar{x}_i L, & \xi_i &= \bar{\xi}_i U, & b &= \bar{b}R, & g &= \bar{g}G, \\ X_i &= \bar{X}_i X, & f &= \bar{f}\Phi, & \varepsilon &= \bar{\varepsilon}\pi. \end{aligned} \right\} \quad (11.2)$$

Here  $\Theta$  is a characteristic time of the flow; L is a characteristic length; U is the mean velocity of the molecules, determined by the expression

$$U^2 = \frac{1}{n} \int \xi^2 f d\xi;$$

G is a mean peculiar velocity:

$$G = \frac{1}{n^2} \int |\xi - \xi_1| f f_1 d\xi d\xi_1;$$

X is a characteristic external force, acting on a molecule; R is a dimension characteristic of molecular collisions, equal to the diameter of the molecules for hard spheres, for example; and  $\Phi$  is a characteristic value of the distribution function, equal to

$$\Phi = \frac{n_0}{c^3},$$

where  $n_0$  is a characteristic number of molecules per unit volume, and c is the mean thermal velocity of the molecules, since the characteristic volume of velocity space, in which the distribution function is defined, does not depend on the velocity of the gas as a

whole. Evidently  $c^3$  is the characteristic volume of velocity space over which the integration of the collision integral is carried out. Usually,  $G$  and  $c$  are of the same order.

By substituting the dimensionless quantities (11.2) into the Boltzmann equation (11.1), we obtain (dropping the bars over the dimensionless quantities)

$$\frac{L}{U\Theta} \frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} + \frac{XL}{mU^2} X_i \frac{\partial f}{\partial \xi_i} = \frac{\pi R^2 n_0 G L}{U} \int (f' f'_i - f f_i) g b \, db \, d\epsilon \, d\xi_i \quad (11.3)$$

or

$$Sh \frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} + \frac{1}{Fr} X_i \frac{\partial f}{\partial \xi_i} = \frac{1}{Kn} \int (f' f'_i - f f_i) g b \, db \, d\epsilon \, d\xi_i. \quad (11.4)$$

It is easy to see that\*

$$Sh = \frac{L}{U\Theta} \quad (11.5)$$

is the analog of the Strouhal number, while

$$Fr = \frac{mU^2}{XL} \quad (11.6)$$

is the Froude number. If the flow occurs in a gravity field, then  $X/m = g_T$  is the acceleration due to gravity, and the Froude number takes the form it usually has in hydrodynamics:

$$Fr = \frac{U^2}{g_T L}.$$

The third similarity parameter appearing in the Boltzmann equation is called the Knudsen number:

$$Kn = \frac{U}{\pi R^2 n_0 G L}. \quad (11.7)$$

Since  $\pi R^2 = \sigma$  is the collision cross section, then (see §1.4),

$$\pi R^2 n_0 G = \sigma n_0 G = v$$

is equal to the number of collisions experienced by a molecule per unit time, while  $v^{-1} = \tau$  is the time between collisions. Therefore,

\*See, for example, L. I. Sedov, "Similarity and Dimensional Methods in Mechanics" [in Russian], Nauka Press, Moscow, 1965.

the Knudsen number may be represented in the form

$$Kn = \frac{\tau U}{L} = \frac{\lambda}{L}, \quad (11.8)$$

where  $\lambda$  is the mean free path of the molecules.

The characteristic velocities  $U$  and  $G$  of the molecules have been introduced to make the corresponding dimensionless quantities of order unity. If we examine a gas flow of Mach number  $M = V/a \leq 1$  the characteristic velocity of the molecules is the thermal velocity, i.e.,  $U \sim G \sim c \sim a$ . When  $M > 1$  we should take  $U \sim V$ , and  $G \sim c \sim a$ .

The Knudsen numbers (and also the mean free paths) are different in different inertial coordinate systems. Let us suppose that we examine the flow over a body in a system of coordinates fixed in the body, with Mach number  $M > 1$ . Then the relevant parameter which defines the relative magnitude of the terms of the Boltzmann equation is the Knudsen number (11.7) in which  $U \sim V$  and, therefore, the mean free path  $\lambda$  is defined in the system of coordinates referred to the body. However, for the character of the oncoming stream, it is usual to use the Knudsen number  $Kn_\infty$  based on the mean free path in the coordinate system fixed in the oncoming stream (the mean free path  $\lambda_\infty$  at the given altitude). In that coordinate system,

$$Kn_\infty \sim \frac{c}{n_0 \sigma G L} \sim \frac{1}{n_0 \sigma L} \sim \frac{\lambda_\infty}{L} \sim \frac{Kn}{M}.$$

The Knudsen number may be related to the Reynolds number. As has been shown in §1.5, the viscosity is equal to

$$\mu \sim \rho a \lambda_\infty.$$

Then

$$Kn_\infty = \frac{\mu}{a \rho L} \frac{V}{V} = \frac{M}{Re}. \quad (11.9)$$

If the molecules interact according to a potential power law  $K/r^{s-1}$ , then the characteristic linear dimension of the collision process is equal to (see §1.3)

$$R = \left( \frac{4(s-1)K}{mG^2} \right)^{1/(s-1)}.$$

In that case, the Knudsen number takes the form

$$Kn = \frac{\frac{2}{s-1}-1}{\pi \left( \frac{4(s-1)K}{m} \right)^{2/(s-1)} n_0 L} \frac{U}{n_0 L}. \quad (11.10)$$

Similarity criteria for mixtures of gases and for gases with internal degrees of freedom are obtained in the same manner. In general, each component of a mixture has different characteristic collision cross sections with molecules of a different type. Therefore, for similarity to prevail for a mixture, it is necessary that the several Knudsen numbers be equal.

For a gas consisting of hard spheres, when there is a change in the L-scale, similarity of the flow (equality of the Knudsen numbers) may be achieved by a change in the diameter  $\sigma$  of the spheres (by a change in the kind of gas), by a change in the gas density  $n_0$ , and also by a change in the ratio  $U/G$ . In the special case when we are simulating a flow in the same gas (i.e., with unchanged collision cross section  $\sigma$ ), and for a fixed value of  $U/G$ , it is necessary that the product  $L n_0$  be the same in both flows. It is evident that this condition is sufficient for similarity of flows of an arbitrary mixture, when the composition of both gases is the same.

In flows with macroscopic gas velocities less than, or of the order of, thermal velocities (subsonic and transonic flows),  $U \sim G$ . Therefore, in such flows we cannot use a change in velocity to preserve similarity. In hypersonic flows ( $U \gg G$ ), and in certain other flows which will be examined in more detail in Chapter VI,  $U$  and  $G$  may be varied independently. To achieve similarity in such flows, we may change the scale of velocities (for example, a change in the velocity and temperature of the oncoming stream).

In a gas consisting of molecules which interact according to a power law, the similarity flow conditions may be attained not only by change in the scale  $L$  of the flow, in the density  $n_0$ , and in the kind of gas (change of  $K$ ), but also by a change in the scale of the velocities  $G$  and  $U$  of the gas molecules in all the flow regimes. In practice, in order to model very low pressure flow regimes which are hard to attain, a change in the scale of the flow is often used.

The Knudsen number characterizes the degree of rarefaction of a gas. At very large Knudsen numbers, the role of molecular collisions becomes negligibly small, and in the limit, when  $Kn \rightarrow \infty$  we may neglect the collision integral. Flows in which collisions of the molecules may be neglected are called free-molecule flows. For free-molecule flows, the Boltzmann equation takes the form

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} - \frac{X_i}{m} \frac{\partial f}{\partial \xi_i} = 0, \quad (11.11)$$

i.e., the distribution function is conserved along the phase trajectory of the molecules. In the other limiting case, when  $Kn \rightarrow 0$ , the distribution function is mainly determined by collisions. As will be seen later, in that case the flow is amenable to a hydrodynamic description (see Chapter III).

In the derivation of the similarity criteria above, we did not consider the initial and boundary conditions. In order to preserve the similarity of flows in the presence of boundaries, additional criteria must be satisfied. We examine, as an example, the flow past a body of characteristic dimension  $L$  of an unbounded gas in equilibrium moving at a velocity  $\mathbf{U}$ . Then the distribution function at infinity is [see formula (9.1)]

$$f_\infty = n_\infty \left( \frac{h_\infty}{\pi} \right)^{3/2} \exp \{-h_\infty(\xi - \mathbf{U})^2\}. \quad (11.12)$$

Choosing the velocity  $h_\infty^{-1/2} = (2kT_\infty/m)^{1/2}$  as a characteristic velocity, and  $n_\infty$  as a characteristic molecular density, we find that three dimensionless parameters appear in (11.12):  $Uh^{1/2}$ , and two angles which define the direction of the velocity of the oncoming stream. The first parameter coincides with the Mach number up to a constant multiplying factor.

On the boundary of the body, condition (9.6) must be satisfied:

$$f_r = \int K(\xi_i, \xi_r) f_i d\xi_i. \quad (11.13)$$

The function  $K$  depends on the coordinates of the point  $\mathbf{x}_w$  of the body, the velocities  $\xi_i, \xi_r$  of the incident and reflected molecules, and the temperature  $T_w$  of the body, the mass  $m_w$  of the molecules of the body, the crystal lattice spacing  $l$ , the effective diameter  $d$  of the collision of the gas molecules with molecules on the surface,

and other parameters which describe the physical and chemical properties and the state of the surface:

$$K = h^{-3/2} \mathcal{H} \left( \frac{x_w}{L}, \xi_r h^{-1/2}, \xi_l h^{-1/2}, \frac{T_w}{T_\infty}, \frac{l}{d}, \frac{m}{m_w}, \dots \right), \quad (11.14)$$

where  $\mathcal{H}$  is a dimensionless function.

Thus, for flow similarity, it is necessary that the Mach and Knudsen numbers be equal for similar flow geometry (or, the Mach and Reynolds numbers) and also the temperature ratio  $T_w / T_\infty$ , etc. If the gas composition and the properties of the body surface are identical, then, to have similar flows, it is sufficient to have equality only of the Mach and Reynolds numbers and the temperature ratios  $T_w / T_\infty$ , as in the hydrodynamic approximation.

### Chapter III

## GENERAL METHODS OF SOLUTION OF THE BOLTZMANN EQUATION

### §3.1. The Conservation Equations

The Boltzmann equation derived in the previous chapter determines the behavior of the distribution function  $f(t, \mathbf{x}, \xi)$ , which is a function of seven independent variables. We know that equations become markedly more difficult to solve as the number of independent variables increases. On the other hand, we saw in §2.1 that a detailed microscopic description by means of a distribution function is unnecessary in the majority of problems. It is therefore natural to attempt to go to a less detailed description using the macroscopic hydrodynamic quantities introduced in §2.1 [see formulas (1.4)-(1.10)].

Let us multiply both sides of the Boltzmann equation [Eq. (2.9) of Chapter II] by some function of velocity  $\varphi(\xi)$  and integrate over all possible molecular velocities ( $-\infty \leq \xi \leq \infty$ ):

$$\int \varphi(\xi) \left( \frac{\partial f}{\partial t} + \xi_l \frac{\partial f}{\partial x_l} + \frac{X_l}{m} \frac{\partial f}{\partial \xi_l} \right) d\xi = I_\varphi, \quad (1.1)$$

where  $I_\varphi$  is the integral introduced in §2.4. Let us transform the left side of (1.1). Since the limits of integration with respect to  $\xi$  do not depend on time, we may interchange the order of the integration and differentiation and we obtain

$$\int \varphi(\xi) \frac{\partial f}{\partial t} d\xi = \frac{\partial}{\partial t} \int \varphi f d\xi. \quad (1.2)$$

Similarly,

$$\int \varphi(\xi) \xi_i \frac{\partial f}{\partial x_i} d\xi = \frac{\partial}{\partial x_i} \int \varphi \xi_i f d\xi \quad (1.3)$$

and

$$\int \varphi(\xi) \frac{\partial f}{\partial \xi_i} d\xi_i d\xi_j d\xi_k = \int \varphi f \left|_{\xi_i=-\infty}^{\xi_i=\infty} \right. d\xi_j d\xi_k - \int f \frac{\partial \varphi}{\partial \xi_i} d\xi. \quad (1.4)$$

It is evident that when the gas energy is finite, there must be very few molecules with very large velocities. Therefore, the function  $f$  must tend to zero at least as rapidly as  $\xi^{-2}$  when  $\xi \rightarrow \infty$ . For a Maxwellian distribution  $f \sim e^{-A^2 \xi^2}$ . It will be assumed everywhere in what follows that the quantity  $\varphi f$  tends to zero sufficiently rapidly so that the first term on the right of (1.4) can be omitted. Then (1.1) can be written in the form

$$\frac{\partial}{\partial t} \int \varphi f d\xi + \frac{\partial}{\partial x_i} \int \xi_i \varphi f d\xi - \frac{X_i}{m} \int f \frac{\partial \varphi}{\partial \xi_i} d\xi = I_\varphi. \quad (1.5)$$

This is the general transport equation.

Let us take the summational invariants  $m$ ,  $m\xi_i$ , and  $\frac{1}{2}m\xi^2$  as the functions  $\varphi$ . It was shown in §2.4 that  $I_\varphi \equiv 0$  for the summational invariants. Taking into account the definition of the hydrodynamic quantities, we transform the integrals appearing in (1.5):

$$m \int \xi_i \xi_j f d\xi = m \int (c_i + u_i)(c_j + u_j) f d\xi = P_{ij} + \rho u_i u_j, \quad (1.6)$$

$$\begin{aligned} \frac{m}{2} \int \xi_i \xi^2 f d\xi &= \frac{m}{2} \int (c_i + u_i)(c^2 + 2c_k u_k + u^2) f d\xi \\ &= q_i + u_j P_{ij} + \rho u_i \left( \frac{3}{2} RT + \frac{1}{2} u^2 \right), \end{aligned} \quad (1.7)$$

where  $R = k/m$  is the gas constant; summation is carried out over the repeated subscripts.

When  $\varphi = m$ , we obtain the equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0. \quad (1.8)$$

When  $\varphi = m\xi_i$ , we obtain the equation of conservation of momentum

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_i u_j + P_{ij}) - \frac{X_i}{m} \rho = 0, \quad (1.9)$$

or, using (1.8),

$$\left( \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \right) u_i = - \frac{1}{\rho} \frac{\partial P_{ij}}{\partial x_j} + \frac{X_i}{m}. \quad (1.9a)$$

Finally, when  $\varphi = \frac{1}{2}m\xi^2$ , we obtain the equation of conservation of energy

$$\begin{aligned} \frac{\partial}{\partial t} \rho \left( \frac{3}{2} RT + \frac{1}{2} u^2 \right) + \frac{\partial}{\partial x_j} \left[ \rho u_j \left( \frac{3}{2} RT + \frac{1}{2} u^2 \right) \right. \\ \left. + u_k P_{kj} + q_j \right] - X_i n u_i = 0, \end{aligned} \quad (1.10)$$

or, taking into account (1.8) and (1.9),

$$\frac{3}{2} R \rho \left( \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \right) T = - \frac{\partial q_j}{\partial x_j} - P_{ij} \frac{\partial u_i}{\partial x_j}. \quad (1.10a)$$

Thus, to determine the thirteen hydrodynamic quantities  $\rho$ ,  $u_i$ ,  $P_{ij}$ , and  $q_i$ , we have only five equations (the temperature  $T$  is expressed in terms of  $P_{ij}$ ; see §2.1), i.e., the system of equations (1.8)-(1.10) is not closed. To close the system we must find additional relations between the quantities appearing in the equations.

In hydrodynamics, the stress tensor  $P_{ij}$  and the heat flux tensor  $q_i$  are expressed in terms of velocity components, density, temperature, and their first derivatives, under the assumption that the stress tensor is proportional to the rate of strain tensor, and that the heat flux vector is proportional to the temperature gradient. This allows us to close the system of equations (1.8)-(1.10). However, as we shall see later, the linear relation between the stress tensor and the heat flux and the gradients of the hydrodynamic quantities is a very special case and is valid only for flows with small Knudsen numbers, i.e., for flows that are close to local equilibrium. In the general case, the flow cannot be described by means of only the hydrodynamic quantities, and the system of equations (1.8)-(1.10) cannot be closed. It is therefore necessary to introduce new functions to describe the flow and to construct equations which they must satisfy under given conditions. Generally speaking, for any flow we can find a finite or an infinite set of macroscopic functions to describe the flow with varying degrees of accuracy, and construct the equations governing them; in other words, we can construct an appropriate macroscopic model of a continuous medium which behaves in some respects in a way similar to a gas composed of molecules. (Since a molecular gas is a

system with an infinite number of degrees of freedom, the corresponding continuous medium which simulates the behavior of the gas in all respects must be defined by an infinite number of parameters.)

We examine below methods of constructing models of continuous media by means of the kinetic Boltzmann equation, i.e., methods of finding the required number of parameters to determine the flow and of constructing the equations which govern them. In principle, the corresponding equations for the macroscopic quantities can also be constructed from phenomenological (macroscopic) considerations, omitting the kinetic stage.\* However, the kinetic coefficients which appear in those equations (the viscosity, thermal conductivity, diffusion coefficient, etc.) cannot be determined from phenomenological theories, and additional considerations or experiments are required to determine them. For example, for a phenomenological derivation of the Navier-Stokes equations, under the assumptions that the stress tensor components are proportional to the rate of strain tensor components, we must introduce 81 unknown coefficients of proportionality. By making the additional assumptions that the medium is isotropic and homogeneous, we can express all these coefficients in terms of two viscosities which must be found in the final analysis by experiment or by means of kinetic theory. As will be seen from what follows, in contrast to the phenomenological theories, the kinetic approach, based on the Boltzmann equation, permits us not only to obtain the macroscopic equations of motion of the medium, but also to express the coefficients which appear in them in terms of the molecular properties of the gas. On the other hand, because of their generality, phenomenological theories allow us to construct equations (models) for continuous media, for which a kinetic theory has not yet been developed.

### §3.2. The Method of Moments

The most universal method which allows us to close the system of macroscopic equations for arbitrary Knudsen numbers in principle is the method of moments.

\*The general phenomenological theory for the construction of models of continuous media on the basis of variational principles has been developed by Sedov. See L. I. Sedov, Uspekhi Mat. Nauk, Vol. 20, No. 5 (1965); Doklady Akad. Nauk SSSR, Vol. 165, No. 3 (1965).

The moment of the  $N$ -th order of the distribution function is defined by the components of a tensor of rank  $N$  as follows:

$$M_{\alpha_1 \dots \alpha_N}^{(N)} = \int \prod_{\beta=1}^N \xi_{\alpha_\beta} f d\xi, \quad \mathcal{M}_{\alpha_1 \dots \alpha_N}^{(N)} = \int \prod_{\beta=1}^N c_{\alpha_\beta} f d\xi \quad (2.1)$$

$$(a_\beta = 1, 2, 3; c_{\alpha_\beta} = \xi_{\alpha_\beta} - u_{\alpha_\beta}).$$

It is evident that the  $M$ -moments of order  $N$  may be expressed in terms of  $\mathcal{M}$  moments of order  $N$  and lower, and vice versa:

$$\begin{aligned} \mathcal{M}_{ij} &= M_{ij} - n u_i u_j, \\ \mathcal{M}_{ijk} &= M_{ijk} - u_i M_{jk} - u_j M_{ki} - u_k M_{ij} + 2 n u_i u_j u_k, \\ \mathcal{M}_{ijkl} &= M_{ijkl} - u_i M_{jkl} - u_j M_{kil} - u_k M_{lij} - u_l M_{ijk} \\ &\quad + u_i u_j M_{kl} + u_i u_k M_{il} + u_i u_l M_{jk} + u_i u_k M_{il} \\ &\quad + u_j u_l M_{ik} + u_k u_l M_{ij} - 3 n u_i u_j u_k u_l. \end{aligned} \quad (2.2)$$

The hydrodynamic quantities introduced in §2.1 are also moments:

$$n(t, \mathbf{x}) = M^0 = \int f d\xi, \quad (2.3)$$

$$n u_i(t, \mathbf{x}) = M_i^{(1)} = \int \xi_i f d\xi \quad (i = 1, 2, 3), \quad (2.4)$$

$$P_{ij}(t, \mathbf{x}) = m \mathcal{M}_{ij}^{(2)} = m \int c_i c_j f d\xi, \quad (2.5)$$

$$q_i = \frac{m}{2} \mathcal{M}_{ijj}^{(3)} = \frac{m}{2} (\mathcal{M}_{i11} + \mathcal{M}_{i22} + \mathcal{M}_{i33}) = \int \frac{m}{2} c_i c^2 f d\xi. \quad (2.6)$$

The continuity equation (1.8) involves moments of order zero and one, the equation of motion (1.9) involves moments of order zero, one, and two, and finally, the energy equation has all the moments (2.3)–(2.6) including a contraction of the third-order moments

$$q_i = \frac{1}{2} m \mathcal{M}_{i11} = \frac{1}{2} m (\mathcal{M}_{i11} + \mathcal{M}_{i22} + \mathcal{M}_{i33}).$$

It is easy to make up equations for  $P_{ij}$ ,  $q_i$ , and all the remaining moments  $M^{(3)}$ , the moments  $M^{(4)}$ , etc. To that effect it is sufficient to replace  $\varphi$  in the general transport equation (1.5) by  $\xi_i \xi_j$ ,  $\xi_i \xi_j^2$ ,  $\xi_i \xi_j \xi_k$ , respectively, etc. Then moments of order  $(N + 1)$  appear on the left side of the equations for moments of order  $N$ . Therefore, the Boltzmann equation may be replaced only by an infinite system of simultaneous equations. The summational invariants  $\psi_i$  were chosen as the functions  $\varphi(\xi)$  in the derivation of Eqs. (1.8)–(1.10). In that case, the integrals  $I_\varphi$  vanish identically. In constructing the transport equations for moments of higher order, the integrals  $I_\varphi$  do not in general go to zero, and in the equation for the moments of order  $N$  there appear not only the unknown moments of order  $(N + 1)$ , but also the unknown integrals  $I_\varphi$ . Therefore, we must first express the integrals  $I_\varphi$  in terms of the moments.

Let us represent the distribution function in the form\*

$$f(t, \mathbf{x}, \xi) = F(\xi, M^{(0)}, M^{(1)}, \dots), \quad (2.7)$$

where the moments  $M^{(N)}$  are functions of  $t$  and  $\mathbf{x}$ . Then, by replacing the distribution function  $f$  given by formula (2.7) in the integrals  $I_\varphi$ , we obtain an infinite system of simultaneous equations for an infinite array of moments. For some very special flows and particular laws of interaction, the system of equations terminates, and the flow is described by a finite number of moments (see §4.1). But, in general, the Boltzmann equation is equivalent only to an infinite system of macroscopic equations. Systems of equations of such complexity are clearly unsuitable for the solution of practical problems.

On the basis of some plausible considerations, an approximate solution of the Boltzmann equation may be sought by representing the distribution function by a finite number of moments. For example, for a given problem, let the distribution function be approximated by the expression

$$f(t, \mathbf{x}, \xi) = F(\xi, A_1, \dots, A_N), \quad (2.8)$$

where  $A_i(t, \mathbf{x})$  are macroscopic quantities depending on  $t$  and  $\mathbf{x}$ . The specific form of the function  $F$  is determined by the nature of

\*See, for example, M. Krook, J. Fluid Mech., Vol. 6, No. 4 (1959).

the flow under examination, the boundary conditions, and the required accuracy of the approximation. Examples of such approximations are given in §§3.3, 3.4, 4.2, and 4.3. By means of the approximating functions (2.8) we express a moment of any order in terms of the function  $A_i$ . Let any  $N$  moments\*  $M_\alpha$  or  $A_\alpha$  be expressed in terms of  $N$  functions  $A_\alpha$ . In principle, we may invert this relationship and express the functions  $A_\alpha$ , and therefore also  $f$ , in terms of the  $N$  chosen moments:

$$f(t, \mathbf{x}, \xi) = F(\xi, M_{\alpha_1}, M_{\alpha_2}, \dots, M_{\alpha_N}). \quad (2.9)$$

We may then express other moments and integrals  $I_\varphi$  in terms of the  $N$  moments. We take any  $N$  moment equations and express all the moments and integrals  $I_\varphi$  which appear in them in terms of the chosen  $N$  moments  $M_\alpha$ . Thus, we obtain  $N$  simultaneous differential equations for the  $N$  moments.

It is clear from that method that there is a great degree of arbitrariness, both in the choice of the approximating function and in the choice of the moments and of the equations which they satisfy, since we may use any  $N$  moment equations for the chosen  $N$  moments.

Since, in the final analysis, we are usually interested only in the hydrodynamic quantities (the first thirteen moments), it is convenient to include these hydrodynamic quantities among the  $N$  determining moments. The five equations of conservation of mass, momentum, and energy [Eqs. (1.8)–(1.10)] are of course included in the moment equations. The choice of additional moments and equations is usually made more or less arbitrarily from considerations of simplicity of solution.

If the approximating function which contains the  $N$  moments is an exact solution of the Boltzmann equation, the solutions of any system composed of  $N$  arbitrarily chosen moment equations are identical. Therefore, the more accurately the chosen function approximates an exact solution of the Boltzmann equation, the less divergence there will be between solutions obtained by different choices of the system of moment equations. We may use this fact to assess the accuracy of a chosen approximating function. To do

\*There may be moments of any order among these  $N$  moments.

this it is sufficient to obtain two solutions with a single approximating function but with a different choice of moment equations. The difference between the solutions thus obtained is a measure of their accuracy.

That lack of uniqueness in choice of the moment equations may be avoided using a method proposed by Tamm.\* The method is similar to the methods of Ritz and Galerkin.

We examine the functional

$$\mathcal{L} = \int \left[ \xi_i \frac{\partial f}{\partial x_i} + \frac{X_i}{m} \frac{\partial f}{\partial \xi_i} - J(f, f) \right]^2 dx d\xi. \quad (2.10)$$

The functional  $\mathcal{L}$  is positive and goes to zero if the function  $f$  satisfies the Boltzmann equation. The form of the approximating function is chosen by requiring that the moments  $M_i(x)$  or the coefficients  $A_i(x)$  which appear in it cause the first variation of the functional (2.10) to go to zero:

$$\delta \mathcal{L} = 0. \quad (2.11)$$

We expect that the functions  $M_i$  or  $A_i$ , which correspond to the minimum of the functional  $\mathcal{L}$  ensure the best approximation to the solution on the average for the chosen form of the approximating function.

### §3.3. The Method of Moments. Expansion of the Distribution Function in Hermite Polynomials

The distribution function  $f(t, x, \xi)$  may be represented in the form (2.7), for example, if we expand  $f$  at each point in the flow by a series of functions of  $\xi$  whose coefficients depend on  $t$  and  $x$ ; we establish the connection between these coefficients and the moments.

1. We know, for example, that a wide class of functions may be expanded, in terms of Hermite polynomials.† Following Grad‡

\*I. E. Tamm, Tr. Fiz. Inst. Akad. Nauk SSSR, No. 29 (1965). The work was done in 1947.

†The choice of Hermite polynomials is mainly for mathematical convenience. If we use polar coordinates  $(\xi, \theta, \varphi)$  instead of rectangular coordinates  $(\xi_1, \xi_2, \xi_3)$  for the molecular velocities, an expansion in terms of Sonine polynomials is more convenient (see §§ 3.8 and 3.11).

‡H. Grad, Commun. Pure Appl. Math., Vol. 2, No. 4 (1949).

we represent the distribution function in the form of the series\*

$$f = f_0 \left( a^{(0)} I f^{(0)} + a_{a_1}^{(1)} H_{a_1}^{(1)} + \dots + \frac{1}{N!} a_{a_1 \dots a_N}^{(N)} H_{a_1 \dots a_N}^{(N)} + \dots \right), \quad (3.1)$$

where the  $a_{a_1 \dots a_N}^N$  are coefficients which depend on  $t$  and  $x$ ,

$$f_0 = n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left\{ - \frac{m}{2kT} v^2 \right\} = n \left( \frac{m}{kT} \right)^{3/2} \omega(v) \quad (3.2)$$

is the local Maxwell function,

$$\omega(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{1}{2} v^2}, \quad v = \sqrt{\frac{m}{kT}} c \quad (3.3)$$

and  $H_{a_1 \dots a_N}^{(N)}(v_1, v_2, v_3)$  are Hermite polynomials in three independent variables determined by the relation

$$H_{a_1 \dots a_N}^{(N)}(v_1, v_2, v_3) = \frac{(-1)^N}{\omega} \frac{\partial^N \omega}{\partial v_{a_1} \dots \partial v_{a_N}}, \quad (a_i = 1, 2, 3). \quad (3.4)$$

The series (3.1) converges on the average, if the integral

$$\int f^2 e^{-\frac{1}{2} v^2} dv$$

exists. That condition is satisfied for functions which tend to zero more rapidly than  $e^{-v^2/4}$  as  $v$  tends to infinity. In what follows we require only the polynomials of lower orders in explicit form:

$$\begin{aligned} H^{(0)} &= 1, \\ H_i^{(1)} &= v_i, \\ H_{ij}^{(2)} &= v_i v_j - \delta_{ij}, \\ H_{ijk}^{(3)} &= v_i v_j v_k - (v_i \delta_{jk} + v_j \delta_{ik} + v_k \delta_{ij}), \\ H_{ijkl}^{(4)} &= v_i v_j v_k v_l - (v_i v_j \delta_{kl} + v_i v_k \delta_{jl} + v_i v_l \delta_{jk} \\ &\quad + v_j v_k \delta_{il} + v_j v_l \delta_{ik} + v_k v_l \delta_{ij}) + (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \end{aligned} \quad (3.5)$$

Since we may change the order of differentiation in (3.4), the polynomials  $H_{a_1 \dots a_N}^{(N)}$ , obtained by permutation of subscripts are

\*A summation is carried out over the repeated subscripts.

identical. It is easy to see also from the definition (3.4) that

$$\frac{\partial H_{a_1 \dots a_N}^{(N)}}{\partial v_{a_v}} = H_{a_1 \dots a_{v-1} a_{v+1} \dots a_N}^{(N-1)}, \quad (v=1, \dots, N) \quad (3.6)$$

and

$$v_{a_v} H_{a_1 \dots a_N}^{(N)} = H_{a_1 \dots a_N a_v}^{(N+1)} + \delta_{a_v a_\mu} H_{a_1 \dots a_{\mu-1} a_{\mu+1} \dots a_N}^{(N-1)}. \quad (3.7)$$

The Hermite polynomials are orthogonal with respect to the weighting function  $\omega$ :

$$\int \omega(v) H_{a_1 \dots a_N}^{(N)}(v) H_{\beta_1 \dots \beta_M}^{(M)}(v) dv = 0 \quad \text{for } M \neq N \quad (3.8)$$

and

$$\int \omega(v) H_{a_1 \dots a_N}^{(N)}(v) H_{\beta_1 \dots \beta_N}^{(N)}(v) dv = \delta_{ab}^{(N)} \quad \text{for } M = N, \quad (3.9)$$

where  $\delta_{ab}^{(N)}$  is the tensor of order  $2N$ :

$$\delta_{ab}^{(1)} = \delta_{ab}, \quad \delta_{ab}^{(2)} = \delta_{a_1 b_1} \delta_{a_2 b_2} + \delta_{a_1 b_2} \delta_{a_2 b_1} + \delta_{a_1 a_2} \delta_{b_1 b_2}, \quad \text{etc.}$$

The components of the tensor  $\delta^{(N)}$  contain  $(2N)! / 2^N N!$  terms representing all the possible different products of the Kronecker symbols obtained by permuting the  $2N$  subscripts, each product appearing only once. Therefore, the polynomials  $H_{a_1 \dots a_N}^{(N)}$  and  $H_{\beta_1 \dots \beta_N}^{(N)}$  are orthogonal, only if  $a_1, \dots, a_N$  and  $\beta_1, \dots, \beta_N$  cannot be obtained from one another by permutation.

Using the orthogonality of the Hermite polynomials, we find

$$a_{a_1 \dots a_N}^{(N)} = \frac{1}{n} \int f H_{a_1 \dots a_N}^{(N)} d\xi. \quad (3.10)$$

The coefficients  $a^{(N)}$  may be expressed in terms of the moments; by substituting (3.5) into (3.10), we obtain

$$a^{(0)} = 1, \quad a_i^{(1)} = 0, \quad a_{ij}^{(2)} = \frac{p_{ij}}{p},$$

$$a_{ijk}^{(3)} = \frac{m \mathcal{M}_{ijk}}{p} \sqrt{\frac{m}{kT}},$$

$$\begin{aligned} a_{ijkl}^{(4)} &= \frac{m \mathcal{M}_{ijkl}}{p} \frac{m}{kT} - \frac{1}{p} (p_{ij} \delta_{kl} + p_{ik} \delta_{jl} + p_{il} \delta_{jk} \\ &+ p_{jk} \delta_{il} + p_{il} \delta_{kj} + p_{kl} \delta_{ij}) - (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \end{aligned} \quad (3.11)$$

where we have introduced the tensor

$$p_{ij} = P_{ij} - \delta_{ij} p. \quad (3.12)$$

To obtain the equations for  $a^{(N)}$ , we multiply the Boltzmann equation (in the absence of external forces) by  $H_{a_1 \dots a_N}^{(N)}$  and integrate over all velocities  $\xi$ . We have

$$\begin{aligned} \int H^{(N)} \left( \frac{\partial f}{\partial t} + \xi_l \frac{\partial f}{\partial x_l} \right) d\xi &= \int \left\{ \frac{df H^{(N)}}{dt} + \sqrt{\frac{kT}{m}} \frac{\partial v_l H^{(N)}}{\partial x_l} f \right. \\ &\quad \left. - f \left( \frac{dH^{(N)}}{dt} + \sqrt{\frac{kT}{m}} \frac{\partial v_l H^{(N)}}{\partial x_l} \right) \right\} d\xi = I_{H^{(N)}}, \end{aligned} \quad (3.13)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}. \quad (3.14)$$

We replace the product  $v_l H_{a_1 \dots a_N}^{(N)}$  in the second term by  $H^{(N+1)}$  and  $H^{(N-1)}$  by means of (3.7). We transform the derivatives of the moments; from (3.4), (3.6), and (3.7) we have

$$\begin{aligned} \frac{dH_{a_1 \dots a_N}^{(N)}}{dt} &= - \frac{\partial H_{a_1 \dots a_N}^{(N)}}{\partial v_{a_v}} \frac{dv_{a_v}}{dt} \\ &= - H_{a_1 \dots a_{v-1} a_{v+1} \dots a_N}^{(N-1)} \left[ \sqrt{\frac{m}{kT}} \frac{du_{a_v}}{dt} + \frac{mv_{a_v}}{2kT} \frac{d}{dt} \left( \frac{kT}{m} \right) \right] \\ &= - \sqrt{\frac{m}{kT}} H_{a_1 \dots a_{v-1} a_{v+1} \dots a_N}^{(N-1)} \frac{du_{a_v}}{dt} \\ &\quad - \frac{m}{2kT} (NH_{a_1 \dots a_N}^{(N)} + \delta_{a_v a_\mu} H_{a_1 \dots a_{v-1} a_{v+1} \dots a_{\mu-1} a_{\mu+1} \dots a_N}^{(N-2)}) \frac{d}{dt} \left( \frac{kT}{m} \right). \end{aligned} \quad (3.15)$$

Similarly,

$$\frac{\partial v_l H^{(N)}}{\partial x_l} = v_l \frac{\partial H^{(N)}}{\partial x_l} + H^{(N)} \frac{\partial v_l}{\partial x_l}, \quad (3.16)$$

identical. It is easy to see also from the definition (3.4) that

$$\frac{\partial H_{\alpha_1 \dots \alpha_N}^{(N)}}{\partial v_{\alpha_v}} = H_{\alpha_1 \dots \alpha_{v-1} \alpha_{v+1} \dots \alpha_N}^{(N-1)}, \quad (v = 1, \dots, N) \quad (3.6)$$

and

$$v_{\alpha_v} H_{\alpha_1 \dots \alpha_N}^{(N)} = H_{\alpha_1 \dots \alpha_N}^{(N+1)} + \delta_{\alpha_v \alpha_\mu} H_{\alpha_1 \dots \alpha_{\mu-1} \alpha_{\mu+1} \dots \alpha_N}^{(N-1)}. \quad (3.7)$$

The Hermite polynomials are orthogonal with respect to the weighting function  $\omega$ :

$$\int \omega(v) H_{\alpha_1 \dots \alpha_N}^{(N)}(v) H_{\beta_1 \dots \beta_M}^{(M)}(v) dv = 0 \quad \text{for } M \neq N \quad (3.8)$$

and

$$\int \omega(v) H_{\alpha}^{(N)}(v) H_{\beta}^{(N)}(v) dv = \delta_{\alpha \beta} \quad \text{for } M = N, \quad (3.9)$$

where  $\delta_{\alpha \beta}^{(N)}$  is the tensor of order  $2N$ :

$$\delta_{\alpha \beta}^{(1)} = \delta_{\alpha \beta}, \quad \delta_{\alpha \beta}^{(2)} = \delta_{\alpha_1 \beta_1} \delta_{\alpha_2 \beta_2} + \delta_{\alpha_1 \beta_2} \delta_{\alpha_2 \beta_1} + \delta_{\alpha_1 \alpha_2} \delta_{\beta_1 \beta_2}, \quad \text{etc.}$$

The components of the tensor  $\delta^{(N)}$  contain  $(2N)! / 2^N N!$  terms representing all the possible different products of the Kronecker symbols obtained by permuting the  $2N$  subscripts, each product appearing only once. Therefore, the polynomials  $H_{\alpha_1 \dots \alpha_N}^{(N)}$  and  $H_{\beta_1 \dots \beta_N}^{(N)}$  are orthogonal, only if  $\alpha_1, \dots, \alpha_N$  and  $\beta_1, \dots, \beta_N$  cannot be obtained from one another by permutation.

Using the orthogonality of the Hermite polynomials, we find

$$a_{\alpha_1 \dots \alpha_N}^{(N)} = \frac{1}{n} \int f H_{\alpha_1 \dots \alpha_N}^{(N)} d\xi. \quad (3.10)$$

The coefficients  $a^{(N)}$  may be expressed in terms of the moments; by substituting (3.5) into (3.10), we obtain

$$a^{(0)} = 1, \quad a_i^{(1)} = 0, \quad a_{ij}^{(2)} = \frac{p_{ij}}{p},$$

$$a_{ijk}^{(3)} = \frac{m \mathcal{M}_{ijk}}{p} \sqrt{\frac{m}{kT}},$$

$$\begin{aligned} a_{ijkl}^{(4)} &= \frac{m \mathcal{M}_{ijkl}}{p} \frac{m}{kT} - \frac{1}{p} (p_{ij} \delta_{kl} + p_{ik} \delta_{jl} + p_{il} \delta_{jk} \\ &+ p_{jk} \delta_{li} + p_{il} \delta_{kj} + p_{kl} \delta_{ij}) - (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \end{aligned} \quad (3.11)$$

where we have introduced the tensor

$$p_{ij} = P_{ij} - \delta_{ij} p. \quad (3.12)$$

To obtain the equations for  $a^{(N)}$ , we multiply the Boltzmann equation (in the absence of external forces) by  $H_{\alpha_1 \dots \alpha_N}^{(N)}$  and integrate over all velocities  $\xi$ . We have

$$\begin{aligned} \int H^{(N)} \left( \frac{\partial f}{\partial t} + \xi_l \frac{\partial f}{\partial x_l} \right) d\xi &= \int \left\{ \frac{df H^{(N)}}{dt} + \sqrt{\frac{kT}{m}} \frac{\partial v_l H^{(N)}}{\partial x_l} f \right. \\ &\quad \left. - f \left( \frac{dH^{(N)}}{dt} + \sqrt{\frac{kT}{m}} \frac{\partial v_l H^{(N)}}{\partial x_l} \right) \right\} d\xi = I_{H^{(N)}}, \end{aligned} \quad (3.13)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_l \frac{\partial}{\partial x_l}. \quad (3.14)$$

We replace the product  $v_l H_{\alpha_1 \dots \alpha_N}^{(N)}$  in the second term by  $H^{(N+1)}$  and  $H^{(N-1)}$  by means of (3.7). We transform the derivatives of the moments; from (3.4), (3.6), and (3.7) we have

$$\begin{aligned} \frac{dH_{\alpha_1 \dots \alpha_N}^{(N)}}{dt} &= \frac{\partial H_{\alpha_1 \dots \alpha_N}^{(N)}}{\partial v_{\alpha_v}} \frac{dv_{\alpha_v}}{dt} \\ &= - H_{\alpha_1 \dots \alpha_{v-1} \alpha_{v+1} \dots \alpha_N}^{(N-1)} \left[ \sqrt{\frac{m}{kT}} \frac{du_{\alpha_v}}{dt} + \frac{mv_{\alpha_v}}{2kT} \frac{d}{dt} \left( \frac{kT}{m} \right) \right] \\ &= - \sqrt{\frac{m}{kT}} H_{\alpha_1 \dots \alpha_{v-1} \alpha_{v+1} \dots \alpha_N}^{(N-1)} \frac{du_{\alpha_v}}{dt} \\ &\quad - \frac{m}{2kT} (NH_{\alpha_1 \dots \alpha_N}^{(N)} + \delta_{\alpha_v \alpha_\mu} H_{\alpha_1 \dots \alpha_{v-1} \alpha_{v+1} \dots \alpha_{\mu-1} \alpha_{\mu+1} \dots \alpha_N}^{(N-2)}) \frac{d}{dt} \left( \frac{kT}{m} \right). \end{aligned} \quad (3.15)$$

Similarly,

$$\frac{\partial v_l H^{(N)}}{\partial x_l} = v_l \frac{\partial H^{(N)}}{\partial x_l} + H^{(N)} \frac{\partial v_l}{\partial x_l}, \quad (3.16)$$

where

$$\begin{aligned} v_i \frac{\partial H_{a_1 \dots a_N}^{(N)}}{\partial x_i} &= -v_i H_{a_1 \dots a_{v-1} a_{v+1} \dots a_N}^{(N-1)} \left[ \sqrt{\frac{m}{kT}} \frac{\partial u_{a_v}}{\partial x_i} + \frac{mv_{a_v}}{2kT} \frac{\partial}{\partial x_i} \left( \frac{kT}{m} \right) \right] \\ &= \sqrt{\frac{m}{kT}} (H_{a_1 \dots a_{v-1} a_{v+1} \dots a_N}^{(N)} + \delta_{ia_\mu} H_{a_1 \dots a_{v-1} a_{v+1} \dots a_{\mu-1} a_{\mu+1} \dots a_N}^{(N-2)}) \\ &\quad \times \frac{\partial u_{a_v}}{\partial x_i} - \frac{m}{2kT} (NH_{a_1 \dots a_N}^{(N+1)} + N\delta_{ia_v} H_{a_1 \dots a_{v-1} a_{v+1} \dots a_N}^{(N-1)} \\ &\quad + \delta_{ia_\mu} H_{a_1 \dots a_{v-1} a_{v+1} \dots a_{\mu-1} a_{\mu+1} \dots a_N}^{(N-1)} + \delta_{a_v a_\mu} \delta_{ia_\tau} \\ &\quad \times H_{a_1 \dots a_{v-1} a_{v+1} \dots a_{\mu-1} a_{\mu+1} \dots a_{\tau-1} a_{\tau+1} \dots a_N}^{(N-3)}) \frac{\partial}{\partial x_i} \left( \frac{kT}{m} \right) \end{aligned}$$

and

$$\begin{aligned} H_{a_1 \dots a_N}^{(N)} \frac{\partial v_i}{\partial x_i} &= -\sqrt{\frac{m}{kT}} H_{a_1 \dots a_N}^N \frac{\partial u_i}{\partial x_i} \\ &- \frac{m}{2kT} (H_{a_1 \dots a_N}^{(N+1)} + \delta_{ia_v} H_{a_1 \dots a_{v-1} a_{v+1} \dots a_N}^{(N-1)}) \frac{\partial}{\partial x_i} \left( \frac{kT}{m} \right). \end{aligned}$$

By substituting these expressions into Eq. (3.13), integrating with respect to  $\xi$ , and dividing by  $n$ , we obtain

$$\begin{aligned} \frac{da_{a_1 \dots a_N}^{(N)}}{dt} &+ a_{a_1 \dots a_{v-1} a_{v+1} \dots a_N}^{(N)} \frac{\partial u_{a_v}}{\partial x_i} \\ &+ \frac{Nm}{2kT} a_{a_1 \dots a_N}^{(N)} \frac{d}{dt} \left( \frac{kT}{m} \right) + \sqrt{\frac{kT}{m}} \frac{\partial a_{a_1 \dots a_N}^{(N+1)}}{\partial x_i} \\ &+ \sqrt{\frac{kT}{m}} a_{a_1 \dots a_N}^{(N+1)} \frac{\partial}{\partial x_i} \ln \left[ n \left( \frac{kT}{m} \right)^{\frac{N+1}{2}} \right] \\ &+ \sqrt{\frac{kT}{m}} \delta_{ia_v} \frac{\partial a_{a_1 \dots a_{v-1} a_{v+1} \dots a_N}^{(N-1)}}{\partial x_i} \\ &+ \left( \sqrt{\frac{kT}{m}} \delta_{ia_v} \frac{\partial}{\partial x_i} \left[ n \left( \frac{kT}{m} \right)^{\frac{N+1}{2}} \right] + \sqrt{\frac{m}{kT}} \frac{du_{a_v}}{dt} \right) \\ &\quad \times a_{a_1 \dots a_{v-1} a_{v+1} \dots a_N}^{(N-1)} + \frac{1}{2} \sqrt{\frac{m}{kT}} \delta_{a_v a_\mu} \\ &\quad \times a_{a_1 \dots a_{v-1} a_{v+1} \dots a_{\mu-1} a_{\mu+1} \dots a_N}^{(N-1)} \frac{\partial}{\partial x_i} \left( \frac{kT}{m} \right) \\ &+ \left[ \frac{m}{2kT} \frac{d}{dt} \left( \frac{kT}{m} \right) \delta_{a_v a_\mu} + \delta_{ia_\mu} \frac{\partial u_{a_v}}{\partial x_i} \right] a_{a_1 \dots a_{v-1} a_{v+1} \dots a_{\mu-1} a_{\mu+1} \dots a_N}^{(N-2)} \end{aligned}$$

$$\begin{aligned} &+ \frac{1}{2} \sqrt{\frac{m}{kT}} \delta_{a_v a_\mu} \delta_{ia_\tau} a_{a_1 \dots a_{v-1} a_{v+1} \dots a_{\mu-1} a_{\mu+1} \dots a_{\tau-1} a_{\tau+1} \dots a_N}^{(N-3)} \\ &\quad \times \frac{\partial}{\partial x_i} \left( \frac{kT}{m} \right) = \frac{1}{n} I_{H^{(N)}}. \end{aligned} \quad (3.17)$$

In the derivation of (3.17) we used relations of the form

$$\int \frac{\partial}{\partial t} (H^{(N)} f) d\xi = \frac{\partial n a^{(N)}}{\partial t}, \quad \int \frac{\partial}{\partial x_i} (H^{(N)} f) d\xi = \frac{\partial n a^{(N)}}{\partial x_i},$$

as well as the continuity equation (1.8)

$$\frac{dn}{dt} = -n \frac{\partial u_i}{\partial x_i}.$$

Using the symmetry relation (4.12) of Chapter II, the integral on the right of (3.17) may be rewritten in the form

$$\begin{aligned} I_{H^{(N)}} &= \int H^{(N)} (f' f'_1 - f f'_1) g b db de d\xi_1 d\xi \\ &= \frac{1}{2} \int f f'_1 [H^{(N)}] g b db de d\xi_1 d\xi_1, \end{aligned} \quad (3.18)$$

where, for brevity, we have introduced the notation

$$[A] = A(\mathbf{v}') + A(\mathbf{v}'_1) - A(\mathbf{v}) - A(\mathbf{v}_1). \quad (3.19)$$

Hence, substituting the expansion (3.1) in place of  $f$ , we obtain

$$I_{H^{(N)}} = \frac{1}{2} \sum_{R, S=0}^{\infty} B_{a_1 \dots a_N}^{(N, R, S)} \beta_1 \dots \beta_R v_1 \dots v_S a_{\beta_1}^{(R)} \dots a_{\beta_R}^{(R)} a_{v_1}^{(S)} \dots a_{v_S}^{(S)}, \quad (3.20)$$

where

$$\begin{aligned} B_{a_1 \dots a_N}^{(N, R, S)} \beta_1 \dots \beta_R v_1 \dots v_S &= \frac{1}{R! S!} \int f_0(\mathbf{v}) f_0(\mathbf{v}_1) \\ &\times [H_{a_1 \dots a_N}^{(N)}] H_{\beta_1 \dots \beta_R}^{(R)}(\mathbf{v}) H_{v_1 \dots v_S}^{(S)}(\mathbf{v}_1) g b db de d\xi_1 d\xi_1. * \end{aligned} \quad (3.21)$$

\* It may be seen from the first form of the integral (3.18) that the coefficients  $B^{(N, 0, 0)} = 0$ . Further, in the six-dimensional space of the vectors  $\mathbf{v}$  and  $\mathbf{v}_1$  the functions  $f(\mathbf{v})f_0(\mathbf{v}_1)$  and  $g = |\mathbf{v}_1 - \mathbf{v}|$  are symmetric, while the functions  $[H^{(N)}]$  and  $H^{(R)}(\mathbf{v}) \times H^{(S)}(\mathbf{v}_1)$  are symmetric or antisymmetric, depending on whether  $N$  and  $R + S$  are even or odd numbers. Therefore, the coefficients  $B^{(N, R, S)} = 0$ , if  $N + R + S$  is an odd number.

Thus, the right side of equations (3.17) contains an infinite number of coefficients  $a^{(N)}$  or moments of any order. The left side of equations (3.17) contains moments of order  $(N + 1)$ , in addition to moments of order  $N$ .

Therefore, if the distribution function may be represented as a series (3.1), the Boltzmann equation may be replaced by an infinite system of simultaneous equations for the macroscopic quantities  $a^{(N)}$  or for the moments.

The series (3.20) terminates for Maxwellian molecules. In fact, as a result of Eq. (2.11) of Chapter II, the expression (3.21) in that case may be written:

$$B_{\alpha_1 \dots \alpha_N}^{(N, R, S)} \beta_1 \dots \beta_R \gamma_1 \dots \gamma_S = \frac{1}{R! S!} \left( \frac{16K}{m} \right)^{1/2} \times \int f_0(\mathbf{v}) f_0(\mathbf{v}_1) [H_{\alpha_1 \dots \alpha_N}^{(N)}] H_{\beta_1 \dots \beta_R}^{(R)}(\mathbf{v}) H_{\gamma_1 \dots \gamma_S}^{(S)}(\mathbf{v}_1) \rho d\rho d\mathbf{v} d\mathbf{v}_1 \quad (3.22)$$

or

$$B_{\alpha_1 \dots \alpha_N}^{(N, R, S)} \beta_1 \dots \beta_R \gamma_1 \dots \gamma_S = n^2 C_{\alpha_1 \dots \alpha_N}^{(N, R, S)} \beta_1 \dots \beta_R \gamma_1 \dots \gamma_S \quad (3.23)$$

where

$$C_{\alpha_1 \dots \alpha_N}^{(N, R, S)} \beta_1 \dots \beta_R \gamma_1 \dots \gamma_S = \frac{1}{R! S!} \left( \frac{16K}{m} \right)^{1/2} \times \int \omega(\mathbf{v}) \omega(\mathbf{v}_1) [H_{\alpha_1 \dots \alpha_N}^{(N)}] H_{\beta_1 \dots \beta_R}^{(R)}(\mathbf{v}) H_{\gamma_1 \dots \gamma_S}^{(S)}(\mathbf{v}_1) \rho d\rho d\mathbf{v} d\mathbf{v}_1. \quad (3.24)$$

The coefficients  $C_{\alpha\beta\gamma}^{(N, R, S)}$  depend only on the law of molecular interaction (on the quantity  $K$ , which appears in the law of interaction).

The relative velocity  $|\mathbf{v} - \mathbf{v}_1|$  does not appear in the integrand of (3.24). The expression  $[H_{\alpha_1 \dots \alpha_N}^{(N)}]$  has polynomials of degree  $N$  in both  $\mathbf{v}$  and  $\mathbf{v}_1$ , and also polynomials in  $\mathbf{v}'$  and  $\mathbf{v}_1'$ . However, as was shown in § 1.3 [formulas (3.10) and (3.11)], for mole-

ber. It is easy to see from considerations of symmetry, moreover, that the coefficients  $B_{\alpha_1 \dots \alpha_N}^{(N, R, S)} \beta_1 \dots \beta_R \gamma_1 \dots \gamma_S$  differ from zero only when the sum  $N + S + R$  for the subscripts  $\alpha$ ,  $\beta$ , and  $\gamma$  is a multiple of two, four, or six. (The subscripts  $\alpha$ ,  $\beta$ , and  $\gamma$  take the values 1, 2, 3.)

cules with a power interaction law, collisions defined by a given value of the parameter  $\rho$  yield a deflection angle  $\chi$  identical for any relative velocities. Therefore, in integrating with respect to  $\mathbf{v}$  or  $\mathbf{v}_1$  for fixed  $\rho$  in (3.24), the velocities  $\mathbf{v}'$  and  $\mathbf{v}_1'$  after collision are linear functions of  $\mathbf{v}$  and  $\mathbf{v}_1$ . Therefore,  $H^{(N)}(\mathbf{v}')$ ,  $H^{(N)}(\mathbf{v}_1')$ , and  $[H^{(N)}]$  are polynomials of degree  $N$  in  $\mathbf{v}$  and  $\mathbf{v}_1$ . The product of the polynomials  $H_{\beta}^{(R)}(\mathbf{v}) H_{\gamma}^{(S)}(\mathbf{v}_1)$  may be regarded as a Hermite polynomial in six independent variables of order  $R + S$  with a weighting function  $\omega(\mathbf{v}) \omega_1(\mathbf{v}_1)$ . That polynomial is orthogonal with a weighting function  $\omega \omega_1$  to any polynomial in  $\mathbf{v}$  and  $\mathbf{v}_1$  of degree lower than  $R + S$ . Therefore,  $C_{\alpha\beta\gamma}^{(N, R, S)} = 0$  when  $N < R + S$ . On the other hand, if the expansion (3.1) is substituted in the first form of the integral (3.18), it is seen that

$$C_{\alpha_1 \dots \alpha_N}^{(N, R, S)} \beta_1 \dots \beta_R \gamma_1 \dots \gamma_S = \frac{2}{R! S!} \left( \frac{16K}{m} \right)^{1/2} \int (\omega(\mathbf{v}) \omega_1(\mathbf{v}_1)) \times (H^{(R)}(\mathbf{v}') H^{(S)}(\mathbf{v}_1') - H^{(R)}(\mathbf{v}) H^{(S)}(\mathbf{v}_1)) H^{(N)}(\mathbf{v}) \rho d\rho d\mathbf{v} d\mathbf{v}_1. \quad (3.25)$$

Now the expression in brackets is a polynomial of degree  $R + S$ , and, therefore,  $C_{\alpha\beta\gamma}^{(N, R, S)} = 0$  when  $R + S < N$ . Thus, the coefficient  $B_{\alpha\beta\gamma}^{(N, R, S)}$  for a Maxwellian gas differs from zero only when  $R + S = N$  and the integral  $I_{H^{(N)}}$  is a finite quadratic sum of coefficients  $a^{(k)}$  of degree no greater than  $N$ .

However, even for Maxwellian molecules the infinite system of simultaneous equations (3.17) does not terminate in general, since the coefficients  $a^{(N+1)}$  appear on the left side of the equation for  $a^{(N)}$ . In order to obtain a closed system with a finite number of equations, we must approximate the distribution function by a finite number of terms of the series. Restricting ourselves to terms of the third order, we represent the distribution function as

$$f = f_0 \left( 1 + \frac{1}{2} a_{ij}^{(2)} H_{ij}^{(2)} + \frac{1}{6} a_{ijk}^{(3)} H_{ijk}^{(3)} \right). \quad (3.26)$$

By expressing the coefficients  $a^{(N)}$  in terms of the moments, and the Hermite functions in terms of the velocities according to formulas (3.11) and (3.5), respectively, we obtain

$$f = f_0 \left( 1 + \frac{p_{ij}}{2p} \frac{m}{kT} c_i c_j + \frac{mc_{ijk}}{6p} \left( \frac{m}{kT} \right)^2 c_i c_j c_k - \frac{q_i}{p} \left( \frac{m}{kT} \right) c_i \right). \quad (3.27)$$

This is the so-called twenty-moment approximation, since the approximation function contains twenty moments  $n$ ,  $u_i$ ,  $T$ ,  $p_{ij}$  and  $\mathcal{M}_{ijk}$ . By substitution of (3.26) and (3.10), from the orthogonality of the Hermite polynomials, we obtain the result that all the  $a^{(N)}$  vanish when  $N > 3$ .

The equations which the above twenty moments satisfy may be obtained by writing out equations (3.17) for  $a^{(2)}$  and  $a^{(3)}$ ; we must set the moments  $a^{(N)}$  with  $N > 3$  equal to zero and express the coefficients  $a^{(N)}$  for  $N \leq 3$  in terms of moments according to (3.11). The result is

$$\begin{aligned} \frac{\partial p}{\partial t} + \frac{\partial}{\partial x_r} (\rho u_r) &= 0, \\ \frac{\partial u_l}{\partial t} + u_r \frac{\partial u_l}{\partial x_r} + \frac{1}{\rho} \frac{\partial P_{lr}}{\partial x_r} &= 0, \end{aligned} \quad (3.28)$$

$$\begin{aligned} \frac{\partial p}{\partial t} + \frac{\partial}{\partial x_r} (u_r p_{lj}) + \frac{2}{3} P_{lr} \frac{\partial u_l}{\partial x_r} + \frac{2}{3} \frac{\partial q_r}{\partial x_r} &= 0, \\ \frac{\partial p_{lj}}{\partial t} + \frac{\partial}{\partial x_r} (u_r p_{lj}) + m \frac{\partial \mathcal{M}_{ljk}}{\partial x_r} - \frac{2}{3} \delta_{lj} \frac{\partial q_r}{\partial x_r} + p_{lr} \frac{\partial u_l}{\partial x_r} + p_{jr} \frac{\partial u_l}{\partial x_r} \\ - \frac{2}{3} \delta_{lj} p_{rs} \frac{\partial u_r}{\partial x_s} + p \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_r}{\partial x_r} \right) &= \frac{p}{n} I_{H_{ij}^{(2)}}, \end{aligned} \quad (3.29)$$

$$\begin{aligned} m \frac{\partial \mathcal{M}_{ijk}}{\partial t} + m \frac{\partial u_r \mathcal{M}_{ijk}}{\partial x_r} + m \left( \mathcal{M}_{rjk} \frac{\partial u_l}{\partial x_r} + \mathcal{M}_{ljk} \frac{\partial u_j}{\partial x_r} + \mathcal{M}_{lji} \frac{\partial u_k}{\partial x_r} \right) \\ + \frac{\partial}{\partial x_l} \left( p_{jk} \frac{kT}{m} \right) + \frac{\partial}{\partial x_j} \left( p_{ik} \frac{kT}{m} \right) + \frac{\partial}{\partial x_k} \left( p_{ij} \frac{kT}{m} \right) \\ + (p_{lr} \delta_{jk} + p_{jr} \delta_{ik} + p_{kr} \delta_{ij}) \frac{\partial}{\partial x_r} \left( \frac{kT}{m} \right) \\ - \frac{1}{\rho} \left( p_{lj} \frac{\partial P_{kr}}{\partial x_r} + p_{lk} \frac{\partial P_{jr}}{\partial x_r} + p_{jk} \frac{\partial P_{lr}}{\partial x_r} \right) \\ + p \left( \delta_{jk} \frac{\partial}{\partial x_l} + \delta_{ik} \frac{\partial}{\partial x_j} + \delta_{lj} \frac{\partial}{\partial x_k} \right) \frac{kT}{m} = \frac{p}{n} \sqrt{\frac{kT}{m}} I_{H_{ijk}^{(3)}}. \end{aligned} \quad (3.30)$$

For the sake of completeness, we have also written the conservation equations (3.28).

Since the coefficients  $a^{(N)} = 0$  when  $N > 3$ , the integrals  $I_{H^{(2)}}$  and  $I_{H^{(3)}}$  are quadratic forms, which contain products  $a^{(S)} a^{(R)}$  of order  $R + S \leq 6$ . For a Maxwellian gas, we noted above that the integrals  $I_{H^{(N)}}$  contain only the products  $a^{(S)} a^{(R)}$ , for which  $S + R = N$ . Therefore, for Maxwellian molecules, since  $a_i^{(1)} \equiv 0$ , the sum (3.20) which appears in the expression for  $I_{H^{(2)}}$ , may contain, only terms of type  $a^{(0)} a^{(2)}$ , and the sum for  $I_{H^{(3)}}$ , only terms of type  $a^{(0)} a^{(3)}$ .

When the appropriate calculations are performed for Maxwellian molecules, the following expressions \* are obtained:

$$\begin{aligned} I_{H_{ij}^{(2)}} &= -6A \left( \frac{8K}{m} \right)^{1/2} n^2 a_{ij}^{(2)}, \\ I_{H_{ijk}^{(3)}} &= -A \left( \frac{8K}{m} \right)^{1/2} n^2 (9a_{ijk}^{(3)} - a_i^{(3)} \delta_{jk} - a_j^{(3)} \delta_{ik} - a_k^{(3)} \delta_{ij}), \\ \text{where } a_i^{(3)} &= a_{i11} + a_{i22} + a_{i33} \text{ is the contraction over the two subscripts and } A = 0.343. \end{aligned} \quad (3.31)$$

In deriving Eqs. (3.29) and (3.30) we need not start from equations (3.17), which were obtained for a distribution function represented by a series of a complete set of functions (Hermite polynomials). Generally speaking, given the approximation function (3.26), Eqs. (3.29) and (3.30) may be obtained by the method discussed in the preceding paragraph by replacing the  $M^{(N)}$ -moments in the sequence of equations (1.5) by the  $\mathcal{M}^{(N)}$ -moments by means of formulas (2.2).

The approximation (3.26) contains twenty moments (all moments of the first, second, and third orders). From a practical point of view it is preferable to have a system of equations containing only the thirteen moments  $n$ ,  $u_j$ ,  $T$ ,  $p_{ij}$ , and  $q_i$ , which have a clear physical meaning and are subject to measurement. We therefore write the distribution function in the form

$$f = f_0 \left( 1 + \frac{1}{2} a_{ij}^{(2)} H_{ij}^{(2)} + b_i H_i^{(3)} \right), \quad (3.32)$$

where  $H_i^{(3)} = v_i (v^2 - 5)$  is the contraction of  $H_{ijk}^{(3)}$  over the two subscripts. As explained in the preceding paragraph, we may generally use any approximating function. Of course, in doing so, we must know for which problem a chosen approximation of the distribution function can secure the desired accuracy. At the end of

\*J. C. Maxwell, The Scientific Papers of James Clark Maxwell "On the Dynamical Theory of Gases," Vol. 2, Cambridge University Press, England, 1890; H. Grad, Commun. Pure Appl. Math., Vol. 2, No. 4 (1949). The last paper also gives approximate values for these integrals for molecules with other interaction laws. The integrals of higher order may be found in a paper by V. D. Perminov and O. G. Fridlander, Zh. Prikl. Mekhan. Tekhn. Fiz., No. 6 (1965).

$$q_i(t) = q_i(0) \exp \left\{ -\frac{2}{3} \int_0^t \frac{d\tau}{\tau_p} \right\} - \int_0^t B_i(s) \exp \left\{ -\int_s^t \frac{d\tau}{\tau_p} \right\} ds. \quad (3.40)$$

Integrating repeatedly by parts, we obtain the following asymptotic expansion for  $\tau_p \rightarrow 0$ :

$$p_{ij}(t) = \left( p_{ij} + \tau_p A_{ij} - \tau_p \frac{\partial A_{ij}\tau_p}{\partial t} + \dots \right) \Big|_{t=0} \exp \left\{ - \int_0^t \frac{d\tau}{\tau_p} \right\} \\ - \left( \tau_p A_{ij} - \tau_p \frac{\partial A_{ij}\tau_p}{\partial t} + \dots \right) \Big|_{t=t}, \quad (3.41)$$

$$q_i(t) = \left( q_i + \frac{3}{2} \tau_p B_i - \frac{9}{4} \tau_p \frac{\partial B_i \tau_p}{\partial t} + \dots \right) \Big|_{t=0} \exp \left\{ -\frac{2}{3} \int_0^t \frac{d\tau}{\tau_p} \right\} \\ - \left( \frac{3}{2} \tau_p B_i - \frac{9}{4} \tau_p \frac{\partial B_i \tau_p}{\partial t} + \dots \right) \Big|_{t=t}. \quad (3.42)$$

It is clear from (3.41) and (3.42) that after a time of the order of several  $\tau_p$  the initial conditions no longer affect the solution and the stress tensor  $p_{ij}$  and the heat flux vector  $q_i$  are completely determined by the state of the gas at time  $t$ . Therefore, the time  $\tau_p$  is the relaxation time of the process.

It was shown in §1.3 [formula (3.18)], that the effective collision cross section for Maxwellian molecules is equal to

$$\sigma^{(v)} = 2\pi \left( \frac{16K}{m} \right)^{1/2} \frac{A^{(v)}}{g}.$$

Therefore,  $1/\tau_p \sim (K/m)^{1/2} n \sim \sigma g n$ , i.e., the relaxation time  $\tau_p$  is of the order of the time between collisions (see §1.4).

For small Knudsen numbers, when the characteristic flow time is much greater than the time between collisions (and, therefore, also than the relaxation time  $\tau_p$ ), the influence of the past history of the flow is negligibly small owing to the rapid exponential decay. Restricting ourselves in this case to the first term of the expansion, we have

$$p_{ij} = -\tau_p A_{ij}, \quad (3.43)$$

$$q_i = -\frac{3}{2} \tau_p B_i. \quad (3.44)$$

By regarding the hydrodynamic quantities  $\rho$ ,  $u_i$ ,  $p$ , and  $T$  and their derivatives as quantities of highest order, we see, from (3.43) and (3.44) and the definition of  $A_{ij}$  and  $B_i$ , that  $p_{ij}$  and  $q_i$  are of order  $\tau_p$ . Retaining the quantities of highest order in  $A_{ij}$  and  $B_i$  and neglecting terms containing  $p_{ij}$  and  $q_i$ , we obtain

$$p_{ij} = -\tau_p p \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_r}{\partial x_r} \right) \quad (3.45)$$

$$q_i = -\frac{15}{4} \tau_p p \frac{\partial}{\partial x_i} \left( \frac{kT}{m} \right). \quad (3.46)$$

It is easy to see that if we put

$$\mu = \tau_p p = \frac{1}{6A} \left( \frac{m}{8K} \right)^{1/2} kT, \quad \lambda = \frac{15}{4} \frac{k}{m} p \tau_p = \frac{5}{8A} \frac{k^2 T}{\sqrt{8Km}}, \quad (3.47)$$

where  $\mu$  and  $\lambda$  are the viscosity and the thermal conductivity, respectively, then expressions (3.45) and (3.46) give the connection of the stress tensor and the heat flux vector to the rate of strain tensor and the temperature gradient, which is ordinarily assumed in the Navier-Stokes equations. Substituting (3.45) and (3.46) into the conservation equations (3.28), we obtain the usual Navier-Stokes equations.

Let us remember that we are considering a Maxwellian gas. Therefore, the expressions obtained for the viscosity and the thermal conductivity are suitable only for a gas composed of molecules which interact according to the Maxwellian law. It was shown in §1.5 from elementary considerations that the viscosity and the thermal conductivity of a Maxwellian gas are proportional to temperature. Now this fact has been confirmed by a rigorous theory.

Retaining two terms in (3.41) and (3.42), we obtain the so-called Burnett approximation:

$$p_{ij} = -\tau_p A_{ij} + \tau_p \frac{\partial \tau_p A_{ij}}{\partial t}, \quad (3.48)$$

$$q_i = -\frac{3}{2} \tau_p B_i + \frac{9}{4} \tau_p \frac{\partial \tau_p B_i}{\partial t}. \quad (3.49)$$

In  $A_{ij}$  and  $B_i$  in the first terms on the right we must retain quantities of order  $\tau_p$ , which contain  $p_{ij}$  and  $q_i$ , and express the latter by formulas (3.45) and (3.46). In the second terms on the right we

should neglect terms containing  $p_{ij}$  and  $q_i$  in  $A_{ij}$  and  $B_i$ , since they are of order  $\tau_p^3$ . With the help of the Navier-Stokes equations we may eliminate partial derivatives with respect to  $t$  with accuracy up to  $\tau_p^3$ . As a result, we obtain the dependence of  $p_{ij}$  and  $q_i$  on the hydrodynamic quantities  $\rho$ ,  $u_i$ ,  $p$ , and  $T$ , and their space derivatives up to the second order:

$$\begin{aligned} p_{ij} = & -2\mu \frac{\partial \overline{u_i}}{\partial x_j} + K_1 \frac{\mu^2}{p} \frac{\partial u_k}{\partial x_k} \frac{\partial \overline{u_i}}{\partial x_j} \\ & + K_2 \frac{\mu^2}{p} \left[ \frac{\partial}{\partial x_l} \left( \frac{X_j}{m} - \frac{1}{\rho} \frac{\partial p}{\partial x_j} \right) - \frac{\partial \overline{u_k}}{\partial x_l} \frac{\partial \overline{u_j}}{\partial x_k} - 2 \frac{\partial \overline{u_i}}{\partial x_k} \frac{\partial \overline{u_k}}{\partial x_j} \right] \\ & + K_3 \frac{\mu^2}{\rho T} \frac{\partial^2 T}{\partial x_l \partial x_j} + K_4 \frac{\mu^2}{\rho p T} \frac{\partial p}{\partial x_l} \frac{\partial T}{\partial x_j} + K_5 \frac{\mu^2}{\rho T^2} \frac{\partial T}{\partial x_l} \frac{\partial T}{\partial x_j} \\ & + K_6 \frac{\mu^2}{p} \frac{\partial \overline{u_i}}{\partial x_k} \frac{\partial \overline{u_k}}{\partial x_j}, \end{aligned} \quad (3.50)$$

$$\begin{aligned} q_i = & -\lambda \frac{\partial T}{\partial x_l} + \theta_1 \frac{\mu^2}{\rho T} \frac{\partial u_j}{\partial x_j} \frac{\partial T}{\partial x_l} \\ & + \theta_2 \frac{\mu^2}{\rho T} \left[ \frac{2}{3} \frac{\partial}{\partial x_l} \left( T \frac{\partial u_j}{\partial x_j} \right) + 2 \frac{\partial u_j}{\partial x_l} \frac{\partial T}{\partial x_j} \right] \\ & + \left( \theta_3 \frac{\mu^2}{\rho p} \frac{\partial p}{\partial x_j} + \theta_4 \frac{\mu^2}{\rho} \frac{\partial}{\partial x_j} + \theta_5 \frac{\mu^2}{\rho T} \frac{\partial T}{\partial x_j} \right) \frac{\partial u_j}{\partial x_l}. \end{aligned} \quad (3.51)$$

For brevity we have introduced the notation

$$\bar{A}_{ij} = \frac{1}{2} (A_{ij} + A_{ji}) - \frac{1}{3} \delta_{ij} A_{kk}.$$

The constants  $K_i$  and  $\theta_i$  have the following values\* for Maxwellian molecules:

$$\begin{aligned} K_1 &= \frac{4}{3} \left( \frac{7}{2} - \frac{T}{\mu} \frac{d\mu}{dT} \right), \quad K_2 = 2, \quad K_3 = 3, \\ K_4 &= 0, \quad K_5 = 3 \frac{T}{\mu} \frac{d\mu}{dT}, \quad K_6 = 8, \\ \theta_1 &= \frac{15}{4} \left( \frac{7}{2} - \frac{T}{\mu} \frac{d\mu}{dT} \right), \quad \theta_2 = -\frac{45}{8}, \quad \theta_3 = -3, \quad \theta_4 = 3, \\ \theta_5 &= 3 \left( \frac{35}{4} + \frac{T}{\mu} \frac{d\mu}{dT} \right). \end{aligned}$$

We note that, for molecules with other interaction laws, expressions (3.50) and (3.51) may also be obtained using other methods (see §§ 3.7 and 3.8), but with other coefficients  $K_i$  and  $\theta_i$ . Thus, for elastic spheres we have

$$\begin{aligned} K_{c1} &= 1.014K_1, \quad K_{c2} = 1.014K_2, \quad K_{c3} = 0.806K_3, \quad K_{c4} = 0.681, \\ K_{c5} &= 0.806K_5 - 0.990, \quad K_{c6} = 0.928K_6, \\ \theta_{c1} &= 1.035\theta_1, \quad \theta_{c2} = 1.035\theta_2, \quad \theta_{c3} = 1.030\theta_3, \quad \theta_{c4} = 0.806\theta_4, \\ \theta_{c5} &= 3 \left( \frac{35}{4} \cdot 0.918 + 0.806 \frac{T}{\mu} \frac{d\mu}{dT} \right) - 0.150. \end{aligned}$$

2. The Navier-Stokes, Burnett, and other equations may also be derived by means of the full moment equations.

Let us examine an infinite sequence of moment equations. The first five equations are the conservation equations (1.8)-(1.10), in which moments of second and third order  $p_{ij}$  and  $q_i$  appear, in addition to the five hydrodynamic quantities  $n$ ,  $u_i$ , and  $T$ . In order to construct the Navier-Stokes, Burnett, etc., equations, we must express the latter in terms of the hydrodynamic quantities and their derivatives.

To find  $p_{ij}$  and  $q_i$  we make use of moment equations of higher order. It is not convenient to write these equations directly for the moments, but instead for the coefficients  $a_{a_1 \dots a_N}^{(N)}$  of the expansion of the distribution function in a series of Hermite polynomials. The coefficients  $a_{a_1 \dots a_N}^{(N)}$  are connected to the moments by the relations (3.11). The functions  $a_{a_1 \dots a_N}^{(N)}$  satisfy (3.17) which (for  $N \geq 2$  and Maxwellian molecules) may be written in the following general form:

$$\begin{aligned} & \frac{\partial a_{a_1 \dots a_N}^{(N)}}{\partial t} + L_{a_1 \dots a_N}^{(N)} (a^{(N+1)}, a^{(N)}, a^{(N-1)}, a^{(N-2)}, a^{(N-3)}) \\ & + \frac{1}{\epsilon} \mathcal{L}_{a_1 \dots a_N}^{(N)} (a^{(N)}) + \frac{1}{\epsilon} D_{a_1 \dots a_N}^{(N)} (a^{(2)}, \dots, a^{(N-2)}) \\ & = \frac{1}{\epsilon} A_{a_1 \dots a_N}^{(N)} a_{a_1 \dots a_N}^{(N)}. \end{aligned} \quad (3.52)$$

where  $\epsilon$  is the ratio of  $\tau_p$  to the characteristic time of the flow. The operator  $L^{(N)}$  is linear with respect to the functions  $a^{(N+1)}$ ,

\*See S. Chapman and T. Cowling, "The Mathematical Theory of Nonuniform Gases," Cambridge University Press, London, 1958.

should neglect terms containing  $p_{ij}$  and  $q_i$  in  $A_{ij}$  and  $B_i$ , since they are of order  $\tau_p^3$ . With the help of the Navier-Stokes equations we may eliminate partial derivatives with respect to  $t$  with accuracy up to  $\tau_p^3$ . As a result, we obtain the dependence of  $p_{ij}$  and  $q_i$  on the hydrodynamic quantities  $\rho$ ,  $u_i$ ,  $p$ , and  $T$ , and their space derivatives up to the second order:

$$\begin{aligned} p_{ij} = & -2\mu \frac{\partial \overline{u_i}}{\partial x_j} + K_1 \frac{\mu^2}{p} \frac{\partial u_k}{\partial x_k} \frac{\partial \overline{u_i}}{\partial x_j} \\ & + K_2 \frac{\mu^2}{p} \left[ \frac{\partial}{\partial x_l} \left( \frac{X_j}{m} - \frac{1}{\rho} \frac{\partial p}{\partial x_j} \right) - \frac{\partial \overline{u_k}}{\partial x_l} \frac{\partial \overline{u_j}}{\partial x_k} - 2 \frac{\partial \overline{u_i}}{\partial x_k} \frac{\partial \overline{u_k}}{\partial x_j} \right] \\ & + K_3 \frac{\mu^2}{\rho T} \frac{\partial^2 T}{\partial x_l \partial x_j} + K_4 \frac{\mu^2}{\rho p T} \frac{\partial p}{\partial x_l} \frac{\partial T}{\partial x_j} + K_5 \frac{\mu^2}{\rho T^2} \frac{\partial T}{\partial x_l} \frac{\partial T}{\partial x_j} \\ & + K_6 \frac{\mu^2}{p} \frac{\partial \overline{u_i}}{\partial x_k} \frac{\partial \overline{u_k}}{\partial x_j}, \end{aligned} \quad (3.50)$$

$$\begin{aligned} q_i = & -\lambda \frac{\partial T}{\partial x_i} + \theta_1 \frac{\mu^2}{\rho T} \frac{\partial u_j}{\partial x_j} \frac{\partial T}{\partial x_i} \\ & + \theta_2 \frac{\mu^2}{\rho T} \left[ \frac{2}{3} \frac{\partial}{\partial x_l} \left( T \frac{\partial u_j}{\partial x_j} \right) + 2 \frac{\partial u_j}{\partial x_l} \frac{\partial T}{\partial x_j} \right] \\ & + \left( \theta_3 \frac{\mu^2}{\rho p} \frac{\partial p}{\partial x_j} + \theta_4 \frac{\mu^2}{\rho} \frac{\partial}{\partial x_j} + \theta_5 \frac{\mu^2}{\rho T} \frac{\partial T}{\partial x_j} \right) \frac{\partial u_j}{\partial x_i}. \end{aligned} \quad (3.51)$$

For brevity we have introduced the notation

$$\bar{A}_{ij} = \frac{1}{2} (A_{ij} + A_{ji}) - \frac{1}{3} \delta_{ij} A_{kk}.$$

The constants  $K_i$  and  $\theta_i$  have the following values\* for Maxwellian molecules:

$$\begin{aligned} K_1 &= \frac{4}{3} \left( \frac{7}{2} - \frac{T}{\mu} \frac{d\mu}{dT} \right), \quad K_2 = 2, \quad K_3 = 3, \\ K_4 &= 0, \quad K_5 = 3 \frac{T}{\mu} \frac{d\mu}{dT}, \quad K_6 = 8, \\ \theta_1 &= \frac{15}{4} \left( \frac{7}{2} - \frac{T}{\mu} \frac{d\mu}{dT} \right), \quad \theta_2 = -\frac{45}{8}, \quad \theta_3 = -3, \quad \theta_4 = 3, \\ \theta_5 &= 3 \left( \frac{35}{4} + \frac{T}{\mu} \frac{d\mu}{dT} \right). \end{aligned}$$

\*See S. Chapman and T. Cowling, "The Mathematical Theory of Nonuniform Gases," Cambridge University Press, London, 1958.

We note that, for molecules with other interaction laws, expressions (3.50) and (3.51) may also be obtained using other methods (see §§ 3.7 and 3.8), but with other coefficients  $K_i$  and  $\theta_i$ . Thus, for elastic spheres we have

$$\begin{aligned} K_{c1} &= 1.014K_1, \quad K_{c2} = 1.014K_2, \quad K_{c3} = 0.806K_3, \quad K_{c4} = 0.681, \\ K_{c5} &= 0.806K_5 - 0.990, \quad K_{c6} = 0.928K_6, \\ \theta_{c1} &= 1.0350, \quad \theta_{c2} = 1.0350, \quad \theta_{c3} = 1.030\theta_3, \quad \theta_{c4} = 0.806\theta_4, \\ \theta_{c5} &= 3 \left( \frac{35}{4} \cdot 0.918 + 0.806 \frac{T}{\mu} \frac{d\mu}{dT} \right) - 0.150. \end{aligned}$$

2. The Navier-Stokes, Burnett, and other equations may also be derived by means of the full moment equations.

Let us examine an infinite sequence of moment equations. The first five equations are the conservation equations (1.8)-(1.10), in which moments of second and third order  $p_{ij}$  and  $q_i$  appear, in addition to the five hydrodynamic quantities  $n$ ,  $u_i$ , and  $T$ . In order to construct the Navier-Stokes, Burnett, etc., equations, we must express the latter in terms of the hydrodynamic quantities and their derivatives.

To find  $p_{ij}$  and  $q_i$  we make use of moment equations of higher order. It is not convenient to write these equations directly for the moments, but instead for the coefficients  $a_{a_1 \dots a_N}^{(N)}$  of the expansion of the distribution function in a series of Hermite polynomials. The coefficients  $a_{a_1 \dots a_N}^{(N)}$  are connected to the moments by the relations (3.11). The functions  $a_{a_1 \dots a_N}^{(N)}$  satisfy (3.17) which (for  $N \geq 2$  and Maxwellian molecules) may be written in the following general form:

$$\begin{aligned} & \frac{\partial a_{a_1 \dots a_N}^{(N)}}{\partial t} + L_{a_1 \dots a_N}^{(N)} (a^{(N+1)}, a^{(N)}, a^{(N-1)}, a^{(N-2)}, a^{(N-3)}) \\ & + \frac{1}{\epsilon} \mathcal{L}_{a_1 \dots a_N}^{(N)} (a^{(N)}) + \frac{1}{\epsilon} D_{a_1 \dots a_N}^{(N)} (a^{(2)}, \dots, a^{(N-2)}) \\ & = \frac{1}{\epsilon} A_{a_1 \dots a_N}^{(N)} a_{a_1 \dots a_N}^{(N)}. \end{aligned} \quad (3.52)$$

where  $\epsilon$  is the ratio of  $\tau_p$  to the characteristic time of the flow. The operator  $L^{(N)}$  is linear with respect to the functions  $a^{(N+1)}$ ,

$a^{(N)}, \dots, a^{(N-3)}$  with coefficients depending on the hydrodynamic quantities and their first derivatives. All coefficients of order  $N$  different from  $a_{\alpha_1 \dots \alpha_N}^{(N)}$  appear in the operator  $\mathcal{L}^{(N)}$ , which is linear with respect to  $a^{(N)}$ . The operator  $D_{\alpha_1 \dots \alpha_N}^{(N)}$  is a linear form of  $a^{(k)} a^{(l)}$ , where  $k, l \leq N-2$ ,  $k+l=N$ , and  $A_{\alpha_1 \dots \alpha_N}^{(N)}$  are coefficients independent of  $a^{(k)}$ .

We integrate the equations (3.52), assuming  $L, \mathcal{L}, D, A$ , and  $\tau_p$  to be known functions of  $t$  for the time being; we have

$$\begin{aligned} a^{(N)}(t) &= a^{(N)}(0) \exp \left\{ - \int_0^t \frac{A^{(N)}}{\epsilon} d\tau \right\} \\ &- \int_0^t Q^{(N)}(s) \exp \left\{ - \int_s^t \frac{A^{(N)}}{\epsilon} d\tau \right\} ds. \end{aligned} \quad (3.53)$$

Here,

$$Q^{(N)} = L^{(N)} + \frac{1}{\epsilon} \mathcal{L}^{(N)} + \frac{1}{\epsilon} D^{(N)}$$

and for brevity the subscripts  $\alpha_1, \dots, \alpha_N$  in  $a^{(N)}, L^{(N)}$ , etc., have been omitted. We integrate, the integral appearing on the right of equation (3.53) by parts  $k$  times in succession; we have

$$\begin{aligned} a^{(N)}(t) &= \left( a^N + P_1^{(N)} - \frac{\epsilon}{A^{(N)}} \frac{\partial P_1^{(N)}}{\partial t} \right. \\ &\dots + (-1)^{k-1} \left. \frac{\epsilon}{A^{(N)}} \frac{\partial P_{k-1}^{(N)}}{\partial t} \right)_{t=0} \exp \left\{ - \int_0^t \frac{A^{(N)}}{\epsilon} d\tau \right\} \\ &- \left( P_1^{(N)} - \frac{\epsilon}{A^{(N)}} \frac{\partial P_1^{(N)}}{\partial t} + \dots + (-1)^{k-1} \frac{\epsilon}{A^{(N)}} \frac{\partial P_{k-1}^{(N)}}{\partial t} \right)_{s=t} \\ &+ (-1)^{k-1} \int_0^t \frac{\epsilon}{A^{(N)}} \frac{\partial P_{k-1}^{(N)}}{\partial t} \exp \left\{ - \int_s^t \frac{A^{(N)}}{\epsilon} d\tau \right\} ds, \end{aligned} \quad (3.54)$$

where

$$P_n^{(N)} = \frac{\epsilon}{A^{(N)}} \frac{\partial P_{n-1}^{(N)}}{\partial t}, \quad P_1^{(N)} = \frac{\epsilon}{A^{(N)}} Q^{(N)}.$$

If the derivatives  $\frac{\partial P_{k-1}^{(N)}}{\partial t}$  are finite, then when the ratio of  $\tau_p$  to the characteristic flow time tends to zero, the functions  $a_{\alpha_1 \dots \alpha_N}^{(N)}$  are the limits of the asymptotically convergent series

$$a^{(N)} = - \frac{\epsilon}{A^{(N)}} Q^{(N)} + \frac{\epsilon}{A^{(N)}} \frac{\partial}{\partial t} \left( \frac{\epsilon}{A^{(N)}} Q^{(N)} \right) - \dots \quad (3.55)$$

We note that in the equations for  $a_{ij}^{(2)}$  and  $a_i^{(3)}$ , i.e., for  $p_{ij}$  and  $q_i$ , the operators  $\mathcal{L}$  and  $D$  are equal to zero, while the operators  $L$  contain terms having  $a^{(0)} = 1$ .

When  $N > 3$ , the expression  $Q^{(N)}$  has the structure (in order of magnitude):

$$\frac{1}{\epsilon} a^{(N)} + \frac{1}{\epsilon} (a^{(N-2)} a^{(2)} + \dots + a^{(2)} a^{(N-2)}) + (a^{(N-3)} + \dots + a^{(N+1)}).$$

Since the determinant of the system

$$A_{\alpha_1 \dots \alpha_N}^{(N)} a_{\alpha_1 \dots \alpha_N}^{(N)} - \mathcal{L}_{\alpha_1 \dots \alpha_N}^{(N)} (a^{(N)}) = 0$$

is not generally equal to zero, it is easy to see that

$$a^{(2)} \sim O(\epsilon), \quad a^{(3)} \sim O(\epsilon) \quad \text{and} \quad a^{(N)} \lesssim \epsilon^2 \quad \text{for } N > 3. \quad (3.56)$$

Therefore, if we expand the values of  $Q^{(N)}$  in expressions (3.55) for  $a_{ij}^{(2)}$  and  $a_i^{(3)}$ , we see that the coefficients  $a^{(4)}$  can make a contribution only in terms of order  $\epsilon^3$  for the stress tensor and the heat flux vector. Therefore, in the Navier-Stokes and Burnett approximations, the twenty-moment equations which are obtained from the exact equations with  $a^{(4)} = 0$ , coincide with the exact equations.

We now examine equations (3.17) for  $a_{ijk}^{(3)}$  in more detail. We write them in the form

$$\begin{aligned} a_i^{(3)} \delta_{jk} + a_j^{(3)} \delta_{ik} + a_k^{(3)} \delta_{ij} - 9 a_{ijk}^{(3)} \\ = \frac{\epsilon}{A^{(3)}} \left( \frac{\partial T}{\partial x_i} \delta_{jk} + \frac{\partial T}{\partial x_j} \delta_{ki} + \frac{\partial T}{\partial x_k} \delta_{ij} \right) + O(\epsilon^2). \end{aligned} \quad (3.57)$$

Taking their contraction, it is easy to verify that

$$\frac{\partial T}{\partial x_i} = - \frac{4}{5} \frac{A^{(3)}}{\epsilon} a_i^{(3)} + O(\epsilon^2). \quad (3.58)$$

$a^{(N)}, \dots, a^{(N-3)}$  with coefficients depending on the hydrodynamic quantities and their first derivatives. All coefficients of order  $N$  different from  $a_{\alpha_1 \dots \alpha_N}^{(N)}$  appear in the operator  $\mathcal{L}^{(N)}$ , which is linear with respect to  $a^{(N)}$ . The operator  $D_{\alpha_1 \dots \alpha_N}^{(N)}$  is a linear form of  $a^{(k)} a^{(l)}$ , where  $k, l \leq N-2$ ,  $k+l=N$ , and  $A_{\alpha_1 \dots \alpha_N}^{(N)}$  are coefficients independent of  $a^{(k)}$ .

We integrate the equations (3.52), assuming  $L, \mathcal{L}, D, A$ , and  $\tau_p$  to be known functions of  $t$  for the time being; we have

$$\begin{aligned} a^{(N)}(t) &= a^{(N)}(0) \exp \left\{ - \int_0^t \frac{A^{(N)}}{\epsilon} d\tau \right\} \\ &- \int_0^t Q^{(N)}(s) \exp \left\{ - \int_s^t \frac{A^{(N)}}{\epsilon} d\tau \right\} ds. \end{aligned} \quad (3.53)$$

Here,

$$Q^{(N)} = L^{(N)} + \frac{1}{\epsilon} \mathcal{L}^{(N)} + \frac{1}{\epsilon} D^{(N)}$$

and for brevity the subscripts  $\alpha_1, \dots, \alpha_N$  in  $a^{(N)}, L^{(N)}$ , etc., have been omitted. We integrate, the integral appearing on the right of equation (3.53) by parts  $k$  times in succession; we have

$$\begin{aligned} a^{(N)}(t) &= \left( a^N + P_1^{(N)} - \frac{\epsilon}{A^{(N)}} \frac{\partial P_1^{(N)}}{\partial t} \right. \\ &\dots + (-1)^{k-1} \left. \frac{\epsilon}{A^{(N)}} \frac{\partial P_{k-1}^{(N)}}{\partial t} \right)_{t=0} \exp \left\{ - \int_0^t \frac{A^{(N)}}{\epsilon} d\tau \right\} \\ &- \left( P_1^{(N)} - \frac{\epsilon}{A^{(N)}} \frac{\partial P_1^{(N)}}{\partial t} + \dots + (-1)^{k-1} \frac{\epsilon}{A^{(N)}} \frac{\partial P_{k-1}^{(N)}}{\partial t} \right)_{s=t} \\ &+ (-1)^{k-1} \int_0^t \frac{\epsilon}{A^{(N)}} \frac{\partial P_{k-1}^{(N)}}{\partial t} \exp \left\{ - \int_\delta^t \frac{A^{(N)}}{\epsilon} d\tau \right\} ds, \end{aligned} \quad (3.54)$$

where

$$P_n^{(N)} = \frac{\epsilon}{A^{(N)}} \frac{\partial P_{n-1}^{(N)}}{\partial t}, \quad P_1^{(N)} = \frac{\epsilon}{A^{(N)}} Q^{(N)}.$$

If the derivatives  $\partial P_{k-1}^{(N)}/\partial t$  are finite, then when the ratio of  $\tau_p$  to the characteristic flow time tends to zero, the functions  $a^{(N)}$  are the limits of the asymptotically convergent series

$$a^{(N)} = - \frac{\epsilon}{A^{(N)}} Q^{(N)} + \frac{\epsilon}{A^{(N)}} \frac{\partial}{\partial t} \left( - \frac{\epsilon}{A^{(N)}} Q^{(N)} \right) - \dots . \quad (3.55)$$

We note that in the equations for  $a_{ij}^{(2)}$  and  $a_i^{(3)}$ , i.e., for  $p_{ij}$  and  $q_i$ , the operators  $\mathcal{L}$  and  $D$  are equal to zero, while the operators  $L$  contain terms having  $a^{(0)} = 1$ .

When  $N > 3$ , the expression  $Q^{(N)}$  has the structure (in order of magnitude):

$$\frac{1}{\epsilon} a^{(N)} + \frac{1}{\epsilon} (a^{(N-2)} a^{(2)} + \dots + a^{(2)} a^{(N-2)}) + (a^{(N-3)} + \dots + a^{(N+1)}).$$

Since the determinant of the system

$$A_{\alpha_1 \dots \alpha_N}^{(N)} a_{\alpha_1 \dots \alpha_N}^{(N)} - \mathcal{L}_{\alpha_1 \dots \alpha_N}^{(N)} (a^{(N)}) = 0$$

is not generally equal to zero, it is easy to see that

$$a^{(2)} \sim O(\epsilon), \quad a^{(3)} \sim O(\epsilon) \text{ and } a^{(N)} \leq \epsilon^2 \text{ for } N > 3. \quad (3.56)$$

Therefore, if we expand the values of  $Q^{(N)}$  in expressions (3.55) for  $a_i^{(2)}$  and  $a_i^{(3)}$ , we see that the coefficients  $a^{(4)}$  can make a contribution only in terms of order  $\epsilon^3$  for the stress tensor and the heat flux vector. Therefore, in the Navier-Stokes and Burnett approximations, the twenty-moment equations which are obtained from the exact equations with  $a^{(4)} = 0$ , coincide with the exact equations.

We now examine equations (3.17) for  $a_{ijk}^{(3)}$  in more detail. We write them in the form

$$\begin{aligned} &a_i^{(3)} \delta_{jk} + a_j^{(3)} \delta_{ik} + a_k^{(3)} \delta_{ij} - 9a_{ijk}^{(3)} \\ &= \frac{\epsilon}{A^{(3)}} \left( \frac{\partial T}{\partial x_i} \delta_{jk} + \frac{\partial T}{\partial x_j} \delta_{ik} + \frac{\partial T}{\partial x_k} \delta_{ij} \right) + O(\epsilon^2). \end{aligned} \quad (3.57)$$

Taking their contraction, it is easy to verify that

$$\frac{\partial T}{\partial x_i} = - \frac{4}{5} \frac{A^{(3)}}{\epsilon} a_i^{(3)} + O(\epsilon^2). \quad (3.58)$$

Substituting (3.58) into (3.57), we have

$$\frac{1}{5}(a_i^{(3)}\delta_{jk} + a_j^{(3)}\delta_{ik} + a_k^{(3)}\delta_{ij}) - a_{ijk}^{(3)} = O(\epsilon^2). \quad (3.59)$$

In deriving the thirteen-moment equations from the twenty-moment equations, a single additional assumption (3.35) has been made, to connect  $a_{ijk}^{(3)}$  with  $a_i^{(3)}$ . By comparing (3.35) with (3.59), we see that this assumption is satisfied to an accuracy of order  $O(\epsilon^2)$ . Since the coefficients  $a_{ijk}^{(3)}$  in the equation for  $a_{ij}^{(2)}$  and  $a_i^{(3)}$  appear only in L, then the substitution of  $a_{ijk}^{(3)}$  defined by formula (3.35) can lead to errors in the expression for  $p_{ij}$  and  $q_i$  of order  $\epsilon^3$  at most. Therefore, for Maxwellian molecules, the expressions for  $p_{ij}$  and  $q_i$  in the Navier-Stokes and Burnett approximations, obtained from the thirteen-moment, twenty-moment, and exact-moment equations, are identical. In other words, the solutions of the Boltzmann equations tend asymptotically to the solutions of the Navier-Stokes or Burnett equations when the Knudsen number tends to zero.\*

The definite information which the twenty-moment or thirteen-moment equations can give at small Knudsen numbers is contained in the Burnett equations. Any additional information obtained from the twenty- and thirteen-moment equations is of the same order as was omitted when these were obtained, i.e.,  $O(\epsilon^3)$ .

We have obtained this result for Maxwellian molecules. For arbitrary non-Maxwellian molecules, an infinite sum of derivatives  $a^{(R)}a^{(S)}$  with  $R, S = 1, 2, \dots$  appears in the operator  $D^{(N)}$ . In particular, terms of the form  $a^{(0)}a^{(k)} \equiv a^{(k)}$  appear for all k. Using the properties of the coefficients  $B^{(N, R, S)}$  (see the footnote on page 117), we find that the system of moment equations has the following structure (we omit coefficients of order unity):

$$\begin{aligned} & \epsilon \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_r}{\partial x_r} \right) + \epsilon O(a^{(2)}, a^{(3)}) \\ &= p_{ij} + a^{(4)} + a^{(6)} + \dots + a^{(2)}a^{(2)} + \dots \end{aligned}$$

\*See also §§ 3.7, 3.11, and 3.12.

$$\begin{aligned} & \epsilon \frac{\partial T}{\partial x_i} + \epsilon O(a^{(2)}, a^{(3)}, a^{(4)}) \\ &= q_i + a^{(5)} + a^{(6)} + \dots + a^{(2)}a^{(3)} + \dots, \\ & \quad \epsilon O(a^{(2)}, a^{(3)}, a^{(4)}, \dots, a^{(2k+1)}) \\ &= a^{(2)} + a^{(4)} + \dots + a^{(2)}a^{(2)} + \dots, \\ & \quad \epsilon O(a^{(2)}, a^{(3)}, a^{(4)}, \dots, a^{(2k)}) \\ &= a^{(3)} + a^{(5)} + \dots + a^{(2)}a^{(3)} + \dots \\ & \quad (k = 2, 3, \dots). \end{aligned} \quad (3.60)$$

From this it may be seen that for non-Maxwellian molecules the coefficients  $a^{(N)} \sim \epsilon$  for any  $N > 0$ . To find the relationships between  $p_{ij}$  and the velocity derivatives, as well as those between  $q_i$  and the temperature derivatives, we must solve an infinite system of equations. In any case, for the given structure of our equations we obtain the relation

$$\begin{aligned} p_{ij} &= -(\tau_p p + a^{(4)} + a^{(6)} + \dots) \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_r}{\partial x_r} \right) \\ &= -\mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_r}{\partial x_r} \right), \\ q_i &= -\left( \frac{15}{4} \tau_p p + a^{(5)} + a^{(7)} + \dots \right) \frac{\partial(kT/m)}{\partial x_i} = -\lambda \frac{\partial T}{\partial x_i} \quad (3.61) \end{aligned}$$

(as above,  $a^{(4)}, a^{(5)}, \dots$  represent the whole set of coefficients of order 4, 5, etc.).

In the twenty- and thirteen-moment approximations, all the  $a^{(N)}$  with  $N > 3$  are neglected. Since in general all  $a^{(N)} \sim \epsilon$  (for Maxwellian molecules  $a^{(N)} \leq \epsilon^2$  when  $N > 3$ ), then in the twenty-moment approximation we determine  $\mu$  and  $\lambda$  up to  $O(\epsilon)$ , i.e., with an error of the same order as these quantities themselves possess. It may be shown that the coefficients  $\mu$  and  $\lambda$  obtained in the thirteen-moment approximation correspond in the first approximation  $[\mu]_1$  and  $[\lambda]_1$  to the values of the viscosities in the Chapman-Enskog method (see § 3.8).

Thus, in the general case when  $\epsilon \rightarrow 0$  we cannot obtain the Navier-Stokes equations from the twenty- or thirteen-moment equations, with exact values of viscosity and of thermal conductivity. However, as has already been noted, the thirteen- or twenty-moment equations may be obtained by the method of moments, if we

postulate the form for the distribution function [for example, (3.34)]. These equations may therefore prove to be sufficiently accurate, for arbitrary Knudsen numbers, for flows for which the postulated distribution function is a good approximation of the true distribution function. Moreover, there is evidently a certain class of flows for which the chosen approximation function is an exact solution of the Boltzmann equation.\*

It is interesting to note that the thirteen-moment equations for unsteady flow are of hyperbolic type,† while the Navier-Stokes equations are parabolic. Similarly, the steady thirteen-moment equations are elliptic (at small flow velocities) or hyperbolic (at large velocities), while the steady Navier-Stokes equations are always elliptic.

### §3.4. The Method of Moments. Discontinuous Distribution Functions

In the preceding section we examined one possible representation of the distribution function as the series (2.7). We may represent the distribution function as an expansion not only in Hermite polynomials, but also in terms of any other functions. The choice of representation for the distribution function is determined first by the rapidity of convergence of the chosen series, since, in order to obtain a system of moment equations that is acceptable in practice, we require the best approximation while retaining the minimum number of terms in the series. However, as we shall see later (see §§4.2, 5.1, 6.5), the distribution function is very frequently discontinuous with respect to velocities at each point of the flow. In that case, the series [in particular the series (3.1) of Hermite polynomials], which represents the distribution function, if it converges at all, does so slowly.

We introduce spherical coordinates so that

$$\xi_x = \xi \cos \theta, \quad \xi_y = \xi \sin \theta \cos \varphi, \quad \xi_z = \xi \sin \theta \sin \varphi, \quad (4.1)$$

\*See §4.1, in which we give exact solutions of the Boltzmann equation, which have the form of a locally Maxwellian distribution. Similarly, we may seek exact solutions which have distribution functions of more general form [for example, the form (3.34)]. However, it is difficult to determine how wide the class of corresponding flows is.

†See H. Grad, Commun. Pure Appl. Math., Vol. 2, No. 4 (1949).

where  $\xi$  is the absolute velocity of the molecules. Let the surface of discontinuity be given by the equation

$$\theta = T(\varphi, x, t). \quad (4.2)$$

The conical surface (4.2) divides the space into two regions at each point  $x$  and time  $t$ . The distribution function may be represented in the form

$$f = \begin{cases} f_A & \text{in region } A, \\ f_B & \text{in region } B. \end{cases} \quad (4.3)$$

If there are no more surfaces of discontinuity in regions A and B, then each of the functions  $f_A$  and  $f_B$  may be expanded in a series in regions A and B, respectively, which, generally speaking, must converge more rapidly than the series which represents the complete distribution function  $f$ .\*

Since the distribution function is different in general in regions A and B, it is easier to construct a function that is a good approximation in each of the regions than in the whole region. We may write the functions  $f_A$  and  $f_B$  in the form (2.8):

$$\left. \begin{aligned} f_A &= F_A(\xi, A_1, \dots, A_m) && \text{in region } A, \\ f_B &= F_B(\xi, B_1, \dots, B_n) && \text{in region } B \end{aligned} \right\}, \quad (m, n) \ll \infty. \quad (4.4)$$

The functions  $F_A$  and  $F_B$  are continuous in regions A and B, respectively, and take account of the discontinuity on the boundary of the region (4.2):  $F_A \neq F_B$ . Substituting the approximation (4.4) into the definition of the moments, we may establish the relationship between the coefficients  $A_i$  and  $B_i$  and the moments. The differential equations for the moments (or for the coefficients  $A_i$  and  $B_i$ ) may be constructed in two ways in a manner similar to that described in §§3.2 and 3.3. By multiplying the Boltzmann equation by the appropriate powers of velocities, we may integrate either over all of velocity space, or in the regions A and B separately. When we integrate over region A (or B), the collision integrals do not vanish even when the Boltzmann equation is multiplied by the summational invariants. We therefore use a mixed method, where

\*If there are more surfaces of discontinuity within regions A and B, the distribution function must be expanded separately in each of the regions of discontinuity.

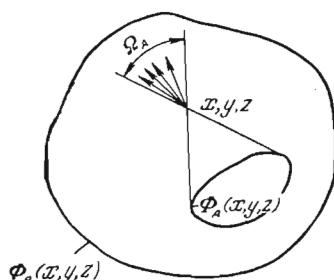


Fig. 12

The equations for the first moments are constructed by integration over all velocity space, and those for the higher moments by integration over the semispaces. All these approaches are equivalent, and the specific form of the approximating functions and the specifications of a definite problem will determine our choice among them (see §§4.2, 4.4).

The simplest form of discontinuous function is the so-called two-sided Maxwell approximation.\* In this approximation, at each point of the flow in regions A and B the distribution function is replaced by its Maxwellian distribution. For region A,

$$f_A = n_A(t, \mathbf{x}) \left( \frac{m}{2\pi k T_A(t, \mathbf{x})} \right)^{3/2} \exp \left\{ -\frac{m}{2kT_A(t, \mathbf{x})} [\xi - \mathbf{u}_A(t, \mathbf{x})]^2 \right\}. \quad (4.5)$$

The corresponding expression for region B is obtained by replacing subscript A by B.

The distribution function (4.5) may give an exact solution for free molecular flow, if molecules are reflected from a wall diffusely with a Maxwellian distribution. Let us examine, for example, the heat transfer (Fig. 12) between the fixed surfaces  $\Phi_A(x, y, z)$  and  $\Phi_B(x, y, z)$ , where the reflected molecules have temperatures  $T_{rA}$  and  $T_{rB}$ , respectively. Then the distribution function (4.5) gives a free molecular solution, if we put  $n_A = n_{rA}$  and  $T_A = T_{rA}$ , for the molecular velocity vectors lying within the solid angle  $\Omega_A$ , and  $n_B = n_{rB}$  and  $T_B = T_{rB}$  for velocities within the solid angle  $\Omega_B$ , equal to  $4\pi - \Omega_A$ , and if  $\mathbf{u}_A \equiv \mathbf{u}_B \equiv 0$ . The situation is the same if the surface is immersed in an infinite stream with temperature  $T_\infty$  and velocity  $\mathbf{U} \equiv (U_1, U_2, U_3)$  at infinity; then within the solid angle  $\Omega_B$  (i.e., for molecules arriving at a given point  $\mathbf{x}$  from infinity) in the free molecular case we must put  $n_B = n_\infty$ ,  $T_B = T_\infty$ , and  $\mathbf{u}_{iB} = \mathbf{U}_i$ .

\*C. Liu and L. Lees, "Rarefied Gas Dynamics," Second Symposium, Academic Press, New York, 1961.

In the other limiting case of very small mean free paths,  $n_A$  may become equal to  $n_B$ ,  $T_A = T_B$ , and  $\mathbf{u}_{iA} = \mathbf{u}_{iB}$ , and the distribution function goes over to a local-Maxwellian function of the Eulerian approximation (see §§3.7 and 3.8).

By means of the function (4.5) we can satisfy exactly the boundary condition of a Maxwellian distribution of reflected molecules (the next section gives greater detail regarding the boundary conditions).

We may obtain a more accurate (but considerably more complex) general approximation for the distribution function from the following examination of the model Boltzmann equation.\*

We examine formula (8.25) of Chapter II for steady flow with  $n = \text{const}$ ; we have

$$f(\mathbf{x}, \xi) = f(\mathbf{x}_w, \xi) \exp \left\{ -\frac{An}{\xi} (s - s_w) \right\} + \frac{An}{\xi} \int_{s_w}^s f_0(l, \xi) \exp \left\{ -\frac{An}{\xi} (s - l) \right\} dl, \quad (4.6)$$

where  $\mathbf{x}_w$  is the intersection of the boundary and the straight line drawn from the point  $\mathbf{x}$  in the direction  $-\xi$ , and  $s$  and  $l$  are parameters which define the position on this straight line ( $l = s$  at the point  $\mathbf{x}$  and  $l = s_w$  at the point  $\mathbf{x}_w$ ). We integrate the integral on the right of (4.6) by parts twice:

$$f(\mathbf{x}, \xi) = f_0(\mathbf{x}, \xi) - \frac{\xi}{An} \frac{df_0(\mathbf{x}, \xi)}{ds} + [f(\mathbf{x}_w, \xi) - f_0(\mathbf{x}_w, \xi)] + \frac{\xi}{An} \frac{df_0(\mathbf{x}_w, \xi)}{ds} \exp \left\{ -\frac{An}{\xi} (s - s_w) \right\} + \frac{\xi}{An} \int_{s_w}^s \frac{d^2 f_0}{dl^2} \exp \left\{ -\frac{An}{\xi} (s - l) \right\} dl. \quad (4.7)$$

We write this expression in the form

$$f(\mathbf{x}, \xi) = f(\mathbf{x}_w, \xi) \exp \left\{ -\frac{An}{\xi} (s - s_w) \right\} +$$

\*S. F. Shen, "Rarefied Gas Dynamics," Third Symposium, Academic Press, 1963.

$$\begin{aligned}
 & + \left[ f_0(\mathbf{x}, \xi) - \frac{\xi}{An} \frac{df_0(\mathbf{x}, \xi)}{ds} \right] \left( 1 - \exp \left\{ -\frac{An}{\xi} (s - s_w) \right\} \right) \\
 & + \left\{ \left[ f_0(\mathbf{x}, \xi) - \frac{\xi}{An} \frac{df_0(\mathbf{x}, \xi)}{ds} \right] - \left[ f_0(\mathbf{x}_w, \xi) - \frac{\xi}{An} \frac{df_0(\mathbf{x}_w, \xi)}{ds} \right] \right\} \\
 & \times \exp \left\{ -\frac{An}{\xi} (s - s_w) \right\} + \frac{\xi}{An} \int_{s_w}^s \frac{d^2 f_0}{dl^2} \exp \left\{ -\frac{An}{\xi} (s - l) \right\} dl. \quad (4.8)
 \end{aligned}$$

In free molecule flow the distribution function does not vary along the trajectory of the particles, and the third and fourth terms on the right side of (4.8) are equal to zero. In the other limiting case of small mean free paths, i.e., when  $An \rightarrow \infty$ , the last terms are also much smaller than the second. Therefore, to a crude approximation we may neglect these terms and approximate the distribution function by the expression

$$\begin{aligned}
 f(\mathbf{x}, \xi) = & f(\mathbf{x}_w, \xi) \exp \left\{ -\frac{An(x)}{\xi} (s - s_w) \right\} \\
 & + \left[ f_0(\mathbf{x}, \xi) - \frac{\xi}{An} \frac{df_0(\mathbf{x}, \xi)}{ds} \right] \left( 1 - \exp \left\{ -\frac{An(x)}{\xi} (s - s_w) \right\} \right). \quad (4.9)
 \end{aligned}$$

As will be shown in §3.6, the expression in the square brackets is none other than the distribution function  $f_{N.S.}$ , corresponding to the Navier-Stokes approximation. Thus, the approximation (4.9) for large mean free paths ( $An \rightarrow 0$ ) goes over to the exact solution for free molecular flow, and for small mean free paths ( $An \rightarrow \infty$ ) — to the Navier-Stokes distribution function. The distribution function (4.9) is in general discontinuous and has a different character in the different regions of velocity space in accordance with the nature of the boundary conditions.

Formula (4.9) is obtained with the aid of the approximate Boltzmann equation. However, instead of the approximate Navier-Stokes distribution function we may put the exact function corresponding to the full Boltzmann equation in the square brackets. While retaining the general structure of the approximation, we may put in the square brackets some other suitable function, for example, the two-sided Maxwellian distribution

$$f(\mathbf{x}, \xi) = f(\mathbf{x}_w, \xi) \exp \left\{ -\frac{An(x)}{\xi} (s - s_w) \right\} +$$

$$+ f_{A, B}(\mathbf{x}, \xi) \left( 1 - \exp \left\{ -\frac{An}{\xi} (s - s_w) \right\} \right), \quad (4.10)$$

where  $f_{A, B}$  is the two-sided Maxwellian distribution (4.5), equal to  $f_A$  in region A and  $f_B$  in region B.

The function (4.9) possesses one appreciable advantage: only the five hydrodynamic quantities appear in it at the point  $\mathbf{x}$ . Therefore, by substituting this approximation into the expression for the stress tensor and the heat flux vector, which appear in the conservation equations (1.8)–(1.10), we obtain a closed system of five equations for the five hydrodynamic unknowns, and take account at the same time of the boundary conditions. In a manner of speaking, these equations are generalized Navier-Stokes equations. The approximation (4.10), like (4.5), contains a large number of unknown parameters, and therefore requires us to use, together with the conservation equations, additional moment equations, the choice of which is somewhat arbitrary.

In the general case, the macroscopic equations obtained are very complex. However, we may gain some idea of the accuracy that they give, even without writing them out. For example, we examine the structure of a shock wave. Since in that case the boundary conditions are set at infinity, the exponential terms vanish, and with the approximation (4.9) we arrive at the Navier-Stokes description of the wave. As was shown in §2.8, the quantity  $\xi/An$  is proportional to the mean free path. Therefore, the exponential terms are appreciable only at distances of the order of several mean free paths from the boundaries. The major part of the flow is described by the Navier-Stokes equations in that approximation. In particular, in the problem of flow over a body in terms of the Navier-Stokes equations, the shock structure is also obtained. However, the Navier-Stokes equations satisfactorily describe the structure of a shock wave only at Mach numbers close to unity (see §4.4).

The particle density at the point  $\mathbf{x}$  appears in the exponents of the exponentials. However, we may allow for the variation of the density over the whole path of the molecules from the boundary to the point  $\mathbf{x}$ . When we derive formula (4.8), we must discard the assumption that  $n = \text{const}$ . In that case, instead of (4.9), we obtain

$$f(\mathbf{x}, \xi) = f(\mathbf{x}_w, \xi) \exp \left\{ -\frac{A}{\xi} \int_{s_w}^s n dl \right\} + f_{h.c.}(\mathbf{x}, \xi) \left( 1 - \exp \left\{ -\frac{A}{\xi} \int_{s_w}^s n dl \right\} \right). \quad (4.11)$$

When there is a strong variation of density in the flow, this approximation should give better results. However, it is not clear whether the considerable complication of the corresponding macroscopic equations is justified and by the refinement obtained.

Similar approximations have up to now been applied only to one-dimensional flows, mainly to the Couette problem. Some examples of these solutions are given in Chapter IV.

### §3.5. Boundary Conditions for the Moment Equations

In §§2.9 and 2.10 we formulated macroscopic boundary conditions for the Boltzmann equation. In a specific problem the moment equations replace the Boltzmann equation up to a certain degree of accuracy. We must, to the same approximation, replace the boundary conditions for the distribution function by a number of macroscopic conditions for the moments. A denumerably infinite set of boundary conditions may be constructed for the moments. In fact, we write down the general microscopic boundary condition (9.6) of Chapter II:

$$f_r(\xi_r) = \int_{\xi \cdot \mathbf{n} < 0} K(\xi_i, \xi_r) f_i(\xi_i) d\xi_i. \quad (5.1)$$

Let us construct the moment equations, for which boundary conditions are being sought, with the aid of the approximating functions

$$f(t, \mathbf{x}, \xi) = F(\xi, A_1, \dots, A_N) = F_1(\xi, M_1, \dots, M_N). \quad (5.2)$$

We multiply the left and right sides of (5.1) by a function  $\varphi_r(\xi_r)$  and integrate with respect to  $\xi_r$ , substituting the approximating function (5.2) for  $f_{r,i}$ . As a result of these operations, we obtain a relation between the functions  $A_j$  or the moments  $M_j$ . The number of functions  $\varphi_r$  may be chosen to be finite or infinite. Correspondingly, we may obtain a finite or an infinite number of con-

ditions for the moments at the boundary. For the function  $\varphi_r$  we may choose, for example, polynomials in  $\xi_r$ , or any other function convenient for the given problem. The family of functions  $\varphi_r$  may be chosen to be complete in the space  $\xi \cdot \mathbf{n} > 0$ .

The question arises as to how many and what kind of conditions we must set for a given problem with given moment equations on each of the sections of the boundary. It is evident that we should not take the number of conditions simply equal to the number of moments or to the order of the equations. It is well known, for example, that the boundary conditions for the Euler equations become different with sub- and supersonic velocities. It is therefore impossible to give any universal recipe. For each approximating distribution function, and for each new system of moment equations, it is necessary to investigate possible ways of specifying the boundary conditions. Since the moment equations are more complicated than the Euler or Navier-Stokes equations in the overwhelming majority of cases, it is easy to see the complexity of this investigation.

In general, the approximating function (5.2) or the discontinuous functions do not coincide with the distribution function of the reflected molecules for any value of the macroscopic parameters which appear in them. In that case the boundary condition may, on the average, be satisfied only approximately.

We call the approximating function adapted to the boundary conditions, if it coincides with the distribution functions for the reflected molecules at each point of the boundary with a certain choice of their macroscopic parameters. For example, consider the flow past a body of an equilibrium stream of gas with velocity  $\mathbf{U}_\infty$ , density  $n_\infty$ , and temperature  $T_\infty$ . And let the molecules be reflected from the surface of the body with a Maxwellian distribution whose temperature equals that of the wall:

$$f_r = n_r \left( \frac{h_w}{\pi} \right)^{3/2} e^{-h_w \xi^2}, \quad \left( h_w = \frac{m}{2kT_w} \right). \quad (5.3)$$

Then, as an approximating function adapted to the boundary conditions we may choose, for example, the generalized two-sided Maxwellian distribution (Fig. 12) given in Eq. (5.4).

$$f(t, x, \xi) = \begin{cases} n_A \left( \frac{h_A}{\pi} \right)^{3/2} \exp \{-h_A(\xi - u_A)^2\} (1 + A_i \xi_i + \\ \quad + A_{ij} \xi_i \xi_j + \dots) & \text{in region } A, \\ n_B \left( \frac{h_B}{\pi} \right)^{3/2} \exp \{-h_B(\xi - u_B)^2\} (1 + B_i \xi_i + \\ \quad + B_{ij} \xi_i \xi_j + \dots) & \text{in region } B. \end{cases} \quad (5.4)$$

If at infinity we put  $n_A = n_B = n_\infty$ ,  $T_A = T_B = T_\infty$ ,  $u_A = u_B = \mathbf{U}_\infty$ , and  $A_i = A_{ij} = \dots = B_i = B_{ij} = \dots = 0$ , then the function (5.4) exactly satisfies the conditions at infinity. Similarly, if at the body we put  $\mathbf{u}_A = 0$ ,  $A_i = A_{ij} = \dots = 0$  and  $T_A = T_w$ , the function (5.4) for molecules leaving the wall goes over exactly to the distribution function of the reflected molecules (5.3), in which  $n_r$  is determined from the condition of no flow across the solid surface

$$\int_{\xi \cdot n > 0} (\xi_r \cdot \mathbf{n}) f_r d\xi_r = - \int_{\xi \cdot n < 0} (\xi_i \cdot \mathbf{n}) f_i d\xi_i. \quad (5.5)$$

This condition relates  $n_A$  on the boundary with the functions  $n_B$ ,  $T_B$ ,  $\mathbf{u}_B$ ,  $B_i$ ,  $B_{ij}$ , etc. We note that the boundary conditions at the wall do not require the functions  $B_i$ ,  $B_{ij}$ ,  $\dots$  to vanish.

The boundary conditions may be satisfied exactly by means of an approximating function adapted to the boundary conditions. However, the form of the distribution function for the reflected molecules is determined by the properties of the surface. Having satisfied the boundary conditions exactly by means of a distribution function adapted to these boundary conditions, we completely determine the boundary values of the moments entering into the approximating function, i.e., in essence, we formulate a boundary problem independently of the differential moment equations themselves, as obtained from the Boltzmann equation by means of the approximating function. It is evident that, in the general case, the moment equations cannot have solutions for such boundary conditions. The question naturally arises: for what adapted approximating function does the boundary value problem prove to be correct for the differential equations corresponding to this function? \*

\*We recall that by means of the same approximating function we may construct an infinite number of different systems of moment equations, obtained by a different choice of functions of the velocities by which the Boltzmann equation is multiplied when constructing the moment equations.

In any case, the following statement is valid. For an approximating function of type (2.7) or (4.4) adapted to the boundary conditions and giving an exact solution of the Boltzmann equation in the free molecular limit, the resulting boundary value problem is correct for the moment equations corresponding to this function, independently of their choice. It is assumed here, of course, that the Boltzmann equation has a solution for the given microscopic boundary conditions and that the approximating function does not bring into the collision integral singularities which are not appropriate to that integral. For example, the generalized two-sided Maxwellian distribution (5.4) is included among functions which satisfy that set of conditions.

The moment equations obtained with the aid of the approximating functions (2.7) and (4.4) are in general inhomogeneous quasi-linear differential equations of the first order. The inhomogeneous part of the equations, which does not depend on the collision integral, is an algebraic function of the desired moments. The type of that system of equations, and, therefore, also the nature of the boundary value problem corresponding to the system, is determined by the differential parts of the moment equations, obtained from the differential operator of the Boltzmann equation. It is evident that the differential part of the moment equations is identical for all Knudsen numbers. According to the definition, the approximating function for a specific choice of moments gives an exact solution of the Boltzmann equation when  $Kn = \infty$ , i.e., when the right-hand side is equal to zero. Therefore, the moments appearing in it must exactly satisfy any system of homogeneous moment differential equations (without integral part) obtained by means of that approximating function. Here the boundary values of the moments are chosen so that the approximating function will exactly satisfy the microscopic conditions. But, since, when  $Kn = \infty$  the homogeneous system of moment equations with these boundary conditions has a solution, then the statement of the boundary value problem which is based on the assertions made above is also valid for the inhomogeneous system (i.e., for arbitrary Knudsen numbers).

In the example cited above, the function (5.4), which was adapted to the microscopic boundary conditions, also gives a solution to the problem for the free molecular regime. It is easy to see that with  $Kn = \infty$  all the coefficients  $A_i = A_{ij} = \dots = B_i = B_{ij} = \dots = 0$  over the whole flow. However, the boundary condi-

tions on the body require that only the coefficients  $A_i = A_{ij} = \dots$  go to zero at the boundary. Therefore, in the general case, when  $Kn \neq \infty$  the coefficients A and B are not equal to zero in the flow field and go to zero when  $Kn = \infty$  only because of the solution of the moment equation.

It should be noted that these assertions are not true for approximating functions which contain the Knudsen number explicitly as a parameter, since, when  $Kn \rightarrow \infty$  it drops out of the equations, and the moment equations with  $Kn = \infty$  may have a different differential part and, therefore, a different formulation of the boundary conditions than with  $Kn \neq 0$ . For example, among such approximating functions are the functions (4.9)–(4.11) examined in the previous section.

### §3.6. Methods of Expansion in Powers of a Small Parameter

It was shown in §2.11 that the Knudsen number which characterizes the degree of rarefaction of a gas enters into the Boltzmann equation. In the limiting cases when  $Kn \gg 1$  and  $Kn \ll 1$ , a small parameter equal to  $\epsilon = Kn^{-1}$  and  $\epsilon = Kn$ , respectively, appears in the Boltzmann equation. It is natural, at large and at small Knudsen numbers, to seek a solution of the Boltzmann equation in the form of an expansion in powers of the small parameter;

$$f(t, x, \xi) = \sum_{k=0}^{\infty} \epsilon^k f^{(k)}(t, x, \xi). \quad (6.1)$$

Before going on to a discussion of methods of expansion in powers of a small parameter for the full Boltzmann equation, we examine certain qualitative special features of such expansions for the model equation (8.22) of Chapter II.

1. We write that equation in dimensionless coordinates in the integral form (8.25) of Chapter II:

$$\begin{aligned} f(t, x, \xi) &= f(t_0, x - \xi(t - t_0), \xi) \exp \left[ -\frac{1}{\epsilon} \int_{t_0}^t n(s, x - \xi(t - s)) ds \right] \\ &+ \frac{1}{\epsilon} \int_{t_0}^t \left\{ n(\tau, x - \xi(t - \tau)) f_0(\tau, x - \xi(t - \tau), \xi) \times \right. \end{aligned}$$

$$\left. \times \exp \left[ -\frac{1}{\epsilon} \int_{\tau}^t n(s, x - \xi(t - s)) ds \right] \right\} d\tau. \quad (6.2)$$

Here the density of particles is referred to a characteristic value  $n_0$ , the time to a certain time  $\Theta$ , the velocity to the characteristic velocity of the molecules U, and  $x$  to the distance which a molecule moving with the characteristic velocity U covers in time  $\Theta$ , i.e., to  $U\Theta$ . We choose  $\Theta$  to be the smaller of the following two times: either a characteristic time of the flow or the time during which a molecule moving with characteristic velocity U travels over the characteristic length L. In steady flow,  $\Theta = L/U$ . The parameter  $\epsilon = (An_0\Theta)^{-1} = \tau/\Theta = Kn$ , where  $\tau$  is the time between collisions.

The integration is carried out along the trajectory of the molecules moving with velocity  $\xi$ . The Maxwellian distribution  $f_0$  depends on  $t$  and  $x$  through the macroscopic quantities  $n(t, x)$ ,  $u(t, x)$ , and  $T(t, x)$ , which appear in it.

We carry out N integrations by parts of the integral which contains  $f_0$  on the right side of (6.2), then \*

$$\begin{aligned} f(t, x, \xi) &= \sum_{k=0}^N \epsilon^k f^{(k)}(t, x, \xi) + \left[ f(t_0, x - \xi(t - t_0), \xi) \right. \\ &- \left. \sum_{k=0}^N \epsilon^k f^{(k)}(t_0, x - \xi(t - t_0), \xi) \right] \exp \left\{ -\frac{1}{\epsilon} \int_{t_0}^t n(s, x - \xi(t - s)) ds \right\} \\ &+ \epsilon^N \int_{t_0}^t \frac{df^{(N)}}{d\tau} \exp \left\{ -\frac{1}{\epsilon} \int_{\tau}^t n(s, x - \xi(t - s)) ds \right\} d\tau, \quad (6.3) \end{aligned}$$

where

$$\begin{aligned} f^{(k)} &= \frac{(-1)}{n} \frac{df^{(k-1)}}{dt}, \\ f^{(0)} &= f_0. \end{aligned}$$

\* M. N. Kogan, Prikl. Mat. Mekhan., No. 4 (1958).

We investigate what happens when the solution of the Boltzmann equation may be represented by the series (6.1) or by the first term of formula (6.3):

$$f(t, \mathbf{x}, \xi) = \sum_{k=0}^N \epsilon^k f^{(k)}(t, \mathbf{x}, \xi). \quad (6.4)$$

Let the distribution function  $f(t_0, \mathbf{x}, \xi) = f(t_0)$  be assigned at the initial instant  $t = t_0$ . The solution (6.3) contains two series of type (6.4): one at the point  $(t, \mathbf{x})$ , the other at the point  $(t_0, \mathbf{x} - \xi(t - t_0))$ . The quantity in square brackets in (6.3) is the difference between the distribution function at the initial time at some point and its expansion in the series (6.4) at the same point. For the series (6.4) to converge at the point  $(t, \mathbf{x})$  it is necessary that the remainder of the series tend to zero as  $N$  increases, and that the distribution function  $f(t_0)$  at the initial time be representable by a series (6.4). Since an arbitrary instant may be taken as the time origin, it is necessary that the distribution function be representable by the series (6.4) at any time and at any point of the flow.

The remainder of the series is expressed in terms of the Maxwellian distribution  $f_0$ , which depends on the hydrodynamic quantities  $n, \mathbf{u}$ , and  $T$ , which depend in turn on the distribution function  $f$ . Since we do not know the properties of the desired distribution function beforehand, it is in general impossible to evaluate the remainder of the series, and, therefore, to establish its convergence in the ordinary sense.

Still, by restricting ourselves to a finite number of terms of the series, and by letting  $\epsilon \rightarrow 0$ , we may expect that the series (6.4) will converge asymptotically, i.e.,

$$\lim_{\epsilon \rightarrow 0} \left( \frac{1}{\epsilon^N} \left| f - \sum_{k=0}^N \epsilon^k f^{(k)} \right| \right) = 0.$$

For asymptotic convergence it is sufficient that  $N + 1$  derivatives with respect to the hydrodynamic quantities be finite. The influence of the initial time, i.e., of the term with the square brackets in (6.3), decays exponentially for  $\epsilon \rightarrow 0$  and for fixed  $t$  and  $\mathbf{x}$ . For a fixed  $\epsilon$  the influence of the initial time decays exponentially as  $t - t_0$  increases. Therefore, as  $\epsilon \rightarrow 0$ , and as we depart from the

initial time, the solution of the Boltzmann equation tends asymptotically to a solution of type (6.4). However, there may be special cases when the series (6.4) converges in the ordinary sense.

Let the series (6.4) converge in the ordinary or in the asymptotic sense. If we restrict ourselves to the first term of the expansion in the series, then far from the time origin or from a boundary we may put

$$f(t, \mathbf{x}, \xi) = f^{(0)}(t, \mathbf{x}, \xi) = f_0(t, \mathbf{x}, \xi) = n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left\{ - \frac{m}{2kT} (\xi - u)^2 \right\}.$$

It is easy to verify that, in this case

$$P_{ij} = \delta_{ij} nkT = \delta_{ij} p \text{ and } q_i = 0. \quad (6.5)$$

Substituting these values into the conservation equations (1.8)–(1.10), which have the same form for the model equation as for the exact Boltzmann equation, since the collision integral drops out, we obtain the ordinary Euler equations:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} &= 0, \\ \left( \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \right) u_i &= - \frac{1}{\rho} \frac{\partial p}{\partial x_i}, \\ \frac{3}{2} \rho R \left( \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \right) T &= - p \frac{\partial u_i}{\partial x_i}. \end{aligned} \quad (6.6)$$

Restricting ourselves to two terms of the series, we have, in dimensionless quantities

$$f(t, \mathbf{x}, \xi) = f_0(t, \mathbf{x}, \xi) + \frac{1}{An} \frac{df_0}{dt}, \quad \left( \frac{d}{dt} = \frac{\partial}{\partial t} + \xi_i \frac{\partial}{\partial x_i} \right). \quad (6.7)$$

The partial derivatives with respect to  $t$  of the hydrodynamic quantities which appear here may be excluded by means of the Euler equations (6.6). In the Euler equations quantities of order  $\epsilon$  have been discarded. Therefore, excluding derivatives with respect to  $t$  from (6.7), we drop quantities of order  $\epsilon^2$ . Carrying through some laborious but uncomplicated calculations, we obtain the result that the distribution function has the form

$$f(t, \mathbf{x}, \xi) = f_0 \left[ 1 + \frac{p_{ij}}{2p} \left( \frac{m}{2T} \right) c_i c_j - \frac{q_i}{p} \left( \frac{m}{kT} \right) \left( 1 - \frac{c^2}{5} \frac{m}{kT} \right) c_i \right], \quad (6.8)$$

where

$$p_{ij} = P_{ij} - \delta_{ij} p = -\mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right), \quad q_i = -\lambda \frac{\partial T}{\partial x_i} \quad (6.9)$$

$$\left( \mu = \frac{kT}{A}, \quad \lambda = \frac{5}{2} \frac{k^2 T}{Am} \right).$$

The expressions (6.9) correspond to the Navier-Stokes approximation. Retaining three terms of the series and eliminating the derivatives with respect to  $t$  from the second term by means of the Navier-Stokes equations, and from the third term by means of the Euler equations, we obtain the distribution function for the Burnett approximation. Substituting into the expressions for the stress tensor and the heat flux vector which appear in the conservation equations (1.8)-(1.10), we obtain the Burnett equations, etc.

It is easy to see that a solution in the form of an infinite or truncated series (6.4), which converges in the ordinary or in the asymptotic sense, cannot represent a general solution. In fact, according to (6.4), the distribution function at some point  $(t, \mathbf{x})$  is completely determined by the hydrodynamic quantities  $n, \mathbf{u}$ , and  $T$  at that point. But the values of the hydrodynamic quantities at any time are determined in terms of the approximation with the help of the Euler, Navier-Stokes, etc., equations according to the values of the hydrodynamic quantities when  $t = t_0$ . Since the hydrodynamic quantities are integrals with respect to  $\xi$  of the distribution function, it is clear that an infinite set of different initial distribution functions leads to the same initial hydrodynamic data. Therefore, the solution in the form (6.4) may at best represent one of the infinite set of solutions which correspond to different initial distribution functions.

On the other hand, the general solution of the model equations (6.3) tends asymptotically to the solution (6.4) as  $t - t_0$  increases and  $\epsilon \rightarrow 0$ . Thus, we may expect that solutions of the type (6.4), and, therefore, the Euler, Navier-Stokes, etc., equations are applicable as  $\epsilon \rightarrow 0$  at internal points of the flow region at some distance from the boundaries or from the initial state (when  $t - t_0 \gg \epsilon$ ). Near the boundaries or the initial time, i.e., when  $t - t_0 \ll \epsilon$ , the second term of the solution (6.3) is of the same order as the first.

The question naturally arises: what initial and boundary

conditions must be satisfied by Euler, Navier-Stokes, etc., equations to be valid at internal points? It is easy to see that a solution of the hydrodynamic equations, obtained from initial hydrodynamic data, calculated from the true initial distribution function, differs by a quantity of order  $\epsilon$  from the asymptotic solution, to which the solution of the Boltzmann equation tends when  $t - t_0 > \epsilon$  and  $\epsilon \rightarrow 0$ , although this latter solution asymptotically satisfies the same hydrodynamic equations. In fact, we write (6.3) and (6.4) respectively, in the form

$$f = f_0 + \frac{(-1)}{n} \frac{d}{dt} \sum_{k=0}^N f^{(k)} + \delta, \quad (6.3a)$$

$$f = f_0 + \frac{(-1)}{n} \frac{d}{dt} \sum_{k=0}^N f^{(k)}. \quad (6.4a)$$

Multiplying (6.4a) by the function  $\psi_i(\xi)$  and integrating with respect to  $\xi$  for  $N = 0, 1, \dots$ , we obtain, respectively, the Euler equations, the Navier-Stokes equations, etc. When we integrate the general solution of (6.3a) with respect to  $\xi$ , we obtain the same hydrodynamic equations which will have integrals of  $\delta$  with respect to  $\xi$  on their right sides. Since  $\delta \rightarrow 0$  when  $t - t_0 \gg \epsilon$ , the exact inhomogeneous equations tend asymptotically to the homogeneous equations of hydrodynamics outside some initial layer.

If the function  $\delta$  were known, the solution of the inhomogeneous hydrodynamic equations with initial hydrodynamic data, calculated according to an assigned initial distribution function, would be the solution of the Boltzmann equation at the internal points.

Since  $\delta \sim f \neq 0$  in a region of order  $\epsilon$ , the solution of the ordinary homogeneous hydrodynamic equations with the same boundary conditions differ from the exact solutions by quantities of order  $\epsilon$ . Therefore, for sufficiently small  $\epsilon$ , i.e., for sufficiently small Knudsen number, the hydrodynamic equations are applicable at internal points of the flow, but with boundary conditions differing by a quantity of order  $\epsilon$  from the true hydrodynamic quantities as assigned at time zero. To obtain the correct initial data, we must investigate the structure of the initial layer. The same situation obtains with the boundary conditions. We shall come back in

Chapter V to the formulation of boundary conditions for the Navier-Stokes equations, i.e., to the investigation of the boundary layer.

Expressions which follow from the relaxation equation were introduced above [see Eq. (6.9)] for the viscosity and thermal conductivity. Comparing those expressions with the corresponding expressions obtained for Maxwellian molecules in §3.3 [Eq. (3.47)], it is easy to see that, by making an appropriate choice of the constant A which appears in the relaxation equation, we may obtain the correct dependence on temperature for the viscosity or the thermal conductivity, but not for both quantities together. The ratio of the viscosity to the thermal conductivity, obtained from the relaxation equation, is smaller than the exact value by a factor of 1.5. Correspondingly, the Prandtl number  $\text{Pr} = \mu c_p / \lambda$  for a gas described by the model equation is equal to unity instead of  $\text{Pr} = 2/3$  for a Maxwellian gas. This fact may be used as a measure of the accuracy of the model equation. Evidently the model equation gives results of greatest accuracy in problems in which processes related to viscosity or to thermal conductivity, appear, but not the two together.

2. We now examine another limiting case corresponding to large Knudsen numbers. It is natural to attempt to expand the distribution function in a series of powers of  $\text{Kn}^{-1} = \epsilon$ . We return to the model equation (8.22) of Chapter II, written in the dimensionless form

$$\frac{df}{dt} = \epsilon n(f_0 - f) = \epsilon J. \quad (6.10)$$

We seek a solution of this equation in the form\*

$$f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \dots . \quad (6.11)$$

Substituting the expansion (6.11) into (6.10) and equating coefficients of equal powers of  $\epsilon$ , we obtain

$$\frac{df^{(0)}}{dt} = 0, \quad (6.12)$$

$$\frac{df^{(1)}}{dt} = J_0, \quad (6.13)$$

$$\frac{df^{(k)}}{dt} = J_{k-1}, \quad (6.14)$$

where  $J_{k-1}$  is the right-hand side of the equation following substitution in it of the functions  $f^{(k-1)}$ ,  $f^{(k-2)}$ , etc. Construction of a solution reduces to the successive solution of these linear differential equations. Although the independent variables in these equations are  $t$  and  $\mathbf{x}$ ,  $\xi$  enters the distribution function as a parameter.

While complex equations for the macroscopic quantities are obtained at small Knudsen numbers as a result of expanding in powers of a small parameter, in the case examined here, i.e., large Knudsen numbers, the expansion in powers of a small parameter leads to a recurrent system of differential equations for the distribution function itself which are comparatively simple in structure. However, the actual solution of these equations is a very complex computational problem, since, when solving the equation for  $f^{(k)}$  we must store the function of seven variables  $f^{(k-1)}$  (or simultaneously solve the whole sequence up to  $f^{(k)}$ , inclusively) and we must evaluate a very complex collision integral. The solution for  $f^{(0)}$ , i.e., for free molecular flows, is an exception. The general solution of that equation is trivial and has the form

$$f^{(0)}(t, \mathbf{x}, \xi) = f(t_0, \mathbf{x} - \xi(t - t_0), \xi). \quad (6.15)$$

Methods of solution of the equation presented above will be examined in detail in Chapters IV and VI during a study of free molecular and closely related flows. At each step of the solution we satisfy arbitrary initial and boundary conditions. In that sense, we may expect that the solution obtained in the series form (6.11) gives a general solution of the Boltzmann equation, but the question of the region of convergence of the method (to say nothing of the difficulties of accomplishing it in practice) remains open at the present time. Moreover, as will be shown in Chapter IV (§2), an expansion of the type (6.11) does not occur in certain cases.

In the next two sections we examine the small parameter method for flows with small Knudsen numbers. We do not deal any further with the small parameter method for large Knudsen numbers when discussing general methods of solution of the full Boltzmann equation, since the special form of the collision integral of the model equation has no relevance in the discussion of the method.

\*G. Jaffe, Ann. Physik, 6: 195 (1930).

It will be shown in § 6.5 that this method is equivalent to a certain method of successive approximations.

### §3.7. Hilbert's Method of Expansion in Terms of a Small Parameter

We now examine methods of seeking solutions of the exact Boltzmann equation in the form of a power series in a small parameter (the Knudsen number).

1. We write the Boltzmann equation in the dimensionless variables of § 2.11 and § 3.6:

$$\frac{df}{dt} = \frac{1}{\epsilon} J(f, f) \quad \left( \frac{d}{dt} = \frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x_1} \right), \quad (7.1)$$

where the quadratic form

$$J(f, \varphi) = \frac{1}{2} \int (f' \varphi'_1 + f'_1 \varphi' - f \varphi_1 - f_1 \varphi) g b db d\theta d\xi_1$$

and  $\epsilon$  is a small parameter of the order of the Knudsen number.

In Eq. (7.1), the dimensional time  $t^*$  and length  $x^*$  are referred, respectively, to the characteristic time of the flow  $\Theta$  and the characteristic flow length  $L$ . These characteristic scales are determined by the boundary and initial conditions of the problem. Neither Eq. (7.1) nor the boundary conditions contain other characteristic scales, beside  $\Theta$  and  $L$  and the parameter  $\epsilon$ . It is therefore clear from dimensional considerations <sup>†</sup> that the solution of (7.1) can depend only on the following variables:  $x = x^*/L$ ,  $t = t^*/\Theta$ ,  $x/\epsilon$ ,  $t/\epsilon$ ,  $x\epsilon$ ,  $t\epsilon$ , etc. In what follows we do not consider motions on the large scales  $L/\epsilon$ ,  $T/\epsilon$ , etc. Therefore, the distribution function may be represented in the form

$$f = f(t_1, x_1, t_L, x_L, \epsilon),$$

where  $t_L = t$ ,  $x_L = x$ ,  $t_1 = t/\epsilon = t^*/\tau$ ,  $x_1 = x/\epsilon = x^*/\lambda$ , and  $\lambda$  and  $\tau$  are the mean free path and the time between collisions, respectively. As in § 2.3, we call the variables  $(t_L, x_L)$  the L-scale variables, and the variables  $(t_1, x_1)$  the  $\lambda$ -scale variables.

<sup>†</sup>Compare § 2.3.

As a result of the structure of the distribution function, the derivatives become

$$\frac{\partial}{\partial t} = \frac{1}{\epsilon} \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_L}, \quad \frac{\partial}{\partial x} = \frac{1}{\epsilon} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_L}, \quad \frac{d}{dt} = \frac{1}{\epsilon} \frac{d}{dt_1} + \frac{d}{dt_L}.$$

We seek a solution of (7.1) as a power series of the small parameter  $\epsilon$ :

$$f(t_1, x_1, t_L, x_L, \epsilon) = f^{(0)}(t_1, x_1, t_L, x_L) + \epsilon f^{(1)}(t_1, x_1, t_L, x_L) + \dots$$

We substitute this expansion into (7.1); taking account of the expressions given for the derivatives, we equate the coefficients of the powers of  $\epsilon$  to zero:

$$\begin{aligned} \frac{df^{(0)}}{dt_1} &= J(f^{(0)}, f^{(0)}), \\ \frac{df^{(1)}}{dt_1} + \frac{df^{(0)}}{dt_L} &= 2J(f^{(0)}, f^{(1)}), \\ &\dots \end{aligned} \quad (7.2)$$

This infinite sequence of equations describes the motion both on the L-scale and on the  $\lambda$ -scale. It is easy to see that the derivatives  $df^{(v)}/dt_1$  may be different from zero only in small regions with characteristic dimensions of order  $\lambda$  or during a time of order  $\tau$ . If this were not so, the functions  $f^{(v)}$  would increase without limit. These thin regions, which are called Knudsen layers, are always close to the boundaries and near time zero. Shock waves also belong to these regions.

Our first problem is the construction of a solution on the L-scale, i.e., outside the Knudsen layer. We shall come back later (see paragraph 2 of this section, and also § 5.1) to an examination of the Knudsen layer.

Outside the Knudsen layer, i.e., when  $t_1 \rightarrow \infty$ , we have

$$f(t_1, x_1, t_L, x_L, \epsilon) \rightarrow f(t_L, x_L, \epsilon)$$

and

$$f(t_L, x_L, \epsilon) = f^{(0)}(t_L, x_L) + \epsilon f^{(1)}(t_L, x_L) + \dots = f^{(0)}(1 + \epsilon \varphi^{(1)} + \dots). \quad (7.3)$$

In order that the functions  $f^{(v)}$  do not increase without limit when  $t_1 \rightarrow \infty$  (in order to avoid the appearance of secular terms), it is necessary that the following conditions be satisfied (we now have only the variables  $t_L, x_L$ , and, therefore, we shall omit the subscript L for brevity):

$$J(f^{(0)}(t, x), f^{(0)}(t, x)) = 0, \quad (7.4)$$

$$2J(f^{(0)}, f^{(0)}\varphi^{(1)}) = \frac{df^{(0)}}{dt}, \quad (7.5)$$

$$2J(f^{(0)}, f^{(0)}\varphi^{(k)}) = \frac{df^{(0)}\varphi^{(k-1)}}{dt} - \sum_{l=1}^{k-1} J(f^{(0)}\varphi^{(l)}, f^{(0)}\varphi^{(k-l)}). \quad (7.6)$$

Thus, we have obtained the system of equations (7.4)–(7.6), which describes the flow in the L-scale outside the Knudsen initial and boundary layers and the shock waves.\*

It was shown in Chapter II that the unique solution of (7.4) is the Maxwell distribution function

$$f^{(0)} = f_0 = n^{(0)} \left( \frac{m}{2\pi k T^{(0)}} \right)^{3/2} e^{-\frac{m}{2kT^{(0)}}(\xi - u^{(0)})^2}.$$

All the equations, apart from the first, are linear inhomogeneous integral equations. It is evident that the five summational invariants satisfy the homogeneous equation

$$J(f^{(0)}, f^{(0)}\varphi^{(k)}) = \int f^{(0)} f_1^{(0)} (\varphi_1^{(k)'} + \varphi^{(k)} - \varphi_1^{(k)} - \varphi^{(k)}) g b db de d\xi_1 = 0. \quad (7.7)$$

It is easy to see that (7.7) has no other solutions. In fact, we multiply the integral (7.7) by a solution  $\varphi^{(k)}$  and integrate it with respect to  $\xi$ . Repeating the arguments of §2.4, we find that

$$I_{\varphi^{(k)}} = \frac{1}{4} \int (\varphi_1^{(k)'} + \varphi^{(k)} - \varphi_1^{(k)} - \varphi^{(k)})^2 f^{(0)} f_1^{(0)} g b db de d\xi_1 = 0.$$

\*In treating the Hilbert and Enskog–Chapman methods below, we mainly follow Grad [H. Grad, "Handbuch der Physik," Vol. 12 (1958)].

Therefore, the solution must make the terms in the bracket go to zero. But the collisions do not have any invariants beside the five given in §1.3. Therefore, the general solution of (7.7) has the form

$$\varphi^{(k)} = \sum_{r=0}^4 \gamma_r^{(k)} \psi_r, \quad (7.8)$$

where  $\psi_r$  are the summational invariants ( $\psi_0 = m$ ,  $\psi_{1,2,3} = m\xi_{1,2,3}$ , and  $\psi_4 = \frac{1}{2}m\xi^2$ ). The coefficients  $\gamma_r^{(k)}$  are functions of  $t$  and  $x$ .

It is known from the theory of integral equations that an inhomogeneous equation has a solution only if the right-hand side of the equation is orthogonal to the eigenfunctions of the homogeneous equation. If the conditions of orthogonality are fulfilled, the solution may be represented in the form

$$\varphi^{(k)} = \bar{\varphi}^{(k)} + \sum_{r=0}^4 \gamma_r^{(k)} \psi_r, \quad (7.9)$$

where  $\bar{\varphi}^{(k)}$  is a particular solution of the inhomogeneous equation. In order to make the choice of the functions  $\bar{\varphi}^{(k)}$  unique, we apply the five conditions:

$$\int \psi_r \bar{\varphi}^{(k)} f^{(0)} d\xi = 0. \quad (7.10)$$

The condition that the right-hand side of the integral equation be orthogonal to the eigenfunctions of the equation (the summational invariants) has the form

$$\int \psi_r \frac{df^{(0)}}{dt} d\xi = 0. \quad (7.11)$$

In §3.1 we obtained the conservation equations from conditions like these. If we put  $f = f^{(0)} = f_0$  in the conservation equations, then, as was shown in a previous paragraph [see (6.5) and (6.6)], these equations reduce to the Euler equations (6.6) for the five macroscopic quantities which appear in  $f_0$ , i.e., for  $n^{(0)}$  (or  $\rho^{(0)}$ ,  $u_i^{(0)}$ , and  $T^{(0)}$ ). Generally speaking, these quantities do not coincide with the hydrodynamic density  $n$ , the velocity  $u_i$ , and the temperature  $T$  at a given point.

In a similar manner the conditions of congruence for  $f^{(k+1)}$  are reduced to

$$\int \psi_r \frac{df^{(0)}\varphi^{(k)}}{dt} d\xi = 0, \quad (7.12)$$

since the integrals

$$\int \psi_r J(f^{(0)}\varphi^{(l)}, f^{(0)}\varphi^{(k-l)}) d\xi = 0$$

are similar to those examined in §2.4 because of their symmetry properties.

The conditions (7.12) are none other than the transfer equations (1.5), in which  $\varphi = \psi_r$  and  $f = f^{(0)}\varphi^{(k)}$ :

$$\frac{\partial}{\partial t} \int \psi_r f^{(0)}\varphi^{(k)} d\xi + \frac{\partial}{\partial x_i} \int \xi_i \psi_r f^{(0)}\varphi^{(k)} d\xi = 0 \quad (7.13)$$

$$(i = 1, 2, 3; r = 0, \dots, 4).$$

By substituting the function  $\varphi^{(k)}$  from (7.9) and taking account of condition (7.10), we now obtain

$$\left( \int \psi_r \psi_\mu f^{(0)} d\xi \right) \frac{\partial \gamma_\mu^{(k)}}{\partial t} + \left( \int \xi_i \psi_r \psi_\mu f^{(0)} d\xi \right) \frac{\partial \gamma_\mu^{(k)}}{\partial x_i}$$

$$+ \left( \int \psi_r \psi_\mu \frac{df^{(0)}}{dt} d\xi \right) \gamma_\mu^{(k)} + \frac{\partial}{\partial x_i} \int \xi_i \psi_r \psi_\mu \bar{\varphi}^{(k)} f^{(0)} d\xi = 0. \quad (7.14)$$

We introduce the notation

$$\Gamma_r = \int \psi_r f d\xi \text{ and } \Gamma_r^{(k)} = \int \psi_r f^{(0)}\varphi^{(k)} d\xi = \sum_{\mu=0}^4 \gamma_\mu^{(k)} \int \psi_r \psi_\mu f^{(0)} d\xi. \quad (7.15)$$

where

$$\Gamma_r = \sum_{k=0}^{\infty} \epsilon^k \Gamma_r^{(k)} \quad (7.16)$$

and

$$\gamma_r^{(0)} = \Gamma_r^{(0)}, \quad \Gamma_0 = \rho, \quad \Gamma_i = \rho u_i, \quad \text{and} \quad \Gamma_4 = \rho \left( \frac{1}{2} u^2 + \frac{3}{2} \frac{k}{m} T \right).$$

It follows from (7.15) that there exists between  $\Gamma_r^{(k)}$  and  $\gamma_r^{(k)}$  a single mutual relationship with a determinant different from zero. Using relation (7.15), we may rewrite the equations (7.14) for  $\Gamma_r^{(k)}$  instead of  $\gamma_r^{(k)}$ . Thus, linear differential equations are obtained for  $\gamma_r^{(k)}$  or  $\Gamma_r^{(k)}$  when  $k \geq 1$ .

If the initial and boundary conditions for all the  $\gamma_r^{(k)}$  (or the  $\Gamma_r^{(k)}$ ) are given, then we may successively find these quantities at any time by means of the Euler differential equations (7.11) and the linear inhomogeneous equations (7.14), and, likewise,  $\bar{\varphi}^{(k)}$  by means of the integral equation (7.5) and (7.6), i.e., we may construct all the  $\varphi^{(k)}$ , and, therefore, the complete solution of (7.3).

Thus, if the solution of the Boltzmann equation may be represented by the series (7.3), then the construction of that solution reduces to the solution of the nonlinear Euler equations and of a recurrent system of linear differential and integral equations. The solution is completely determined if one assigns the function  $\gamma_r^{(k)}(0, x)$  or  $\Gamma_r^{(k)}(0, x)$  at time zero.

Let us try to solve the Boltzmann equation with a given distribution function at time  $t = 0$ .\* In terms of dimensionless variables the statement of the Boltzmann equation and of the initial distribution function includes a fixed value of the Knudsen number (the parameter  $\epsilon_0$ ) for any specific problem.† When we seek a solution of the Boltzmann equation in the form of a power series in  $\epsilon$ , we must eventually replace  $\epsilon$  by its fixed value  $\epsilon_0$ . It is easy to see that the parameter  $\epsilon$  may be introduced into the initial distribution function in infinitely many ways, which satisfy the unique

\*The system of equations (7.4)-(7.6) describes the flow outside the Knudsen layer. Therefore, the solution (7.3) is inapplicable inside these layers and we cannot use it to go to the boundaries or to time zero. A solution (7.3) is sought only for initial and boundary conditions assigned outside the Knudsen layer, for example, on its boundary. Therefore, we now imply precisely those conditions. We shall see somewhat later how to relate those conditions to the true conditions of the problem at the boundaries and at time zero. It is evident that to do this we must solve the Boltzmann equation inside the Knudsen layers.

†The mean free path  $\lambda$  and, therefore,  $\epsilon_0$ , do not enter into the true initial distribution function. However,  $\epsilon_0$  may enter into the initial distribution function outside the Knudsen layer, since that function is obtained from a solution of the Boltzmann equation in the Knudsen layer which contains  $\lambda$ .

conditions that, when  $\epsilon = \epsilon_0$ , the initial function  $f(0, x, \xi, \epsilon)$  must coincide with the given one. When we introduce the small parameter  $\epsilon$  into the initial distribution function, it may be represented in the form of the series

$$f(0, x, \xi, \epsilon) = f^{(0)}(0, x, \xi) + \epsilon f^{(1)}(0, x, \xi) + \dots \quad (7.17)$$

Correspondingly,

$$\begin{aligned} \Gamma_r(0, x, \epsilon) &= \int \psi_r f(0, x, \xi, \epsilon) d\xi \\ &= \Gamma_r^{(0)}(0, x) + \epsilon \Gamma_r^{(1)}(0, x) + \epsilon^2 \Gamma_r^{(2)}(0, x) + \dots \end{aligned} \quad (7.18)$$

Therefore, the initial values of the hydrodynamic quantities  $\Gamma_r$  may be distributed more or less arbitrarily among the functions  $\Gamma_r^{(k)}$  (or  $\psi_r^{(k)}$ ). In particular, we may put all the  $\Gamma_r^{(k)} = 0$  for  $k > 0$ , i.e., we may consider that at time zero,  $\Gamma_r^{(0)}(0, x) = \Gamma_r(0, x)$ . From these initial data, by the above procedure, we can construct the complete solution of the Boltzmann equation, represented as a series (7.3). In that case, the solution is completely determined by the hydrodynamic quantities  $\Gamma_r$  at time zero.

The question naturally arises: to what extent are such hydrodynamic initial conditions special, and what is the relationship between solutions constructed from initial hydrodynamic quantities  $\Gamma_r$ , "smeared" differently among the function  $\Gamma_r^{(k)}$ ? The answer to this question is given by Hilbert's uniqueness theorem, which states that all solutions obtained for various distributions among  $\Gamma_r^{(k)}$  of the same initial hydrodynamic initial data  $\Gamma_r$  are identical for a fixed  $\epsilon = \epsilon_0$ .

In fact, let us examine an equation which differs from (7.1) only in notation:

$$\frac{dF}{dt} = \frac{1}{\mu} J(F, F). \quad (7.19)$$

Let us find a solution of that equation as a power series in  $\mu$  under the following initial conditions:

$$\Gamma_r^{(0)} = \Gamma_r(0, x, \epsilon), \quad \Gamma_r^{(k)} = 0 \text{ when } k \geq 1. \quad (7.20)$$

The solution is a function of the two parameters  $\mu$  and  $\epsilon$ , i.e.,  $F = F(t, x, \xi, \epsilon, \mu)$ . That solution is constructed on the basis of the hydrodynamic conditions (7.20), and is valid, in particular, when  $\mu = \epsilon$ . Putting  $\mu$  equal to a fixed value of  $\epsilon$  and expanding  $F(t, x, \xi, \epsilon, \mu)$  and  $\Gamma_r(0, x, \epsilon)$  in a power series in  $\epsilon$ , we find that, as a function of  $\epsilon$ , the solution which has been found satisfies the initial conditions in which  $\Gamma_r^{(k)} \neq 0$  when  $k \geq 1$ , which proves the theorem.

Thus, the solution of the Boltzmann equation  $f_0(t, x, \xi)$  represented as a series (7.3) is uniquely determined by assigning the hydrodynamic quantities  $\Gamma_r(0, x)$  at time zero. Since the initial time does not differ from other instants of time, the distribution function is also uniquely determined at  $t = 0$  if the five hydrodynamic quantities  $\Gamma_r(0, x)$  are assigned. But any time may be taken as the initial time and, therefore, at any instant of time  $t \neq 0$ , the distribution function is uniquely determined by the value of the hydrodynamic quantities at that time.

The five hydrodynamic quantities are five integrals of the distribution function with respect to  $\xi$ . It is evident that a denumerable infinity of distribution functions exists whose integrals are equal to the same hydrodynamic quantities, i.e., in general, the distribution function is not determined by assigning the five hydrodynamic quantities. Therefore, the solutions of the Boltzmann equation represented as a power series in a small parameter are singular in that sense. It is evident that a rather limited class of solutions of the Boltzmann equation can be represented in the form of a power series in  $\epsilon$ . That class of solutions of the Boltzmann equation is called the Hilbert class of normal solutions. Solutions belonging to this class are designated as  $f_H$ . Furthermore, if a distribution function belongs to that class at some time  $t$ , then it belongs to it at all subsequent times.

The fact that a solution of the Boltzmann equation can be represented as a series (7.3) only for a special class of functions may also be seen from the following. Suppose that we are seeking a solution of Eq. (7.1) with an initial distribution function  $f(0, x, \xi)$  independent of  $\epsilon$ . Then, when  $t = 0$ , all the functions  $\Gamma_r^{(k)} = 0$  for  $k > 0$ . But, since (7.14) is inhomogeneous, then, when  $t > 0$ , the functions  $\Gamma_r^{(k)} \neq 0$  and the solution  $f(t, x, \xi, \epsilon)$  are functions of  $\epsilon$ . In the general case, the solution remains a function of  $\epsilon$  even for

$t = 0$ , i.e., the solution obtained, is not equal to the original distribution function. In order for the solution to coincide with the given initial function when  $t = 0$ , it is necessary to impose conditions on the latter; those conditions define the class of solutions representable in the form of the series (7.3).

Thus, outside a Knudsen layer, a solution of the Boltzmann equation represented by a power series in  $\epsilon$  tends to a Hilbert solution determined completely at every point of the flow by the values of the hydrodynamic (thermodynamic) quantities.

2. As we have already noted, a Hilbert solution is constructed on the basis of some initial and boundary conditions outside a Knudsen layer, those conditions being different from the true conditions for the distribution function at the initial time and on the boundaries. We now examine in what way a solution of the Boltzmann equation which satisfies the true boundary and initial conditions goes over to a Hilbert solution as we proceed away from a boundary or from the initial time, i.e., we investigate the solution inside the Knudsen layer. We restrict our examination to the initial value problem for the linearized Boltzmann equation. We consider the linear Boltzmann equation [see (8.2) of Chapter II]\*

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial t} + \xi_i \frac{\partial\varphi}{\partial x_i} = \frac{1}{\epsilon} L(\varphi), \quad (7.21)$$

where the linear operator  $L$  is equal to

$$L(\varphi) = \int f_1^{(0)} (\varphi' + \varphi'_i - \varphi - \varphi_i) g b \, db \, d\xi \, d\xi_i.$$

We seek a solution of (7.21) in the form

$$\varphi_m(t, x, \xi, \epsilon) = e^{-\frac{\lambda_m t}{\epsilon}} \sum_{k=0}^{\infty} \epsilon^k \varphi_m^{(k)}(t, x, \xi). \quad (7.22)$$

As above, we substitute the solution (7.22) into the Boltzmann equation and equate coefficients of the various powers of  $\epsilon$ ; we have†

\*The investigation given here is due to Grad [H. Grad, Phys. Fluids, Vol. 6, No. 2 (1963)]. See also the paper by Sirovich [L. Sirovich, Phys. Fluids, Vol. 6, No. 2 (1963)] and § 3.11.

†Compare this with equations (7.2);  $t$  and  $x$  are coordinates of the L-scale.

$$L(\varphi_m^{(0)}) - \lambda_m \varphi_m^{(0)} = 0, \quad (7.23)$$

$$L(\varphi_m^{(k)}) - \lambda_m \varphi_m^{(k)} = - \left( \frac{\partial \varphi_m^{(k-1)}}{\partial t} + \xi_i \frac{\partial \varphi_m^{(k-1)}}{\partial x_i} \right), \quad (k = 1, 2, \dots), \quad (7.24)$$

where  $\lambda_m$  are the eigenvalues of Eq. (7.23). We designate the eigenfunction corresponding to  $\lambda_m$  by  $\varphi_m$ .\* We assume the functions  $\varphi_m$  to be normalized, i.e., we put

$$\int \phi_m^2 f^{(0)} d\xi = 1.$$

We introduce the new macroscopic variables

$$\sigma_m(t, x, \epsilon) = e^{\frac{\lambda_m t}{\epsilon}} \int \phi_m \varphi_m f^{(0)} d\xi = \sum_{k=0}^{\infty} \epsilon^k \sigma_m^{(k)}, \quad (7.25)$$

where

$$\sigma_m^{(k)}(t, x) = \int \varphi_m^{(k)} \phi_m f^{(0)} d\xi. \quad (7.26)$$

The solutions of Eqs. (7.23) and (7.24) may be represented in the form

$$\varphi_m^{(0)} = \sigma_m^{(0)} \phi_m, \quad \varphi_m^{(k)} = \bar{\varphi}_m^{(k)} + \sigma_m^{(k)} \phi_m, \quad (7.27)$$

where  $\bar{\varphi}_m^{(k)}$  is a particular solution of equations (7.24), subject to the condition

$$\int \bar{\varphi}_m^{(k)} \phi_m f^{(0)} d\xi = 0.$$

For the inhomogeneous equations (7.24) to be soluble, the right sides must be orthogonal to the eigenfunction  $\varphi_m$ . Multiplying (7.24) by  $\phi_m f^{(0)}$ , and integrating with respect to  $\xi$ , we obtain

$$\frac{\partial \sigma_m^{(0)}}{\partial t} = 0, \quad \frac{\partial \sigma_m^{(k)}}{\partial t} = - \int \xi_i \phi_m f^{(0)} \frac{\partial \bar{\varphi}_m^{(k)}}{\partial x_i} d\xi, \quad (k = 1, 2, \dots), \quad (7.28)$$

\*For simplicity, we only examine the case of nondegenerate eigenvalues, i.e., the case where a single eigenfunction corresponds to each  $\lambda_m \neq 0$ . The more general case is examined in Grad's original work.

where we have used the fact that the eigenfunctions of the operator  $L$  possess the following property: \*

$$\int \xi_l f^{(0)} \phi_m^2 d\xi = 0.$$

It follows from the first equation that  $\sigma^{(0)}$  does not depend on  $t$ . Therefore, the  $\sigma_m^{(k)}$  have the form

$$\sigma_m^{(k)}(t, \mathbf{x}) = t^k A_m^{(k)}(\mathbf{x}) + B_m^{(k)}(\mathbf{x}), \quad (7.29)$$

i.e., near  $t = 0$  the expansion in powers of  $\epsilon$  is equivalent to an expansion in powers of  $t$ .

If all the  $\sigma_m^{(k)}(0, \mathbf{x})$ , are given at the initial time  $t = 0$ , then, as in the Hilbert method, we may find all the  $\sigma_m^{(k)}$  and the  $\bar{\varphi}_m$  for  $t > 0$  in succession, i.e., we may construct a solution of (7.22) for each eigenvalue  $\lambda_m$ . The Hilbert solution corresponds to  $\lambda = 0$ . As was true in the Hilbert method, the solution  $\varphi_m$  is a function of  $\epsilon$  (in particular, for  $t = 0$ ).

Let a distribution function which does not depend on  $\epsilon$  be given at  $t = 0$ :

$$\varphi(0, \mathbf{x}, \xi) = \varphi(\mathbf{x}, \xi). \quad (7.30)$$

Bearing in mind that the system of eigenfunctions  $\phi_m$  is complete, we seek a solution of (7.21) with the initial condition (7.30) in the form of a superposition of solutions (7.22):

$$\varphi = \sum_m \varphi_m = \sum_m e^{-\frac{\lambda_m t}{\epsilon}} \sum_k \epsilon^k \varphi_m^{(k)}. \quad (7.31)$$

We must find the initial conditions for the functions  $\varphi_m^{(k)}$ , from the given function  $\varphi(\mathbf{x}, \xi)$ , i.e., to relate  $\varphi(\mathbf{x}, \xi)$  and  $\sigma_m^{(k)}(0, \mathbf{x})$ . We change the order of the summation in (7.31) and introduce the notation:

$$\varphi = \sum_k \epsilon^k \varphi^{(k)}, \quad \varphi^{(k)} = \sum_m e^{-\frac{\lambda_m t}{\epsilon}} \varphi_m^{(k)}. \quad (7.32)$$

Since the distribution function does not depend on  $\epsilon$  at  $t = 0$ , then  $\varphi^{(k)}(0) = 0$  when  $k > 0$ . Therefore, all the integrals

$$\int f_0 \phi_m \varphi^{(k)} d\xi = 0 \text{ when } k > 0 \quad (7.33)$$

are equal to zero. The conditions (7.33) permit us to establish the desired relationship between  $\varphi(\mathbf{x}, \xi)$  and  $\sigma_m^{(k)}(0, \mathbf{x})$ . When  $t = 0$ ,

$$\varphi^{(0)} = \sum_m \varphi_m^{(0)} = \sum_m \sigma_m^{(0)} \phi_m. \quad (7.34)$$

Making use of the orthogonality of the eigenfunctions  $\phi_m$  with respect to the weighting function  $f^{(0)}$ , we obtain

$$\int f^{(0)} \phi_m \varphi^{(0)} d\xi = \sigma_m^{(0)}. \quad (7.35)$$

We transform conditions (7.33). From (7.27) and (7.32), we have

$$\int f^{(0)} \phi_m \varphi^{(k)} d\xi = \sum_j \int f^{(0)} \phi_m \varphi_j^{(k)} d\xi + \sigma_m^{(k)}. \quad (7.36)$$

To find a particular solution  $\bar{\varphi}_m^{(k)}$  we need to know the functions  $\varphi_m^{(j)}$  for  $j < k$ . Making use of the conditions (7.35), we can find  $\sigma_m^{(0)}(t, \mathbf{x})$  from (7.28) and, therefore, also  $\varphi_m^{(0)}$ , in accordance with (7.27). By solving Eq. (7.24), we find  $\bar{\varphi}_m^{(1)}$ . Then, conditions (7.36) permit us to establish the initial conditions for  $\sigma_m^{(1)}$ , for which it is necessary to solve Eq. (7.28) and, therefore, to find  $\varphi_m^{(1)}$  in accordance with (7.27). By solving (7.24), we find  $\bar{\varphi}_m^{(2)}$ , from (7.36) we find  $\sigma_m^{(2)}$ , etc.

Although each  $\varphi_m$ -solution at  $t = 0$  is a function of  $\epsilon$ , by combining them it proves possible to satisfy initial conditions, which do not depend on  $\epsilon$ . The procedure described allows us in principle to satisfy arbitrary initial conditions and, therefore, the solution (7.31) gives a general solution of Eq. (7.21), while the Hilbert procedure would permit us to satisfy only special initial conditions and to construct only a special class of solutions of the Boltzmann equation. But it may be seen from (7.31) that when  $t \gg \epsilon$ , the general solution tends exponentially to  $\varphi_0$ , i.e., to the Hilbert solution. But this limiting Hilbert solution differs by a quantity of order  $\epsilon$  from the solution which would have been ob-

\*See Grad's paper cited above.

tained directly with the same initial conditions. In fact, if the initial distribution function (7.30) does not depend on  $\epsilon$ , it is necessary in the Hilbert method to assume that  $\sigma^{(k)}(0, \mathbf{x}) = 0$  when  $k > 0$ , while in constructing solution (7.31),  $\sigma_0^{(k)} \neq 0$ \* in the general case. To find the correct initial conditions for the Hilbert solution, we need to find  $\sigma_0^{(k)}$  according to the above-described procedure; in fact, this implies that we must construct the full solution (7.31).

**3.** All the preceding operations on the series have been formal, since the convergence of the series was not proved. It is evident, as was also the case for the model equation, that the series converge only for a special class of problems. It can be shown, however, for the linear Boltzmann equation for molecules with a sufficiently rapid decay in potential and with a finite interaction parameter ( $s \geq 5$ ), that a truncated series gives an asymptotic solution of the Boltzmann equation when  $\epsilon \rightarrow 0$ .†

Let  $\varphi$  be a solution of the linear Boltzmann equation, satisfying the initial condition (7.30). That solution may be represented in the form

$$\varphi = \varphi_H(t, \mathbf{x}, \xi, \epsilon) + \varphi_\mu(t_1, \mathbf{x}, \xi, \epsilon), \quad (7.37)$$

where

$$\varphi_H = \sum_k \epsilon^k \varphi_0^{(k)}(t, \mathbf{x}, \xi)$$

is the Hilbert solution,

$$\varphi_\mu = \sum_k \epsilon^k \varphi_\mu^{(k)}(t_1, \mathbf{x}, \xi)$$

is the remainder which decays exponentially with  $t_1$  ( $t_1 = t/\epsilon$ ). Because of the linearity of the equation, the function  $\varphi_\mu$  is also a solution of the Boltzmann equation. The solution converges asymptotically in the sense that

$$\lim_{\epsilon \rightarrow 0} \left( \frac{1}{\epsilon^N} |\varphi_B - \varphi^{(N)}| \right) = 0, \quad (7.38)$$

\*In the description of the Hilbert method,  $\sigma_0^{(k)}$  was denoted by  $\Gamma^{(k)}$ .

†See Grad's paper cited above, and also the end of §3 and §§11 and 12 of the present chapter.

where  $\varphi_B$  is the solution of the Boltzmann equation, and  $\varphi^{(N)} = \varphi_H^{(N)} + \varphi_\mu^{(N)}$ ; the subscript N indicates that in each of the summations, appearing in (7.37), only terms up to and including  $\epsilon^N$  have been retained. The asymptotic solution  $\varphi^{(N)}$  must satisfy the same initial condition as the exact solution  $\varphi_B$ .

When  $t \gg \epsilon$ , the exponentially decaying terms tend to zero, and any component of the Hilbert series is an asymptotic solution of the Boltzmann equation. If we restrict ourselves to a single term of the series, we obtain the Euler approximation; if we take two terms, we obtain the Navier-Stokes approximation. However, near  $t = 0$ , the exponential terms are of the same order as the terms of the Hilbert series. Therefore, to establish the correct initial conditions for the Euler, Navier-Stokes, or higher approximations, we must investigate the structure of the initial layer, i.e., to find  $\varphi_\mu$ .

Evidently, solutions of the Boltzmann equation possess similar properties when boundaries are present. As was the case for the model equation, away from boundaries, the solution must tend to the Hilbert solution, or asymptotically, when  $\epsilon \rightarrow 0$ , to the Euler or Navier-Stokes approximations. To establish the boundary conditions, we must investigate the structure of a layer of thickness  $O(\epsilon)$  near the wall.\* However, these statements have not been proved rigorously at the present time, even for the linearized Boltzmann equation.

**4.** For the complete nonlinear Boltzmann equation with an arbitrary initial distribution function we seek a solution in a form analogous to expression (7.37):†

$$f(t, \mathbf{x}, \xi) = f_H(t, \mathbf{x}, \xi, \epsilon) + f_\mu(t_1, \mathbf{x}, \xi, \epsilon), \quad \left( t_1 = \frac{t}{\epsilon} \right), \quad (7.39)$$

where the first term on the right side of  $f_H$  is the Hilbert solution which depends only on the coordinates  $t$  and  $\mathbf{x}$  of the L-scale, and  $f_\mu$  is a correction which depends on the time  $t_1$  of the  $\lambda$ -scale. The function  $f_H$ , like the Hilbert solution, itself satisfies the Boltzmann equation. Therefore, the function  $f_\mu$  satisfies the equation

\*See §5.1.

†See Grad's paper cited above and the work of V. V. Sturninskii [Doklady Akad. Nauk SSSR, Vol. 158, No. 2 (1964)].

$$\frac{\partial f_\mu}{\partial t_1} + \epsilon \xi_l \frac{\partial f_\mu}{\partial x_l} = J(f_H, f_\mu) + J(f_\mu, f_\mu). \quad (7.40)$$

The variable  $t$  also appears in that equation, through  $f_H$ , in addition to  $t_1$ . In order to eliminate it, we replace the variable  $t$  by  $t_1$  in  $f_H$ :

$$f_H(t, x, \xi, \epsilon) = f_H(t_1 \epsilon, x, \xi, \epsilon).$$

We expand the functions  $f_H$  and  $f_\mu$  in a power series in  $\epsilon$ :

$$f_H = \sum_{k=0}^{\infty} \epsilon^k f_H^{(k)}(t_1, x, \xi) \text{ and } f_\mu = \sum_{k=0}^{\infty} \epsilon^k f_\mu^{(k)}(t_1, x, \xi). \quad (7.41)$$

It is easy to see that this expansion of  $f_H$  in terms of  $\epsilon$  differs from the Hilbert expansion. In particular, the first term of the expansion  $f_H^{(0)}(x, \xi)$ , which is the first term of the expansion of the term  $f_H^{(0)}(t, x, \xi)$  in the Hilbert series with respect to  $t$ , is the local Maxwellian distribution which is independent of time. By substituting the expansion (7.41) into (7.40) and equating coefficients of like powers of  $\epsilon$ , we obtain

$$\frac{\partial f_\mu^{(0)}}{\partial t_1} - J(f_H^{(0)}, f_\mu^{(0)}) - J(f_\mu^{(0)}, f_\mu^{(0)}) = 0, \quad (7.42)$$

$$\frac{\partial f_\mu^{(1)}}{\partial t_1} - J(f_H^{(0)}, f_\mu^{(1)}) - 2J(f_\mu^{(0)}, f_\mu^{(1)}) = -\xi_l \frac{\partial f_\mu^{(0)}}{\partial x_l} + J(f_H^{(1)}, f_\mu^{(0)}), \quad (7.43)$$

. . . . .

We see, however, that the solution of the first equation of this recurrent system is itself an independent problem.

The problem is simplified appreciably for the special case of initial conditions when the initial distribution function  $f(0, x, \xi)$  is locally Maxwellian  $f_0(x, \xi)$ . In that case,  $f_\mu^{(0)}(0, x, \xi) = 0$ , and since Eq. (7.42), which is satisfied by the function  $f_\mu^{(0)}(t, x, \xi)$ , is homogeneous, the function  $f_\mu^{(0)}$  is identically equal to zero at all times. The functions  $f_\mu^{(k)}(t, x, \xi)$  for  $k > 0$  satisfy the linear equations:

$$\frac{\partial f_\mu^{(1)}}{\partial t_1} + J(f_H^{(0)}, f_\mu^{(1)}) = 0, \quad (7.44)$$

$$\frac{\partial f_\mu^{(k)}}{\partial t_1} + J(f_H^{(0)}, f_\mu^{(k)}) = -\xi_l \frac{\partial f_\mu^{(k-1)}}{\partial x_l} - \sum_{r=1}^{k-1} J(f_H^{(r)} + f_\mu^{(r)}, f_\mu^{(k-2)}). \quad (7.45)$$

The analysis of the boundary layer is a more difficult problem.

5. In the analysis given above, the introduction of the different scales plays an important role. For example, if, in a problem with initial conditions, we were to seek a distribution function of the form

$$f(t_1, x, \xi, \epsilon) = f^{(0)}(t_1, x, \xi) + \epsilon f^{(1)}(t_1, x_1, \xi) + \dots, \quad (7.46)$$

then, since the functions  $f^{(v)}$  depend only on a single time, we would evidently have

$$\xi_l \cdot \frac{\partial f}{\partial x_l} = \frac{1}{\epsilon} \left( J - \frac{\partial f}{\partial t_1} \right) \quad \left( t_1 = \frac{t}{\epsilon} \right), \quad (7.47)$$

$$\left. \begin{aligned} \frac{\partial f^{(0)}}{\partial t_1} - J(f^{(0)}, f^{(0)}) &= 0, \\ \frac{\partial f^{(1)}}{\partial t_1} - 2J(f^{(0)}, f^{(1)}) &= -\xi_l \frac{\partial f^{(0)}}{\partial x_l}, \\ \dots &\dots \end{aligned} \right\} \quad (7.48)$$

It is clear that the series (7.46) can converge only for  $t \leq \epsilon$ . When  $t \sim 1$  and greater,  $f^{(1)}$  increases without limit, i.e., there are secular terms in the expansion.

The expansion (7.46) is sometimes called a Hilbert solution, and one speaks of the Hilbert method as a method suitable only to describe processes occurring on the  $\lambda$ -scale.

### §3.8. The Enskog — Chapman Method. Derivation of Equations of Hydrodynamics

It was shown in the previous section that the Hilbert solution of the Boltzmann equation given as a power series of a small parameter  $\epsilon$  (the Knudsen number) is completely determined when the hydrodynamic quantities  $\Gamma_r(0, x)$  are specified at time zero. But, if the distribution function  $f(t, x, \xi)$  at an arbitrary time  $t$  is

expressed in terms of the hydrodynamic quantities for  $t = 0$ , then even the hydrodynamic quantities  $\Gamma_r(t, x)$ , which are integrals of the distribution function with respect to  $\xi$ , are determined when  $\Gamma_r(0, x)$  is assigned. Therefore, we may exclude the distribution function from consideration and establish a direct connection between the hydrodynamic quantities and the various times. The stress tensor and the heat flux vector are also integrals of the distribution function. Therefore,  $P_{ij}$  and  $q_i$  can be expressed in the final analysis in terms of the hydrodynamic quantities  $\Gamma_r(t, x)$ . If  $P_{ij}$  and  $q_i$  have been expressed in terms of the hydrodynamic quantities, then, by substituting these relations into the conservation equations (1.8)–(1.10), we obtain a closed system of five equations for the five hydrodynamic quantities.

The objective of the Enskog–Chapman method is to establish the above relationship and to obtain a closed system of hydrodynamic equations. We first establish the desired relationship, making some changes in the argument of the Hilbert method, as described in the previous paragraph. Later on we give a derivation of the equations of hydrodynamics by a method close to the original Enskog–Chapman method.

1. As was shown in the previous paragraph, at the initial time  $t = 0$ , without loss of generality, we may put\*

$$\Gamma_r^{(0)}(0, x) = \Gamma_r(0, x), \quad \Gamma_r^{(k)}(0, x) = 0 \text{ when } k > 0. \quad (8.1)$$

Therefore, when  $t = 0$ , the hydrodynamic quantities  $\Gamma_r^{(0)} = \Gamma_r$  enter into the function  $f^{(0)} = f_0$ .

The functions  $\Gamma_r^{(k)}$  (or, equivalently,  $v_r^{(k)}$ ) satisfy the inhomogeneous differential equations (7.14). Therefore, when  $t > 0$ , the functions  $\Gamma_r^{(k)}$  are not equal to zero, and the  $\Gamma_r^{(0)}$  are no longer equal to the hydrodynamic quantities  $\Gamma_r$ . In particular, the derivative  $\partial\Gamma_r^{(k)}/\partial t$  is not equal to zero when  $t = 0$ .

To find the function  $f^{(1)}$ , we must solve the integral equation (7.5), on the right side of which appears the derivative of  $f^{(0)}$ :

\*We recall here that we are concerned not with the true initial conditions, but with conditions outside the Knudsen layer.

$$\frac{df^{(0)}}{dt} = \frac{\partial f^{(0)}}{\partial t} + \xi_l \frac{\partial f^{(0)}}{\partial x_l} = \sum_r \frac{\partial f^{(0)}}{\partial \Gamma_r^{(0)}} \left( \frac{\partial \Gamma_r^{(0)}}{\partial t} + \xi_l \frac{\partial \Gamma_r^{(0)}}{\partial x_l} \right). \quad (8.2)$$

The derivatives with respect to time may be eliminated by means of the Euler equations which the functions  $\Gamma_r^{(0)}$  must satisfy, as a result of solubility conditions. Then the right side of (7.5) is expressed in terms of the  $\Gamma_r^{(0)}$ , and their space derivatives. Since the coordinates  $x_i$  appear as parameters in the integral equation (7.5), it is evident that the solution  $f^{(1)}$  of that equation will be a function of the  $\Gamma_r^{(0)}$  and their spatial derivatives. The general solution of those equations may be represented in the form (7.9), in which the quantities  $\gamma_r^{(1)}$  are replaced by  $\Gamma_r^{(1)}$  according to (7.15).

It is evident that the first time and space derivatives of  $\Gamma_r^{(1)}$  and the second derivatives of  $\Gamma_r^{(0)}$  enter into the right side of the integral equation (7.6) for  $f^{(2)}$ . The partial derivatives with respect to  $t$  of  $\Gamma_r^{(0)}$  and  $\Gamma_r^{(1)}$  may be eliminated by means of the Euler equations (7.14) for  $\Gamma_r^{(1)}$ . Then the general solution of (7.6) for  $f^{(2)}$  is expressed in terms of  $\Gamma_r^{(0)}$ , and  $\Gamma_r^{(1)}$  and their space derivatives and, according to (7.9), in terms of  $\Gamma_r^{(2)}$ . Continuing this process we may express  $f^{(n)}$  in terms of  $\Gamma_r^{(0)}, \dots, \Gamma_r^{(n)}$  and their space derivatives up to order  $(n - k)$  for the functions  $\Gamma_r^{(k)}$ . But, when  $t = 0$ , all the  $\Gamma_r^{(k)}$  for  $k > 0$  and their space derivatives are equal to zero, while  $\Gamma_r^{(0)} = \Gamma_r$ . Therefore, when  $t = 0$ , the functions  $f^{(n)}$  are expressed in terms of the hydrodynamic quantities and their space derivatives up to and including order  $n$ . But the time  $t = 0$  is not exceptional. Any time may be taken as the initial time. Therefore, the dependence of  $f^{(n)}$  on the hydrodynamic quantities is valid at any instant of time. Substituting that dependence into the definitions of  $P_{ij}$  and  $q_i$  in terms of  $f$ , we express the latter in terms of hydrodynamic quantities and their derivatives up to order  $n$ . Then, by expressing  $P_{ij}$  and  $q_i$  in the conservation equations in terms of  $\Gamma_r$  and their derivatives, we obtain a closed system of equations for the hydrodynamic quantities.

If we restrict ourselves to a single term of the series and require that  $n^{(0)} = n$ ,  $T^{(0)} = T$ , and  $u_r^{(0)} = u_r$ , we obtain:

$$f = f^{(0)} = f_0 = n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left\{ -\frac{m}{2kT} c^2 \right\} \quad (8.3)$$

and

$$P_{ij} = p\delta_{ij} \text{ and } q_i = 0.$$

The conservation equations (1.8)–(1.10) take the form of the Euler equations [see (6.6)].

If we restrict ourselves to two terms of the series, i.e., we put

$$f = f^{(0)} + \epsilon f^{(1)} = f^{(0)}(1 + \epsilon\varphi^{(1)}), \quad (8.4)$$

we must solve the integral equation (7.5) for  $\varphi^{(1)}$ . Substituting the function  $f^{(0)}$  (in which the  $\Gamma_r^{(0)}$  appear rather than the  $\Gamma_r$ ) into the right side of Eq. (7.5), eliminating the derivatives with respect to time with the aid of the Euler equations for  $\Gamma_r^{(0)}$ , and then putting  $n^{(0)} = n$ ,  $T^{(0)} = T$ , and  $u_r^{(0)} = u_r$ , we obtain

$$\frac{df^{(0)}}{dt} \Big|_{t=0} = f^{(0)} \left\{ \frac{m}{kT} \left( c_i c_j - \frac{1}{3} c^2 \delta_{ij} \right) \frac{\partial u_i}{\partial x_j} + \frac{1}{2T} \frac{\partial T}{\partial x_i} c_i \left[ \left( \frac{m}{kT} \right) c^2 - 5 \right] \right\}. \quad (8.5)$$

Therefore, the integral equation (7.5) takes the form

$$\begin{aligned} & \int f^{(0)} f_1^{(0)} (\varphi_i^{(1)'} + \varphi^{(1)'} - \varphi_i^{(1)} - \varphi^{(1)}) g b db de d\xi_i \\ &= \frac{1}{2} f^{(0)} \left\{ \frac{1}{2T} c_i \left[ \left( \frac{m}{kT} \right) c^2 - 5 \right] \frac{\partial T}{\partial x_i} \right. \\ & \quad \left. + \frac{m}{kT} \left( c_i c_j - \frac{1}{3} c^2 \delta_{ij} \right) \frac{\partial u_i}{\partial x_j} \right\}. \end{aligned} \quad (8.6)$$

It is easy to see that the solution of that equation must have the form

$$\varphi^{(1)} = -A_i \frac{\partial T}{\partial x_i} - B_{ij} \frac{\partial u_i}{\partial x_j}, \quad (8.7)$$

where the coefficients  $A_i$  and  $B_{ij}$  are functions of  $c_i$ ,  $n$ , and  $T$ .

Substituting (8.7) into Eq. (8.6), and equating coefficients of the various gradients, we obtain the following integral equations for  $A_i$  and  $B_{ij}$ :

$$\begin{aligned} & \frac{1}{2} f^{(0)} \frac{c_i}{2T} \left[ \left( \frac{m}{kT} \right) c^2 - 5 \right] \\ &= - \int f^{(0)} f_1^{(0)} (A'_{ii} + A'_i - A_{ii} - A_i) g b db de d\xi_i, \end{aligned} \quad (8.8)$$

$$\begin{aligned} & \frac{1}{2} f^{(0)} \frac{m}{kT} \left( c_i c_j - \frac{1}{3} c^2 \delta_{ij} \right) \\ &= - \int f^{(0)} f_1^{(0)} (B'_{ii} + B'_{ij} - B_{ii} - B_{ij}) g b db de d\xi_i. \end{aligned} \quad (8.9)$$

Thus, the integral equation (8.6) has been replaced by the twelve equations – for the three components of the vector  $A_i$  and the nine components of the tensor  $B_{ij}$ .

The vector  $\mathbf{A}$  depends on  $c_i$ ,  $n$ , and  $T$ . The only vector which can be formed from these quantities is the vector  $\mathbf{c}$ . Therefore,

$$A_i = c_i A(c, n, T). \quad (8.10)$$

We sum the diagonal equations (8.9), i.e., the three equations with subscripts  $i = j$ . The left part goes to zero, and

$$J[f^{(0)}, f^{(0)}(B_{11} + B_{22} + B_{33})] = 0. \quad (8.11)$$

Using the symmetry properties of § 2.4, it may easily be found that the integral (8.11) is equal to zero only when

$$B_{11} + B_{22} + B_{33} = 0. \quad (8.12)$$

Similarly, subtracting the  $ij$ -th equation of (8.9) from the  $ji$ -th, we obtain

$$J[f^{(0)}, f^{(0)}(B_{ij} - B_{ji})] = 0, \quad (8.13)$$

and, therefore,

$$B_{ij} = B_{ji}. \quad (8.14)$$

Therefore, the tensor  $B_{ij}$  is a symmetrical nondivergent tensor. But, from the quantities  $\mathbf{c}$ ,  $n$ , and  $T$ , on which this tensor depends, we may form the unique symmetric nondivergent tensor  $c_i c_j - \frac{1}{3} c^2 \delta_{ij}$ ; therefore,

$$B_{ij} = \left( c_i c_j - \frac{1}{3} c^2 \delta_{ij} \right) B(c, n, T). \quad (8.15)$$

Thus, the general solution of (8.6) has the form

$$\varphi^{(1)} = -A_i \frac{\partial T}{\partial x_i} - B \left( c_i c_j - \frac{1}{3} c^2 \delta_{ij} \right) \frac{\partial u_i}{\partial x_j} + \gamma_r \psi_r, \quad (8.16)$$

where  $\gamma_r \psi_r$  is the general solution of the homogeneous equation, and  $\psi_r$  are the summational invariants. To determine the coefficients  $\gamma_r$  we must subject the function (8.16) to the conditions (7.10). To carry out the quadratures, it is somewhat more convenient to rewrite (8.16) in the equivalent form

$$\varphi^{(1)} = -A c_l \frac{\partial T}{\partial x_l} - B \left( c_l c_j - \frac{1}{3} c^2 \delta_{lj} \right) \frac{\partial u_l}{\partial x_j} + \alpha + \beta_r c_r + \gamma c^2, \quad (8.16a)$$

and, instead of multiplying by  $\psi_r = m, m \xi_i$ , and  $\frac{1}{2} m \xi^2$ , we multiply by 1,  $c_i$ , and  $c^2$ . Since all the integrals (7.10) are equal to zero, these operations are equivalent. We have:

for multiplication by unity:

$$\int (\alpha + \gamma c^2) f^{(0)} d\xi = 0, \quad (8.17)$$

for multiplication by  $c_i$ :

$$\int \left( -A \frac{\partial T}{\partial x_i} + \beta_i \right) c^2 f^{(0)} d\xi = 0, \quad (8.18)$$

for multiplication by  $c^2$ :

$$\int (\alpha + \gamma c^2) c^2 f^{(0)} d\xi = 0. \quad (8.19)$$

We have used the fact that the integral of a function of  $c$ , multiplied by  $c_i^2$ , is equal to the integral of that function, multiplied by  $\frac{1}{3} c^2$ .

It follows from conditions (8.17) and (8.19) that  $\alpha$  and  $\gamma$  are equal to zero, and from condition (8.18) that  $\beta_i = \beta \partial T / \partial x_i$  and  $\beta$  may be included in  $A$ , i.e.,  $A$  may be replaced by  $A^* = A - \beta$ , for which the conditions

$$\int \left( -A^* \frac{\partial T}{\partial x_l} \right) c^2 f^{(0)} d\xi = 0 \quad (8.20)$$

are satisfied. Thus, for  $f$  in the second approximation, we have the following expression (without change of notation, we go over to dimensionless variables so that  $\epsilon$  drops out of the formulas, but  $A^*$  and  $B$  are of order  $\epsilon$ ):

$$f = f^{(0)} \left\{ 1 - A^* c_l \frac{\partial T}{\partial x_l} - B \left( c_l c_j - \frac{1}{3} c^2 \delta_{lj} \right) \frac{\partial u_l}{\partial x_j} \right\}. \quad (8.21)$$

We substitute the expression (8.21) into the definition of the heat flux vector; we have

$$q_l = \int f c_l \frac{mc^2}{2} d\xi = -\frac{m}{6} \frac{\partial T}{\partial x_l} \int A^* c^4 f^{(0)} d\xi = -\lambda \frac{\partial T}{\partial x_l}. \quad (8.22)$$

Comparing (8.22) with the ordinary Navier-Stokes representation for the heat flux vector, we find that  $\lambda$  is the thermal conductivity, equal to

$$\lambda = \frac{m}{6} \int A^* c^4 f^{(0)} d\xi. \quad (8.23)$$

Similarly, for the stress tensor we have

$$P_{lj} = m \int c_l c_j f d\xi = \delta_{lj} p - m \int B c_l c_j \left( c_k c_l - \frac{1}{3} c^2 \delta_{kl} \right) \frac{\partial u_k}{\partial x_l} f^{(0)} d\xi$$

If  $F(c)$  is an arbitrary function of  $c$ , then <sup>†</sup>

$$\begin{aligned} \int F(c) c^4 d\xi &= \int_0^\infty \int_0^{2\pi} \int_0^{\pi/2} F(c) c^6 \cos^4 \theta \sin \theta dc d\varphi d\theta \\ &= \frac{1}{5} \int_0^\infty \int_0^{2\pi} \int_0^{\pi/2} F(c) c^6 \sin \theta dc d\varphi d\theta = \frac{1}{5} \int F(c) c^4 d\xi. \end{aligned} \quad (8.24)$$

Further, it is evident that

$$\int c_i^2 c^2 F d\xi = \frac{1}{3} \int c^4 F d\xi,$$

whence, taking account of (8.24), we have

$$\int c_i^2 c^2 F d\xi = \frac{1}{6} \int c^4 F d\xi - \frac{1}{2} \int c_i^4 F d\xi = \frac{1}{15} \int c^4 F d\xi. \quad (8.25)$$

Taking into account the equalities (8.24) and (8.25), we obtain

$$P_{lj} = p \delta_{lj} - \mu \left( \frac{\partial u_l}{\partial x_j} + \frac{\partial u_j}{\partial x_l} - \frac{2}{3} \delta_{lj} \frac{\partial u_k}{\partial x_k} \right), \quad (8.26)$$

<sup>†</sup>In spherical coordinates

$d\xi = c^2 \sin \theta dc d\theta d\varphi$ .

where

$$\mu = \frac{m}{15} \int B f^{(0)} c^4 d\xi.$$

It is easy to see that (8.26) is the ordinary Navier-Stokes expression for the stress tensor and  $\mu$  is the viscosity. To obtain numerical values of the viscosity  $\mu$  and the thermal conductivity  $\lambda$  we must still solve the integral equations (8.8) and (8.9) for A and B. To solve these equations, we expand the functions A(c) and B(c) in a series of Sonine polynomials:<sup>\*</sup>

$$A(c) = \sum_{r=1}^{\infty} a_r S_{3/2}^{(r)}\left(\frac{mc^2}{2kT}\right), \quad B(c) = \sum_{r=0}^{\infty} b_r S_{5/2}^{(r)}\left(\frac{mc^2}{2kT}\right).$$

We shall introduce the notation

$$a_s = \int f^{(0)} \frac{c^2}{2T} \left[ \frac{m}{2kT} c^2 - \frac{5}{2} \right] S_{3/2}^{(s)} d\xi,$$

$$\beta_s = \sum_{i,j} \int f^{(0)} \frac{m}{2kT} \left( c_i c_j - \frac{1}{3} c^2 \delta_{ij} \right)^2 S_{5/2}^{(s)} d\xi,$$

$$[\varphi, \psi] = - \int f^{(0)} f_1^{(0)} (\varphi'_1 + \varphi' - \varphi_1 - \varphi) \psi(\xi) g b d\theta d\varepsilon d\xi d\xi_1.$$

Multiplying (8.8) by  $c_i S_{3/2}^{(s)}(mc^2/2kT)$ , and combining them, after replacing  $A_i = c_i A(c)$  by the series given above, we obtain

$$a_s = \sum_r a_r \sum_l [c_i S_{3/2}^{(r)}, c_l S_{3/2}^{(s)}] \quad (s = 1, 2, \dots).$$

\*The Sonine polynomials are defined by the expression

$$S_{(m)}^{(n)}(x) = \sum_{p=0}^n \frac{(-x)^p (m+n)!}{p! (n-p)! (m+p)!}.$$

In particular,  $S_m^{(0)} = 1$ ,  $S_m^{(1)} = m+1-x$ .

The Sonine polynomials satisfy the conditions of orthogonality

$$\int_0^\infty x^m e^{-x} S_{(m)}^{(p)}(x) S_{(m)}^{(q)}(x) dx = \frac{(m+p)!}{p!} \delta_{pq}.$$

Thus, to determine the coefficients  $a_r$  of the expansion, we obtain an infinite system of equations. Similarly, to find the coefficients  $b_r$ , we obtain the system

$$\beta_s = \sum_r b_r \sum_{i,j} [(c_i c_j - \frac{1}{3} c^2 \delta_{ij}) S_{5/2}^{(r)}, (c_i c_j - \frac{1}{3} c^2 \delta_{ij}) S_{5/2}^{(s)}] \quad (s = 1, 2, \dots).$$

In practice, the infinite series of Sonine polynomials for A(c) and B(c) are replaced by finite sums. Then, to determine the coefficients  $a_r$  and  $b_r$ , finite systems of linear equations are obtained. However, the solution of those systems is quite laborious. The viscosity and the thermal conductivity are found most simply, if only one term is retained in the expansion in Sonine polynomials. The transfer coefficients found in this way are called the first approximation coefficients and are usually denoted by  $[\mu]_1$  and  $[\lambda]_1$ . We give the values of these coefficients for the simplest models of molecules.\*

Solid spheres of diameter d:

$$[\mu]_1 = \frac{15}{16} \frac{\sqrt{k m T}}{\sqrt{\pi} d^2}, \quad [\lambda]_1 = \frac{5}{2} c_v [\mu]_1;$$

where  $c_v$  is the specific heat of the gas at constant volume.

Maxwellian molecules:

$$[\mu]_1 = \frac{1}{6A} \left( \frac{m}{8K} \right)^{1/2} kT, \quad [\lambda]_1 = \frac{5}{2} c_v [\mu]_1.$$

We note that in the first approximation for any molecules

$$[\lambda]_1 = \frac{5}{2} c_v [\mu]_1.$$

From those expressions it is seen that the dependence of the transport coefficients on temperature is the same in the strict theory as was established in § 1.5 from elementary kinetic considerations. The viscosity and the thermal conductivity of a pure

\*Values of the transport coefficients for different molecular models, both in the first approximation, and in higher approximations, for a pure gas and for gaseous mixtures, may be found in the already frequently cited monographs of Chapman and Cowling, and Hirschfelder, Curtiss, and Bird.

gas are determined quite accurately by the first approximation. They are determined to poorer accuracy for a mixture. The diffusion coefficient, and especially the thermal diffusion coefficient, is determined even less accurately.

Comparing the expression given for the transport coefficients of Maxwellian molecules in the first approximation to the exact values obtained in §3.3 [see formula (3.47)], we note that they coincide. It is easy to show that this is not accidental. In fact, we saw in §3.3 that the distribution function in the thirteen-moment approximation has the form

$$f = f^{(0)} \left[ 1 + \frac{p_{ij}}{p} \frac{m}{2kT} c_i c_j - \frac{2}{3} \frac{q_i}{p} \frac{m}{kT} c_i \left( \frac{5}{2} - \frac{mc^2}{2kT} \right) \right],$$

where

$$q_i = -\lambda \frac{\partial T}{\partial x_i}, \quad p_{ij} = -\mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_r}{\partial x_r} \right).$$

Comparing this expression with the distribution function (8.21), and equating coefficients of various powers of  $c_i$ , we find that

$$A^*(c) = \text{const} \left( \frac{5}{2} - \frac{m}{2kT} c^2 \right) \text{ and } B(c) = \text{const}.$$

Therefore, the thirteen-moment approximation distribution function is obtained from the distribution function (8.21) if only one term is retained in the expansions of the functions  $A(c)$  and  $B(c)$  in Sonine polynomials. Therefore, the thirteen-moment approximation for arbitrary molecules leads to the Navier-Stokes equations with inexact values of viscosity and thermal conductivity obtained in the first approximation.

On the other hand, we saw in §3.3 that the thirteen-moment approximation distribution function for Maxwellian molecules differs from the exact case by quantities of order  $\epsilon^2$ . Therefore, the distribution function (8.21) must agree with the thirteen-moment case to that accuracy, i.e., the series for  $A(c)$  and  $B(c)$  in terms of Sonine polynomials for Maxwellian molecules must contain only the first terms. The validity of that assertion may also be checked by direct substitution of the truncated series in Eqs. (8.8) and (8.9).

Continuing the above process, we can similarly find  $f^{(2)}$  and derive the Burnett equations, etc. However, leaving aside the extreme complexity of those equations, it should be borne in mind that the conclusions obtained in the preceding paragraph for the Hilbert method are applicable to the Chapman-Enskog method. The convergence of the method is in general asymptotic as  $\epsilon \rightarrow 0$ . Therefore, the Burnett equations improve the solution where the Navier-Stokes equations have good accuracy, but in general it is not safe to say that we may progress in the direction of larger values of  $\epsilon$  by means of the Burnett equations, i.e., toward more rarefied gases, in the cases when the Navier-Stokes equations are already unsuitable.\* As we have pointed out, there may of course be cases when a Hilbert series, and therefore also an Enskog-Chapman series, converges in the ordinary sense, and when the Burnett equations and the higher approximations permit us to progress toward higher Knudsen numbers, but such cases are very special.

Like the Hilbert method, the Enskog-Chapman method does not permit the solution of problems with an arbitrary initial distribution function  $f$ , since it takes into account only the initial hydrodynamic quantities (the first moments of  $f$ ). Furthermore, when  $t \gg \epsilon$ , the solution of the Boltzmann equation tends to an Enskog-Chapman solution for which the hydrodynamic initial conditions differ by a quantity of order  $\epsilon$  from the initial hydrodynamic quantities calculated from the initial distribution function  $f$ . To establish the correct initial and boundary conditions we must investigate the structure of the initial or boundary layers:  $0 \ll t \leq O(\epsilon)$ .†

2. We may arrive at the derivation of the hydrodynamic equations in a somewhat different way.

Since it has been shown above that the distribution function depends on the coordinates and on the time only through the medium

\*We recall that for asymptotic series there is always a maximum number of terms of the series giving the best approximation for a given value of the argument. With a further increase in the number of terms, the approximation does not improve, but gets worse.

†Apropos of this, for the linearized Boltzmann equation, see H. Grad, "Handbuch der Physik, Vol. 12 (1958); Phys. Fluids, Vol. 6, No. 2 (1963); Commun. Pure Appl. Math., Vol. 18, No. 1/2 (1965); L. Sirovich, Phys. Fluids, Vol. 6, No. 2 (1963). See also §§ 3.7 and 5.1.

of the hydrodynamic quantities and their derivatives, the distribution function may be represented in the form

$$f = f_s^{(0)}(\xi, \Gamma_r) + \epsilon f_s^{(1)}(\xi, \Gamma_r, \nabla \Gamma_r) + \epsilon^2 f_s^{(2)}(\xi, \Gamma_r, \nabla \Gamma_r, \nabla^2 \Gamma_r) + \dots, \quad (8.27)$$

where the symbols  $\nabla \Gamma_r, \nabla^2 \Gamma_r$ , etc., denote the first, second, etc., space derivatives of  $\Gamma_r$ , respectively. The meaning of the subscript  $s$  in the distribution function will be clear from what follows. In addition, we may put

$$\frac{\partial \Gamma_r}{\partial t} = \Phi_r^{(0)}(\Gamma_r) + \epsilon \Phi_r^{(1)}(\Gamma_r, \nabla \Gamma_r) + \epsilon^2 \Phi_r^{(2)}(\Gamma_r, \nabla \Gamma_r, \nabla^2 \Gamma_r) + \dots. \quad (8.28)$$

In accordance with these definitions, the time derivative which appears in the Boltzmann equation is equal to

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{\partial f_s^{(0)}}{\partial \Gamma_r} \Phi_r^{(0)} + \epsilon \left( \frac{\partial f_0^{(0)}}{\partial \Gamma_r} \Phi_r^{(1)} + \frac{\partial f_s^{(1)}}{\partial \Gamma_r} \Phi_r^{(0)} + \frac{\partial f_s^{(1)}}{\partial \nabla \Gamma_r} \nabla \Phi_r^{(0)} \right) + \dots \\ &= \sum_{\mu, v=0}^{\infty} \epsilon^{\mu+v} \left( \frac{\partial f_0^{(\mu)}}{\partial \Gamma_r} \Phi_r^{(v)} + \frac{\partial f_s^{(\mu)}}{\partial \nabla \Gamma_r} \nabla \Phi_r^{(v)} + \frac{\partial f_s^{(\mu)}}{\partial \nabla^2 \Gamma_r} \nabla^2 \Phi_r^{(v)} + \dots \right). \end{aligned} \quad (8.29)$$

Substituting that expression and (8.27) into the Boltzmann equation, and equating coefficients of the various powers of  $\epsilon$ , we obtain

$$\begin{aligned} \sum_{\mu+v=k-1} \left( \frac{\partial f_0^{(\mu)}}{\partial \Gamma_r} \Phi_r^{(v)} + \frac{\partial f_s^{(\mu)}}{\partial \nabla \Gamma_r} \nabla \Phi_r^{(v)} + \frac{\partial f_s^{(\mu)}}{\partial \nabla^2 \Gamma_r} \nabla^2 \Phi_r^{(v)} + \dots \right) + \xi_i \frac{\partial f_s^{(k-1)}}{\partial x_i} \\ = \sum_{v+\mu=k} J(f_s^{(\mu)}, f_s^{(v)}). \end{aligned} \quad (8.30)$$

These equations are similar to (7.4) and (7.6). In particular, for  $k = 0, 1$ , and  $2$ , we have

$$J(f_s^{(0)}, f_s^{(0)}) = 0, \quad (8.31)$$

$$2J(f_s^{(0)}, f_s^{(1)}) = \frac{\partial f_s^{(0)}}{\partial \Gamma_r} \Phi_r^{(0)} + \xi_i \frac{\partial f_s^{(0)}}{\partial x_i}, \quad (8.32)$$

$$\begin{aligned} 2J(f_s^{(0)}, f_s^{(2)}) &= \frac{\partial f_s^{(0)}}{\partial \Gamma_r} \Phi_r^{(1)} + \frac{\partial f_s^{(1)}}{\partial \Gamma_r} \Phi_r^{(0)} + \frac{\partial f_s^{(1)}}{\partial \nabla \Gamma_r} \nabla \Phi_r^{(0)} \\ &\quad + \xi_i \frac{\partial f_s^{(1)}}{\partial x_i} - J(f_s^{(1)}, f_s^{(1)}), \end{aligned} \quad (8.33)$$

.....

It follows from (8.31) that  $f_s^{(0)}$  is a Maxwellian function of the five macroscopic hydrodynamic quantities:

$$f_s^{(0)} = n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left\{ - \frac{m}{2kT} c^2 \right\}, \quad (8.34)$$

such that

$$\int \psi_i f_s^{(0)} d\xi = \Gamma_i, \quad \int \psi_i f_s^{(k)} d\xi = 0 \text{ when } k > 0. \quad (8.35)$$

From this it follows that

$$\int \psi_i \frac{\partial f_s^{(0)}}{\partial \Gamma_r} d\xi = \delta_{ir}, \quad \int \psi_i \frac{\partial f_s^{(k)}}{\partial \Gamma_r} d\xi = 0, \quad \int \psi_i \frac{\partial f_s^{(k)}}{\partial \nabla \Gamma_r} d\xi = 0. \quad (8.35a)$$

For Eqs. (8.32) and (8.33), etc., to be soluble, their right sides must be orthogonal to  $\psi_r$ . Taking account of (8.35) and (8.35a), we obtain the following existence conditions for the solution

$$\Phi_r^{(k)} = - \frac{\partial}{\partial x_i} \int \psi_r \xi_i f_s^{(k)} d\xi. \quad (8.36)$$

Substituting these values for  $\Phi_r^{(k)}$  into the expansion (8.28), we obtain

$$\frac{\partial \Gamma_r}{\partial t} + \frac{\partial}{\partial x_i} \sum_k \epsilon^k \int \psi_r \xi_i f_s^{(k)} d\xi = 0. \quad (8.37)$$

It is easy to see that those equations are none other than the general conservation equations [Eqs. (1.8)-(1.10)].

From the solubility condition (8.36) for  $f_s^{(0)}$  we find the functions  $\Phi_r^{(0)}$ . Substituting these into the right side of the linear integral equation (8.32) and solving that equation, we find  $f_s^{(1)}$ . Knowing  $f_s^{(1)}$ , we find the functions  $\Phi_r^{(1)}$  from (8.36), by substituting these into Eq. (8.33) we find  $f_s^{(2)}$ , etc. The functions  $f_s^{(k)}$  thus obtained depend on  $\Gamma_r$  and their space derivatives up to and including the  $k$ -th order.

By substituting  $f_s^{(0)}, f_s^{(1)}, f_s^{(2)}$ , etc., the functions thus obtained, in succession into the conservation equation (8.37), we obtain the Euler, Navier-Stokes, Burnett, etc., equations, respectively.

It should be noted that the functions  $f_s^{(k)}$  which appear in the expansions (8.27) are different from the functions  $f^{(k)}$ , which ap-

pear in the Hilbert expansion (7.3). In fact, the functions  $f_s^{(k)}$  are functions of the complete  $\Gamma_r$  and their derivatives. But the latter in turn are series of powers of  $\epsilon$ . In order to go over to the Hilbert functions  $f^{(k)}$ , we must expand the functions  $f_s^{(k)}$  in powers of  $\epsilon$ , and afterwards gather terms with equal powers of  $\epsilon$  in the expansion (8.27). The coefficients of  $\epsilon$  are equal to the Hilbert functions  $f^{(k)}$ . The difference between the functions  $f_s^{(k)}$  and  $f^{(k)}$  is very important for an understanding of the relationship between the Hilbert and Enskog-Chapman methods. It is precisely because the functions  $f_s^{(k)}$  are functions of the complete hydrodynamic quantities  $\Gamma_r$  that we may apply the conditions (8.35), while in the Hilbert method, according to (7.15),  $\Gamma_r \neq \Gamma_r^{(0)}$  and  $\Gamma_r^{(k)} \neq 0$ . Therefore,

$$f_s^{(0)}(\Gamma_r) = f_s^{(0)}(\Gamma_r^{(0)}) + \frac{\partial f_s^{(0)}(\Gamma_r^{(0)})}{\partial \Gamma_r} (\epsilon \Gamma_r^{(1)} + \epsilon^2 \Gamma_r^{(2)} + \dots), \quad (8.38)$$

$$\begin{aligned} f_s^{(1)}(\Gamma_r, \nabla \Gamma_r) &= f_s^{(1)}(\Gamma_r^{(0)}, \nabla \Gamma_r^{(0)}) + \frac{\partial f_s^{(1)}(\Gamma_r^{(0)})}{\partial \Gamma_r} (\epsilon \Gamma_r^{(1)} + \dots) \\ &\quad + \frac{\partial f_s^{(1)}(\Gamma_r^{(0)})}{\partial \nabla \Gamma_r} (\epsilon \nabla \Gamma_r^{(1)} + \dots), \end{aligned} \quad (8.39)$$

$$f_s^{(2)}(\Gamma_r, \nabla \Gamma_r, \nabla^2 \Gamma_r) = f_s^{(2)}(\Gamma_r^{(0)}, \nabla \Gamma_r^{(0)}, \nabla^2 \Gamma_r^{(0)}) + \dots \quad (8.40)$$

Substituting these expressions into the expansion (8.27) and collecting terms with equal powers of  $\epsilon$ , we obtain the following expressions for the functions  $f^{(k)}$  which are coefficients of  $\epsilon^k$ :

$$f^{(0)} = f_s^{(0)}(\Gamma_r^{(0)}), \quad (8.41)$$

$$f^{(1)} = f_s^{(1)}(\Gamma_r^{(0)}, \nabla \Gamma_r^{(0)}) + \frac{\partial f_s^{(0)}(\Gamma_r^{(0)})}{\partial \Gamma_r} \Gamma_r^{(1)}, \quad (8.42)$$

$$\begin{aligned} f^{(2)} &= f_s^{(2)}(\Gamma_r^{(0)}, \nabla \Gamma_r^{(0)}, \nabla^2 \Gamma_r^{(0)}) + \frac{\partial f_s^{(0)}(\Gamma_r^{(0)})}{\partial \Gamma_r} \Gamma_r^{(2)} \\ &\quad + \frac{\partial f_s^{(1)}(\Gamma_r^{(0)})}{\partial \Gamma_r} \Gamma_r^{(1)} + \frac{\partial f_s^{(1)}(\Gamma_r^{(0)})}{\partial \nabla \Gamma_r} \nabla \Gamma_r^{(1)}. \end{aligned} \quad (8.43)$$

The Hilbert functions  $f^{(k)}$  satisfy equations (7.6) while the Enskog-Chapman functions  $f_s$  satisfy equations (8.30). We shall show that the functions  $f^{(k)}$  determined by formulas (8.41), (8.42), (8.43), etc., satisfy equations (7.6), if the functions  $f_s^{(k)}$  satisfy equations (8.30), i.e., we shall show that  $f^{(k)}$  are in fact Hilbert functions.

We substitute  $f^{(k)}$ , represented in the form (8.41), (8.42), etc., into Eqs. (7.4), (7.5), and (7.6); we have

$$J[f_s^{(0)}(\Gamma_r^{(0)}), f_s^{(0)}(\Gamma_r^{(0)})] = 0, \quad (8.44)$$

$$\begin{aligned} 2J &\left[ f_s^{(0)}(\Gamma_r^{(0)}), f_s^{(1)}(\Gamma_r^{(0)}, \nabla \Gamma_r^{(0)}) + \frac{\partial f_s^{(0)}(\Gamma_r^{(0)})}{\partial \Gamma_r} \Gamma_r^{(1)} \right] \\ &= \frac{\partial f_s^{(0)}(\Gamma_r^{(0)})}{\partial \Gamma_r} \frac{\partial \Gamma_r^{(0)}}{\partial t} + \xi_i \frac{\partial f_s^{(0)}(\Gamma_r^{(0)})}{\partial x_i}, \end{aligned} \quad (8.45)$$

$$\begin{aligned} 2J &\left[ f_s^{(0)}(\Gamma_r^{(0)}), f_s^{(2)}(\Gamma_r^{(0)}, \nabla \Gamma_r^{(0)}, \nabla^2 \Gamma_r^{(0)}) + \frac{\partial f_s^{(0)}(\Gamma_r^{(0)})}{\partial \Gamma_r} \Gamma_r^{(2)} \right. \\ &\quad \left. + \frac{\partial f_s^{(1)}(\Gamma_r^{(0)})}{\partial \Gamma_r} \Gamma_r^{(1)} + \frac{\partial f_s^{(1)}(\Gamma_r^{(0)})}{\partial \nabla \Gamma_r} \nabla \Gamma_r^{(1)} \right] = - \frac{\partial f_s^{(1)}(\Gamma_r^{(0)}, \nabla \Gamma_r^{(0)})}{\partial \Gamma_r} \frac{\partial \Gamma_r^{(0)}}{\partial t} \\ &\quad + \frac{\partial f_s^{(1)}(\Gamma_r^{(0)}, \nabla \Gamma_r^{(0)})}{\partial \nabla \Gamma_r} \frac{\partial \nabla \Gamma_r^{(0)}}{\partial t} + \frac{\partial^2 f_s^{(0)}(\Gamma_r^{(0)})}{\partial \Gamma_r \partial \Gamma_s} \frac{\partial \Gamma_s^{(0)}}{\partial t} \Gamma_r^{(1)} + \frac{\partial f_s^{(0)}(\Gamma_r^{(0)})}{\partial \Gamma_r} \frac{\partial \Gamma_r^{(1)}}{\partial t} \\ &- J \left[ f_s^{(1)}(\Gamma_r^{(0)}, \nabla \Gamma_r^{(0)}) + \frac{\partial f_s^{(0)}(\Gamma_r^{(0)})}{\partial \Gamma_r} \Gamma_r^{(1)}, f_s^{(1)}(\Gamma_r^{(0)}, \nabla \Gamma_r^{(0)}) + \frac{\partial f_s^{(0)}(\Gamma_r^{(0)})}{\partial \Gamma_r} \Gamma_r^{(1)} \right], \\ &\quad \cdot \end{aligned} \quad (8.46)$$

It is clear that (8.44) coincides with (8.31). Therefore, the function  $f^{(0)}$  satisfies Eq. (7.4), if the function  $f_s^{(0)}$  satisfies Eq. (8.31).

According to the definition (7.2) of the quadratic form

$$J(f, C_1 \varphi_1 + C_2 \varphi_2) = C_1 J(f, \varphi_1) + C_2 J(f, \varphi_2) \quad (8.47)$$

and

$$J(f, \varphi) = J(\varphi, f). \quad (8.48)$$

Then

$$\frac{\partial}{\partial \Gamma_r} J[f_s^{(0)}(\Gamma_r^{(0)}), f_s^{(0)}(\Gamma_r^{(0)})] = 2J \left[ f_s^{(0)}(\Gamma_r^{(0)}), \frac{\partial f_s^{(0)}(\Gamma_r^{(0)})}{\partial \Gamma_r} \right]$$

and, therefore,

$$2J \left[ f_s^{(0)}(\Gamma_r^{(0)}), \frac{\partial f_s^{(0)}(\Gamma_r^{(0)})}{\partial \Gamma_r} \Gamma_r^{(1)} \right] = 2\Gamma_r^{(1)} J \left[ f_s^{(0)}(\Gamma_r^{(0)}), \frac{\partial f_s^{(0)}(\Gamma_r^{(0)})}{\partial \Gamma_r} \right] = 0. \quad (8.49)$$

The functions  $\Gamma_r^{(0)}$  in the Hilbert method satisfy the compatibility conditions (7.13):

$$\frac{\partial \Gamma_r^{(0)}}{\partial t} = - \frac{\partial}{\partial x_i} \int \Psi_r \xi_i f^{(0)}(\Gamma_r^{(0)}) d\xi.$$

It then follows from (8.37) and (8.36) that

$$\frac{\partial \Gamma_r}{\partial t} = \Phi_r^{(0)}(\Gamma_r^{(0)}). \quad (8.50)$$

Substituting (8.49) and (8.50) into Eq. (8.45), we verify that it coincides with Eq. (8.32). Therefore, function  $f^{(1)}$  satisfies Eq. (7.5), whenever the functions  $f_s^{(0)}$  and  $f_s^{(1)}$  satisfy Eqs. (8.31) and (8.32). Carrying out similar but more cumbersome calculations, it can be shown that Eq. (8.46) or Eq. (7.6) is satisfied when  $f_s^{(0)}$ ,  $f_s^{(1)}$ , and  $f_s^{(2)}$  satisfy Eqs. (8.31), (8.32), (8.33), etc.

Thus, the functions  $f^{(k)}$ , determined by the expressions (8.41), (8.42), etc., are in fact terms of a Hilbert series.

The procedure for transition from a Hilbert expansion to an Enskog-Chapman expansion given at the beginning of this section, differs somewhat from that just described. There, we made use of the solutions of equations (7.6), from which we eliminated the derivatives with respect to time by means of the compatibility conditions (7.13), and then put  $\Gamma_r^{(k)} = 0$  when  $k > 0$ , which was legitimate, since the examination was made at a certain chosen instant of time  $t = 0$ . It can easily be verified that the solutions  $f^{(k)}$  of Eq. (7.6) are then identical with the solutions  $f_s^{(k)}$  of equations (8.36). In fact, if we express the functions  $f^{(k)}$  in terms of  $f_s^{(k)}$  according to formulas (8.41), (8.42), etc., and substitute into Eq. (7.6), we arrive at Eqs. (8.44), (8.45), etc., as was shown above. Replacing the time derivatives in those equations by means of the compatibility equations (7.13), using (8.37) and (8.36), and then putting  $\Gamma_r^{(k)} = 0$  when  $k \neq 0$ , we obtain equations (8.30). But, as was shown above, the series constructed from solutions of equations (8.30) is an Enskog-Chapman series. Thus, the procedure described at the beginning of the section leads to the same Enskog-Chapman series and, therefore, to the same hydrodynamic equations: to the Euler, Navier-Stokes, Burnett, etc., equations. In that, the Hilbert and Enskog-Chapman methods are equivalent. In the Hilbert method (if we do not aim to obtain a closed system of five hydrodynamic equations for the five hydrodynamic quantities) the quantities  $\Gamma_r^{(k)}$  satisfy the equations which may be obtained from the hydrodynamic equations if the hydrodynamic quantities in them are expanded in powers of  $\epsilon$ , and the coefficients of equal powers of  $\epsilon$  are equated. If, for example, we restrict consideration to quantities

of order  $\epsilon$ , then

$$f_s = f_s^{(0)} + \epsilon f_s^{(1)} \text{ and } \Gamma_r = \Gamma_r^{(0)} + \epsilon \Gamma_r^{(1)}. \quad (8.51)$$

The equations of hydrodynamics in that approximation, i.e., the Navier-Stokes equations, may be written in the form [see (8.37)]

$$\frac{\partial \Gamma_r}{\partial t} + \frac{\partial}{\partial x_i} \int \psi_r \xi_i f_s^{(0)} d\xi + \epsilon \frac{\partial}{\partial x_i} \int \psi_r \xi_i f_s^{(1)} d\xi = 0, \quad (8.52)$$

where  $f_s^{(0)}$  and  $f_s^{(1)}$  are functions of the complete  $\Gamma_r$ .

We substitute for  $f_s^{(0)}$  and  $f_s^{(1)}$  according to formulas (8.38) and (8.39); neglecting quantities of order  $\epsilon^2$ , we obtain

$$\begin{aligned} \frac{\partial \Gamma_r}{\partial t} + \frac{\partial}{\partial x_i} \int f_s^{(0)}(\Gamma_r^{(0)}) \psi_r \xi_i d\xi + \epsilon \frac{\partial}{\partial x_i} \Gamma_r^{(1)} \int \frac{\partial f_s^{(0)}(\Gamma_r^{(0)})}{\partial \Gamma_r} \psi_r \xi_i d\xi \\ + \epsilon \frac{\partial}{\partial x_i} \int f_s^{(1)}(\Gamma_r^{(0)}) \psi_r \xi_i d\xi = 0. \end{aligned} \quad (8.53)$$

Taking into account (8.41) and (8.42), which relate the Hilbert functions  $f^{(k)}$  to the functions  $f_s^{(k)}$ , and equating coefficients of  $\epsilon^0$  and  $\epsilon^1$  to zero, we obtain

$$\frac{\partial \Gamma_r^{(0)}}{\partial t} + \frac{\partial}{\partial x_i} \int \psi_r \xi_i f^{(0)} d\xi = 0, \quad \frac{\partial \Gamma_r^{(1)}}{\partial t} + \frac{\partial}{\partial x_i} \int \psi_r \xi_i f^{(1)} d\xi = 0. \quad (8.54)$$

It is clear that these equations coincide with Eqs. (7.13) or (7.14) for  $\gamma_r^{(0)}$  and  $\gamma_r^{(1)}$  (or  $\Gamma_r^{(0)}$  and  $\Gamma_r^{(1)}$ ) in the Hilbert method.

In the Enskog-Chapman method the parameter  $\epsilon$  enters into the solution in a more complex fashion which is not analytical, generally speaking. The solution obtained to the same approximation (with the same number of terms of the expansion) by the Enskog-Chapman method may be shown to be more accurate than in the Hilbert method.

Yet there are examples where the higher approximations in the Enskog-Chapman method do not have solutions, while the Hilbert method permits us to construct a solution for any approximation.

3. The method of expansion in powers of a small parameter may be given a somewhat more geometrical pictorial interpretation.\* We write the dimensionless Boltzmann equation in the integral form (7.4) of Chapter II:

$$f(t, \mathbf{x}_0 + \xi(t - t_0), \xi) - f(t_0, \mathbf{x}_0, \xi) = \frac{1}{\epsilon} \int_{t_0}^t J(s, \mathbf{x}_0 + \xi(s - t_0), \xi) ds. \quad (8.55)$$

If  $\tau = t - t_0 = O(\epsilon)$ , then the left side of the equation gives the difference in distribution functions at points whose distance apart is of order  $\epsilon$ . Let the flow in the neighborhood of the point  $(t_0, \mathbf{x}_0)$  be such that the distribution function may be represented for all  $\xi$  in the form

$$f(t_0 + \tau, \mathbf{x}_0 + \xi\tau, \xi) = f(t_0, \mathbf{x}_0, \xi) + \frac{df}{dt} \Big|_{\tau=0} \tau + \frac{1}{2} \frac{d^2 f}{dt^2} \Big|_{\tau=0} \tau^2 + \dots \quad (8.56)$$

Let the variation of the distribution function in the first approximation be negligible in a length or in a time of order  $\epsilon$ . Then the distribution function in that approximation must satisfy the condition:

$$\int_{t_0}^t J ds = 0, \quad \text{or} \quad J = 0. \quad (8.57)$$

It is easy to see that the condition (8.57) is identical to Eq. (7.4) and, therefore, if the flow is such that a variation in the distribution function over a length of order  $\epsilon$  can be neglected, then the distribution function in the first approximation must be the Maxwellian:  $f = f_0$ . In the second approximation we represent the distribution function in the form

$$f = f_0(1 + \varphi), \quad (8.58)$$

where  $\varphi$  is a small correction, and we retain the second term in the expansion (8.56). Again putting  $\tau = O(\epsilon)$  from (8.55), and neglecting squares of  $\varphi$ , we have

$$\frac{df_0}{dt} \Big|_{\tau=0} \tau + \frac{df_0 \varphi}{dt} \Big|_{\tau=0} \tau = \frac{2}{\epsilon} \int_{t_0}^t J(f_0, f_0 \varphi) ds. \quad (8.59)$$

Hence it may be seen that  $\varphi \sim O(\epsilon)$ , i.e.,  $\varphi = \epsilon f^{(1)}$ . Neglecting quantities  $O(\epsilon^2)$ , we obtain an equation identical to (7.5):

$$\frac{df_0}{dt} = 2J(f_0, f_0 f^{(1)}). \quad (8.60)$$

Since the time  $t_0$  is arbitrary, this condition is fulfilled at any point.

In the third approximation,

$$f = f_0(1 + \epsilon f^{(1)} + \varphi), \quad (8.61)$$

where  $\varphi$  is a small correction, whose squares may be neglected. We retain the third term of the expansion on the left side of Eq. (8.55). We have

$$\begin{aligned} & \frac{df_0}{dt} \Big|_{t_0} \tau + \epsilon \frac{df_0 f^{(1)}}{dt} \Big|_{t_0} \tau + \frac{df_0 \varphi}{dt} \Big|_{t_0} \tau + \frac{1}{2} \frac{d^2 f_0}{dt^2} \Big|_{t_0} \tau^2 \\ & + \frac{1}{2} \frac{d^2 f_0 f^{(1)}}{dt^2} \Big|_{t_0} \tau^2 + \frac{1}{2} \frac{d^2 f_0 \varphi}{dt^2} \Big|_{t_0} \tau = \int_{t_0}^{t+\tau} [2J(f_0, f_0 f^{(1)})]_s \\ & + \epsilon J(f_0 f^{(1)}, f_0 f^{(1)})]_s + \frac{1}{\epsilon} 2J(f_0, f_0 \varphi)]_s ds. \end{aligned} \quad (8.62)$$

According to (8.60),

$$2J(f_0, f_0 f^{(1)})]_s = \frac{df_0}{dt} \Big|_s = \frac{df_0}{dt} \Big|_{t_0} + \frac{d^2 f_0}{dt^2} \Big|_{t_0} (s - t_0).$$

Substituting this expression into (8.62), we find that  $\varphi = \epsilon^2 f^{(2)}$ ; neglecting quantities of order  $\epsilon^3$ , we obtain

$$\frac{df_0 f^{(1)}}{dt} - J(f_0 f^{(1)}, f_0 f^{(1)}) = 2J(f_0, f_0 f^{(2)}). \quad (8.63)$$

That expression evidently coincides with Eq. (7.6) for  $k = 2$ . Continuing in a similar manner, we can construct a distribution function to any approximation.

The interpretation given for the small-parameter method indicates that it is possible to represent the distribution function in the form of an expansion in powers of  $\epsilon$ , because we can represent it in the form of expansion (8.56) in powers of  $x$  and  $t$  for  $\tau = O(\epsilon)$ .

It should be noted that, in order to construct a solution, it is necessary that the derivatives  $d/dt$  along the trajectories of the

\* M. N. Kogan, Zh. Prikl. Mekhan. i Tekhn. Fiz., No. 1 (1965).

molecules exist, while the partial derivatives  $\partial/\partial t$  and  $\partial/\partial x_i$  may or may not exist.

4. From the derivation of the Navier-Stokes equations given above, the impression may have been created that those equations are applicable only when the Navier-Stokes (viscous) terms are small in comparison with the Eulerian terms. It would then be invalid, for example, to investigate flows in a boundary layer or Stokes flow at small Reynolds numbers by means of the Navier-Stokes equations. We show that this is not the case.

The distribution function was expanded in a series of powers of a small parameter, equal to the ratio of the mean free path to a characteristic flow dimension (the Knudsen number). In a steady boundary layer there are two characteristic flow dimensions: the length  $L$  and the boundary layer thickness  $\delta$ , equal to

$$\delta \sim \frac{L}{\sqrt{Re}} \sim L \sqrt{\frac{\mu}{\rho L U}} \sim L \sqrt{\frac{m a \lambda}{\rho L U}} \sim L \sqrt{\frac{Kn}{M}} \sim L \sqrt{Kn}.$$

Here,  $U$  is the characteristic macroscopic velocity of the flow and  $a$  is the velocity of sound; the viscosity  $\mu$  has been substituted according to formula (5.7) of Chapter I; the Knudsen number  $Kn = \lambda/L$  and the Mach number  $M = U/a$  have been determined according to the characteristic parameters  $\lambda$  and  $a$  within the boundary layer.

From the treatment of the small-parameter method just given, it is clear that the nature of the variation of the distribution function along the trajectory of the particles is important. For the overwhelming majority of trajectories (excluding trajectories making a small angle of order  $\delta/L$  with the plate), the characteristic length over which the distribution function varies is of order  $\delta$ . Therefore, in this case, the small parameter is  $\epsilon = \lambda/\delta = \sqrt{Kn}$ . We seek a solution of the Boltzmann equation, as was done above, in the form of an expansion in powers of  $\epsilon$ . Therefore, the stress tensor  $p_{ij}$  and the heat flux vector  $q_i$ , determined by the function  $f^{(1)}$ , will be of order  $\epsilon$  as above, i.e., of order  $\sqrt{Kn}$  in the case examined. According to the Enskog-Chapman method, the values obtained for  $p_{ij}$  and  $q_i$  must be substituted in the general conservation equations (1.9) and (1.10). As an example we write the equation of conservation of longitudinal momentum:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{1}{\rho} \left( \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} \right).$$

Since the derivative  $\partial/\partial x \sim 1/L$  and  $\partial/\partial y \sim 1/\delta$ , the terms  $u \partial u / \partial x \sim u^2 L$  and  $\partial p_{xy} / \partial y \sim \epsilon / \delta$  are of the same order. Thus, in spite of the fact that, as in the general case, the function  $f^{(1)}$  is small in comparison with  $f_0$ , the term in the boundary layer due to the function  $f^{(1)}$  turns out to be of the same order as the terms determined by the function  $f^{(0)}$ .

In other words, if we go over to dimensionless coordinates in the Boltzmann equation and in the conservation equations, if we refer  $x$  and  $y$  to a characteristic length  $\delta$ , all the arguments of the Hilbert and Enskog-Chapman methods remain unchanged. The Navier-Stokes equations have their usual form, and the viscosity terms are proportional to the small parameter  $\epsilon = \lambda/\delta = \sqrt{Kn}$ . The equality of that term to the Euler terms, which do not have the small multiplying factor in front of them, is due to the smallness of the longitudinal derivative, which is of order  $\epsilon$  in these variables. We should then have no doubts as to the applicability of the Navier-Stokes equations, just as there was no doubt of their applicability when examining Couette flow, in which one of the derivatives was exactly equal to zero, while the characteristic distance between the plates may be both of order  $L$  and of order  $\delta \sim L\sqrt{\lambda/L}$ .

Further, the characteristic flow dimension may be of order  $\lambda^\alpha$ , where  $\alpha < 1$ . Then  $\epsilon \sim \lambda^{1-\alpha}$  and the expansion in powers of  $\epsilon$  for given values of  $\lambda$  clearly converges more slowly as  $\alpha$  approaches unity.

The situation is similar in a Stokes flow at small Reynolds numbers. Here also,  $f^{(1)}$  is less than  $f^{(0)}$  by a factor  $\epsilon$ , and all the arguments which apply to the Navier-Stokes equations remain in force. However, the inertia terms turn out to be small in comparison with the viscous terms, because of the smallness of the macroscopic velocities.

### §3.9. Derivation of the Equation of Hydrodynamics for a Mixture of Gases

It was shown in §2.6 that the behavior of a mixture of  $N$  gases is described by a simultaneous system of  $N$  Boltzmann equa-

tions for the N-distribution functions  $f_k(t, x, \xi^k)$  for each component of the mixture\*

$$\begin{aligned} \frac{df_k}{dt} &= \frac{\partial f_k}{\partial t} + \xi_r^k \frac{\partial f_k}{\partial x_r} + \frac{X_r^k}{m^k} \frac{\partial f_k}{\partial \xi_r^k} \\ &= \frac{1}{\epsilon} \sum_{l=1}^N \int (f'_k f'_l - f_k f_l) g_{kl} b db de d\xi^l = \frac{1}{\epsilon} \sum_{l=1}^N J(f_k, f_l) \quad (9.1) \\ &\quad (k = 1, \dots, N). \end{aligned}$$

We assume the equations to be written in dimensionless form so that they contain the small parameter  $\epsilon$ . In the final hydrodynamic equations, without any explicit statement, we return to dimensional quantities, so that the small parameter does not appear explicitly.

Before we proceed to derive the hydrodynamic equations, we introduce a definition of the hydrodynamic properties of a mixture of gases.

In analogy with §2.1 [formulas (1.4)-(1.11)], we introduce the following notation:

$$n^k = \int f_k d\xi^k \quad (9.2)$$

for the number of particles (concentration) of the k-th component of the gas per unit volume,

$$\rho = \sum_{k=1}^N m^k n^k \quad (9.3)$$

for the mass density of the gas,

$$u^k = \frac{1}{n^k} \int \xi^k f_k d\xi^k \quad (9.4)$$

for the mean velocity of the k-th component of the gas,

$$u = \frac{1}{\rho} \sum_{k=1}^N n^k m^k u^k \quad (9.5)$$

\*There is no summation over the subscript k.

for the mean velocity of the mixture,

$$v^k = u^k - u = \frac{1}{n^k} \int (\xi^k - u) f_k d\xi^k = \frac{1}{n^k} \int c^k f_k d\xi^k \quad (9.6)$$

for the diffusion rate of the k-th component of the gas,  $c^k = \xi^k - u$ , where  $u$  is the thermal or peculiar velocity of a molecule of the k-th component,

$$\frac{3}{2} kT = \frac{1}{n} \sum_k \int \frac{m^k c^{k2}}{2} f_k d\xi^k, \quad \left( n = \sum_{k=1}^N n^k \right) \quad (9.7)$$

for the temperature of the mixture,

$$P_{ij}^k = m^k \int c_i^k c_j^k f_k d\xi^k \quad (9.8)$$

for the partial stress tensor of the k-th component,

$$P_{ij} = \sum_{k=1}^N P_{ij}^k, \quad P_{ij} = p_{ij} - \delta_{ij} p, \quad p = \frac{1}{3} (P_{11} + P_{22} + P_{33}) \quad (9.9)$$

for the stress tensor of the mixture,

$$q^k = \int c^k \frac{m^k c^{k2}}{2} f_k d\xi^k \quad (9.10)$$

for the partial heat flux of the k-th component, and

$$q = \sum_{k=1}^N q^{(k)} \quad (9.11)$$

for the heat flux in the mixture.

The collision integrals on the right side of equations (9.1) possess symmetry properties similar to those established in §2.4. In fact, let us examine the integral

$$I_{\varphi_k}^{kl} = \int \varphi_k (f'_k f'_l - f_k f_l) g_{kl} b db de d\xi^k d\xi^l, \quad (9.12)$$

where  $\varphi_k = \varphi(\xi^k)$ . It is evident that the integral (9.12) does not change if we interchange the velocities before and after a collision, i.e.,

$$I_{\varphi_k}^{kl} = \int \varphi'_k (f_k f_l - f'_k f'_l) g'_{kl} b' db' de' d\xi'^k d\xi'^l. \quad (9.13)$$

But, for an elastic collision we have

$$g_{kl} = g'_{kl}, \quad b = b', \quad \varepsilon = \varepsilon', \quad d\xi^k d\xi^l = d\xi'^k d\xi'^l. \quad (9.14)$$

Combining (9.12) and (9.13), and taking account of relation (9.14), we obtain

$$I_{\varphi_k}^{kl} = \frac{1}{2} \int (\varphi_k - \varphi'_k)(f'_k f'_l - f_k f_l) g_{kl} b db de d\xi^k d\xi^l. \quad (9.15)$$

The subscripts  $k$  and  $l$  appear in this expression in an unsymmetrical manner.\* However, if we sum these integrals over  $k$  and  $l$ , then  $k$  and  $l$  appear symmetrically, and we have the following equality

$$\sum_{k,l}^N I_{\varphi_k}^{kl} = \sum_{k,l}^N I_{\varphi_l}^{kl} \quad (9.16)$$

or, in abbreviated form,

$$\begin{aligned} & \sum_{k,l}^N \int (\varphi_k - \varphi'_k)(f'_k f'_l - f_k f_l) g_{kl} b db de d\xi^k d\xi'^l \\ &= \sum_{k,l}^N (\varphi_l - \varphi'_l)(f'_l f'_k - f_l f_k) g_{kl} b db de d\xi^l d\xi^k, \end{aligned} \quad (9.16a)$$

whence

$$\sum_{k,l}^N I_{\varphi_k}^{kl} = \frac{1}{4} \sum_{k,l}^N \int (\varphi_k + \varphi_l - \varphi'_k - \varphi'_l)(f'_k f'_l - f_k f_l) g_{kl} b db de d\xi^k d\xi'^l. \quad (9.17)$$

Making use of these symmetry relations, it is easy to derive the continuity equations for the mixture. Integrating the  $k$ -th equation (9.1) with respect to  $\xi^k$ , we have

$$\frac{\partial n^k}{\partial t} + \frac{\partial n^k u_r}{\partial x_r} = 0. \quad (9.18)$$

\*The functions  $f_k \neq f_l$ .

The collision integral disappears as a consequence of relation (9.15), in which we must put  $\varphi_k = 1$ . The vanishing of the right side can also be interpreted physically, since the integral over  $\xi^k$  of the sum of the collision of  $k$ -particles with particles of all other types gives the total change in the number of  $k$ -particles per unit volume resulting from molecular collisions. Since the particles do not undergo chemical changes in the collisions, the total number of particles of a given type does not change as a result of the collisions.

Multiplying the  $k$ -th equation (9.18) by  $m^k$  and summing over  $k$ , we obtain the equation of conservation of mass for the mixture of gases:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_r}{\partial x_r} = 0. \quad (9.19)$$

If we multiply the  $k$ -th equation of (9.1) by  $m^k \xi_r^k$ , and integrate with respect to  $\xi^k$ , the collision integral does not vanish, because some of the momentum of the  $k$ -molecules may be transferred upon collision to molecules of other types. However, if we combine the momentum equations, the right side goes to zero, since the total momentum of molecules of all types is conserved during collisions. This fact also follows from the symmetry relation (9.17). By simple calculations we obtain the following equations of conservation of momentum for the mixture:

$$\frac{\partial u_l}{\partial t} + u_r \frac{\partial u_l}{\partial x_r} = -\frac{1}{\rho} \frac{\partial P_{lr}}{\partial x_r} + \frac{1}{\rho} \sum_{k=1}^N n^k X_l^k, \quad (l = 1, 2, 3). \quad (9.20)$$

Similarly, multiplying the  $k$ -th equation of (9.1) by  $\frac{1}{2} m^k \xi^{k2}$ , integrating with respect to  $\xi^k$ , and summing over  $k$ , we obtain the equation of conservation of energy for the mixture:

$$\begin{aligned} \frac{3}{2} k n \left( \frac{\partial T}{\partial t} + u_r \frac{\partial T}{\partial x_r} \right) &= -\frac{\partial q_r}{\partial x_r} - P_{tr} \frac{\partial u_l}{\partial x_r} + \sum_{k=1}^N n^k X_r^k v_r^k \\ &+ \frac{3}{2} k T \frac{\partial}{\partial x_r} \sum_{k=1}^N n^k v_r^k. \end{aligned} \quad (9.21)$$

The system of  $5 + N$  equations (9.18)–(9.21) contains the five hydrodynamic quantities for the mixture –  $\rho$ ,  $u_r$ ,  $T$ , and the

$N$  densities  $n^k$ , and  $3N$  additional unknowns  $v_r^k$ ; the components of the stress tensor  $P_{ij}$  and of the heat flux vector  $q_i$ . The velocities  $u_r^k = u_r + v_r^k$ , by definition.

In order to close this system of equations, we must find the dependence of  $P_{ij}$ ,  $q_i$ , and  $v_r^k$  on the  $5 + N$  hydrodynamic quantities  $\rho$ ,  $u_r$ ,  $T$ , and  $n^k$ . As shown above, Eq. (9.19) follows from equations (9.18), but the unknown  $n$  which appears there is expressed in terms of  $n^k$  as  $n = \sum n^k$ . Therefore, there are  $4 + N$  independent equations and  $4 + N$  unknowns.

To obtain the desired relations, we need the solution of the Boltzmann equations (9.1). As was done in the previous paragraph, we seek a solution in the form

$$f_k(t, x, \xi) = f_k^{(0)}(n^k, u_i, T, \xi) + \epsilon f_k^{(1)}(n^k, u_i, T, \nabla n^k, \nabla u_i, \nabla T, \xi) + \dots \quad (9.22)$$

and, in addition, we put

$$\frac{\partial n^k}{\partial t} = \Phi_{n^k}^{(0)}(n^k, u_i, T) + \epsilon \Phi_{n^k}^{(1)}(n^k, u_i, T, \nabla n^k, \nabla u_i, \nabla T) + \dots, \quad (9.22a)$$

$$\frac{\partial u_r}{\partial t} = \Phi_{u_r}^{(0)}(n^k, u_i, T) + \epsilon \Phi_{u_r}^{(1)}(n^k, u_i, T, \nabla n^k, \nabla u_i, \nabla T) + \dots, \quad (9.22b)$$

$$\frac{\partial T}{\partial t} = \Phi_T^{(0)}(n^k, u_i, T) + \epsilon \Phi_T^{(1)}(n^k, u_i, T, \nabla n^k, \nabla u_i, \nabla T) + \dots, \quad (9.22c)$$

where the symbol  $\nabla A$  indicates the dependence on derivatives with respect to the position coordinates  $\partial A / \partial x_i$ .

Substituting expressions (9.22) into the Boltzmann equations (9.1) and equating coefficients of equal powers of  $\epsilon$ , we obtain

$$\sum_{l=1}^N J(f_k^{(0)}, f_l^{(0)}) = 0, \quad (9.23)$$

$$\begin{aligned} & \sum_{l=1}^N [J(f_k^{(0)}, f_l^{(1)}) + J(f_k^{(1)}, f_l^{(0)})] \\ &= \frac{\partial f_k^{(0)}}{\partial \Gamma_\mu} \Phi_{\Gamma_\mu}^{(0)} + \xi_r \frac{\partial f_r^{(0)}}{\partial x_r} + \frac{X_r^k}{m^k} \frac{\partial f_k^{(0)}}{\partial \xi_r^k} = P_k^{(1)}, \end{aligned} \quad (9.24)$$

$$\sum_{l=1}^N [J(f_k^{(0)}, f_l^{(2)}) + J(f_k^{(2)}, f_l^{(0)})] =$$

$$\begin{aligned} &= -J(f_k^{(1)}, f_l^{(1)}) + \frac{\partial f_k^{(0)}}{\partial \Gamma_\mu} \Phi_{\Gamma_\mu}^{(1)} + \frac{\partial f_k^{(1)}}{\partial \Gamma_\mu} \Phi_{\Gamma_\mu}^{(0)} + \frac{\partial f_k^{(1)}}{\partial \nabla \Gamma_\mu} \nabla \Phi_{\Gamma_\mu}^{(0)} \\ &+ \xi_r \frac{\partial f_k^{(1)}}{\partial x_r} + \frac{X_r^k}{m^k} \frac{\partial f_k^{(1)}}{\partial \xi_r^k} = P_k^{(2)}. \end{aligned} \quad (9.25)$$

Here, for brevity, the set of macroscopic quantities  $n^k$ ,  $u_i$ , and  $T$  is denoted by  $\Gamma_\mu$ .

The equations (9.23) are satisfied for the Maxwellian distribution

$$f_k^{(0)} = n^k \left( \frac{m^k}{2\pi kT} \right)^{3/2} \exp \left\{ - \frac{m^k}{2kT} [\xi^k - u]^2 \right\}. \quad (9.26)$$

Since the full hydrodynamic quantities appear in  $f_k^{(0)}$ , then

$$\int f_k^{(0)} d\xi^k = n^k \text{ and } \int f_k^{(v)} d\xi^k = 0 \text{ when } v > 0, \quad (9.27)$$

$$\sum_{k=1}^N \int m^k \xi_i^k f_k^{(0)} d\xi^k = \rho u_i \quad \text{and} \quad \sum_{k=1}^N \int m^k \xi_i^k f_k^{(v)} d\xi^k = 0 \quad \text{when } v > 0, \quad (9.28)$$

$$\sum_{k=1}^N \int \frac{m^k c^{k2}}{2} f_k^{(0)} d\xi^k = \frac{3}{2} k n T \text{ and } \sum_{k=1}^N \int \frac{m^k c^{k2}}{2} f_k^{(v)} d\xi^k = 0 \quad \text{when } v > 0. \quad (9.29)$$

We represent the functions  $f_k^{(v)}$  as follows:

$$f_k^{(v)} = f_k^{(0)} \varphi_k^{(v)}, \quad (v = 1, \dots). \quad (9.30)$$

Then Eqs. (9.24), (9.25), etc. may be written in the symmetrical form (we shall omit the subscript  $v$  for brevity)

$$\sum_{l=1}^N \int f_k^{(0)} f_l^{(0)} (\varphi'_k + \varphi'_l - \varphi_k - \varphi_l) g_{kl} b db d\varepsilon d\xi^l = P_k. \quad (9.31)$$

It is easy to see that this system of homogeneous equations has the solution

$$\varphi_k = \alpha^k m^k + \beta_r m^k \xi_r^k + \gamma m^k \xi^k$$

or

$$\varphi_k = \alpha^k m^k + \beta_r m^k c_r^k + \gamma m^k c^{k2}. \quad (9.32)$$

It is also easy to show that there are no other solutions. In fact, by multiplying the  $k$ -th homogeneous equation of (9.31) by its solution  $\varphi_k(\xi^k)$  and integrating with respect to  $\xi^k$ , by analogy with (9.15), we have

$$\begin{aligned} I_{\varphi_k}^k &= \sum_{l=1}^N \int \varphi_k f_k^{(0)} f_l^{(0)} (\varphi'_k + \varphi'_l - \varphi_k - \varphi_l) g_{kl} b db d\xi^l d\xi^k \\ &= \sum_{l=1}^N \int \varphi'_k f_k^{(0)'} f_l^{(0)'} (\varphi_k + \varphi_l - \varphi'_k - \varphi'_l) g'_{kl} b' db' d\xi^l d\xi^k \\ &= \frac{1}{2} \sum_{l=1}^N \int (\varphi_k - \varphi'_k) f_k^{(0)} f_l^{(0)} (\varphi'_k + \varphi'_l - \varphi_k - \varphi_l) g_{kl} b db d\xi^l d\xi^k. \quad (9.33) \end{aligned}$$

Summing the integrals (9.33) over  $k$ , and using the fact that the expressions thus obtained are symmetrical with respect to  $k$  and  $l$ , we obtain, by analogy with (9.17),

$$\begin{aligned} \sum_{k=1}^N I_{\varphi_k}^k &= \frac{1}{2} \sum_{k,l} \int f_k^{(0)} f_l^{(0)} (\varphi_k - \varphi'_k) (\varphi'_k + \varphi'_l - \varphi_k - \varphi_l) g_{kl} b db d\xi^l d\xi^k \\ &= \frac{1}{2} \sum_{k,l} \int f_k^{(0)} f_l^{(0)} (\varphi_l - \varphi'_l) (\varphi'_k + \varphi'_l - \varphi_k - \varphi_l) g_{kl} b db d\xi^l d\xi^k \\ &= -\frac{1}{4} \sum_{k,l} \int f_k^{(0)} f_l^{(0)} (\varphi'_k + \varphi'_l - \varphi_k - \varphi_l)^2 g_{kl} b db d\xi^l d\xi^k. \quad (9.34) \end{aligned}$$

The summational invariants which appear in the solution (9.32) make the quadratic term in the brackets go to zero. Therefore, the solution (9.32) is a general solution of the homogeneous system of equations (9.31).

Exactly in the same way, by multiplying the inhomogeneous equations by the summational invariants  $\psi_{k,\mu} = m^k, m^k c_1^k$ , and  $m^k c^{k2}$ , we obtain

$$\sum_{l=1}^N \frac{1}{2} \int \{(\psi_{k,\mu} - \psi'_{k,\mu}) f_k^{(0)} f_l^{(0)} (\varphi'_k + \varphi'_l - \varphi_k - \varphi_l)\}$$

$$\times g_{kl} b db d\xi^l d\xi^k = \int \psi_{k,\mu} P_k d\xi^k.$$

$$\begin{aligned} \sum_{k,l} \frac{1}{4} \int \{(\psi_{k,\mu} + \psi_{l,\mu} - \psi'_{k,\mu} - \psi'_{l,\mu}) (\varphi'_k + \varphi'_l - \varphi_k - \varphi_l) f_k^{(0)} f_l^{(0)}\} \\ \times g_{kl} b db d\xi^l d\xi^k = \sum_{k=1}^N \int \psi_{k,\mu} P_k d\xi^k. \end{aligned}$$

Therefore, the inhomogeneous equations have solutions only when their right sides satisfy the following equations:

$$\int m^k P_k d\xi^k = 0, \quad \sum_{k=1}^N \int m^k \xi_i^k P_k d\xi^k = 0, \quad \sum_{k=1}^N \int \frac{m^k \xi_i^{k2}}{2} P_k d\xi^k = 0. \quad (9.35)$$

If we limit ourselves to the first term of the expansion (9.22) for the distribution function in  $\epsilon$ , i.e., we put  $f_k = f_k^{(0)}$ , then

$$u_r^k = \frac{1}{n^k} \int \xi_r^k f_k^{(0)} d\xi^k = u_r, \quad (9.36)$$

$$P_{lj} = \sum_{k=1}^N m^k \int c_l^k c_j^k f_k^{(0)} d\xi^k = \delta_{lj} p, \quad (9.37)$$

$$q_l = \sum_{k=1}^N \frac{m^k}{2} \int c^{k2} c_l^k f_k^{(0)} d\xi^k = 0. \quad (9.38)$$

There are no diffusion components in this approximation; nor is there any heat flux or stress tensor component except the pressure. By substituting these quantities into the conservation equations (9.18)-(9.21), we obtain the following system of Euler equations for the mixture:

$$\frac{\partial n^k}{\partial t} + \frac{\partial n^k u_r}{\partial x_r} = 0, \quad (9.39)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_r}{\partial x_r} = 0, \quad (9.40)$$

$$\frac{\partial u_l}{\partial t} + u_r \frac{\partial u_l}{\partial x_r} = -\frac{1}{\rho} \frac{\partial p}{\partial x_r} + G_l, \quad (9.41)$$

$$\frac{3}{2} kn \left( \frac{\partial T}{\partial t} + u_r \frac{\partial T}{\partial x_r} \right) = - p \frac{\partial u_r}{\partial x_r}, \quad (9.42)$$

where

$$G_i = \frac{1}{\rho} \sum_{k=1}^N n^k X_i^k$$

is the force per unit mass of mixture. For example, if the mixture is in a gravitational field, then  $G_i$  is the acceleration of gravity, and the equations (9.40)–(9.42) for the mixture coincide with the equations for a one-component gas. The concentration of the components  $n^k$  of the mixture may be found independently by means of equations (9.39), after the overall pattern of the gas flow is determined.

To determine the functions  $f_k^{(1)}, f_k^{(2)}, \dots$ , it is necessary to solve Eqs. (9.24), (9.25), etc., in succession or, equivalently, equations (9.31). For these equations to have solutions, their right sides must satisfy conditions (9.35). From the first condition (9.35), using (9.27) for the right side of Eq. (9.24), we have

$$\Phi_{\Gamma_\mu}^{(0)} \frac{\partial}{\partial \Gamma_\mu} \int f_k^{(0)} d\xi^k + \frac{\partial}{\partial x_r} \int \xi_r f_k^{(0)} d\xi^k = \Phi_{n^k}^{(0)} + \frac{\partial}{\partial x_r} \int \xi_r f_k^{(0)} d\xi^k = 0. \quad (9.43)$$

From the second condition of (9.35), by means of (9.28), we have

$$\begin{aligned} & \Phi_{\Gamma_\mu}^{(0)} \frac{\partial}{\partial \Gamma_\mu} \sum_{k=1}^N \int m^k \xi_l f_k^{(0)} d\xi^k + \frac{\partial}{\partial x_r} \sum_{k=1}^N m^k \int \xi_l \xi_r f_k^{(0)} d\xi^k \\ & + \sum_{k=1}^N \int X_r^k \xi_l \frac{\partial f_k^{(0)}}{\partial \xi_r} d\xi^k = \sum_{k=1}^N \Phi_{n^k}^{(0)} m^k u_i + \Phi_{u_i}^{(0)} \rho \\ & + \frac{\partial}{\partial x_r} \sum_{k=1}^N m^k \int \xi_l \xi_r f_k^{(0)} d\xi^k - \sum_{k=1}^N X_i^k n^k = 0. \end{aligned} \quad (9.44)$$

Finally, from the third condition of (9.35), using (9.29), we obtain

$$\Phi_{\Gamma_\mu}^{(0)} \frac{\partial}{\partial \Gamma_\mu} \sum_{k=1}^N \int \frac{m^k \xi^{k2}}{2} f_k^{(0)} d\xi^k + \frac{\partial}{\partial x_r} \sum_{k=1}^N \int \frac{m^k \xi^{k2}}{2} \xi_r f_k^{(0)} d\xi^k$$

$$\begin{aligned} & + \sum_{k=1}^N \int X_r^k \frac{\xi^{k2}}{2} \frac{\partial f_k^{(0)}}{\partial \xi_r} d\xi^k = \sum_{k=1}^N \Phi_{n^k}^{(0)} \frac{3}{2} kT + \frac{3}{2} kn \Phi_T^{(0)} \\ & + \sum_{k=1}^N \Phi_{n^k}^{(0)} m^k \frac{u^2}{2} + \Phi_{u_i}^{(0)} \rho u_i + \frac{\partial}{\partial x_r} \sum_{k=1}^N \int \frac{m^k \xi^{k2}}{2} \xi_r f_k^{(0)} d\xi^k \\ & - \sum_{k=1}^N X_r \int \xi_r f_k^{(0)} d\xi^k = 0. \end{aligned} \quad (9.45)$$

Hence, after some simple transformations,

$$\Phi_{n^k}^{(0)} = - \frac{\partial}{\partial x_r} \int \xi_r f_k^{(0)} d\xi^k, \quad (9.43a)$$

$$\Phi_{u_i}^{(0)} = - u_r \frac{\partial u_i}{\partial x_r} + \frac{1}{\rho} \sum_{k=1}^N X_i^k n^k - \frac{1}{\rho} \frac{\partial}{\partial x_r} \sum_{k=1}^N m^k \int c_i^k c_r^k f_k^{(0)} d\xi^k, \quad (9.44a)$$

$$\begin{aligned} \Phi_T^{(0)} = & - \frac{2}{3kn} \left[ \frac{3}{2} kn u_r \frac{\partial T}{\partial x_r} + \frac{\partial u_i}{\partial x_r} \sum_{k=1}^N m^k \int c_i^k c_r^k f_k^{(0)} d\xi^k \right. \\ & \left. + \frac{\partial}{\partial x_r} \sum_{k=1}^N \int \frac{m^k c^{k2}}{2} c_r^k f_k^{(0)} d\xi^k \right]. \end{aligned} \quad (9.45a)$$

Similarly, for the right side of Eq. (9.25), the conditions of solubility give

$$\Phi_{n^k}^{(1)} = - \frac{\partial}{\partial x_r} \int \xi_r f_k^{(1)} d\xi^k, \quad (9.46)$$

$$\Phi_{u_i}^{(1)} = - \frac{1}{\rho} \frac{\partial}{\partial x_r} \sum_{k=1}^N m^k \int c_i^k c_r^k f_k^{(1)} d\xi^k, \quad (9.47)$$

$$\begin{aligned} \Phi_T^{(1)} = & \frac{2}{3kn} \left[ \frac{3}{2} kT \frac{\partial}{\partial x_r} \sum_{k=1}^N \int c_r^k f_k^{(1)} d\xi^k \right. \\ & + \sum_{k=1}^N X_r^k \int c_r^k f_k^{(1)} d\xi^k - \frac{\partial u_i}{\partial x_r} \sum_{k=1}^N m^k \int c_i^k c_r^k f_k^{(1)} d\xi^k \\ & \left. - \frac{\partial}{\partial x_r} \sum_{k=1}^N \int \frac{m^k c^{k2}}{2} c_r^k f_k^{(1)} d\xi^k \right]. \end{aligned} \quad (9.48)$$

Then, according to the definition (9.22),

$$\frac{\partial n^k}{\partial t} = -\frac{\partial}{\partial x_r} \int \xi_r^k (f_k^{(0)} + \epsilon f_k^{(1)} + \dots) d\xi^k, \quad (9.49)$$

$$\begin{aligned} \frac{\partial u_i}{\partial t} &= -u_r \frac{\partial u_i}{\partial x_r} + \frac{1}{\rho} \sum_{k=1}^N X_i^k n^k \\ &- \frac{1}{\rho} \frac{\partial}{\partial x_r} \sum_{k=1}^N m^k \int c_l^k c_r^k (f_k^{(0)} + \epsilon f_k^{(1)} + \dots) d\xi^k, \end{aligned} \quad (9.50)$$

$$\begin{aligned} \frac{\partial T}{\partial t} &= \frac{2}{3kn} \left[ -\frac{3}{2} kn u_r \frac{\partial T}{\partial x_r} - \frac{\partial u_i}{\partial x_r} \sum_{k=1}^N m^k \int c_l^k c_r^k (f_k^{(0)} + \epsilon f_k^{(1)} + \dots) d\xi^k \right. \\ &+ \frac{3}{2} kT \frac{\partial}{\partial x_r} \sum_{k=1}^N \int c_r^k (\epsilon f_k^{(1)} + \dots) d\xi^k + \sum_{k=1}^N X_r^k \int c_r^k (\epsilon f_k^{(1)} + \dots) d\xi^k \\ &\left. - \frac{\partial}{\partial x_r} \sum_{k=1}^N \frac{m^k}{2} \int c_l^k c_r^k (\epsilon f_k^{(1)} + \dots) d\xi^k \right]. \end{aligned} \quad (9.51)$$

It is easy to see that Eqs. (9.49), (9.50), and (9.51) are none other than the conservation equations (9.18), (9.20), and (9.21).

Thus, the right sides of equations (9.31) for  $\nu = 1, 2, \dots$  satisfy the solubility conditions, if we substitute in them the functions  $\Phi_{\Gamma_\mu}^{(0)}, \Phi_{\Gamma_\mu}^{(1)}$  from expressions (9.43a)-(9.45a), (9.46)-(9.48).

Then the right side of Eq. (9.31) for  $\nu = 1$  takes the form

$$\begin{aligned} P_k^{(1)} &= f_k^{(0)} \left[ \frac{n}{n^k} d_r^k c_r^k + \frac{m^k}{kT} \left( c_l^k c_r^k - \frac{1}{3} \delta_{lr} c^{k2} \right) \frac{\partial u_i}{\partial x_r} \right. \\ &\left. + \left( \frac{m}{2kT} c^{k2} - \frac{5}{2} \right) c_r^k \frac{\partial \ln T}{\partial x_r} \right], \end{aligned} \quad (9.52)$$

where

$$d_r^k = \frac{\partial}{\partial x_r} \left( \frac{n^k}{n} \right) + \left( \frac{n^k}{n} - \frac{m^k n^k}{\rho} \right) \frac{\partial \ln p}{\partial x_r} - \frac{n^k}{p} X_r^k + \frac{m^k n^k}{p\rho} \sum_{l=1}^N n^l X_r^l. \quad (9.53)$$

It is easy to see that we obtain the same expression for  $P_k^{(1)}$ , if we write formally

$$\frac{df_k^{(0)}}{dt} = \frac{\partial f_k^{(0)}}{\partial \Gamma_\mu} \frac{\partial \Gamma_\mu}{\partial t} + \xi_i \frac{\partial f_k}{\partial \Gamma_\mu} \frac{\partial \Gamma_\mu}{\partial x_i} + \frac{X_i}{m} \frac{\partial f^{(0)}}{\partial \xi_i}$$

and then eliminate the partial derivatives  $\partial \Gamma_\mu / \partial t$  by means of the Euler equations (9.39)-(9.42). Then the solubility conditions are automatically fulfilled by that formal operation. This was just how we proceeded in fact in deriving the Navier-Stokes equations. The arguments given above serve as a basis for this formal procedure.

This operation is formal because, in fact, the hydrodynamic quantities  $\Gamma_\mu$  do not satisfy the Euler equations to approximations higher than order zero. It may be shown that elimination of the partial derivatives with respect to  $t$  by means of the Euler equations is valid only up to an accuracy  $\epsilon^2$ , i.e., when only two terms are retained in the expansion for  $f$ . In this last case it is sufficient to find  $f^{(1)}$  up to  $\epsilon$ . Since the solutions of the exact hydrodynamic equations to higher approximations differ from solutions of the Euler equation by quantities of order  $\epsilon$ , then, by eliminating derivatives with respect to  $t$  by means of the Euler equations, we incur an error of order  $\epsilon$ . However, the method described above for obtaining the right side indicates that this is not so, and that the expression for the right side is valid to any order. The function  $f^{(1)}$  obtained in that way is the same when any number of terms is retained in the expansion for  $f$ .

From the form of the right side of (9.52), arguments similar to those given in the previous section allow us to conclude that the solution of Eq. (9.31) for  $\nu = 1$  must have the form

$$\begin{aligned} \varphi_k &= -A^k c_r^k \frac{\partial \ln T}{\partial x_r} - B^k \left( c_s^k c_r^k - \frac{1}{3} \delta_{sr} c^{k2} \right) \frac{\partial u_s}{\partial x_r} \\ &+ c_r^k \sum_{\nu=1}^N C_\nu^k d_r^\nu + \alpha^k + \beta_r c_r^k + \gamma c^{k2}, \end{aligned} \quad (9.54)$$

where  $A^k, B^k, C_\nu^k$  are functions of  $T, c^k$ , and all the  $n^l$ . It is clear from the definition of the  $d_i^k$  that

$$\sum_{k=1}^N d_i^k = 0.$$

Therefore, one of the coefficients  $C_v^k$  may be assigned arbitrarily, for example, put equal to zero. The  $N + 4$  unknowns  $\alpha^k$ ,  $\beta_r$ , and  $\gamma$  are determined by the conditions

$$\int f_k d\xi^k = \int f_k^{(0)} \alpha \xi^k = n^k \quad \text{or} \quad \int f_k^{(0)} \varphi_k d\xi^k = 0, \quad (9.55)$$

$$\sum_{k=1}^N m^k \int \xi_i^k f_k d\xi^k = \sum_{k=1}^N m^k \int \xi_i^k f_k^{(0)} \varphi_k d\xi^k = \rho u_i$$

or

$$\sum_{k=1}^N \int m^k \xi_i^k f_k^{(0)} \varphi_k d\xi^k = 0, \quad (9.56)$$

$$\sum_{k=1}^N \int \frac{m^k c^{k2}}{2} f_k d\xi^k = \sum_{k=1}^N \int \frac{m^k c^{k2}}{2} f_k^{(0)} d\xi^k = \frac{3}{2} k T n$$

or

$$\sum_{k=1}^N \int \frac{m^k c^{k2}}{2} f_k^{(0)} \varphi_k d\xi^k = 0. \quad (9.57)$$

From condition (9.55) we have

$$\int (\alpha^k + \gamma c^{k2}) f_k^{(0)} d\xi^k = 0. \quad (9.58)$$

Since conditions (9.55) and (9.56) must be fulfilled simultaneously, we may replace  $\xi_i^k$  by  $c_i^k$  in the latter. Then condition (9.56) gives

$$\sum_{k=1}^N m^k \int \left( -A^k \frac{\partial \ln T}{\partial x_i} + \sum_{v=1}^N C_v^k d_i^v + \beta_i \right) c^{k2} f_k^{(0)} d\xi^k = 0. \quad (9.59)$$

Similarly, from condition (9.57), we have

$$\sum_{k=1}^N \frac{m^k}{2} \int (\alpha^k + \gamma c^{k2}) c^{k2} f_k^{(0)} d\xi^k = 0. \quad (9.60)$$

It follows from the  $(N + 1)$ -th homogeneous equation in (9.58) and (9.60) that  $\alpha^k$  and  $\gamma$  vanish. According to (9.59), we may represent the constant  $\beta_i$  in the form  $\beta \partial \ln T / \partial x_i$  and include it in the variables  $A^k$ .

Thus, we may represent the distribution function in the Navier-Stokes approximation in the form

$$f_k = f_k^{(0)} \left[ 1 - A^k c_i^k \frac{\partial \ln T}{\partial x_i} - B^k \left( c_s^k c_r^k - \frac{1}{3} \delta_{rs} c^{k2} \right) \frac{\partial u_s}{\partial x_r} + c_r^k \sum_{v=1}^N C_v^k d_i^v \right]. \quad (9.61)$$

Substituting this distribution function into the integral equations (9.31), and equating the coefficients  $\partial \ln T / \partial x_r$ ,  $\partial u_s / \partial x_r$ , and  $d_i^k$  to zero, we obtain integral equations for  $A^k$ ,  $B^k$ , and  $C_v^k$ . These equations are somewhat more awkward than the integral equations for  $A$  and  $B$  in the previous section for a single gas. But they are similar in principle, and the same methods of solution are applicable to them. We seek the functions  $A$ ,  $B$ ,  $C$  in the form of series of Sonine polynomials, thus reducing the problem to the solution of systems of algebraic equations (see the previous section). Detailed calculations and specific values of these quantities for various molecular interaction laws may be found, for example, in the monographs of Hirschfelder, Curtiss, and Bird, and Chapman and Cowling.

Assuming that the coefficients  $A^k$ ,  $B^k$ ,  $C_v^k$  are known, we may express the stress tensor, the heat flux vector, and the diffusion velocity in terms of the hydrodynamic quantities and their derivatives and thereby close the system of conservation equations (9.18)-(9.21).

Substituting the solution of (9.61) into the definition of the diffusion velocity (9.6), we have

$$v_r^k = \frac{1}{n^k} \int c_i^k f_k d\xi^k = \frac{n^2}{n^k \rho} \sum_{v=1}^N m^v D^{kv} d_i^v - \frac{1}{n^k m^k} D^k \frac{\partial \ln T}{\partial x_r}, \quad (9.62)$$

where

$$D^{kv} = \frac{\rho}{3nm^v} \sqrt{\frac{m^k}{2kT}} \int C_v^k (c^k) c^{k2} f_k^{(0)} d\xi^k, \quad (9.63)$$

$$D^k = \frac{m^k}{3} \sqrt{\frac{m^k}{2kT}} \int A^k(c^k) c^{k2} f_k^{(0)} d\xi^k \quad (9.64)$$

are the diffusion and thermal-diffusion coefficients, respectively.

Similarly, for the stress-tensor components, we have

$$\begin{aligned} P_{ij} &= \sum_{k=1}^N m^k \int c_i^k c_j^k f_k d\xi^k \\ &= p\delta_{ij} - \sum_{k=1}^N m^k \int B^k \left( c_s^k c_r^k - \frac{1}{3} \delta_{rs} c^{k2} \right) \frac{\partial u_s}{\partial x_r} f_k^{(0)} d\xi^k. \end{aligned}$$

Taking into account Eqs. (8.24) and (8.25), we obtain

$$P_{ij} = p\delta_{ij} - \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right), \quad (9.65)$$

where

$$\mu = \frac{1}{15} \sum_{k=1}^N m^k \int B^k(c^k) c^{k4} f_k^{(0)} d\xi^k \quad (9.66)$$

is the viscosity coefficient.

Finally, for the heat-flux vector components we have

$$\begin{aligned} q_i &= \sum_{k=1}^N \int \frac{m^k c^{k2}}{2} c_i^k f_k d\xi^k = - \frac{\partial \ln T}{\partial x_i} \sum_{k=1}^N \int A^k(c^k) \frac{m^k c^{k2}}{2} c_i^k f_k^{(0)} d\xi^k \\ &\quad + \sum_{v=1}^V d_i^v \sum_{k=1}^N \frac{m^k}{2} \int C_v^k(c^k) c^{k2} c_i^k f_k^{(0)} d\xi^k \end{aligned} \quad (9.67)$$

or

$$q_i = -\lambda_T \frac{\partial \ln T}{\partial x_i} + \sum_{v=1}^V \lambda^v d_i^v, \quad (9.68)$$

where

$$\lambda = \sum_{k=1}^N \frac{m^k}{6} \int A^k(c^k) c^{k4} f_k^{(0)} d\xi^k. \quad (9.69)$$

$$\lambda^v = \sum_{k=1}^N \frac{m^k}{6} \int C_v^k(c^k) c^{k4} f_k^{(0)} d\xi^k. \quad (9.70)$$

By substituting expressions (9.62), (9.65), and (9.67) into the conservation equations (9.18)-(9.21), we obtain a closed system of equations for the hydrodynamic quantities  $\rho$ ,  $u_r$ ,  $T$ , and  $n^k$ .

It is interesting to note that the phenomenon of thermal diffusion was first discovered by Enskog and Chapman from a similar derivation, and was later verified experimentally. We note also that the thermal-diffusion coefficient is equal to zero for Maxwellian molecules.

### §3.10. Derivation of the Hydrodynamic Equations, with Account of Internal Degrees of Freedom of the Molecules. The Relaxation Equations\*

It was assumed in the preceding sections that molecules possess only translational degrees of freedom. In fact, molecules undergoing collisions may exchange translational, rotational, vibrational, and other forms of energy. We showed in §2.6 that the study of a gas with quantized internal degrees of freedom may be reduced to that of a reacting mixture of gases, each of which consists of molecules in a given quantum state  $v$ .

If  $f_v(t, x, \xi^v)$  is the distribution function for molecules in the  $v$ -th quantum state with internal energy  $E^v$ , then the Boltzmann

\* Many papers have been devoted to a strict derivation from the Boltzmann equation, of the hydrodynamic equations for gases with internal degrees of freedom. Among them are: C. S. Wang Chang and G. F. Uhlenbeck, University of Michigan Engineering Research, Rept. No. CM-681 (1951); see also "Studies in Statistical Mechanics," Vol. 2 (1964); N. Taxman, Phys. Rev., 110: 1235 (1958); E. V. Samuilov, "Physical Gas Dynamics," Izd. Akad. Nauk SSSR, Moscow, 1959; L. Monchick and E. A. Mason, J. Chem. Phys., 35: 1676 (1961) and 36: 1622 (1962); V. N. Zhigulev, Inzh. Zh. No. 1 (1963); S. V. Vallander and E. A. Nagnibed, Vestn. Leningrad Univ., No. 12 (1963); S. V. Vallander, I. A. Egorova, and M. A. Rydalevskaya, in: "Aerodynamics of Rarefied Gases," Vol. 2, Leningrad Univ. Press (1965); M. N. Kogan, Zh. Prikl. Mekhan. Tekhn. Fiz., No. 1 (1965).

The monographs of Chapman and Cowling and of Hirschfelder, Curtiss, and Bird give the results of earlier work, in which internal degrees of freedom are accounted for by replacing the molecules by roughened spheres, ellipsoids, etc.

Below we follow M. N. Kogan's paper quoted above.

equations take the form (see §2.6; for simplicity no external force field is considered)

$$\begin{aligned} \frac{df_v}{dt} &= \frac{\partial f_v}{\partial t} + \xi^v \frac{\partial f_v}{\partial x_r} \\ &= \sum_{\mu, k, l} \int (f_k f_l - f_\mu f_v) g_{\mu\nu} \sigma_{\mu\nu}^{kl} (\xi^v, \xi^\mu, \xi^k, \xi^l) d\xi^k d\xi^l d\xi^\mu. \end{aligned} \quad (10.1)$$

Since all the components of the mixture thus formed consist of molecules which are chemically the same, differing only in internal energy, the masses of all the molecules are the same. Therefore, the definitions (9.2)-(9.11) of the previous section take the form

$$n^v = \int f_v d\xi^v \quad (10.2)$$

for the number of particles in the  $v$ -th state, per unit volume,

$$\rho = m \sum_v n^v = mn \quad (10.3)$$

for the gas density,

$$u^v = \frac{1}{n^v} \int \xi^v f_v d\xi^v \quad (10.4)$$

for the velocity of the  $v$ -th component of the gas,

$$u = \frac{1}{n} \sum_v n^v u^v \quad (10.5)$$

for the mean gas velocity,

$$v^v = u^v - u = \frac{1}{n^v} \int f_v (\xi^v - u) d\xi^v = \frac{1}{n^v} \int c^v f_v d\xi^v \quad (10.6)$$

for the diffusion velocity of the component in the  $v$ -th state,

$$\frac{3}{2} kT = \frac{m}{2n} \sum_v \int c^{v2} f_v d\xi^v \quad (10.7)$$

for the translational temperature of the gas, this being a measure of the energy of the translational degrees of freedom,

for the stress tensor, and

$$q_i = \sum_v \int c_i^v c_j^v f_v d\xi^v \quad (10.9)$$

for the heat-flux vector.

In addition to these quantities, it is also necessary to introduce the mean internal energy of the molecules:

$$\varepsilon = \frac{1}{n} \sum_v n^v E^v. \quad (10.10)$$

The conservation equations in this case differ from Eqs. (9.19), (9.20) only in that we must write the total mean energy as  $\frac{3}{2}kT + \varepsilon$  instead of the mean energy of translational motion of the molecules  $\frac{3}{2}kT$  in the energy equation; we have:

$$\frac{\partial n}{\partial t} + \frac{\partial n u_r}{\partial x_r} = 0, \quad (10.11)$$

$$\left( \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial x_r} \right) u_l = - \frac{1}{nm} \frac{\partial P_{lr}}{\partial x_r}, \quad (10.12)$$

$$n \left( \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial x_r} \right) \left( \frac{3}{2} kT + \varepsilon \right) = - \frac{\partial q_r}{\partial x_r} - P_{ir} \frac{\partial u_i}{\partial x_r}. \quad (10.13)$$

The external forces are not taken into account. We also write the sum

$$\sum_v n^v v_r^v = \sum_v \int (\xi^v - u) f_v d\xi^v = 0.$$

Equations (9.18) do not hold here, since molecules pass from one state to another without conservation of the number of molecules in the  $v$ -th state during collisions with molecules in other states. The total number of molecules of all kinds does not change.

In order to close the system of equations (10.11)-(10.13), we need to express  $P_{ij}$ ,  $q_i$ , and  $\varepsilon$  in terms of the hydrodynamic quantities  $n$ ,  $u_r$ , and  $T$ . Furthermore, in general, the internal energy

is not a function of a single temperature, but, according to (10.10), depends on the distribution of particles among all available states. Therefore, we also require equations to determine the functions  $n^\nu$ . Each  $\nu$ -state of a molecule may be characterized by one or several quantum numbers  $\nu_1, \nu_2, \nu_3$ , etc., which define the rotational, vibrational, and other levels of energy.

The transition probabilities for the various energy levels during molecular collisions may be of different orders. To define the transition rates of the various kinds of energy we define the collision cross sections

$$\sigma_{v\mu}^{kl} = \frac{1}{\epsilon_l} \sigma_{v\mu}^{*kl},$$

where  $\sigma^*$  are quantities of order unity, and  $\epsilon_i$  is a parameter which characterizes the order of the transition probability; it is proportional to the ratio of the relaxation time for a given type of transition  $\theta_i$  to the characteristic flow time  $\Theta$ . Let  $\theta_1 \leq \theta_2 \leq \theta_3 \leq \dots$ . Usually the most probable transition is the transfer of translational energy from particle to particle without change of internal energy. We therefore assume that  $\theta_1$  refers to that form of transition and is of the same order of magnitude as the time between collisions. At moderately high temperatures, the probability of exchange of translational and rotational energy is next.

As the temperature increases the probability of exciting molecular rotations decreases, and at temperatures of the order of  $25-30 \cdot 10^3$  °K, the probability of rotational transition becomes comparable to the probability of vibrational excitation. Only a few collisions are required to excite rotations. Several thousand, and even several tens of thousands of collisions are required to excite vibrations at low temperatures. Along with the exchange of translational and vibrational energy between molecules, the so-called resonance transitions are sometimes important; here, molecules exchange vibrational quanta, without change in the total translational energy of the colliding molecules. Cases of "linkup" of several processes are possible, when different types of transitions occur with comparable probability. It is assumed here that the transitions may be divided into groups so that we have the inequalities

$$\theta_1 \ll \theta_2 \ll \dots \ll \theta_{n-1} \ll \theta_n = \Theta \ll \theta_{n+1} \ll \dots,$$

where  $\Theta$  is a characteristic time for the problem.

We divide the collision integrals in the equations (10.1) into groups corresponding to different characteristic relaxation times. In dimensionless variables we may write

$$\frac{df_v}{dt} = \frac{1}{\epsilon_1} \Sigma_{(1)}^\nu + \frac{1}{\epsilon_2} \Sigma_{(2)}^\nu + \dots, \quad (\epsilon_i = \frac{\theta_i}{\Theta}), \quad (10.14)$$

where collision integrals with coefficients  $\epsilon_i$  of the same order\* have been combined under the summation signs  $\Sigma_{(1)}^\nu, \Sigma_{(2)}^\nu$ , etc.

We integrate equations (10.14) after introducing the new variable  $\tau = t/\epsilon_1$  (see the end of §3.8):

$$f_v(t + \tau \epsilon_1, x + \xi^\nu \tau \epsilon_1, \xi^\nu) - f_v(t, x, \xi^\nu) \\ = \int_0^\tau \Sigma_{(1)}^\nu d\tau + \frac{\epsilon_1}{\epsilon_2} \int_0^\tau \Sigma_{(2)}^\nu d\tau + \dots, \quad (10.15)$$

where  $\tau = O(1)$ .

1. Derivation of the Euler Equations. We assume that the variation in the distribution function during the time between collisions or in a length of the order of the mean free path may be neglected in the first approximation (see §3.8).

**1.1.** For example, let  $\epsilon_1 \ll \epsilon_2 \sim 1 \ll \epsilon_3$ .† If  $\epsilon_i \sim 1$ , then the characteristic time of the  $i$ -th type of transition is of the order of the characteristic flow time. Then the third type of transition may be neglected, i.e., we may consider that the  $\nu_3$ -states are

\*If  $\theta_{k_1} < \theta_{k_2} < \theta_{k_3} < \dots$ , then  $k_1$ -transitions may occur in the presence of  $k_2$ -transitions;  $k_2$ -transitions may occur in the presence of  $k_3$ -transitions, etc. But, in the presence of  $k_1$ -transitions the energy levels corresponding to  $k_2, k_3$ , etc., do not change. For example, let  $\theta_2$  be the characteristic time for excitation of rotations, and  $\theta_3 \gg \theta_2$  be the characteristic time for excitation of vibrations. It is then evident that during collisions leading to change in the vibrational levels of the molecules ( $\theta_3$ -transitions), the rotational levels may also change ( $\theta_2$ -transitions). But only collisions with change of rotational levels are combined in the sum with coefficient  $\epsilon_2^{-1}$ .

†This case was examined in the paper cited above, by S. V. Vallander and E. A. Nagnibed.

frozen. By summing equations (10.1) over all the  $\nu_3$ -states belonging to a single state  $\nu_2$ , we obtain a system of equations of the same form for distribution functions of translational velocities  $f_{\nu}$  of the molecules in the different  $\nu_2$ -states.\*

From Eq. (10.15), neglecting quantities of order  $\epsilon_1$ , we find in this case that in the first approximation  $\Sigma_1^{\nu} = 0$  and, therefore,

$$f_{\nu}^{(0)} = n^{\nu} \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left( -\frac{m}{2kT} c^{\nu 2} \right), \quad (10.16)$$

i.e., the translational degrees of freedom of each of the  $\nu$ -gases are in equilibrium at the same temperature  $T$ . In that case there is no diffusion of the  $\nu$ -components, i.e.,  $u_r^{\nu} = u_r$ . Substituting the distribution function in the form (10.16) into the definitions (10.8) and (10.9) for the stress tensor and the heat-flux vector we obtain

$$P_{ij} = \delta_{ij} p \text{ and } q_i = 0.$$

Then the conservation equations (10.11)-(10.13) take the form

$$\frac{\partial n}{\partial t} + \frac{\partial n u_r}{\partial x_r} = 0, \quad (10.17)$$

$$\left( \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial x_r} \right) u_t = - \frac{1}{mn} \frac{\partial p}{\partial x_t}, \quad p = nkT, \quad (10.18)$$

$$\left( \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial x_r} \right) \left( \frac{3}{2} kT + \varepsilon \right) = - kT \frac{\partial u_r}{\partial x_r}. \quad (10.19)$$

This is the system of Euler equations for a gas, allowing for internal degrees of freedom of the molecules. But this system of equations is not closed, since, in addition to the five hydrodynamic quantities  $n$ ,  $u_r$ , and  $T$  (the pressure  $p$  is related to  $n$  and  $T$  by the

\*It is convenient to introduce the notation

$$f_{\nu} \equiv f_{\nu_2 \nu_3}, \quad \sigma_{\mu\nu}^{kl} = \sigma_{\mu_2 \mu_3 \nu_2 \nu_3}^{k_2 k_3 l_2 l_3}.$$

It is assumed that the  $\nu_2$ -transitions do not depend on the  $\nu_3$ -states, i.e., that

$$\sigma_{\mu\nu}^{kl} = \sigma_{\mu_2 \mu_3 \nu_2 \nu_3}^{k_2 k_3 l_2 l_3} = \sigma_{\mu_2 \nu_2}^{k_2 l_2}$$

independently of  $\mu_3$  and  $\nu_3$ . Since we are interested only in  $\nu_2$ -states in what follows, we shall write  $\nu$  in place of  $\nu_2$ .

equation of state  $p = knT$ ) the internal energy of the molecules also appears; to determine it we need to know the distribution of particles among the  $\nu$ -levels of energy, i.e., all the  $n^{\nu}$ .\* To find these we integrate equations (10.1) with respect to  $\xi^{\nu}$ . The integral of  $\Sigma_{(1)}^{\nu}$  is evidently equal to zero, since this sum contains only collision integrals which take into account transfer of translational energy and the state of the molecules does not change in that process. Replacing  $f_{\nu}$  by the equilibrium distribution function (10.16), we obtain the following semimacroscopic equations for the number of particles in a given  $\nu$ -state (the subscript 2 in  $\nu$ ,  $\mu$ ,  $k$ , and  $l$  has been omitted):

$$\frac{\partial n^{\nu}}{\partial t} + \frac{\partial u_r n^{\nu}}{\partial x_r} = \sum_{\mu, k, l} (n^k n^l a_{\mu k l}^{\nu} - n^{\nu} n^{\mu} a_{\mu k l}^{\nu}), \quad (10.20)$$

where

$$a_{\mu k l}^{\nu} = \frac{1}{n^{\nu} n^{\mu}} \int f_{\nu}^{(0)} f_{\mu}^{(0)} \sigma_{\nu \mu}^{k l} (\xi^{\nu}, \xi^{\mu}, \xi^k, \xi^l) g_{\nu \mu} d\xi^k d\xi^l d\xi^{\mu} d\xi^{\nu}$$

is the probability of transition of the  $\nu$ - and  $\mu$ -molecules into the states  $k$  and  $l$  averaged over all velocities.

The coefficients  $a$  depend only on the translational temperature  $T$ . Therefore, the equilibrium of the gas with respect to translational velocities allows us to find  $n^{\nu}$  with less detailed knowledge of the transition probabilities. Since  $\epsilon_2 \sim 1$ , it has been assumed that the occupation numbers of the levels  $n^{\nu_2}$  vary over their full range in the flow. Therefore, it is necessary, in general, to solve the system of equations (10.17)-(10.19) along with equations (10.20), i.e., in the general case one cannot write a single additional equation for the internal energy  $\varepsilon$ .

To simplify the system of equations (10.20), or to reduce it to a single equation for the internal energy, we need to make additional assumptions about the transition probabilities  $a_{\mu k l}^{\nu}$ .

\*The quantity  $n^{\nu_2}$  is the number of molecules in the  $\nu_2$ -state for all possible states  $\nu_3$ . Since the  $\nu_3$ -states are frozen, the number of molecules located in some  $\nu_3$ -state is determined by the initial and boundary conditions.

For example, let us examine a gas where the transition probabilities possess the following properties.\* The most probable transitions are those for which the quantum number changes by one, i.e., transitions of the type  $a_{v\mu}^{v+1}, \mu = a_v^{v+1}$ . According to the principle of detailed balancing, the probability of deactivation of a given level is greater than the probability of excitation of that level, by a factor of  $e^{\Delta E/kT}$ , i.e.,

$$a_v^{v-1} = a_{v-1}^v e^{\Delta E^v/kT},$$

where

$$\Delta E^v = E^v - E^{v-1}.$$

Let us assume, in addition, that

$$a_v^{v-1} = v a_1^0$$

and that the energy levels satisfy the distribution function of a harmonic oscillator, i.e.,

$$E^v = v \hbar \omega,$$

where  $\omega$  is the frequency.

Neglecting transitions where the quantum number changes by more than one, we obtain

$$\frac{\partial n^v}{\partial t} + \frac{\partial n^v u_r}{\partial x_r} = \sum_{\mu} [vn^{\mu} a_1^0 (n^{v-1} e^{-\hbar \omega / kT} - n^v) + (v+1) n^{\mu} a_1^0 (n^{v+1} - n^v e^{-\hbar \omega / kT})].$$

Summing over  $\mu$  on the right side, and taking into account that

$\sum_{\mu} n^{\mu} = n$ , we find

$$\begin{aligned} \frac{\partial n^v}{\partial t} + \frac{\partial n^v u_r}{\partial x_r} &= n a_1^0 [vn^{v-1} e^{-\hbar \omega / kT} - vn^v \\ &+ (v+1) n^{v+1} - (v+1) n^v e^{-\hbar \omega / kT}]. \end{aligned} \quad (10.21)$$

\*Vibrational energy transitions possess these properties under specific conditions. See, for example, R. N. Schwartz, Z. I. Slawsky, and K. F. Herzfeld, Chem. Phys., Vol. 20, No. 10 (1952).

Multiplying each of the equations (10.21) by  $E^v = v \hbar \omega$ , and summing over  $v$ , we obtain

$$\left( \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial x_r} \right) \varepsilon = n a_1^0 (1 - e^{-\hbar \omega / kT}) [\varepsilon^{(0)}(T) - \varepsilon], \quad (10.22)$$

where

$$\varepsilon^{(0)}(T) = \hbar \omega (e^{\hbar \omega / kT} - 1)^{-1}$$

is the equilibrium energy of the system of oscillators at temperature  $T$ . This is the so-called relaxation equation, which is very often written in the form

$$\left( \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial x_r} \right) \varepsilon = \frac{\varepsilon^{(0)}(T) - \varepsilon}{\tau(n, T)}, \quad (10.22a)$$

where  $\tau(n, T)$  is the relaxation time.

Thus, the Euler equations (10.17)-(10.19) are closed by a relaxation equation of type (10.22a) only when very special assumptions are made about the transition probabilities.

These relaxation equations for the internal energy are very often used in practice. However, it is clear from what has been said above, that they are approximate for arbitrary transition probabilities.

**1.2.** Let us now examine a gas with three types of transitions  $\epsilon_1 \leq \epsilon_2 \ll \epsilon_3 \sim 1$ . In that case, it follows from equations (10.15) in the first approximation that  $\Sigma_{(1)}^v + \Sigma_{(2)}^v = 0$ , whence

$$f_v^{(0)} = n^{v_3} \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{m}{2kT}} c^{v_2} \frac{e^{-E^{v_2}/kT}}{\sum_{v_2} e^{-E^{v_2}/kT}}, \quad (10.23)$$

where

$$n^{v_3} = \sum \int f_v^{(0)} d\xi^v.$$

The macroscopic functions which appear here must satisfy the Euler equations (10.17)-(10.19), in which the internal energy is equal to

$$\varepsilon = \varepsilon_2^{(0)} + \frac{1}{n} \sum_{\nu_3} n^{\nu_3} E^{\nu_3}, \quad \varepsilon_2^{(0)} = \frac{\sum_{\nu_2} E^{\nu_2} e^{-E^{\nu_2}/kT}}{\sum_{\nu_2} e^{-E^{\nu_2}/kT}}.$$

The translational degrees of freedom and the  $\nu_2$ -levels are in equilibrium; the  $\nu_3$ -states are relaxing. The  $\nu_2$ -states may correspond, for example, to rotational energy levels, and the  $\nu_3$ -states to vibrational levels. It is clear that all the arguments remain valid if there are several types of transitions with  $\epsilon_i \ll \epsilon_3$ .

To determine the number of particles  $n^{\nu_3}$  for a  $\nu_3$ -gas, i.e., a gas whose molecules are located in the  $\nu_3$ -state and in arbitrary states  $\nu_2$ , we integrate equations (10.1) over  $\xi^\nu$  and sum the equations relating to the  $\nu_3$ -state. Since the sums  $\Sigma_{(1)}^\nu$  and  $\Sigma_{(2)}^\nu$  do not include transitions of  $\nu_3$ -energy into other forms of energy, they vanish identically in the summation. Then,

$$\begin{aligned} \frac{\partial n^{\nu_3}}{\partial t} + \frac{\partial u_r n^{\nu_3}}{\partial x_r} &= \left( \sum_{\nu_2} e^{-E^{\nu_2}/kT} \right)^{-2} \sum_{\nu_2, \mu, k, l}^{\nu_3} (a_{\nu_2 \mu}^{k \mu} n^k n^l e^{-(E^k + E^l)/kT} \\ &\quad - a_{\nu_2 \mu}^{k l} n^{\nu_3} n^{\mu_3} e^{-(E^{\nu_2} + E^{\mu_3})/kT}), \end{aligned} \quad (10.24)$$

where we have used the fact that

$$n^\nu = n^{\nu_2} e^{-E^{\nu_2}/kT} \left( \sum_{\nu_2} e^{-E^{\nu_2}/kT} \right)^{-1},$$

since each of the  $\nu_3$ -gases has an equilibrium distribution with respect to  $\nu_2$ -energy levels.

Summing on the right side with respect to  $\nu_2$ , we obtain equations of type (10.20), in which the subscripts  $\nu$ ,  $\mu$ ,  $k$ , and  $l$  are replaced by  $\nu_3$ ,  $\mu_3$ ,  $k_3$ , and  $l_3$ , respectively, and where

$$a_{\nu_3 \mu_3}^{k_3 l_3} = \left( \sum_{\nu_2} e^{-E^{\nu_2}/kT} \right)^{-2} \sum_{\nu_2 k_2 l_2} a_{\nu_2 \nu_3 \mu_2 \mu_3}^{k_2 k_3 l_2 l_3} \exp \left( -\frac{E^{\nu_2} + E^{\mu_2}}{kT} \right). \quad (10.24a)$$

It is possible to simplify this system of equations further or to reduce it to a single equation for the internal energy by making

additional assumptions regarding the transition probabilities  $a_{\nu_3 \mu_3}^{k_3 l_3}$ .

If the  $\epsilon_2$ -transitions include  $\nu_3$ -energy resonance transitions and  $\epsilon_2 \ll \epsilon_3$ , then we can obtain a single additional equation for the internal energy with any assumption on the probability of transition from the  $\nu_3$ -energy to translation. In fact, in this case, the  $\nu_3$ -levels have an equilibrium distribution, and

$$f_v^{(0)} = n \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{mv^2}{2kT}} \frac{e^{-E^{\nu_3}/kT}}{\sum_{\nu_2} e^{-E^{\nu_2}/kT}} \frac{e^{-E^{\nu_3}/kT_3}}{\sum_{\nu_3} e^{-E^{\nu_3}/kT_3}}, \quad (10.25)$$

where  $T_3$  is the temperature characterizing the mean  $\nu_3$ -energy, which is in general not equal to  $T$ . The temperature  $T_3$  is the single additional unknown. The mean internal energy  $\varepsilon$  may be represented in the form

$$\varepsilon = \varepsilon_2 + \varepsilon_3 = \frac{1}{n} \sum_{\nu_2} n^{\nu_2} E^{\nu_2} + \frac{1}{n} \sum_{\nu_3} n^{\nu_3} E^{\nu_3}. \quad (10.26)$$

Multiplying equations (10.24) by  $E^{\nu_3}$ , respectively, and adding, we obtain, on the left,

$$\frac{\partial}{\partial t} \sum_{\nu_3} n^{\nu_3} E^{\nu_3} + \frac{\partial}{\partial x_r} \left( u_r \sum_{\nu_3} E^{\nu_3} n^{\nu_3} \right) = \frac{\partial n \varepsilon_3}{\partial t} + \frac{\partial n u_r \varepsilon_3}{\partial x_r} = n \left( \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial x_r} \right) \varepsilon_3.$$

Substituting in the right side of (10.24)

$$n^{\nu_3} = n \frac{e^{-E^{\nu_3}/kT_3}}{\sum_{\nu_3} e^{-E^{\nu_3}/kT}},$$

we find that it depends only on  $T$ ,  $n$ , and  $T_3$ :

$$n \left( \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial x_r} \right) \varepsilon_3 = \sum_{\nu_3} E^{\nu_3} \sum_{\mu, k, l} (n^k n^l a_{\nu_3 \mu_3}^{k \mu} - n^{\nu_3} n^{\mu_3} a_{\nu_3 \mu_3}^{k_3 l_3}). \quad (10.27)$$

Thus, an equation is obtained for the single additional unknown  $T_3$ .

If the probabilities of nonresonant transitions are such that a separate equation is obtained for the internal energy, the pres-

ence of resonance transitions clearly does not change that equation, since the internal energy does not change when resonance transitions occur. In particular, if  $\epsilon_2 \sim 1$  for resonance transitions, while all the other transition probabilities of the internal energy are much smaller, then

$$\left( \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial x_r} \right) \varepsilon = 0. \quad (10.28)$$

In approximate calculations it is customary to use a relaxation equation of the form (10.22a) for the internal energy, with empirical or semiempirical dependences of the relaxation time on  $n$  and  $T$ . Therefore, resonance transitions do not appear in such calculations.

2. Derivation of the Navier - Stokes Equations. We now turn to the construction of the Navier - Stokes equations for a gas with internal degrees of freedom. In that case, we can no longer neglect the variation of the distribution function over a length of the order of the mean free path, or a time of the order of the time between collisions. But we consider the variations to be small, so that the distribution function can be represented in the form (see the end of § 8)

$$f(t + \tau \epsilon_1, x + \xi \tau \epsilon_1, \xi) = f(t, x, \xi) + \frac{df}{dt} \Big|_t \epsilon_1 \tau. \quad (10.29)$$

**2.1.** Let us first examine the case when all the  $\epsilon_i \ll 1$ .\* Then, according to (10.15), we have in the first approximation

$$\sum_i \frac{1}{\epsilon_i} \Sigma_{(i)}^v = 0. \quad (10.30)$$

The solution of that equation is evidently the distribution function which corresponds to equilibrium between translational and internal energies:

$$f_v^{(0)} = n \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{m}{2kT} c^{v2}} \frac{e^{-E^v/kT}}{\sum_v e^{-E^v/kT}}. \quad (10.31)$$

\*This is the case examined in the great majority of papers devoted to a derivation of the Navier - Stokes equations allowing for internal degrees of freedom of the molecules.

It is easy to see that in that approximation the mean internal energy is equal to

$$\varepsilon^{(0)}(T) = \frac{\sum_v E^v e^{-E^v/kT}}{\sum_v e^{-E^v/kT}} \quad (10.32)$$

and the hydrodynamic quantities satisfy the Euler equations (10.17)-(10.19), in which we must put  $\varepsilon = \varepsilon^0(T)$  in accordance with (10.32).

It is clear that the presence of resonance transitions does not change the distribution function in that case.

When we allow for the variation in the distribution function during a time or in a length of order  $\epsilon_1$ , the distribution function must differ little from the equilibrium function (10.31):

$$f_v = f_v^{(0)}(1 + \varphi_v). \quad (10.33)$$

Substituting (10.33) into (10.15), and dropping terms of order  $\epsilon_1^2$ , in analogy with (8.60) we obtain

$$\frac{df_v^{(0)}}{dt} = \frac{1}{\epsilon_1} \sum_{(1)}^v J_{v\mu kl}(f, \varphi) + \frac{1}{\epsilon_2} \sum_{(2)}^v J_{v\mu kl}(f, \varphi) + \dots, \quad (10.34)$$

where

$$J_{v\mu kl}(f, \varphi) = \int f_v^{(0)} f_\mu^{(0)} (\varphi_k + \varphi_l - \varphi_v - \varphi_\mu) g_{v\mu} \sigma_{v\mu}^{kl} d\xi^k d\xi^l d\xi^\mu.$$

Substituting the equilibrium function (10.31) into the left side of Eq. (10.34), and eliminating the partial derivatives with respect to time with the aid of the Euler equations, we have

$$\begin{aligned} \frac{df_v^{(0)}}{dt} = f_v^{(0)} & \left\{ \left( \frac{m}{2kT} c^{v2} - \frac{5}{2} + \frac{E^v - \varepsilon^{(0)}(T)}{kT} \right) c_r^v \frac{\partial \ln T}{\partial x_r} \right. \\ & + \frac{m}{kT} \left( c_r^v c_l^v - \frac{1}{3} \delta_{rl} c^{v2} \right) \frac{\partial u_l}{\partial x_r} + \left[ \frac{1}{3} \frac{m}{kT} c^{v2} - 1 \right. \\ & \left. \left. - \frac{k}{\frac{3}{2}k + \frac{d\varepsilon^{(0)}(T)}{dT}} \left( \frac{m}{2kT} c^{v2} - \frac{3}{2} + \frac{E^v - \varepsilon^{(0)}(T)}{kT} \right) \right] \frac{\partial u_r}{\partial x_r} \right\}. \end{aligned} \quad (10.35)$$

It is easy to verify that the homogeneous integral equations (10.34) have only five eigenfunctions  $\psi_1 = 1$ ,  $\xi_r^v$ ,  $\frac{1}{2}m\xi_r^{v2} + E^v$ . The form of the left side of equations (10.34) allows us to look for a solution in the form

$$\begin{aligned}\varphi_v &= -A^v c_r^v \frac{\partial \ln T}{\partial x_r} - B^v \left( c_i^v c_r^v - \frac{1}{3} \delta_{ir} c^{v2} \right) \frac{\partial u_i}{\partial x_r} \\ &\quad - D^v \frac{\partial u_r}{\partial x_r} + \alpha + \beta_r c_r^v + \gamma \left( \frac{mc^{v2}}{2} + E^v \right),\end{aligned}\quad (10.36)$$

where  $A^v$ ,  $B^v$ , and  $D^v$  are functions of  $c^v$ ,  $T$ , and  $E^v$ .

In order to determine the arbitrary functions  $\alpha$ ,  $\beta_r$ , and  $\gamma$ , we require that

$$\sum_v \int f_v d\xi^v = \sum_v \int f_v^{(0)} d\xi^v = n \quad \text{or} \quad \sum_v \int f_v^{(0)} \varphi_v d\xi^v = 0,\quad (10.37)$$

$$\sum_v \int \xi_r^v f_v d\xi^v = \sum_v \int \xi_r^v f_v^{(0)} d\xi^v = n u_r \quad \text{or} \quad \sum_v \int \xi_r^v f_v^{(0)} \varphi_v d\xi^v = 0.\quad (10.38)$$

$$\begin{aligned}\sum_v \int \left( \frac{mc^{v2}}{2} + E^v \right) f_v d\xi^v &= \sum_v \int \left( \frac{mc^{v2}}{2} + E^v \right) f_v^{(0)} d\xi^v \\ &= n \left[ \frac{3}{2} kT + \varepsilon^{(0)}(T) \right]\end{aligned}\quad (10.39)$$

or

$$\sum_v \int \left( \frac{mc^{v2}}{2} + E^v \right) f_v^{(0)} \varphi_v d\xi^v = 0.$$

It is important to note that under those conditions the temperature  $T$  appearing here is not a measure of the mean energy of translational motion of the molecules defined by formula (10.7). It follows from the conditions (10.37)-(10.39) that  $\alpha$ ,  $\beta_r$ , and  $\gamma$  may be represented in the form

$$\alpha = \alpha^* \frac{\partial u_r}{\partial x_r}, \quad \beta_r = \beta^* \frac{\partial \ln T}{\partial x_r}, \quad \gamma = \gamma^* \frac{\partial u_r}{\partial x_r}.$$

This permits us to include  $\alpha$ ,  $\beta_r$ , and  $\gamma$  in  $A^v$  and  $D^v$ , by putting

$$A^{v*}(c^v, T, E^v) = -\beta^* + A^v(c^v, T, E^v).$$

$$D^{v*}(c^v, T, E^v) = D^v - \alpha^* - \gamma^* \left( \frac{mc^{v2}}{2} + E^v \right).$$

The functions  $A^v*$  and  $D^v*$  are subject to conditions (10.37)-(10.39). Since only those new functions appear in what follows, we shall write them without asterisks for simplicity.

By substituting the solutions in the form (10.36) into the integral equations (10.34) and equating coefficients of  $\partial \ln T / \partial x_r$ ,  $\partial u_i / \partial x_r$ ,  $\partial u_r / \partial x_r$ , we obtain integral equations for the determination of  $A^v$ ,  $B^v$ , and  $D^v$ ; to solve these we must know the specific form of the molecular interaction laws and of the transition probability  $\sigma_{vu}^{kl}$ . But some qualitative conclusions may be drawn without solving the equations.

In order to obtain the Navier-Stokes equations, we must express the stress tensor  $P_{ij}$ , the heat-flux vector  $q_i$ , and the internal energy  $\varepsilon$ , which appear in the conservation equations (10.11)-(10.13) in terms of  $n$ ,  $u_i$ , and  $T$  and their derivatives, by means of our solution (10.36).

It follows from condition (10.39) that the factor  $\frac{3}{2}kT + \varepsilon$  in the energy equation (10.13) may be replaced by  $\frac{3}{2}kT + \varepsilon^{(0)}(T)$ , where  $\varepsilon^{(0)}(T)$  is defined by (10.32).

By substituting the solution (10.36) into the definition of the stress tensor and of the heat-flux vector, we obtain

$$P_{ij} = p \delta_{ij} - \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) - \zeta \frac{\partial u_k}{\partial x_k} \delta_{ij}, \quad (10.40)$$

$$\left. \begin{aligned} q_i^{(1)} &= \sum_v m \int c_i^v \frac{c^{v2}}{2} f_v d\xi^v = -\lambda_1 \frac{\partial T}{\partial x_i}, \\ q_i^{(2)} &= \sum_v \int c_i^v E^v f_v d\xi^v = -\lambda_2 \frac{\partial T}{\partial x_i}, \end{aligned} \right\} \quad (10.41)$$

where

$$\mu = \frac{m}{15} \sum_v \int B^v c^{v4} f_v^{(0)} d\xi^v; \quad \zeta = \frac{m}{3} \sum_v \int D^v c^{v2} f_v^{(0)} d\xi^v;$$

$$\lambda_1 = \frac{m}{6T} \sum_v \int A^v c^{v4} f_v^{(0)} d\xi^v; \quad \lambda_2 = \frac{1}{3T} \sum_v E^v \int A^v c^{v2} f_v^{(0)} d\xi^v.$$

We note that the quantity  $p$  in (10.40) is equal to  $nkT$ , but is not equal to  $\frac{1}{3}(P_{11} + P_{22} + P_{33})$ . The appearance of a second viscosity  $\xi$ , in addition to the shear viscosity  $\mu$ , is due to the presence of the internal degrees of freedom. For a gas composed of monatomic molecules,  $\xi = 0$ . The coefficients  $\mu$  and  $\lambda$  depend on the transition probabilities. In general, therefore,  $\mu$  and  $\lambda$  for a polyatomic gas differ from the corresponding values for a monatomic gas or for a gas with unexcited molecules.

If  $\epsilon_1/\epsilon_i = \alpha_i \ll 1$  when  $i \geq 2$  and  $\epsilon \rightarrow 0$ , then the viscosity and thermal conductivity may be determined approximately without knowledge of the levels  $E^\nu$  and of the effective cross sections  $\sigma$ . In fact, we write the Boltzmann equations in the form

$$\epsilon_1 \frac{df_\nu}{dt} = \sum_{(1)}^{\nu} J(f, f) + \alpha \sum_{(2)}^{\nu} J(f, f).$$

We expand the functions  $f_\nu$  in the series

$$f_\nu = f_\nu^{(0)} + (f^{10} + \alpha f^{11} + \dots) + \dots$$

In determining the transport coefficients we calculate the term  $\epsilon_1 f^{10}$ , omitting terms of order  $\alpha f^{11}$  and above. For the determination of the function  $f_\nu^{10} = f_\nu^{00} \varphi_\nu$ , instead of (10.34), we obtain the equations

$$\begin{aligned} \frac{df_\nu^{00}}{dt} &= \sum_{(1)}^{\nu} J(f^{00}, f^{10}) \\ &= \sum_{\mu} \int f_\nu^{00} f_\mu^{00} (\varphi'_\mu + \varphi'_\nu - \varphi_\mu - \varphi_\nu) g_{\nu\mu} b db d\varepsilon d\xi^\mu, \end{aligned} \quad (10.34a)$$

where the functions  $f_\nu^{00}$  are solutions of the equations

$$\sum_{(1)}^{\nu} J(f^{00}, f^{00}) = 0.$$

The last equations require only that the functions  $f_\nu^{00}$  be equilibrium functions with respect to velocities for arbitrary  $n^\nu$  and, therefore, the problem of determining the functions  $\varphi_\nu$  has been reduced to the mixture problem examined in the preceding section. The left sides of equations of (10.34a) take the form (9.52), where

for an equilibrium mixture of molecules of equal mass

$$d_r^\nu = \frac{\partial}{\partial x_r} \left( \frac{n^\nu}{n} \right) = \frac{\partial}{\partial x_r} \left( \frac{e^{-E^\nu/kT}}{\sum_v e^{-E^\nu/kT}} \right) = \frac{n^\nu}{n} \frac{E^\nu - \varepsilon^0(T)}{kT} \frac{\partial \ln T}{\partial x_r}.$$

Following substitution of these expressions into (9.52), the left sides of equations (10.34a) take the form (10.35), where the term with  $\partial u_r / \partial x_r$  is missing. It follows from the form of the equations thus obtained (and this should have been expected for a mixture of molecules differing only in internal energy) that, to the accuracy of this analysis, the transfer of momentum and of translational energy of the molecules must be the same as for a monatomic gas (i.e., with the same  $\mu$  and  $\lambda_1$  and  $\xi = 0$ ). To determine  $\lambda_2$ , we note that  $A^\nu$  may be represented in the form

$$A^\nu = A_1(c, T) + A_2(c, T) [E^\nu - \varepsilon^0(T)],$$

where  $A_1$  and  $A_2$  do not depend on  $\nu$ . Using condition (10.38) for  $\lambda_2$ , we have

$$\begin{aligned} \lambda_2 &= \frac{1}{3T} \sum_{\nu} E^\nu \int A^\nu c^{\nu 2} f_\nu^{(0)} d\xi - \frac{\varepsilon^0(T)}{3T} \sum_{\nu} \int A^\nu c^{\nu 2} f_\nu^{(0)} d\xi \\ &= \frac{1}{3T} \left( \sum_{\nu} n^\nu E^{\nu 2} - n \varepsilon^{(0) 2} \right) \int A_2 c^2 \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{m}{2kT} c^2} d\xi \\ &= \text{const}(T) \frac{d\varepsilon^0(T)}{dT} = c_{vi} \text{const}(T), \end{aligned}$$

where  $c_{vi}$  is the specific heat of the internal degrees of freedom;  $\lambda_2$  gives the Eucken correction\* to the thermal conductivity of a monatomic gas taking into account the presence of internal degree of freedom of the molecules.

**2.2.** We now examine the more general case where, in addition to those examined ( $\epsilon_i \ll 1$ ,  $i \leq n-1$ ), there is another form of internal energy with a relaxation time of the order of the characteristic flow time, i.e.,  $\epsilon_n \sim 1$ , or, equivalently, we take the characteristic flow time equal to the longest relaxation time  $\epsilon_n$  of the gas examined in the previous case. For simplicity we assume that there are three characteristic times in all,  $\epsilon_1 \leq \epsilon_2 \ll \epsilon_3 \sim 1$ .

\* A. Eucken, Physik Z., 14: 324 (1913).

This does not reduce the generality of our treatment, since all the  $\epsilon_i \neq 1$  or the  $\epsilon_n$  may be included in  $\epsilon_2$ .

As in the analogous case examined above, each  $v$ -state is characterized by two quantum numbers:  $v_2$  and  $v_3$ . In the first approximation, the translational degrees of freedom and the  $v_2$ -states are in equilibrium, and the function  $f_v^{(0)}$  is given by formula (10.23). By representing the distribution function in the form (10.33) in the next approximation, we obtain an equation for  $\varphi_v$ :

$$\left( \frac{d}{dt} - \frac{1}{\epsilon_3} \Sigma_{(3)}^v \right) f_v^{(0)} = \frac{1}{\epsilon_1} \sum_{(1)}^v J(f, \varphi) + \frac{1}{\epsilon_2} \sum_{(2)}^v J(f, \varphi). \quad (10.42)$$

Here,  $\Sigma_{(3)}^v(f_v^{(0)})$  is the sum of the collision integrals in the equation for  $f_v$ , which includes only collisions with transitions of  $v_3$ -energy; the functions  $f_v^{(0)}$  are substituted for  $f_v$ . The integrals have the same value as in (10.34), with the only difference that now  $\varphi_v$  and  $\varphi_k$  are referred to the same  $v_3$ -state, and  $\varphi_\mu$  and  $\varphi_l$  — to the other  $v_3$ -state, since the sums  $\Sigma_{(1)}^v$  and  $\Sigma_{(2)}^v$  do not include collision integrals with transitions of  $v_3$ -energy into other forms of energy. Hence, it follows that  $\varphi_v = \alpha_{v_3}$ , where  $\alpha_{v_3}$  is any constant depending on  $v_3$  and the macroscopic flow parameters, and is a solution of the homogeneous equation. Therefore, here there are not five eigenfunctions, as in the previous case, but  $N_{v_3} + 4$  functions  $\psi_j = \alpha_{v_3}, \xi_i^v, \frac{1}{2}mc^{v2} + E^v$ ;  $N_{v_3}$  is the number of  $v_3$ -states. The macroscopic quantities  $n^{v_3}$ ,  $u_r$ , and  $T$  satisfy the Euler equations (10.17)-(10.20) in the first approximation. In equations (10.20) we must replace the subscripts  $v$ ,  $\mu$ ,  $k$ , and  $l$  by  $v_3$ ,  $\mu_3$ ,  $k_3$ , and  $l_3$ . By substituting the functions  $f_v^{(0)}$  in the form (10.23) into the left side of equations (10.42), and eliminating the partial derivatives with respect to  $t$  by means of the Euler equations, we obtain

$$\begin{aligned} \frac{df_v^{(0)}}{dt} - \Sigma_{(3)}^v(f_v^{(0)}) &= f_v^{(0)} \left\{ \left( \frac{m}{2kT} c^{v2} - \frac{5}{2} + \frac{E^{v_2} - \epsilon_2^{(0)}(T)}{kT} \right) \frac{\partial \ln T}{\partial x_r} c_r^v \right. \\ &+ \frac{m}{kT} \left( c_l^v c_r^v - \frac{1}{3} \delta_{lr} c^{v2} \right) \frac{\partial u_l}{\partial x_r} + \left[ \frac{1}{3} \frac{m}{kT} c^{v2} - 1 - k \left( \frac{m}{2kT} c^{v2} - \frac{3}{2} \right. \right. \\ &\left. \left. + \frac{E^{v_2} - \epsilon_2^{(0)}(T)}{kT} \right) \left( \frac{3}{2} k + \frac{d\epsilon_2^{(0)}}{dT} \right)^{-1} \right] \frac{\partial u_r}{\partial x_r} + c_r^v \frac{\partial}{\partial x_r} \ln \left( \frac{n^{v_3}}{n} \right) \\ &+ \frac{1}{n^{v_3}} \Sigma_{(3)}^v(a) - \left( \frac{m}{2kT} c^{v2} - \frac{3}{2} + \frac{E^{v_2} - \epsilon_2^{(0)}(T)}{kT} \right) \times \end{aligned}$$

$$\left. \times \sum_{v_3} E^{v_3} \Sigma_{(3)}^{v_3}(a) \left[ nT \left( \frac{3}{2} k + \frac{d\epsilon_2^{(0)}(T)}{dT} \right) \right]^{-1} \right\} - \Sigma_{(3)}^v(f_v^{(0)}). \quad (10.43)$$

Here for brevity the right side of Eq. (10.20) for  $n^{v_3}$  has been denoted by the symbol  $\Sigma_{(3)}^{v_3}(a)$ .

This form of the left side allows us to seek a solution in the following form:

$$\begin{aligned} \varphi_v &= -A^v \frac{\partial \ln T}{\partial x_r} c_r^v - B^v \left( c_r^v c_l^v - \frac{1}{3} \delta_{lr} c^{v2} \right) \frac{\partial u_l}{\partial x_r} - D^v \frac{\partial u_r}{\partial x_r} \\ &+ c_r^v \sum_{v_3} F_{v_3}^v \frac{\partial}{\partial x_r} \left( \frac{n^{v_3}}{n} \right) + G^v + a_{v_3} + \beta_r c_r^v + \gamma \left( \frac{mc^{v2}}{2} + E^v \right). \quad (10.44) \end{aligned}$$

The functions  $A$ ,  $B$ ,  $D$ ,  $F$ , and  $G$  depend on  $T$ , the energy levels, the composition of the mixture of  $v_3$ -gases, i.e., on all the  $n^{v_3}$ , and the velocities of the molecules  $c^v$ . The quantities  $a_{v_3}$ ,  $\beta_r$ , and  $\gamma$  do not depend on  $c^v$ . We subject the solution to the conditions defining  $a_{v_3}$ ,  $\beta_r$ , and  $\gamma$ :

$$\sum_{v_2} \int f_v d\xi^v = \sum_{v_2} \int f_v^{(0)} d\xi^v = n^{v_3} \quad \text{or} \quad \sum_{v_2} \int f_v^{(0)} \varphi_v d\xi^v = 0, \quad (10.45)$$

$$\sum_v \int \xi_r f_v d\xi^v = \sum_v \int \xi_r f_v^{(0)} d\xi^v \quad \text{or} \quad \sum_v \int \xi_r f_v^{(0)} \varphi_v d\xi^v = 0, \quad (10.46)$$

$$\begin{aligned} \sum_v \int \left( \frac{mc^{v2}}{2} + E^v \right) f_v d\xi^v &= \sum_v \int \left( \frac{mc^{v2}}{2} + E^v \right) f_v^{(0)} d\xi^v \\ &= n \left( \frac{3}{2} kT + \epsilon_2^{(0)}(T) + \frac{1}{n} \sum_{v_3} n^{v_3} E^{v_3} \right) = n \left( \frac{3}{2} kT + \epsilon \right) \quad (10.47a) \end{aligned}$$

or

$$\sum_v \int \left( \frac{mc^{v2}}{2} + E^v \right) f_v^{(0)} \varphi_v d\xi^v = 0. \quad (10.47b)$$

These conditions, like conditions (10.37)-(10.39), introduce a temperature  $T$ , different from the mean energy of translational motion of the molecules, as determined by formula (10.7).

The conditions (10.45) and (10.47) indicate that  $\alpha_\nu$  and  $\gamma$  may be included in  $D^\nu$  and  $G^\nu$ , while condition (10.46) allows us to include  $\beta_r$  in  $A^\nu$  and  $F_{\nu_3}^\nu$ .

Under conditions (10.45)–(10.47), the conservation equations (10.11)–(10.13) do not change their form. The internal energy in (10.13) is determined by the condition (10.47a). Only the expressions for  $P_{ij}$  and  $q_i$  change. In comparing the solution with the preceding one, we see that the expression for the heat flux has a term depending on the diffusion of the components of the  $\nu_3$ -gas, proportional to  $\partial n^{\nu_3}/\partial x_r$ . In addition, the viscosities  $\mu$  and  $\xi$  and the thermal conductivities  $\lambda_1$  and  $\lambda_2$  are in general different from those examined above even for the same gas, but for a smaller characteristic dimension of the phenomenon.

Since, in general,  $G^\nu \neq 0$ , the diagonal components of the stress tensor have terms containing  $G^\nu$  and independent of the velocity gradients. These terms may be combined either with the terms with volume viscosity, or with the pressure, by putting the pressure equal to

$$p = knT + \frac{1}{3} \sum_v \int mc^{v2} G^\nu f_v^{(0)} d\xi^\nu.$$

We call this additional pressure the *relaxation pressure*, since it occurs only when the relaxation time of one of the internal degrees of freedom is of the same order as the characteristic flow time (in our case,  $\theta_3 \sim \Theta$  or  $\epsilon_3 \sim 1$ ). There is no additional pressure either when the above-mentioned internal degree of freedom is in equilibrium (i.e.,  $\epsilon_3 \ll 1$ ), or when it is frozen (i.e.,  $\epsilon_3 \gg 1$ ). In addition, the relaxation pressure is equal to zero also in the case when there is only one relaxing degree of freedom, (i.e., in our case, if  $E^{\nu_2} = 0$ ), since, in this case,  $D^\nu = 0$ , and, instead of (10.47b) we have the condition

$$\sum_v \int \frac{mc^{v2}}{2} f_v^{(0)} \varphi_v d\xi^\nu = \sum_v \int \frac{mc^{v2}}{2} G^\nu f_v^{(0)} d\xi^\nu = 0$$

and, therefore, the relaxation pressure vanishes.\*

\* Apparently, attention was first directed to the relaxation pressure in a paper by L. I. Mandel'shtam and M. A. Leontovich [Zh. Eksperim. i Teor. Fiz., Vol. 7, No. 3 (1937)].

In the equations of motion, the terms which contain the relaxation pressure are of the same order as the terms containing the shear and volume viscosities. A relaxation pressure may appear, for example, when the rotational degrees of freedom are in equilibrium, but the vibrational degrees of freedom are relaxing.

The system of equations (10.11)–(10.13) is not closed, since the equations contain the  $N_{\nu_3}$  unknowns  $n^{\nu_3}$ , besides the variables  $n$ ,  $u_i$ , and  $T$ . To construct equations for  $n^{\nu_3}$ , we shall integrate equations (10.14) with respect to  $\xi^\nu$  and sum the equations obtained over all the states  $\nu_2$ , belonging to a single  $\nu_3$ -state; we have

$$\frac{\partial n^{\nu_3}}{\partial t} + \frac{\partial}{\partial x_r} \sum_{\nu_2} n^\nu u_r^\nu = \sum_{\nu_2} \int \Sigma_{(3)}^\nu d\xi^\nu. \quad (10.48)$$

The sums  $\Sigma_{(1)}^\nu$  and  $\Sigma_{(2)}^\nu$  vanish, since the number of molecules in a given  $\nu_3$ -state does not change as a result of exchange of translational and  $\nu_2$ -energies.

We transform the sum on the left side of the equation:

$$\begin{aligned} \sum_{\nu_2} n^\nu u_r^\nu &= \sum_{\nu_2} \int \xi_r^\nu f_v d\xi^\nu = \sum_{\nu_2} \left( \int c_r^\nu f_v d\xi^\nu + u_r \int f_v d\xi^\nu \right) \\ &= \sum_{\nu_2} \int c_r^\nu f_v^{(0)} \varphi_v d\xi^\nu + u_r n^{\nu_3}. \end{aligned}$$

Substituting  $\varphi_v$  according to (10.44), we find

$$\sum_{\nu_2} n^\nu u_r^\nu = u_r n^{\nu_3} + L^{\nu_3} \frac{\partial T}{\partial x_r} + \sum_{\nu_3} M^{\nu_3} \frac{\partial}{\partial x_r} \left( \frac{n^{\nu_3}}{n} \right), \quad (10.49)$$

where  $L^{\nu_3}$  and  $M^{\nu_3}$  depend on  $T$ , the energy levels, and the local composition of the mixture of  $\nu_3$ -gases.

The right side of (10.48) may be written in the following

However, the phenomenological approach used in this paper does not allow conclusions to be made regarding the magnitude of this pressure, or even to distinguish the cases when it is in fact equal to zero. That information may be obtained only from kinetic theory (see Kogan, loc. cit.).

form:

$$\begin{aligned} & \sum_{v_2, \mu, k, l} \int (f_k f_l - f_\mu f_v) \sigma_{v\mu}^{kl} g_{v\mu} d\xi^k d\xi^l d\xi^\mu d\xi^v \\ = & \sum_{v_2, \mu, k, l} \int \{ [f_k^{(0)} f_l^{(0)} (1 + \varphi_k + \varphi_l) - f_v^{(0)} f_\mu^{(0)} (1 + \varphi_v + \varphi_\mu)] \sigma_{v\mu}^{kl} g_{v\mu} \} \\ & \times d\xi^k d\xi^l d\xi^\mu d\xi^v = \sum_{v_2, \mu, k, l} \left[ n^{k_3} n^{l_3} \left( b_{kl}^{v\mu} + d_{kl}^{v\mu} \frac{\partial u_r}{\partial x_r} \right) e^{-(E^{k_2} + E^{l_2})/kT} \right. \\ & \quad \left. - n^{v_3} n^{\mu_3} \left( b_{v\mu}^{kl} + d_{v\mu}^{kl} \frac{\partial u_r}{\partial x_r} \right) e^{-(E^{v_2} + E^{\mu_2})/kT} \right]. \end{aligned}$$

In carrying out this integration we remember that the transition probabilities cannot depend on orientation and are functions of the relative velocity of the colliding molecules. The coefficients  $b_{kl}^{v\mu}$  differ from the corresponding coefficients  $a_{kl}^{v\mu}$  in Eq. (10.24), since the portion  $G^v$  is included in them.

Summing over  $v_2$  in the last expression, we finally obtain

$$\begin{aligned} & \frac{\partial n^{v_3}}{\partial t} + \frac{\partial u_r n^{v_3}}{\partial x_r} + \frac{\partial}{\partial x_r} \left[ L^{v_3} \frac{\partial T}{\partial x_r} + \sum_{v_3} M^{v_3} \frac{\partial}{\partial x_r} \left( \frac{n^{v_3}}{n} \right) \right] \\ = & \sum_{\mu_3, k_3, l_3} \left[ n^{k_3} n^{l_3} \left( b_{k_3 l_3}^{v_3 \mu_3} + d_{k_3 l_3}^{v_3 \mu_3} \frac{\partial u_r}{\partial x_r} \right) - n^{v_3} n^{\mu_3} \left( b_{v_3 l_3}^{k_3 \mu_3} + d_{v_3 l_3}^{k_3 \mu_3} \frac{\partial u_r}{\partial x_r} \right) \right]. \quad (10.50) \end{aligned}$$

where  $b_{k_3 l_3}^{v_3 \mu_3}$  and  $d_{k_3 l_3}^{v_3 \mu_3}$  are related to  $b_{kl}^{v\mu}$  and  $d_{kl}^{v\mu}$  by relations of type (10.24a). These equations close the system (10.11)-(10.13) with the above values of  $\epsilon$ ,  $q_i$ , and  $P_{ij}$ .

Further simplifications of the system (10.50) are possible only when special assumptions are made regarding the transition probabilities.

If the  $\epsilon_2$ -transitions also include resonance transitions of the  $v_3$ -states, the distribution function  $f_v^{(0)}$  may be written in the form (10.25). In that case, the resonance transitions are included in the sum  $\Sigma_{(2)}^v$  in Eq. (10.42). We may write the left side of that equation in the form

$$\begin{aligned} & \frac{df_v^{(0)}}{dt} - \Sigma_v^{(3)}(f_v^{(0)}) \\ = & f_v^{(0)} \left\{ \left( \frac{m}{2kT} c^{v2} - \frac{5}{2} + \frac{E^{v_2} - \epsilon_2^{(0)}}{kT} \right) \frac{\partial \ln T}{\partial x_r} c_r^v + \frac{m}{kT} \left( c_i^v c_r^v - \frac{1}{3} \delta_{ir} c^{v2} \right) \frac{\partial u_i}{\partial x_r} \right\} \end{aligned}$$

$$\begin{aligned} & + \left[ \frac{1}{3} \frac{m}{kT} c^{v2} - 1 - k \left( \frac{3}{2} k + \frac{d\epsilon_2^{(0)}}{dT} \right)^{-1} \left( \frac{m}{2kT} c^{v2} - \frac{3}{2} + \frac{E^{v_2} - \epsilon_2^{(0)}}{kT} \right) \right] \frac{\partial u_r}{\partial x_r} \\ & + \frac{E^{v_2} - \epsilon_2^{(0)}}{kT_3^2} \frac{\partial T_3}{\partial x_r} c_r^v + \left[ \frac{E^{v_3} - \epsilon_3^{(0)}}{kT_3^2} \frac{dT_3}{d\epsilon_3^{(0)}} \right. \\ & \left. - \frac{1}{T} \left( \frac{m}{2kT} c^{v2} - \frac{3}{2} + \frac{E^{v_2} - \epsilon_2^{(0)}}{kT} \right) \left( \frac{3}{2} k + \frac{d\epsilon_2^{(0)}}{dt} \right)^{-1} \right] \sum_{v_3} E^{v_3} \Sigma^{(3)}(a) \Big\} - \Sigma_v^{(3)}(f_v^{(0)}). \quad (10.51) \end{aligned}$$

Here we understand  $\sum_{v_3} E^{v_3} \Sigma^{(3)}(a)$  to be the right side of Eq. (10.27). That expression is a function of  $T_3$ .

It may be seen by comparing expressions (10.51) and (10.43) that the solution of Eq. (10.42) with those changes can differ from the solution of (10.44) only by the presence of a term with  $\partial T_3 / \partial x_r$  and the absence of a term proportional to  $\partial n_3 / \partial x$ .

The number of arbitrary constants is decreased in that case, since the function  $\varphi_v = \alpha v_3$  is not a solution of the homogeneous equation which includes resonance transitions. The solution in that case will be  $\varphi_v = \sigma E^{v_3}$ , since the  $v_3$ -energy of the colliding molecules is conserved in resonance and other  $\epsilon_1$ - and  $\epsilon_2$ -transitions. Therefore, the general solution of the homogeneous equation has the form

$$\varphi_v = a + \beta_k c_k^v + \gamma \left( \frac{mc^{v2}}{2} + E^v \right) + \sigma E^{v_3}. \quad (10.52)$$

The presence of six arbitrary constants allows us to impose six conditions on  $\varphi_v$ , so that

$$\sum_v \int f^v d\xi^v = \sum_v \int f_v^{(0)} d\xi^v = n, \quad (10.53)$$

$$\sum_v \int \xi^v f_v d\xi^v = \sum_v \int \xi^v f_v^{(0)} d\xi^v = nu, \quad (10.54)$$

$$\begin{aligned} \sum_v \int \left( \frac{mc^{v2}}{2} + E^v \right) f_v d\xi^v &= \sum_v \int \left( \frac{mc^{v2}}{2} + E^v \right) f_v^{(0)} d\xi^v \\ &= n \left[ \frac{3}{2} kT + \epsilon_2^{(0)}(T) + \epsilon_3^{(0)}(T_3) \right], \quad (10.55) \end{aligned}$$

$$\sum_{v_1} E^{v_1} \sum_{v_2} \int f_v d\xi^v = \sum_{v_3} E^{v_3} \sum_{v_2} \int f_v^{(0)} d\xi^v = n \epsilon_3^{(0)}(T_3), \quad (10.56)$$

where

$$\varepsilon_3^{(0)} = \sum_{v_3} E^{v_3} e^{-E^{v_3}/kT_3} \left( \sum_{v_3} e^{-E^{v_3}/kT_3} \right)^{-1}.$$

Here, as above, the temperature T is not a measure of the mean energy of translational motion of the molecules.

Equations (10.11)-(10.13) remain unchanged under the conditions (10.53)-(10.56). The expression for  $q_r$ , has a term proportional to  $\partial T_3 / \partial x_r$ , as was pointed out above. The expression for  $P_{ij}$  remains of the same form as in the previous case. The internal energy is

$$\epsilon = \frac{3}{2} kT + \varepsilon_2^{(0)} + \varepsilon_3^{(0)}.$$

The transfer coefficients have different values in the general case.

The equations (10.11)-(10.13) include two temperatures: T and  $T_3$ . Integrating equations (10.14) with respect to  $\xi^v$ , multiplying by  $E^{v_3}$ , and adding, we obtain [see the derivation of Eqs. (10.27) and (10.50)]:

$$n \left( \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial x_r} \right) \varepsilon_3^{(0)} - \frac{\partial}{\partial x_r} \left( L \frac{\partial T}{\partial x_r} + M \frac{\partial T_3}{\partial x_r} \right) = b + d \frac{\partial u_r}{\partial x_r}, \quad (10.57)$$

where b and d depend on n, T, and  $T_3$ . This equation closes the system (10.11)-(10.13).

If only the  $v_3$ -states are present,\* so that only the resonance transitions of  $v_3$ -energy appear in the  $\epsilon_2$ -transitions, then  $\varepsilon_2^{(0)}(T) = 0$  and the temperature T has its usual meaning as a measure of the translational energy of the molecules,  $p = nkT$ , and the bulk viscosity and the relaxation pressure are absent.

It is important to note that in general, for the same gas, the transfer coefficients have different values, depending on the ratio of the characteristic flow time to the characteristic relaxation time, i.e., depending on which of the  $\epsilon_n = 1$ . The viscosity, thermal conductivity, etc., of the gas at some point of the flow depend not only on the temperature and pressure at that point, but

\*This case was examined in the above-mentioned paper by V. N. Zhigulev.

also on the characteristic dimension or characteristic time of the flow.

Only the structure of the equation of a relaxing gas has been obtained above. To find specific values of the transfer coefficients we need to know the transition probabilities, and it is a laborious task to solve the appropriate integral equations for the functions A, B, D, etc. The solution of those integral equations is complicated by the fact that their kernels are functions not only of the velocities c of the molecules, but also of the discrete energy levels  $E_v$ . Therefore, the functions A, B, etc., must be expanded not as Sonine polynomials in  $c^2$ , but in terms of more complex functions, containing the energy levels.\*

In many cases of practical importance we may simplify these integral equations while retaining sufficient accuracy. The probabilities of transfer of internal energy are usually roughly an order of magnitude less than the probabilities of transfer of translational energy, i.e.,  $\epsilon_i \gg \epsilon_1$  with  $i = 2, 3, \dots$ , then in the integral equations for A, B, D, and so on, we need retain only the sum  $\Sigma_{(1)}$ , omitting all the integrals containing probabilities of transfer of internal energy (see the derivation of the Eucken correction on page 217). The integral equations then obtained differ from the integral equations of a monatomic gas only as regards the inhomogeneous part. The solution of those equations is considerably simpler. The functions A, B, etc., may be represented, as was done in § 3.8 as Sonine polynomials in  $c^2$ . In that approximation the viscosity, thermal conductivity, and diffusion coefficients do not depend directly on the probabilities of transfer of internal forms of energy, which are in equilibrium. To determine the transfer coefficients we need to know only the effective cross section for the relaxing degree of freedom. The inhomogeneous parts of the equations will differ as before, depending on which of the  $\epsilon_n$  is of order unity and, therefore, the transfer coefficients will be different when they describe processes with different characteristic times, in the same gas.

\*See, for example, L. Waldmann and E. Z. Trübenbacher, Naturforschung, Vol. 170, No. 5 (1962); V. M. Kuznetsov, Inzh. Zh., Vol. 5, No. 5 (1965); and the other papers cited at the beginning of this section.

### §3.11. Solution of the Linearized Boltzmann Equation

1. We examine the linearized Boltzmann equation

$$\begin{aligned} \frac{d\varphi}{dt} &= \frac{\partial\varphi}{\partial t} + \xi_i \frac{\partial\varphi}{\partial x_i} = \int f_{01}(\varphi' + \varphi'_i - \varphi - \varphi_i) g b db de d\xi_1 \\ &= \int f_{01}(\varphi' - \varphi'_i - \varphi - \varphi_i) B(g, \vartheta) d\vartheta de d\xi_1 = L(\varphi), \\ f &= f_0(1 + \varphi). \end{aligned} \quad (11.1)$$

As was shown in §2.8, the integral operator  $L(\varphi)$  is a Fredholm operator with a symmetric kernel. The natural way to solve this equation is to expand the solution in eigenfunctions of the integral operator. The eigenfunctions of that operator are solutions of the equation

$$\lambda\varphi = L(\varphi). \quad (11.2)$$

It is easy to prove some general properties of the integral equation (11.2).

a. The eigenfunctions are orthogonal with respect to a weighting function  $f_0$ . We examine the integral

$$\begin{aligned} I_{\varphi_j} &= \int f_0 \varphi_j L(\varphi_i) d\xi \\ &= \int f_0 f_{01} \varphi_j (\varphi'_i + \varphi'_{i1} - \varphi_i - \varphi_{i1}) B(g, \vartheta) d\vartheta de d\xi_1 d\xi. \end{aligned}$$

where  $\varphi_i$  and  $\varphi_j$  are eigenfunctions corresponding to the eigenvalues  $\lambda_i$  and  $\lambda_j$ .

By arguments analogous to those presented in §2.4, it is easy to show that

$$\begin{aligned} I_{\varphi_j} &= -\frac{1}{4} \int f_0 f_{01} (\varphi'_i + \varphi'_{i1} - \varphi_i - \varphi_{i1})(\varphi'_j + \varphi'_{j1} - \varphi_j - \varphi_{j1}) \\ &\quad \times B(g, \vartheta) d\vartheta de d\xi_1 d\xi. \end{aligned}$$

Therefore,

$$\lambda_i \int f_0 \varphi_i \varphi_j d\xi = \int L(\varphi_i) f_0 \varphi_j d\xi = \int L(\varphi_j) f_0 \varphi_i d\xi = \lambda_j \int f_0 \varphi_i \varphi_j d\xi.$$

whence the orthogonality of the eigenfunctions follows at once when  $\lambda_i \neq \lambda_j$ .

b. Five eigenfunctions — the summational invariants — correspond to the eigenvalue  $\lambda = 0$ . This statement is self-evident.

c. The eigenvalues are negative. We multiply Eq. (11.2) by  $f_0 \varphi$  and integrate with respect to  $\xi$ . Then, according to the relation just obtained, we have

$$\begin{aligned} \lambda \int \varphi^2 d\xi &= \int \varphi L(\varphi) f_0 d\xi \\ &= -\frac{1}{4} \int f_0 f_{01} (\varphi'_i + \varphi' - \varphi_i - \varphi)^2 B(g, \vartheta) d\vartheta de d\xi_1 d\xi. \end{aligned}$$

The properties of the integral operator have been investigated in detail only for Maxwellian and hard-sphere molecules.\* Molecules whose interaction potential falls off more rapidly than for Maxwellian molecules ( $s > 5$ ), are called "hard," and molecules with a slower falloff in potential, "soft." For pseudo-Maxwellian molecules the linear operator has a complete discrete spectrum of eigenvalues. For molecules of finite interaction radius, either somewhat harder or somewhat softer than Maxwellian, the operator  $L$  has a continuous spectrum of eigenvalues.†

In fact, for molecules of finite interaction radius, the operator  $L(\varphi)$  may be written in the form (see §2.7)

$$L(\varphi) = -\varphi J_2(\xi) + J_1(\xi).$$

\*The theory of eigenfunctions for the linearized Boltzmann equation evidently was presented in the papers of Wang Chang and Uhlenbeck (C. S. Wang Chang and G. E. Uhlenbeck, "On the Propagation of Sound in Monatomic Gases," University of Michigan Press, Ann Arbor, Michigan, 1952), and Mott-Smith (1954). Those original papers were not available to the author, however. A statement of some of their results may be found, for example, in: G. Uhlenbeck and G. Ford, "Lectures on Statistical Mechanics," 1963; L. Waldmann, Handbuch der Physik, Vol. 12 (1958); L. Sirovich, Phys. Fluids, Vol. 6, No. 1 (1963); E. P. Gross and E. A. Jackson, Phys. Fluids, Vol. 2, No. 4 (1959).

†H. Grad, "Rarefied Gas Dynamics," Third Symposium, Academic Press, New York, 1966.

It is evident that the equation

$$\varphi J_2(\xi) = -\lambda_2 \varphi$$

has in general a continuous spectrum of eigenvalues

$$\lambda_2 = -J_2(\xi).$$

Even if the equation

$$J_1(\varphi) = \lambda_1 \varphi$$

has a discrete spectrum, the complete operator  $L(\varphi)$  has a continuous spectrum of eigenvalues  $\lambda = \lambda_1 + \lambda_2$ :

$$L(\varphi) = -\varphi J_2 + J_1 = \lambda_2 \varphi + \lambda_1 \varphi = \lambda \varphi.$$

For pseudo-Maxwellian molecules [see formula (2.13) of §2.2], the operator

$$J_2(\xi) = \text{const} = A_n$$

does not depend on  $\xi$ . In that case, if the operator  $J_1$  has a discrete spectrum, the complete operator has a discrete spectrum.

For hard molecules, the spectrum extends from some finite value  $v_0$  to infinity, while for soft molecules it goes from  $v_0$  to 0. This last fact is important for the proof of the asymptotic convergence of the general solution of the Boltzmann equation to the Hilbert normal solution (see §3.7). It is precisely because of the fact that the eigenvalues reach zero for soft molecules that it is impossible to prove that when  $t \rightarrow \infty$ , the general solution of the Boltzmann equation tends to a solution corresponding to the eigenvalue  $\lambda = 0$  (to the normal solution).

For molecules with an infinite interaction radius, the spectrum of eigenvalues has been investigated only for Maxwellian molecules, for which the spectrum is discrete and complete. The nature of the spectrum of eigenvalues for molecules with other interaction laws may be assessed only from the above-mentioned results for corresponding molecules with a finite interaction radius. The eigenvalues and the eigenfunctions may be found, for example, as follows.\*

\*H. Grad, "Handbuch der Physik, Vol. 12 (1958).

We introduce the notation of §3.3:

$$\omega(v) = \frac{1}{(2\pi)^{2/3}} e^{-\frac{1}{2} v^2}, \quad v = \sqrt{\frac{m}{kT_0}} c, \quad f_0 = n_0 \left(\frac{m}{kT_0}\right)^{3/2} \omega(v),$$

$$[\varphi] = \varphi' + \varphi'_1 - \varphi - \varphi_1.$$

In that notation, Eq. (11.2) takes the form

$$\lambda \varphi = n_0 \int \omega(v_1) [\varphi] g b db d\varepsilon dv_1. \quad (11.3)$$

We multiply both sides of the equation by  $\omega(v) H_{\alpha_1}^{(N)} \dots \alpha_N^{(N)}$  and integrate with respect to  $(v)$ ; we have

$$\lambda a_{\alpha_1}^{(N)} \dots \alpha_N^{(N)} = n_0 \int \omega(v) \omega(v_1) [\varphi] g b db d\varepsilon dv dv_1, \quad (11.4)$$

where

$$a_{\alpha_1}^{(N)} \dots \alpha_N^{(N)} = \int \omega(v) \varphi H_{\alpha_1}^{(N)} \dots \alpha_N^{(N)} dv = \frac{1}{n_0} \int f_0 \varphi H_{\alpha_1}^{(N)} \dots \alpha_N^{(N)} d\xi. \quad (11.5)$$

This definition of the coefficients  $a^{(N)}$  of the expansion of the function  $\varphi$  in Hermite polynomials coincides with the definition (3.10). We represent  $\varphi$  in a series of Hermite polynomials similar to the series (3.11):

$$\varphi = \sum_{m=0}^{\infty} \frac{1}{m!} a_{\alpha_1}^{(m)} \dots \alpha_m^{(m)} H_{\alpha_1}^{(m)} \dots \alpha_m^{(m)}. \quad (11.6)$$

If that series is substituted into the integral on the right side of Eq. (11.4), the integral also becomes an infinite series in  $a^{(m)}$  with coefficients known for a given molecular interaction law. We thus obtain an infinite system of simultaneous equations for the determination of  $a^{(N)}$ .

As was shown in §3.3, the series which represents the integral for Maxwellian molecules contains only a finite number of terms containing the coefficients  $a^{(m)}$  when  $m = N$ ; in that case we have

$$\lambda a_{\alpha_1}^{(N)} \dots \alpha_N^{(N)} = n_0 B_{\alpha_1}^{(N)} \dots \alpha_N^{(N)} \beta_1^{(N)} \dots \beta_N^{(N)}, \quad (11.7)$$

where

$$B_{\alpha_1 \dots \alpha_N \beta_1 \dots \beta_N}^{(N)} = \int \omega(v) \omega(v_1) H_{\alpha_1 \dots \alpha_N}^{(N)} [H_{\beta_1 \dots \beta_N}^{(N)}] g b d\theta d\varepsilon d\mathbf{v} d\mathbf{v}_1.$$

Knowing the quantities  $B^{(N)}$ , we obtain systems of homogeneous equations for the components  $a^{(N)}$  of each of the tensors of order  $N$ , individually. Each such system determines its own eigenvalues  $\lambda$ . To each such eigenvalue there correspond specific values  $a^{(N)}$ , which give solutions of the system (11.7); substitution into (11.6) gives the corresponding eigenfunction. Therefore, the eigenfunctions may be divided into groups, in each of which the eigenfunctions are expressed only in terms of Hermite polynomials of order  $N$ .

Polar coordinates are frequently used in applications instead of rectangular coordinates in velocity space. Therefore, the eigenfunctions are expressed in terms of Sonine polynomials rather than Hermite polynomials.\*

In polar coordinates the eigenfunctions  $\varphi_{rlm}$  of the equation

$$L(\varphi_{rlm}) = \lambda_{rlm} \varphi_{rlm}$$

for Maxwellian molecules have the form†

$$\varphi_{rlm} = \left[ \frac{2^{l+1} \Gamma(r + l + \frac{3}{2})(l + |m|)!}{\sqrt{\pi} r! (2l + 1)(l - |m|)!} \right]^{-\frac{1}{2}} S'_{l+\frac{1}{2}} \left( \frac{\xi^2}{2} \right) \xi^l P_l^m(\cos \theta) \exp(im\chi), \quad (11.8)$$

where  $S'_{l+\frac{1}{2}}$  are Sonine polynomials,  $P_l^m$  are Legendre polynomials, and  $\theta$  and  $\chi$  are the polar and azimuthal angles of the spherical coordinates in velocity space.

2. In what follows we examine only one-dimensional problems for the linearized Boltzmann equation. We choose the polar axis of the spherical coordinates in velocity space parallel to the single real space coordinate, which we call  $x_1$ . Since the distribution function is symmetrical with respect to the polar axis, in the

one-dimensional case we are not interested in its dependence on the angle  $\chi$ .

In that case, the Boltzmann equation for Maxwellian molecules takes the form

$$\left( \frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x_1} \right) \varphi = \int f_{01}[\varphi] B(\theta) d\theta d\varepsilon d\xi_1. \quad (11.9)$$

In what follows it is convenient to go over to dimensionless variables:

$$\left. \begin{aligned} \bar{t} &= \frac{t}{\tau}, & \bar{x}_1 &= \frac{x_1}{\tau} \sqrt{\frac{m}{kT_0}}, & \bar{\xi} &= \sqrt{\frac{m}{kT_0}} \xi, & \bar{B} &= n_0 \tau B, \\ f_{01} &= n_0 \left( \frac{m}{kT_0} \right)^{3/2} \omega(\xi_1). \end{aligned} \right\} \quad (11.10)$$

Here,  $\tau$  is a characteristic time. Retaining the same notation for the dimensionless variables as was used for the dimensional variables, we write Eq. (11.9) in the form

$$\left( \frac{\partial}{\partial \bar{t}} + \xi_1 \frac{\partial}{\partial \bar{x}_1} \right) \varphi = \int \omega[\varphi] B(\theta) d\theta d\varepsilon d\xi_1 = L(\varphi). \quad (11.11)$$

The eigenfunctions of the operator  $L(\varphi)$  are equal to\*

$$\psi_{rl} = \left( \frac{\sqrt{\pi} r! (2l + 1)}{2^{l+1} \Gamma(r + l + \frac{3}{2})} \right)^{1/2} S'_{l+\frac{1}{2}} \left( \frac{1}{2} \xi^2 \right) P_l(\cos \theta) \xi^l, \quad (11.12)$$

where  $\cos \theta = \xi_1 / \xi$ . The eigenfunctions are orthogonal with the weighting function  $\omega$ :

$$\int \omega \psi_{rl} \psi_{r'l'} d\xi = \delta_{rr'} \delta_{ll'}. \quad (11.13)$$

The corresponding eigenvalues are

$$\begin{aligned} \lambda_{rl} = 2\pi \int B(\theta) & \left[ \left( \cos \frac{\theta}{2} \right)^{2r+l} P_l \left( \cos \frac{\theta}{2} \right) + \left( \sin \frac{\theta}{2} \right)^{2r+l} \right. \\ & \times P_l \left( \sin \frac{\theta}{2} \right) - (1 + \delta_{r0} \delta_{l0}) \right] d\theta. \end{aligned} \quad (11.14)$$

\* See the definition of Sonine polynomials on page 172.

† G. Uhlenbeck and G. Ford, "Lectures on Statistical Mechanics," 1963; L. Waldmann, "Handbuch der Physik," Vol. 12 (1958).

\* See L. Waldmann, "Handbuch der Physik," Vol. 12, 1958; L. Sirovich, Phys. Fluids, Vol. 6, No. 1 (1963).

Table 3

$r \backslash l$	0	1	2	3	4	5	6
0	0	0	1	1.5	1.8731	2.1828	2.4532
1	0	0.6667	1.1667	1.5704	1.9106	2.2066	2.4703
2	0.6667	1	1.3422	1.6670	1.8633	2.2415	2.4936
3	1	1.2281	1.4915	1.7631	2.0288	2.2824	2.5215
4	1.2281	1.4037	1.6193	1.8533	2.0917	2.3262	2.5525
5	1.4037	1.5475	1.7310	1.9371	2.1533	2.3710	
6	1.5475	1.6698	1.8302	2.0148			
7	1.6698	1.7767					

It follows from (11.14) that

$$\lambda_{rl} \leq 0 \text{ and } \lambda_{r0} = \lambda_{r-1,1}. \quad (11.15)$$

Values of  $\lambda_{rl}/\lambda_{02}$  for Maxwellian molecules are given in Table 3.\*

The eigenvalues increase (in absolute magnitude) with increase of  $r + l$ . It is clear that the eigenvalue  $\lambda = 0$  must be five-fold in the general case, and triply degenerate in the one-dimensional case, and the corresponding eigenfunctions must be the summational invariants. We note also that the eigenvalue  $\lambda_{02} = \lambda_{30} = \lambda_{21}$  is triply degenerate, and that the eigenvalues  $\lambda_{r0} = \lambda_{r-1,1}$  are doubly degenerate. The first eigenfunctions in the general case have the form

$$\left. \begin{aligned} \psi_{00} &= 1, & \psi_{01} &= \xi_1, & \psi_{10} &= \sqrt{\frac{3}{2}} \left( 1 - \frac{1}{3} \xi^2 \right), \\ \psi_{02} &= \frac{\sqrt{3}}{2} \left( \xi_1^2 - \frac{1}{3} \xi^2 \right), \\ \psi_{11} &= \sqrt{\frac{5}{2}} \left( 1 - \frac{1}{5} \xi^2 \right) \xi_1. \end{aligned} \right\} \quad (11.16)$$

The perturbed distribution function may be expanded in terms of the eigenvalues:

$$\varphi = \sum_{r,l}^{\infty} a_{rl} \psi_{rl}. \quad (11.17)$$

\* Detailed tables of eigenvalues are given in: Z. Alterman, K. Frankowski, and C. L. Pakeris, *Astrophys. J. Suppl. Ser.*, Vol. 7, No. 291 (1962).

where

$$a_{rl} = \int \omega \varphi \psi_{rl} d\xi. \quad (11.18)$$

Substituting the eigenfunctions (11.16) into the definition (11.18), we obtain

$$\begin{aligned} a_{00} &= n_1, & a_{01} &= u_1, & a_{10} &= -\sqrt{\frac{3}{2}} T_1, & a_{02} &= \frac{\sqrt{3}}{2} p_{11}, \\ a_{11} &= -\sqrt{\frac{2}{5}} S_1, \end{aligned} \quad (11.19)$$

where  $n_1$ ,  $u_1$ ,  $T_1$ ,  $p_{11}$ , and  $S_1$  are the dimensionless perturbed density, velocity, temperature, stress tensor, and heat-flux vector, defined by the relations

$$\begin{aligned} n &= n_0(1 + n_1), & u_1 &= u \sqrt{\frac{m}{kT_0}}, & T_1 &= T_0(1 + T_1), \\ p_{11} &= \frac{P_{11} - p_0}{p_0}, & S_1 &= \frac{\sqrt{m}}{(kT_0)^{3/2}} q_1. \end{aligned} \quad (11.20)$$

Substitution of the series (11.17) into the collision integral obviously gives

$$L(\varphi) = \sum_{r,l}^{\infty} \lambda_{rl} \psi_{rl} a_{rl}. \quad (11.21)$$

The validity of the relation

$$\begin{aligned} \xi_l \psi_{rl} &= (l+1) \left\{ \psi_{r, l+1} \left[ \frac{2(r+l+\frac{3}{2})}{(2l+1)(2l+3)} \right]^{1/2} - \right. \\ &\quad \left. - \psi_{r-1, l+1} \left[ \frac{2r}{(2l+3)(2l+1)} \right]^{1/2} \right\} + l \left\{ \psi_{r, l-1} \left[ \frac{2(r+l+\frac{1}{2})}{(2l+1)(2l-1)} \right]^{1/2} - \right. \\ &\quad \left. - \psi_{r+1, l-1} \left[ \frac{2(r+1)}{(2l-1)(2l+1)} \right]^{1/2} \right\} \end{aligned} \quad (11.22)$$

may be verified by direct substitution. We now substitute the distribution function in the form (11.17) into the Boltzmann equation, and, taking account of relations (11.21) and (11.22), we equate coefficients for identical values of  $\psi_{rl}$  to zero. After some simple

calculations we obtain an infinite system of simultaneous equations for the functions  $a_{rl}(t, \mathbf{x})$ :

$$\left( \frac{\partial}{\partial t} - \lambda_{rl} \right) a_{rl} + \frac{\partial}{\partial x_1} \left[ l \left\{ a_{r,l-1} \sqrt{\frac{2(r+l+\frac{1}{2})}{(2l-1)(2l+1)}} - \right. \right. \right. \\ \left. \left. \left. - a_{r+1,l-1} \sqrt{\frac{2(r+l)}{(2l-1)(2l+1)}} \right\} + (l+1) \left\{ a_{r,l+1} \sqrt{\frac{2(r+l+\frac{3}{2})}{(2l+3)(2l+1)}} - \right. \right. \\ \left. \left. - a_{r-1,l+1} \sqrt{\frac{2r}{(2l+3)(2l+1)}} \right\} \right] = 0. \quad (11.23)$$

We write out the first three equations corresponding to  $\lambda_{00} = \lambda_{10} = \lambda_{01} = 0$ ; using relations (11.19), we obtain

$$\left. \begin{aligned} \frac{\partial \rho_1}{\partial t} + \frac{\partial u_1}{\partial x_1} &= 0, \\ \frac{\partial u_1}{\partial t} + \frac{\partial \rho_{11}}{\partial x_1} + \frac{\partial}{\partial x_1} (\rho_1 + T_1) &= 0, \\ \frac{\partial T_1}{\partial t} + \frac{2}{3} \frac{\partial S_1}{\partial x_1} + \frac{2}{3} \frac{\partial u_1}{\partial x_1} &= 0. \end{aligned} \right\} \quad (11.24)$$

It is evident that these equations are none other than the linearized conservation equations written in dimensionless form.

To obtain a qualitative representation of the nature of the solutions of the linearized equation, we examine the change in a distribution function assigned at time zero. We carry out a spectral analysis of the solution.

We seek a solution of the form \*

$$a_{rl} = \hat{a}_{rl} e^{\sigma t - i k x_1}, \quad (11.25)$$

where  $\hat{a}_{rl}$  are constants,  $\sigma$  is the frequency of oscillations,  $k = 2\pi/L$  is the wave number, and  $L$  is the wavelength.

Substituting (11.25) into equations (11.23), we obtain an infinite system of algebraic equations to determine  $\hat{a}_{rl}$ :

			$a_{00}$	$a_{20}$	$a_{10}$	$a_{01}$	$a_{11}$
$\sigma$	$-lk$	$0$	$0$	$0$	$0$	$0$	$0$
$-lk$	$0$	$\sqrt{\frac{2}{3}} lk$	$-\sqrt{\frac{2}{3}} lk$	$0$	$0$	$0$	$0$
$0$	$\sqrt{\frac{2}{3}} lk$	$0$	$-\sqrt{\frac{5}{3}} lk$	$0$	$0$	$0$	$0$
$0 - \sqrt{\frac{2}{3}} lk$	$0$	$\sigma - \lambda_{00}$	$\sqrt{\frac{8}{15}} lk$	$-\frac{3}{\sqrt{5}} lk$	$0$	$0$	$0$
$0$	$0 - \sqrt{\frac{5}{3}} lk$	$\sqrt{\frac{8}{15}} lk$	$\sigma - \lambda_{11}$	$0$	$\frac{2}{\sqrt{3}} lk$	$-\sqrt{\frac{28}{15}} lk$	$0$
$1$	$0$	$0$	$0$	$0$	$0$	$0$	$0$
$2$	$0$	$0$	$0 - \lambda_{00}$	$0$	$\sqrt{\frac{18}{35}} lk$	$-\frac{4}{\sqrt{7}} lk$	$0$
$0$	$0$	$0$	$0$	$\sigma - \lambda_{20}$	$0$	$-\sqrt{\frac{7}{3}} lk$	$0$
$0$	$0$	$0$	$0$	$0$	$\sqrt{\frac{18}{35}} lk$	$0$	$\sqrt{\frac{16}{15}} lk$
$3$	$0$	$0$	$0$	$0$	$0$	$0$	$\sqrt{\frac{81}{35}} lk$

B(a)

\* See L. Sirovich, Phys. Fluids, Vol. 6, No. 1 (1963).

$$(\sigma - \lambda_{rl}) \hat{a}_{rl} - ik \left\{ l \left[ \hat{a}_{r, l-1} \sqrt{\frac{2(r+l+\frac{1}{2})}{(2l-1)(2l+1)}} - \right. \right. \\ \left. - a_{r+1, l-1} \sqrt{\frac{2(r+1)}{(2l-1)(2l+1)}} \right] + (l+1) \left[ \hat{a}_{r, l+1} \sqrt{\frac{2(r+l+\frac{3}{2})}{(2l+3)(2l+1)}} - \right. \\ \left. \left. - \hat{a}_{r-1, l+1} \sqrt{\frac{2r}{(2l+3)(2l+1)}} \right] \right\} = 0. \quad (11.26)$$

The dispersion equation, which determines the dependence of the oscillation frequency  $\sigma$  on the wave number, is given by the determinant of the system

$$D(\sigma, k) = 0. \quad (11.27)$$

We write down part of this determinant in the form of Table 4.

It is easy to see that when  $k = 0$  and  $\sigma = \lambda_{rl}$ , the system (11.26) has nontrivial solutions and, therefore,

$$\sigma = \lambda_{rl} \quad (11.28)$$

are the roots of the dispersion equation. Here the roots  $\sigma = 0$  and  $\sigma = \lambda_{02} = \lambda_{30} = \lambda_{21}$  are triply degenerate while the roots  $\sigma = \lambda_{r0} = \lambda_{r-1,1}$  are doubly degenerate.

We examine the behavior of the roots of the dispersion equation for small wave numbers (for long waves). We first examine the behavior of roots close to the triply degenerate root  $\sigma = 0$ . We represent  $\sigma$  in the form

$$\sigma = ia_1 k + a_2 k^2 + ia_3 k^3 + a_4 k^4 + ia_5 k^5 + \dots \quad (11.29)$$

It is easy to see that  $\alpha_1$  is determined by the coefficient of  $k^3$  in the determinant

$$\begin{vmatrix} \sigma & -ik & 0 \\ -ik & \sigma & \sqrt{\frac{2}{3}} ik \\ 0 & \sqrt{\frac{2}{3}} ik & \sigma \end{vmatrix} = 0. \quad (11.30)$$

Similarly,  $\alpha_2$  and  $\alpha_3$  are determined by the coefficients of  $k^4$  and  $k^5$ , respectively, in the determinants

$$\begin{vmatrix} \sigma & -ik & 0 & 0 & 0 \\ -ik & \sigma & \sqrt{\frac{2}{3}} ik & -\frac{2}{\sqrt{3}} ik & 0 \\ 0 & \sqrt{\frac{2}{3}} ik & \sigma & 0 & \sqrt{\frac{5}{3}} ik \\ 0 & -\frac{2}{\sqrt{3}} ik & 0 & -\lambda_{02} & 0 \\ 0 & 0 & -\sqrt{\frac{5}{3}} ik & 0 & \lambda_{11} \end{vmatrix} = 0 \quad (11.31)$$

and

$$\begin{vmatrix} \sigma & -ik & 0 & 0 & 0 \\ -ik & \sigma & \sqrt{\frac{2}{3}} ik & -\frac{2}{\sqrt{3}} ik & 0 \\ 0 & \sqrt{\frac{2}{3}} ik & \sigma & 0 & -\sqrt{\frac{5}{3}} ik \\ 0 & -\frac{2}{\sqrt{3}} ik & 0 & \sigma - \lambda_{02} & \sqrt{\frac{8}{15}} ik \\ 0 & 0 & -\sqrt{\frac{5}{3}} ik & \sqrt{\frac{8}{15}} ik & \sigma - \lambda_{11} \end{vmatrix} = 0. \quad (11.32)$$

It is easy to verify that if we seek a solution in the form (11.25) for the Euler, Navier-Stokes, and thirteen-moment Grad equations, the dispersion equations reduce, respectively, to the determinants (11.30), (11.31), and (11.32). The last two rows in determinant (11.31) come from the equations which determine the relation of the stress tensor  $p_{11}$  and the heat-flux vector  $S_1$ , with the gradients of velocity and temperature, respectively. Here the viscosity and the thermal conductivity are inversely proportional to  $\lambda_{02}$  and  $\lambda_{11}$ .

Determining the values of  $\alpha_1, \alpha_2, \dots$  by means of (11.30)–(11.32), we obtain

$$\sigma = \frac{k^2}{\lambda_{11}} + O(k^4). \quad (11.33)$$

$$\sigma = \pm i \sqrt{\frac{5}{3}} \left[ k + k^3 \left( \frac{1}{10\lambda_{11}^2} + \frac{8}{15\lambda_{02}^2} - \frac{2}{5\lambda_{11}\lambda_{02}} \right) \right] + k^2 \left( \frac{1}{3\lambda_{11}} + \frac{2}{3\lambda_{02}} \right) + O(k^4). \quad (11.34)$$

The expressions (11.33) and (11.34) determine three branches of the roots of the dispersion equation, these being called the hydrodynamic roots. The term  $O(k)$  in (11.34) determines the unattenuated sound waves in the Euler approximation. Taking into account terms  $O(k^2)$ , the expressions (11.33) and (11.34) determine the attenuated waves in the Navier-Stokes approximation, and when terms  $O(k^3)$  are taken into account, those in the thirteen-moment approximation (the eigenvalues are negative).

Similarly, we may find other branches of the roots of the dispersion equation near the root  $\sigma = \lambda_{11}$ . It should be noted that in the thirteen-moment approximation, in addition to the hydrodynamic branches (11.33) and (11.34) there are two other branches, obtained from the determinant (11.32):

$$\sigma = \lambda_{11} - \frac{k^2}{\lambda_{11}} \left[ \frac{5}{3} + \frac{8\lambda_{11}}{15(\lambda_{11} - \lambda_{02})} \right] + O(k^4), \quad (11.35)$$

$$\sigma = \lambda_{02} - \frac{k^2}{\lambda_{02}} \left[ \frac{4}{3} + \frac{8\lambda_{02}}{15(\lambda_{02} - \lambda_{11})} \right] + O(k^4). \quad (11.36)$$

A qualitative picture of the roots of the dispersion equation are shown in Fig. 13. The direction of increase of the parameter  $k$  in the complex  $\sigma$ -plane along each branch is indicated by an arrow. Thus, there is a succession of waves with a steadily increasing rate of attenuation (with increase of the absolute value of  $\lambda_{11}$ ). Therefore, the solution of the Boltzmann equation for the initial value problem that we are examining may be represented in the form of a sum of solutions, which approach zero at different rates. Here the hydrodynamic perturbations are equilibrated more slowly than the others (for  $k = 0$ , the attenuation decrement of hydrodynamic perturbations is equal to zero;  $\sigma = 0$ ). The characteristic decay time for the remaining perturbations is of the order of the time between collisions.

The analysis given here refers only to small-wave number  $k$  (to long wavelengths). It is interesting to examine the dispersion

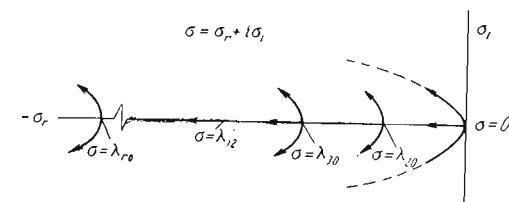


Fig. 13

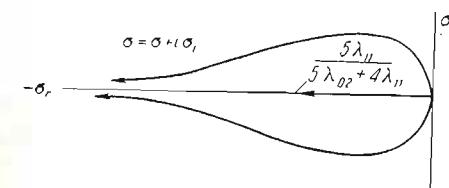


Fig. 14

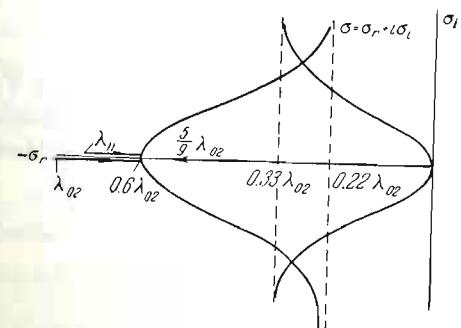


Fig. 15

relations for arbitrary wave numbers. In order to present a qualitative picture, we find the asymptotic values when  $k \rightarrow \infty$ .

It is very difficult to do this for the Boltzmann equation in general. However, it is natural to expect that when  $k \rightarrow \infty$ , the attenuation must increase, i.e., the sonic branches in the complex  $\sigma$ -plane must go off to infinity when  $k \rightarrow \infty$  along the negative real axis.

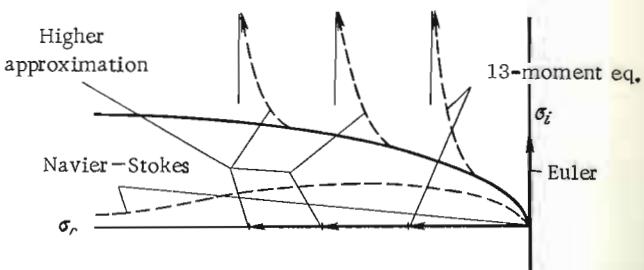


Fig. 16

There is no attenuation, as we have already noted, in the Euler approximation. In the Navier-Stokes approximation, when  $k \rightarrow \infty$ , we have from (11.31):

$$\left. \begin{aligned} \sigma &= \frac{5\lambda_{11}}{5\lambda_{02} + 4\lambda_{11}} + O\left(\frac{1}{k^2}\right), \\ \sigma &= \frac{4k^2}{3\lambda_{02}} + O(1), \quad \sigma = \frac{5k^2}{3\lambda_{02}} + O(1). \end{aligned} \right\} \quad (11.37)$$

The dispersion relation for this case is shown in Fig. 14. It may be seen that the sonic branches give a correct qualitative picture (an increase of attenuation) with decrease of wavelength (when  $k \rightarrow \infty$ ).

In the thirteen-moment approximation, when  $k \rightarrow \infty$ , we have:

$$\left. \begin{aligned} \sigma &= \frac{5}{9}\lambda_{02} + O\left(\frac{1}{k^2}\right), \quad \sigma = \pm \sqrt{4.54}ik + 0.33\lambda_{02} + O(k^{-1}), \\ \sigma &= \pm \sqrt{0.66}ik + 0.22\lambda_{02} + O(k^{-1}). \end{aligned} \right\} \quad (11.38)$$

The corresponding dispersion picture is shown in Fig. 15. Therefore, in this approximation, the damping coefficient for sound waves tends to a finite value as  $k \rightarrow \infty$ .

The Euler approximation corresponds to selection in Table 4 of the matrix  $\alpha, \beta < 1$ , and the thirteen-moment approximation to the matrix  $\alpha, \beta < 2$ . Selection of the matrix  $\alpha, \beta < 3$  corresponds to a "26-moment theory," and so on.

It may be shown\* that for any arbitrarily large values of  $\alpha = \beta$ , the dispersion picture corresponding to the "truncated" system is similar to the thirteen-moment one in the sense that the real part of roots of the dispersion equation is bounded:

$$0 > \sigma_r > \lambda_r,$$

where  $\lambda_r$  are quantities whose absolute value increases as  $r$  increases.

It follows from what has been said that the dispersion picture for large  $k$ , as given by the Navier-Stokes equations, is qualitatively more correct than the picture given by the thirteen-moment equations. The thirteen-moment equations, like the higher approximations, improve upon the Navier-Stokes picture only for small  $k$ . The general dispersion picture is given in Fig. 16.

3. The theory of eigenfunctions described above refers to Maxwellian molecules. For non-Maxwellian molecules with a finite interaction radius, we noted at the beginning of this section that the full operator  $L(\varphi)$  has a continuous spectrum of eigenvalues. However, we may transform that operator in such a way that the spectrum of the new operator turns out to be discrete.

We examine the Boltzmann equation for hard spherical molecules [see § 2.2, formula (2.10)]

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \xi_l \frac{\partial f}{\partial x_l} = \frac{d^2}{2} \int (f' f'_1 - f f_1) |\mathbf{g} \cdot \mathbf{k}| d\Omega d\xi_1 \quad (11.39)$$

or for the perturbed distribution function

$$\begin{aligned} f_0(\xi) \left( \frac{\partial \varphi}{\partial t} + \xi_l \frac{\partial \varphi}{\partial x_l} \right) &= \\ &= \frac{d^2}{2} \int |\mathbf{g} \cdot \mathbf{k}| f_0 f_{01} (\varphi'_1 + \varphi' - \varphi_1 - \varphi) d\Omega d\xi_1 = L(\varphi). \end{aligned} \quad (11.39a)$$

It was shown in § 2.8, for an arbitrary molecular interaction law, that the integral  $L(\varphi)$  may be reduced to the form

$$L(\varphi) = -k(\xi) \varphi + \int K(\xi, \xi_1) \varphi_1 d\xi_1. \quad (11.40)$$

\* L. Sirovich, Phys. Fluids, Vol. 6, No. 1 (1963).

We find specific expressions for the functions  $k(\xi)$  and  $K(\xi, \xi_1)$  for hard spherical molecules.\*

We denote the direction cosines of the unit vector  $\mathbf{k}$  by  $\alpha_{1,2,3}$ . Then

$$\mathbf{g} \cdot \mathbf{k} = \sum_{i=1}^3 \alpha_i (\xi_{ii} - \xi_i). \quad (11.41)$$

In addition, we introduce the dimensionless velocity

$$v = \sqrt{\frac{m}{2kT_0}} \xi = h_0^{1/2} \xi$$

and rewrite the operator  $L(\varphi)$  of Eq. (11.39a) in the form

$$L(\varphi) = -\frac{n_0^2 h_0 d^2}{2\pi^3} \int |w| (\varphi + \varphi_i - \varphi' - \varphi'_i) e^{-(v^2 + v_{ii}^2)} d\Omega dv_1, \quad (11.42)$$

where

$$w = (\mathbf{g} \cdot \mathbf{k}) h_0^{1/2} = R \cdot \mathbf{k} = \sum_{i=1}^3 \alpha_i (v_{ii} - v_i), \quad (R = v_1 - v).$$

It is easy to see that the function  $k(\xi)$  is equal to

$$k(\xi) = \frac{h_0^2 h_0 d^2}{2\pi^3} e^{-v^2} \int e^{-v_{ii}^2} |w| d\Omega dv_1 = \frac{n_0^2 h_0 d^2}{\pi^2} e^{-v^2} \int e^{-v_{ii}^2} R dv_1. \quad (11.43)$$

We now examine the integrals

$$G' = \int |w| \varphi' e^{-v_{ii}^2} d\Omega dv_1 \text{ and } G'_1 = \int |w| \varphi'_i e^{-v_{ii}^2} d\Omega dv_1. \quad (11.44)$$

Since

$$\xi' = \xi + \mathbf{k}(\mathbf{g} \cdot \mathbf{k}) \text{ and } \xi'_i = \xi_i - \mathbf{k}(\mathbf{g} \cdot \mathbf{k}) \quad (11.44a)$$

during a collision of hard-sphere molecules, the functions  $\varphi'$  and  $\varphi'_i$  appear in  $L$  in a fully symmetrical manner (the integration with respect to  $\Omega$  is carried out over a complete sphere) and, therefore,

$$G' = G'_1. \quad (11.45)$$

Furthermore, it is easy to verify the identity

$$G' = 3 \int_0^1 G' r^2 dr, \quad (11.46)$$

where  $r$  is a new variable. If  $r$  is the radius, then  $r^2 dr d\Omega = d\alpha_1 d\alpha_2 d\alpha_3$  is a volume element in the space  $0 \leq \alpha^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \leq 1$ .

Then (11.46) may be written in the form

$$G' = 3 \int_{0 < \alpha^2 < 1} \left| \frac{(\alpha \cdot R)}{\alpha} \right| e^{-v_{ii}^2} \varphi' dv_1 d\alpha, \quad (da = d\alpha_1 d\alpha_2 d\alpha_3), \quad (11.47)$$

where

$$\varphi' = \varphi(v') = \varphi\left(v_i + \frac{\alpha_i w}{\alpha^2}\right).$$

We replace the variables  $v_{ii}$  by the new variables

$$\lambda_i = \frac{v_{ii} - v_i}{\alpha^2}. \quad (11.48)$$

Then

$$G' = 3 \int_{0 < \alpha^2 < 1} |(\alpha \cdot \lambda)| \alpha^7 \exp \left\{ -[(v_1 + \lambda_1 \alpha^2)^2 + (v_2 + \lambda_2 \alpha^2)^2 + (v_3 + \lambda_3 \alpha^2)^2] \right\} \varphi' d\lambda d\alpha, \quad (11.49)$$

where

$$\varphi' = \varphi[v_1 + \alpha_1(\alpha \cdot \lambda), v_2 + \alpha_2(\alpha \cdot \lambda), v_3 + \alpha_3(\alpha \cdot \lambda)].$$

Finally, we shall replace the  $\alpha_i$  by the variables  $\beta_i$  according to the formulas

$$\beta_i = \alpha_i(\alpha \cdot \lambda). \quad (11.50)$$

In terms of the new variables, after some simple but tedious manipulations, we have

$$G' = 3 \int_{0 < \beta^2 < \beta \cdot \lambda} \frac{\beta^7}{(\lambda \cdot \beta)^{9/2}} \exp \left\{ - \left[ \left( v_1 + \lambda_1 \frac{\beta^2}{(\lambda \cdot \beta)} \right)^2 + \left( v_2 + \lambda_2 \frac{\beta^2}{(\lambda \cdot \beta)} \right)^2 + \left( v_3 + \lambda_3 \frac{\beta^2}{(\lambda \cdot \beta)} \right)^2 \right] \right\} \varphi(v_i + \beta_i) d\beta d\lambda. \quad (11.51)$$

\*D. Hilbert, "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen,"  
New York, 1953.

In this integral the integration with respect to  $\lambda$  may be carried out:

$$G' = 2\pi \int \frac{1}{\beta} \exp \left\{ -\frac{(\beta \cdot (\beta + \mathbf{v}))^2}{\beta^2} \right\} \varphi(v_i + \beta_i) d\beta. \quad (11.52)$$

We designate  $v_i + \beta_i = v_{1i}$ ; then finally we have

$$G' = 2\pi \int \frac{1}{R} \exp \left\{ -\frac{(v_1 \cdot R)^2}{R^2} \right\} \varphi_1 d\mathbf{v}_1. \quad (11.53)$$

Taking account of (11.42), (11.43), (11.45), and (11.53), we obtain the following expression for L:

$$\begin{aligned} L = & -\frac{n_0^2 h_0 d^2}{\pi^2} e^{-v^2} \varphi \int R e^{-v_1^2} d\mathbf{v}_1 \\ & -\frac{n_0^2 h_0 d^2}{\pi^2} e^{-v^2} \int \left\{ R e^{-v_1^2} - \frac{2}{R} e^{-(v_1 \cdot R)^2/R^2} \right\} \varphi_1 d\mathbf{v}_1. \end{aligned} \quad (11.54)$$

Thus, the linearized Boltzmann equation (11.39a) for hard-sphere molecules may be written in the form

$$\begin{aligned} h_0^{1/2} \frac{\partial \varphi}{\partial t} + v_i \frac{\partial \varphi}{\partial x_i} = & -n_0 d^2 \sqrt{\pi} \left\{ M(\mathbf{v}) e^{-v^2} \varphi(\mathbf{v}) \right. \\ & \left. + \frac{1}{\pi} \int \varphi(\mathbf{v}_1) e^{-v_1^2} \left( R - \frac{2}{R} e^{D^2} \right) d\mathbf{v}_1 \right\}, \end{aligned} \quad (11.55)$$

where

$$\left. \begin{aligned} M(\mathbf{v}) &= \frac{1}{\pi} \int R e^{-v_1^2} d\mathbf{v}_1 = 1 + \frac{\sqrt{\pi}}{2} \left( 2v + \frac{1}{v} \right) e^{v^2} \operatorname{erf} v, \\ D^2 &= v_i^2 - \frac{(v_1 \cdot R)^2}{R^2} = \frac{v_1^2 v^2 - (v_1 \cdot \mathbf{v})^2}{R^2} = \left( \frac{v v_1 \sin \theta}{R} \right)^2 \end{aligned} \right\} \quad (11.56)$$

and  $\theta$  is the angle between  $\mathbf{v}$  and  $\mathbf{v}_1$ .

It is convenient to replace the function  $\varphi$  by the new unknown function \*

$$\psi(\mathbf{v}) = e^{-v^2} \sqrt{M(\mathbf{v})} \varphi(\mathbf{v}), \quad (11.57)$$

which satisfies the equation

$$\begin{aligned} & \frac{e^{v^2}}{M(\mathbf{v})} \left( h_0^{1/2} \frac{\partial \psi}{\partial t} + v_i \frac{\partial \psi}{\partial x_i} \right) \\ & = -d^2 n_0 \sqrt{\pi} \left[ \psi(\mathbf{v}) + \frac{1}{\pi} \int \psi(\mathbf{v}_1) K(\mathbf{v}, \mathbf{v}_1) d\mathbf{v}_1 \right], \end{aligned} \quad (11.58)$$

where

$$K(\mathbf{v}, \mathbf{v}_1) = \left( R - \frac{2}{R} e^{D^2} \right) (M(\mathbf{v}) M(\mathbf{v}_1))^{-1/2}.$$

It is appropriate to seek a solution of (11.58) in the form of an expansion in terms of eigenfunctions of the integral equation

$$\psi(\mathbf{v}) = \frac{\lambda}{\pi} \int \psi(\mathbf{v}_1) K(\mathbf{v}, \mathbf{v}_1) d\mathbf{v}_1. \quad (11.59)$$

In accordance with the comments made at the beginning of this section, the operator on the right of Eq. (11.55) has a continuous spectrum, while the operator on the right side of Eq. (11.58) has a discrete spectrum.

To find the eigenvalues and the eigenfunctions of Eq. (11.59), use may be made, for example, of Galerkin's method, in which we expand the function  $\psi(\mathbf{v})$  in terms of some complete system of functions, let us say in terms of Hermite polynomials, or in terms of the eigenfunctions (11.8) of the Boltzmann integral operator for Maxwellian molecules.

A considerable number of eigenfunctions has been obtained in the references cited earlier.\*

### §3.12. Model Equations for the Linearized Boltzmann Equation

In § 2.8, from physical reasoning, we obtained an approximate model equation for the complete (not linearized) Boltzmann equation. The theory of eigenvalues described in the previous sec-

\*C. L. Pekeris, Z. Alterman, L. Finkelstein, and K. Frankowski, Phys. Fluids, Vol. 5, No. 12 (1962).

\*See the paper by Alterman et al., cited on page 232, and the paper by Pekeris et al., cited on page 244.

tion allows us to develop a regular process for constructing similar models for the linear Boltzmann equation.\*

For simplicity, let us examine the linear one-dimensional Boltzmann equation written in dimensionless variables (11.10) as in the previous section:

$$\frac{\partial \varphi}{\partial t} + \xi_1 \frac{\partial \varphi}{\partial x_1} = L(\varphi) = \sum_{r,l}^{\infty} \lambda_{rl} a_{rl} \psi_{rl}, \quad (\varphi = \sum_{r,l} a_{rl} \psi_{rl}). \quad (12.1)$$

The linear integral operator on the right side has been represented for Maxwellian molecules in the form of a series of eigenfunctions according to formula (11.21).

In what follows, it is more convenient to drop the summation over the two subscripts, and go to a summation over a single subscript (see the paper by Sirovich and Thurber):

$$\varphi = \sum_{r,l}^{\infty} a_{rl} \psi_{rl} = \sum_l a_l \psi_l, \quad (12.2)$$

where

$$r = r(l), \quad l = l(r), \quad \psi_{r(l)l(l)} = \psi_l, \quad a_l = \int \omega \varphi \psi_l d\xi. \quad (12.3)$$

The choice of the specific form of relations (12.3) determines the sequence of terms in the theories (12.2). Then for Maxwellian molecules,

$$L(\varphi) = \sum_{i=0}^{\infty} \lambda_i^* a_i \psi_i, \quad \lambda_i^* = \lambda_{r(i)l(i)}. \quad (12.4)$$

For arbitrary molecules, the distribution function may also be represented in the form

$$\varphi = \sum_l a_l \psi_l, \quad (a_l = \int \omega \varphi \psi_l d\xi), \quad (12.5)$$

where the  $\psi_i$  are eigenfunctions of the Maxwell operator  $L(A)$ . However, in that case,

\*E. P. Gross and E. A. Jackson, Phys. Fluids, Vol. 2, No. 4 (1959); L. Sirovich and J. K. Thurber, J. Acoust. Soc. Am., Vol. 37, No. 2 (1965).

$$L(\varphi) = \sum_{i,j=0}^{\infty} \lambda_{ij}^* a_i \psi_j, \quad (\lambda_{ij}^* = \int \omega \psi_j L(\psi_i) d\xi). \quad (12.6)$$

For Maxwellian molecules,

$$\lambda_{ij}^* = 0 \quad \text{for } i \neq j.$$

One way of constructing the models is to replace  $L$  approximately by the expression

$$L(\varphi) \approx L^N = \sum_{i,j=0}^N \lambda_{ij}^* a_i \psi_j + C \sum_{i=N+1}^{\infty} a_i \psi_i, \quad C = \text{const.} \quad (12.7)$$

Adding the following sum to the right side of (12.7),

$$C \sum_{i=0}^N a_i \psi_i = C \sum_{i,j=0}^N a_i \psi_i \delta_{ij},$$

we reduce the expansion (12.7) to the form

$$L^N = \sum_{i,j=0}^N (\lambda_{ij}^* - \delta_{ij} C) a_i \psi_j + C \varphi. \quad (12.8)$$

Then the model equations may be written in the form

$$\frac{\partial \varphi}{\partial t} + \xi_1 \frac{\partial \varphi}{\partial x_1} = C \varphi + \sum_{i,j=0}^N (\lambda_{ij}^* - \delta_{ij} C) a_i \psi_j. \quad (12.9)$$

In this way we may construct an infinite multitude of models by suitable choice of the constants  $C$  and  $N$  and of the corresponding sequence of terms in the series (12.2), i.e., by choice of the relations  $r(i)$  and  $l(i)$ .

Let the eigenfunctions  $\psi_{rl}$  be polynomials of degree  $2r + l$ . For example, we may choose  $r(i)$  and  $l(i)$  in such a way that in the series (12.2) the terms are arranged in increasing powers of the polynomials, while all the terms with polynomials of the single power  $2r + l$  are arranged in increasing values of the subscript  $l$ .

With that arrangement of terms, the series (12.2) may be written in the form

$$\varphi = \sum_{i=0}^{\infty} a_i \psi_i = n_1 + \xi_1 u_1 + \left( \frac{\xi^2}{2} - \frac{3}{2} \right) T_1 + \frac{3}{2} \left( \xi_1^2 - \frac{\xi^2}{3} \right) p_{11} + \left( 1 - \frac{\xi^2}{5} \right) \xi_1 s_1 + \dots, \quad (12.10)$$

where  $a_i$  and  $\psi_i$  have been replaced according to (11.16) and (11.19). If we put  $N = 3$  in this sequence of terms in Eq. (12.9), then for Maxwellian molecules we obtain the equation

$$\frac{\partial \varphi}{\partial t} + \xi_1 \frac{\partial \varphi}{\partial x_1} = C \left[ \varphi - n_1 - \xi_1 u_1 - \left( \frac{1}{2} \xi^2 - \frac{3}{2} \right) T_1 \right], \quad (12.11)$$

since, for these molecules,  $\lambda_{00} = \lambda_{01} = \lambda_{10} = 0$ .

It is easy to see that Eq. (12.11) is a linearized variation of the model equation (8.22) of Chapter II.

It may be seen from Table 4 of the previous section that this model is obtained by replacing by  $C$  all the  $\lambda_{ij}$  which appear in the table. When we do this the determinant (11.30), which corresponds to the Euler equations, does not change. In the Navier-Stokes determinant (11.31) we have put  $\lambda_{02} = \lambda_{11} = C$ . As we saw in the previous section,  $\lambda_{02}$  and  $\lambda_{11}$  are inversely proportional to the viscosity and the thermal conductivity, respectively. Since  $\lambda_{11}/\lambda_{02} = \frac{2}{3}$  (see Table 4), then, putting  $\lambda_{02} = \lambda_{11} = C$ , we must obtain, in the Navier-Stokes equations, the ratio of the viscosity and thermal conductivity, distorted by the factor  $\frac{2}{3}$ , i.e., the Prandtl number  $Pr = 1$  instead of  $Pr = 2/3$ , as must be the case for a monatomic gas. We obtained that result earlier in the derivation of the Navier-Stokes equations from the nonlinear model equation in § 3.6.

If we put  $N = 5$  in Eq. (12.9), we obtain a model equation in which the undistorted thirteen-moment determinant (11.32) remains:

$$\begin{aligned} \frac{\partial \varphi}{\partial t} + \xi_1 \frac{\partial \varphi}{\partial x_1} &= C \left[ \varphi - n_1 - \xi_1 u_1 - \left( \frac{1}{2} \xi^2 - \frac{3}{2} \right) T_1 \right] \\ &+ \frac{3}{2} (\lambda_{02} - C) \left( \xi_1^2 - \frac{\xi^2}{3} \right) p_{11} + (\lambda_{11} - C) \left( 1 - \frac{\xi^2}{5} \right) \xi_1 s_1. \end{aligned} \quad (12.12)$$

From that equation we may obtain the exact Euler, Navier-Stokes, Burnett, and thirteen-moment equations.

Equation (12.12) is sometimes called the tri-relaxation equation, since it contains the three relaxation times  $\lambda_{02}^{-1}$ ,  $\lambda_{11}^{-1}$ , and  $C^{-1}$ , in contrast to Eq. (12.11), which contains only the one relaxation time  $C^{-1}$ .

Since the one-relaxation equation (12.11) has been widely used, because of its relative simplicity, it is interesting to examine its dispersion pattern.\* To be definite we put  $C = \lambda_{02}$ .

We seek a solution of (12.11) in the form

$$\left. \begin{aligned} \varphi(t, x_1, \xi) &= \varphi(\xi) e^{\sigma t - lkx_1}, & n_1 &= n_1 e^{\sigma t - lkx_1}, \\ u_1 &= u_1 e^{\sigma t - lkx_1}, & T_1 &= T_1 e^{\sigma t - lkx_1}, \end{aligned} \right\} \quad (12.13)$$

where the quantities  $n_1$ ,  $u_1$ , and  $T_1$  on the right are constants, equal, respectively, to

$$n_1 = \int \omega \varphi(\xi) d\xi, \quad u_1 = \int \omega \xi_1 \varphi(\xi) d\xi, \quad T_1 = \int \left( \frac{\xi^2}{3} - 1 \right) \omega \varphi(\xi) d\xi. \quad (12.14)$$

Substituting the solution (12.13) into Eq. (12.11), we have

$$(\sigma - lk\xi_1 - \lambda_{02}) \varphi = -\lambda_{02} \left[ n_1 + \xi_1 u_1 + \left( \frac{\xi^2}{2} - \frac{3}{2} \right) T_1 \right], \quad (12.15)$$

whence

$$\varphi = -\lambda_{02} \frac{n_1 + \xi_1 u_1 + \left( \frac{\xi^2}{2} - \frac{3}{2} \right) T_1}{\sigma - lk\xi_1 - \lambda_{02}}. \quad (12.16)$$

Multiplying this equation successively by  $\omega$ ,  $\xi_1 \omega$ , and  $(\frac{1}{3} \xi^2 - 1)$ , and integrating with respect to  $\xi$ , we obtain a system of three homogeneous equations for  $n_1$ ,  $u_1$ , and  $T_1$ ; by equating their determinant to zero we obtain the dispersion relation of Eq. (12.11). Without dwelling on the analysis of this very complicated equation,† we point out one result of great interest to us which follows directly from Eq. (12.16). It may be seen from (12.16) that the real part of  $\sigma$  is bounded as  $k \rightarrow \infty$  (if this were not true,  $\varphi \rightarrow 0$  at all times, including  $t = 0$ ). Moreover, the same conclusion may be reached

\*See L. Sirovich and J. K. Thurber, "Rarefied Gas Dynamics," Third Symposium, Academic Press, New York, 1963.

†This analysis was presented in the paper by Sirovich and Thurber just cited.

for a model equation formed with any finite  $N$ . If we derive the Navier-Stokes equations from Eq. (12.11), they differ from the Navier-Stokes equations obtained from the Boltzmann equation only in that we must put  $\lambda_{11} = \lambda_{02}$  in the latter. Therefore, the dispersion pattern for the Navier-Stokes equations obtained from the model equation will be qualitatively the same as for the Navier-Stokes equations obtained from the exact equation [see formulas (11.33), (11.34), (11.37) and Fig. 14 of the previous section].

We saw in the previous section that as  $k \rightarrow \infty$  the Navier-Stokes equations give a dispersion pattern which is qualitatively more plausible than the thirteen-moment equations and the higher approximations. It seems that the Navier-Stokes equations reflect to some extent the behavior of the complete Boltzmann equation as  $k \rightarrow \infty$ . However, we see now that as  $k \rightarrow \infty$  the Navier-Stokes approximation of the model equation gives a qualitatively different picture from that given by the model equation itself. Therefore, the behavior of the solutions of the Navier-Stokes equations as  $k \rightarrow \infty$  is fortuitous.

Even the model equation distorts the dispersion pattern of the Boltzmann equation as  $k \rightarrow \infty$ . However, it is true that the model equation of order  $N$  is a much better approximation to the solution than the corresponding system of moment equations.

In fact, let us return to Table 4. As has already been noted, the Euler equations correspond to the determinant  $\alpha, \beta < 1$ , the thirteen-moment equations to the determinant  $\alpha, \beta < 2$ , and so on. Therefore, the moment equations take into account only the elements of the corresponding determinants, without taking account at all of the remaining infinite part of the table. The model equations of the corresponding order are formed by replacing  $\lambda_{ij}$  in the table by  $C$ , beginning with a certain value. Therefore, the model equation takes account correctly of the elements of the appropriate determinant, and approximately (with an accuracy worse as  $i, j$  become larger) of the remaining part of the table.

Thus, for  $N = 3$ , the Euler part of the table is taken into account exactly and the remaining part approximately. The approximate calculation based on the remaining part of the table permits us, although only qualitatively, to account for dissipative processes, and to examine free molecular flows by means of the one-relaxation equation, while the Euler equations do not permit this.

We may therefore expect that, for the same value of  $N$ , by means of the model equations, we may obtain a better qualitative description of a phenomenon for a wide range of wave numbers (Knudsen numbers) than by means of the corresponding moment equations. This is confirmed by our investigation of sound waves (see § 4.5).

The accuracy of the model equation in each specific problem may be increased by appropriate choice of the relations  $r(i)$  and  $l(i)$ , which determine the sequence of terms of the series (12.2) and (12.4).

### §3.13. The Method of Discrete Velocities

We examine the Boltzmann equation in the absence of external forces:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} = \int (f' f'_1 - f f_1) g b db de d\xi_1 = J(\xi). \quad (13.1)$$

Let us choose a set of values of the velocity  $\xi^\nu$ , where  $\nu = 1, \dots, N$ . By means of some approximation formula we define the distribution function in terms of its values corresponding to the velocities  $\xi^\nu$ , i.e., we put

$$f(t, x, \xi) = F(\xi, f^{(1)}, \dots, f^{(N)}), \quad f^{(\nu)} = f(t, x, \xi^\nu). \quad (13.2)$$

If we substitute this approximation into the collision integral, the latter also depends on  $f^\nu$  and  $\xi$ :

$$J(\xi) = J(\xi, f^{(1)}, \dots, f^{(N)}). \quad (13.3)$$

The velocity  $\xi$  enters as a parameter in the differential operator of Eq. (13.1). Therefore, for the chosen set of velocities  $\xi^\nu$ , we may write the equations

$$\frac{\partial f^{(\nu)}}{\partial t} + \xi_i^{(\nu)} \frac{\partial f^{(\nu)}}{\partial x_i} = J(\xi^\nu, f^{(1)}, \dots, f^{(\nu)}) \quad (\nu = 1, 2, \dots, N). \quad (13.4)$$

Thus, the integro-differential Boltzmann equation has been replaced by a system of simultaneous nonlinear differential equations of the first order for the  $N$ -functions  $f^{(\nu)}$ .

The general boundary condition (9.6) of Chapter II may also be replaced for  $N_1(t, x) \leq N$ , by the conditions:

$$f_r(\xi^v) = \int K(\xi_i, \xi^v) F(\xi_i, f^{(1)}, \dots, f^{(N)}) d\xi_i, \quad (13.5)$$

where  $v$  takes values corresponding to those velocities of the set  $\xi^v$ , which are directed into the flow at any given point of the boundary.

For example, we may choose Lagrange or Chebyshev polynomials as the approximation functions (13.2).

As in the method of moments, instead of looking for a distribution function which depends on the seven variables,  $t$ ,  $\mathbf{x}$ , and  $\xi$ , the problem has been reduced to finding a system of functions of the four variables  $t$  and  $\mathbf{x}$ . However, the equations obtained by the method of discrete coordinates always possess a simple linear differential operator, while in the method of moments, as a rule, quasilinear equations are obtained. In the method of discrete coordinates, no difficulties arise in specifying boundary conditions for the equations obtained (compare §5 of the present chapter). The right sides of the moment equations are often simpler (especially for Maxwellian molecules) than in the method of discrete velocities. The same approximation functions may be used, in principle, in both methods. Let the distribution function be represented in terms of moments by the approximation

$$f(t, \mathbf{x}, \xi) = F(\xi, M_i) \quad (i = 1, 2, \dots, N).$$

Assigning the  $N$  chosen values of  $\xi$ , we obtain  $N$  equations, from which we may choose  $N$  moments  $M_i$  through the  $N$  eigenfunctions of the distribution function  $f^{(v)}$ .

The method of discrete velocities may also be given a somewhat different form.\* We may consider  $N$  groups of molecules, in each of which all the molecules ( $n^v$  per unit volume) have a single velocity  $\xi^v$ . In collisions between molecules of group  $v$  and molecules of group  $\mu$  we allow only those discrete values of the impact parameter  $b(v, \mu)$  and those discrete azimuthal angles  $\epsilon(v, \mu)$  for which the molecules assume velocities within the set  $\xi^v$  as a result of collisions.

\* M. Krook, *Astrophys. J.*, 128:485 (1955); S. Chandrasekhar, "Radiative Transfer," Dover, New York, 1953.

In that case, the Boltzmann equation (13.4) is replaced by a system of differential equations of the form

$$\frac{\partial n^v}{\partial t} + \xi_l \frac{\partial n^v}{\partial x_l} = \sum_{k, l} n^k n^l g_{k, l} \sigma(\xi^k, \xi^l | \xi^v) - \sum_{\mu} n^v n^{\mu} g_{v\mu} \sigma(g_{v\mu}), \quad (13.6)$$

where  $\sigma(\xi^k, \xi^l | \xi^v)$  is the effective collision cross section of the  $\xi^k$  and  $\xi^l$  molecules, in which one of the molecules assumes the velocity  $\xi_v$ , and  $\sigma(g_{kl}) = \sum_v \sigma(\xi^k, \xi^l | \xi^v)$  is the total interaction cross section.

In that case, the boundary condition (13.5) takes the form

$$n_r^v = \sum_{\mu} K(\xi_r^v, \xi^{\mu}) n^{\mu}, \quad (13.7)$$

where the summation is performed over all the velocities  $\xi^{\mu}$ , directed toward the boundary at a given point; the functions  $K$  satisfy the normalization condition

$$\sum_v K(\xi_r^v, \xi^{\mu}) \frac{(\xi_r^v \cdot n)}{(\xi^{\mu} \cdot n)} = 1.$$

It is clear that the second form of the method is suitable for computer calculations with a large number of discrete velocities. With a good approximating function, the method of moments and the first form of the method of discrete velocities may achieve good accuracy with a small number of moments or discrete velocities.

The method of moments is evidently preferable for small Knudsen numbers, and the method of discrete coordinates for large Knudsen numbers. However, it is not yet possible to select with confidence the classes of problems for which these or other methods are preferable. Only the simplest problems in a very crude formulation have yet been solved by the method of discrete velocities.

For example, in the Couette flow problem at small Mach number\* it was assumed that the molecules can move only in eight

\* J. E. Broadwell, *J. Fluid Mech.*, 19:401 (1964). The linearized Couette flow has been

directions with a velocity of identical magnitude. In the shock-wave structure problem\* it was assumed that there are six allowable directions with identical velocities. In essence, the crude methods of elementary kinetic theory (see §1.5 and §6.6) are the simplest examples of the application of the method of discrete coordinates. It often turns out that a very small number of discrete velocities is sufficient to obtain crude results. However, to have confidence in the accuracy of a given calculation it is necessary to solve a very large number of simultaneous equations.

Indeed, suppose that we solve three-dimensional steady problems. Even if we take ten points (which is obviously too few) for each velocity coordinate, the problem reduces to the solution of thousands of simultaneous partial differential equations.

For one-dimensional steady problems the equations reduce to ordinary equations.

### §3.14. Integral Methods

Methods which use an integral form of the Boltzmann equation are called integral methods. As was noted in §2.7, we may construct a great many integral equations equivalent to the Boltzmann equation. However, two of these (see §2.7) have been most widely used:

$$f(t, \mathbf{x}, \xi) = f(t_0, \mathbf{x} - \xi(t-t_0), \xi) + \int_{t_0}^t J(s, \mathbf{x} - \xi(t-s), \xi) ds \quad (14.1)$$

and

$$\begin{aligned} f(t, \mathbf{x}, \xi) &= f(t_0, \mathbf{x} - \xi(t-t_0), \xi) \exp \left\{ - \int_{t_0}^t J_2(s, \mathbf{x} - \xi(t-s), \xi) ds \right\} \\ &+ \int_{t_0}^t J_1(\tau, \mathbf{x} - \xi(t-\tau), \xi) \exp \left\{ - \int_{\tau}^t J_2(s, \mathbf{x} - \xi(t-s), \xi) ds \right\} d\tau, \quad (14.2) \end{aligned}$$

examined in a more accurate formulation by means of the method of discrete coordinates in the paper: B. Hamel and M. Wachman, "Rarefied Gas Dynamics," Fourth Symposium, Academic Press, 1965.

\*J. E. Broadwell, Phys. Fluids, Vol. 7, No. 8 (1964).

where

$$J = J_1 - f J_2,$$

$$J_1(t, \mathbf{x}, \xi) = \int f' f'_1 g b db de d\xi_1,$$

$$J_2(t, \mathbf{x}, \xi) = \int f_1 g b db de d\xi_1.$$

The integrals  $J_1$  and  $J_2$  converge only for molecules with a finite interaction radius. Therefore, the integral equation (14.2) is applicable only to such molecules.

The natural method of seeking solutions of integral equations is the method of iteration. This method has been applied to both forms of the integral equations in order to prove existence of solutions of the Boltzmann equation when a distribution function is assigned at time zero.\* As has already been noted, the convergence of the method for a finite time interval has been proved only for the spatially homogeneous case and for molecules with a finite interaction radius (for spheres by Carleman, and for pseudo-Maxwellian molecules by Morgenstern). If the initial distribution function depends on  $\mathbf{x}$ , then the convergence of the successive approximations may be proved only for a small time interval (Grad).

The integral method of iteration has also been used to solve a number of nearly free-molecular flow problems (see §§4.2 and 6.5).

It is easy to see that the solution of steady and unsteady problems by means of Eqs. (14.1) and (14.2) is completely identical. Therefore, instead of examining successive approximations, we may examine the unsteady process of flow establishment.†

Let a solution be desired for a certain steady problem. At time  $t = 0$  we assign some distribution function  $f(0, \mathbf{x}, \xi)$ , which may not satisfy the boundary conditions in the general case. It is clear from physical considerations, that the distribution function must tend to some steady solution of the Boltzmann equation as time goes on; that solution satisfies the assigned boundary conditions (however, this has not been proved rigorously, any more than the uniqueness of such a process of flow establishment).

\*See the papers by Carleman, Morgenstern, and Grad, cited in §2.9.  
†M. N. Kogan, Zh. Prikl. Mekhaniki i Tekhn. Fiz., No. 3 (1963).

We choose a sufficiently small time interval  $\Delta t$ , during which the collision integral changes only by a quantity of order  $\Delta t$ . Then, neglecting quantities of order  $(\Delta t)^2$ , Eq. (14.1) may be written in the difference form:

$$f(t_n, \mathbf{x}, \xi) = f(t_{n-1}, \mathbf{x} - \xi \Delta t, \xi) + J_{n-1}(t_{n-1}, \mathbf{x}, \xi) \Delta t. \quad (14.3)$$

As in the method of discrete velocities (see the previous section), a calculation may be made for a set of values  $\xi^\nu$  ( $\nu = 1, \dots, m$ ). The chosen  $m$  directions may be regarded as characteristics and a solution may be constructed in a manner similar to the solution of hyperbolic differential equations.

Two processes lead to a change in the collision integral. On the one hand, molecules arrive at a given point in space from other regions of the flow. If  $L$  is a characteristic flow dimension, and  $\xi$  is the characteristic velocity of the molecules, the characteristic time for this process is  $\Theta_1 = L/\xi$ . On the other hand, even if the distribution function were homogeneous throughout the space, it would change as a result of collisions of the molecules. The characteristic time for this process is the relaxation time, or the time between collisions of the molecules,  $\Theta_2 = \tau_p = \lambda/\xi$ , where  $\lambda$  is the characteristic mean free path of the molecules. Therefore,  $\Delta t$  must be less than the smaller of the times  $\Theta_1$  and  $\Theta_2$ , and in practice we use the calculation process defined by formula (14.3) only when the Knudsen numbers are not too small. The process (14.3) is analogous to the very simple Euler method of numerical integration of ordinary differential equations. Using more complicated approximations for the collision integral, it is easy to construct analogs of the more accurate methods, for instance, of the Runge-Kutta type.

In carrying out these types of numerical processes, one encounters two difficulties in practice. First, the calculation of the collision integral is very complex, since we must not only carry out the quadratures, but also calculate incidentally the outcome of molecular collisions. Second, and this is the basic difficulty, at each step we must store the distribution function, which depends, in general, on seven independent variables (the number  $m$  of chosen velocities must be very large to attain sufficient accuracy).

A reduction of the required computer storage may be achieved by introducing an effective approximating function

$$f(t, \mathbf{x}, \xi) = F(\xi, A_1, \dots, A_N) = F_1(\xi, M_1, \dots, M_N). \quad (14.4)$$

As in the method of moments (see §3.2), we multiply Eq. (14.3) by a function of the velocities  $\varphi(\xi)$ , and integrate with respect to  $\xi$ ; we obtain equations of type

$$M_i(t_n, \mathbf{x}) = \int f_{n-1} \varphi(\xi) d\xi + I_{\varphi, n-1} \cdot \Delta t. \quad (14.5)$$

For Maxwellian molecules, we showed in §3.3 that the integrals  $I$  may be expressed comparatively simply in terms of the moments. By substituting for  $f_{n-1}$  in terms of moments by means of the approximating function (14.4), and applying some suitable quadrature formula, we finally find the dependence of the moments at the  $n$ -th step at the point  $\mathbf{x}$  on the moments at the  $(n-1)$ -st step at the same point and at a set of neighboring points.

It is easy to see that if we write

$$f_{n-1}(t_{n-1}, \mathbf{x} - \xi \Delta t, \xi) = f(t_{n-1}, \mathbf{x}, \xi) - \xi_i \frac{\partial f_{n-1}}{\partial x_i} \Delta t,$$

we go over to the differential-moment method of §3.2. Indeed, a transition from the integral-moment method to the differential method can always be found. In different situations, any of several approaches may prove preferable. In the integral approach we follow the evolution of the distribution function along the trajectory of the molecules and, in constructing the computational scheme, it is easy to take correct account of the physics of the phenomena (regions of influence, breakdown of the distribution function, etc.). In the differential approach, the differential equations obtained are different for different approximating functions and for different functions  $\varphi(\xi)$  used to construct the moment equations. To construct a correct computational scheme it is necessary to investigate the properties (to find the characteristics) of complex systems of differential equations for each approximating function.

In the integral approach the boundary conditions enter in a natural fashion [the point  $(t_{n-1}, \mathbf{x} - \xi \Delta t, \xi)$  may lie on the boundary]. In §3.5 we mentioned the difficulties connected with the formulation of boundary value problems for differential moment equations. Since we may always find a transition from the integral approach to the differential one, we can establish the correct boundary conditions for the differential moment equations by means of the integral methods.

The use of the flow establishment method to solve steady problems is convenient, but not inevitable. By means of Eqs. (14.3) or (14.5), we may relate the values of the distribution functions or of the moments at the nodes of some mesh. The flow establishment method is in fact the simplest explicit computational scheme for solving the complex system of algebraic equations obtained. The convergence of the method must be established in each specific case.

The approaches described may also be applied to Eq. (14.2).\*

### §3.15. Monte Carlo Methods

Monte Carlo methods, applied to problems of kinetic theory, are defined as experimental methods of investigation, in which a mathematical simulation of the phenomena under examination is performed on high-speed computers in lieu of carrying out an actual physical experiment. The real molecules are replaced by their statistical models, and the motion of one or several of the chosen particles is traced by the computers. The appropriate mathematical experiment may be set up in a multitude of ways. We examine two types of experiments below.

1. One possible way of applying the Monte Carlo method is as follows.

Let the problem be, for example, to find the distribution function of some steady flow. Let the boundary of the region of the flow under study consist in part of solid surfaces, and in part of surfaces through which the gas may flow in or out. On the solid surfaces there is a given probability that a particle of velocity  $\xi_i$  will reflect with velocity  $\xi_r$  (§§ 2.9, 2.10). On the remaining parts of the boundary we regard the distribution function as being assigned for molecules whose velocities are directed into the domain of the flow.

Let the distribution function be given at the initial moment, let us say  $t = 0$ . We are required to follow the change of the distribution function during the establishment of steady flow, assuming, of course, that such a limiting state exists with the given bound-

\*R. G. Barantsev, Doklady Akad. Nauk SSSR, Vol. 151, No. 5 (1963); B. V. Kuksenko, Doklady Akad. Nauk SSSR, Vol. 151, No. 5 (1963).

ary and initial conditions. However, it is not possible to follow simultaneously the motion of a large number of molecules of the gas (of the order of Avogadro's number). Instead, we choose a certain particle which we call the test particle, and we follow its motion.

We divide the phase space into cells, whose number is a compromise between the required accuracy and the available memory storage of the computer. Each cell may be labeled by six numbers  $(\nu_1, \nu_2, \nu_3, \mu_1, \mu_2, \mu_3)$ . For example, the first three numbers determine the coordinates of the center of the cell in physical space, while the last three numbers define its center in velocity space.

If the distribution function is known at the point  $\mathbf{x} = (\nu_1, \nu_2, \nu_3)$ , it is easy to find the number of particles  $N_{\nu_1 \nu_2 \nu_3 \mu_1 \mu_2 \mu_3} \equiv N_{\nu_i \mu_i}$  in the cell

$$\nu_i - 1/2 \Delta x_i \leq x_i \leq \nu_i + 1/2 \Delta x_i,$$

which possess velocities in the range

$$\mu_i - 1/2 \Delta \xi_i \leq \xi_i \leq \mu_i + 1/2 \Delta \xi_i,$$

where  $\Delta x_i$  and  $\Delta \xi_i$  are the lengths of the edges of the cells in the physical and velocity spaces, respectively. If the distribution function  $f(t, \mathbf{x}, \xi)$  is assumed constant within the cells, it is evident that

$$N_{\nu_i \mu_i} = f(t, \nu_i, \mu_i) \Delta x \Delta \xi, \quad (\Delta A = \Delta A_1 \Delta A_2 \Delta A_3). \quad (15.1)$$

The assignment of the numbers  $N$  in all the cells of phase space is equivalent to an approximate assignment of the distribution function. We consider all the numbers  $N_{\nu_i \mu_i}$  given at time zero.

Now we choose the test particle in a random manner. To do this we must draw six random numbers with probability density proportional to the distribution function.\* In other words, the den-

Finally, standard programs give random numbers with a uniform distribution law in the range  $0 \leq \xi \leq 1$ . Methods of obtaining random quantities with a given distribution have been described in handbooks on the Monte Carlo method (see, for example, P. Buslenko and Yu. A. Shreider, "Methods of Statistical Testing" [in Russian], Nauk. i Tekhn. literatury, Moscow, 1961).

sity of the distribution of random numbers must be such that the number of points falling in each of the cells of phase space as a result of a sufficiently large number of draws, is proportional to the numbers  $N$  assigned to them.

We follow the motion of the test particle amid the particles with the given distribution (the field particles). We assume that the characteristic dimension of cells in physical space  $\Delta x$  is much smaller than the mean free path of the molecules. In its motion, the test molecule traverses some cells without collisions, while it experiences collisions in other cells, thus changing its velocity. When the test particle passes from one physical cell to another, it also passes into another cell of phase space, corresponding to the same velocity. When the test molecule undergoes a collision, it passes to another cell of phase space, corresponding to the same cell of physical space, but to a different velocity.

A molecule entering some physical cell with velocity  $(\mu_1, \mu_2, \mu_3)$  has a probability of traveling a distance  $L$  without collision equal to (see §1.4)

$$W_L = \exp \left\{ -\frac{L}{\lambda_{v_i \mu_i}} \right\}, \quad (15.2)$$

where  $\lambda_{v_i \mu_i}$  is the free path of the test molecule with velocity  $\mu$  relative to all the other groups of molecules existing in the cell  $(v_1, v_2, v_3)$ ,

$$\lambda_{v_1 v_2 v_3 \mu_1 \mu_2 \mu_3} = \frac{\sqrt{\mu_1^2 + \mu_2^2 + \mu_3^2}}{\sum_{\mu'_1 \mu'_2 \mu'_3} g_{\mu_1 \mu_2 \mu_3 \mu'_1 \mu'_2 \mu'_3} \sigma_{\mu_1 \mu_2 \mu_3 \mu'_1 \mu'_2 \mu'_3} N_{v_1 v_2 v_3 \mu'_1 \mu'_2 \mu'_3}}. \quad (15.3)$$

When a random quantity  $L$  with probability density (15.2) is drawn, two outcomes are possible. If it turns out that  $L/\sqrt{\mu_1^2 + \mu_2^2 + \mu_3^2}$  is greater than the residence time  $t_{v_i \mu_i}$  of the molecule in the cell being examined, the molecule passes to a neighboring cell without change of velocity, and the probability of traveling a path length  $L$  is drawn afresh. But if it turns out that  $L/\sqrt{\mu_1^2 + \mu_2^2 + \mu_3^2} < t_{v_i \mu_i}$ , then a collision occurs in the cell under study. (Molecules with a given velocity traversing a cell at different locations have a different time of flight. We therefore understand  $t_{v_i \mu_i}$  to be the mean time.)

In order to determine the velocity of the molecule  $\mu_j$ , with which a collision has occurred, we must draw a random number with probability density

$$W_{v_j \mu_j} = g_{\mu_i \mu_j} \sigma_{\mu_i \mu_j} N_{v_i \mu_i}. \quad (15.4)$$

Then two random numbers are drawn which determine the collision parameters (the impact parameter and the collision plane). When all those numbers have been chosen, we calculate the result of the collision according to the laws of mechanics.

The occupation numbers of the cells of phase space  $N_{v_i \mu_i}$  are proportional to the probability of finding molecules with velocity  $(\mu_1, \mu_2, \mu_3)$  in the range  $\Delta \xi_i$  in the corresponding cell  $(v_1, v_2, v_3)$  of physical space. On the other hand, it is clear that the probability of finding molecules with a given velocity in a given cell of physical space is proportional to the number of arrivals of molecules with this velocity at the cell and to the residence time of a molecule in the cell. Therefore, if by following the test molecule we sum the time spent by the test particle in the corresponding physical cell with the appropriate velocity, the ratio of this time to the total test time determines a certain distribution function.

The question naturally arises: for how long must we follow the chosen test molecule?

It may be seen from what has been said above that the statistical model of the paths and of the collisions in the method under examination is exactly the same as that used in deriving the Boltzmann equation. Therefore, we may expect that if the given distribution function of the field particles were a solution of the Boltzmann equation for our problem, then, by observing the test molecule for a sufficient time, and remembering the time that it spends in the cells of phase space, we should obtain the same distribution function in the limit.

In the problem examined above, the given initial distribution function is not a solution of the problem. The actual distribution function begins to be built up. It is therefore natural to limit the time during which the chosen test molecule is followed to a time in which the distribution function changes only slightly. Therefore, for simulation of the physical process, the tracking time  $\Delta t$  must

satisfy the inequality

$$\frac{\Delta x}{\xi} \ll \Delta t < \min\left(\frac{\lambda}{\xi}, \frac{L}{\xi}\right),$$

where  $\xi$  is the characteristic velocity of the molecules,  $L$  is the characteristic dimension of the flow, and  $\lambda$  is the mean free path. When a tracking time  $\Delta t$  has elapsed, we must choose a new test molecule by the same method and repeat the above process.

However, test calculations (see, for example, §§4.2 and 4.4) show that condition (15.5) is unnecessarily rigorous, and that the process converges even for comparatively large tracking time. It is clear that the tracking time may be increased, if the initial distribution function is close to the desired one.

We call the process of tracking a single test molecule a cycle. The time spent by each test particle in the cells is summed. When the ratio of this time to the total test time (the time for all the cycles) is established in all the cells, we may take these new established quantities as the new distribution function and repeat the whole process again. The calculation ends when the distribution function obtained does not differ, up to a specified accuracy, from the distribution function of the preceding approximation.

The rate of convergence of the process depends, to a considerable extent, on how successfully the initial distribution function has been chosen.

So far we have not considered the possibility of a test particle hitting the boundary of a region. If the test particle hits the boundary, we stop tracking that particle and a new test particle which starts from the boundary, is chosen. If the molecule strikes a solid wall, a new molecule is chosen at the point of impact with a probability density corresponding to the law of re-emission.

If the test molecule is incident on a permeable part of the boundary, then a draw is made (with a probability density corresponding to the known distribution function of particles entering the volume), not only of the velocity of an incoming particle, but also of the point of entrance. The total number of incoming and outgoing particles is the same in this procedure. But at any point

the boundary the flux of entering and leaving particles is generally not the same.

The method described above has been designed by analogy (15.5) processes occurring in a real gas. It is based on the same statistical hypothesis on which the Boltzmann equation rests. However, a strict theory of the method, based on a systematic examination of the Markov processes occurring here, has not yet been worked out. In the examples of the method presently available, the plausibility of the results obtained (see §§4.2, 4.4, and 6) justifies the chosen formulation of the mathematical experiments. The convergence of the method for each problem has been verified during the calculations.

The results of calculations according to the Monte Carlo method are averages of some large number of random quantities. Like any mean statistical quantities, the results of the Monte Carlo method are subject to fluctuations which increase as the number of averaged quantities decreases. The accuracy of the method increases in inverse proportion to the square root of the number of rays. Therefore, to obtain great accuracy an amount of computation unacceptable in practice may be required. The results of the Monte Carlo method should be regarded as the results of an experiment, always subject to a certain scatter due to errors of measurement.

One of the main obstacles to widespread use of the method is the small operating memory of contemporary computers. In fact, it is necessary to store both the desired distribution function in each phase cell and the distribution function of the previous approximation. In addition, it is convenient to calculate the quantities  $t_{v,\mu_i}$  and  $\lambda_{v,\mu_i}$  in all the cells beforehand for a given approximation (to avoid calculating them afresh every time a particle falls into the cell). Even if we take only ten points in each position and velocity coordinate, the number of cells for a three-dimensional problem turns out to be of order  $10^6$ . In practice, during calculations the number of stored numbers is larger by a factor of 10 or 100. Therefore, at present, calculations have been only for one-dimensional problems, with one position and two velocity coordinates.

The required storage may be reduced substantially for pseudo-Maxwellian molecules or, more accurately, for molecules

whose collision cross sections vary in inverse proportion to the relative velocity of the colliding molecules.\*

In fact, since  $\sigma = \sigma_0 g_0 / g$ , in this case, according to (15.3), the mean free path is determined by the expression

$$\lambda_{v_1 v_2 v_3 \mu_1 \mu_2 \mu_3} = \frac{\sqrt{\mu_1^2 + \mu_2^2 + \mu_3^2}}{g_0 \sigma_0 \sum_{\mu'_1 \mu'_2 \mu'_3} N_{v_1 v_2 v_3 \mu'_1 \mu'_2 \mu'_3}} = \frac{\mu}{g_0 \sigma_0 N_{v_1 v_2 v_3}}, \quad (15.3a)$$

and to determine the probable mean free path of the test molecule we need store only the total number of molecules  $N_{v_1 v_2 v_3}$ , present in each geometrical cell, and not their distribution over the different velocities (i.e., the number  $N_{v_1 v_2 v_3 \mu_1 \mu_2 \mu_3}$ ).

The probability of collision of a  $\mu$ -test molecule with a  $\mu_j$ -field molecule of this kind is equal to

$$W_{v_i \mu_j \mu} = N_{v_i \mu_j} = f(v_i, \mu_j) \Delta x \Delta \xi_j. \quad (15.4a)$$

If the motion of the test particle occurs according to the distribution function  $f$ , it falls into a cell with a frequency proportional to  $f(\mu)\mu$ . Its residence time in the cell is proportional to  $1/\mu$ , and, therefore, the probability of observing the molecule in the cell is proportional to

$$f(\mu)\mu(1/\mu) = f(\mu).$$

Suppose that we have stored the velocity  $\mu_j^k$  in a given space cell. Suppose that at the next,  $(k+1)$ -th, incidence of the test molecule in the cell it has a velocity  $\mu$ . Then, if we store the velocity  $\mu$  in the cell with probability  $g = 1/\mu$ , i.e., we put  $\mu_j^{(k+1)} = \mu$  and we put  $\mu_j^{(k+1)} = \mu_j^k$  with probability  $1-g$ , then, on the average (for a sufficiently large number of incidences of the test molecule in the cell), the probability of finding a molecule with velocity  $\mu_j$  will be proportional to the function  $f(\mu_j)$ . But the latter, according to (15.4a), is proportional also to the probability of collision of a  $\mu$ -molecule with a  $\mu_j$ -molecule. Therefore, in each geometrical cell it is sufficient to store only one velocity  $\mu_j^k$  according to the above rule. If a collision occurs for the  $(k+1)$ -th

\* V. I. Vlasov, Doklady Akad. Nauk SSSR, Vol. 167, No. 5 (1966).

incidence of a test molecule in a given cell, then it occurs with a molecule possessing velocity  $\mu_j^k$ .

Thus, for molecules with collision cross sections inversely proportional to the relative velocity, it is sufficient in each geometrical cell to store only the total number of molecules in this cell and a single velocity, whereas, in general, we must store the whole distribution function. If in a three-dimensional case, for arbitrary molecules, we store only 10 velocities along each velocity coordinate, then we must store  $10^3$  numbers in each space cell. Therefore, for pseudo-Maxwellian molecules the storage requirement may be reduced by roughly three orders of magnitude. This makes it reasonable to calculate complex two- and three-dimensional flows by the Monte Carlo method on contemporary computers.

The simplest application of the Monte Carlo method is given in §§4.2, 4.4, and 6.6.

2. We now examine another possible calculation scheme, in which the motion of a large number of molecules is followed simultaneously.

We saw in §2.11 that, to conserve simulation of flows, it is necessary to conserve the Knudsen number. Similarity may be satisfied if we keep the product of the number of particles and the collision cross section. In other words, by examining a gas with a small number of large spheres, we model the motion of a gas with a large number of small particles. However, the reduction of the number of particles is not unlimited. In the first place, the fluctuations increase as the number of particles is reduced. Secondly, when we keep the product  $n\sigma = \text{const}$  and reduce the number of particles, there is an increase in the volume occupied by the molecules, and the gas represented by them cannot be considered ideal from a certain time onward. Thirdly, by replacing several small molecules by a single large molecule, we in fact replace a continuous distribution function by a discontinuous one. In particular, we lose molecules with very large velocities.

Let us assume that the whole space is divided into cubical cells in each of which there is still a sufficiently large number of molecules. We choose a certain cell and consider it to be the central one. Let the positions and velocities of the molecules in all

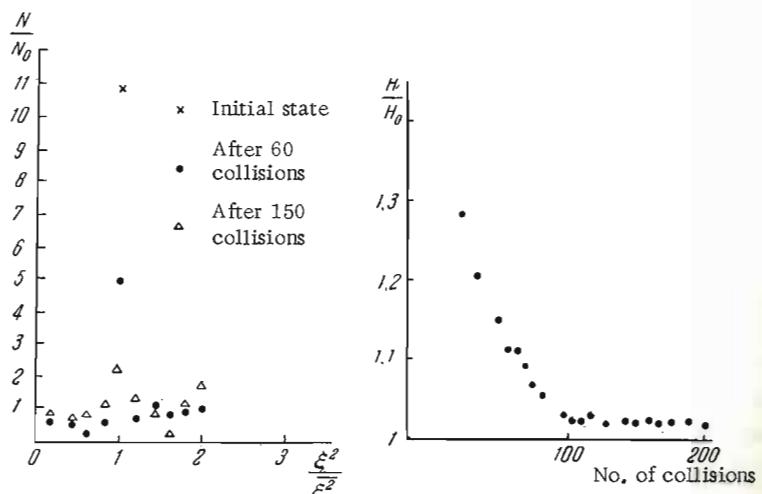


Fig. 17

Fig. 18

the cells be identical at time zero, i.e., the distribution function is periodic. If at time zero the distribution of the molecules differs from a Maxwellian, then the gas begins to approach equilibrium as a result of collisions. We follow the motion of molecules in the central cell. If any molecule leaves the cell, then, because of the periodicity, a particle with the same velocity enters through the opposite face. The motion of molecules in the central cell completely determines the motion in the whole space.

The papers of Alder and Wainwright\* give computer calculations for 32 and 100 hard-sphere molecules in a cell. At time zero all the molecules had velocities identical in magnitude. The directions of the velocities were chosen at random. By means of their calculations we can follow the process of establishing equilibrium.

We show below the results of calculations for 100 spheres, occupying less than  $\frac{1}{14}$  of the volume. Figure 17 shows the ratio of the number of particles with a given velocity  $N(\xi)$  in the process under examination to the number of particles  $N_0(\xi)$  with the same velocity at equilibrium. The x axis shows the ratio of the energy of the particles to the mean energy of the molecules. When 150

\*B. Alder and T. Wainwright, J. Chem. Phys., 27: 1208 (1957); T. Wainwright and B. Alder, Nuovo Cimento, Vol. 9, Suppl. 1, p. 116 (1958).

collisions have been drawn, each molecule has experienced a total of three collisions on the average. It may be seen from the graph presented that even after two or three collisions, the distribution of molecules is already very close to equilibrium. However, because of the small number of particles examined, there are considerable fluctuations in a cell.

Knowing the distribution function, we may calculate the Boltzmann H-function at each instant of time, and follow its approach to equilibrium. Figure 18 shows the variation of the ratio of the H-function to its equilibrium value with time. When each molecule has experienced two collisions, the H-function is already very close to equilibrium, but the limiting value of the H-function is somewhat higher than the equilibrium value. This is due to the above-mentioned absence of molecules with very large velocities (velocities greater than average by a factor of 4-5).

The fluctuations of the H-function are considerably smaller than those of the distribution function. This is natural and typical for the Monte Carlo method. The overall characteristics (the H-function in this case) are the outcome of averaging of a considerably larger number of random numbers than local properties (in this case the number of particles with a given velocity). Therefore, the Monte Carlo methods are especially economical for calculation of total characteristics. In particular, the example presented shows that replacement of a large number of molecules in a real gas by a comparatively small number permits us to obtain satisfactory accuracy for overall properties.

### §3.16. The Maximum Probability Principle

It was shown in §2.5 that the Maxwellian distribution is the most probable when we are given the number of particles and their momentum and energy. By solving the corresponding very simple variational problem, we obtained the distribution function for the equilibrium case. It is natural to try to find the distribution function as the most probable function also for nonequilibrium processes.\*

\*P. S. Epstein, Verhandl. Deut. Phys. Ges., 21: 96 (1919); A. Einstein, Ann. Physik, Vol. 69, No. 4 (1922); P. S. Epstein, Phys. Rev., 23: 710-733 (1924); A. M. Kogan, Doklady Akad. Nauk SSSR, Vol. 158, No. 5 (1964); J. P. Guiraud, "Rarefied Gas Dynamics," Third Symposium, Academic Press, New York, 1963.

The probability of any distribution is determined by the Boltzmann H-function (see §2.5)

$$H = \int f \ln f d\xi. \quad (16.1)$$

We seek an extremum of the functional (16.1) under a set of additional conditions (moments):

$$M_{a_1 \dots a_N}^{(N)} = \int \xi^{a_1} \dots \xi^{a_N} f d\xi, \quad (a_1, \dots, a_N = 1, 2, 3). \quad (16.2)$$

Conditions (16.2) may include both equilibrium ( $n, u_i, T$ ), and non-equilibrium ( $P_{ij}, q_i$ , etc.) flow properties.

As was done in §2.5, we introduce Lagrange multipliers, defining the functional

$$F = \int \left( f \ln f + f \sum \lambda_{a_1 \dots a_N}^{(N)} \xi^{a_1} \dots \xi^{a_N} \right) d\xi. \quad (16.3)$$

The Euler equation of the variational problem takes the form

$$\ln f + 1 + \sum \lambda_{a_1 \dots a_N}^{(N)} \xi^{a_1} \dots \xi^{a_N} = 0. \quad (16.4)$$

Therefore, the most probable distribution has the form

$$f = \exp \left\{ - \left( 1 + \sum \lambda_{a_1 \dots a_N}^{(N)} \xi^{a_1} \dots \xi^{a_N} \right) \right\}. \quad (16.5)$$

To find the Lagrange multipliers it is necessary to substitute that distribution function into the conditions (16.2). The relations obtained allow us to express the multipliers  $\lambda^{(N)}$  in terms of the given moments  $M^{(N)}$ .

Having the distribution function, expressed in terms of the moments which determine the problem, we may then use the method of moments (see §§3.2 and 3.4) to construct a system of equations which the moments or the Lagrange multipliers  $\lambda^{(N)}$  must satisfy.

As we showed earlier, the accuracy of the method of moments is determined by the correct choice of the form of the distribution function. The formal procedure described above makes it possible to find the most probable distribution function for any specific problem, in terms of the given macroscopic parameters

of the problem. However, formal use of the method immediately encounters several well-known difficulties.

The first difficulty is of a mathematical nature. Suppose that we examine a problem in which the highest moment determining the problem is an odd moment, for example, the heat flux  $q_i$ . Then, when the distribution function  $f$  in the form (16.5) is substituted into conditions (16.2), we obtain divergent integrals of the type

$$\int_{-\infty}^{+\infty} \xi^m \exp \left\{ - (a_0 + a_1 \xi + \dots + a_{2n+1} \xi^{2n+1}) \right\} d\xi,$$

and the problem has no solution.

If the highest moments are even, as was the case in §2.5 when we constructed an equilibrium distribution function, then the divergence due to the odd moments is suppressed by the higher even moments.

The second, more basic, difficulty is in choice of the required finite number of moments. It is evident that if, for example, in describing a flow with a nonzero heat flux, we do not include  $q_i$  among the determining parameters, not only is the distribution function obtained insufficiently accurate, but it cannot lead to qualitatively correct results. Generally speaking, in any nondegenerate problem there is an infinite number of nonzero moments.

If the most probable distribution function found by the choice of a finite number of determining moments is the true distribution function of the problem, it is clear that it must satisfy the Boltzmann equation exactly and not on the average through a finite number of moment equations. If the most probable distribution function is not exact and does not satisfy the Boltzmann equation, we must elucidate the degree of approximation that it gives, i.e., we must explain the general meaning of the choice of the most probable function.

We examine a physical volume around a point  $x$  at which we seek the distribution function  $f(t, x, \xi)$ . Let its greatest characteristic dimension  $\delta$  be such that within the volume we can consider the determining moments to be constant to within a certain accuracy.

If the majority of the molecules passes through the chosen volume without collisions, it is clear that the distribution function at the point  $\mathbf{x}$  is determined by processes (collisions) outside the volume. Therefore, the probability of a state at the point  $\mathbf{x}$  is in no way determined by the parameters of the flow at that point. There is an infinite number of distribution functions which correspond to the same macroscopic determining parameters (if the number of the latter is finite). Therefore, when the determining parameters are given at the point  $\mathbf{x}$ , the arriving molecules may possess a distribution with an arbitrary rather than maximum probability.

The collisions within our volume are the single mechanisms that may make the distribution function the most probable when the determining macroscopic parameters are given in the volume. Therefore, for us to be able to speak of the most probable distribution function, the number of molecules entering the volume in a time of the order of the time between collisions must be much smaller than the number of molecules in the volume. The number of molecules with velocities in the range  $d\xi$  about the velocity  $\xi$  in the volume being examined is of order  $f(t, \mathbf{x}, \xi)d\xi\delta^3$ , while the number of molecules entering the volume with that velocity in the time between collisions is proportional to  $f d\xi\delta^2\xi\tau$  or  $f\delta^2\lambda d\xi$ , where  $\lambda$  is the mean free path. Therefore, the requirement formulated above reduces to the condition

$$\delta \gg \lambda. \quad (16.6)$$

If some molecule, let us say with velocity  $\xi$ , leaves the volume, and is replaced by another molecule with the same velocity, the process of establishment of the most probable distribution does not change [when condition (16.6) is fulfilled and disorder is assumed]. The variation of the number of molecules in the volume with velocity  $\xi$  during time  $\tau$  is equal to

$$\tau \int_S (\xi \cdot \mathbf{n}) f dS \sim \xi \tau \delta^2 \frac{df}{dl} \delta. \quad (16.7)$$

where the integration is carried out over the surface  $S$  of the volume.

The quantity in (16.7) must be much smaller than the number of molecules with the same velocity inside the volume  $f\delta^3$ , i.e.,

$$1 \gg \frac{\lambda}{f} \frac{df}{dl}. \quad (16.8)$$

Therefore, we may speak of the distribution function as most probable only for flows with small Knudsen numbers, i.e., for flows close to local equilibrium. In that case, all the moments, with the exception of the hydrodynamic ones, may be considered small, and we may linearize expression (16.5) relative to the equilibrium distribution function, i.e., we may write it in the form

$$f = n \left( \frac{h}{\pi} \right)^{3/2} \exp \{-h^2(\xi - \mathbf{u})^2\} (A + B_i \xi_i + \lambda_i^{(2)} \xi_i \xi_j + \dots). \quad (16.9)$$

After the linearization, there are no longer any divergent integrals, such as occurred above.

Let us take, for example, the thirteen moments  $n, u_i, T, P_{ij}$ , and  $q_i$  as determining parameters. For ease of calculation it is convenient to go to a system of coordinates moving with the velocity of the gas at the point  $\mathbf{x}$ . Then  $u_i = 0, \xi = c_i$ , and the conditions (16.2) take the form

$$\left. \begin{aligned} n &= \int f dc, & u_i = 0 &= \int c_i f dc, & \frac{3}{2} kT &= \int \frac{mc^2}{2} f dc, \\ P_{ij} &= m \int c_i c_j f dc, & q_i &= \int \frac{mc^2}{2} c_i f dc. \end{aligned} \right\} \quad (16.10)$$

The expression (16.9) may be written for this case in the form

$$f = n \left( \frac{h}{\pi} \right)^{3/2} e^{-hc^2} (A + B_i c_i + \lambda_i^{(2)} c_i c_j + \lambda_i^{(3)} c_i c^2). \quad (16.11)$$

Substituting (16.11) into conditions (16.10), we obtain, after some simple manipulations:

$$A = 1, \quad B_i = -\frac{q_i}{p} \left( \frac{m}{kT} \right), \quad \lambda_i^{(2)} = \frac{p_{ij}}{2p} \left( \frac{m}{kT} \right), \quad \lambda_i^{(3)} = \frac{1}{5} \frac{q_i}{p} \left( \frac{m}{kT} \right)^2.$$

The most probable distribution function (16.11) takes the form

$$f = n \left( \frac{h}{\pi} \right)^{3/2} e^{-hc^2} \left[ 1 + \frac{p_{IJ}}{2p} \left( \frac{m}{kT} \right) c_I c_J - \frac{q_I}{p} \left( \frac{m}{kT} \right) \left( 1 - \frac{c^2}{5} \frac{m}{kT} \right) c_I \right]. \quad (16.12)$$

The expression obtained coincides with the distribution function of the Grad thirteen-moment approximation [see formula (3.34)]. Its substitution into the moment equation leads to the thirteen-moment equations, and, therefore, also to all the derivations following from them (see §3.3). However, it should be noted that, in deriving the thirteen-moment equations, the quantities  $p_{ij}$  and  $q_i$  are arbitrary in expression (16.12), whereas they must be assumed to be small in the derivation we gave. In §3.3, for Maxwellian molecules with small  $p_{ij}$  and  $q_i$ , we obtained expressions for  $p_{ij}$  and  $q_i$  in the Navier-Stokes and the Burnett approximations from the thirteen-moment equations (3.27) and (3.28). However, as we have seen, the higher approximations for  $p_{ij}$  and  $q_i$ , obtained from the thirteen-moment equations, differ from the corresponding approximations obtained from the full moment equations. Therefore, the results obtained by means of the maximum probability principle with thirteen determining parameters  $n$ ,  $u_i$ ,  $T$ ,  $p_{ij}$ , and  $q_i$  for Maxwellian molecules do not go beyond the limits of the Burnett approximation of the Enskog method. All other results which may be obtained from the corresponding thirteen-moment equations are out of order.

Moreover, we saw in §§3.3 and 3.8 that, for arbitrary non-Maxwellian molecules, we could not even obtain the Navier-Stokes equations from the thirteen- and twenty-moment equations with correct values of the transport coefficients (values of the viscosity and thermal conductivity may be found only in the first approximation in the Enskog sense; see §3.8). As was shown in §3.3, Maxwellian molecules are exceptional, since for them, at small Knudsen numbers ( $Kn = \epsilon \rightarrow 0$ ) the higher moments (more exactly, the coefficients  $a^{(N)}$  of the expansion of the distribution function in a series of Hermite polynomials, which are a linear function of the moments) are of order  $\epsilon^2$  and higher. For arbitrary molecules the whole infinite sequence of moments (more exactly, of coefficients  $a^{(N)}$ ) is of order  $\epsilon$ , as are also  $p_{ij}$  and  $q_i$ . Therefore, although we are usually interested only in the first thirteen moments in practical applications, we have no right, in choosing the determinant parameters, to limit ourselves only to these moments, and

it is necessary to take into account an infinite number of determining parameters, even to obtain the distribution function in the Navier-Stokes approximation, and, therefore, to the bracketed term of expression (16.11), it is necessary to add an infinite number of terms of the form  $\lambda_{a_1}^{(N)} \dots a_N c_{a_1} \dots c_{a_N}$ .

While the equilibrium state of the gas is determined by a finite number of macroscopic parameters, states near equilibrium are in general determined by an infinite number of macroscopic parameters.

Thus, for arbitrary molecules, results which may be validly obtained in principle by means of the maximum probability (entropy) principle, are equivalent, when an infinite number of moments is taken into account, to calculation of terms of order  $\epsilon$  in the Enskog series, i.e., approximately the Navier-Stokes series. Results obtained by taking account only of thirteen moments do not allow us to obtain even the exact equations of the Navier-Stokes approximation.

The use of the most probable distribution function with arbitrary Knudsen numbers is not much more justified than the use of an arbitrarily chosen function.

Evidently, in some cases the maximum entropy principle may be used for approximate calculation of transport coefficients entering into the equations of the Navier-Stokes type for complex systems, for which no rigorous theory has been developed.

A problem closely related to the preceding one has been suggested by P. Ehrenfest.\* We examine a nonequilibrium process, say the heat transfer in a gas located between flat plates at different temperatures. If at time zero the distribution function of the gas molecules between the plates is assigned arbitrarily, it is clear that the flow establishment process will begin, and that a steady state will be established as  $t \rightarrow \infty$ . The question arises whether the steady distribution function is in some sense the most probable. For a limited range of problems to which the thermodynamics of irreversible processes has been applied, an answer

\*P. Ehrenfest and T. Ehrenfest, Enzyk. Math. Wiss., Vol. 4, No. 32, Leipzig (1911).

is given, to a certain extent, by the so-called principle of minimum entropy production.\* However, for arbitrary Knudsen numbers, an analogous principle has not yet been established. In the general case, we may speak only of the probability of the whole system, and not of the distribution function at a point. In any case, the appropriate variational principle, suitable for all Knudsen numbers, must yield the Boltzmann equation, i.e., the solution of the variational problem must also satisfy the Boltzmann equation.

### §3.17. The Kinetic Theory and Nonequilibrium Thermodynamics

In thermodynamics, to characterize the state of a macroscopic system, the concept of entropy  $\mathcal{S}$  is of fundamental importance.

Entropy may change both as the result of a flux through the boundary of the system, and by its creation within the system:

$$d\mathcal{S} = d\mathcal{S}_e + d\mathcal{S}_i, \quad (17.1)$$

where  $d\mathcal{S}_e$  is the outward flux of entropy and  $d\mathcal{S}_i$  is the entropy production rate inside the system. The second law of thermodynamics states that

$$d\mathcal{S}_i \geq 0. \quad (17.2)$$

Thermodynamics and statistical physics deal with reversible processes for which  $d\mathcal{S}_i = 0$ , i.e., processes in which a system goes from some equilibrium states to other equilibrium states through a sequence of equilibrium states. Nonequilibrium thermodynamics (or the thermodynamics of irreversible processes, quasi-thermodynamics), like kinetic theory, studies nonequilibrium processes. The objective of the present section is to show the relationship between these two disciplines.

We examine the basic concepts of nonequilibrium thermodynamics† schematically for the example of mixing of ideal mono-

\*I. Prigogine, "Introduction to the Thermodynamics of Irreversible Processes," 2nd ed., Wiley, New York, 1962.

†A systematic account of the thermodynamics of nonequilibrium processes may be found in the monographs: I. Prigogine, cited above; S. R. de Groot and P. Mazur, "Nonequilibrium Thermodynamics," Wiley, New York, 1962.

tomic gases in the absence of external forces and chemical reactions.

The thermodynamics of nonequilibrium processes is a phenomenological science and concerns itself with macroscopic quantities. For the mixing of ideal nonreacting gases, the conservation equations (9.18)–(9.21) must be satisfied; they may be written in the form

$$\frac{\partial n^k}{\partial t} + \frac{\partial n^k u_r^k}{\partial x_r} = 0, \quad (17.3)$$

$$\frac{dp}{dt} + p \frac{\partial u_r}{\partial x_r} = 0, \quad \left( \frac{d}{dt} = \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial x_r} \right), \quad (17.4)$$

$$p \frac{du_t}{dt} = - \frac{\partial P_{tr}}{\partial x_r}, \quad (17.5)$$

$$p \frac{dU}{dt} = - \frac{\partial q_r}{\partial x_r} - P_{tr} \frac{\partial u_t}{\partial x_r}, \quad (17.6)$$

where  $U$  is the internal energy per unit mass of the gas; for a monoatomic gas, it is equal to the energy of translational motion of the molecules:

$$U = \frac{3}{2} \frac{knT}{\rho}. \quad (17.7)$$

In nonequilibrium thermodynamics, the important assumption is made that the entropy of a system in which irreversible processes occur is the same function of the state of the system as for a system in equilibrium. Therefore, the entropy change along the trajectory of a macroscopic system is determined by Gibbs's formula:

$$T \frac{d\mathcal{S}}{dt} = \frac{dU}{dt} + p \frac{dV}{dt} - \sum_k \mu^k \frac{dc^k}{dt}, \quad (17.8)$$

where  $V = \rho^{-1}$  is the specific volume,  $\mu^k$  is the thermodynamic potential, and  $c^k = \rho^k / \rho$  is the mass concentration of the  $k$ -th component of the gas. For an ideal gas, the thermodynamic potential is equal to

$$\mu^k = \frac{kT}{m^k} \ln n^k \left( \frac{h^k}{\pi} \right)^{3/2} \quad \left( h^k = \frac{m^k}{2kT} \right). \quad (17.9)$$

The expression (17.8) may be taken as a definition of entropy in nonequilibrium thermodynamics.

For the entropy we may write the explicit conservation equation:

$$\frac{\partial \sigma}{\partial t} + \operatorname{div} J_{\sigma} = \sigma, \quad (17.10)$$

where  $J_{\sigma}$  is the total flux of entropy, and  $\sigma$  is the entropy source strength, or the production of entropy per unit volume per unit time.

By means of the continuity equation (17.4), this equation may be reduced to the form

$$\rho \frac{d\sigma}{dt} = -\operatorname{div} J'_{\sigma} + \sigma, \quad (17.11)$$

where

$$J'_{\sigma} = J_{\sigma} - \rho u \sigma.$$

For the entropy flux  $J'_{\sigma}$  the postulate is made that it is due only to the heat flux  $J_{q_i} = q_i$  and to the diffusion fluxes of matter

$$J'_{\sigma} = \frac{1}{T} \left( J_q - \sum_k \mu^k J^k \right), \quad (17.12)$$

where

$$J^k = \rho^k (u^k - u) \quad (17.13)$$

are the diffusion fluxes.

Substituting this expression for the entropy flux into (17.11), replacing  $d\sigma/dt$  in the latter by use of Gibbs's relation (17.8), and eliminating the total derivatives of  $U$ ,  $\rho$ , and  $c^k$  by means of the conservation equations (17.3)-(17.6) for the mixture of ideal gases, we obtain

$$\sigma = J_q \operatorname{grad} \frac{1}{T} - \sum_k J^k \operatorname{grad} \frac{\mu^k}{T} - \frac{1}{T} p_{tr} \frac{\partial u_l}{\partial x_r}. \quad (17.14)$$

Therefore, the entropy production may be written in the form

$$\sigma = \sum_i J_i X_i, \quad (17.15)$$

where  $J_i$  are the fluxes, and  $X_i$  are the thermodynamic forces associated with them.

Relation (17.14) or (17.15) allows us to determine the thermodynamic forces associated with the corresponding fluxes. The fluxes are assumed to be linearly related to the associated thermodynamic forces, i.e.,

$$J_i = \sum_{j=1}^n L_{ij} X_j \quad (j = 1, \dots, n). \quad (17.16)$$

The fluxes  $J_i$  may be heat fluxes, diffusion fluxes, etc., while the thermodynamic forces  $X_i$  are gradients of temperature, concentration, velocity, etc. The relation (17.16) assumes that any gradient in general can create any flux.

For systems satisfying these assumptions, it has been shown that the phenomenological kinetic coefficients  $L_{ij}$  satisfy the Onsager symmetry relations:

$$L_{ij} = L_{ji}. \quad (17.17)$$

We note that the Onsager relations are satisfied only when (17.16) contains the associated fluxes and thermodynamic forces in accordance with expression (17.15). By comparison of the postulates of quasi-thermodynamics with the kinetic results obtained in the previous sections, it is easy to see that the assumed linear relation between fluxes and gradients is valid only for the Navier-Stokes approximations. Therefore, nonequilibrium thermodynamics is applicable only to describe states close to equilibrium, and the information derived by its use cannot exceed that given by taking account of the first term of the expansion with respect to deviation from equilibrium.

This result is often reached by comparing the expressions derived above for the entropy flux and the entropy production with the corresponding expressions obtained from the kinetic definition of entropy (see §2.5)

$$\rho \mathcal{S} = -k \sum_k \int f_k (\ln f_k - 1) d\xi.$$

However, that comparison is invalid, since the relations (17.8) and (17.11) may be regarded as definitions of entropy and of entropy production in nonequilibrium thermodynamics, these being independent of the kinetic definition of entropy.

A very important, and evidently a unique, result of the thermodynamics of nonequilibrium processes is the set of Onsager relations, which allow us to relate different phenomena. It is easy to verify that the Onsager relations are satisfied also in kinetic theory within the limits of the Navier-Stokes approximation. To do this it is sufficient, in expression (9.62), (9.65), and (9.67), to separate the coefficients of the thermodynamic forces defined by relation (17.14).

In contrast to kinetic theory, the thermodynamics of irreversible processes gives no information on the value of the kinetic coefficients. However, the methods of the thermodynamics of irreversible processes are applicable to a very wide class of phenomena (chemical reactions, phase transitions, crystals, bodies in the presence of magnetic fields, etc.). The kinetic theory has only been satisfactorily developed at present for rarefied gases.

## Chapter IV

# SOLUTION OF THE BOLTZMANN EQUATION FOR DEGENERATE FLOWS

## §4.1. Exact Solutions of the Boltzmann Equation

Because of the complex structure of the collision integral, only a small number of exact solutions of the Boltzmann equation have been obtained at the present time. In spite of the fact that a large proportion of those solutions describe highly artificial situations, they are very valuable as standard solutions for confirmation of approximate methods of calculation, as well as giving valuable information on the qualitative behavior of solutions of the Boltzmann equation.

1. A most important exact solution is the absolute Maxwellian distribution which describes a gas in a state of equilibrium in the absence of external forces:

$$f_0 = n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left\{ -\frac{m}{2kT} \xi^2 \right\}. \quad (1.1)$$

This function makes both the collision integral and the left side of the Boltzmann equation go to zero, since the macroscopic quantities  $n$  and  $T$  appearing there do not depend on the coordinates.

The Boltzmann barometric formula is a natural generalization of that solution. We write the Boltzmann equation for a gas in a steady state in a force field which possesses a potential  $\varphi$ :

$$\xi_l \frac{\partial f}{\partial x_l} - \frac{1}{m} \frac{\partial \varphi}{\partial x_l} \frac{\partial f}{\partial \xi_l} = J(f, f). \quad (1.2)$$

We seek a solution of that equation in the form

$$f = A(x) e^{-B(x)\xi^2}. \quad (1.3)$$

Substituting (1.3) into equations (1.2), we have

$$\xi_i \left( \frac{\partial A}{\partial x_i} + 2 \frac{AB}{m} \frac{\partial \varphi}{\partial x_i} \right) - \xi_i \xi^2 A \frac{\partial B}{\partial x_i} = 0, \quad (1.4)$$

since the function (1.3) makes the collision integral go to zero. Since the velocity  $\xi$  is an independent variable, it follows from (1.4) that

$$B = \text{const} \quad \text{and} \quad \ln A + \frac{2B}{m} \varphi = \ln C = \text{const}. \quad (1.5)$$

Therefore,

$$f = Ce^{-\frac{2B}{m}\varphi} e^{-B\xi^2}. \quad (1.6)$$

To determine the constants  $B$  and  $C$ , we use the relations

$$n = \int f d\xi = C \frac{\pi^{3/2}}{B^{3/2}} e^{-\frac{2B}{m}\varphi}, \quad (1.7)$$

$$\frac{3}{2} kT = \frac{1}{n} \int \frac{m\xi^2}{2} f d\xi = \frac{1}{n} \frac{\pi^{3/2}}{B^{3/2}} \frac{m}{2B} Ce^{-\frac{2B}{m}\varphi}, \quad (1.8)$$

whence

$$B = \frac{m}{2kT} \quad \text{and} \quad C = n_0 \left( \frac{m}{2\pi kT} \right)^{3/2}, \quad (1.9)$$

where  $n_0$  is the density of the gas at the point of zero potential of the forces,  $\varphi = 0$ . We have, finally,

$$f = n_0 \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{\varphi}{kT}} e^{-\frac{m}{2kT}\xi^2} \quad (1.10)$$

and

$$n = n_0 e^{-\frac{\varphi}{kT}}. \quad (1.11)$$

In particular, in the field of the Earth's gravity,  $\varphi = gh$ , where  $g$  is the acceleration of gravity and  $h$  is the altitude; we obtain the

well-known Boltzmann barometric formula

$$n = n_0 e^{-\frac{gh}{kT}}, \quad (1.12)$$

where  $n_0$  is the density at the Earth's surface.

It should be noted, however, that the solution (1.10) is an exact solution of the Boltzmann equation either in an infinite domain or in the presence of boundaries which do not disturb the distribution. We note that conditions (1.5) and (1.9) require that the temperature be constant. The solution obtained is a special case of a locally Maxwellian distribution.

The general locally Maxwellian distribution

$$f_0 = n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left\{ -\frac{m}{2kT} c^2 \right\} \quad (c = \xi - u), \quad (1.13)$$

i.e., the distribution in which the hydrodynamic quantities  $n$ ,  $\mathbf{u}$ , and  $T$  depend on the coordinates and on time, also makes the collision integral go to zero. However, the left side of the Boltzmann equation imposes conditions on the change of the hydrodynamic quantities with respect to  $t$  and  $\mathbf{x}$  which define a class of exact solutions.\*

We write the Boltzmann equation in the absence of a force field in the form

$$\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} + g_i \frac{\partial f}{\partial \xi_i} = J, \quad (1.14)$$

where  $g_i = X_i/m$  is the acceleration of a molecule.

We seek a locally Maxwellian solution of Eq. (1.14). It is convenient to write the locally Maxwellian function in the form

$$\ln f = \gamma_0 + \gamma_i \xi_i + \gamma_4 \xi^2, \quad (i = 1, 2, 3), \quad (1.15)$$

where

$$2 \frac{k}{m} T = -\frac{1}{\gamma_4}, \quad u_i = -\frac{\gamma_i}{2\gamma_4}, \quad \ln \left( \frac{\rho}{2\pi kT} \right)^{3/2} = \gamma_0 - \frac{\gamma^2}{4\gamma_4} \quad (1.16)$$

\* T. Carleman, "Problèmes mathématiques dans la théorie cinétique des gaz," Almqvist and Wiksell, Uppsala, 1957; H. Grad, Commun. Pure Appl. Math., Vol. 2, No. 4 (1949); O. G. Friedlander, Prikl. Mat. Mekhan., Vol. 29, No. 5 (1965).

$$(\gamma^2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2).$$

We divide Eq. (1.14) by  $f$  and substitute solution (1.15) into it. Then, by equating terms with different powers of  $\xi$ , we obtain

$$\frac{\partial \gamma_0}{\partial t} + g_i \gamma_i = 0, \quad (1.17)$$

$$\frac{\partial \gamma_1}{\partial t} + 2\gamma_4 g_i + \frac{\partial \gamma_0}{\partial x_i} = 0, \quad (1.18)$$

$$\frac{\partial \gamma_4}{\partial t} \delta_{ij} + \frac{1}{2} \left( \frac{\partial \gamma_i}{\partial x_j} + \frac{\partial \gamma_j}{\partial x_i} \right) = 0, \quad (1.19)$$

$$\frac{\partial \gamma_4}{\partial x_i} = 0. \quad (1.20)$$

We return in these equations to the physical variables  $\rho$ ,  $u_i$ , and  $T$ . From Eq. (1.20) we have

$$\frac{\partial T}{\partial x_i} = 0, \quad (1.20a)$$

i.e., the temperature is constant in space, but can change with time.

Since neither  $\gamma_4$  nor  $T$  depends on  $x_i$ , then Eq. (1.19) takes the form

$$\delta_{ij} \frac{\partial \ln T}{\partial t} + \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = 0. \quad (1.21)$$

We add the equations for  $i = j = 1, 2, 3$ . It is clear that the contraction gives

$$\frac{\partial \ln T}{\partial t} = -\frac{2}{3} \frac{\partial u_k}{\partial x_k}. \quad (1.22)$$

Then condition (1.21) takes the form

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} = 0. \quad (1.19a)$$

Equation (1.18) in physical variables becomes

$$\frac{\partial u_i}{\partial t} - u_i \frac{\partial \ln T}{\partial t} + \frac{RT}{\rho} \frac{\partial \rho}{\partial x_i} - u_j \frac{\partial u_j}{\partial x_i} - g_i = 0. \quad (1.23)$$

We multiply (1.21) by  $u_j$  and sum over  $j$ ; we have

$$u_i \frac{\partial \ln T}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial u_j}{\partial x_i} = 0. \quad (1.24)$$

By definition,  $p = \rho RT$ , and since  $T$  does not depend on  $x_i$ , then

$$\frac{\partial}{\partial x_i} \frac{\rho}{\rho} = \frac{1}{\rho} \frac{\partial \rho}{\partial x_i} - \frac{\rho}{\rho^2} \frac{\partial \rho}{\partial x_i} = \frac{1}{\rho} \frac{\partial \rho}{\partial x_i} - \frac{RT}{\rho} \frac{\partial \rho}{\partial x_i} = 0. \quad (1.25)$$

Substituting (1.24) and (1.25) into Eq. (1.23), we obtain

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \rho}{\partial x_i} + g_i. \quad (1.18a)$$

Finally, we write Eq. (1.17) in physical coordinates:

$$\frac{1}{\rho} \frac{\partial \rho}{\partial t} - \frac{3}{2} \frac{\partial \ln T}{\partial t} + \frac{u^2}{2RT} \frac{\partial \ln T}{\partial t} - \frac{u_i}{RT} \frac{\partial u_i}{\partial t} + \frac{u_i}{RT} g_i = 0. \quad (1.26)$$

Using (1.22) and (1.18a), we transform this equation into the form

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u_k}{\partial x_k} + u_j \frac{\partial \rho}{\partial x_j} - \frac{\rho}{RT} \left( \frac{u^2}{3} \frac{\partial u_k}{\partial x_k} - \frac{u_j}{2} \frac{\partial u^2}{\partial x_j} \right) = 0. \quad (1.27)$$

It is easy to verify that the contents of the bracket are equal to zero. In fact, by multiplying equations (1.19a) by  $u_j u_i$  and summing with respect to  $i$  and  $j$ , we obtain

$$u_{ij} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} u_i u_j \frac{\partial u_k}{\partial x_k} = u_j \frac{\partial u^2}{\partial x_j} - \frac{2}{3} u^2 \frac{\partial u_k}{\partial x_k} = 0. \quad (1.28)$$

Therefore, Eq. (1.27) may be written in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0. \quad (1.17a)$$

It is easy to see that Eqs. (1.17a) and (1.18a) are none other than the equations of continuity and the equations of motion in the Euler approximation. The energy equation in the Euler approximation has the form (see §3.6)

$$\frac{3}{2} kn \left( \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \right) T = -p \frac{\partial u_k}{\partial x_k}, \quad (1.29)$$

which clearly coincides with Eq. (1.22) for a temperature independent of the space coordinates. Conditions (1.20a) and (1.19a) evidently agree with the requirement that the heat-flux vector and the stress tensor vanish in the Navier-Stokes approximation.

Thus, solutions of the Euler equation which satisfy the condition that the heat-flux vector and the stress tensor are zero, and, therefore, are simultaneously solutions of the Navier-Stokes equations, are exact solutions of the Boltzmann equation with a locally Maxwellian distribution function.

We now analyze the nature of possible locally Maxwellian flows defined by the system of equations (1.17)-(1.20).

Let us differentiate Eq. (1.19) with respect to  $x_k$ ; since  $\partial\gamma_4/\partial x_k = 0$ , then

$$\frac{\partial\gamma_i}{\partial x_j \partial x_k} + \frac{\partial\gamma_j}{\partial x_i \partial x_k} = 0. \quad (1.30)$$

Adding to (1.30) and subtracting from it the equations obtained from (1.30), we obtain by cyclic permutation of the subscripts,

$$\frac{\partial\gamma_i}{\partial x_j \partial x_k} = 0. \quad (1.31)$$

Therefore, the  $\gamma_i$  have the form

$$\gamma_i(t, \mathbf{x}) = a_i(t) + b_{ij}(t) x_j. \quad (1.32)$$

Substituting expression (1.32) into Eq. (1.19), we find that

$$b_{ij}(t) = \begin{cases} -b_{ij} & \text{when } i \neq j, \\ -\frac{\partial\gamma_4}{\partial t} & \text{when } i = j. \end{cases}$$

Therefore, (1.32) may be written in the vector form

$$\mathbf{v}(t, \mathbf{x}) = \mathbf{a}(t) - \frac{\partial\gamma_4}{\partial t} \mathbf{x} + (\boldsymbol{\omega}(t) \times \mathbf{x}), \quad (1.33)$$

where  $\boldsymbol{\omega}(t)$  is the vector angular velocity of rotation of the gas as a solid body.

Since the vector  $\mathbf{v}$ , according to relations (1.16), is proportional to the velocity  $\mathbf{u}$ , it follows from (1.33) that the possible motions are a superposition of translational motion, radial expansion, and solid body rotation.

We write Eq. (1.18) in vector form:

$$\frac{\partial\mathbf{v}}{\partial t} + 2\gamma_4 \mathbf{g} + \operatorname{grad} \gamma_0 = 0. \quad (1.18b)$$

Taking the curl of this expression, and representing  $\mathbf{v}$  in the form (1.33), we obtain

$$\operatorname{curl} \mathbf{g}(t, \mathbf{x}) = -\frac{1}{\gamma_4} \frac{\partial \boldsymbol{\omega}(t)}{\partial t}, \quad (1.34)$$

whence

$$\mathbf{g}(t, \mathbf{x}) = -\frac{1}{m} \nabla \varphi - \frac{1}{2} \left( \frac{1}{\gamma_4} \frac{\partial \boldsymbol{\omega}(t)}{\partial t} \times \mathbf{x} \right), \quad (1.35)$$

where  $\varphi(t, \mathbf{x})$  is the potential. By substituting in Eq. (1.18b) for the functions  $\mathbf{g}$  and  $\mathbf{v}$  from expressions (1.35) and (1.33), respectively, after integration we obtain the following expression for  $\gamma_0(t, \mathbf{x})$ :

$$\gamma_0(t, \mathbf{x}) = \frac{2}{m} \gamma_4(t) \varphi(t, \mathbf{x}) + \frac{1}{2} \frac{\partial^2 \gamma_4}{\partial t^2} x^2 - \left( \frac{\partial \mathbf{a}}{\partial t} \cdot \mathbf{x} \right) + b(t), \quad (1.36)$$

where  $b(t)$  is an arbitrary function of integration.

Now substituting the expressions found for  $\mathbf{g}$  and  $\gamma_0$  into Eq. (1.17), we obtain

$$\begin{aligned} & 2\gamma_4 \frac{\partial \varphi}{\partial t} - (\mathbf{v} \cdot \operatorname{grad} \varphi) + 2 \frac{\partial \gamma_4}{\partial t} \varphi \\ &= m \left\{ -\frac{1}{2} \frac{\partial^3 \gamma_4}{\partial t^3} x^2 + \left( \frac{\partial^2 \mathbf{a}}{\partial t^2} \cdot \mathbf{x} \right) - \frac{\partial b(t)}{\partial t} + \frac{1}{2} \left[ \left( \frac{1}{\gamma_4} \frac{\partial \boldsymbol{\omega}}{\partial t} \times \mathbf{x} \right) \cdot \mathbf{v} \right] \right\}. \end{aligned} \quad (1.37)$$

Equation (1.37) may be regarded as a partial differential equation of the first order, defining a potential compatible with locally Maxwellian flows when  $\gamma_4, \mathbf{a}, \boldsymbol{\omega}$ , and  $b$  are arbitrarily assigned functions of time, since Eqs. (1.17)-(1.20) do not impose conditions.

Conversely, if we regard the potential as given, then Eq. (1.37) imposes a relation on the functions  $\gamma_4, \mathbf{a}, \boldsymbol{\omega}$ , and  $b$ . After we find the potential or the functions  $\gamma_4, \mathbf{a}, \boldsymbol{\omega}$  and  $b$ , the density and velocity of the flow are found according to formulas (1.36) and (1.33).

In a steady flow, we have, from (1.33) and (1.37),

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}, \quad (\mathbf{v} \cdot \operatorname{grad} \varphi) = 0,$$

i.e., the gas can rotate as a whole (transport with constant velocity is irrelevant), while the force must be perpendicular to the velocity. When  $\omega = \gamma = 0$  the potential is arbitrary, and we come back to the barometric formula.

2. The locally Maxwellian distribution makes the collision integral vanish identically. Therefore, if we remain in the class of locally Maxwellian solutions, we cannot obtain information about the behavior of solutions of the Boltzmann equation for dissipative processes. We now examine a class of exact solutions where the collision integral is not equal to zero.\*

We seek solutions of the Boltzmann equation

$$\frac{\partial f}{\partial t} + \xi_l \frac{\partial f}{\partial x_l} = \int (f' f'_l - f f_{l'}) g b db de d\xi_l \quad (1.38)$$

in the form

$$f(t, x, \xi) = F\left[\chi(t), t\left(\xi - \frac{x}{t}\right)\right], \quad (1.39)$$

where  $\chi(t)$  is as yet an unknown function. By substituting (1.39) into Eq. (1.38), we obtain

$$\begin{aligned} \chi' \frac{\partial F\left[\chi, t\left(\xi - \frac{x}{t}\right)\right]}{\partial \chi} &= \int \left\{ F\left[\chi, t\left(\xi' - \frac{x}{t}\right)\right] F\left[\chi, t\left(\xi'_l - \frac{x}{t}\right)\right] \right. \\ &\quad \left. - F\left[\chi, t\left(\xi - \frac{x}{t}\right)\right] F\left[\chi, t\left(\xi - \frac{x}{t}\right)\right] \right\} g b db de d\xi_l. \end{aligned} \quad (1.40)$$

We introduce the new variables

$$A_l = t\left(\xi_l - \frac{x_l}{t}\right). \quad (1.41)$$

We consider a hard sphere gas. It is clear that spherical molecules, colliding with "velocities"  $t\xi - x$  and  $t\xi_l - x$ , will have velocities  $t\xi' - x$  and  $t\xi'_l - x$  after a collision, where  $\xi'$  and  $\xi'_l$  are the velocities after collision, of molecules which had velocities  $\xi$  and  $\xi_l$  before collisions and the same values of the parameters  $b$  and  $e$ .

\* A. A. Nikol'skii, Doklady Akad. Nauk SSSR, Vol. 151, Nos. 2 and 3 (1963).

Therefore, Eq. (1.40) may be written in the form

$$\chi' \frac{\partial F(\chi, A)}{\partial \chi} = \frac{1}{t^4} \int [F(t, A') F(t, A'_l) - F(t, A) F(t, A_l)] \times |A_l - A| b db de dA_l. \quad (1.42)$$

By putting

$$\chi' = \frac{1}{t^4}, \quad (1.43)$$

we obtain, finally,

$$\frac{dF}{d\chi} \equiv \int (F' F'_l - FF_l) |A_l - A| b db de dA_l. \quad (1.44)$$

Thus, relation (1.39) distinguishes a class of solutions of the complete Boltzmann equation, in which a spatially homogeneous solution of the Boltzmann equation (1.44) is associated with the distribution function at each point.

According to (1.43),

$$\chi = C - \frac{1}{3} \frac{1}{t^3}, \quad (1.45)$$

where  $C$  is a constant. Therefore, the solution of the Boltzmann equation (1.39) may be written in the form

$$f(t, x, \xi) = F\left(C - \frac{1}{3} \frac{1}{t^3}, t\xi - x\right). \quad (1.46)$$

Unfortunately, at present we do not have available any solutions of the spatially uniform equation (1.44) with a nonzero collision integral.

However, we may obtain one interesting consequence even without solution.

Let a distribution function be assigned, let us say at time  $t = 1$ , by the relation

$$f(t, x, \xi) = F\left(C - \frac{1}{3}, \xi - x\right), \quad (1.47)$$

where  $F(C - \frac{1}{3}, A)$  is the state of the gas at time  $\chi = C - \frac{1}{3}$ ; it is different from the equilibrium state and homogeneous in space. In accordance with the H-theorem, when left to itself a state that is uniform throughout the space tends to the equilibrium Maxwellian

state, i.e., when  $x \rightarrow \infty$ , the function  $F$  tends to the absolute Maxwellian distribution. But it follows from (1.45) and (1.46) that, when  $t$  changes from 1 to  $\infty$ , the variable  $x$  varies only from  $C - \frac{1}{2}$  to  $C$  and, therefore, the state of the gas given by relation (1.47) does not reach the Maxwellian distribution after an infinite time  $t$ . Of course, the result obtained does not contradict the H-theorem in any way. In fact, the class of solutions examined describes the scattering of the gas. If we isolate an arbitrary region in space, a flux of the H-function passes continuously through its boundary, i.e., the conditions necessary for the H-theorem are not satisfied (see § 2.5).

3. It was noted in Chapter III that the infinite system of moment equations obtained by means of the full system of functions is equivalent to the Boltzmann equation. In particular, Grad's system of equations, constructed in § 3.3 by means of Hermite polynomials, is one such system. One method of constructing exact solutions of the Boltzmann equation is to find solutions for the equivalent system of equations for macroscopic quantities (moments).

Let us examine at time  $t = 0$  the state of a gas uniform in space, whose macroscopic velocity is equal to zero; the density is  $\rho$ , and the temperature  $T$ .\* Let all higher moments at that same time be different from zero, in particular,  $p_{ij} \neq 0$  and  $q_i \neq 0$ . It is evident that the velocity, density, and temperature of the gas do not change for  $t > 0$ . The variation of the stress tensor  $p_{ij}$  and of the heat-flux vector  $q_i$  with time for a Maxwellian gas may be obtained directly from formulas (3.39) and (3.40) of § 3.3. In fact, in that case, all the derivatives with respect to the space coordinates are equal to zero and, therefore, the quantities  $A_{ij}$  and  $B_i$  are also equal to zero. Therefore, taking into account that  $\tau_p$  is constant for this problem, we have

$$p_{ij}(t) = p_{ij}(0) e^{-\frac{t}{\tau_p}}, \quad q_i(t) = q_i(0) e^{-\frac{2}{3} \frac{t}{\tau_p}}, \dots \quad (1.48)$$

Analogous expressions are obtained for the higher moments. We examined the same problem in § 2.8 for the model equation. We obtained the result there [see formula (8.26) of Chapter II] that the distribution function tends to equilibrium exponentially. There-

\*H. Grad, Commun. Pure Appl. Math., Vol. 2, No. 4 (1949).

fore, the moments also decay exponentially with the same relaxation time. In the exact formulation we find that the stress tensor decays with a relaxation time  $\tau_p$ , the heat-flux vector with a relaxation time  $\frac{3}{2}\tau_p$ , and so on, i.e., the relaxation time is different for different processes. Therefore, the model equation is often justifiably called the single-relaxation equation.

Galkin\* has discussed a more general class of solutions, which also permits truncation of the infinite system of equations and the obtaining of a closed system of equations for a finite number of moments.

For convenience, let us again write the equations for the moments up to the third order, inclusively (see § 3.3):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_r} (\rho u_r) = 0, \quad (1.49)$$

$$\frac{\partial u_l}{\partial t} + u_r \frac{\partial u_l}{\partial x_r} + \frac{1}{\rho} \frac{\partial p}{\partial x_l} + \frac{1}{\rho} \frac{\partial p_{lr}}{\partial x_r} = 0, \quad (1.50)$$

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_r} (\rho u_r) + \frac{2}{3} P_{lr} \frac{\partial u_l}{\partial x_r} + \frac{2}{3} \frac{\partial q_r}{\partial x_r} = 0, \quad (1.51)$$

$$\begin{aligned} \frac{\partial p_{ij}}{\partial t} + \frac{\partial}{\partial x_r} (u_r p_{ij}) + m \frac{\partial \mathcal{M}_{ljr}}{\partial x_r} - \frac{2}{3} \delta_{ij} \frac{\partial q_r}{\partial x_r} + p_{lr} \frac{\partial u_l}{\partial x_r} \\ + 2 p_{lr} \frac{\partial u_j}{\partial x_r} + 2 p \frac{\partial \bar{u}_i}{\partial x_r} + \alpha p_{ij} = 0, \end{aligned} \quad (1.52)$$

$$\begin{aligned} \frac{\partial \mathcal{M}_{ijk}}{\partial t} + \frac{\partial}{\partial x_r} (u_r \mathcal{M}_{ijk} + \mathcal{M}_{ljk} + \mathcal{M}_{lkr} + \mathcal{M}_{lir} \frac{\partial u_k}{\partial x_r} + \mathcal{M}_{irk} \frac{\partial u_j}{\partial x_r} \\ + \mathcal{M}_{rjk} \frac{\partial u_l}{\partial x_r} - \frac{1}{m\rho} (P_{ij} \frac{\partial P_{kr}}{\partial x_r} + P_{ik} \frac{\partial P_{jr}}{\partial x_r} + P_{jk} \frac{\partial P_{ir}}{\partial x_r}) \\ + \frac{\alpha}{6} (9\mathcal{M}_{ijk} - 2\delta_{jk}q_i - 2\delta_{ik}q_j - 2\delta_{ij}q_k) = 0. \end{aligned} \quad (1.53)$$

Here,

$$\alpha = \frac{kn}{\mu_0}, \quad \mu_0 = \frac{\mu}{T} = \text{const}, \quad \bar{A}_{ij} = \frac{1}{2} (A_{ij} + A_{ji}) - \frac{1}{3} \delta_{ij} A_{kk}.$$

Let us seek a solution in which all the moments except  $u_i$  depend only on the time  $t$ , while  $u_i$  depends linearly on  $x_j$ :

\*V. S. Galkin, Prikl. Mat. Mekhan., Vol. 22, No. 3 (1958).

$$u_i = \psi_{ij}(t) x_j. \quad (1.54)$$

Since only the space derivatives of the fourth moments  $\mathcal{M}_{ijkl}(t)$  appear in equations (1.53), the system of equations becomes closed. Also, it is clear from equations (3.17) of Chapter III that the velocities  $u_i$  appear in the equations of higher orders only through their space derivatives, and, therefore, the space coordinates are eliminated from the equations. The moments of order  $N + 1$  also drop out of the equations of order  $N \geq 4$ . For Maxwellian molecules, the velocity  $u_i$  and the moments of higher order do not affect the terms which determine the collision integral. Thus, for this class of flows, the moments of order higher than the third may be found successively from a recurrent system of ordinary differential equations in time.

Let us determine the lower moments from the system (1.49)–(1.53).

From the continuity equation (1.49) we have

$$\rho = \rho_0 \exp\left(-\int_0^t I(t) dt\right), \quad (I(t) = \psi_{11} + \psi_{22} + \psi_{33}). \quad (1.55)$$

Substituting (1.55) into the momentum equations (1.50), we obtain

$$\frac{d\psi_{ij}}{dt} + \psi_{ir}\psi_{jr} = 0. \quad (1.56)$$

Finally, Eqs. (1.51)–(1.53) give

$$\frac{dp}{dt} + \frac{5}{3} pI + \frac{2}{3} p_{ij}\psi_{ij} = 0. \quad (1.57)$$

$$\frac{dp_{ij}}{dt} + p_{ij}I + 2\overline{p_{ir}\psi_{jr}} + 2\overline{p\psi_{ij}} + a_0 p_{ij} \exp\left(-\int_0^t I dt\right) = 0 \\ \left(a_0 = \frac{k n_0}{\mu_0}\right), \quad (1.58)$$

$$\frac{d\mathcal{M}_{ijk}}{dt} + \mathcal{M}_{ijr}\psi_{kr} + \mathcal{M}_{irk}\psi_{jr} + \mathcal{M}_{rjk}\psi_{ir} + \mathcal{M}_{ijk}I \\ + \frac{1}{6} a_0 \exp\left(-\int_0^t I dt\right) (9\mathcal{M}_{ijk} - 2\delta_{ik}q_j - 2\delta_{jk}q_i - 2\delta_{ij}q_k) = 0. \quad (1.59)$$

#### §4.2] COUETTE FLOW

Thus, for the class of flows under examination, the system of Eqs. (1.49)–(1.53) has separated into a series of systems. Equations (1.56) determine the  $\psi_{ij}$ . When the functions  $\psi_{ij}$  have been found, we find  $p$  and  $p_{ij}$  from Eqs. (1.57) and (1.58), and we find  $\mathcal{M}_{ijk}$  independently from the system of equations (1.59). Equation (1.55) determines the density.

The simplest solutions of the type examined are for shear flow\*

$$u_1 = \text{const} \cdot x_2, \quad u_2 = u_3 = 0, \quad \rho = \text{const} \quad (1.60)$$

and for one-dimensional expansion†

$$u_1 = \frac{x_1}{t}, \quad u_2 = u_3 = 0, \quad \rho = \frac{\rho_0}{t}. \quad (1.61)$$

Comparison of these exact solutions with the corresponding solutions of the Navier–Stokes and Burnett equations shows that in many cases the Burnett equations prove to be considerably more accurate than the Navier–Stokes equations as the Knudsen number increases. At large Knudsen number the expansions about free molecular flows (i.e., with respect to  $1/\text{Kn}$ ) prove to be quite accurate in a much wider region than one would expect (right up to Knudsen numbers  $\text{Kn} \sim 1$ ).

The moments computed from the equations presented here do not allow us to determine the corresponding distribution function, which is a solution of the Boltzmann equation, since the distribution function is determined in general by an infinite number of moments, and the moments of higher order do not vanish.

#### §4.2. Couette Flow

The Couette problem of shear flow and heat transfer between parallel infinite plates moving relative to each other is one of the

V. S. Galkin, Prikl. Mat. Mekhan., Vol. 20, No. 3 (1956); C. Truesdell, J. Rational Mech. Anal., Vol. 5, No. 1 (1956).

V. S. Galkin, Prikl. Mat. Mekhan., Vol. 28, No. 1 (1964). For the general class of uniform motions, see A. A. Nikol'skii, Inzh. Zh., Vol. 5, No. 6 (1965); V. S. Galkin, Mekhan. Zhidkosti i Gaza, No. 5 (1966).

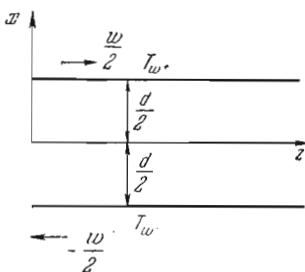


Fig. 19

fer between rotating coaxial cylinders. Experiments with cylinders allow us to compare the results of theory and experiment for various gas pressures between the cylinders.

The main purpose of the present section is to compare the various methods of solution of the Boltzmann equation in the simple example of Couette flow.\* We must remember, however, that when we examine a method intended for real three-dimensional flows, the Couette flow may display singularities which are not characteristic of real flows. These singularities are due to the fact that in an arbitrarily rarefied gas the number of collisions for molecules moving at glancing angles  $\theta$  to the wall is proportional to  $1/\theta$ .

1. The Free-Molecular Flow Regime ( $Kn \rightarrow \infty$ ). In order to explain certain qualitative peculiarities of the flow, we begin with an examination of a free-molecular flow.

We take the  $x$  axis perpendicular to the plate (Fig. 19), and choose the origin of coordinates so that the plates are located in the planes  $x = \pm d/2$ . Let the velocities of the plates be, respectively,  $\pm \frac{1}{2}w$  and their temperatures  $T_{w^\pm}$ . The flow between the plates will be free-molecular when  $\lambda/d \rightarrow \infty$ , where  $\lambda$  is the mean free path of the molecules. In that case, we may neglect the collision integral in the Boltzmann equation, and the equation takes the form

$$\frac{df(t, x, \xi)}{dt} = 0 \quad (2.1)$$

\*Here we examine only the flow between infinite plates. The reader may find a discussion of the results for flow between coaxial cylinders and the corresponding bibliography in the book by Shidlovskii (V. P. Shidlovskii, "Introduction to Dynamics of Rarefied Gas," Nauka Press [in Russian], 1965).

or for the one-dimensional steady flow under consideration,

$$\frac{\partial f(x, \xi)}{\partial x} = 0. \quad (2.2)$$

Equations (2.1) and (2.2) have the following solutions:

$$f(t, x, \xi) = f(t_0, x - \xi(t - t_0), \xi) \text{ and } f(x, \xi) = f(\xi). \quad (2.3)$$

i.e., the distribution function is constant along molecular trajectory.

We assume the simplest model for the interaction of the molecules with the solid plates (see § 2.10). We assume that a fraction  $\alpha_\tau$  of the molecules is re-emitted with a Maxwellian distribution (in the system of coordinates fixed with respect to a plate) and a fraction  $(1 - \alpha_\tau)$  is reflected specularly:

$$\begin{aligned} f\left(\pm \frac{d}{2}, \xi, \xi_x \leq 0\right) &= (1 - \alpha_\tau) f^\pm\left(\pm \frac{d}{2}, -\xi_x, \xi_y, \xi_z\right) \\ &+ \alpha_\tau n^\pm \left(\frac{h^\pm}{\pi}\right)^{3/2} \exp\left\{-h^\pm \left[\xi_x^2 + \xi_y^2 + \left(\xi_z \mp \frac{w}{2}\right)^2\right]\right\} \\ &\quad \left(h^\pm = \frac{m}{2kT^\pm}\right), \end{aligned} \quad (2.4)$$

where the temperatures of the reflected molecules  $T^\pm$  are related to the wall temperatures  $T_{w^\pm}$  through the accommodation coefficients  $\alpha_e$  and  $\alpha_\tau$ ;  $f^\pm$  is the distribution function for molecules with velocities  $\xi_x \geq 0$ , respectively.

We first solve the problem, by assuming that the temperature of the reflected molecules  $T^\pm$  and one of the quantities  $n^\pm$  is assigned. To be definite, we assign  $n^-$ . This defines the density level.

According to (2.3), the velocity distribution function is the same at all points  $x$  of the flow and is equal to

$$\begin{aligned} f^\pm(x, \xi) &= f^\pm\left(\mp \frac{d}{2}, \xi\right) = f\left(\mp \frac{d}{2}, \xi, \xi_x \geq 0\right) \\ &= (1 - \alpha_\tau) f^\pm\left(\pm \frac{d}{2}, -\xi_x, \xi_y, \xi_z\right) \\ &+ \alpha_\tau n^\pm \left(\frac{h^\pm}{\pi}\right)^{3/2} \exp\left\{-h^\pm \left[\xi_x^2 + \xi_y^2 + \left(\xi_z \pm \frac{w}{2}\right)^2\right]\right\}. \end{aligned} \quad (2.5)$$

The single unknown quantity  $n^+$  must be chosen from the surface condition

$$u_x \left( -\frac{d}{2} \right) = u_x \left( +\frac{d}{2} \right) = u_x(x) = \int_{\xi_x > 0} \xi_x f^+(x, \xi) d\xi + \int_{\xi_x < 0} \xi_x f^-(x, \xi) d\xi = 0.$$

The number of particles  $N^-$  which cross a surface element normal to the  $x$  axis downward per unit time is equal to the number of particles  $N^+$  which cross the element upward. Since the accommodation coefficients  $\alpha_\tau$  on both the plates are assumed to be the same, the number of particles going upward and downward, reflected specularly and diffusely, are equal, i.e.,

$$\begin{aligned} & \int_{\xi_x > 0} \xi_x n^+ \left( \frac{h^+}{\pi} \right)^{3/2} \exp \left\{ -h^+ \left[ \xi_x^2 + \xi_y^2 + \left( \xi_z + \frac{w}{2} \right)^2 \right] \right\} d\xi \\ & + \int_{\xi_x < 0} \xi_x n^- \left( \frac{h^-}{\pi} \right)^{3/2} \exp \left\{ -h^- \left[ \xi_x^2 + \xi_y^2 + \left( \xi_z - \frac{w}{2} \right)^2 \right] \right\} d\xi = 0 \end{aligned}$$

or

$$\frac{n^-}{\sqrt{\pi h^-}} = \frac{n^+}{\sqrt{\pi h^+}}. \quad (2.6)$$

On the right side of formula (2.5) we replace  $f^\mp$  by  $f^\pm$ . Then

$$\begin{aligned} & f^\pm(x, \xi, \xi_x \geq 0) \\ & = \frac{(1 - \alpha_\tau)}{2 - \alpha_\tau} n^\mp \left( \frac{h^\mp}{\pi} \right)^{3/2} \exp \left\{ -h^\mp \left[ \xi_x^2 + \xi_y^2 + \left( \xi_z \mp \frac{w}{2} \right)^2 \right] \right\} \\ & + \frac{1}{2 - \alpha_\tau} n^\pm \left( \frac{h^\pm}{\pi} \right)^{3/2} \exp \left\{ -h^\pm \left[ \xi_x^2 + \xi_y^2 + \left( \xi_z \pm \frac{w}{2} \right)^2 \right] \right\}. \end{aligned} \quad (2.7)$$

For purely specular reflection,  $f^+(\xi_x) = f^-(\xi_x)$ , where this function may be assigned arbitrarily. However, with an arbitrarily small number of diffusely reflected particles ( $\alpha_\tau \ll 1$ ), the velocity distribution consists of two Maxwellian functions.

It may be seen from (2.7) that the distribution function at each point of the flow is discontinuous with respect to the velocities when  $\xi_x = 0$ .

#### § 4.2] COUETTE FLOW

Let us now find the relation between the temperatures  $T^\mp$  of the reflected molecules and the temperatures of the walls  $T_{w^\pm}$ . Let us examine the lower plate, for example. From formula (10.7) of Chapter II, in a coordinate system fixed with respect to the wall,

$$E^+ = (1 - \alpha_e) E^- + \alpha_e E_{w^+}, \quad (2.8)$$

where  $E^+$  and  $E^-$  are the energies of the reflected and incident molecules, respectively, and  $E_{w^+}$  is the energy which the molecules would carry away, if they were reflected with a Maxwellian distribution corresponding to the wall temperature  $T_{w^+}$ . The energy carried away per unit area is equal to [see Eq. (10.8) of Chapter II]

$$E^+ = (1 - \alpha_\tau) E^- + \alpha_\tau \frac{mn^+}{2h^+ \sqrt{\pi h^+}}. \quad (2.9)$$

Similarly,

$$E_{w^-} = (1 - \alpha_\tau) E^- + \alpha_\tau \frac{mn_{w^-}}{2h_{w^-} \sqrt{\pi h_{w^-}}}. \quad (2.10)$$

For  $E^-$ , according to (2.7), we have (in a coordinate system fixed with respect to the lower wall),

$$\begin{aligned} E^- &= \frac{m}{2} \int_{\xi_x < 0} \xi_x \xi^2 f^- d\xi \\ &= \frac{1 - \alpha_\tau}{2 - \alpha_\tau} \frac{mn^+}{2} \left( \frac{h^+}{\pi} \right)^{3/2} \int_{\xi_x < 0} |\xi_x| \xi^2 \exp \left\{ -h^+ [\xi_x^2 + \xi_y^2 + \xi_z^2] \right\} d\xi \\ &+ \frac{1}{2 - \alpha_\tau} \frac{mn^-}{2} \left( \frac{h^-}{\pi} \right)^{3/2} \int_{\xi_x < 0} |\xi_x| \xi^2 \exp \left\{ -h^- [\xi_x^2 + \xi_y^2 + (\xi_z + w)^2] \right\} d\xi \\ &= \frac{1 - \alpha_\tau}{2 - \alpha_\tau} \frac{mn^+}{2h^+ \sqrt{\pi h^+}} + \frac{1}{2 - \alpha_\tau} \frac{mn^-}{2h^- \sqrt{\pi h^-}} \left( 1 + \frac{mw^2}{4kT^-} \right). \end{aligned} \quad (2.11)$$

To determine  $n_{w^-}$  we must still calculate the number  $N^-$  of molecules striking the lower wall [see Eq. (10.12) of Chapter II]:

$$\frac{n_{w^-}}{2 \sqrt{\pi h_{w^-}}} = \frac{n^+}{2 \sqrt{\pi h^+}}. \quad (2.12)$$

Substituting expressions (2.9)–(2.11) into Eq. (2.8), and using relations (2.6) and (2.12), we obtain

$$\left[ \frac{1}{h^+} - (1 - \alpha_e) \frac{1 - \alpha_\tau}{2 - \alpha_\tau} \frac{1}{h^+} - \frac{1 - \alpha_e}{2 - \alpha_\tau} \frac{1}{h^-} - \alpha_e \frac{1}{h_{w^-}} - \frac{1 - \alpha_e}{2 - \alpha_\tau} \frac{w^2}{2} \right] \alpha_\tau = 0. \quad (2.13)$$

This equation contains two unknown quantities:  $T^\pm$ . It is evident that a completely similar equation, in which we only interchange the plus and minus superscripts, is satisfied on the upper plate. Those two equations allow us to express  $T^\mp$  in terms of  $T_{w^\pm}$ . It is not difficult to find the solution of that system. However, because it is cumbersome, we do not write down the solution for the general case.

When  $\alpha_e = 1$ , the reflected molecules are at the wall temperature  $T^\pm = T_{w^\pm}$ . For purely specular reflection ( $\alpha_\tau = 0$ ), the temperature of the molecules is evidently arbitrary. When the wall temperatures are equal ( $T_{w^-} = T_{w^+} = T_w$ ), the temperatures of the reflected molecules are also equal:

$$T^- = T^+ = T_w + \frac{1 - \alpha_e}{\alpha_e (2 - \alpha_\tau)} \frac{mw^2}{4k}. \quad (2.14)$$

In the case  $\alpha_e = 0$ , the energy of the reflected molecules in a coordinate system fixed with respect to the wall, is equal to the energy of the incident molecules. Therefore, the fact that the gas temperature increases as  $\alpha_e \rightarrow 0$  seems strange at first sight. However, it is easy to see that, for diffuse reflection and for any  $\alpha_e$ , the walls do work on the gas. For a finite  $\alpha_e$ , the increase in the energy of the gas due to that work is balanced by a loss of energy through the walls. The smaller the accommodation coefficient  $\alpha$ , the more difficult it is to transfer energy to the walls, and the higher the gas temperature. When  $\alpha_e \rightarrow 0$ , the gas temperature  $T^\mp \rightarrow \infty$ .

We now calculate the density, velocity, and temperature of the gas between the plates, as well as the friction stress  $P_{xz}$  and the heat flux  $q_x$ . Those quantities do not depend on  $x$  and are equal to

$$n = \int_{\xi_x > 0} f^+ d\xi + \int_{\xi_x < 0} f^- d\xi = \frac{1}{2}(n^+ + n^-). \quad (2.15)$$

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$$u_z = \int_{\xi_x > 0} \xi_z f^+ d\xi + \int_{\xi_x < 0} \xi_z f^- d\xi = \frac{w}{2} \frac{(n^- - n^+)}{n^- + n^+} = \frac{w}{2} \frac{\sqrt{T^+} - \sqrt{T^-}}{\sqrt{T^-} + \sqrt{T^+}}, \quad (2.16)$$

$$\begin{aligned} \frac{3}{2} kT &= \frac{m}{2n} \int_{\xi_x > 0} c^2 f^+ d\xi + \frac{m}{2n} \int_{\xi_x < 0} c^2 f^- d\xi \\ &= \frac{3k}{2} \sqrt{T^- T^+} + \frac{mw^2}{2} \frac{\sqrt{T^- T^+}}{(\sqrt{T^-} + \sqrt{T^+})^2}, \end{aligned} \quad (2.17)$$

$$\begin{aligned} P_{xz} &= m \int_{\xi_x > 0} c_x c_z f^+ d\xi + m \int_{\xi_x < 0} c_x c_z f^- d\xi \\ &= - \frac{\alpha_\tau}{2 - \alpha_\tau} \frac{mw}{4\sqrt{\pi}} \left( n^- \sqrt{\frac{2kT^-}{m}} + n^+ \sqrt{\frac{2kT^+}{m}} \right), \end{aligned} \quad (2.18)$$

$$\begin{aligned} q_x &= \frac{m}{2} \int_{\xi_x > 0} c_x c^2 f^+ d\xi + \frac{m}{2} \int_{\xi_x < 0} c_x c^2 f^- d\xi \\ &= \frac{\alpha_\tau}{2 - \alpha_\tau} \frac{m}{2\sqrt{\pi}} \left[ n^+ \left( \frac{2kT^+}{m} \right)^{3/2} - n^- \left( \frac{2kT^-}{m} \right)^{3/2} \right] \\ &\quad + \frac{\alpha_\tau}{2 - \alpha_\tau} \frac{mw^2}{8\sqrt{\pi}} \left( n^+ \sqrt{\frac{2kT^+}{m}} + n^- \sqrt{\frac{2kT^-}{m}} \right) \frac{\sqrt{T^+} - \sqrt{T^-}}{\sqrt{T^+} + \sqrt{T^-}}. \end{aligned} \quad (2.19)$$

Thus, the velocity and temperature of the gas (including the gas near the walls) is not equal to the velocity and temperature of the walls. The friction stress and the heat transfer do not depend on the distance between the plates, i.e., they are determined by the difference of velocity and of temperature, respectively, and not by their gradients.

It was shown above [formula (2.7)] that, in a free-molecule flow (i.e., with mean free path  $\lambda \rightarrow \infty$ ) the distribution function is discontinuous with respect to the velocities when  $\xi_x = 0$ . Let us point out, using the model equation (8.22) of Chapter II as an example, that, for the degenerate one-dimensional configuration examined here, with any finite  $\lambda$ , the distribution function must be continuous when  $\xi_x = 0$ . We consider molecules whose velocity vector makes a small angle with the plane  $x = \text{const}$ . No matter how large the mean free path  $\lambda$  (or the time  $\tau$  between collisions) and the velocity of a molecule may be, we can always choose  $\xi_x$

so small that in a time  $t$  of the order of several  $\tau$ , the molecule travels a distance  $\xi_x t$  in the direction of the  $x$  axis much smaller than the distance between the plates  $d$ , and undergoes a finite number of collisions. We may regard the distance to be so small that the macroscopic quantities which appear in  $f_0$  of Eq. (8.22) of Chapter II are constant, i.e., the function  $f_0$  is also constant. Then for trajectories inclined at a small angle to the plane  $x = \text{const}$ , the solution (8.26) of Chapter II is valid:

$$f(t, x, \xi) = f_0(x, \xi) + e^{-t/\tau} [f(0, x, \xi) - f_0(x, \xi)].$$

Therefore, for the model equation when  $\xi_x \rightarrow 0$ , the distribution function tends to a Maxwellian distribution appropriate to the flow parameters at any given point  $x$ . On the other hand, molecules which move at large angles with respect to the walls, i.e., with  $\xi_x > \xi d/\lambda = \xi/Kn$  undergo few collisions when  $\lambda > d$ , so that the distribution function of the molecules which move upward and downward is close to the distribution functions of the molecules which have been re-emitted from the lower and upper plates, respectively. The transition from the distribution law given by the walls to a local-equilibrium occurs in the velocity interval  $0 < \xi_x < \xi/Kn$ , i.e., the thickness of the region over which the discontinuity is spread is of order  $1/Kn$ . It is just because the distribution function for this problem at large Knudsen numbers varies sharply in a small velocity interval of order  $1/Kn$  that it is difficult to approximate it. Therefore, the moment methods are less accurate when  $Kn \gg 1$ . The appearance of this singularity is linked to the degenerate geometry of the problem. For example, in the problem of heat transfer between concentric spheres, this singularity disappears, and the distribution function is discontinuous for large finite Knudsen numbers.

It is easy to see that the solution of the complete Boltzmann equation has the same properties for molecules with a finite interaction radius. In that case, when  $\xi_x \rightarrow 0$ , the distribution function tends to  $J_1/J_2$  (see §2.7).

2. Weakly Perturbed Flows (Linear Theory,  $0 < Kn \leq \infty$ ). Let us now examine a Couette flow of arbitrary Knudsen number, but with a small relative velocity of the plates and a small temperature range between the plates  $T_w^+$ . Under those assumptions, the problem is linearized. However, even for

the linearized Boltzmann equation, the problem turns out to be complex. Therefore, we first examine the problem with the aid of the model equation. Hopefully, the qualitative conclusions obtained by means of that equation are valid also for the Boltzmann equation. The model equation may be treated as a Boltzmann equation for molecules with an interaction law close to Maxwellian.

The model equation (8.22) of Chapter II for our one-dimensional steady flow has the form

$$\xi_x \frac{\partial f}{\partial x} = An(f_0 - f). \quad (2.20)$$

If  $w$  is small and  $T^+$  and  $T^-$  differ only slightly, then the distribution function  $f$  differs only slightly from the Maxwellian distribution

$$f = f_{00}(1 + \varphi), \quad f_{00} = n_0 \left(\frac{h_0}{\pi}\right)^{3/2} e^{-h_0 \xi^2}, \quad (2.21)$$

where  $\varphi(x, \xi)$  is a small correction. For convenience we introduce the dimensionless velocity

$$v = \xi \sqrt{h_0}. \quad (2.22)$$

According to (2.21), the particle density and the temperature may be represented in the form

$$n(x) = n_0[1 + v(x)], \quad T(x) = T_0[1 + \tau(x)], \quad (2.23)$$

where

$$v(x) = \frac{1}{n_0} \int f_{00} \varphi d\xi, \quad \tau(x) = \frac{2}{3} \frac{1}{kn_0 T_0} \int \frac{m \xi^2}{2} f_{00} \varphi d\xi - v(x). \quad (2.24)$$

By definition, the flow velocity is equal to

$$u_z = \frac{1}{n_0} \int f_{00} \xi_z \varphi d\xi. \quad (2.25)$$

In this notation we have for  $f_0$  in the linear approximation

$$f_0 = f_{00} \left[ 1 + v + 2u_z v + \left(v^2 - \frac{3}{2}\right) \tau \right], \quad (2.26)$$

where  $u_1 = u_z \sqrt{h_0}$  is the dimensionless macroscopic velocity.

Substituting  $f$  from (2.21) and  $f_0$  from (2.26) into Eq. (2.20), we write it in the following linearized form:

$$\frac{v_x}{\alpha} \frac{\partial \varphi}{\partial x_1} = -\varphi + v + 2v_z u_1 + \left(v^2 - \frac{3}{2}\right) \tau, \quad (2.27)$$

where

$$x_1 = \frac{x}{d}, \quad \alpha = A n_0 d \sqrt{h_0} = \frac{p_0 d \sqrt{h_0}}{\mu_0}.$$

Here we use the relation  $\mu_0 = k T_0 / A$  from §3.6, where  $\mu$  is the viscosity. The quantity  $\alpha$  is inversely proportional to the Knudsen number.

We assume for simplicity that the molecules reflected from the wall have a Maxwellian distribution in a system of coordinates fixed with respect to the plate, with a temperature corresponding to the wall temperature (the accommodation coefficients are equal to unity):

$$f_r(x_1 = \pm \frac{1}{2}, \xi_x \leq 0) = n^\pm \left(\frac{h^\mp}{\pi}\right)^{3/2} \exp\{-h^\mp(\xi^2 - w\xi_z)\}. \quad (2.28)$$

We note again that  $n^\pm$  is not equal to  $n(\mp \frac{1}{2}d)$  and  $T_w^\mp = T^\pm$  is not equal to  $T(\mp \frac{1}{2}d)$ . Putting

$$n^\pm = n_0(1 + v^\pm) \text{ and } T^\pm = T_0(1 + \tau^\pm) \quad (2.29)$$

and linearizing expression (2.28), we obtain

$$\varphi(x_1 = \pm \frac{1}{2}; v_x \leq 0) = v^\mp \pm v_z w_1 + \left(v^2 - \frac{3}{2}\right) \tau^\mp, \quad (2.30)$$

where  $w_1 = w\sqrt{h_0}$ .

It is easy to see that our problem may be divided into two parts: a pure shear problem with  $T_w^+ = T_w^-$ , for which  $v = \tau = 0$ , and a problem of heat transfer with  $w_1 = 0$ , for which  $u_1 = 0$ .

We examine the shear problem.\* Equation (2.27) for that problem takes the form

\*See E. P. Gross, E. A. Jackson, and S. Ziering, Ann. Phys. N. Y., Vol. 1, No. 2 (1957); D. R. Willis, "Rarefied Gas Dynamics," First Symposium, Pergamon Press, New York, 1960.

$$\frac{1}{\alpha} v_x \frac{\partial \varphi}{\partial x_1} = -\varphi + 2v_z u_1. \quad (2.31)$$

We seek a solution of the form

$$\varphi(x_1, v_x, v_y, v_z) = v_z w_1 \psi(x_1, v_x).$$

Recalling the definition of the macroscopic velocity (2.25), we obtain the equation

$$\begin{aligned} \frac{1}{\alpha} v_x \frac{\partial \psi}{\partial x_1} &= -\psi + g(x_1), \\ g(x_1) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-v_x^2} \psi(x_1, v_x) dv_x = \frac{2u_1}{w_1}. \end{aligned} \quad (2.32)$$

The boundary conditions for  $\psi$  take the form

$$\psi\left(x_1 = \pm \frac{1}{2}, v_x \leq 0\right) = \pm 1. \quad (2.33)$$

We denote by  $\psi^+(x_1, v_x)$  the part of the distribution function corresponding to  $v_x > 0$ , and by  $\psi^-(x_1, v_x)$  the distribution function for  $v_x < 0$ . Considering  $g(x_1)$  as a known function, we integrate Eq. (2.32) respectively from the upper and the lower boundaries to an arbitrary point  $x_1$ , taking account of the boundary condition (2.33):

$$\begin{aligned} \psi^\mp(x_1, v_x) &= \pm \exp\left[-\frac{\alpha}{v_x}\left(x_1 \mp \frac{1}{2}\right)\right] \\ &+ \frac{\alpha}{v_x} \int_{\mp \frac{1}{2}}^{x_1} g(s) \exp\left[-\frac{\alpha}{v_x}(x_1 - s)\right] ds. \end{aligned} \quad (2.34)$$

This is an integral form of Eq. (2.32), which, clearly, may also be obtained by linearizing Eq. (8.25) of Chapter II.

In the free-molecule flow, i.e., when  $\alpha \rightarrow 0$ , the solution is obviously

$$\psi^-(x_1) = 1, \quad \psi^+(x_1) = -1 \quad (2.35)$$

and, correspondingly,

$$\varphi^-(x_1) = v_z w_1, \quad \varphi^+(x_1) = -v_z w_1. \quad (2.36)$$

The solution at each point  $x_1$  is discontinuous with respect to the velocities of the molecules. It must be true that in a flow without collision, the distribution function must be the same at all points between the plates. Therefore, all the macroscopic quantities must also be constant. According to (2.25), the velocity is equal to

$$u_1 = \frac{w_1}{2} g(x_1) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 e^{-v_x^2} dv_x - \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-v_x^2} dv_x = 0, \quad (2.37)$$

i.e., the velocity between the plates is equal to the arithmetic mean of the wall velocities. In particular, the gas velocity at the wall is not equal to the velocity of the wall.

The friction stress is equal to

$$P_{xz} = m \int \xi_x (\xi_z - u_z) f d\xi = \frac{2kn_0 T_0}{\pi^{3/2}} \int v_x v_z e^{-v^2} \varphi dv. \quad (2.38)$$

Replacing  $\varphi$  by its value from (2.36), we have

$$P_{xz} (Kn = \infty) = -\frac{kn_0 T_0 w_1}{\sqrt{\pi}} = -p_0 w \sqrt{\frac{m}{2\pi k T_0}}. \quad (2.39)$$

This simple result, which may also be obtained from (2.18), is important for an understanding of the physics of the phenomenon. It is easy to verify that, in the hydrodynamic Navier-Stokes approximation for the linear problem being examined,

$$P_{xz} (Kn \rightarrow 0) = -\mu \frac{w}{d}, \quad (2.40)$$

the viscosity  $\mu$  does not depend on the pressure. Therefore, in the hydrodynamic approximation (i.e., for large  $\alpha$ ), the stress is proportional to the velocity gradient and does not depend on the pressure. In the limit of low pressures, the velocity gradient in the gas is equal to zero but, in spite of that there is a friction stress proportional to the pressure and to the relative velocity of the plates.

The presence of a stress in the flow with constant hydrodynamic magnitudes once again indicates that, in the general case, the stress tensor and the heat-flux vector cannot be expressed in terms of the five hydrodynamic quantities and their derivatives of

any order and, therefore, the general conservation equations [Eqs. (1.8)-(1.10) of Chapter III] cannot be closed. Solutions may be constructed by the method of successive approximations for flows close to free-molecule flow, i.e., for small  $\alpha$ , by taking as the best approximation the present free-molecule solution.

The method of successive approximations may be constructed in two ways.

First, we may use the integral form (2.34), and substitute the free-molecule value  $g(s) = 0$  into the right side; then,

$$\psi^\mp = \pm \exp \left[ -\frac{\alpha}{v_x} \left( x_1 \mp \frac{1}{2} \right) \right]. \quad (2.41)$$

Along with the integral form (2.34), we may introduce the integral form which is a linearization of (8.24) of Chapter II:

$$\psi^\mp = \pm 1 \mp \frac{\alpha}{v_x} \int_{\pm 1/2}^{x_1} [-\psi(s, v_x) + g(s)] ds. \quad (2.42)$$

Substituting the free-molecule values (2.35) into the right side of the equation, we obtain

$$\psi^\mp = \pm 1 \mp \frac{\alpha}{v_x} \left( x_1 \mp \frac{1}{2} \right). \quad (2.43)$$

The second method of successive approximations is equivalent to expanding the distribution function in a series in  $\alpha$ . In this case, the singularity mentioned above appears. This singularity is related to the degenerate geometry of the flow, and from its existence we cannot draw the conclusion that the expansion in  $\alpha$  is inapplicable for near free-molecule flows with different geometries. It may be seen from (2.43) that the singularity does not allow us to find the next approximation or to calculate the flow velocity. Yet we can calculate the friction stress  $P_{xz}$  since, in this case,  $\psi$  is multiplied by  $v_x$  and the singularity vanishes:

$$P_{xz} = -\frac{p_0 w_1}{\sqrt{\pi}} \left( 1 - \frac{\sqrt{\pi}}{2} \alpha \right). \quad (2.44)$$

It is easy to see that the friction stress calculated by means of (2.41) agrees with (2.44), up to quantities of higher order.

The velocity itself is finite, according to (2.41), and, as  $\alpha \rightarrow 0$ , is equal to

$$\begin{aligned} \frac{2u_1}{w_1} &= g(x_1) = \frac{1}{\sqrt{\pi}} \left\{ J_0 \left[ \alpha \left( \frac{1}{2} - x_1 \right) \right] - J_0 \left[ \alpha \left( \frac{1}{2} + x_1 \right) \right] \right\} \\ &\simeq \frac{1}{\sqrt{\pi}} \left\{ \alpha \left( \frac{1}{2} - x_1 \right) \ln \left[ \alpha \left( \frac{1}{2} - x_1 \right) \right] - \alpha \left( \frac{1}{2} + x_1 \right) \ln \left[ \alpha \left( \frac{1}{2} + x_1 \right) \right] \right\}. \quad (2.45) \end{aligned}$$

where

$$J_m(x) = \int_0^\infty v^m e^{-(v+\frac{x}{v})} dv, \quad \frac{dJ_m}{dx} = -J_{m-1}. *$$

When  $x \rightarrow 0$ ,

$$J_0(x) \rightarrow \frac{\sqrt{\pi}}{2} + x \ln x.$$

Therefore, the dependence of the velocity on  $\alpha$  (on Knudsen number) for  $\alpha \ll 1$  is not analytic.

To find the velocity for arbitrary  $\alpha$ , we may construct an integral equation directly for the velocity  $u_1$  (or  $g$ ), omitting the distribution function. In fact, by substituting (2.34) into the definition of  $g(x_1)$ , we obtain the integral equation

$$\begin{aligned} \sqrt{\pi} g(x_1) &= J_0 \left[ \alpha \left( \frac{1}{2} - x_1 \right) \right] - J_0 \left[ \alpha \left( \frac{1}{2} + x_1 \right) \right] \\ &+ \alpha \int_0^{1/2} [J_{-1}[\alpha(x_1 - s)] - J_{-1}[\alpha(x_1 + s)]] g(s) ds. \quad (2.46) \end{aligned}$$

That equation has been solved numerically. Figure 20 shows the results of the numerical solution, and also compares them with values of the velocity calculated from formula (2.45), for small  $\alpha$ .

The successive approximation process may be continued.<sup>11</sup> It may be shown<sup>†</sup> that the process converges for any finite  $\alpha$ . However, the convergence deteriorates as  $\alpha$  increases. In practice,

\*The properties of this integral have been studied in the paper: M. Abramowitz, J. Math. Phys., 32: 188 (1953).

†D. R. Willis, "Rarefied Gas Dynamics," Second Symposium, Academic Press, New York, 1961.

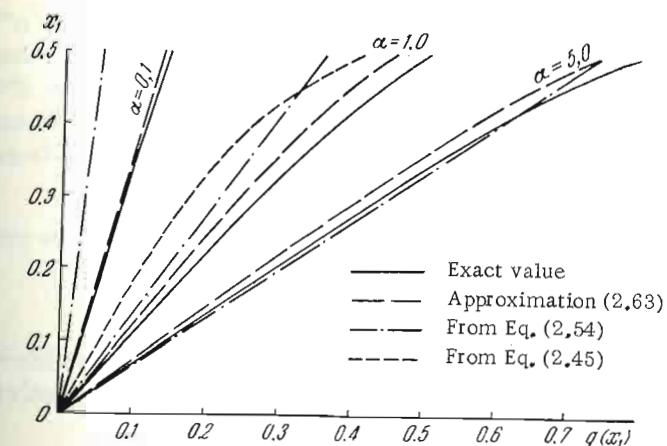


Fig. 20

this method is applicable only when  $\alpha < 1$ , since too many approximations are required for large  $\alpha$ .

We can find a solution of Eq. (2.20) for our problem in the form of a Hilbert series [see formulas (6.3) and (6.4) of Chapter III]:

$$f = f_0 + \frac{(-1)}{An} \frac{df_0}{dt} + \frac{(-1)^2}{An} \frac{d}{dt} \left( \frac{1}{An} \frac{df_0}{dt} \right) + \dots$$

In our case, according to (2.26),

$$\frac{df_0}{dt} = \xi_x \frac{\partial f_0}{\partial x} = 2\xi_x v_z f_{00} \frac{du_1}{dx}$$

and for  $\psi$  we obtain a series in  $v_x$  whose coefficients depend on  $x_1$ :

$$\psi = 2 \frac{u_1}{w_1} - \frac{2}{aw_1} \frac{du_1}{dx} v_x + \dots \quad (2.47)$$

If the series (2.47) converges with respect to  $v_x$  for all  $v_x$ , it follows from the boundary conditions (2.33) with  $v_x = 0$ , that

$$u_1 \left( \pm \frac{1}{2} \right) = \pm \frac{w_1}{2} \quad \text{or} \quad u_z \left( \pm \frac{d}{2} \right) = \pm \frac{w}{2},$$

i.e., at the wall the gas satisfies a no-slip condition, which can in fact occur only for  $\alpha \rightarrow \infty$ .

The series (2.47) cannot converge with respect to  $\alpha^{-1}$  for any  $v_x$ , since  $v_x$  may always be chosen large enough so that the terms of the series will not decrease as  $n \rightarrow \infty$  unless  $d^m u_i / dx_1^m$  goes to zero, after a certain value of  $m$ . However, it is easy to see that we cannot satisfy Eq. (2.32) by means of a finite polynomial.

The same conclusion is reached by examining the general series in  $v_x$ :

$$\psi = a_0(x_1) + a_1(x_1)v_x + \dots$$

Moreover, we can try to satisfy Eq. (2.32) and the boundary conditions with a solution in the form of series in  $v_x$  for each of the functions

$$\psi^\pm = a_0^\pm(x_1) + a_1^\pm(x_1)v_x + \dots, \quad (2.48)$$

i.e., with solutions which are discontinuous in  $v_x$ . If those series converge, then, by substituting them into Eq. (2.32), and equating coefficients of equal powers of  $v_x$ , we obtain

$$a_0^+ = a_0^- = g = J_0(0)(a_0^- + a_0^+) + J_1(0)(a_1^+ - a_1^-) \\ + J_2(0)(a_2^+ + a_2^-) + \dots,$$

$$a_1^\pm = \frac{(-1)}{\alpha} \frac{da_0^\pm}{dx_1}, \quad a_n^\pm = \frac{(-1)}{\alpha} \frac{da_{n-1}^\pm}{dx_1} = \frac{(-1)^n}{\alpha^n} \frac{d^n a_0^\pm}{dx_1^n},$$

$$J_n(0) = \int_0^\infty v^n e^{-v^2} dv.$$

Thus, the functions  $a_n^+$  are equal to  $a_n^-$ , and once again we arrive at the Hilbert series (2.47), by means of which, as noted, we cannot satisfy the boundary conditions.

Nevertheless, if we do not attempt to satisfy Eq. (2.32) exactly, we may use polynomials of type (2.48) as distribution functions to find an average solution by means of the method of moments.

We examine the simplest approximation

$$\psi^\pm = a_0^\pm(x), \quad (2.49)$$

corresponding to the linearization of the two-sided Maxwellian distribution

$$f^\pm = n_0 \left( \frac{h_0}{\pi} \right)^{3/2} \exp \left[ -h_0 [\xi_x^2 + \xi_y^2 + (\xi_z - u_z^\pm)^2] \right].$$

To find the two functions  $a_0^\pm$  we must construct two moment equations. We designate the moment of order  $n$  by  $M_n$ :

$$M_n = \frac{1}{V\pi} \int_{-\infty}^{+\infty} \psi v_x^n e^{-v_x^2} dv_x. \quad (2.50)$$

By multiplying Eq. (2.32) successively by  $\pi^{-1/2} e^{-v_x^2}$  and  $\pi^{-1/2} v_x e^{-v_x^2}$  and integrating with respect to  $v_x$  from  $-\infty$  to  $+\infty$ , we obtain

$$\frac{dM_1}{dx_1} = 0, \quad \frac{1}{\alpha} \frac{dM_2}{dx_1} = -M_1. \quad (2.51)$$

Therefore,

$$M_1(x_1) = c_1 \text{ and } M_2(x_1) = -\alpha c_1 x_1 + c_2, \quad (2.52)$$

where  $c_1$  and  $c_2$  are constants.

On the other hand, by substituting  $\psi$  into (2.50) with  $n = 1$  and  $n = 2$ , we have

$$M_1 = \frac{1}{2V\pi} (a_0^+ - a_0^-), \quad M_2 = \frac{1}{2} (a_0^+ + a_0^-).$$

Therefore,

$$\psi^\pm = a_0^\pm = (\pm \sqrt{\pi} - 2\alpha x_1) c_1 + 2c_2,$$

and, by determining  $c_1$  and  $c_2$  from the boundary conditions (2.33), we obtain, finally,

$$\psi^\pm = \frac{2\alpha x_1 \mp \sqrt{\pi}}{\alpha + V\pi}. \quad (2.53)$$

When  $\alpha \rightarrow 0$ , the solution tends to the exact free-molecule solution  $\psi^\pm = \mp 1$ . When  $\alpha \rightarrow \infty$ , the solution tends to the exact Navier-Stokes solution  $\psi = 2x_1$ .

For arbitrary Knudsen numbers we have

$$u_1 = \frac{w_1}{2} g = \frac{w_1 a}{a + V\pi} x_1, \quad (2.54)$$

$$P_{xz} = -\frac{p_0 w_1}{a + V\pi} = -\frac{p_0 w \sqrt{h_0}}{\frac{p_0 d}{\mu} \sqrt{h_0} + V\pi}. \quad (2.55)$$

In formula (2.55), the constant A which appears in the model equation has been expressed in terms of  $\mu$  by the formula  $\mu = kT_0/A$ , obtained in § 3.6. In the limiting cases we have

$$P_{xz}(a \rightarrow 0) = \frac{p_0 w_1}{V\pi}, \quad P_{xz}(a \rightarrow \infty) = -\mu \frac{w}{d},$$

i.e., the exact values for the free-molecule flow and for a continuous medium, respectively.

Figure 21 shows comparison of values of  $P_{xz}/P_{xz}^\infty$ , where  $P_{xz}^\infty$  is the value of  $P_{xz}$  when  $Kn=\infty$ , and  $1-g(\frac{1}{2})$ , obtained by numerical solution of the integral equation (2.46) and from formulas (2.54) and (2.55), which correspond to the simplest approximation (2.49). In the latter case,

$$\frac{P_{xz}}{P_{xz}^\infty} = 1 - g\left(\frac{1}{2}\right) = \frac{V\pi}{a + V\pi}.$$

Figure 20 also presents velocity profiles calculated from formula (2.54).

It is clear from the comparison presented that even this simplest approximation gives satisfactory values of the stress (with maximum error of 7%). The values of the velocities are approximated less accurately (with an error up to 25%).

More accurate solutions may be obtained by means of more complex approximations of the type (2.48)\* or by successive substitution of the solution found for  $g(x)$  into the right side of Eq. (2.46).†

As  $a \rightarrow \infty$ , the distribution function must tend to the Enskog-Chapman distribution (2.47). The approximation (2.49) allows us

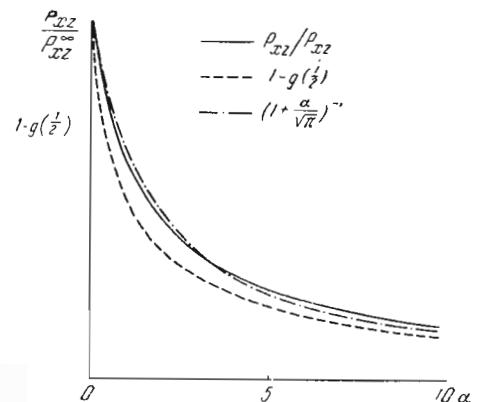


Fig. 21

to take into account only the first Euler term of that expansion. For the distribution function to go over to the Navier-Stokes function when  $\alpha \rightarrow \infty$ , we require the four-moment approximation

$$\psi^\pm = a_0^\pm + a_1^\pm v_x. \quad (2.48a)$$

In that approximation we require four moment equations to find the four unknown quantities  $a_0^\pm$  and  $a_1^\pm$ . Those equations may be obtained by integration of Eq. (2.32) when multiplied by some powers of  $v_x$ , either over the whole of velocity space or over the half-spaces. We present only some results of the calculations\* without giving the computations.

Table 5 illustrates the following methods: (1) exact solution; (2) four-moment approximation; (3) solution obtained by iteration of the four-moment approximation; (4) two-moment approximation (2.49); (5) integrated two-moment approximation.

We see on Table 5 that the approximation (2.48a) is considerably more accurate than the two-moment approximation. The analogous six-moment approximation differs only a little from the four-moment case, as calculations have shown.† It should be noted

\*E. P. Gross and S. Ziering, Phys. Fluids, Vol. 1, No. 3 (1958).

†E. P. Gross and S. Ziering, Phys. Fluids, Vol. 2, No. 4 (1959).

\*D. R. Willis, Phys. Fluids, Vol. 5, No. 2 (1962).

†See the first paper by Gross and Ziering cited on page 308.

Table 5

$\alpha$	Method				
	1	2	3	4	5
Slope of the velocity profile $d/dx(u_1/w_1)$ at the point $x = 0$					
0.01	0.02509	0.01294	0.02516	0.00561	0.02510
0.1	0.1320	0.1134	0.1310	0.0532	0.1277
1.0	0.4444	0.4890	0.4603	0.3607	0.4065
$1 - (P_{xz}/P_{xz}^\infty)$					
0.1	0.008719	0.008746		0.005610	0.008741
0.1	0.07412	0.07846		0.05340	0.07626
1.0	0.3992	0.4101		0.3637	0.4113

also that the simplest approximation (2.49) reduces to a linear velocity profile for all Knudsen numbers, i.e., by means of this approximation we cannot detect the boundary Knudsen layer which is exhibited satisfactorily by the four-moment approximation.

We can also obtain better accuracy by the two-moment approximation, with more correct account of the nonanalytical dependence of  $f$  on  $\alpha$  [approximation (4.10) of Chapter III].\*

For the linear problem examined here, that approximation takes the form

$$\psi^\pm = \mp \exp\left[-\frac{\alpha}{v_x}\left(x_1 \pm \frac{1}{2}\right)\right] \pm a_0^\pm \left\{1 - \exp\left[-\frac{\alpha}{v_x}\left(x_1 \pm \frac{1}{2}\right)\right]\right\}. \quad (2.56)$$

For small  $\alpha$ , the main term of that expression is the first, which coincides with the solution (2.41), and deals quite accurately with the nonanalytical character of the distribution function for near free-molecule flows.

To find the functions  $a_0^\pm$ , we use the moment equations (2.51) and their solutions (2.52). The problem differs from that examined above since the moments  $M_1$  and  $M_2$  are related differently to the functions  $a_0^\pm$ . In order to establish that relation for the present case, we substitute expression (2.56) into the definition (2.50) of the moments  $M_i$ , and thus obtain:

\* See S. F. Shen, "Rarefied Gas Dynamics," Third Symposium, Academic Press, New York, 1963.

#### § 4.2] COUETTE FLOW

$$M_1 = \frac{a_0^+}{V\pi} \left\{ \frac{1}{2} - J_1 \left[ a \left( \frac{1}{2} + x_1 \right) \right] \right\} - \frac{a_0^-}{V\pi} \left\{ \frac{1}{2} - J_1 \left[ a \left( \frac{1}{2} - x_1 \right) \right] \right\} \\ - \frac{1}{V\pi} \left\{ J_1 \left[ a \left( \frac{1}{2} + x_1 \right) \right] + J_1 \left[ a \left( \frac{1}{2} - x_1 \right) \right] \right\} = c_1, \quad (2.57)$$

$$M_2 = \frac{a_0^+}{V\pi} \left\{ \frac{V\pi}{4} - J_2 \left[ a \left( \frac{1}{2} + x_1 \right) \right] \right\} \\ + \frac{a_0^-}{V\pi} \left\{ \frac{V\pi}{4} - J_2 \left[ a \left( \frac{1}{2} - x_1 \right) \right] \right\} \\ + \frac{1}{V\pi} \left\{ J_2 \left[ a \left( \frac{1}{2} - x_1 \right) \right] - J_2 \left[ a \left( \frac{1}{2} + x_1 \right) \right] \right\} = -ac_1x_1 + c_2. \quad (2.58)$$

The function  $\psi^\pm$  in the form (2.56) automatically satisfies the boundary conditions (2.33). The constants  $c_1$  and  $c_2$  are found as eigenvalues.

For  $x_1 = \frac{1}{2}$ , according to (2.57) and (2.58), we have

$$\frac{a_0^+}{V\pi} \left[ \frac{1}{2} - J_1(a) \right] = c_1 + \frac{1}{V\pi} [J_1(a) + J_1(0)], \\ \frac{a_0^+}{V\pi} \left[ \frac{V\pi}{4} - J_2(a) \right] = -\frac{1}{2} ac_1 + c_2 + \frac{1}{V\pi} [J_2(a) - J_2(0)].$$

Eliminating  $a_0^\pm$ , we obtain

$$\frac{\frac{1}{2} - J_1(a)}{\frac{V\pi}{4} - J_2(a)} = \frac{c_1 + \frac{1}{V\pi} [J_1(a) + J_1(0)]}{-\frac{1}{2} ac_1 + c_2 + \frac{1}{V\pi} [J_2(a) - J_2(0)]}. \quad (2.59)$$

Similarly, for  $x_1 = -\frac{1}{2}$ ,

$$-\frac{a_0^-}{V\pi} \left[ \frac{1}{2} - J_1(a) \right] = c_1 + \frac{1}{V\pi} [J_1(a) + J_1(0)], \\ \frac{a_0^-}{V\pi} \left[ \frac{V\pi}{4} - J_2(a) \right] = \frac{1}{2} ac_1 + c_2 - \frac{1}{V\pi} [J_2(a) - J_2(0)]$$

and

$$\frac{\frac{1}{2} - J_1(\alpha)}{\frac{V\pi}{4} - J_2(\alpha)} = \frac{c_1 + \frac{1}{V\pi} [J_1(\alpha) + J_1(0)]}{-\frac{1}{2} \alpha c_1 - c_2 + \frac{1}{V\pi} [J_2(\alpha) - J_2(0)]}. \quad (2.60)$$

It follows from (2.59) and (2.60), that

$$c_2 = 0 \text{ and } c_1 = -\frac{1}{V\pi \left[ 1 + \frac{1}{2} \alpha \frac{\frac{1}{2} - J_1(\alpha)}{\frac{V\pi}{4} - J_2(\alpha)} \right]}. \quad (2.61)$$

In those relations we used the result

$$J_0(0) = \frac{V\pi}{2}, \quad J_1(0) = \frac{1}{2}, \quad J_2(0) = \frac{V\pi}{4}.$$

Knowing  $c_1$  and  $c_2$ , we find  $a_0^\pm$ , and thereby complete the solution. After simple manipulations, we obtain final expressions for the stress and the velocity:

$$P_{xz} = -\frac{p_0 w_1}{V\pi} \frac{\frac{V\pi}{4} - J_2(\alpha)}{\frac{V\pi}{4} - J_2(\alpha) + \frac{\alpha}{2} \left[ \frac{1}{2} - J_1(\alpha) \right]}, \quad (2.62)$$

$$u_1 = \frac{w_1}{2V\pi} \left\{ J_0 \left[ a \left( \frac{1}{2} - x_1 \right) \right] - J_0 \left[ a \left( \frac{1}{2} + x_1 \right) \right] + a_0^+ \left( \frac{V\pi}{2} - J_0 \left[ a \left( \frac{1}{2} + x_1 \right) \right] \right) + a_0^- \left( \frac{V\pi}{2} - J_0 \left[ a \left( \frac{1}{2} - x_1 \right) \right] \right) \right\}, \quad (2.63)$$

where

$$a_0^+ = \frac{k_1 \left( \frac{V\pi}{4} - J_2 \left[ a \left( \frac{1}{2} - x_1 \right) \right] \right) + k_2 \left( \frac{1}{2} - J_1 \left[ a \left( \frac{1}{2} - x_1 \right) \right] \right)}{\Delta V\pi},$$

$$a_0^- = -\frac{k_1 \left( \frac{V\pi}{4} - J_2 \left[ a \left( \frac{1}{2} + x_1 \right) \right] \right) - k_2 \left( \frac{1}{2} - J_1 \left[ a \left( \frac{1}{2} + x_1 \right) \right] \right)}{\Delta V\pi},$$

$$\Delta = \frac{1}{\pi} \left\{ \left( \frac{1}{2} - J_1 \left[ a \left( \frac{1}{2} + x_1 \right) \right] \right) \left( \frac{V\pi}{4} - J_2 \left[ a \left( \frac{1}{2} - x_1 \right) \right] \right) + \left( \frac{1}{2} - J_1 \left[ a \left( \frac{1}{2} - x_1 \right) \right] \right) \left( \frac{V\pi}{4} - J_2 \left[ a \left( \frac{1}{2} + x_1 \right) \right] \right) \right\},$$

$$k_1 = c_1 + \frac{1}{V\pi} \left\{ J_1 \left[ a \left( \frac{1}{2} + x_1 \right) \right] + J_1 \left[ a \left( \frac{1}{2} - x_1 \right) \right] \right\},$$

$$k_2 = -\alpha c_1 x_1 - \frac{1}{V\pi} \left\{ J_2 \left[ a \left( \frac{1}{2} - x_1 \right) \right] - J_2 \left[ a \left( \frac{1}{2} + x_1 \right) \right] \right\}.$$

It is easy to see that as  $\alpha \rightarrow 0$ , those formulas go over to formulas (2.44) and (2.45), respectively, as obtained by the method of successive approximations. The velocity field calculations by formula (2.63) are presented on Fig. 20. As expected, the accuracy of the approximation (2.63) is greater than that of the approximation (2.49). This compensates for the somewhat increased complication of the calculations.

For small  $\alpha$  (i.e., for large Knudsen numbers), the solution may be improved considerably by means of iteration. For example, if we substitute the solution for  $g = 2u_1/w_1$  found by means of approximation (2.49) into the right side of the integral equation (2.34), then the solution obtained after quadrature proves to be considerably more accurate, as is seen on Table 5.

For simplicity, we have demonstrated the different moment methods and their accuracy on the model equation. All those methods are also applicable to the linearized Boltzmann equation. In principle, the solution is constructed as for the model equation. For an arbitrary molecular interaction law, the basic difficulty lies in calculation of the moments from the collision integral. In what follows we examine only Maxwellian molecules for which this difficulty is easily overcome.

The Boltzmann equation for the one-dimensional problem examined here has the form

$$\xi_x \frac{\partial f}{\partial x} = J(x, \xi). \quad (2.64)$$

We approximate the distribution function by a two-sided Maxwellian distribution or, in the linearized formulation, by formula (2.49). That approximation contains two unknown functions and, therefore, two moment equations are required. The first five moment equations [the equations of conservation of mass, momentum, and energy (1.8)-(1.10) of Chapter III] for the problem examined obviously give

$$u_x = \text{const} = 0, \quad P_{xx} = \text{const}, \quad P_{xy} = \text{const}, \quad P_{xz} = \text{const}, \quad q_x = \text{const}. \quad (2.65)$$

It is easy to see that, with approximation (2.49), i.e., when

$$f^\pm = f_{00} [1 + v_z w_1 a_0^\pm(x)], \quad (2.49a)$$

the conditions  $u_x = P_{xy} = q_x = 0$  and  $P_{xx} = p_0$  are identically satisfied. The fifth condition has the form

$$P_{xz} = m \int f^\pm \xi_x (\xi_z - u_z) d\xi = \frac{1}{2} \frac{p_0 w_1}{V\pi} (a_0^+ - a_0^-) = c_1, \quad (2.66)$$

where  $c_1$  is a constant of integration, which must be found from the boundary conditions.

To determine the two functions  $a_0^+$  and  $a_0^-$ , we still require one higher-order moment equation. The choice of that equation is arbitrary. We multiply Eq. (2.64) by  $\xi_x \xi_z$ , and integrate with respect to  $\xi$ ; we have:

$$\frac{d}{dx} m \int f \xi_x^2 \xi_z d\xi = m \int \xi_x \xi_z J d\xi. \quad (2.67)$$

That equation is equivalent to the second equation (2.51). In the notation of §3.3, the integral appearing on the right is equal to

$$\begin{aligned} m \int \xi_x \xi_z J d\xi &= m \int \xi_x (\xi_z - u_z) J d\xi = kT I_{H_{xz}^{(2)}} \\ &= -6 \cdot 0.343 \left( \frac{8B}{m} \right)^{1/2} n_0 P_{xz} = -\frac{p}{\mu} P_{xz}, \end{aligned}$$

for Maxwellian molecules, where  $\mu$  is the viscosity. Then Eq. (2.67) takes the form

$$\frac{d}{dx} m \int f \xi_x^2 \xi_z d\xi = -\frac{p}{\mu} P_{xz}, \quad (2.68)$$

and, therefore,

$$\int f \xi_x^2 \xi_z d\xi = -\frac{p}{\mu m} P_{xz} x + c_2, \quad (2.69)$$

where  $c_2$  is a constant of integration. The integral appearing on the left for the distribution function (2.49a) is equal to

$$\int f \xi_x^2 \xi_z d\xi = \frac{n_0 w_1}{8h_0^{3/2}} (a_0^+ + a_0^-). \quad (2.70)$$

Then relation (2.69) takes the form

$$\frac{n_0 w_1}{8h_0^{3/2}} (a_0^+ + a_0^-) = -\frac{p}{m\mu} c_1 x + c_2. \quad (2.71)$$

By finding the constants  $c_1$  and  $c_2$  from the boundary conditions  $a^+(d/2) = 1$  and  $a_0^-(d/2) = 1$ , after simple manipulations, we again arrive at the solution (2.53):

$$\psi^\pm = a_0^\pm = \frac{2ax_1 \mp \sqrt{h_0}}{a + \sqrt{h_0}}, \quad (1.53a)$$

where, as above,

$$a = \frac{p_0 d \sqrt{h_0}}{\mu}.$$

A similar analysis for elastic-sphere molecules\* leads to the same relations, but since the values of the viscosity for molecules with different interaction laws are different, the flows are not identical.

As shown in §3.6, the dependence of the viscosity on temperature, obtained from the model equation, is the same as for Maxwellian molecules. In any given case, we may therefore choose the constant A in the relaxation equation in such a way that the solution of the Boltzmann equation for Maxwellian molecules and the solution of the model equation agree completely: namely,  $A = kT/\mu$ .

We have presented a detailed analysis of the shear problem in the linear approximation. All our methods are equally applicable to the problem of heat transfer between flat plates in the linear approximation. For that problem, the approximating function, which properly goes to the free-molecule and Navier-Stokes limits, has a somewhat more complicated form [cf. (2.48a)]:

$$\varphi = a_0^\pm + a_1^\pm v^2 + a_2^\pm v_x \left( v^2 - \frac{5}{2} \right). \quad (2.72)$$

A solution of the linear heat-transfer problem by the method of moments has been given by Gross and Ziering.<sup>†</sup> Since the methods of solution of the linearized heat-transfer problem differ only in details from the methods examined above for the solution of the shear problem, we do not dwell on them, and go on to examine some nonlinear problems.

\*E. P. Gross and S. Ziering, Phys. Fluids, Vol. 2, No. 6 (1959).

<sup>†</sup>See the paper by Gross and Ziering just cited and the paper by Ziering [S. Ziering, Phys. Fluids, Vol. 3, No. 4 (1960)].

**3. Nonlinear Problems. The Moment Method.** We first examine the solution of the Couette problem for arbitrary Knudsen numbers by the method of moments. We consider the full Boltzmann equation. To simplify the calculation of moments from the collision integral, we consider the gas to be Maxwellian. In the nonlinear approximation the shear problem cannot be separated from the heat-flux problem between the plates.

We take the simplest approximating function in the form of the two-sided Maxwellian distribution\*

$$\begin{aligned} f^{\pm}(x, \xi, \xi_x \geq 0) \\ = n_{1,2}(x) h_{1,2}^{3/2}(x) \exp\{-h_{1,2}(x)[\xi_x^2 + \xi_y^2 + (\xi_z - u_{1,2}(x))^2]\}, \end{aligned} \quad (2.73)$$

where  $n_{1,2}$ ,  $h_{1,2}$ , and  $u_{1,2}$  are six unknown functions; the subscripts 2 and 1 refer to the distribution functions for  $\xi_x \geq 0$ , respectively (see §3.4).

To determine those six unknown functions we need six moment equations. From the first five moment equations (1.8)-(1.10) of §2.1 we have

$$\left. \begin{aligned} \rho u_x &= \text{const}, & P_{xx} &= \text{const}, & P_{xy} &= \text{const}, & P_{xz} &= \text{const}, \\ q_x - P_{xz} u_z &= \text{const}. \end{aligned} \right\} \quad (2.74)$$

Since  $u_x = 0$  at the walls,

$$u_x(x) \equiv 0. \quad (2.75)$$

The equation  $P_{xy} = \text{const} = 0$  is satisfied identically. We form two additional moment equations by multiplying the Boltzmann equation by  $m\xi_x\xi_z$  and  $\frac{1}{2}m\xi^2\xi_x$ , and integrating over all velocities. For Maxwellian molecules we have [see §3.3, formulas (3.31) and (3.47)]:

$$\frac{d}{dx} \int m f \xi_x^2 \xi_z d\xi = -\frac{p}{\mu} P_{xz}, \quad (2.76)$$

$$\frac{d}{dx} \int \frac{m}{2} f \xi_x^2 \xi^2 d\xi = -\frac{p}{\mu} \left( P_{xz} u_z + \frac{2}{3} q_x \right). \quad (2.77)$$

\*C. Liu and L. Lees, "Rarefied Gas Dynamics," Second Symposium, Academic Press, New York, 1961.

Substituting the approximating function (2.73) into the integrals appearing here and into the definitions of  $u_x$ ,  $u_z$ ,  $P_{xx}$ ,  $P_{xz}$ , and  $q_x$  we obtain six equations to determine the six unknown functions.

The continuity equation

$$\bar{n}_1 \bar{V} \bar{T}_1 = \bar{n}_2 \bar{V} \bar{T}_2. \quad (2.78a)$$

The momentum equations

$$\bar{n}_1 (\bar{u}_2 - \bar{u}_1) \bar{V} \bar{T}_1 = \alpha_1, \quad (2.78b)$$

$$\bar{n}_1 \bar{T}_1 + \bar{n}_2 \bar{T}_2 = \alpha_2. \quad (2.78c)$$

The energy equation

$$\bar{n}_1 \bar{V} \bar{T}_1 [\bar{T}_2 - \bar{T}_1 + \frac{\kappa}{4} M^2 (\bar{u}_2^2 - \bar{u}_1^2)] = \alpha_2 \alpha_3. \quad (2.78d)$$

The stress tensor equation

$$\frac{d}{dx_1} (\bar{n}_1 \bar{u}_1 \bar{T}_1 + \bar{n}_2 \bar{u}_2 \bar{T}_2) + \frac{1}{\sqrt{2\pi\kappa}} \frac{\text{Re}}{M} \alpha_1 (\bar{n}_1 + \bar{n}_2) = 0. \quad (2.78e)$$

The heat-flux equation

$$\begin{aligned} \frac{d}{dx_1} (\bar{n}_1 \bar{T}_1^2 + \bar{n}_2 \bar{T}_2^2) + \frac{\kappa M^2}{5} \frac{d}{dx_1} (\bar{n}_1 \bar{T}_1 \bar{u}_1^2 + \bar{n}_2 \bar{T}_2 \bar{u}_2^2) \\ - \frac{2}{5} \kappa M^2 \frac{\bar{n}_1 \bar{u}_1 + \bar{n}_2 \bar{u}_2}{\bar{n}_1 + \bar{n}_2} \frac{d}{dx_1} (\bar{n}_1 \bar{T}_1 \bar{u}_1 + \bar{n}_2 \bar{T}_2 \bar{u}_2) \\ - \frac{4}{15} \sqrt{\frac{\kappa}{2\pi}} \left( \frac{\text{Re}}{M} \right) M^2 \alpha_1 (\bar{n}_1 \bar{u}_1 + \bar{n}_2 \bar{u}_2) \\ + \frac{4}{15} \sqrt{\frac{2}{\pi\kappa}} \left( \frac{\text{Re}}{M} \right) \alpha_2 \alpha_3 (\bar{n}_1 + \bar{n}_2) = 0, \end{aligned} \quad (2.78f)$$

where  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  are constants of integration, and  $\kappa = c_p/c_v$ .

In equations (2.78), dimensionless quantities (designated by a bar above the letter) have been introduced, and  $x_1 = x/d$ . We have chosen  $n^+$ ,  $T^+$ ,  $w$ , and  $d$  as characteristic dimensions. In addition, we have introduced the Mach and Reynolds numbers

$$M = \frac{w}{\sqrt{\kappa R T^+}} \quad \text{and} \quad Re = \frac{m n^+ w d}{\mu (T^+)}. \quad (2.79)$$

One boundary condition ( $u_x \equiv 0$ ) has already been used. For the distribution function of the reflected molecules in the form (2.4) with  $\alpha_r = \alpha_e = 1$ , we have also the five boundary conditions:

$$\left. \begin{array}{l} \bar{u}_1 = \frac{1}{2}, \\ \bar{T}_1 = \frac{T^-}{T^+} \end{array} \right\} \text{when } x_1 = \frac{1}{2}; \quad \left. \begin{array}{l} \bar{u}_2 = -\frac{1}{2}, \\ \bar{T}_2 = 1, \\ \bar{n}_2 = 1 \end{array} \right\} \text{when } x_1 = -\frac{1}{2}. \quad (2.80)$$

It is easy to see that the parameter  $M/Re$  is proportional to the Knudsen number, such that  $Re/M=0$  relates to free-molecule flow, while  $Re/M \rightarrow \infty$  relates to Navier-Stokes flow.

At small Mach numbers ( $M^2 \ll 1$ ), the system (2.78) splits into two parts. The velocity variables  $\bar{u}_{1,2}$  drop out of Eqs. (2.78d) and (2.78f), so that the system of equations (2.78a), (2.78c), (2.78d), and (2.78f) gives a solution of the heat-transfer problem for any ratio  $T^-/T^+$ . When that problem has been solved, the functions  $\bar{u}_{1,2}$  are determined from Eqs. (2.78b) and (2.78e). For an arbitrary Mach number all the equations (2.78) must be solved simultaneously.

Figure 22 shows the results of calculations of the friction stress, obtained by means of equations (2.78). It may be seen from Fig. 22 that the curves corresponding to different temperature ratios diverge very strongly for large  $Re/M$ , i.e., near the Navier-Stokes regime. For Navier-Stokes flow it may easily be shown that (e.g., Liu and Lees, loc. cit.)

$$\frac{P_{xz}}{P_{xz}^\infty} = \sqrt{\frac{\pi \kappa}{2} \frac{M}{Re}} \left[ 1 + \frac{T^-}{T^+} + \frac{\kappa-1}{6} Pr M^2 \right], \quad (2.81)$$

where  $P_{xz}^\infty$  is the corresponding free-molecule value, obtained with the same values of  $n^+$ ,  $T^+$ , and  $T^-$ ;  $Pr = c_p u / \lambda$  is the Prandtl number.

It is clear that at large values of  $Re/M$ , the curves on Fig. 22, corresponding to various wall temperature ratios and various Mach numbers, must draw close together, if we construct them in terms of the variable

$$\left( \frac{Re}{M} \right)^* = 2 \left( \frac{Re}{M} \right) \left[ 1 + \frac{T^-}{T^+} + \frac{\kappa-1}{6} Pr M^2 \right]^{-1}. \quad (2.82)$$

The curves of Fig. 22, redrawn in terms of that variable, are shown on Fig. 23. The correlation of the data has been improved considerably. Figures 24 and 25 show velocity profiles for  $M=3$  and for the two temperature ratios:  $T^-/T^+ = 4$  and 1. It is interesting to note that for equal wall temperatures, the velocity profiles are nearly linear. For all Knudsen numbers (except  $Re/M=\infty$ ), a discontinuity in the velocities is observed at the wall. However, the approximation assumed for the distribution function is evidently too coarse to show the structure of the Knudsen boundary layer (see §5.1).

Thus, the method of moments with the simplest approximating function (2.73) explains the qualitative pattern of the flow between flat plates at arbitrary Knudsen numbers and plate temperature ratios and for Mach numbers of order unity.\* However, the accuracy of the results obtained is completely determined by how well the approximating function was chosen. In the analysis of linear problems we saw that the two-sided Maxwellian approximation misses a whole series of effects. There is no reason to expect greater accuracy when applying the approximation to nonlinear problems. To obtain accurate solutions, some algorithm of successive refinement of the distribution function is required. But further progress in this direction is naturally associated with further appreciable complication of the moment equations obtained and with an increase in their number. By examining the Couette flow as the simplest model for testing the methods intended for the solution of complex practical problems, it is easy to visualize the difficulties which arise in solving those problems by moment methods with a sufficiently accurate approximating function.

4. Nonlinear Problems. The Monte Carlo Method. The Monte Carlo method (see §3.15) is very promising for the solution of complex problems with an accuracy sufficient for practical purposes. There is a multitude of possible schemes for application of the method of statistical trials. We present one of these for the problem of heat transfer between flat plates.† The distribution function for this problem depends on the three variables:  $x$ ,  $\xi_x$ , and  $\xi_R = (\xi_y^2 + \xi_z^2)^{1/2}$ .

\* It has been shown in the work of Liu and Lees that, at large Mach numbers, the system (2.78) has no solution for small  $Re/M$ .

† J. K. Haviland and M. L. Lavin, Phys. Fluids, Vol. 5, No. 11 (1962).

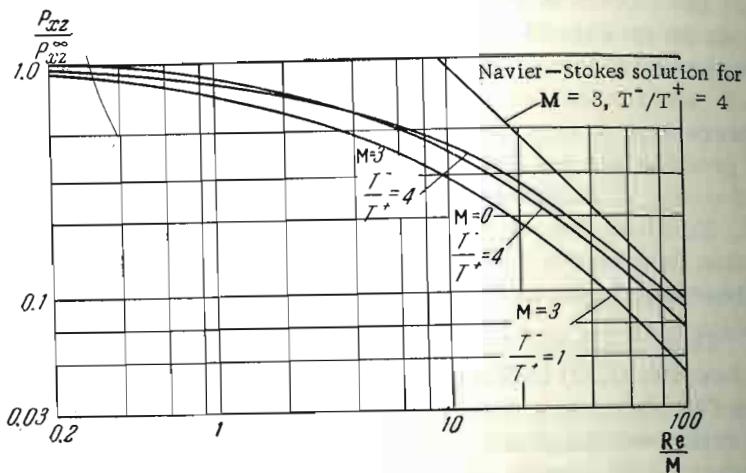


Fig. 22

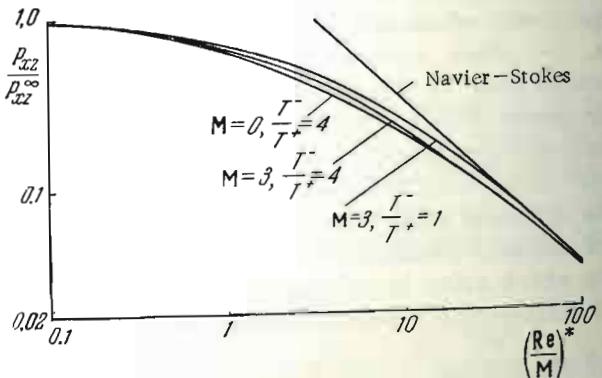


Fig. 23

We subdivide this three-dimensional phase space into cells, each of which corresponds to definite values of  $x_i$ ,  $\xi_{xj}$ , and  $\xi_{Rk}$ . The assignment of the numbers  $N_{ijk}$  in each cell determines the distribution function, for which  $N_{ijk}$  is the number of molecules in the element  $\Delta x$  near the point  $x_i$  with velocities  $\xi_{xj}$  and  $\xi_{Rk}$  in the element of velocity space  $\Delta \xi_x \Delta \xi_R$ .

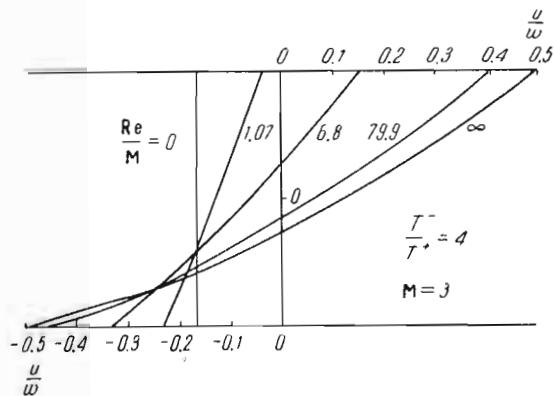


Fig. 24

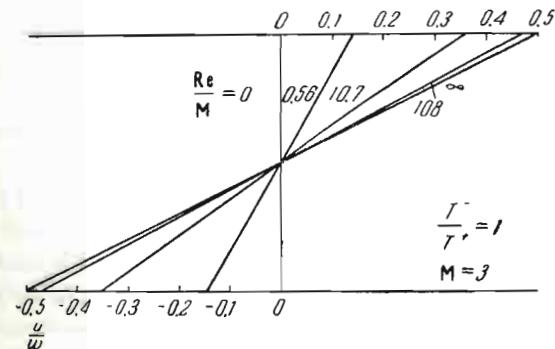


Fig. 25

Let some original distribution be given, i.e., let the corresponding numbers  $N_{ijk}$  be assigned. We call the molecules of that distribution field molecules. We examine the motion of a particle which we call a test particle. Let the test particle enter some cell of phase space. With a probability depending on the molecular interaction law and the distribution function of the field particles, the particle may either experience a collision in the cell, or pass through it without a collision. In the first case, as a result of the collision, the particle acquires another velocity, i.e., falls into a cell with another velocity, but with the same coordinate  $x_i$ . In the second case the particle enters the adjoining  $x_i$  cell, but with the

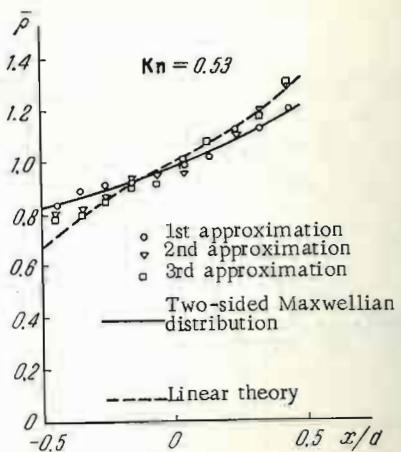


Fig. 26

same velocity. In the first case, the residence time of the test particle in the cell is  $\tau < \Delta x / \xi_x$  and, in the second case,  $\tau = \Delta x / \xi_x$ . By using the computer to draw random numbers, distributed with a density proportional to the probability of collisions, we determine the various "lifetimes"  $\tau$  of a test molecule in a given cell. The tracking of a test molecule begins when it leaves one of the walls and ends when it returns to the wall. Thereafter, a new test molecule is chosen, whose velocity is determined by drawing random numbers with a probability depending on the law of interaction of the molecules with the wall. By following the movement of a sufficiently large number of test molecules, and by storing the time spent by these molecules in each cell, we in fact store a new distribution function. The molecules corresponding to the new distribution function are taken as field molecules, and the calculation of the next approximation begins.

Figure 26 shows the results of a calculation of heat transfer carried out for Maxwellian molecules, with a plate temperature ratio of 4:1. Here,  $\bar{\rho}$  is the mean density,  $Kn = \lambda/d$  is the Knudsen number, where the mean free path  $\lambda$  is determined from the formula

$$\lambda = \frac{1}{1.203} \cdot \frac{16}{5\bar{\rho}} \frac{m\mu}{\sqrt{2\pi m k T}}.$$

For Maxwellian molecules a difficulty arises because the molecules interact at arbitrarily great distances from one another. Therefore, we must limit the radius of interaction of the molecules, by rejecting collisions leading to deflections less than some small angle.

The original distribution function is taken to be the distribution corresponding to free-molecule flow. If we take into account the unavoidable fluctuations which are peculiar to the method of statistical testing, and decrease in inverse proportion to the square root of the number of tests, we may consider the iterations to converge. To obtain more reliable results, it is necessary to reduce the statistical scatter. But to reduce the fluctuations by one order it is necessary to increase the number of draws by two orders. Then it is necessary to increase the time of calculation by two orders. Figure 26 shows a comparison of the results of these calculations, with values obtained by means of the method of moments described above for the nonlinear Boltzmann equation with a two-sided Maxwellian approximating function, and by means of a linear equation with an approximating function with eight adjustable functions\*

$$\varphi = a_0^\pm(x) + a_1^\pm(x)v_x + a_2^\pm v^2 + a_3^\pm(x)v_x v^2.$$

As may be seen from this comparison, the Monte Carlo results are rather closer to the solution of the linear equation with a more detailed distribution function, than to the solution of the nonlinear equation with a coarser two-sided Maxwellian function. It is possible that in the case examined with  $T^-/T^+ = 4$ , the influence of nonlinear effects is still not very important and that the choice of the approximating function is decisive.

In this one-dimensional problem, the distribution function is axisymmetric. The space between the plates is divided into ten strips. In each strip the velocity space is divided into 288 cells. Thus, there are 2880 cells in all. Since we must store both the distribution function of the previous approximation, and the new distribution function, the minimum storage capacity required for the calculation is equal to 5760. As pointed out in § 3.15, the re-

\*The latter results have been copied from Ziering's paper [S. Ziering, Phys. Fluids, Vol. 3, No. 4 (1960)].

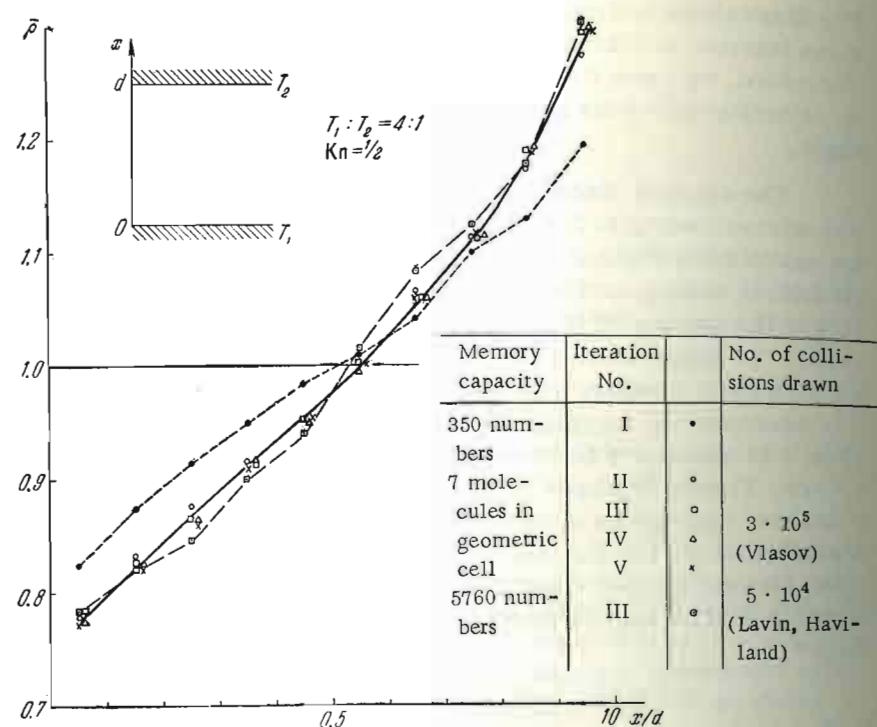


Fig. 27

quired storage volume may be considerably reduced for pseudo-Maxwellian molecules. That problem has been solved by Vlasov.\* As pointed out in §3.15, it is sufficient in principle for pseudo-Maxwellian molecules to store only the velocity of a single molecule in a geometrical cell (strip). However, test calculations show that the calculation proceeds considerably better, if we store several velocities in each geometrical cell. Vlasov's results, presented in Fig. 27, were obtained by storing the velocities of seven molecules in each geometrical cell. A total of 350 numbers was stored.

As may be seen from the graph, the agreement between the results of Vlasov and those of Haviland and Lavin is quite satisfactory.

\*V. I. Vlasov, Doklady Akad. Nauk SSSR, Vol. 167, No. 5 (1966).

**5. Nonlinear Problems for the Model Equation.** We can estimate the relative accuracy of different methods by means of the Boltzmann model equation, for which it is much simpler to obtain the exact solution.\* The model equation for the Couette problem may be written in the form

$$f^\pm(t, x, \xi) = f\left(\mp \frac{d}{2}, \xi\right) \exp \left\{ -\frac{A}{\xi_x} \int_{\mp \frac{d}{2}}^x n(s) ds \right\} + \frac{A}{\xi_x} \int_{\mp \frac{d}{2}}^x n(s_1) f_0(n, T, u_z, \xi) \exp \left\{ -\frac{A}{\xi_x} \int_{s_1}^x n(s) ds \right\} ds_1, \quad (2.83)$$

where

$$f\left(\mp \frac{d}{2}, \xi \geq 0\right) = n^\pm \left(\frac{h^\pm}{\pi}\right)^{3/2} \exp \left\{ -h^\pm \left[\xi_x^2 + \xi_y^2 + \left(\xi_z \pm \frac{w}{2}\right)^2\right] \right\}. \quad (2.84)$$

Substituting these expressions into the definitions of  $n$ ,  $u_z$ , and  $T$  in terms of the distribution function, we obtain the following integral equations:

$$\begin{aligned} \sqrt{\pi} \bar{n}(\omega) &= \bar{n}^+ J_0(\omega) + \bar{n}^- J_0[S_-(\alpha - \omega)] \\ &+ \int_0^\alpha \bar{n}(\omega') S(\omega') J_{-1}[S(\omega)|\omega - \omega'|] d\omega', \end{aligned} \quad (2.85a)$$

$$\begin{aligned} 2 \sqrt{\pi} \bar{u}_z \bar{n}(\omega) &= -\bar{n}^+ J_0(\omega) + \bar{n}^- J_0[S_-(\alpha - \omega)] \\ &+ 2 \bar{u}_z \int_0^\alpha \bar{n}(\omega') S(\omega') J_{-1}[S(\omega')|\omega - \omega'|] d\omega', \end{aligned} \quad (2.85b)$$

$$\begin{aligned} \frac{3}{2} \sqrt{\pi} \frac{\bar{n}(\omega)}{S_-^2(\omega)} &= C + \bar{n}^+ J_0(\omega) + \frac{\bar{n}^-}{S_-^2} J_0[S_-(\alpha - \omega)] \\ &+ \int_0^\alpha \frac{\bar{n}(\omega')}{S(\omega')} J_{-1}[S(\omega')|\omega - \omega'|] d\omega'. \end{aligned} \quad (2.85c)$$

\*D. R. Willis, "Rarefied Gas Dynamics," Third Symposium, Academic Press, New York, 1963.

Here, the following notation is introduced:

$$\omega = A \sqrt{h^+} \int_{-\frac{d}{2}}^x n(s) ds, * \quad a = \omega \left( \frac{d}{2} \right) = A \sqrt{h^+} n_0 d, \quad \bar{n}(\omega) = \frac{n(\omega)}{n_0},$$

$$S(\omega) = \sqrt{\frac{T^+}{T(\omega)}}, \quad S_- = \sqrt{\frac{T^+}{T^-}},$$

$$\bar{n}^+ = \frac{n^+}{n_0}, \quad \bar{n}^- = \frac{n^-}{n_0}, \quad \bar{u}_z = \frac{u_z}{w},$$

$$C = \bar{n}^+ J_2(\omega) + \frac{\bar{n}^-}{S_-^2} J_2[S_-(a - \omega)] + \int_0^a \frac{\bar{n}(\omega')}{S(\omega')} J_1[S(\omega') | \omega - \omega' |] d\omega', \quad (2.86)$$

where  $n_0$  is the mean density of the gas between the plates,  $C$  is a constant proportional to  $P_{xx}$ . The integrals  $J_n(x)$  were determined in (2.45).

We have three simultaneous integral equations for the three macroscopic variables  $\bar{n}$ ,  $S$ , and  $\bar{u}_z$ . They also contain the three constants  $\bar{n}^+$ ,  $\bar{n}^-$ , and  $n_0$ ; specification of any of these determines the degree of rarefaction of the gas between the plates. We consider the specified quantity to be  $n_0$ . Then, to determine the other two constants, we have the following two conditions:

the velocity  $u_x$  is equal to zero:

$$0 = \bar{n}^+ J_1(\omega) - \frac{\bar{n}^-}{S_-} J_1[S_-(a - \omega)] \\ + \int_0^a \text{sign}(\omega - \omega') \bar{n}(\omega') J_0[S(\omega') | \omega - \omega' |] d\omega' \quad (2.87)$$

and the definition of the mean density

$$n_0 = \frac{1}{d} \int_{-\frac{d}{2}}^{+\frac{d}{2}} n(x) dx \quad \text{or} \quad a = \int_0^u \frac{d\omega}{n(\omega)}. \quad (2.88)$$

\*This transformation is analogous to that of Dorodnitsyn in boundary layer theory [A. A. Dorodnitsyn, Prikl. Mat. Mekhan., Vol. 6, No. 6 (1942)].

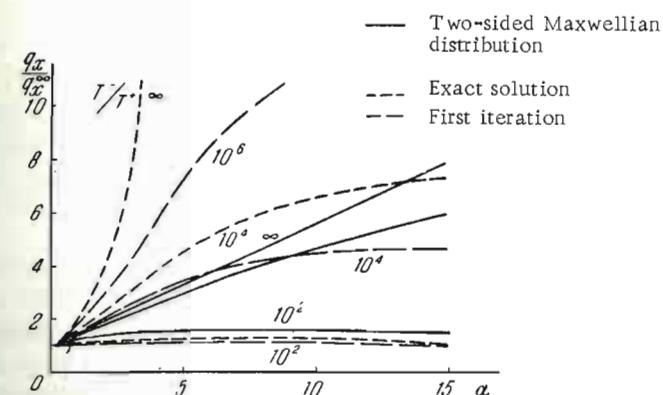


Fig. 28

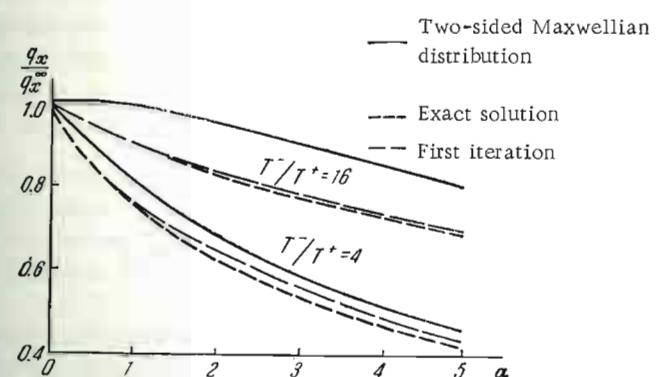


Fig. 29

The relations (2.86) and (2.87) are also integral equations. However, they follow from Eq. (2.85a), which is obtained by differentiation of Eq. (2.87) with respect to  $\omega$ . The latter, in turn, may be obtained by differentiation of Eq. (2.86).

The resulting system of integral equations must generally be solved by numerical means.

Figures 28 and 29 give the results of calculations of heat transfer (with  $w = 0$ ) between plates for a wide range of temperature drops. The value of the heat flux  $q_x$  is referred to the free-molecule value  $q_x^\infty$ , computed for the same value  $n_0$ :

$$q_x^\infty = q_x(Kn = \infty) = \frac{mn_0}{V\pi} \frac{\sqrt{h^-} - \sqrt{h^+}}{h^- h^+}. \quad (2.89)$$

It should be noted that a comparison of flows with equal values of  $n_0$ , i.e., with an identical number of molecules between the plates, is physically more justified than a comparison for equal values of  $n^+$ , as given in Figs. 22 and 23.

Having an exact solution of the model equation, we may estimate the accuracy of the approximate methods of solution of the Boltzmann equation, by applying those methods to the model equation and comparing the results obtained with those of the exact solution. We apply the method of moments with a two-sided Maxwellian approximation function (2.73) to the model equation. Since, for the first five moment equations the right side goes to zero, both for the exact and for the model Boltzmann equation, relations (2.74) are valid also in that case. By multiplying the model equation (2.20) by  $m\xi_x \xi_z$  and  $\frac{1}{2}m\xi_x \xi^2$ , and integrating, we obtain, instead of (2.76) and (2.77):

$$\frac{d}{dx} \int m f \xi_x^2 \xi_z d\xi = -An P_{xz}, \quad (2.90)$$

$$\frac{d}{dx} \int \frac{m}{2} f \xi_x^2 \xi^2 d\xi = -An(q_x + u_z P_{xz}). \quad (2.91)$$

When we examined the linearized shear problem with a two-sided Maxwellian approximation, we saw that by proper choice of the constants A we can make the solutions of the model equation and those of the Boltzmann equation identical. We see by comparing Eqs. (2.76), (2.77), and (2.90) and (2.91), that this cannot be done in the nonlinear case.

In the heat-transfer problem, since  $u_z$  and  $P_{xz}$  are equal to zero, one equation is sufficient, (2.77) or (2.91). In that case, the solutions may again be made identical by appropriate choice of the constant A. However, while in the shear problem this required a value of A equal to  $kT/\mu$ , in the heat-transfer problem we must put A equal to  $\frac{2}{3}kT/\mu$ , which agrees with the discussion in §3.6. This fact, in particular, indicates the accuracy which may be expected when we replace the Boltzmann equation by the model equation in cases when the viscosity and the thermal conductivity are

both important. Thus, in the heat-transfer problem for the model equation, we use directly the solutions of equations (2.78) for the exact Boltzmann equation. The solution of equations (2.78) was constructed in terms of the dimensionless parameter  $Re/M$ . It is easy to establish the relationship between that parameter and the parameter  $\alpha$  which appears in the solution of the model equation. In fact, putting  $A = \frac{2}{3}kT^+/\mu(T^+)$ , we have

$$a = An_0 d \sqrt{h^+} = \frac{2}{3} \frac{kT^+}{\mu^+} n_0 d \sqrt{h^+} = \frac{1}{3} \sqrt{\frac{2}{\pi}} \frac{n_0}{n^+} \frac{Re}{M}.$$

Figures 28 and 29 compare solutions of the model equation obtained by the moment method with the exact solution. The graphs show that the accuracy of the moment method decreases as the temperature drop increases. The figures also show results of calculations of the heat transfer by the method of successive approximations. Those results were obtained by substituting the free-molecule solution into the right side of Eqs. (2.85a) and (2.85c) and carrying out the appropriate quadratures. The agreement between these results and the exact solution for large  $\alpha$  is much better than might be expected.

We should draw attention to an interesting special feature of flows with large temperature or velocity differences. Let us examine, for example, the heat transfer at sufficiently large Knudsen number between plates with temperatures  $T_1$  and  $T_2$ , where we let  $T_2 \gg T_1$ . We distinguish two kinds of molecules. Molecules which leave the "hot" plate are type-2 molecules, and molecules which leave the cold plate are type-1 molecules. Let  $v_1$ ,  $v_2$  and  $n_1$ ,  $n_2$  be the mean velocities and densities of the molecules of type 1 and 2, respectively. Obviously,  $v_2 \gg v_1$ . From the continuity of the flow we have

$$v_1 n_1 = v_2 n_2. \quad (2.92)$$

Molecules of each type undergo collisions among themselves and also with molecules of the other type. We designate by  $\lambda_{ij}$  the mean free path of molecules of type i with respect to molecules of type j. Let the collision cross section with a relative molecular velocity of order  $v_1$  be equal to  $\sigma$ . We consider a Maxwellian gas. As noted in Chapter I, for Maxwellian molecules the collision cross section varies inversely with the relative velocity of the molecules. Therefore, for a relative velocity of order  $v_2$ , the col-

lision cross section will be equal to  $\sigma v_1/v_2 \ll \sigma$ . Taking this and relation (2.92) into account, we obtain the following estimates for the respective mean free paths:

$$\left. \begin{aligned} \lambda_{11} &\sim \frac{v_1}{n_1 \sigma v_1} = \frac{1}{n_1 \sigma}, & \lambda_{12} &\sim \frac{v_1 v_2}{n_2 \sigma v_1 v_2} = \frac{1}{n_2 \sigma} = \frac{1}{n_1 \sigma} \frac{v_2}{v_1}, \\ \lambda_{21} &\sim \frac{v_2 v_2}{n_1 \sigma v_1 v_2} = \frac{1}{n_1 \sigma} \frac{v_2}{v_1}, & \lambda_{22} &\sim \frac{v_2 v_2}{n_2 \sigma v_1 v_2} = \frac{1}{n_1 \sigma} \frac{v_2^2}{v_1^2}. \end{aligned} \right\} \quad (2.93)$$

Thus,

$$\lambda_{22} \gg \lambda_{12} \sim \lambda_{21} \gg \lambda_{11}. \quad (2.94)$$

Therefore, the slow molecules collide many times among themselves before they collide with the fast molecules.

For very large temperature ratios it may turn out, for example, that

$$\lambda_{22} \gg \lambda_{12} = \lambda_{21} \gg d \text{ and } \lambda_{11} \ll d. \quad (2.95)$$

In that case, the molecules leaving the hot wall reach the cold wall without collisions, as in free-molecule flow. The cold molecules also do not collide with the hot molecules, but collide repeatedly among themselves.

When  $\lambda_{11} \ll d$ , the distribution function of the cold molecules (type 1), at distances of several mean free paths  $\lambda_{11}$  from the cold wall, must be close to local-Maxwellian. At the same time, the hot molecules may either not experience any collisions [as in case (2.95)], or experience only a few collisions with molecules of type 1. Therefore, the distribution function of the hot molecules is close to the distribution of molecules reflected from the wall. In the comparison presented in Figs. 28 and 29, the two-sided Maxwellian distribution was used in the moment method. From the qualitative analysis just presented, it is seen that this distribution can be a satisfactory approximation only for the distribution function of the hot molecules (leaving the hot wall), and is a poor approximation for the distribution function of molecules leaving the cold wall, which is close to a local-Maxwellian distribution, as pointed out above. This is evidently also the reason for the large discrepancy

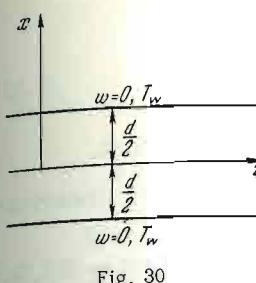


Fig. 30

were to examine a gas composed of hard spheres ( $\sigma = \text{const}$ ), then

$$\lambda_{11} \sim \lambda_{12} \sim \lambda_{21} \sim \frac{1}{n_1 \sigma} \ll \lambda_{22} \sim \frac{1}{n_1 \sigma} \frac{v_2}{v_1}. \quad (2.96)$$

In that case we would not observe the phenomenon just mentioned. The corresponding calculations require the full Boltzmann equation.

#### §4.3. Poiseuille Flow. The Knudsen Paradox

Let us examine the flow between two infinite parallel fixed plates (Fig. 30).\*

Let the temperature of the plates be constant and equal to  $T_w$ . We assume that the flow takes place under the influence of a small pressure gradient and that the walls reflect molecules with a Maxwellian law with a temperature equal to that of the wall (i.e., that the accommodation coefficient  $\alpha_e = 1$ ).

We describe the flow by the model equation, which here takes the form

$$\xi_x \frac{\partial f}{\partial x} + \xi_z \frac{\partial f}{\partial z} = An(f_0 - f). \quad (3.1)$$

At the walls, with the assumptions made,

$$\begin{aligned} f\left(\pm \frac{d}{2}, z, \xi_x \leq 0\right) &= f_0(z, \xi) \\ &= n_w(z) \left(\frac{h_w}{\pi}\right)^{3/2} e^{-h_w \xi^2}. \end{aligned} \quad (3.2)$$

\*K. Takao, "Rarefied Gas Dynamics," Second Symposium, Academic Press, New York, 1961. C. Cercignani, "Rarefied Gas Dynamics," Third Symposium, Academic Press, New York, 1963.

We seek a solution of Eq. (3.1) of the following form:

$$f = f_{00}(1 + \varphi), \quad f_{00} = n_0 \left( \frac{h_w}{\pi} \right)^{3/2} e^{-h_w \xi^2}, \quad (3.3)$$

where  $\varphi(x, y, \xi)$  is a small correction, and  $n_0$  is a constant.

By substituting (3.3) into Eq. (3.1), we obtain, after linearization,

$$\frac{v_x}{a} \frac{\partial \varphi}{\partial x_1} + \frac{v_z}{a} \frac{\partial \varphi}{\partial z_1} = -\varphi + v + 2v_z u_1 + \left( v^2 - \frac{3}{2} \right) \tau. \quad (3.4)$$

Here we have introduced the same notation as was used in deriving Eq. (2.27) in the previous section:

$$\begin{aligned} v_{x,z} &= \xi_{x,z} \sqrt{h_w}, \quad x_1 = x/d, \quad z_1 = z/d, \quad u_1 = u_z \sqrt{h_w}, \\ a &= A n_0 d \sqrt{h_w}, \quad T = T_w(1 + \tau), \quad n = n_0(1 + v), \quad v = \frac{1}{\pi^{3/2}} \int e^{-v^2} \varphi d\mathbf{v}, \\ \tau &= \frac{2}{3} \frac{1}{\pi^{3/2}} \int e^{-v^2} v^2 \varphi d\mathbf{v} - v, \quad u_z = \frac{1}{\pi^{3/2} \sqrt{h_w}} \int e^{-v^2} v_z \varphi d\mathbf{v}. \end{aligned} \quad (3.5)$$

The pressure may be represented in the form

$$p = p_0(1 - Kz) \quad (p_0 = kT_w n_0). \quad (3.6)$$

The quantity

$$-Kp_0 = \frac{dp}{dz}$$

is the pressure gradient.

On the other hand, by definition,

$$p = \frac{1}{3} \int m c^2 f d\xi = p_0 + \frac{1}{3} \int m \xi^2 f_{00} \varphi d\xi. \quad (3.7)$$

Hence, it may be seen that the function  $\varphi$  must have the form

$$\varphi = -Kz + v_z \psi(x, z, v_x, v_y, v_z). \quad (3.8)$$

Following substitution of the solution (3.8) into Eq. (3.4), it takes the form

$$\frac{v_x}{a} \frac{\partial \psi}{\partial x_1} + \frac{v_z}{a} \frac{\partial \psi}{\partial z_1} - K \frac{d}{a} = -\psi + 2u_1 \quad (3.9)$$

$$\left( u_1 = \frac{1}{\pi^{3/2}} \int e^{-v^2} v_z^2 \psi d\mathbf{v} \right).$$

At the walls ( $x_1 = \pm 1/2$ ), according to (3.2),

$$\psi \left( \pm \frac{d}{2}; v_x \leq 0 \right) = v = -Kz, \quad \psi \left( \pm \frac{d}{2}; v_x \geq 0 \right) = 0. \quad (3.10)$$

Since neither the equation (3.9), nor the boundary conditions (3.10) contain  $z$ ,  $\psi$  does not depend on  $z$ . Then Eq. (3.9) may be written in the form

$$\frac{v_x}{a} \frac{\partial \psi}{\partial x_1} = -\psi + \frac{Kd}{a} + 2u_1(x_1). \quad (3.11)$$

We write that equation in integral form:

$$\psi(x_1, v_x \geq 0) = a \int_{-1/2}^{x_1} \left[ \frac{Kd}{a} + 2u_1(s) \right] e^{-a \frac{x_1-s}{v_x}} \frac{ds}{v_x}. \quad (3.12)$$

Multiplying by  $\pi^{-3/2} v_z^2 e^{-v^2}$  and integrating with respect to  $v$ , we obtain

$$u_1(x_1) = \frac{a}{\sqrt{\pi}} \int_{-1/2}^{+1/2} \left[ \frac{Kd}{2a} + u_1(s) \right] J_{-1}(a|x_1 - s|) ds, \quad (3.13)$$

where  $J_n(x)$  are integrals, introduced in the previous section.

The function  $J_{-1}(a|x_1 - s|)$  has a logarithmic singularity when  $s = x_1$ . We can therefore replace  $u_1(s)$  on the right side of Eq. (3.13) in the first approximation by  $u_1(x_1)$ . Then, taking into account that

$$J_n(x) = -\frac{dJ_{n+1}(x)}{dx},$$

we easily obtain

$$u_1(x_1) \approx \frac{Kd}{2a} \frac{\sqrt{\pi} - J_0 \left[ a \left( \frac{1}{2} - x_1 \right) \right] - J_0 \left[ a \left( \frac{1}{2} + x_1 \right) \right]}{J_0 \left[ a \left( \frac{1}{2} - x_1 \right) \right] + J_0 \left[ a \left( \frac{1}{2} + x_1 \right) \right]}. \quad (3.14)$$

The function  $J_0(x)$  possesses the following asymptotic properties:\*

$$J_0(x) \rightarrow \frac{\sqrt{\pi}}{2} + x \ln x \quad \text{for } x \rightarrow 0$$

and

$$J_0(x) \rightarrow \frac{\pi}{3} e^{\frac{1}{2}-3} \sqrt{\frac{x}{2}} \left( 1 + \frac{5}{36} \sqrt{\frac{1}{x}} \right) \quad \text{for } x \rightarrow \infty.$$

Therefore, in the crude approximation (3.14), the velocity  $u_1$  tends to infinity both when  $\alpha \rightarrow 0$  (i.e., when  $Kn \rightarrow \infty$ ), and also when  $\alpha \rightarrow \infty$  (i.e., when  $Kn \rightarrow 0$ ). Thus, the velocity is a minimum for some value  $\alpha$ . The volume flow rate, equal to

$$Q = d \int_{-1/2}^{+1/2} u_1(x_1) dx_1$$

behaves analogously (the velocity is measured in units of the thermal velocity of the molecules,  $h_w^{-\frac{1}{2}}$ ).

The presence of a minimum flow rate at a certain pressure (when  $0 < \alpha < \infty$ ) is also obtained from a numerical solution of Eq. (3.13).† This phenomenon was first observed experimentally‡ and is known as the Knudsen paradox. The variation of the quantity  $2Q/Kd^2$  with  $\alpha$  is shown in Fig. 31, while Fig. 32 gives velocity profiles, obtained from formula (3.14).

For large pressures (with  $\alpha \gg 1$ ) we have the Poiseuille solution, according to which the flow rate is proportional to the pressure:

$$\frac{2Q}{Kd^2} = \frac{a}{b},$$

At small pressures Knudsen observed a logarithmic increase of flow rate with reduction of pressure.

The solution of Eq. (3.13) gives the same qualitative results. A quantitative comparison is difficult, since the Knudsen tests

\*M. Abramowitz, J. Math. Phys., 32: 188 (1953).

†See the work of Cercignani cited above.

‡M. Knudsen, Ann. Physic., 28: 75 (1909).

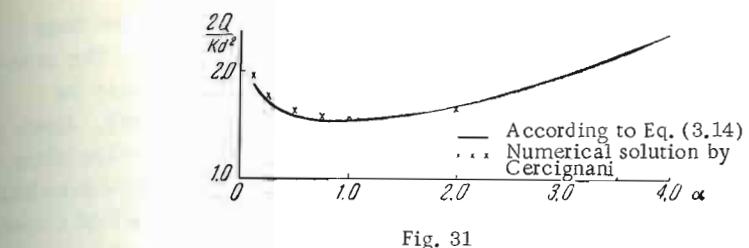


Fig. 31

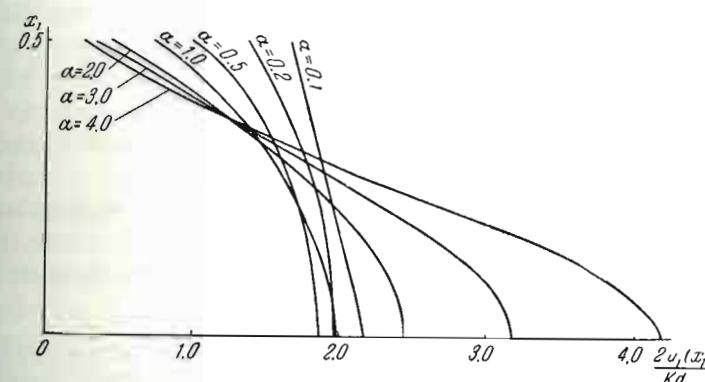


Fig. 32

were conducted in circular tubes, while the solution presented refers to a plane configuration.\* Moreover, for this degenerate geometry, the flow rate tends to infinity as  $\alpha \rightarrow 0$ , while the flow rate remains finite in a tube of finite cross section in the free-molecule limit. Therefore, even for a very narrow channel of finite width, as  $\alpha \rightarrow 0$ , the magnitude of the flow rate will deviate from the solution obtained above, and tend toward a finite free-molecule limit.

#### §4.4. The Structure of a Shock Wave

The problem of the structure of shock waves, like that of Couette flow, is one of the classical problems of kinetic theory. Couette flow and the flow through shock waves are the simplest representatives of characteristic types of flow arising near a solid

\*M. Knudsen, "Theory of Gases," 1934.

surface. Flows similar to flow through a shock wave arise near blunt bodies at small Knudsen numbers. The thickness of the compression region at sufficiently small Knudsen numbers may be commensurate with a characteristic dimension of the body. However, the study of the structure of a shock wave is essential also when the shock wave thickness is negligibly small in comparison with a characteristic dimension of the body. In strong shock waves, we find a whole series of processes which have an appreciable influence on the entire flow. A large number of papers whose results have been summarized in a monograph by Zel'dovich and Raizer,\* are devoted to the study of shock waves; they discuss dissociation, ionization, radiation, relaxation of excited internal molecular degrees of freedom, and similar processes for monatomic gases and gas mixtures within the framework of the Navier-Stokes theory. On the other hand, it is known that the Navier-Stokes equations, strictly speaking, are not applicable to describe the structure of strong shock waves, whose thickness is of the order of the molecular mean free path. Below we examine the structure of shock waves from the kinetic point of view by means of the Boltzmann equation for monatomic gases.

1. Let the  $x$  axis be directed along the flow perpendicular to a shock wave. Let  $u_1, n_1, T_1$  and  $u_2, n_2, T_2$  be the mean velocity, density, and temperature of a gas, respectively, before and after the shock wave.

The gas is in equilibrium at infinity before and after the shock, so that the corresponding distribution functions are Maxwellian:

$$f(x \rightarrow -\infty) = f_1(\xi) = n_1 F_1 = n_1 \left( \frac{h_1}{\pi} \right)^{3/2} \exp \left\{ -h_1 [(\xi_x - u_1)^2 + \xi_y^2 + \xi_z^2] \right\}. \quad (4.1a)$$

$$f(x \rightarrow \infty) = f_2(\xi) = n_2 F_2 = n_2 \left( \frac{h_2}{\pi} \right)^{3/2} \exp \left\{ -h_2 [(\xi_x - u_2)^2 + \xi_y^2 + \xi_z^2] \right\}. \quad (4.1b)$$

Since the mass, momentum, and energy of the gas are conserved in the shock,

\* Ya. B. Zel'dovich and Yu. P. Raizer, "The Physics of Shock Waves and of High-Temperature Hydrodynamic Phenomena," Nauka Press, 1966. English translation (R. F. Probstein and W. D. Hayes eds.), Academic Press, New York.

$$m \int \xi_x f_1 d\xi = m \int \xi_x f_2 d\xi \quad \text{or} \quad \rho_1 u_1 = \rho_2 u_2 = j. \quad (4.2a)$$

$$m \int \xi_x^2 f_1 d\xi = m \int \xi_x^2 f_2 d\xi \quad \text{or} \quad j u_1 + k n_1 T_1 = j u_2 + k n_2 T_2, \quad (4.2b)$$

or

$$\left. \begin{aligned} \frac{m}{2} \int \xi_x \xi_x^2 f_1 d\xi &= \frac{m}{2} \int \xi_x \xi_x^2 f_2 d\xi \\ j \left( \frac{u_1^2}{2} + \frac{5}{2} \frac{k}{m} T_1 \right) &= j \left( \frac{u_2^2}{2} + \frac{5}{2} \frac{k}{m} T_2 \right). \end{aligned} \right\} \quad (4.2c)$$

These are the well-known Rankine-Hugoniot conditions.

For the one-dimensional problem under examination, the Boltzmann equation takes the form

$$\xi_x \frac{\partial f}{\partial x} = J(x, \xi). \quad (4.3)$$

The investigation of the structure of the shock wave requires the solution of Eq. (4.3) with the boundary conditions (4.1). As in the case of the Couette problem, there is as yet no exact solution for this problem. For an approximate solution of the problem it is natural to use the method of moments.

We represent the approximating distribution function in the form\*

$$f(x, \xi) = a_1(x) F_1(\xi) + a_2(x) F_2(\xi) + a_3(x) F_3(\xi), \quad (4.4)$$

where  $F_{1,2}(\xi)$  are determined by formulas (4.1),

$$F_3(\xi) = \frac{h_3^2}{\pi^{3/2}} (\xi_x - u_3) \exp \left\{ -h_3 [(\xi_x - u_3)^2 + \xi_y^2 + \xi_z^2] \right\} \quad (4.5)$$

and  $u_3$  and  $T_3$  are certain constants. When  $a_3 \equiv 0$ , the distribution (4.4) goes over to the bimodal distribution of Tamm and Mott-Smith.<sup>†</sup>

\* H. Salwen, C. Grosch, and S. Ziering, Phys. Fluids, Vol. 7, No. 2 (1964).

<sup>†</sup> This distribution was introduced by Tamm in an investigation of the structure of a shock wave in 1947, in the reference cited on page 112. The same distribution was used independently by Mott-Smith [H. M. Mott-Smith, Phys. Rev., 82: 885 (1951)].

It follows from the boundary conditions (4.1) that

$$\begin{aligned} a_1(-\infty) &= n_1, \quad a_2(+\infty) = n_2, \quad a_3(+\infty) \\ &= a_2(-\infty) = a_3(-\infty) = a_3(+\infty) = 0. \end{aligned} \quad (4.6)$$

The conservation equations (1.8)–(1.10) of Chapter III for this problem take the form

$$m \int \xi_x f d\xi = mnu(x) = mn_1 u_1 = j, \quad (4.7a)$$

$$m \int \xi_x^2 f d\xi = ju(x) + P_{xx}(x) = ju_1 + kn_1 T_1, \quad (4.7b)$$

$$\frac{m}{2} \int \xi_x \xi_x^2 f d\xi = j \left( \frac{3}{2} \frac{k}{m} T + \frac{1}{2} u^2 \right) + uP_{xx} + q_x = j \left( \frac{5}{2} \frac{k}{m} T_1 + \frac{1}{2} u_1^2 \right). \quad (4.7c)$$

Substituting the distribution function (4.4) into the left side of equations (4.7), we obtain

$$u_1 a_1 + u_2 a_2 + \frac{1}{2 \sqrt{h_3}} a_3 = n_1 u_1, \quad (4.8a)$$

$$m \left( \frac{1}{2h_1} + u_1^2 \right) a_1 + m \left( \frac{1}{2h_2} + u_2^2 \right) a_2 + \frac{mu_3}{\sqrt{h_3}} a_3 = ju_1 + kn_1 T_1, \quad (4.8b)$$

$$\begin{aligned} mu_1 \left( \frac{u_1^2}{2} + \frac{5}{4h_1} \right) a_1 + mu_2 \left( \frac{u_2^2}{2} + \frac{5}{4h_2} \right) a_2 \\ + m \left( \frac{5}{8h_3^{3/2}} + \frac{3}{4} \frac{u_3^2}{h_3^{1/2}} \right) a_3 = j \left( \frac{5}{2} \frac{k}{m} T_1 + \frac{1}{2} u_1^2 \right). \end{aligned} \quad (4.8c)$$

The Rankine–Hugoniot conditions (4.2) follow from the boundary conditions (4.6) and relations (4.8). Using conditions (4.2), relations (4.8) may be rewritten in the form

$$u_1 a_1 + u_2 a_2 = n_1 u_1 - \frac{1}{2 \sqrt{h_3}} a_3, \quad (4.9a)$$

$$\frac{m}{u_1} \left( \frac{1}{2h_1} + u_1^2 \right) (u_1 a_1 + u_2 a_2) = ju_1 + ku_1 T_1 - \frac{mu_3}{\sqrt{h_3}} a_3, \quad (4.9b)$$

$$m \left( \frac{u_1^2}{2} + \frac{5}{4h_1} \right) (u_1 a_1 + u_2 a_2) = j \left( \frac{5}{2} \frac{k}{m} T_1 + \frac{u_1^2}{2} \right) - \frac{m}{4 \sqrt{h_3}} \left( \frac{5}{2h_3} + 3u_3^2 \right). \quad (4.9c)$$

For those equations to have solutions, the following conditions of compatibility must be satisfied:

$$u_3 = \frac{1}{2u_1} \left( u_1^2 + \frac{kt_1}{m} \right) \text{ and } 5 \frac{kt_3}{m} = \left( u_1^2 + 5 \frac{kt_1}{m} \right) - 3u_3^2. \quad (4.10)$$

These determine the hitherto arbitrary parameters  $u_3$  and  $T_3$ .

If the compatibility conditions (4.10) are satisfied, the equations (4.9) reduce to a single equation, for example, to Eq. (4.9a). Therefore, to determine the three unknown functions  $a_1$ ,  $a_2$ , and  $a_3$  it is necessary to form two further moment equations. As noted, we can judge the accuracy of a chosen approximating function by comparing the solutions obtained by means of the same approximating function with the use of different moment equations. We consider three groups of supplementary moment equations, formed by multiplying the Boltzmann equation (4.3) by  $Q_1 = \xi_x^2$ ,  $\xi_x^2$ ,  $\xi_x^3$  and  $Q_2 = \xi_x^3$ ,  $\xi_x \xi_x^2$ ,  $\xi_x^2$ , respectively. The left-hand side differential part of the moment equations for the distribution function (4.4) can be written in the form

$$\frac{d}{dx} \int \xi_x Q f(x, \xi) d\xi = \sum_{i=1}^3 J_i(Q) \frac{da_i}{dx}, \quad (4.11)$$

where

$$J_i(Q) = \int \xi_x Q F_i d\xi$$

are integrals whose calculation for the chosen functions  $F_i$  and  $Q$  presents no difficulty.

The calculation of moments of the integral part of the Boltzmann equation is somewhat more difficult. We transform the right side, by using the symmetry property of (4.11) of Chapter II:

$$I_Q = \int (Q' - Q) f f_1 g b db de d\xi_1 d\xi_2 = \sum_{i,j=1}^3 J_{ij}(Q) a_i a_j, \quad (4.12)$$

where

$$J_{ij}(Q) = \int (Q' - Q) F_i F_j g b db de d\xi_1 d\xi_2. \quad (4.13)$$

Therefore, the desired moment equations have the form

$$\sum_{l=1}^3 J_l(Q_\mu) \frac{da_l}{dx} = \sum_{l, j=1}^3 J_{lj}(Q_\mu) a_l a_j. \quad (4.14)$$

To find the free functions  $a_i(x)$ , we must solve the system of two nonlinear differential equations (4.14), simultaneously with the algebraic equation (4.9a).

For Maxwellian molecules the integrals (4.13) may be expressed in terms of integrals whose values we already know. In fact, the functions  $Q$  may be expressed in terms of Hermite polynomials. In the notation of §3.3, we have, for example,

$$\begin{aligned} \xi_x^2 &= c_x^2 + 2c_x u_x + u_x^2 = \frac{kT}{m} v_x^2 + 2v_x \sqrt{\frac{kT}{m}} u + u^2 \\ &= \frac{kT}{m} (H_{xx}^{(2)} + 1) + 2v_x \sqrt{\frac{kT}{m}} u + u^2. \end{aligned}$$

Since the integrals of the collision integral, weighted by the summational invariants 1 and  $v_x$ , are equal to zero, the corresponding integral may be expressed in terms of the known integral (3.31) of Chapter III:

$$\xi_x^2 = \frac{kT}{m} I_{H_{xx}^{(2)}} = -6 \frac{A}{m} \left( \frac{8K}{m} \right)^{1/2} n p_{xx}. \quad (4.15)$$

The quantities  $n$  and  $p_{xx}$  appearing here may be represented in the form of integrals of the distribution function and, therefore, in the form of linear functions of  $a_i(x)$  with coefficients depending on  $n_i$ ,  $u_i$ , and  $h_i$ . Thus, the integral (4.15) will be a quadratic function  $a_i$ . By equating coefficients of equal  $a_i a_j$  in (4.12) and (4.15), we find the values of the integrals  $J_{ij}(Q)$ .<sup>\*</sup> Knowing the values of the integrals  $J_{ij}(Q)$ , we can solve the system of equations (4.14) and (4.9a) numerically. We do not give details of the solution but, before we give the results of calculations to determine (4.4), we dwell in somewhat greater detail on the special case of the bimodal

distribution function ( $a_3 \equiv 0$ ).<sup>\*</sup> In that case, the equations (4.9) reduce to the equation

$$un = u_1 a_1 + u_2 a_2 = n_1 u_1. \quad (4.16)$$

As the single supplementary moment equation we take the equation with  $Q = \xi_x^2$ . Using (4.15), we have

$$u_1 \left( 3 \frac{kT_1}{m} + u_1^2 \right) \frac{da_1}{dx} + u_2 \left( 3 \frac{kT_2}{m} + u_2^2 \right) \frac{da_2}{dx} = -6 \frac{A}{m} \left( \frac{8K}{m} \right)^{1/2} n p_{xx}. \quad (4.17)$$

Transforming the right side of the equation, we have:

$$\begin{aligned} n p_{xx} &= n P_{xx} - n p = mn \int c_x^2 f d\xi - \frac{mn}{3} \int c^2 f d\xi \\ &= \frac{2}{3} mn \int \xi_x^2 f d\xi - \frac{2}{3} m (un)^2 - \frac{1}{3} mn \int (\xi_y^2 + \xi_z^2) f d\xi = \frac{2}{3} ma_1 a_2 (u_1 - u_2)^2. \end{aligned}$$

Eliminating the function  $a_2(x)$  from (4.17) by means of (4.16), and substituting the expression just obtained into the right side, we obtain the equation

$$\frac{d}{dx} \left( \frac{a_1}{n_1} \right) = -\beta \left[ \frac{a_1}{n_1} - \left( \frac{a_1}{n_1} \right)^2 \right], \quad (4.18)$$

where

$$\begin{aligned} \bar{x} &= 4MA \left( \frac{8K}{m} \right)^{1/2} \frac{n_1}{u_1} x = \frac{32}{15} \sqrt{\frac{3}{10\pi}} \frac{x}{\lambda}, \\ \beta &= \frac{5}{2} \left( \frac{u_1}{u_2} - 1 \right)^2 \left[ \left( \frac{u_1}{u_2} \right)^2 - 1 \right]^{-1} = \frac{90M}{3+M^2} \frac{(M^2-1)^2}{16M^4-(3+M^2)^2}, \end{aligned} \quad (4.19)$$

$$\lambda = \frac{16}{5} \frac{\mu}{\rho V 2\pi R T_1} = \frac{4}{15} \frac{\sqrt{kT_1}}{An_1 \sqrt{4\pi K}} \quad (4.20)$$

is the molecular mean free path ahead of the shock, and

$$M = u_1 \sqrt{\frac{3m}{5kT_1}} \quad (4.21)$$

is the Mach number of the oncoming stream.

\*Tables of these integrals have been given in the above-cited paper by Salwen, Grosch and Ziering (see p. 337).

<sup>\*</sup>H. M. Mott-Smith, Phys. Rev., 82: 885 (1951).

The solution of (4.18) is easily found:

$$a_1(\bar{x}) = \frac{n_1}{1 + e^{\beta\bar{x}}}. \quad (4.22)$$

Then

$$\frac{n}{n_1} = \frac{a_1 + a_2}{n_1} = \frac{1 + \left(\frac{n_2}{n_1}\right) e^{\beta\bar{x}}}{1 + e^{\beta\bar{x}}} = \frac{1 + \frac{4M^2}{3+M^2} e^{\beta\bar{x}}}{1 + e^{\beta\bar{x}}} \quad (4.23)$$

and

$$\begin{aligned} \frac{u}{u_1} &= \frac{n_1}{n}, \quad \frac{T}{T_1} = \left(1 + \frac{5}{9} M^2\right) \frac{a_1}{n} + \frac{T_2}{T_1} \left(1 + \frac{5}{4} M^2\right) \frac{a_2}{n} - \frac{5}{9} \left(\frac{u}{u_1}\right)^2 M^2 \\ (M_2 &= u_2 \sqrt{\frac{3m}{5kT_2}}). \end{aligned} \quad (4.24)$$

The structure of the shock with the distribution (4.4) and with the bimodal distribution is shown in Figs. 33 and 34.

The point at which the profile of any quantity in the shock has its maximum slope is taken as the center of the profile. We designate the center of the profile as  $x_\Gamma$ , where the subscript  $\Gamma$  indicates the hydrodynamic quantity (density, velocity, temperature, and so on), whose profile is being considered.

It is clear from the curves presented that the center of the temperature profile is displaced forward with respect to the velocity center, while the latter is located ahead of the center of the density profile.

We see in formulas (4.23) and (4.24) that the velocity and density profiles for the bimodal distribution function are symmetric with respect to inversion about their centers. For the symmetry characteristics of the profile we may introduce the quantity

$$S_\Gamma = \frac{[\Gamma(\infty) - \Gamma(x_\Gamma)] - [\Gamma(x_\Gamma) - \Gamma(-\infty)]}{\Gamma(\infty) - \Gamma(-\infty)}.$$

It is evident that  $S_n = S_u = 0$  for the bimodal distribution, while the corresponding profiles for the distribution (4.4) possess a weak asymmetry  $S_n \sim 3.5\%$  and  $S_u \sim 1.2\%$ .

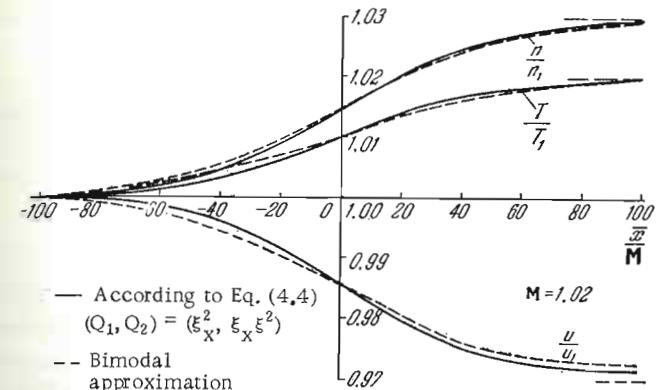


Fig. 33

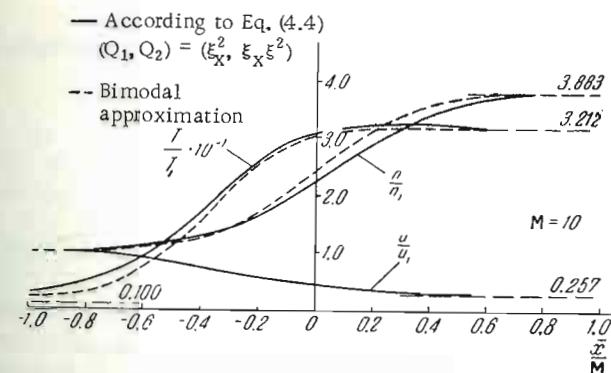


Fig. 34

It is interesting to note that the temperature profile corresponding to the distribution (4.4) has a maximum. Calculations indicate that when there is a change of sign of the temperature gradient, the sign of the heat flux does not change. It is evident that effects like these cannot occur within the framework of Navier-Stokes theory. At the same time, we cannot assert with confidence that the observed weak maximum is not caused by an inaccuracy in the approximation.

One of the average characteristics of the shock, which is convenient for comparison with experimental data, is its thickness.

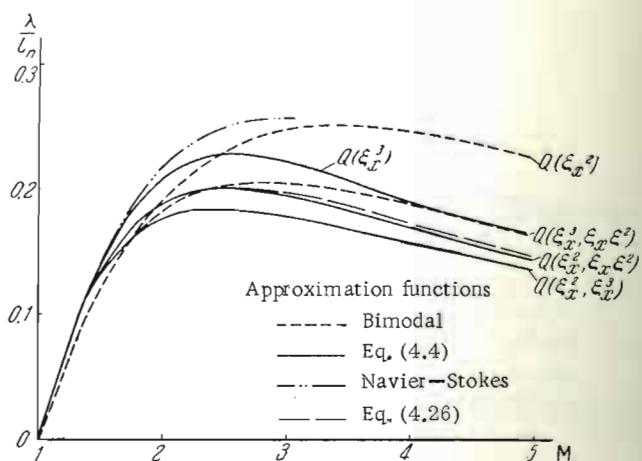


Fig. 35

Naturally, the concept of thickness is arbitrary. We take the definition of shock thickness given by Prandtl:

$$L_\Gamma = \frac{\Gamma_2 - \Gamma_1}{\left(\frac{d\Gamma}{dx}\right)_{\max}}. \quad (4.25)$$

The shock thicknesses determined in this way for different values of  $\Gamma$  are, in general, different. Thus, for the bimodal distribution, the temperature thickness of the shock is somewhat smaller than the density and velocity thicknesses, which are equal.

Figure 35 shows the variation of shock-wave thickness (relative to the molecular mean free path ahead of the shock) with Mach number, both for the bimodal distribution, and for the distribution (4.4), with different choices of  $Q(\xi)$ . The shock-wave thickness is of the order of the mean free path in both theories. With increase of Mach number the shock thickness first falls, and then increases as the Mach number.

At small Mach numbers the solutions corresponding to the distribution (4.4) agree quite well among themselves and with the solution introduced for comparison, which is obtained from the Navier-Stokes theory for Maxwellian molecules. At large Mach numbers the separation of the curves for selected moment equa-

#### § 4.4] THE STRUCTURE OF A SHOCK WAVE

tions (for various  $Q$ ) is considerably less than for the bimodal distribution. Therefore, the addition of a single term to the Tamm-Mott-Smith distribution allows the increase of accuracy. However, the remaining discrepancy between the curves is still considerable. Increased accuracy may be obtained either by adding a further term, or by means of a more efficient choice of the function  $F_3$ .

The best accuracy may be attained even in the bimodal approximation, by varying the functions  $F_1$  and  $F_2$ . Results close to those found by means of the distribution (4.4) have been obtained, for example, with the bimodal distribution function, which introduces a different scale of velocities for the molecules in different directions, in which\*

$$\left. \begin{aligned} F_1 &= \frac{1}{\pi^{3/2}} h_1^{3/2} \exp \left\{ -h_1 [(\xi_x - u_1)^2 + \xi_y^2 + \xi_z^2] \right\}, \\ F_2 &= \frac{1}{\pi^{3/2}} h_{22} h_{11}^{1/2} \exp \left\{ -h_{11} [(\xi_x - Q)^2 + h_{22} (\xi_y^2 + \xi_z^2)] \right\}, \end{aligned} \right\} \quad (4.26)$$

where  $Q$ ,  $h_{11}$ , and  $h_{22}$  are three functions subject to definition together with  $a_1(x)$  and  $a_2(x)$ . The thickness of the shock wave as calculated for this distribution and for Maxwellian molecules, is shown in Fig. 35. With this distribution function a maximum in the temperature curve is also obtained.

The shock-wave structure has been examined only for Maxwellian molecules. For other interaction laws the treatment of the solution remains the same, but the calculation of moments of the collision integral is more complicated. For the bimodal distribution the qualitative conclusions derived above are valid even for other interaction laws.†

The shock-wave thickness increases without bound as  $M \rightarrow \infty$  for any  $s$  in a power-law molecular interaction law. Only when  $s \rightarrow \infty$  (i.e., for hard spheres) does the shock-wave thickness tend to a finite limit. This result may be explained qualitatively as follows. The shock-wave thickness may be crudely

\* L. H. Holway, "Rarefied Gas Dynamics," Fourth Symposium, Academic Press, New York, 1965.

† See C. Muckenfuss, Phys. Fluids, Vol. 5, No. 11 (1962).

assumed to be proportional to the molecular mean free path of the oncoming stream in the gas behind the discontinuity in density. We denote that length by  $\lambda_{12}$ . Then (see § 1.4)

$$\lambda_{12} \sim \frac{u_1}{n_2 \sigma_{12} u_{12}},$$

where  $u_{12}$  is the velocity of the molecules in the oncoming stream relative to molecules of the gas behind the shock, and  $\sigma_{12}$  is the collision cross section for a relative molecule velocity  $u_{12}$ . For large Mach numbers

$$u_{12} \approx u_1$$

and the expression for the mean free path may be written in the form

$$\begin{aligned}\lambda_{12} &\sim \frac{1}{n_2 \sigma_{12}} \sim \left( \frac{n_1}{n_2} \right) \frac{\sigma_1}{\sigma_{12}} \frac{1}{n_1 \sigma_1} \sim \varepsilon \frac{\sigma_1}{\sigma_{12}} \lambda_\infty \\ &\left( \lambda_\infty \sim \frac{1}{n_1 \sigma_1}, \quad \varepsilon = \frac{n_1}{n_2} \right).\end{aligned}$$

If the molecules are rigid, they may be modeled by hard spheres, and  $\sigma_{12} = \sigma_1$ . In that case,

$$\lambda_{12} \sim \varepsilon \lambda_\infty.$$

As the Mach number of the shock  $M \rightarrow \infty$  in a perfect gas, the ratio of the densities  $\varepsilon$  tends to

$$\varepsilon = \frac{\gamma - 1}{\gamma + 1},$$

where  $\gamma$  is the ratio of the specific heats. Therefore, for hard spheres, the shock wave thickness is smaller than the mean free path in the oncoming stream, and tends to a finite limit as  $M \rightarrow \infty$ .

In the limiting case of soft molecules, modeled by Maxwellian molecules (see § 1.3),

$$\sigma_{12} \sim \sigma_1 \frac{\bar{c}_1}{u_{12}} \sim \sigma_1 \frac{a_1}{u_1} \sim \frac{\sigma_1}{M}.$$

Here,  $\bar{c}_1$  is the mean thermal velocity of the molecules, roughly equal to the velocity of sound  $a_1$  in the oncoming stream. In that

case, the shock-wave thickness is

$$\lambda_{12} \sim \varepsilon M \cdot \lambda_\infty,$$

i.e., it increases as the Mach number.

We note that the mean free path  $\lambda_{22}$  of the molecules relative to each other behind the shock is proportional to  $\lambda_{12}$ . Therefore, as  $M \rightarrow \infty$  for all molecules, except hard spheres, the mean free path of the molecules behind the shock becomes larger than ahead of the shock, in spite of the increased density at the shock.

We have already mentioned a large number of effects (radiation, diffusion of components of the mixture, and so on), generated in shock waves (especially in strong shocks); they have been widely studied within the framework of the Navier-Stokes theory. It is therefore important to know the region of applicability of the results thus obtained and to estimate their accuracy. The possibility of describing the structure of shock waves by means of the Navier-Stokes equations is also very attractive, because of the fact that in many cases the shock waves occupy a comparatively narrow region of the flow, in other parts of which the Navier-Stokes description is completely valid.

Calculation of the shock-wave structure according to the Navier-Stokes theory reduces to the solution of equations (4.7), in which we must substitute, in accordance with the approximation being considered [see, for example, formulas (3.45) and (3.46) of Chapter III]:

$$p_{xx} = -\frac{4}{3} \mu \frac{\partial u}{\partial x} \quad \text{and} \quad q_x = -\lambda \frac{\partial T}{\partial x},$$

where  $\mu$  and  $\lambda$  are functions of  $T$ , which depend on the molecular interaction law. In the general case, that system of equations can only be solved numerically.\* The system (4.7) with boundary conditions

$$\begin{aligned}n(-\infty) &= n_1, & T(-\infty) &= T_1, & u(-\infty) &= u_1, \\ n(\infty) &= n_2, & T(\infty) &= T_2, & n(\infty) &= u_2\end{aligned}$$

\*For a Prandtl number of  $Pr = 3/4$  the Navier-Stokes equations for the structure of a shock have an analytical solution [see R. Becker, Z. Phys., 8: 321 (1922)].

has a solution for any shock Mach number. However, according to the Enskog-Chapman theory, the Navier-Stokes equations themselves are applicable only for flows, whose characteristic dimension is much larger than the mean free path, i.e., in the case examined only for weak shock waves. It is clear from the comparison made in Fig. 35 that the Navier-Stokes solution agrees satisfactorily with the solution corresponding to the distribution (4.4) at small Mach numbers, and diverges strongly at large Mach numbers. It is natural to expect that at large Mach numbers the solution corresponding to the distribution (4.4) is more accurate. However, we cannot reach a final conclusion without an exact solution. Therefore, we must turn to a comparison with experiment.

In comparing theory with experiment, it should be borne in mind that, in addition to errors connected with the approximate nature of the theoretical results being compared, a discrepancy between the theoretical and experimental data may also be due to poor correspondence between the molecular interaction law assumed in the theory and the true molecular interaction law in the experiment. The constants appearing in the theoretical molecular interaction laws are usually taken from macroscopic experiments. The shock-wave thickness is very sensitive to the choice of molecular interaction model. Therefore, experimental data on the thicknesses of shocks are very suitable for determination of molecular interaction laws. In order to compare theoretical and experimental data on shock-wave structure, we require interaction laws taken from independent tests, for example, from experiments to determine viscosity. However, experimental data on viscosity are available only for temperatures lower than those in strong shock waves.

In the comparison\* presented in Fig. 36 for argon, a Lennard-Jones law has been assumed in the theoretical calculations, with coefficients taken from experiments conducted up to temperatures of 1100°K. This temperature corresponds to shock waves up to  $M = 3$ . For  $M > 3$  the interaction law taken should be regarded as an extrapolation. The experimental data were obtained by means of a thin wire probe in a low-density wind tunnel up to  $M = 2$  † and by an optical method based on the reflection by the

\*L. M. Schwartz and F. D. Hornig, Phys. Fluids, Vol. 6, No. 12 (1963).

†F. S. Sherman and L. Talbot, "Rarefied Gas Dynamics," First Symposium, Pergamon Press, 1960.

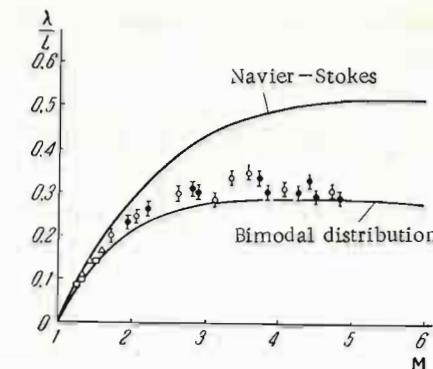


Fig. 36

shock wave of a beam directed toward it at a certain angle in a shock tube.\* The thickness of the shock wave is referred to the mean free path, determined by formula (4.20), in which  $\mu$  corresponds to a Lennard-Jones interaction law.

Taking into account the comments made above, we can only conclude from the comparison presented here that the Navier-Stokes theory agrees with experiment at low Mach numbers. At large Mach numbers the experimental data are much closer to the Tamm-Mott-Smith theory. The Navier-Stokes profiles of density and velocity possess a considerable asymmetry ( $S \sim 10-20\%$ ) while the asymmetry of the temperature profile is lower than according to Tamm-Mott-Smith.

2. The relative accuracy of the Navier-Stokes solution may be estimated also by means of the model equation, for which we can construct an exact solution.†

We write the relaxation equation for one-dimensional flow in integral form [see formula (8.25) of §2.8]:

$$f^\pm = f(x, \xi, \xi_x \geq 0) = \int_{-\infty}^x \frac{An f_0}{\xi_x} \exp \left\{ - \int_\tau^x \frac{An}{\xi_x} ds \right\} d\tau. \quad (4.27)$$

\*M. Linser and D. F. Hornig, Phys. Fluids, Vol. 6, No. 12 (1963).

†H. W. Liepmann, R. Narasimha, and M. T. Chahine, Phys. Fluids, Vol. 5, No. 11 (1962).

Since the integration is carried out from plus and minus infinity, the exponentially decaying term containing the initial distribution function vanishes.

We integrate Eq. (4.27) over all  $\xi$ ; we have

$$\begin{aligned} n = & \frac{1}{V\pi} \int_{-\infty}^x An^2 h^{1/2} J_{-1} \left( \int_{\tau}^x Ands, u \right) d\tau \\ & + \frac{1}{V\pi} \int_x^{\infty} An^2 h^{1/2} J_0 \left( \int_{\tau}^x Ands, -u \right) d\tau, \end{aligned} \quad (4.28a)$$

where \*

$$J_v(x, u) = \int_0^{\infty} \xi_x^v e^{-[(\xi_x - u)^2 + \frac{x}{\xi_x}]} d\xi_x.$$

Similarly, by multiplying the equation by  $\xi_x$  and  $\frac{1}{2}mc^2$  and integrating with respect to  $\xi$ , we obtain equations for  $u$  and  $T$ :

$$\begin{aligned} un = & \frac{1}{V\pi} \int_{-\infty}^x An^2 h^{1/2} J_0 \left( \int_{\tau}^x Ands, u \right) d\tau \\ & - \frac{1}{V\pi} \int_x^{\infty} An^2 h^{1/2} J_0 \left( \int_{\tau}^x Ands, -u \right) d\tau, \end{aligned} \quad (4.28b)$$

$$\begin{aligned} \frac{3}{2} knT = & \frac{m}{2V\pi} \int_{-\infty}^x An^2 h^{-1/2} J_{-1} \left( \int_{\tau}^x Ands, u \right) d\tau \\ & + \frac{m}{2V\pi} \int_x^{\infty} An^2 h^{-1/2} J_{-1} \left( \int_{\tau}^x Ands, -u \right) d\tau - \frac{mn u^2}{2} \\ & + \frac{m}{2V\pi} \int_{-\infty}^x An^2 h^{1/2} J_1 \left( \int_{\tau}^x Ands, u \right) d\tau \\ & + \frac{m}{2V\pi} \int_x^{\infty} An^2 h^{1/2} J_1 \left( \int_{\tau}^x Ands, -u \right) d\tau. \end{aligned} \quad (4.28c)$$

\*Properties of this integral and tables of its values have been given in: M. T. Chahine and R. Narasimha, "Rarefied Gas Dynamics," Fourth Symposium, Academic Press, 1965.

Thus, we obtain three integral equations for the three macroscopic quantities  $n$ ,  $u$ , and  $T$ . Those equations are solved by the method of successive approximations. The Navier-Stokes solution is taken as the starting point. The Navier-Stokes values of  $n$ ,  $u$ , and  $T$  are substituted into the right side, and new values of those quantities are found; those are again substituted into the right side, and so on. The analysis of ten successive approximations \* indicates a sufficiently rapid convergence of this process. Figure 37 gives a comparison of the profiles of temperature and velocity, calculated according to the Navier-Stokes formula, with the "exact" solution just described. The Navier-Stokes equations obtained from the model equation (see § 3.6) were used, i.e., equations corresponding to Prandtl number unity. The calculations are made for a Sutherland dependence of viscosity and thermal conductivity on temperature. It was shown in § 3.6 that it follows from the relaxation equation that

$$\mu = \frac{k\tau}{A} \text{ and } \lambda = \frac{5}{2} \frac{k^2 T}{Am}.$$

In order that the solution of the model equation (4.27) also correspond to the Sutherland law, the quantity  $A$  is regarded as a function of temperature.

From examination of the curves presented, we draw the same conclusions as above: the Navier-Stokes theory gives a quite accurate description of the structure of a shock wave at Mach numbers close to unity, and the accuracy of the theory falls off with increasing Mach number. The greatest discrepancies are observed in the temperature profile. The exact solution of the model equation does not give the maximum in the temperature curve noted above.

It is interesting to note that in the high-pressure region the Navier-Stokes theory is in good agreement with the exact solution† for the structure of the shock wave even at large Mach numbers.

\*M. T. Chahine and R. Narasimha, "Rarefied Gas Dynamics," Fourth Symposium, Academic Press, 1965.

†This fact was used by Vaglio-Laurin in presenting his theory, in which the Navier-Stokes solutions for the high-pressure region were joined up with a solution of the kinetic equation in the low-pressure region (Seminar at the Akad. Nauk SSSR Computing Center, January 1964).

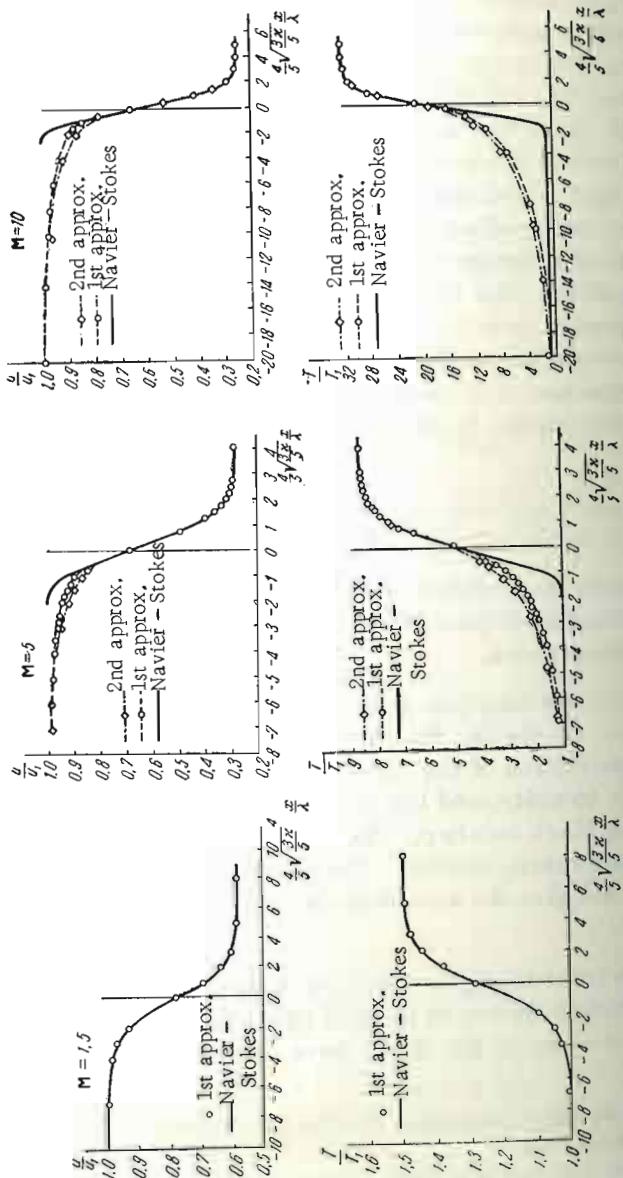


Fig. 37

We also note that the profile obtained by the exact solution at large  $M$  slopes slightly toward the oncoming stream, as observed experimentally by Ivanov.\*

3. In order to obtain greater accuracy than in the Navier-Stokes theory, attempts have been made to use the Burnett and the Grad thirteen-moment equations. However, it has turned out that the Burnett equations have no solution for Mach numbers  $M > 2.1$  and the Grad equations for Mach numbers  $M > 1.65$ .†

We could attempt to obtain a more exact solution by retaining a greater number of terms in the series of Hermite polynomials. However, it can be shown‡ for molecules with a finite interaction radius that, in general, the series of Hermite polynomials representing the distribution function in the shock wave does not converge.

In fact, the expansion in Hermite polynomials (see § 3.3)

$$f = f_0 (a^{(0)} H^{(0)} + a_{\alpha_1}^{(1)} H_{\alpha_1}^{(1)} + \dots), \quad f_0 = n \left( \frac{h}{\pi} \right)^{3/2} e^{-hc^2} \quad (4.29)$$

converges on the average if the function  $f(\xi)$  tends to zero as  $\xi \rightarrow \infty$  more rapidly than  $e^{-h\xi^2/2}$ . We show that the distribution function in the shock wave does not satisfy that condition for shock-wave Mach numbers greater than  $M = 1.851$ .

For molecules with a finite interaction radius, the Boltzmann equation may be written in integral form [see formula (7.5) of § 2.7; in the formula the substitution  $t = x/\xi_x$  has been made]:

$$f(x, \xi) = f(x_0, \xi) \exp \left\{ -\frac{1}{\xi_x} \int_{x_0}^x J_2(s, \xi) ds \right\} \\ + \frac{1}{\xi_x} \int_{x_0}^x J_1(\tau, \xi) \exp \left\{ -\frac{1}{\xi_x} \int_{\tau}^x J_2(s, \xi) ds \right\} d\tau. \quad (4.30)$$

We consider the distribution function  $f(x, \xi)$  at some point  $x$  inside the wave front for molecules moving upstream ( $\xi_x < 0$ ); by choos-

\*A. V. Ivanov, Zh. Prikl. Mekhan. Tekhn. Fiz., No. 6 (1964).

†H. Grad, Commun. Pure Appl. Math., Vol. 5, No. 3 (1962).

‡See L. H. Holway, Phys. Fluids, Vol. 7, No. 6 (1964).

ing the point  $x_0$  at infinity, we obtain

$$f(x, \xi, \xi_x < 0) = \int_x^\infty \frac{J_1(\tau, \xi)}{-\xi_x} \exp \left\{ \frac{1}{\xi_x} \int_x^\tau J_2(s, \xi) ds \right\} d\tau. \quad (4.31)$$

Since the integrand is positive,

$$f(x, \xi) \geq \int_{x_1}^\infty \frac{J_1(\tau, \xi)}{-\xi_x} \exp \left\{ \frac{1}{\xi_x} \int_x^\tau J_2(s, \xi) ds \right\} d\tau \quad (4.32)$$

for  $x_1 > x$ . When  $x_1 \rightarrow \infty$ , the distribution function  $f$  tends to the equilibrium distribution function behind the shock

$$f_{02} = n_2 \left( \frac{h_2}{\pi} \right)^{3/2} e^{-h_2 \xi^2},$$

while the integral  $J_1$  tends to  $f_{02} J_2(\infty, \xi)$ . Since the integral  $J_2$  is finite, then, according to (4.32),

$$f(x, \xi, \xi_x < 0) \geq C \exp(-h_2 \xi^2), \quad (4.33)$$

where  $C$  is a positive constant.

On the other hand, for points  $x$  lying upstream of the shock-wave center,  $f$  tends to

$$f_{01} = n_1 \left( \frac{h_1}{\pi} \right)^{3/2} e^{-h_1 \xi^2}$$

as  $x \rightarrow -\infty$ . In order that for these points the solution be represented by the series (4.29), which converges on the average, it is necessary that

$$\lim_{\xi \rightarrow \infty} f(x, \xi) \leq C_1 \exp \left( -\frac{1}{2} h_1 \xi^2 \right), \quad (4.34)$$

where  $C_1 > 0$ . When the inequality (4.33) holds, the condition (4.34) can be satisfied only when

$$h_2 \geq \frac{1}{2} h_1 \quad \text{or} \quad T_2 \leq 2T_1. \quad (4.35)$$

According to the Hugoniot conditions,  $T_2 = 2T_1$  at a Mach number of 1.851.

Therefore, a solution in the form of the series (4.29) which converges on the average exists only when  $M < 1.851$ .

Of course, from the fact that when  $M > 1.851$  the solution cannot be represented in the form of the series (4.29), it does not follow, for example, that the thirteen- or twenty-moment equations, or equations of higher order, cannot have solutions at those Mach numbers. Thus, the Navier-Stokes equations have a solution at any shock-wave Mach number, and the distribution function corresponding to them for Maxwellian molecules is identical with the thirteen-moment distribution function. It follows from what has been proved, however, that one cannot hope to obtain a more exact solution by retaining more terms of the series in the expansion in terms of Hermite polynomials.

In the region where solutions of the Grad equations exist, they show worse agreement with experiment than do the solutions of the Navier-Stokes and Burnett\* equations.

4. In principle, exact solutions for the shock-wave structure may be obtained by the Monte Carlo method (see § 3.15).† However, that method requires too much computing time on existing computers. Therefore, the results obtained are of a tentative nature. The method of computation is analogous to that described in § 4.2 for the problem of heat flow between two infinite walls. For this one-dimensional problem, the phase space can again be described by three coordinates: a geometrical coordinate  $x$ , the longitudinal velocity of the molecules  $\xi_x$ , and the radial velocity of the molecules  $\xi_R$  (the distribution function has an axis of symmetry  $\xi_x$ ). The phase space is divided into cells by subdividing each of the above coordinates into equal parts  $\Delta x$ ,  $\Delta \xi_x$ , and  $\Delta \xi_R$ . Each cell, labeled by the three subscripts  $i$ ,  $j$ , and  $k$ , corresponds to specific values of  $x_i$ ,  $\xi_{xj}$ , and  $\xi_{Rk}$ . The distribution function is completely determined in the entire space, if we specify in each cell numbers  $N_{ijk}$ , equal to the number of molecules located in the element of space  $\Delta x$  with center at  $x_i$  and  $\Delta \xi_x$ ,  $\Delta \xi_R$  with velocities  $\xi_{xj}$  and  $\xi_{Rk}$ , respectively.

Although the region occupied by the shock wave is infinite, to apply the Monte Carlo method we must insert the Rankine-

\*See the paper by Sherman and Talbot cited previously.

†J. K. Haviland, "Rarefied Gas Dynamics," Third Symposium, Academic Press, 1963.

Hugoniot conditions at some finite distance, which may be chosen by trial and error in such a way that the solution will not change as the boundaries are moved further away. The results presented below were obtained for boundaries located at three shock-wave thicknesses from one another, as determined from formula (4.25).

Let some distribution function be given, i.e., let the numbers  $N_{ijk}$  be assigned in all the cells of phase space. The molecules of the given distribution are field molecules. We now take a test molecule entering the flow region through one of the boundaries with velocity  $(\xi_x, \xi_R)$ . The number of molecules entering through the forward and rear boundaries must be chosen proportional to the number of molecules entering through these boundaries according to the distributions  $f_1$  and  $f_2$  at infinity. The numbers  $\xi_x$  and  $\xi_R$  are determined by random numbers, chosen with the probability density corresponding to the Maxwellian distributions  $f_1$  and  $f_2$ . A test molecule entering the flow region intersects the boundary at some cell. There are then two possibilities for it: either it passes through the cell and intersects the next boundary, or it experiences a collision. The probability of the two results is determined by the molecular interaction law and the density of field molecules. With an appropriate probability density we choose a random number to determine the time  $\tau$  of free flight of the molecule to the next collision. If  $\tau > \Delta x / \xi_x$ , the first of the above results occurs. In that case, the residence time of the molecule in the cell examined is equal to  $\Delta x / \xi_x$ . The molecule enters the next cell (with respect to  $x$ ), where again a draw is made for the time of free flight  $\tau$ . If  $\tau < \Delta x / \xi_x$ , a collision occurs in the first cell. As a result of the collision the test molecule acquires a new velocity, i.e., it falls into a cell with other velocity subscripts. In that cell a draw is again made for the time of free flight, and so on. The chosen test molecule is followed in that way until it leaves through one of the boundaries. Then a new test molecule is chosen and the process is begun afresh. By storing the time spent by the test molecules in each cell, we determine a new distribution function. The required number of test molecules is determined from the condition that the distribution function should not change as their number is increased. Thereafter, molecules corresponding to the new distribution function are assumed to be field molecules, and the calculation is begun afresh. To obtain a new distribution function requires one iteration. For these calculations we require a large

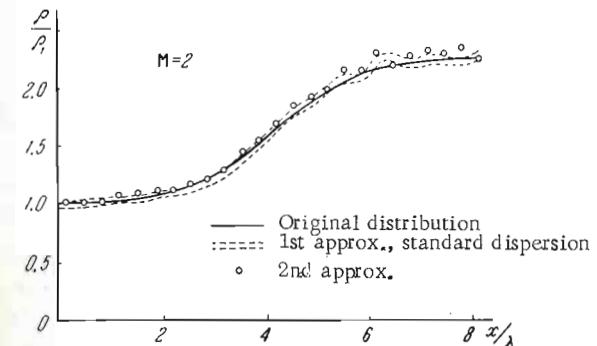


Fig. 38

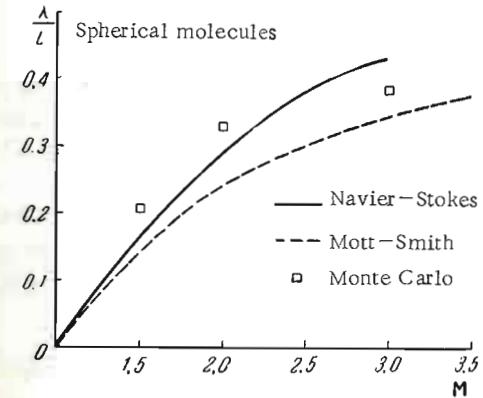


Fig. 39

operating memory, since it is necessary to store simultaneously the original distribution function and that being recalculated.

The calculations presented here were carried out on an IBM-709 computer for hard-sphere molecules. A bimodal Mott-Smith distribution was chosen as the original distribution function. Twenty-five cells were chosen in  $x$ , and 180 velocity cells for each  $x$ , i.e., for a single distribution function, 4500 values had to be stored.

Figure 38 shows a typical density profile, and Fig. 39 makes a comparison of the shock-wave thickness as calculated by the Monte Carlo method with the results of other methods.

As suggested earlier, the calculated points lie close to the Navier-Stokes curve for  $M$  numbers close to unity and closer to the Mott-Smith curve for large  $M$ . However, the fact that the number of calculations was so restricted, and that the convergence of the method has not been proved, does not allow us to regard the solutions obtained as being accurate. Nevertheless, the results obtained indicate the possibility of statistical modeling of complex molecular flows. The calculation scheme used is analogous to some extent to the method of successive approximations, in which the distribution function of the previous approximation is substituted into the right side of the Boltzmann equation. The convergence of the method depends to an appreciable extent on an effective choice of the original distribution function.

#### §4.5. Acoustic Oscillations

Until comparatively recently, acoustic oscillations of a monoatomic gas were studied in the hydrodynamic approximation. However, such an analysis is valid only for wavelengths large compared to the molecular mean free path, or for frequencies much lower than the frequency of collision of the molecules. To describe high-frequency (ultrasonic) oscillations over a wide range of frequencies we must begin with the linearized Boltzmann equation

$$\frac{\partial \varphi}{\partial t} + \xi_1 \frac{\partial \varphi}{\partial x_1} = \int \omega (\varphi'_1 + \varphi' - \varphi_1 - \varphi) B(\vartheta) d\vartheta de d\xi_1 = L(\varphi). \quad (5.1)$$

That equation has been written in the dimensionless coordinates used in §§3.11 and 3.12, namely,

$$\left. \begin{aligned} f &= f_0(1 + \varphi), \quad f_0 = n_0 \left( \frac{m}{kT_0} \right)^{3/2} \omega(\xi), \quad \bar{t} = \frac{t}{\tau}, \\ \bar{x}_1 &= \frac{x_1}{\tau} \sqrt{\frac{m}{kT_0}}, \quad \bar{\xi}_i = \sqrt{\frac{m}{kT_0}} \xi_i, \quad \bar{B} = n_0 \tau B. \end{aligned} \right\} \quad (5.2)$$

For simplicity, the primes above the dimensionless quantities in Eq. (5.1) have been omitted.

The dispersion equations have already been examined in §§3.11 and 3.12.

#### §4.5] ACOUSTIC OSCILLATIONS

We examine two methods for investigating sound waves.\* In the first of these, the perturbed distribution function  $\varphi$  is represented in the form of a truncated series in eigenfunctions of the operator  $L(\varphi)$  for Maxwellian molecules:

$$\varphi = \sum_{i=1}^N a_i \psi_i, \quad (a_i = \int \omega \psi_i d\xi). \quad (5.3)$$

By substituting  $\varphi$  in the form (5.3) into the Boltzmann equation (5.1), and multiplying it in succession by  $\omega \psi_\mu$  and integrating with respect to  $\xi$ , we obtain a system of  $N$  linear equations for the  $N$  moments  $a_\mu$ :

$$\frac{\partial a_\mu}{\partial t} + \sum_{i=1}^N A_{\mu i} \frac{\partial a_i}{\partial x_1} = \sum_{i=1}^N a_i \lambda_{i\mu}, \quad (5.4)$$

where

$$A_{\mu i} = \int \omega \xi_1 \psi_i \psi_\mu d\xi, \quad \lambda_{i\mu} = \int \omega \psi_\mu L(\psi_i) d\xi.$$

For Maxwellian molecules  $\lambda_{i\mu} = 0$  when  $i \neq \mu$ , and the system (5.4) may be simplified

$$\frac{\partial a_\mu}{\partial t} + \sum_{i=1}^N A_{\mu i} \frac{\partial a_i}{\partial x_1} = a_\mu \lambda_{ii}, \quad (\lambda_i = \lambda_{ii}). \quad (5.4a)$$

For hard-sphere molecules it is more convenient to use the Boltzmann equation in the form (11.58) of §3.11.

Seeking a solution of the form

$$a_i = \hat{a}_i e^{i\sigma t + kx_1}, \quad (5.5)$$

we obtain a system of linear homogeneous equations to determine the constants  $\hat{a}_i$ . The determinant of that system equated to zero determines the relationship between  $\sigma$  and  $k$  (dispersion relation).

\* Use has been made also of the thirteen-moment equations [I. I. Moiseev-Ol'khovskii, Doklady Akad. Nauk SSSR, Vol. 118, No. 3 (1958)].

In that way an investigation was made of acoustic oscillations for Maxwellian molecules using up to 483 moments\* and for hard spheres using up to 105 moments.†

The second method consists of using the model equations [see § 3.12, formula (12.9)]:

$$\left( \frac{\partial}{\partial t} + \xi_i \frac{\partial}{\partial x_i} - \lambda_{N+1, N+1} \right) \varphi = \sum_{i,j}^N a_i \beta_{ij} \psi_j, \quad (5.6)$$

where

$$\beta_{ij} = \lambda_{ij} - \delta_{ij} \lambda_{N+1, N+1}.$$

We seek a solution for  $\varphi$  in the form

$$\varphi = g(\xi) e^{i\sigma t + kx_i}, \quad a_i = \hat{a}_i e^{i\sigma t + kx_i}, \quad (5.7)$$

where the  $\hat{a}_i$  are constants. Then Eq. (5.6) takes the form

$$(i\sigma + k\xi_i - \lambda_{N+1, N+1}) g = \sum_{i,j}^N \hat{a}_i \beta_{ij} \psi_j, \quad (5.8)$$

whence

$$g(\xi) = \frac{\sum_{i,j}^N \hat{a}_i \beta_{ij} \psi_j}{i\sigma + k\xi_i - \lambda_{N+1, N+1}}. \quad (5.9)$$

By multiplying that equation in succession by  $\omega \psi_k$  and integrating with respect to  $\xi$ , we obtain a system of equations to determine the  $\hat{a}_k$ :

$$\hat{a}_k = \int \omega g(\xi) \psi_k d\xi = \sum_{i,j}^N \hat{a}_i \beta_{ij} \int \frac{\omega \psi_j \psi_k d\xi}{i\sigma + k\xi_i - \lambda_{N+1, N+1}} = \sum_{i,j}^N \hat{a}_i \beta_{ij} c_{jk}. \quad (5.10)$$

\*C. L. Pekeris, Z. Alterman, and L. Finkelstein, "Symposium on the Numerical Treatment of Ordinary Differential Equations," Birkhäuser-Verlag, Basel, 1960.

†C. L. Pekeris, Z. Alterman, L. Finkelstein, and K. Frankowski, Phys. Fluids, Vol. 5, No. 12 (1962).

When we equate the determinant of that system to zero we obtain the dispersion equation.\*

We note that by substituting the expansion (5.3) into the integral term of the Boltzmann equation, we obtain the equation for Maxwellian molecules

$$\left( \frac{\partial}{\partial t} + \xi_i \frac{\partial}{\partial x_i} \right) \varphi = \sum_i^N \lambda_i a_i \psi_i.$$

It is analogous to Eq. (5.6). Representing the solution of that equation in the form (5.7), we obtain a system of equations of type (5.10) in the same way as above. Therefore, the methods examined differ only as regards the original equations.

Since an analytical search for the roots of the dispersion equation is very complicated, a computer program was written, and the roots of the dispersion equation were found for  $N = 3, 5, 8$ , and  $11$ .† Only the hydrodynamic branch (see §§ 3.11 and 3.12) was investigated.

It was noted in § 3.12 that a different sequence of terms of the series (5.3) may be chosen. In the calculations presented, the order of the sequence of eigenfunctions in the series (5.3) was chosen so that the  $\lambda_{ij}$  corresponding to them are ordered in increasing magnitude.

In §§ 3.11 and 3.12 we were interested in the dependence of  $\sigma$  on  $k$ , i.e., the dependence of the frequency of oscillations on the wavelength. That approach is equivalent to the solution of the problem of damping of an initial perturbation. In experimental investigations forced oscillations are usually studied. Therefore, for comparison with experiment, we must consider  $\sigma$  to be a given real quantity, and  $k$  a desired complex quantity:

$$k = a + i\beta.$$

\*A table of the quantities  $c_{ij}$  has been given by L. Sirovich and J. K. Thurber, "Rarefied Gas Dynamics," Third Symposium, Academic Press, New York, 1963.

†L. Sirovich and J. K. Thurber, J. Acoust. Soc. Am., 37(2): 329-339 (1965).

The quantity  $\alpha$  characterizes the attenuation of the wave (the attenuation decrement), while  $\beta$  is the wavelength (wave number). The velocity of propagation of the wave,  $a$ , is equal to

$$a = \frac{\sigma}{\beta}.$$

As a parameter determining the flow regime, it is natural to take the ratio of the frequency of collisions to the frequency of oscillations.

We saw in §1.5 that the viscosity  $\mu$  and the pressure  $p$  are proportional, respectively, to

$$\mu \sim nmc\lambda \text{ and } p \sim nmc^2.$$

Therefore, the frequency of collisions is

$$\frac{1}{\tau} \sim \frac{c}{\lambda} \sim \frac{p}{\mu}.$$

Therefore, we shall introduce the parameter

$$r = \frac{p}{\sigma\mu}. \quad (5.11)$$

Figures 40-43 show a comparison of the results of various calculations of the sound velocity  $a$  and the attenuation decrement  $\alpha$  with experimental results.\* The sound velocity is referred to the adiabatic sound velocity  $a_0 = (\nu p / \rho)^{1/2}$ , and the attenuation decrement to the corresponding wave number  $\beta_0 = \alpha / a_0$ .

The tests were conducted in argon, helium, neon, krypton, and xenon. The comparison shows that the Navier-Stokes equation has satisfactory accuracy only at low oscillation frequency. The solution obtained by means of an expansion in series (5.3) proves to be more accurate than the Navier-Stokes solution, up to values  $r \sim 1$ . However, for still smaller values of  $r$ , it also diverges markedly from the experimental values.

For each model equation (for each  $N$ ) there is some critical frequency  $\sigma_0(N)$ , such that for  $\sigma > \sigma_0(N)$  the model equation does

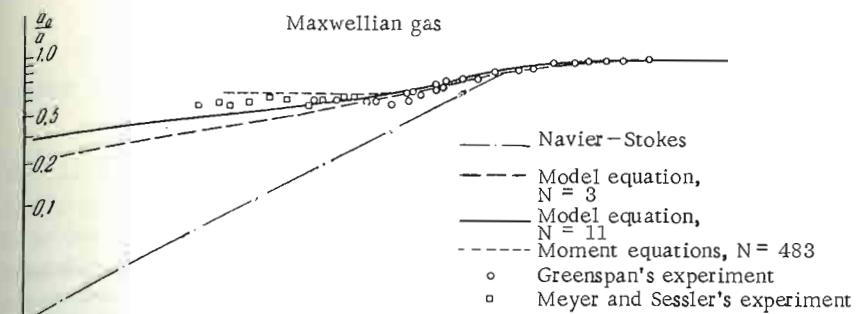


Fig. 40

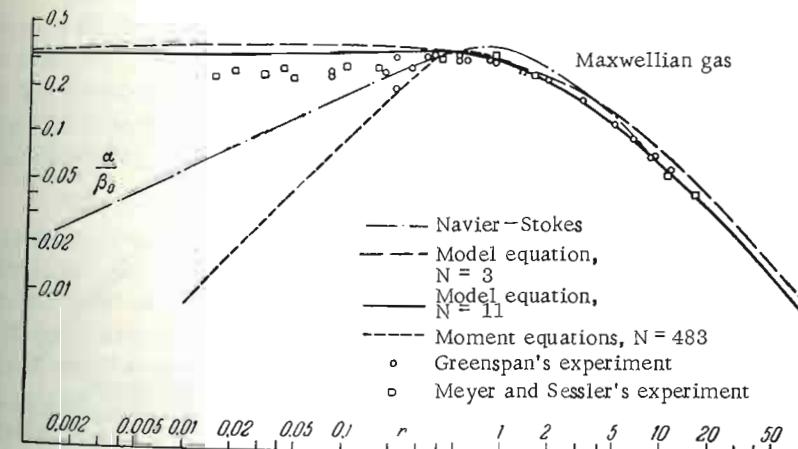


Fig. 41

not have a solution in the form of plane waves (compare with §3.12). The critical frequency increases with  $N$ . For Maxwellian molecules the critical frequencies correspond to  $r = 0.701$  with  $N = 3$ , and  $r = 0.413$  with  $N = 11$ . For hard spheres the corresponding critical values are  $r = 0.701$  and  $r = 0.290$ . However, in the complex  $k$ -plane, the corresponding branches of the solution of the dispersion equation may be continued analytically into the high-frequency region.\* The analytical continuation of the solution of the

\*M. Greenspan, J. Acoust. Soc., 28: 644 (1956); E. Meyer and G. Z. Sessler, Z. Phys., 149: 15 (1957).

\*See the paper by Sirovich and Thurber cited above.

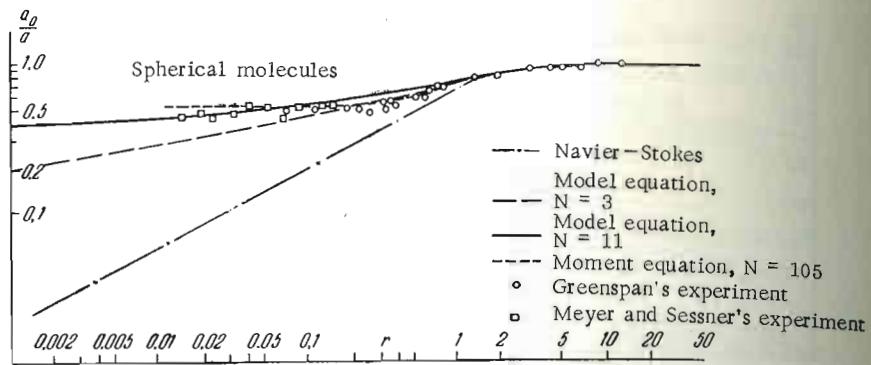


Fig. 42

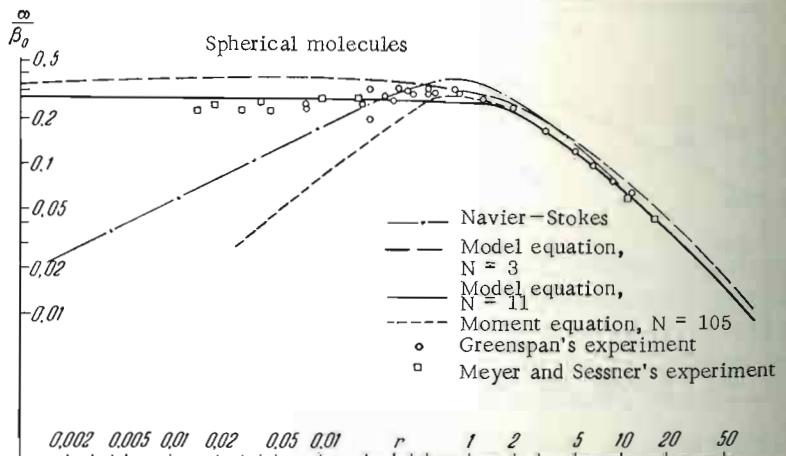


Fig. 43

model equation for smaller  $N$  is close to the solution (in the region between the corresponding critical frequencies of the model equation for larger  $N$ ). We may therefore assume that the analytical extrapolation determines the sound waves with an accuracy which increases with  $N$ . However, if this supposition is valid, then, as may easily be seen from the method of construction of the model equations, at each finite  $N$  we may point out still larger frequencies for which we may hope to obtain only a qualitatively correct picture.

Analytical continuation of the corresponding solutions is shown in Figs. 40-43 for values of  $r$  below critical.

The solutions thus constructed for the model equation with  $N = 11$  are in quite good agreement with experiment over a wide range of frequencies, which validates the assumption made above to some extent. Thus, we may expect that the solution of the model equation with a single relaxation time ( $N = 3$ ) diverges from the other solutions and from experiment in the Navier-Stokes region, since it corresponds to a distorted Prandtl number. However, it gives a satisfactory qualitative picture in the whole range of frequencies.

The somewhat better agreement with experiment for calculations with hard-sphere molecules, than for Maxwellian molecules, is also fully understandable. In fact, the temperature dependence of the viscosity, for example for helium ( $\mu \sim T^{0.647}$ ), is closer to the dependence given by spherical molecules ( $\mu \sim T^{0.5}$ ) than to the dependence corresponding to Maxwellian molecules ( $\mu \sim T$ ).

The difference between results obtained by means of the expansion in series (5.3) and by means of the model equations is very striking. In spite of the fact that many more moments were taken in the first method (up to 483 for Maxwellian molecules and up to 105 for hard spheres) than in the model equations (up to 11), the first method turned out to be quite unsuitable in the transition and Knudsen regions (for large oscillation frequencies). Yet, for low frequencies, the accuracy increases with increase in the number of terms in the series (5.3). This gives us reason to think (although this has not been demonstrated rigorously) that the series (5.3) converges only for values of  $r$  larger than some critical value.

As for the other problems, the absence of exact solutions of the Boltzmann equation does not allow us to draw firm conclusions as to the nature of acoustic oscillations as  $r \rightarrow 0$ . It should be borne in mind here that experimental investigation of those regimes is difficult. Sound waves are usually generated by means of an oscillating plate. As may be seen from Figs. 41 and 43, for small  $r$  the characteristic distance of attenuation is of the order of the wavelength:

$$\lambda_0 = \frac{2\pi}{\beta_0} = \frac{2\pi a_0}{\sigma},$$

which becomes less than the mean free path as  $\sigma \rightarrow \infty$ . On the other hand, at distances less than the mean free path from the oscillating plate, no sound waves can be formed, and it is not possible to measure perturbations caused by molecules which experienced their last collision with the plate.

## Chapter V

# FLows AT SMALL KNUDSEN NUMBERS

### §5.1. Slip and Temperature Jump

It was shown in Chapter III (see §§3.6-3.8) that, at internal points of the flow, the Hilbert-Enskog-Chapman expansion gives a solution which converges asymptotically to a solution of the Boltzmann equation for Knudsen numbers tending to zero. However, for an arbitrarily small Knudsen number, there is a region near the boundary in which that series is not a solution of the Boltzmann equation. As we saw in §§3.6-3.8 (and this will be proved again in a somewhat different manner below), the thickness of that region, called the Knudsen layer, is of the order of the mean free path  $\lambda$ .

Let us examine the flow inside the Knudsen layer in the Navier-Stokes approximation, i.e., let us restrict ourselves to two terms of the Hilbert or Enskog-Chapman series. Our objective will be to establish what kind of fictitious macroscopic boundary conditions should hold for the Navier-Stokes equations at that solid wall, so that, when they are fulfilled, the solution of the Navier-Stokes equation outside the Knudsen layer coincides (within the accuracy of the Navier-Stokes approximation) with the solution of the Boltzmann equation under the given actual kinetic conditions at the wall.

The nature of the interaction of the molecules with the wall determines the boundary conditions for the distribution function at the inner boundary of the Knudsen layer (Fig. 44). Since the Navier-Stokes distribution function is a solution of the Boltzmann equation only at some distance from the wall, in order to establish

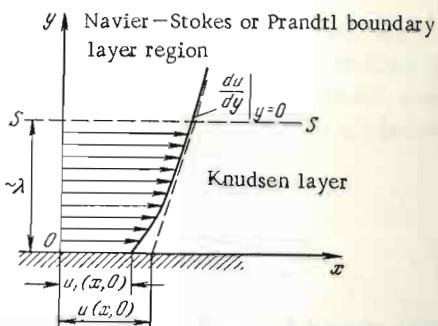


Fig. 44

the boundary conditions for the Navier-Stokes equations, it is necessary to investigate the Knudsen layer.

Let the velocity of the wall be zero. Clearly, the presence of a stationary wall with any law of molecular reflection (except completely specular reflection) must slow down the gas. On the other hand, there is no basis for our regarding the velocity of the gas at the wall as equal to the velocity of the wall. Similarly, there is no basis for us to regard the gas temperature at the wall as equal to the wall temperature.

Let the variation of the true gas velocity at the wall be as shown by the solid curve in Fig. 44. Let the line SS on Fig. 44 be located in a region where the solution of the Boltzmann equation is approximated with the required accuracy by the Navier-Stokes approximation. If we know the velocity and temperature of the gas along that line, then, by solving the Navier-Stokes equations, we can construct a solution in the whole region outside the Knudsen layer. Then, by continuing that solution of the Navier-Stokes equations inside the Knudsen layer (the dotted line on Fig. 44), we can determine fictitious values of velocity and temperature at the wall. In general, the velocity and temperature obtained in this way are not equal, either to the actual velocity and temperature of the gas at the wall, or to the velocity and temperature of the wall. The difference between the fictitious velocity and the wall velocity is called the *slip velocity*, and the corresponding temperature difference is called the *temperature jump*.

It is clear from the above method of obtaining the fictitious velocities and temperatures that, by taking them as boundary con-

ditions at the wall for the Navier-Stokes equations, we obtain a solution which coincides with the actual solution outside the Knudsen layer. Since, as a rule, we are not interested in the details of the flow when examining flows at small Knudsen numbers in the Knudsen layer,\* the slip velocity and the temperature jump are sufficient to calculate the flow in the Navier-Stokes approximation. But, as we have seen, to find those quantities we need the value of the actual velocity and temperature at the boundary of the Knudsen layer (roughly on the line SS); to determine it we must solve the Boltzmann equation inside the layer with a specified law of molecular reflection at the wall. At present, that problem has only been solved for the model equation.

However, before passing to the construction of that solution, we note that attempts are frequently made to establish the slip conditions without solving the Boltzmann equation. We might try to find the slip conditions, by using the conservation laws as we did in deriving the Hugoniot conditions for shock waves. The control surfaces might be, for example, the line SS and the wall. However, it is easy to see that this derivation cannot be carried through. Since the Navier-Stokes equations are valid on and beyond the line SS (Fig. 44), the solution of the external problem determines the fluxes of mass, momentum, and energy across the line SS in terms of the still unknown values of velocity and temperature on that line. However, at the second control surface (at the wall) we cannot find these fluxes, since we have only assigned the law of molecular reflection, i.e., the relation between the distribution function for the reflected molecules and the distribution function for the incident molecules, but the latter cannot be found in the layer without solving the Boltzmann equation. Therefore, we proceed as follows.† The solution of the Navier-Stokes equations is continued to the wall, i.e., it is assumed that the line SS coincides with the wall and that the velocity and temperature on it are equal to the desired

\*See the next section for more details regarding the general picture of flow at small Knudsen numbers.

†See, for example, the discussion of this question in the monographs: G. N. Patterson, "Molecular Flow of Gases," Wiley, New York, 1956; M. F. Shirokov, "Physical Basis of Gas Dynamics," Fizmatgiz, 1958; V. P. Shidlovskii, "Introduction to the Dynamics of Rarefied Gases," Nauka Press, 1965.

fictitious velocity and temperature at the wall. Since we can express the derivatives of the hydrodynamic quantities on that line in terms of those quantities from the solution of the Navier-Stokes equations, we can also express the distribution function in the Navier-Stokes approximation in terms of them. By taking that distribution function as the distribution function of the molecules incident on the wall, and using the law of molecular reflection, we can now write the fluxes of mass, energy, and momentum at the lower control surface, namely at the wall. By equating those fluxes to the fluxes through the upper control surface (through the line SS), we obtain equations to determine the unknown hydrodynamic quantities on the line SS, i.e., the slip conditions which are then included in the solution of the external problem in the Navier-Stokes approximation.

It is easy to see that this scheme for the flow near the wall appreciably distorts the actual picture described above. In addition, this scheme is not self-consistent. By taking the distribution function of the incident molecules to be the same as on the line SS, we have essentially assumed that the distribution function does not change in the Knudsen layer. At the same time it is assumed that whatever the distribution function of the reflected molecules is, molecules arriving from the wall to the line SS acquire the Navier-Stokes distribution. That scheme leads to an overspecified system of equations (we have an equation of mass conservation, three equations of momentum conservation, and an equation of conservation of energy, and only four unknowns;  $u_x$ ,  $u_y$ ,  $u_z$ , and  $T$  — on the line SS), which in turn leads not only to quantitative, but also to qualitative errors. Thus, for example, because of the overspecification of the problem, we must introduce a jump in pressure at the wall (a fifth unknown), although, as we see below, the pressure is constant across the Knudsen layer within the limit of our approximation.

Sometimes an attempt is made to improve matters, by taking the distribution function of the molecules incident at the wall, not in the Navier-Stokes approximation, but in the Burnett or thirteen-moment approximation. However, it is clear that this cannot correct the situation, since the flow pattern itself is incorrect, because it does not account for variations of the distribution function in the Knudsen layer; to find the correct slip conditions

we must solve the Boltzmann equation in the Knudsen layer.

1. Let us examine that problem for the model equation. We restrict the examination to the plane case.\*

We write the model equation in dimensionless variables in the following integral forms [see formula (6.3) of Chapter III]:

$$f(\mathbf{x}, \xi) = f_0(\mathbf{x}, \xi) + \exp \left\{ -\frac{1}{\epsilon} \int_{l(x_1)}^{l(x)} n \left[ \mathbf{x} - \frac{\xi}{\xi} (l(\mathbf{x}) - l) \right] \frac{dl}{\xi} \right\} [f(\mathbf{x}_1) - f_0(\mathbf{x}_1)] - \int_{l(x_1)}^{l(x)} \frac{df_0}{dl} \exp \left\{ -\frac{1}{\epsilon} \int_l^{l(x)} n \left[ \mathbf{x} - \frac{\xi}{\xi} (l(\mathbf{x}) - l) \right] \frac{dl}{\xi} \right\} dl, \quad (1.1)$$

$$f(\mathbf{x}, \xi) = f_0(\mathbf{x}, \xi) - \epsilon \frac{\xi}{n(\mathbf{x})} \frac{df_0(\mathbf{x}, \xi)}{dl} + \exp \left\{ -\frac{1}{\epsilon} \int_{l(x_1)}^{l(x)} n \left[ \mathbf{x} - \frac{\xi}{\xi} (l(\mathbf{x}) - l) \right] \frac{dl}{\xi} \right\} [f(\mathbf{x}_1) - f_0(\mathbf{x}_1) + \epsilon \frac{\xi}{n(\mathbf{x}_1)} \frac{df_0(\mathbf{x}_1)}{dl}] + \epsilon \int_{l(x_1)}^{l(x)} \frac{d}{dl} \left( \frac{\xi}{\xi} \frac{df_0}{dl} \right) \exp \left\{ -\frac{1}{\epsilon} \int_l^{l(x)} n \left[ \mathbf{x} - \frac{\xi}{\xi} (l(\mathbf{x}) - l) \right] dl \right\} dl, \quad (1.2)$$

where  $\mathbf{x}$  is referred to the characteristic length  $L$  of the flow outside the Knudsen layer,  $\xi$  is referred to the characteristic velocity of the molecules  $U \sim (kT_0/m)^{1/2}$ , and  $n$  to the characteristic density  $n_0$ ;  $l$  is the length measured along the trajectory of a molecule, and

$$\epsilon = Kn = \frac{U}{An_0 L}.$$

The first term in those equations corresponds to the Euler approximation.† In the Navier-Stokes region the distribution function is given by the first two terms of formula (1.2). If the points  $\mathbf{x}$  and  $\mathbf{x}_1$  both lie in the Navier-Stokes region, the square brackets in (1.2) vanish up to quantities of order  $\epsilon^2$ , and the ratio of the remaining term to the Navier-Stokes term is of order  $\epsilon$ . Now let the point  $\mathbf{x}_1$  be located at the wall ( $y_1 = 0$ ); we consider molecules

\* This problem apparently was first correctly formulated by Welander [P. Welander, Arkiv Fysik 7(6): 507 (1954)].

† See § 3.6.

moving away from the wall ( $\xi_y > 0$ ). Then,

$$f(x_1, \xi) = f_r(x_1, \xi)$$

is the distribution function of the molecules reflected from the wall. Since the Navier-Stokes representation of the distribution function is invalid inside the Knudsen layer, the square brackets do not vanish. However, because of the exponential term which multiplies the bracket, the contribution of the wall to the distribution function falls off with increasing distance from the wall.

We determine the thickness  $\delta_k$  of the Knudsen layer from the condition that inside the layer the term determining the influence of the wall and the Navier-Stokes term are of the same order. When  $y > \delta_k$ , we may consider the Navier-Stokes representation to be valid.

We assume that the law of reflection of molecules from the wall has the form (see §§2.9, 2.10)

$$f_r = f_r(0, \xi_x, \xi_y > 0, \xi_z) = n_r \left( \frac{h_r}{\pi} \right)^{3/2} e^{-h_r \xi^2}, \quad (1.3)$$

i.e., we assume that the molecules are reflected diffusely with a Maxwellian distribution. The temperature of the reflected molecules is related to the wall temperature through the accommodation coefficient

$$E_r = E_t - \alpha_e (E_i - E_w). \quad (1.4)$$

The quantity  $n_r$  is found from the condition that no gas flows through the wall.

We now make the natural assumption that the derivatives with respect to  $x$  and  $y$  of the hydrodynamic quantities, and, therefore, also of  $f_0$ , are of the same order inside the layer as outside the layer. That assumption is supported by the solution obtained for the layer.

We take the point  $x_1$  at a distance  $\delta_k$  from the wall and examine the molecules which proceed toward the wall ( $\xi_y < 0$ ). Then the square brackets in (1.1) are of order  $\epsilon df_0/dy$ , and the distribution function of the molecules incident on the wall at  $y = 0$  has the form

$$f_t = f_0(0, \xi) + O\left(\epsilon \frac{df_0}{dy}\right), \quad (1.5)$$

i.e., the distribution function of the incident molecules is close to the equilibrium distribution function, which corresponds to the hydrodynamic quantities at the wall,  $n(0)$ ,  $u_x(0)$ , and  $T(0)$ .

It is easy to see that in that case the distribution function of the reflected molecules  $f_r$  is also close to equilibrium. In fact,

$$n(0) = \int_{\xi_y < 0} f_t d\xi + \int_{\xi_y > 0} f_r d\xi = \frac{n(0)}{2} + \frac{n_r}{2} + O\left(\epsilon \frac{df_0}{dy}\right), \quad (1.6a)$$

$$n(0) u_x(0) = \int_{\xi_y < 0} \xi_x f_t d\xi + \int_{\xi_y > 0} \xi_x f_r d\xi = \frac{n(0) u_x(0)}{2} + O\left(\epsilon \frac{df_0}{dy}\right), \quad (1.6b)$$

$$\begin{aligned} \frac{3}{2} kT(0) n(0) &= \frac{m}{2} \int_{\xi_y < 0} c^2 f_t d\xi + \frac{m}{2} \int_{\xi_y > 0} c^2 f_r d\xi \\ &= \frac{3}{4} kT(0) n(0) + \frac{3}{4} kT_r n_r + O\left(\epsilon \frac{df_0}{dy}\right). \end{aligned} \quad (1.6c)$$

We see from these relations that  $n_r$  and  $T_r$  differ from  $n(0)$  and  $T(0)$  by quantities of order  $\epsilon$ . We also see that as a result of relation (1.6b),  $u_x(0) = O(\epsilon)$ . Since  $T_r$  only differs from  $T(0)$  by a quantity of order  $\epsilon$ , it follows from (1.4), with  $\alpha_e \neq 0$ , that  $T(0)$  differs from the wall temperature  $T_w$  by a quantity of order  $\epsilon$ .

Now let the point  $x_1$  be at the wall, and the point  $x$  an arbitrary point within the layer. We consider molecules leaving the wall ( $\xi_y > 0$ ). In accordance with the estimates made above, the square brackets in (1.2) are of order  $\epsilon df_0/dy$ . By equating the square bracket to the Navier-Stokes term, we obtain

$$\epsilon \frac{\xi}{n} \frac{df_0}{dl} \sim \epsilon \frac{\xi}{n} \frac{df_0}{dt} \exp \left\{ -\frac{1}{\epsilon} \int_{l(x_1)}^{l(x)} n dl \right\},$$

whence it follows that  $\delta_k = O(\epsilon)$ . Here, the remaining terms in Eqs. (1.1) and (1.2) are of the same order as the Navier-Stokes term.

It then follows from (1.1) and (1.2) that the distribution function has the form

$$f(x, \xi) = f_0(x, \xi) + O(\epsilon)$$

everywhere inside the layer. Since the hydrodynamic quantities which appear in  $f_0$  vary across the layer by a quantity of order  $\epsilon$  (the derivatives with respect to  $u$ ,  $n$ , and  $T$  are of order unity), the distribution function inside the layer may be represented in the form

$$f(x, \xi) = f_{00}(x, \xi)(1 + \phi), \quad (\phi = O(\epsilon)), \quad (1.7)$$

where  $f_{00}(x, \xi)$  is an equilibrium distribution function constant across the layer. Although the distribution function inside the layer is close to the Maxwellian distribution, it cannot be represented by an Enskog series, and to determine it we must solve the Boltzmann equation inside the layer.

Thus, there are two regions: the first has a characteristic dimension  $L$  (unity in dimensionless variables), and the other has a characteristic dimension  $\epsilon L$  (in dimensionless quantities,  $\epsilon$ ) in the direction perpendicular to the wall, and a dimension  $L$  along the wall. We must find the asymptotic solution (as  $\epsilon \rightarrow 0$ ) of the Boltzmann equation

$$\xi_x \frac{\partial f}{\partial x} + \xi_y \frac{\partial f}{\partial y} = \frac{n}{\epsilon} (f_0 - f). \quad (1.8)$$

It is convenient to introduce the variable  $y_1 = y/\epsilon$  in the Knudsen layer:

$$\xi_y \frac{\partial f}{\partial y_1} + \epsilon \xi_x \frac{\partial f}{\partial x} = n(f_0 - f). \quad (1.9)$$

In both regions we seek the solution as a series in  $\epsilon$ . The outer expansion is

$$f = F^{(0)}(x, y, \xi) + \epsilon F^{(1)}(x, y, \xi) + \dots \quad (1.10)$$

and the inner expansion is

$$f = f^{(0)}(x, y_1, \xi) + \epsilon f^{(1)}(x, y_1, \xi) + \dots \quad (1.11)$$

The expansion (1.10) is none other than the Hilbert series (see §3.7).

The boundary conditions at the wall ( $y_1 = 0$ ) for the functions  $f^{(k)}(x, y_1, \xi)$  of the inner expansion follow immediately from the condition that no gas crosses the wall

$$\int_{\xi_y < 0} \xi_y f_i(x, 0, \xi) d\xi + \int_{\xi_y > 0} \xi_y f_r(x, 0, \xi) d\xi = 0, \quad (1.12)$$

which for the law of molecular reflection (1.3) takes the form

$$\int_{\xi_y < 0} \xi_y f_i d\xi = \frac{n_r}{2V\pi h_r}, \quad (1.12a)$$

where  $h_r$  is determined by relation (1.4):

$$(a_e - 1) \int_{\xi_y < 0} \xi_y \xi^2 f_i d\xi = \frac{n_r}{V\pi h_r} \left( \frac{a_e}{h_w} - \frac{1}{h_r} \right). \quad (1.13)$$

By expanding  $f_i$ ,  $n_r$ , and  $h_r$  in series in  $\epsilon$ , and equating coefficients of equal powers of  $\epsilon$ , we find the relations between  $f_r^{(k)}$ ,  $n_r^{(k)}$ , and  $h_r^{(k)}$ .

The boundary conditions for the outer expansion as  $y \rightarrow 0$  and for the inner expansion as  $y_1 \rightarrow \infty$  are found from the matching conditions.

We write the outer expansion in the coordinates of the inner expansion (i.e., we replace  $y$  by  $y_1$ ):

$$f = F^{(0)}(x, \epsilon y_1, \xi) + \epsilon F^{(1)}(x, \epsilon y_1, \xi) + \dots \quad (1.14)$$

and expand the functions  $F^{(k)}(x, \epsilon y_1, \xi)$  in a series in  $\epsilon$  with  $y_1 = \text{const}$ ; we have

$$f = F^{(0)}(x, 0, \xi) + \epsilon \left[ F^{(1)}(x, 0, \xi) + \frac{\partial F^{(0)}(x, 0, \xi)}{\partial y} y_1 \right] + \dots \quad (1.15)$$

It is evident that as  $y_1 \rightarrow \infty$  the inner expansion must coincide with the outer expansion (1.15), i.e.,

$$f^{(0)}(y_1 \rightarrow \infty) = F^{(0)}(0), \quad f^{(1)}(y_1 \rightarrow \infty) = F^{(1)}(0) + y_1 \frac{\partial F^{(0)}(0)}{\partial y}, \text{ etc.} \quad (1.16)$$

The aim of this section is to establish boundary conditions for the Navier-Stokes equations.

If we restrict ourselves to two terms (Navier-Stokes approximation) in the Hilbert expansions (1.10) and (1.15), then, up to terms of order  $\epsilon^2$ , we may replace them by the corresponding Enskog-Chapman expansions by means of formulas (8.41) and (8.42) of Chapter III:

$$f = F^{(0)}(x, y, \xi) + \epsilon F^{(1)}(x, y, \xi) = F^{(0)}(x, y, \xi) - \epsilon \frac{\xi}{n(y)} \frac{dF^{(0)}(x, y, \xi)}{dy} \quad (1.10a)$$

and

$$\begin{aligned} f &= F^{(0)}(x, 0, \xi) + \epsilon \left[ F^{(1)}(x, 0, \xi) + \frac{\partial F^{(0)}(x, 0, \xi)}{\partial y} y_1 \right] \\ &= F^{(0)}(x, 0, \xi) + \epsilon \left( y_1 \frac{\partial F^{(0)}(x, 0, \xi)}{\partial y} - \frac{\xi}{n(0)} \frac{dF^{(0)}(x, 0, \xi)}{dy} \right). \end{aligned} \quad (1.15a)$$

Here and below, hydrodynamic quantities without subscripts refer to the outer (Navier-Stokes) flow. The hydrodynamic quantities in the layer are denoted by the subscript 1.

We can also restrict ourselves to two terms in the Navier-Stokes approximation in the expansion (1.11). Instead of seeking the functions  $f^{(0)}$  and  $f^{(1)}$  with boundary conditions (1.16), on the basis of the estimates presented above, we can seek the function  $\varphi(x, y_1)$ , determined by expression (1.7), in which we put, for example,

$$f_{00} = n(x, 0) \left( \frac{h(x, 0)}{\pi} \right)^{3/2} \exp \{-h(0, x)\xi^2\}.$$

Substituting the distribution function (1.7) into Eq. (1.9) and linearizing it, we obtain

$$\begin{aligned} &\frac{1}{n(x, 0) \sqrt{h(x, 0)}} \left[ v_y \frac{\partial \varphi}{\partial y_1} + \epsilon v_x \left( \frac{\partial \varphi}{\partial x} + \frac{\partial \ln n(x, 0)}{\partial x} \right. \right. \\ &\left. \left. + \left( \frac{3}{2} - v^2 \right) \frac{\partial \ln h(x, 0)}{\partial x} \right) \right] = -\varphi + v + 2v_x u_1' v_1 + \left( v^2 - \frac{3}{2} \right) \tau. \end{aligned} \quad (1.17)$$

where the notation is similar to that introduced in § 4.2:

$$\begin{aligned} v_x &= \xi_x \sqrt{h(x, 0)}, \quad v_y = \xi_y \sqrt{h(x, 0)}, \\ n_1(x, y_1) &= n(x, 0)[1 + v(x, 0)], \\ T_1(x, y_1) &= T(x, 0)[1 + \tau(x, y_1)]. \end{aligned}$$

$$\left. \begin{aligned} u(x, 0) &= u_x(x, 0) h^{1/2}(x, 0), \quad u_1(x, y_1) = u_{x1}(x, y_1) h^{1/2}(x, 0), \\ v(x, y_1) &= \frac{1}{n(x, 0)} \int f_{00} \varphi d\xi, \\ \tau(x, y_1) &= \frac{2}{3} \frac{h(x, 0)}{n(x, 0)} \int \xi^2 f_{00} \varphi d\xi - v, \\ u_1(x, y_1) &= \frac{1}{n(x, 0)} \int f_{00} v_x \varphi d\xi. \end{aligned} \right\} \quad (1.18)$$

The derivatives  $\partial \varphi / \partial x$  and  $\partial \varphi / \partial y_1$  are of the same order (order  $\epsilon$ ), but the small parameter  $\epsilon$  multiplies  $\partial \varphi / \partial x$ . Therefore we may neglect the term with  $\partial \varphi / \partial x$ .

The Boltzmann equation (1.9) is written in dimensionless variables, with  $x$  and  $y$  referred to the characteristic flow dimension  $L$  outside the Knudsen layer.

If the flow takes place at large Reynolds numbers (but small Knudsen numbers,  $\epsilon = \lambda / L$ ), then the outer flow is a boundary layer flow whose characteristic dimension  $L$  is the boundary layer thickness  $\delta$  in the  $y$  direction, and  $L_\infty \sim L \sqrt{Re_\infty}$  in the direction  $x$ . Since

$$\begin{aligned} L &\sim \delta \sim \frac{L_\infty}{\sqrt{Re_\infty}}, \quad \mu \sim \rho a \lambda, \\ Re_\infty &= \frac{\rho V_\infty L_\infty}{\mu} \sim \frac{V_\infty L_\infty}{a \lambda L} \sim \frac{M}{\epsilon} \sqrt{Re_\infty}, \end{aligned}$$

$L_\infty \sim LM/\epsilon$  in this case, and the derivatives  $\partial \ln n / \partial x$  and  $\partial \ln h / \partial x$  are of order  $\epsilon/M$  and may be neglected.

If  $Re_\infty \leq 1$ , the characteristic dimensions in the outer Navier-Stokes flow in the directions  $x$  and  $y$  are the same and  $L = L_\infty$ . In that case, the derivatives  $\partial \ln n / \partial x$  and  $\partial \ln h / \partial x$  are of order unity must be retained. But in that kind of flow, the Navier-Stokes equations give

$$\frac{1}{p} \frac{\partial p}{\partial x} = \frac{\partial \ln T}{\partial x} + \frac{\partial \ln n}{\partial x} \sim \frac{\mu}{p} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \sim M\epsilon.$$

Therefore, to the required accuracy, we may put

$$\frac{\partial \ln n(x, 0)}{\partial x} = -\frac{\partial \ln T(x, 0)}{\partial x} = \frac{\partial \ln h(x, 0)}{\partial x}$$

and eliminate the derivative of  $\ln n$  from Eq. (1.17).

Thus, the equation for  $\varphi$  takes the form

$$\begin{aligned} \frac{v_y}{n(x, 0) \sqrt{h(x, 0)}} \frac{\partial \varphi}{\partial y_1} &= -\varphi + v + 2v_x u_1(y_1) \\ &+ \left(v^2 - \frac{3}{2}\right) \tau - \frac{\epsilon(5/2 - v^2) v_x}{n(x, 0) \sqrt{h(x, 0)}} \frac{d \ln h(x, 0)}{dx}. \end{aligned} \quad (1.19)$$

That equation is analogous to the equation for Couette flow [Eq. (2.27) of Chapter IV].

The boundary condition at the wall (1.3) for the function  $\varphi$  takes the form

$$\varphi_r(x, 0) = v_r + \left(v^2 - \frac{3}{2}\right) \tau_r. \quad (1.20)$$

When  $y_1 \rightarrow \infty$ , neglecting terms of order  $\epsilon^2$ , we have, from (1.16) and (1.15a),

$$\begin{aligned} \varphi(y_1 \rightarrow \infty) &= 2u(x, 0)v_x - \epsilon \left( \frac{v_y}{n(x, 0) \sqrt{h(x, 0)}} - y_1 \right) \frac{1}{f_{00}(x)} \frac{\partial F^0(x, 0)}{\partial y} \\ &- \epsilon \frac{v_x}{n(x, 0) \sqrt{h(x, 0)} f_{00}(x)} \frac{\partial F^0(x, 0)}{\partial x}. \end{aligned} \quad (1.21)$$

We represent  $\varphi$  in the form of the sum

$$\varphi = \chi(x, y_1, \xi) + v_x \psi(x, y_1, \xi). \quad (1.22)$$

Then for  $\chi$  and  $\psi$ , from (1.19) we obtain the equation

$$\frac{v_y}{n(x, 0) \sqrt{h(x, 0)}} \frac{\partial \chi}{\partial y_1} = -\chi + v + \left(v^2 - \frac{3}{2}\right) \tau \quad (1.23)$$

and

$$\begin{aligned} \frac{v_y}{n(x, 0) \sqrt{h(x, 0)}} \frac{\partial \psi}{\partial y_1} \\ = -\psi + 2u_1 - \frac{\epsilon}{n(x, 0) \sqrt{h(x, 0)}} \left(\frac{5}{2} - v^2\right) \frac{d \ln h(x, 0)}{dx}. \end{aligned} \quad (1.24)$$

Correspondingly, the boundary conditions take the form

$$\chi_r(x, v) = v_r + \left(v^2 - \frac{3}{2}\right) \tau_r, \quad (1.25)$$

$$\psi_r(x, v) = 0 \quad (1.26)$$

at the wall, and

$$\chi = -\epsilon \left( \frac{v_y}{n(x, 0) \sqrt{h(x, 0)}} - y_1 \right) \left( \frac{5}{2} - v^2 \right) \frac{d \ln h(x, 0)}{dy}, \quad (1.27)$$

$$\begin{aligned} \psi = 2u(x, 0) - 2\epsilon \left( \frac{v_y}{n(x, 0) \sqrt{h(x, 0)}} - y_1 \right) \frac{\partial u(x, 0)}{\partial y} \\ - \frac{\epsilon}{n(x, 0) \sqrt{h(x, 0)}} \left( \frac{5}{2} - v^2 \right) \frac{d \ln h(x, 0)}{dx} \end{aligned} \quad (1.28)$$

when  $y_1 \rightarrow \infty$ .

Equation (1.23) is of the same form as in the linear problem of heat transfer between infinite plates (see §4.2). The problems of finding the temperature jump and the velocity jump are separated.

2. We begin with the latter problem. We write Eq. (1.24) in the integral form

$$\psi(y_2, v_y > 0) = 2 \int_0^{y_2} u_1(s) e^{-\frac{y_2-s}{v_y}} \frac{ds}{v_y} - \left(\frac{5}{2} - v^2\right) \frac{d \ln h(x, 0)}{dx_2} \left(1 - e^{-\frac{y_2}{v_y}}\right), \quad (1.29)$$

$$\psi(y_2, v_y < 0) = -2 \int_{y_2}^{\infty} u_1(s) e^{-\frac{y_2-s}{v_y}} \frac{ds}{v_y} - \left(\frac{5}{2} - v^2\right) \frac{d \ln h(x, 0)}{dx_2}, \quad (1.30)$$

where

$$\begin{cases} y_2 = y_1 n(x, 0) \sqrt{h(x, 0)}, \\ x_2 = \epsilon^{-1} x n(x, 0) \sqrt{h(x, 0)}. \end{cases} \quad (1.31)$$

The variable  $x$  enters as a parameter.

We multiply Eqs. (1.29) and (1.30) by  $\pi^{-3/2} v_x^2 e^{-v^2}$  and integrate over the molecular velocities; using definition (1.18) for  $u_1(x, y_1)$ , we obtain

$$\begin{aligned} u_1(y_2) &= \frac{1}{\sqrt{\pi}} \int_0^{y_2} u_1(s) J_{-1}(y_2 - s) ds + \frac{1}{\sqrt{\pi}} \int_{y_2}^{\infty} u_1(s) J_{-1}(s - y_2) ds \\ &+ \frac{1}{2\sqrt{\pi}} \left[ \frac{1}{2} J_0(y_2) - J_2(y_2) \right] \frac{d \ln h(x, 0)}{dx_2}, \end{aligned} \quad (1.32)$$

where

$$J_n(x) = \int_0^\infty \lambda^n e^{-(\lambda^2 + \frac{x}{\lambda})} d\lambda \quad \text{and} \quad \frac{dJ_n(x)}{dx} = -J_{n-1}(x).$$

It follows from condition (1.28) that when  $y \rightarrow \infty$ ,

$$u_1(y_2) = u(x, 0) + y_2 \frac{\partial u(x, 0)}{\partial y_2}. \quad (1.33)$$

On the other hand,

$$u_1(y_2) = u_1(x, 0) + \int_0^{y_2} \frac{\partial u_1(x, y_1)}{\partial y_2} dy_2. \quad (1.34)$$

The velocity  $u(x, 0)$  is the wall boundary condition for the Navier-Stokes equations, and is called the *slip velocity*. It is not equal to the true gas velocity at the wall  $u_1(x, 0)$ . According to (1.33) and (1.34), the slip velocity is equal to

$$u(x, 0) = u_1(x, 0) + \int_0^\infty \left( \frac{\partial u_1(y_2)}{\partial y_2} - \frac{\partial u(0)}{\partial y_2} \right) dy_2. \quad (1.35)$$

It is convenient to introduce the new variable

$$G(y_2) = u_1(y_2) - \frac{\partial u(0)}{\partial y_2} y_2. \quad (1.36)$$

Then (1.35) may be written in the form

$$u(x, 0) = G(0) + \int_0^\infty \frac{dG}{dy_2} dy_2 = G(\infty). \quad (1.37)$$

Equation (1.32) for the variable  $G$  takes the form

$$\begin{aligned} G(y_2) \sqrt{\pi} &= \int_0^{y_2} G(s) J_{-1}(y_2 - s) ds + \int_{y_2}^\infty G(s) J_{-1}(s - y_2) ds \\ &+ J_1(y_2) \theta_1 + \left[ \frac{1}{4} J_0(y_2) - \frac{1}{2} J_2(y_2) \right] \theta_2, \end{aligned} \quad (1.38)$$

where we define

$$\theta_1 = \frac{\partial u(x, 0)}{\partial x_2} \quad \text{and} \quad \theta_2 = \frac{\partial \ln h(x, 0)}{\partial x_2}.$$

We integrate the integrals in (1.38) by parts; we have

$$\begin{aligned} G(0) J_0(y_2) - J_1(y_2) \theta_1 - \frac{1}{2} \left[ \frac{1}{2} J_0(y_2) - J_2(y_2) \right] \theta_2 \\ = - \int_0^{y_2} \frac{dG(s)}{ds} J_0(y_2 - s) ds + \int_{y_2}^\infty \frac{dG(s)}{ds} J_0(s - y_2) ds. \end{aligned} \quad (1.39)$$

We obtain a somewhat more convenient form of the equation by integrating it with respect to  $y_2$ :

$$\begin{aligned} G(0) J_1(0) - G(0) J_1(y_2) - \theta_1 J_2(0) + \theta_1 J_2(y_2) \\ - \theta_2 \left[ \frac{1}{4} J_1(0) - \frac{1}{2} J_3(0) \right] + \theta_2 \left[ \frac{1}{4} J_1(y_2) - \frac{1}{2} J_3(y_2) \right] \\ = \int_0^\infty \frac{dG}{ds} J_1(|y_2 - s|) ds - \int_0^\infty \frac{dG}{ds} J_1(s) ds. \end{aligned}$$

When  $y_2 \rightarrow \infty$ , all the terms containing  $y_2$  tend to zero and, therefore,

$$\begin{aligned} -G(0) J_1(y_2) + \theta_1 J_2(y_2) + \theta_2 \left[ \frac{1}{4} J_1(y_2) - \frac{1}{2} J_3(y_2) \right] \\ = \int_0^\infty \frac{dG}{ds} J_1(|y_2 - s|) ds. \end{aligned} \quad (1.40)$$

We represent the function  $G$  as the sum

$$G(y_2) = \theta_1 G_1(y_2) + \theta_2 G_2(y_2), \quad (1.41)$$

so that the functions  $G_1$  and  $G_2$  satisfy, respectively, the equations

$$-G_1(0) J_1(y_2) + J_2(y_2) = \int \frac{dG_1}{ds} J_1(|y_2 - s|) ds \quad (1.42)$$

and

$$-G_2(0) J_1(y_2) + \left[ \frac{1}{4} J_1(y_2) - \frac{1}{2} J_3(y_2) \right] = \int_0^\infty \frac{dG_2}{ds} J_1(|y_2 - s|) ds. \quad (1.43)$$

Equation (1.42) was first obtained in the paper by Welander cited above. There are different methods of analytical solution of Eq. (1.42).\* However, the solution is given in such complicated quadratures that a direct numerical solution by the method of iteration is simpler. Here  $G_1(0)$  and  $G_2(0)$  are found by quadratures from (1.38) when  $y_2 = 0$ .

According to (1.37) and (1.41), we have, for the slip velocity,

$$u(x, 0) = G_1(\infty) \frac{\partial u(x, 0)}{\partial y_2} + G_2(\infty) \frac{d \ln h(x, 0)}{dx}. \quad (1.44)$$

The coefficients which appear above have the numerical values†

$$G_1(\infty) = 1.012, \quad G_2(\infty) = -0.42. \quad (1.45)$$

Returning to dimensional quantities, and replacing the quantity  $A$  in  $\epsilon$  according to the formula  $A = kT_0/\mu_0$ , we obtain (compare § 4.2)

$$u_x(x, 0) = \frac{\mu(x, 0)}{n(x, 0)} \sqrt{\frac{2}{mkT(x, 0)}} \left( 1.012 \frac{\partial u_x(x, 0)}{\partial y} + 0.42 \frac{1}{\sqrt{h(x, 0)}} \frac{\partial \ln T(x, 0)}{\partial x} \right). \quad (1.46)$$

This is the so-called slip condition at the wall. The velocity profile in the Knudsen layer is shown in Fig. 45.

If a Prandtl boundary layer forms the outer flow outside the Knudsen layer, the second term in (1.46) may be neglected, since the  $y$  derivative is larger than the  $x$  derivative by a factor of  $\sqrt{Re} \gg 1$ . The second term may become comparable to the first when the Knudsen number is small, and the Reynolds number is of order unity, i.e., at small Mach number. In that case, the variables  $x$  and  $y$  are comparable.

If we put a uniformly heated body in a rarefied gas, then, according to (1.46), the gas begins to move from the cold parts of the body to the warmer parts.

\*See D. R. Willis, Phys. Fluids, Vol. 5, No. 2 (1962); C. Cercignani, Ann. Phys., 20: 219 (1962); S. Albertoni, C. Cercignani, and L. Gotusso, Phys. Fluids, Vol. 6, No. 7 (1963).

†Numerical solution of the integral equation for  $G_2$  was carried out at the request of the author by M. Istomina. The calculations were carried out with comparatively low accuracy. To assess the accuracy, the equation for  $G_1$  was also solved. A value of  $G_1(\infty) = 1.009$  was found, instead of the value  $G_1(\infty) = 1.012$ , obtained in the papers mentioned in the preceding footnote.

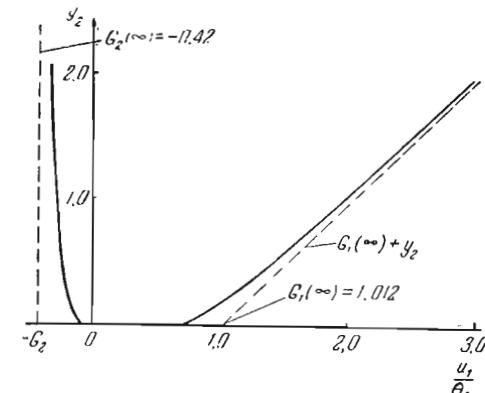


Fig. 45

3. To construct the temperature profile in the layer and to determine the temperature jump, we must solve Eq. (1.23) with boundary conditions (1.25) and (1.27).\* In a manner similar to (1.29) and (1.30), we have

$$\chi(y_2, v_y > 0) = \left[ v_2 + \left( v^2 - \frac{3}{2} \right) \tau_r \right] e^{-\frac{y_2}{v_x}} + \int_0^{y_2} \left[ v + \left( v^2 - \frac{3}{2} \right) \tau \right] e^{-\frac{y_2-s}{v_y}} \frac{ds}{v_y}, \quad (1.47)$$

$$\chi(y_2, v_y < 0) = \int_{-\infty}^{y_2} \left[ v + \left( v^2 - \frac{3}{2} \right) \tau \right] e^{-\frac{y_2-s}{v_y}} \frac{ds}{v_y}. \quad (1.48)$$

Integrating over the velocities, we obtain integral equations for the macroscopic quantities  $v$  and  $\tau$ :

$$\begin{aligned} \sqrt{\pi} \tau &= v_r \left[ \frac{2}{3} J_2(y_2) - \frac{1}{3} J_0(y_2) \right] + \tau_r \left[ \frac{2}{3} J_4(y_2) - \frac{2}{3} J_2(y_2) + \frac{5}{6} J_0(y_2) \right] \\ &\quad + \int_0^\infty v \left[ \frac{2}{3} J_1(|y_2-s|) - \frac{1}{3} J_{-1}(|y_2-s|) \right] ds \\ &\quad + \int_0^\infty \tau \left[ \frac{2}{3} J_3(|y_2-s|) - \frac{2}{3} J_1(|y_2-s|) + \frac{5}{6} J_{-1}(|y_2-s|) \right] ds, \end{aligned} \quad (1.49)$$

\*See the paper by Welander cited above.

$$\begin{aligned} \sqrt{\pi} v &= v_r J_0(y_2) + \tau_r \left[ J_2(y_2) - \frac{1}{2} J_0(y_2) \right] + \int_0^\infty v J_{-1}(|y_2 - s|) ds \\ &\quad + \int_0^\infty \tau \left[ J_1(|y_2 - s|) - \frac{1}{2} J_{-1}(|y_2 - s|) \right] ds. \end{aligned} \quad (1.50)$$

From the condition (1.27) at infinity, we have

$$\tau = y \frac{\partial \ln T(x, 0)}{\partial y} \text{ and } v = -y \frac{\partial \ln T(x, 0)}{\partial y}. \quad (1.51)$$

We introduce the new variables

$$G_\tau = \tau - y_2 \frac{\partial \ln T(x, 0)}{\partial y_2} \text{ and } G_v = v + y_2 \frac{\partial \ln T(x, 0)}{\partial y_2}. \quad (1.52)$$

In the new variables, instead of Eqs. (1.49) and (1.50), we obtain

$$\begin{aligned} \sqrt{\pi} G_\tau &= v_r \left[ \frac{2}{3} J_2(y_2) - \frac{1}{3} J_0(y_2) \right] + \tau_r \left[ \frac{2}{3} J_4(y_2) - \frac{2}{3} J_2(y_2) + \frac{5}{6} J_0(y_2) \right] \\ &\quad + \int_0^\infty G_v \left[ \frac{2}{3} J_1(|y_2 - s|) - \frac{1}{3} J_{-1}(|y_2 - s|) \right] ds \\ &\quad + \int_0^\infty G_\tau \left[ \frac{2}{3} J_3(|y_2 - s|) - \frac{2}{3} J_1(|y_2 - s|) + \frac{5}{6} J_{-1}(|y_2 - s|) \right] ds \\ &\quad + \frac{\partial \ln T(x, 0)}{\partial y_2} \left[ \frac{2}{3} J_5(y_2) - \frac{4}{3} J_3(y_2) + \frac{7}{6} J_1(y_2) \right], \end{aligned} \quad (1.53)$$

$$\begin{aligned} \sqrt{\pi} G_v &= v_r J_0(y_2) + \tau_r \left[ J_2(y_2) - \frac{1}{2} J_0(y_2) \right] \\ &\quad + \int_0^\infty G_v J_{-1}(|y_2 - s|) ds + \int_0^\infty G_\tau \left[ J_1(|y_2 - s|) - \frac{1}{2} J_{-1}(|y_2 - s|) \right] ds \\ &\quad + \frac{\partial \ln T(x, 0)}{\partial y_2} \left[ J_3(y_2) - \frac{3}{2} J_1(y_2) \right]. \end{aligned} \quad (1.54)$$

By definition, the temperature jump is equal to

$$\Delta T = T(x, 0) - T_w(x) = -T(x, 0) \tau_w, \quad (T_w = T(x, 0)(1 + \tau_w)). \quad (1.55)$$

To determine  $\tau_r$ ,  $\tau_w$ , and  $v_r$ , we use the no-flow condition (1.4), and the condition that the energy flux does not change across the layer

$$v_r + \frac{1}{2} \tau_r = -2 \int_0^\infty v J_0(s) ds - 2 \int_0^\infty \tau \left[ J_2(s) - \frac{1}{2} J_0(s) \right] ds, \quad (1.56)$$

$$E_r - E_w = \frac{k n(x, 0) T(x, 0)}{\sqrt{\pi h(x, 0)}} (\tau_r - \tau_w) = \frac{1 - \alpha_e}{\alpha_e} (E_i - E_r), \quad (1.57)$$

$$E_i - E_r = \lambda \frac{\partial T(x, 0)}{\partial y}, \quad (1.58)$$

where  $\lambda$  is the thermal conductivity. From relation (1.57) we have, for the temperature jump,

$$\Delta T = -T(x, 0) \tau_w = \frac{\sqrt{\pi h(x, 0)}}{k n(x, 0)} \frac{1 - \alpha_e}{\alpha_e} \lambda \frac{\partial T(x, 0)}{\partial y} - T(x, 0) \tau_r. \quad (1.59)$$

It is clear from the form of Eqs. (1.53) and (1.54) and conditions (1.56)-(1.58) that

$$T(x, 0) \tau_r = C \frac{\partial T(x, 0)}{\partial y_2} = \epsilon \frac{C}{n(x, 0) \sqrt{h(x, 0)}} \frac{\partial T(x, 0)}{\partial y},$$

where  $C$  is a constant obtained from solution of the integral equations (1.53) and (1.54). Expressing the quantity  $A$  in the definition of  $\epsilon$  in terms of  $\lambda$  according to the formula (see §3.6)

$$A = \frac{5}{2} \frac{k^2 T(x, 0)}{\lambda m}, \quad \left( \epsilon = \frac{U}{A n_0 L} \right),$$

we obtain

$$T(x, 0) \tau_r = C_1 \lambda \frac{\sqrt{\pi h(x, 0)}}{k n(x, 0)} \frac{\partial T(x, 0)}{\partial y},$$

where  $C_1$  is a new constant. Therefore, the expression for the temperature jump may be rewritten in the form

$$\Delta T = \frac{\sqrt{\pi h(x, 0)}}{2 k n(x, 0)} \frac{2 - \alpha \alpha_e}{\alpha_e} \lambda \frac{\partial T(x, 0)}{\partial y}, \quad (1.60)$$

where the constant  $\alpha$  is obviously related to the coefficient  $C_1$ . In Welander's paper the solution of the integral equations (1.53) and (1.54), expressed in somewhat different variables, yielded the value  $\alpha = 0.827$ .

We note that formula (1.60) is unsuitable when  $\alpha_e \rightarrow 0$ , since then the estimates made in formulating the problem of the Knudsen

layer at the beginning of the paragraph lose their validity [see the analysis of relations (1.6)].

4. In solving the problem of flow over a body at small Knudsen numbers within the framework of the Navier-Stokes equations, the slip conditions (1.46) and temperature jump condition (1.60) must be applied at the wall. We note again that the temperature  $T(x, 0)$  and the velocity  $u_x(x, 0)$  are conditions at the wall for the Navier-Stokes equations, and are not equal to the actual temperature  $T_1(x, 0)$  and the velocity  $u_{x1}(x, 0)$  of the gas at the wall which bounds the Knudsen layer.

The quantities  $\rho$ ,  $u_i$ , and  $T$ , obtained by means of the Navier-Stokes equation with the slip boundary conditions at the wall [ $u_{yw} = 0$ ,  $u_{xw} = u_x(x, 0)$ ,  $T_w = T(x, 0)$ ], differ from the corresponding quantities obtained by means of the Boltzmann equation with the kinetic boundary condition (1.3) by quantities of order  $\epsilon^2$  outside the layer, and by quantities of order  $\epsilon$  inside the Knudsen layer.

It is sometimes felt that the flow around a body at small Knudsen numbers may be represented in the form of layers: an external layer described by the Euler equations, under it a layer described by the Navier-Stokes equations, beyond that a layer described by the Burnett equations, and so on. In fact, if we examine the flow in the Burnett approximation, it may easily be seen that the Knudsen layer has a thickness of order  $\epsilon \ln \epsilon$ , i.e., greater than in the Navier-Stokes approximation, since, naturally, the influence of the wall is of the order of the Burnett terms at large distances from the wall.

We can establish the boundary conditions for the Burnett equations as we did for the Navier-Stokes equations. But we must retain three terms in the expansions (1.10) and (1.11).

### §5.2. The Boundary Layer with Slip and Temperature Jump

1. In the preceding section we derived the conditions (slip conditions), which must be applied on solid boundaries in solving the Navier-Stokes equations. In hydrodynamics it is usual to assume the so-called no-slip conditions, i.e., to consider that the gas velocity and temperature are equal to the velocity and tem-

perature of the wall. We examine the changes caused in boundary layer theory by the slip and temperature jump.

It was shown in the preceding section that the hydrodynamic quantities  $\rho$ ,  $u_i$ , and  $T$  outside the boundary layer, obtained by means of the Navier-Stokes equations with the slip boundary conditions (1.46) and (1.60), differ from the corresponding Boltzmann (exact) values by quantities of order  $\epsilon^2$ . We recall that when a Prandtl boundary layer is the outer flow on a Knudsen layer, the characteristic dimension  $\epsilon$  is the thickness of the boundary layer  $\delta$ , which is proportional to the square root of the mean free path  $\lambda$ . Therefore,  $\epsilon = \lambda / \delta$  is also of order  $\sqrt{\lambda}$ , i.e., the Enskog-Chapman expansion behaves as  $\sqrt{\lambda}$ .

When investigating a Prandtl boundary layer we are interested, in the final analysis, in the forces acting on the wall and in the energy transmitted to the wall.

The momentum and energy transmitted to the wall are equal to

$$\left. \begin{aligned} m \int \xi_y^2 f d\xi &= m \int c_y^2 f d\xi = P_{yy}, \\ m \int \xi_y \xi_x f d\xi &= m \int c_y c_x f d\xi = P_{xy}, \\ m \int \xi_y \xi_x^2 f d\xi &= \frac{m}{2} \int c_y c_x^2 f d\xi + mu_x \int c_y c_x f d\xi = q_y + u_x P_{xy}. \end{aligned} \right\} (2.1)$$

At the wall those quantities may be found directly from a solution of the Navier-Stokes equations with slip boundary conditions. On the other hand, the Navier-Stokes equations determine them outside the Knudsen layer (where the solution differs from the exact solution by quantities of order  $\epsilon^2$ ), and the solution may be continued inside the Knudsen layer (where the hydrodynamic quantities  $\rho$ ,  $u$ , and  $T$  are found with an error of order  $\epsilon$  by means of the Navier-Stokes equations) by means of the Boltzmann equation. By comparing the results of those two methods, we estimate the errors which arise in computing the friction and heat transfer at the wall directly from the boundary layer equations with slip conditions.

We examine the general conservation equations (see §3.1), valid for any Knudsen number and, therefore, valid also within the Knudsen layer to any approximation:

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho} \left( \frac{\partial P_{xy}}{\partial y} + \frac{\partial P_{xx}}{\partial x} \right), \quad (2.2)$$

$$u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} = -\frac{1}{\rho} \left( \frac{\partial P_{yy}}{\partial y} + \frac{\partial P_{xy}}{\partial x} \right), \quad (2.3)$$

$$\begin{aligned} & \frac{3}{2} R\rho \left( u_x \frac{\partial T}{\partial x} + u_y \frac{\partial T}{\partial y} \right) \\ & = -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - P_{yy} \frac{\partial u_y}{\partial y} - P_{xy} \frac{\partial u_y}{\partial x} - P_{xy} \frac{\partial u_x}{\partial y} - P_{xx} \frac{\partial u_x}{\partial x}. \end{aligned} \quad (2.4)$$

Within the Knudsen layer  $u_x \sim O(\sqrt{\lambda})$  and  $\partial u_x / \partial x \sim O(\sqrt{\lambda})$ . Then, since the thickness of the Knudsen layer is  $O(\lambda)$ ,\* it follows from the continuity equation that

$$\frac{\partial u_y}{\partial y} \sim \frac{\partial u_x}{\partial x} \sim O(\lambda^{1/2}), \quad u_y \sim O(\lambda^{3/2}).$$

Furthermore,

$$\begin{aligned} \frac{\partial u_x}{\partial y} & \sim O\left(\frac{1}{\sqrt{\lambda}}\right), \quad \frac{\partial T}{\partial y} \sim O\left(\frac{1}{\sqrt{\lambda}}\right), \quad \frac{\partial T}{\partial x} \sim O(1). \\ P_{yy} & = O(1), \quad P_{xx} = O(1), \quad P_{xy} \sim O(\sqrt{\lambda}), \\ q_y & \sim O(\sqrt{\lambda}), \quad q_x \sim O(\lambda). \end{aligned}$$

Taking these estimates into account, we have, from (2.2)-(2.4):

$$\frac{\partial P_{xy}}{\partial y} = \frac{\partial P_{xx}}{\partial x} + O(\lambda), \quad (2.2a)$$

$$\frac{\partial P_{yy}}{\partial y} = O(\sqrt{\lambda}). \quad (2.3a)$$

$$\frac{\partial (q_y + u_x P_{xy})}{\partial y} = u_x \frac{\partial P_{xy}}{\partial y} + O(\sqrt{\lambda}) = -u_x \frac{\partial P_{xx}}{\partial x} + O(\sqrt{\lambda}) = O(\sqrt{\lambda}). \quad (2.4a)$$

It follows from relations (2.3a) and (2.4a) that  $P_{yy}$  and  $q_y + u_x P_{xy}$  change across the Knudsen layer by an amount of order  $\lambda^{3/2}$ . The magnitude of  $P_{xx}$  inside the Knudsen layer, obtained by means

\*It was shown in the preceding section that the thickness of the Knudsen layer is of order  $\delta_k \sim \epsilon L$ . But, since all the quantities have been referred to the characteristic flow length, which in this case  $L \sim \delta \sim \sqrt{\lambda}$  and  $\epsilon \sim \sqrt{\lambda}$ , the thickness of the Knudsen layer is of order  $\lambda$ .

of the Navier-Stokes equations, differs from the corresponding exact value by a quantity of order  $\sqrt{\lambda}$ . Therefore, according to (2.2a), the value of the stress  $P_{xy}$  at the wall, obtained by means of the Navier-Stokes equations, differs from the corresponding Boltzmann value by a quantity of order  $\lambda^{3/2}$ , if the values of  $P_{xy}$  are the same at the boundary of the Knudsen layer. In an exactly similar manner,  $P_{yy}$  and  $q_y + u_x P_{xy}$  are determined by means of the Navier-Stokes equations with an error not exceeding  $\lambda^{3/2}$ , if they have been found up to that accuracy outside the Knudsen layer.

Outside the Knudsen layer the Navier-Stokes equations and the estimates of the Prandtl boundary layer are valid. If we consider the Burnett representation for the stress tensor and the heat-flux vector to be valid, then

$$P_{xy} = \mu \frac{\partial u_x}{\partial y} + O(\lambda^{3/2}), \quad q_y = \lambda_T \frac{\partial T}{\partial y} + O(\lambda^{3/2}), \quad (2.5)$$

where  $O(\lambda^{3/2})$  is an estimate of the Burnett term in the Prandtl boundary layer (see § 3.3).\*

Thus, by finding  $P_{xy}$ ,  $q_y$ , and  $q_y + u_x P_{xy}$  at the wall by means of the Navier-Stokes equations, with the slip conditions, we make errors of order  $\lambda^{3/2}$ , i.e., errors of the same order as the omission of the Burnett terms in the hydrodynamic equations.

The pressure  $p \sim O(1)$  is found outside the layer with an error  $\epsilon^2 \sim \lambda$ . According to (2.3a), it is found also at the wall with the same accuracy. Some papers introduce a wall pressure jump of order  $\sqrt{\lambda}$  which is a consequence of our incorrect flow scheme, as we pointed out at the beginning of the preceding section.

The slip velocity and the temperature jump are of order  $\sqrt{\lambda} \sim Re_\infty^{-1/2}$ .† Therefore, by replacing the no-slip conditions by

\*The quantities  $u_x$  and  $T$  are found outside the layer up to an error  $\epsilon^2 \sim \lambda$ . Therefore, the error in  $\partial u_x / \partial y$  and  $\partial T / \partial y$  is of order  $\sqrt{\lambda}$ . But  $\mu \sim \lambda$  and, therefore, the errors in  $P_{xy}$  and  $q_y$  are of order  $\lambda^{3/2}$ .

†The Reynolds number, as is usual in boundary layer theory, is determined from the parameters of the oncoming stream and a characteristic length  $L_\infty$  of the flow outside the Prandtl boundary layer (the length of the plate).

the slip conditions, we obtain corrections of order  $\sqrt{\lambda}$  to the velocity  $u_x$  and to the temperature  $T$  which, according to (2.5), must lead to corrections of order  $\lambda$  in  $P_{xy}$  and  $q_y$ . Since the Navier-Stokes theory permits us to obtain those quantities within an error of order  $\lambda^{3/2}$ , we may calculate the conditions of slip and the corrections to  $P_{xy}$  and  $q_y$  due to them within the framework of this theory.

Note that the quantity  $u_x P_{xy}$  is also of order  $\lambda$  and, according to (2.1), it must therefore be included in the expressions for the energy flux at the wall within the framework of boundary layer theory with slip.

In deriving the Prandtl boundary layer equations on a flat plate in the Navier-Stokes equations, we neglect quantities of order  $Re_\infty^{-1} \sim \lambda$  in comparison with unity. Since the calculation of slip leads to corrections of order  $\sqrt{\lambda}$ , we may use the ordinary boundary layer equations when calculating the boundary layer on a flat plate allowing for slip.

In deriving the ordinary boundary layer equations for a curved surface we neglect both quantities of order  $Re_\infty^{-1}$  and quantities of order  $Re_\infty^{-1/2} K$ , where  $K$  is the curvature of the surface. But for a finite curvature, the latter are of order  $\sqrt{\lambda}$  and, therefore, give the same contribution as does the slip. Therefore, when allowing for slip in the boundary layer equations on a curved surface, it is necessary to retain terms of order  $Re_\infty^{-1/2}$  which account for the longitudinal and transverse curvature of the wall.

In exactly the same way, we must take account of other second-order effects in Prandtl boundary layer theory, when their contributions are of the same order as the slip. Since the boundary layer thickness is of order  $\sqrt{\lambda}$ , the displacement effect of the boundary layer leads to variations of the same order in the external flow and, therefore, to corresponding variations in the boundary conditions at the outer edge of the boundary layer. In hypersonic flow over bluff or blunt bodies, second-order effects also arise, due to the steep gradient of enthalpy and entropy (vortex interaction) at the edge of the boundary layer. However, the scope of this monograph does not allow us to dwell on all the second-order effects.\* Following the general plan of this book, among all the

\*The interested reader is referred to the papers: W. D. Hayes and R. F. Probstein,

second-order effects, we examine the slip and temperature jump, which are most closely connected with kinetic theory.

Among second-order effects we should also include phenomena which occur near the leading edge of a flat plate (at distances of order  $\lambda$  from the nose), which cannot be correctly accounted for within the framework of Navier-Stokes theory, and which require solutions of the Boltzmann equation. We show at the end of this section that the contribution to the drag and to the heat flux from that small region is of the same order as from the slip.

The interaction of the boundary layer with the external flow depends appreciably on the Mach number. The thickness of the boundary layer is proportional to a power of the Mach number, which depends on the law of variation of viscosity with temperature. The nature of the interaction of the boundary layer with the external flow depends on the body shape. For blunt bodies, in the range of applicability of the boundary layer equations, there is only a weak interaction at any Mach number. On slender bodies, if the Mach number is increased at a fixed Reynolds number, the interaction becomes strong, exhibits an appreciably nonlinear character, and a separate examination of the various second-order effects is impossible. In that case, the interaction becomes a first-order effect, and the effect of slip, especially on cooled bodies, is much smaller than the effect of the interaction.\* If at a fixed Mach number the Reynolds number tends to infinity, or, equivalently, the Knudsen number tends to zero, as was the case when we derived the slip conditions, then we may always attain conditions for which the interaction of the boundary layer with the external flow is weak, the problem may be linearized, and each of the above-mentioned second-order effects may be studied separately. For that particular asymptotic case, we examine the influence of slip and of temperature jump, without taking into account the interaction of the boundary layer with the external flow, the body curvature, or other secondary effects.

"Hypersonic Flow Theory," 2nd ed., Academic Press, New York, 1966; M. Van Dyke, "Rarefied Gas Dynamics," Third Symposium, Academic Press, New York, 1963; L. G. Loitsyanskii, "The Laminar Boundary Layer," Fizmatgiz, 1962; M. Van Dyke, "Perturbation Method in Fluid Mechanics," Academic Press, New York, 1964.

\*See, for example, V. S. Galkin, Inzh. Zh., Vol. 3, No. 1 (1963).

Thus, we examine the problem of the flow of a gas in a boundary layer, which differs from the classical case only in that the no-slip conditions at the wall are replaced by the slip conditions (1.46) and (1.60):

$$u_x(x, 0) = 1.012 \frac{\mu_w}{\rho_w(x)} \sqrt{\frac{2m}{kT_w}} \frac{\partial u_x(x, 0)}{\partial y} = \frac{\beta(x)}{\rho_w} \frac{\partial u_x(x, 0)}{\partial y}, \quad (2.6)$$

$$T(x, 0) = T_w(x) + \frac{2 - a\alpha_e}{2a_e} \sqrt{\frac{\pi m}{2kT_w}} \frac{\lambda_{Tw} m}{k\rho_w} \frac{\partial T(x, 0)}{\partial y}, \quad (2.7)$$

where  $\mu_w$  and  $\lambda_{Tw}$  are the viscosity and the thermal conductivity, evaluated at the wall temperature  $T_w$ . In that formulation, the problem has been examined in a number of papers.\*

Condition (2.7) may be rewritten in the form

$$T(x, 0) = T_w(x) + \frac{C\beta}{\rho_w} \frac{\partial T(x, 0)}{\partial y}, \quad (2.7a)$$

where

$$C = \frac{3}{8} \frac{2 - a\alpha_e}{\alpha_e} \frac{\sqrt{\pi}}{1.012} \frac{x}{Pr}, \quad \left( x = \frac{c_p}{c_v} \right).$$

The slip conditions may be interpreted geometrically by stating that the velocity  $u_x$  vanishes, and the gas temperature becomes equal to the wall temperature not at the actual wall where  $y = 0$ , but at effective boundaries located, respectively, at

$$y_{eff} = -\frac{\beta(x)}{\rho_w(x)} \text{ and } y_{eff} = -C \frac{\beta(x)}{\rho_w(x)}.$$

Since, in general,  $C \neq 1$ , while  $\beta$  and  $\rho$  depend on  $x$ , the effective walls for the temperature and the velocity are different and are curved, i.e.,  $y_{eff} = y(x)$ . However, for the qualitative examination which interests us here, we may put  $C = 1$ . With a Prandtl number  $Pr = 2/3$ ,  $\alpha_e = 1$ ,  $a = 0.827$ , for a monatomic gas  $C = 1.9$ , i.e., in fact, the displacement of the "thermal" effective wall is al-

\*S. H. Maslen, J. Aeron. Sci., No. 6; V. P. Shidlovskii, Izv. Akad. Nauk SSSR, Otd. Tekhn. Nauk, No. 9 (1958); A. I. Bumimovich, Izv. Akad. Nauk SSSR, Otd. Tekhn. Nauk, Mekhan. i Mashinostr., No. 5 (1959); V. P. Myasnikov, Izv. Akad. Nauk SSSR, Otd. Tekhn. Nauk, Mekhan. i Mashinostr., No. 5 (1959); V. S. Galkin and M. D. Ladyzhenskii, Doklady Akad. Nauk SSSR, Vol. 154, No. 6 (1964).

most twice as large as the displacement of the "velocity" wall. When  $C = 1$ , both effective walls coincide. The "waviness" of the effective wall (i.e., the dependence of  $y_{eff}$  on  $x$ ) is due to the variation with  $x$  of the density  $\rho_w$  and the ratio  $\mu_w/\sqrt{T_w}$ . If  $\mu \sim \sqrt{T}$ , as is the case for hard-sphere molecules,  $\mu/\sqrt{T}$  and, therefore,  $\beta$ , do not depend on  $x$ . The quantity  $\beta$  is also constant for an isothermal wall. The waviness of the effective wall due to variation in density is easily avoided by going to Dorodnitsyn variables:\*

$$\xi = \int_0^x p(x) dx, \quad \eta = \int_0^y \rho(x, y) dy. \quad (2.8)$$

In those variables the slip conditions with  $C = 1$  take the form

$$u_x(x, 0) = \beta \frac{\partial u_x(x, 0)}{\partial \eta}, \quad T(x, 0) = T_w + \beta \frac{\partial T(x, 0)}{\partial \eta}, \quad u_y = 0. \quad (2.9)$$

With  $\beta = \text{const}$  in those variables, the displacement of the effective wall is constant:  $\eta = -\beta$ .

As we know, the boundary layer equations have the form (see, for example, Loitsyanskii's monograph)†

$$\frac{\partial \rho u_x}{\partial x} + \frac{\partial \rho u_y}{\partial y} = 0, \quad (2.10)$$

$$\rho \left( u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u_x}{\partial y} \right), \quad (2.11)$$

$$\rho u_x \frac{\partial}{\partial x} \left( c_p T + \frac{u_x^2}{2} \right) + \rho u_y \frac{\partial}{\partial y} \left( c_p T + \frac{u_x^2}{2} \right) = \frac{\partial}{\partial y} \left( \lambda_T \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial y} \left( \mu u_x \frac{\partial u_x}{\partial y} \right) \quad (2.12)$$

or, in Dorodnitsyn variables (2.8),

$$p \frac{\partial \rho u_x}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial \rho u_x}{\partial \eta} + \rho \frac{\partial \rho u_y}{\partial \eta} = 0, \quad (2.10a)$$

$$\rho \left( u_x p \frac{\partial u_x}{\partial \xi} + u_x \frac{\partial \eta}{\partial x} \frac{\partial u_x}{\partial \eta} + u_y \rho \frac{\partial u_x}{\partial \eta} \right) = -\frac{\partial p}{\partial x} + \rho \frac{\partial}{\partial \eta} \left( \rho \mu \frac{\partial u_x}{\partial \eta} \right) \quad (2.11a)$$

\*A. A. Dorodnitsyn, Prikl. Mat. i Mekhan., Vol. 6, No. 6 (1942).

†L. G. Loitsyanskii, "The Laminar Boundary Layer," Fizmatgiz, 1962.

$$\begin{aligned} u_x p \frac{\partial}{\partial \xi} \left( c_p T + \frac{u_x^2}{2} \right) + u_x \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} \left( c_p T + \frac{u_x^2}{2} \right) + \rho u_y \frac{\partial}{\partial \eta} \left( c_p T + \frac{u_x^2}{2} \right) \\ = \frac{\partial}{\partial \eta} \left( \lambda_{\tau 0} \frac{\partial T}{\partial \eta} \right) + \frac{\partial}{\partial \eta} \left( u_x \mu_0 \frac{\partial u_x}{\partial \eta} \right). \end{aligned} \quad (2.12a)$$

If  $u_x^*(\xi, \eta)$ ,  $u_y^*(\xi, \eta)$ , and  $T^*(\xi, \eta)$  constitute a solution of the boundary layer equations with no-slip boundary conditions  $u_x^*(\xi, 0) = 0$ ,  $u_y^*(\xi, 0) = 0$ , and  $T^*(\xi, 0) = T_w(\xi)$ , it is evident that the functions  $u_x^*(\xi, \eta + \beta)$ ,  $u_y^*(\xi, \eta + \beta)$ , and  $T^*(\xi, \eta + \beta)$  also form a solution of Eqs. (2.10a)-(2.12a), satisfying the no-slip conditions at  $\eta = -\beta$ . Since we are interested in a linear perturbation of the basic solution due to a slip of order  $\beta \sim \epsilon \sim \sqrt{\lambda}$ , the solution above may be expanded in a series in  $\beta$ , of which we retain only the first terms:

$$\left. \begin{aligned} u_x(\xi, \eta) &= u_x^*(\xi, \eta) + \frac{\partial u_x^*(\xi, \eta)}{\partial \eta} \beta, & u_y &= u_y^*(\xi, \eta) + \frac{\partial u_y^*(\xi, \eta)}{\partial \eta} \beta, \\ T &= T^*(\xi, \eta) + \frac{\partial T^*(\xi, \eta)}{\partial \eta} \beta. \end{aligned} \right\} \quad (2.13)$$

When  $\eta = 0$ , by definition,  $u_x^* = u_y^* = 0$  and  $T^* = T_w$ . It follows from the continuity equation (2.10) that  $\partial u_y^*/\partial y = 0$ . Therefore, the solution (2.13) satisfies the slip conditions (2.9) when  $\eta = 0$ , and the conditions satisfied by  $u_x^*$ ,  $u_y^*$ , and  $T^*$  at infinity.

Thus, with  $C = 1$  and  $\beta = \text{const}$ , the solution of the boundary layer equations with slip conditions may be expressed directly by formulas (2.13) in terms of the solution of the same equations without slip. This easily permits us to determine the effect of slip on friction and heat transfer.

The friction when  $\eta = 0$  is equal to

$$\begin{aligned} P_{xy} &= \mu(T) \frac{\partial u_x(x, 0)}{\partial y} = \mu(T) \rho(T) \frac{\partial u_x(x, 0)}{\partial \eta} \\ &= \mu(T) \rho(T) \left( \frac{\partial u_x^*(x, 0)}{\partial \eta} + \beta \frac{\partial^2 u_x^*(x, 0)}{\partial \eta^2} \right) \\ &= \mu^* \rho^* \frac{\partial u_x^*(x, 0)}{\partial \eta} + \beta \left( \mu^* \mu^* \frac{\partial^2 u_x^*(x, 0)}{\partial \eta^2} + \frac{\partial \rho^* \mu^*}{\partial T^*} \frac{\partial T^*}{\partial \eta} \frac{\partial u_x^*(x, 0)}{\partial \eta} \right). \end{aligned} \quad (2.14)$$

The notation here is  $\mu^* = \mu(T^*)$  and  $\rho^* = \rho(\xi, T^*)$ . It follows from the equation of motion (2.11a) that the brackets are equal to  $\partial p/\partial x$ , i.e.,

$$P_{xy} = \mu^* \rho^* \frac{\partial u_x^*(\xi, 0)}{\partial \eta} + \beta \frac{\partial p}{\partial x} = \mu(T_w) \frac{\partial u_x^*(x, 0)}{\partial y} + \beta \frac{\partial p}{\partial x}. \quad (2.15)$$

Therefore, if  $\partial p/\partial x = 0$ , the presence of slip does not alter the friction. With a positive pressure gradient the friction increases; it decreases for a negative pressure gradient. We note that this result is exact when  $\beta = \text{const}$  and the surface is thermally insulated (i.e., if  $\partial T/\partial y = 0$  when  $y = 0$ ) since, in that case, the quantity  $C$  drops out of the picture.

As noted above, the energy flux at the wall is equal to

$$\begin{aligned} q_y &= \lambda_{\tau} \frac{\partial T}{\partial y} + u_x P_{xy} = \lambda_{\tau 0} \frac{\partial T}{\partial \eta} + u_x \mu(T) \rho \frac{\partial u_x}{\partial \eta} \\ &= \lambda_{\tau}^* \rho^* \frac{\partial T^*}{\partial \eta} + \beta \left[ \lambda_{\tau}^* \rho^* \frac{\partial^2 T^*}{\partial \eta^2} + \frac{\partial \lambda_{\tau}^* \rho^*}{\partial T^*} \left( \frac{\partial T^*}{\partial \eta} \right)^2 + \mu^* \rho^* \left( \frac{\partial u_x^*}{\partial \eta} \right)^2 \right] \end{aligned} \quad (2.16)$$

in flow with slip. But it follows from the energy equation (2.12a) that, when  $\eta = 0$ , the square brackets vanish and

$$q_y = \lambda_{\tau}^* \rho^* \frac{\partial T^*}{\partial \eta} = \lambda_{\tau}(T_w) \frac{\partial T^*}{\partial y}, \quad (2.17)$$

i.e., the presence of slip and a temperature jump has no influence on the heat transfer.

That result was obtained by making the important assumption that  $C = 1$ . It has also been shown, in Maslen's paper, that this result is valid for a flat plate with  $C \neq 1$ .

A number of methods have been proposed for calculating friction and heat transfer on a flat plate at any Knudsen number.\* However, those methods are peculiarly approximate and interpolative. According to them, the coefficients of friction and heat transfer increase monotonically with increasing Knudsen number (decreasing Reynolds number), and reach free-molecule values when

\*S. A. Schaaf, Heat Transfer Symposium, University of Michigan, 1959; S. A. Schaaf and F. S. Sherman, J. Aeron. Sci., 21:85 (1959); G. N. Patterson, Molecular Flow of Gases, Wiley, New York, 1956; V. A. Sukhnev, Izv. Akad. Nauk SSSR, Mekhan. i Mashinostr., No. 3 (1962).

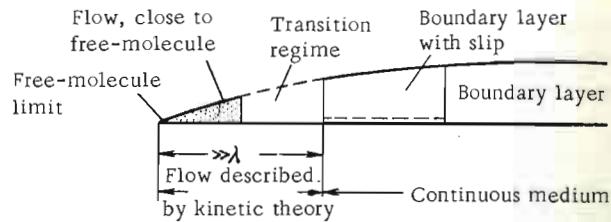


Fig. 46

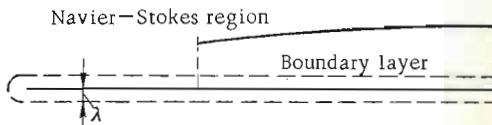


Fig. 47

$\text{Kn} = \infty$ . In fact, at large Mach numbers, as the Knudsen number decreases, the drag and the heat transfer first increase above their free-molecule values, reach a maximum, and then decrease to values corresponding to a continuous medium.\*

2. Of course, the boundary layer equations are not applicable near the leading edge. A picture of the flow over a flat plate at large Reynolds numbers is often given as follows (Fig. 46).† In the region where the Reynolds number based on distance from the leading edge  $\text{Re}_x \gg 1$ , the boundary layer equations with slip conditions are valid up to an accuracy  $\text{Re}_x^{-1/2}$ . Near the leading edge, where  $\text{Re}_x \ll 1$ , free-molecule flow or near free-molecule flow occurs. Between those two limiting regions, there is some kind of intermediate flow regime.

\*M. N. Kogan, Prikl. Mat. i Mekhan., Vol. 26, No. 3 (1962); see also § 6.6.

†See, for example, A. F. Charwat, "Rarefied Gas Dynamics," Second Symposium, Academic Press, New York, 1961; R. L. Chuan and S. A. Waiter, "Rarefied Gas Dynamics," Third Symposium, Academic Press, New York, 1963; R. J. Vidal and C. E. Wittliff, "Rarefied Gas Dynamics," Third Symposium, Academic Press, New York, 1963; S. Ziering, L. Chi, and R. Faute, "Rarefied Gas Dynamics," Fourth Symposium, Academic Press, New York, 1965; J. M. Li and R. E. Street, "Rarefied Gas Dynamics," Fourth Symposium, Academic Press, New York, 1965.

It is easy to see, however, that this picture is not correct. The small Knudsen number flow pattern around any body, including a flat plate, may be represented as consisting of two parts (Fig. 47): a flow region described by the Navier-Stokes equations, and a discrete number of regions whose width is of the order of the mean free path  $\lambda$ , in which the flow can only be described by the Boltzmann equation.\*

Indeed, let  $L$  be the characteristic flow dimension. Then, in formula (1.2) of the previous section,  $\epsilon = \lambda/L = \text{Kn}$  is the flow Knudsen number, where  $\lambda$  is the characteristic mean free path. Generally speaking, the mean free path  $\lambda \sim (n\sigma)^{-1}$  may be different at different points of the flow. The variation of the mean free path depends on the variations of density and, if the collision cross section depends on the relative velocity of the molecules, on the temperature variation. The variation of  $\rho$  and  $T$  depend in turn on the Mach number  $M$  of the oncoming stream and on the ratio of the specific heats,  $\kappa$ . However, for fixed  $M$  and  $\kappa$  we may consider the mean free path to be of the same order throughout the flow.

We examine the flow pattern around a body as  $\epsilon \rightarrow 0$ . Let the point  $\mathbf{x}_1$  in formula (1.2) lie in a region in which the Navier-Stokes equations are not valid. Let us estimate the order of the terms in that equation for some point  $\mathbf{x}$ . If the point  $\mathbf{x}$  is several mean free paths distant from the point  $\mathbf{x}_1$ , it is evident that the term containing the square bracket is negligibly small in comparison with the first two terms corresponding to the Navier-Stokes approximation.

Let  $(d^2 f_0 / dl^2)_m$  be the mean value of the second derivative of  $f_0$  in the interval  $\Delta l$  between the point  $\mathbf{x}_1$  and some point  $\mathbf{x}$ . Then the remaining term (1.2) is of order  $\epsilon^2 (d^2 f / dl^2)_m$ . We assume that, in that interval, the remaining term is of the same order or larger than the Navier-Stokes term, i.e.,

$$\epsilon^2 \left( \frac{d^2 f_0}{dl^2} \right)_m \geq \epsilon \frac{df_0}{dl}.$$

\*M. N. Kogan, paper at the Seventh Conference on Problems and Methods of Hydrodynamics, Jurata, Poland, 1965.

But

$$\frac{df_0}{dl} \sim \Delta l \left( \frac{d^2 f_0}{dl^2} \right)_m.$$

Therefore,  $\varepsilon \geq \Delta l$ . Thus, the remaining term may be of the same order as the Navier-Stokes term only in regions of the order of, or smaller than, the mean free path. Outside those narrow zones, the flow may be described by the Navier-Stokes equations.

It is evident that the density discontinuities and the Knudsen layer at the wall are among such narrow zones. Since the Reynolds number  $\text{Kn} \rightarrow 0$  when  $\text{Re} \rightarrow \infty$  for a fixed Mach number, a Prandtl boundary layer is formed inside the Navier-Stokes region. Then, on the plate, at distances larger than  $\lambda$  from the leading edge, the slip conditions are valid. The fact that in the boundary layer and in the outer flow the derivatives are of different orders, and that, consequently, the expansion in fact is in powers of  $\sqrt{\lambda}$  and not in powers of  $\lambda$  in the boundary layer, clearly does not alter the validity of the statements made above.

The flow structure near the leading edge of a plate at hypersonic stream velocities within the framework of the Navier-Stokes approximation has been examined in a whole series of papers.\* An experimental investigation of those flows has been presented in the papers of Bogdonoff, Nagamatsu, and others.†

It is clear from this picture that the flow along a length of order  $\lambda$  from the leading edge cannot be calculated within the framework of Navier-Stokes theory. On the other hand, we should not expect the friction and pressure near the leading edge of the plate to take the corresponding free-molecule values.‡

\*R. Probstein and Y. S. Pan, "Rarefied Gas Dynamics," Third Symposium, Academic Press, New York, 1963; H. Oguchi, "Rarefied Gas Dynamics," Third Symposium, Academic Press, New York, 1963; see also the papers by Charwat, Chuan and Waiter, Vidal and Wittliff, Ziering, Chi, and Faute; and Li and Street in "Rarefied Gas Dynamics," Fourth Symposium, Academic Press, New York, 1965.

†H. J. Nagamatsu, J. A. Weil, and R. E. Sheer, ARSJ, 32: 533 (1962); S. M. Bogdonoff, in a paper presented at the 7th Conference on Problems and Methods in Hydrodynamics, Jurata, Poland, 1965; W. J. McCroskey, S. H. Bogdonoff, and J. G. McDougall, AIAA, paper No. 66-31 (1966); see also the papers in "Rarefied Gas Dynamics," Fourth Symposium, mentioned directly above.

‡This has been confirmed by the experiments of Bogdonoff.

As will be shown in § 6.6, even in hypersonic nearly free molecule flow, the pressure and friction at the leading edge may be larger than in free-molecule flow.

The Knudsen layer near the wall is always present on any body. In the framework of the Navier-Stokes theory it may be taken into account by introducing the slip conditions (exceptions are the region near the leading edge of the plate noted above, corners, etc.). The slip can only be neglected in the Prandtl boundary layer theory, when not only terms of order  $\text{Re}^{-1}$ , but also terms of order  $\text{Re}^{-1/2}$  are neglected in the Navier-Stokes equations.

It follows from boundary layer theory that the friction coefficient  $C_x \sim 1/\sqrt{\text{Re}}$ . Allowance for the weak interaction of the boundary layer with the external flow, for slip, and other second-order effects, adds terms of order  $1/\text{Re}$ . We saw above that the flow over an element of the plate of length of order  $\lambda$  at the leading edge may be described only by means of the Boltzmann equation. On the other hand, the contribution of that element to the drag of the plate, both in boundary layer theory and in the free-molecule limit, is of order  $1/\text{Re}$ . We may therefore expect the same kind of result from the Boltzmann equation.

Therefore, properly speaking, the term of order  $1/\text{Re}$  cannot be obtained, either from the Navier-Stokes theory, or from the Burnett theory, or from the higher approximations. Boundary layer theory of second, third, etc., order takes into account terms of the same order as those which have been discarded. The only consistent theory within the framework of the Navier-Stokes equations is the Prandtl theory.\*

As we have already noted, at hypersonic velocities the effects of the strong interaction of the boundary layer with the external flow become first-order effects, and may therefore also be examined in closed form within the framework of the Navier-Stokes equations.

\*That is, boundary layer theory without allowing for second-order effects. The second approximation theory gives valid corrections only to local values of friction and of heat transfer for  $x \gg \lambda$  from the leading edge.

We note that the use of the Burnett equations and of higher approximations does not allow us to determine the structure or to penetrate in any way the above-mentioned thin regions of order  $\lambda$  (the Knudsen layer or the shock wave). If we try to calculate the whole flow within the framework of the Burnett approximation, then, as we indicated in the preceding section, the thickness of the layer of inapplicability of these equations does not become thinner than in the Navier-Stokes case, but thicker (of order  $\lambda \ln \lambda$ ). The Burnett equations permit us to obtain a more accurate description of the flow in a region where the Navier-Stokes equations are also applicable, but they do not permit us to progress into the region where the latter are inapplicable.

## Chapter VI

# FLOWS AT LARGE KNUDSEN NUMBERS

### §6.1. Free-Molecule Flows. Flow Past Convex Bodies

Free-molecule flow is defined as the flow obtained in the limit when the Knudsen number  $Kn \rightarrow \infty$ . In that case, the Boltzmann equation takes the form (see also §§2.11 and 4.2)

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial x} + \frac{\mathbf{X}}{m} \cdot \frac{\partial f}{\partial \xi} = 0. \quad (1.1)$$

In such flows, the interaction of the molecules with the wall plays a major role, while the collisions of the molecules among themselves may be neglected. A gas in which the molecules do not collide is called a Knudsen gas. For a given characteristic dimension of the flow, a Knudsen gas may be represented as a gas in which  $n\sigma \rightarrow 0$ , i.e., either the density or the diameter of the molecules tends to zero. Then the mean free path tends to infinity.

The general solution of Eq. (1.1) in the absence of external forces has the form

$$f(t, x, \xi) = f(t_0, x - \xi(t - t_0), \xi), \quad (1.2)$$

i.e., the distribution function is conserved along the straight-line trajectories of the particles. In a field of forces  $\mathbf{X}(x)$  the distribution function is also conserved along the trajectories of the particles:

$$f(t, x(t), \xi(t)) = f(t_0, x_0, \xi_0), \quad (1.3)$$

where the functions  $\mathbf{x}(t)$  and  $\xi(t)$  are solutions of Newton's equations

$$m \frac{d^2 \mathbf{x}}{dt^2} = \mathbf{X}, \quad (\xi = \frac{d\mathbf{x}(t)}{dt}) \quad (1.4)$$

with the initial conditions

$$\mathbf{x}(t_0) = \mathbf{x}_0 \text{ and } \xi(t_0) = \xi_0. \quad (1.5)$$

We examine the flow of an infinite stream of Knudsen gas over a convex body in the absence of external forces. Let  $f_\infty(t, \mathbf{x}, \xi)$  be the distribution function of the oncoming stream. Since we neglect molecular collisions, if we place a motionless convex body in the stream, we do not alter the distribution function of the molecules impinging on the body,\* i.e., the distribution function  $f_i$  of the molecules incident on the body is equal to

$$f_i(t, \mathbf{x}_w, \xi_i) = f_\infty(t, \mathbf{x}_w, \xi_i), \quad (1.6)$$

where  $\mathbf{x}_w$  is a point of the body surface.

We are usually interested in the total flux of molecules, and in the momentum and energy carried to the body by the molecules. Since molecules reflected from one part of a convex body do not strike the body elsewhere in the absence of external forces, each element of the surface may be considered isolated. If  $\mathbf{n}(\mathbf{x}_w)$  is the outward normal to the body, then

$$\left. \begin{aligned} N_i(t, \mathbf{x}_w) &= - \int_{\xi \cdot \mathbf{n} < 0} (\xi \cdot \mathbf{n}) f_\infty(t, \mathbf{x}_w, \xi) d\xi, \\ P_i(t, \mathbf{x}_w) &= - \int_{\xi \cdot \mathbf{n} < 0} m (\xi \cdot \mathbf{n}) f_\infty(t, \mathbf{x}_w, \xi) \xi d\xi, \\ E_i(t, \mathbf{x}_w) &= - \int_{\xi \cdot \mathbf{n} < 0} \frac{m \xi^2}{2} (\xi \cdot \mathbf{n}) f_\infty(t, \mathbf{x}_w, \xi) d\xi, \end{aligned} \right\} \quad (1.7)$$

where  $N_i$ ,  $P_i$ , and  $E_i$  are the number of molecules, the momentum, and the energy brought to unit surface area in unit time. To deter-

\*If the body changes its geometry with time, then even on a convex body molecules which have been reflected from the body at earlier times may be incident upon it. [See, for example, N. T. Pashchenko, Prikl. Mat. i Mekhan., No. 4 (1959).]

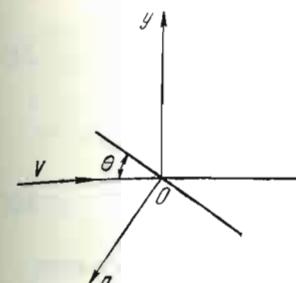


Fig. 48

mine the total momentum and energy transmitted to the surface element, we must also take account of the reactive momentum and the energy of the reflected molecules:

$$P_r(t, \mathbf{x}_w) = \int_{\xi \cdot \mathbf{n} > 0} (\xi \cdot \mathbf{n}) f_r(t, \mathbf{x}_w, \xi) \xi d\xi,$$

$$E_r(t, \mathbf{x}_w) = \int_{\xi \cdot \mathbf{n} > 0} \frac{m \xi^2}{2} (\xi \cdot \mathbf{n}) f_r(t, \mathbf{x}_w, \xi) d\xi. \quad (1.8)$$

The distribution function for the reflected molecules,  $f_r(t, \mathbf{x}_w, \xi)$  is determined by the law of interaction of the molecules with the surface (see § 2.10), and must satisfy the condition of impermeability

$$N_r(t, \mathbf{x}_w) = \int_{\xi \cdot \mathbf{n} > 0} (\xi \cdot \mathbf{n}) f_r(t, \mathbf{x}_w, \xi) d\xi = N_i(t, \mathbf{x}_w). \quad (1.9)$$

We examine, as an example, the steady flow over a body of a uniform equilibrium stream of velocity  $V$ , density  $n_\infty$ , and temperature  $T_\infty$ . The distribution function of that flow is equal to

$$f_\infty = n_\infty \left( \frac{h_\infty}{\pi} \right)^{3/2} \exp \{-h_\infty(\xi - V)^2\}, \quad h_\infty = \frac{m}{2kT_\infty}. \quad (1.10)$$

We examine (Fig. 48) an element of the body surface inclined at an angle of attack  $\theta$  to the stream (the vector  $\mathbf{V}$  forms an angle  $\pi/2 - \theta$  with the inward normal to the surface).

Substituting the distribution function (1.10) into expression (1.7), we have, after simple quadratures:

$$N_i = \frac{n_\infty}{2V\pi h_\infty} \chi(S_0), \quad (\chi(x) = e^{-x^2} + \sqrt{\pi}x(1 + \operatorname{erf} x)), \quad (1.11)$$

$$P_{ni} = \frac{\rho_\infty V^2}{2\sqrt{\pi}S^2} \left[ S_0 e^{-S_0^2} + \sqrt{\pi} \left( \frac{1}{2} + S_0^2 \right) (1 + \operatorname{erf} S_0) \right], \quad (1.12)$$

$$P_{it} = \frac{\rho_\infty V^2 \cos \theta}{2\sqrt{\pi}S} \chi(S_0), \quad (1.13)$$

$$E_i = \frac{n_\infty k T_\infty}{2 V \pi h_\infty} \left[ (S^2 + 2) e^{-S_\theta^2} + \sqrt{\pi} \left( S^2 + \frac{5}{2} \right) S_\theta (1 + \operatorname{erf} S_\theta) \right]. \quad (1.14)$$

Here,  $S$  is the so-called velocity ratio:

$$S = V \sqrt{h_\infty} = \frac{V}{v_{r\infty}} = \sqrt{\frac{\kappa}{2}} M_\infty, \quad S_\theta = S \sin \theta, \quad M_\infty = \frac{V}{a_\infty}, \quad (1.15)$$

where  $M_\infty$  is the Mach number  $v_{r\infty} = h_\infty^{-\frac{1}{2}}$  is the most probable velocity in the Maxwellian distribution,  $\kappa$  is the ratio of the specific heats, and

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\lambda^2} d\lambda. \quad (1.16)$$

In particular, when  $V = 0$ , i.e., for a body immersed in a motionless gas, we have

$$N_i = \frac{n_\infty}{2 V \pi h_\infty}, \quad P_{ni} = \frac{m N_i}{2} \sqrt{\frac{\pi}{h_\infty}}, \quad P_{ri} = 0, \quad E_i = \frac{m N_i}{h_\infty}. \quad (1.17)$$

If we are not interested in the flow field, and the problem consists only of determining the forces acting on the body and the energy transmitted to it, then in free-molecule flow there is no need to know the distribution function of the reflected particles. As we noted in §2.10, the momentum and energy transmitted to the surface are completely determined, if the accommodation coefficients are given [compare formulas (10.16) of §2.10]

$$\left. \begin{aligned} a_n(\xi, \vartheta, T_w) &= \frac{\bar{P}_{ni} + \bar{P}_{nr}}{P_{ni}}, & a_r(\xi, \vartheta, T_w) &= \frac{\bar{P}_{ri} - \bar{P}_{rr}}{P_{ri}}, \\ a_e(\xi, \vartheta, T_w) &= \frac{\bar{E}_i - \bar{E}_r}{E_i}, \end{aligned} \right\} \quad (1.18)$$

where  $\bar{P}_{ni}$ ,  $\bar{P}_{ri}$ , and  $\bar{E}_i$  are the momentum and energy brought by one molecule of a beam moving with velocity  $\xi$  at an angle  $\vartheta$  to the surface, and  $\bar{P}_{nr}$ ,  $\bar{P}_{rr}$ , and  $\bar{E}_r$  are, respectively, the average momentum and energy carried away by reflected molecules, and  $T_w$  is the wall temperature.

When the distribution function of the molecules of the incident stream is homogeneous [as in this case, where it is given by

expression (1.10)], the momentum and energy of the reflected molecules determined by relations (1.18) may be integrated for each surface element, and in place of (1.18) new average accommodation coefficients depending on  $T$ ,  $V_\infty$ , and  $\theta$  are introduced:

$$\left. \begin{aligned} a_n(\theta, V, T_\infty, T_w) &= \frac{P_{ni} + P_{nr}}{P_{ni}}, \\ a_r(\theta, V, T_\infty, T_w) &= \frac{P_{ri} - P_{rr}}{P_{ri}}, \\ a_e(\theta, V, T_\infty, T_w) &= \frac{E_i - E_r}{E_i}, \end{aligned} \right\} \quad (1.19)$$

where, now,  $P_{ni}$ ,  $P_{ri}$ , and  $E_i$  are the momentum and energy brought to unit area of the body by all the molecules incident on a given element, and  $P_{nr}$ ,  $P_{rr}$ , and  $E_r$  are the momentum and energy carried away by those molecules upon reflection. These accommodation coefficients immediately determine the fraction of the momentum and energy of the incident molecules transmitted to the wall. The form (1.19) of accommodation coefficients is convenient when the flow is far from equilibrium with the wall, for example, at large  $S$ . When the gas is in equilibrium with the wall, i.e.,  $V = 0$  and  $T_\infty = T_w$ , the momentum and energy of the reflected particles must be equal to the momentum and energy of the incident molecules, for any value of the accommodation coefficients. Therefore, the accommodation coefficients are frequently introduced in the form

$$\left. \begin{aligned} a_n(\theta, V, T_\infty, T_w) &= \frac{P_{ni} - P_{nw}}{P_{ni} - P_{nw}}, \\ a_r(\theta, V, T_\infty, T_w) &= \frac{P_{ri} - P_{rw}}{P_{ri} - P_{rw}}, \\ a_e(\theta, V, T_\infty, T_w) &= \frac{E_i - E_w}{E_i - E_w}, \end{aligned} \right\} \quad (1.20)$$

where  $P_{nw}$  and  $E_w$  are the normal momentum and energy of the reflected molecules, with a Maxwellian distribution, corresponding to the wall temperature;

$$f_w = n_w \left( \frac{h_w}{\pi} \right)^{3/2} e^{-h_w \xi^2}. \quad (1.21)$$

The expression for  $P_w$  and  $E_w$  may obviously be obtained from formulas (1.17), by replacing  $T_\infty$  by  $T_w$ :

$$P_{nw} = \frac{mN_i}{2}, \quad E_w = \frac{mN_i}{Vh_w}. \quad (1.22)$$

As already pointed out in § 2.10, the nature of the interaction of the molecules with the surface depends on the adsorption time of the molecules on the surface. The fraction of the surface occupied by the adsorbed molecules is proportional to the adsorption time, and to the number of molecules incident on the surface. If that fraction is not negligibly small, then the accommodation coefficients may depend also on  $n_\infty$ , i.e., on the pressure.

The value of the accommodation coefficients (1.20) may fluctuate between zero and unity. The values of those coefficients are interrelated. That relation is determined by the interaction law.

Knowing the accommodation coefficients, for example, in the form (1.20), and using (1.11)-(1.14) and (1.22), we obtain for the total momentum and energy transmitted per unit area:

$$\begin{aligned} p(\theta, V_\infty, T_\infty, T_w) &= P_{ni} + P_{nr} = (2 - \alpha_n) P_{ni} + \alpha_n P_{nw} \\ &= \frac{\rho_\infty V^2}{2S^2} \left\{ \left( \frac{2 - \alpha_n}{V^\pi} S_\theta + \frac{\alpha_n}{2} \sqrt{\frac{T_w}{T_\infty}} \right) e^{-S_\theta^2} + \left[ (2 - \alpha_n) \left( S_\theta^2 + \frac{1}{2} \right) \right. \right. \\ &\quad \left. \left. + \frac{\alpha_n}{2} \sqrt{\frac{\pi T_w}{T_\infty}} S_\theta \right] (1 + \operatorname{erf} S_\theta) \right\}, \end{aligned} \quad (1.23)$$

$$\tau(\theta, V, T_\infty, T_w) = P_{ti} - P_{tr} = \alpha_r P_{ti} = \alpha_r \frac{\rho_\infty V^2 \cos \theta}{2V\pi S} \chi(S_\theta). \quad (1.24)$$

$$\begin{aligned} q(\theta, V, T_\infty, T_w) &= E_i - E_r = \alpha_e (E_i - E_w) = \\ &= \alpha_e \frac{n_\infty k T_\infty}{2V\pi h_\infty} \left\{ \left( S^2 + \frac{5}{2} - 2 \frac{T_w}{T_\infty} \right) \chi(S_\theta) - \frac{1}{2} e^{-S_\theta^2} \right\}. \end{aligned} \quad (1.25)$$

The accommodation coefficients (1.19) or (1.20) are in general functions of the same variables  $\theta$ ,  $V$ ,  $T_\infty$ ,  $n_\infty$ , and  $T_w$ , as are  $p$ ,  $\tau$ , and  $q$ . The introduction of the accommodation coefficients would be completely futile if they were not conservative. One feels that the accommodation coefficients are comparatively slowly changing functions of their arguments. Therefore, it is frequently assumed, for example, that the accommodation coefficients are

constant over the body surface (independent of  $\theta$ ), or are constant over a wide range of velocity and temperature of the oncoming stream, and so on.

Formula (1.25) determines the flux of energy transmitted to the wall by the translational degrees of freedom of the molecules. If the molecules of the oncoming stream are in equilibrium at temperature  $T_\infty$ , then there must be an energy equal to  $\frac{1}{2}kT_\infty$  in each internal degree of freedom. The mean internal energy of a molecule possessing  $j^B$  degrees of freedom, is equal to  $\frac{1}{2}j^BkT_\infty$ . For a perfect gas  $j^B = (5 - 3\nu)/(\nu - 1)$ . Therefore,

$$E_t^B = N_t \frac{5 - 3\nu}{\nu - 1} \frac{kT_\infty}{2} \quad \text{and} \quad E_w^B = N_t \frac{5 - 3\nu}{\nu - 1} \frac{kT_w}{2}. \quad (1.26)$$

In many papers the flux of internal energy of the molecules is taken into account by replacing the energy  $E_i$  in formula (1.25) by  $E_i + E_i^B$  and  $E_w$  by  $E_w + E_w^B$ ; then,

$$q = \alpha_e \frac{n_\infty k T_\infty}{2V\pi h_\infty} \left\{ \left( S^2 + \frac{\nu}{\nu - 1} - \frac{1}{2} \frac{\nu + 1}{\nu - 1} \frac{T_w}{T_\infty} \right) \chi(S_\theta) - \frac{1}{2} e^{-S_\theta^2} \right\}. \quad (1.27)$$

In other words, it is assumed that the exchange of internal energy of the molecules with the wall may be characterized by the accommodation coefficients

$$\alpha_e^B = \frac{E_t^B - E_r^B}{E_t^B - E_w^B} \quad (1.28)$$

and, moreover, that  $\alpha_e^B = \alpha_e$ . However, formula (1.27) does not take account of possible transition of translational energy of the molecules into internal energy upon collision with the wall. For example, if  $T_\infty \sim T_w$ , then, according to (1.28), the internal energy of the molecules at collision does not vary for any accommodation coefficient  $\alpha_e^B$ . But if the translational energy of the molecules is much greater than the internal energy (for example, in hypersonic flow, i.e., when  $S \gg 1$ ), then it is evident that some fraction of the translational energy may be imparted to the internal degrees of freedom of the molecules upon collision with the wall. The possibility of such transitions may be crudely taken into account by introducing an additional accommodation coefficient, and by writing (cf. § 2.10)

$$E_r^n = E_i^n - \alpha_e(E_i - E_w) - \alpha_e^B(E_i^B - E_w^B), \quad (1.29)$$

where  $E_i^n = E_i + E_i^B$  and  $E_r^n = E_r + E_r^B$  is the total energy of the incident and reflected molecules.

Each of the accommodation coefficients appearing here consists of two parts. The coefficient  $\alpha_e$ , for example, may be represented in the form  $\alpha_e = \alpha_e^{TT} - \alpha_e^{TB}$ , where  $\alpha_e^{TT}$  determines the fraction of translational energy of the molecules transferred to the wall, while the coefficient  $\alpha_e^{TB}$  is the fraction of the translational energy that goes into excitation of the internal degrees of freedom of the reflected molecules. Similarly,  $\alpha_e^B = \alpha_e^{BB} - \alpha_e^{BT}$ , where  $\alpha_e^{BB}$  determines the fraction of internal energy transmitted to the wall and  $\alpha_e^{BT}$  the fraction of internal energy going to increase of the translational energy of the reflected molecules.

In this case, we obtain for  $q$  the expression

$$\begin{aligned} q &= E_i^n - E_r^n = \alpha_e(E_i - E_w) + \alpha_e^B(E_i^B - E_w^B) \\ &= \frac{n_\infty k T_\infty}{2 V \pi h_\infty} \left\{ \left[ \alpha_e \left( S^2 + \frac{5}{2} - 2 \frac{T_w}{T_\infty} \right) \right. \right. \\ &\quad \left. \left. + \alpha_e^B \frac{5 - 3x}{2(x-1)} \left( 1 - \frac{T_w}{T_\infty} \right) \right] \chi(S_\theta) - \frac{1}{2} \alpha_e e^{-S_\theta^2} \right\}. \end{aligned} \quad (1.30)$$

For hypersonic velocities, when the translational energy of the incident molecules is much greater than their internal energy, the term containing  $\alpha_e^B$  in (1.30) may be neglected in the first approximation, and formula (1.30) goes over to formula (1.25) for a monatomic gas. However, the accommodation coefficient  $\alpha_e$  must be reduced in this case because of the fraction of energy going to excite internal degrees of freedom.

If  $\alpha_e = \alpha_e^B$ , formula (1.30) coincides with formula (1.27) for a monatomic gas, and also with the reduced accommodation coefficient. We can assume, as we did for freely colliding molecules, that the rotational degrees of freedom of the molecules are most easily excited in a single collision with the wall.\*

In the derivation of formulas (1.23)-(1.25), (1.27), or (1.30), no specific form for the distribution function of the reflected particles was assumed. In this approach it is not possible to calculate the flow field, or to compute the collisions of molecules with each other.

Since the probabilities of reflection of the molecules have not been studied extensively, the simplest reflection schemes are ordinarily used.

The specular-diffuse reflection [see formula (10.3) of § 2.10] has been most widely used. In that case,

$$\begin{aligned} p &= (2 - \alpha_r) P_{nt} + \alpha_r P_{nr} \\ &= \frac{\rho_\infty V^2}{2 S^2 \sqrt{\pi}} \left\{ \left[ S_\theta e^{-S_\theta^2} + \sqrt{\pi} \left( \frac{1}{2} + S_\theta^2 \right) (1 + \operatorname{erf} S_\theta) \right] (2 - \alpha_r) \right. \\ &\quad \left. + \alpha_r \frac{\sqrt{\pi}}{2} \sqrt{\frac{T_r}{T_\infty}} \chi(S_\theta) \right\}, \end{aligned} \quad (1.31)$$

$$\tau = \alpha_r \frac{\rho_\infty V^2 \cos \theta}{2 \sqrt{\pi} S} \chi(S_\theta), \quad (1.32)$$

$$\begin{aligned} q &= E_i - E_r = \alpha_r \frac{n_\infty k T_\infty}{2 \sqrt{\pi} h_\infty} \left\{ \left[ (S^2 + 2) e^{-S_\theta^2} \right. \right. \\ &\quad \left. \left. + \sqrt{\pi} \left( S^2 + \frac{\pi}{2} \right) S_\theta (1 + \operatorname{erf} S_\theta) \right] - 2 \frac{T_r}{T_\infty} \right\}. \end{aligned} \quad (1.33)$$

The temperature  $T_r$  may be defined in terms of the energy accommodation coefficient  $\alpha_e$ :

$$E_r = E_i - \alpha_e(E_i - E_w). \quad (1.34)$$

In that case, even for a body at constant surface temperature  $T_w = \text{const}$ , and  $\alpha_e = \text{const}$ , the temperature of the reflected molecules is different at different points.

Generally speaking,  $T_r$  and  $\alpha_e$  are determined by the same parameters (see above), and we may immediately establish the connection between  $T_r$  and  $V$  in the form  $T_r = \beta(\theta, V, T_\infty, T_w)$ . Preference should be given to the more conservative value.

In a number of papers, the quantity  $T_r$  is considered given, or taken to be constant over the body surface. If  $T_w = \text{const}$ , then the value of  $T_r$  which has been assumed constant is found from Eq. (1.34), in which  $E_i$ ,  $E_r$ , and  $E_w$  are, respectively, the total fluxes over the whole body, and  $\alpha_e$  is some average accommodation coefficient. Similar assumptions allow us to simplify the calculations substantially.

The determination of the overall aerodynamic characteristics of bodies immersed in flows reduces to very complex quadratures.

\* See A. I. Erofeev, Zh. Prikl. Mekhan. i Tekhn. Fiz., No. 3 (1966).

At the present time, the aerodynamic coefficients of a considerable number of different bodies \* have been calculated using models of interaction or assumptions about the accommodation coefficients.

We present formulas only for the simplest of these.

For a flat plate (monatomic gas):

$$X = \frac{1}{2} \rho_{\infty} V^2 A \left\{ \left[ \frac{2(2-a_n)}{\sqrt{\pi S}} \sin^2 \theta + \frac{2a_r}{\sqrt{\pi S}} \cos^2 \theta \right] e^{-S_\theta^2} + \frac{a_n}{S} \sqrt{\frac{\pi T_w}{T_\infty}} \sin^2 \theta + 2 \sin \theta \left[ (2-a_n) \left( \sin^2 \theta + \frac{1}{S^2} \right) + a_r \cos^2 \theta \right] \operatorname{erf} S_\theta \right\}, \quad (1.35)$$

$$Y = \frac{1}{2} \rho_{\infty} V^2 A \left\{ \frac{2(2-a_n-a_r)}{\sqrt{\pi S}} \cos \theta \sin \theta e^{-S_\theta^2} + \frac{a_n}{S} \sqrt{\frac{\pi T_w}{T_\infty}} \sin \theta \cos \theta + \cos \theta \left[ 2(2-a_r-a_n) \sin^2 \theta + \frac{2-a_n}{S^2} \right] \operatorname{erf} S_\theta \right\}, \quad (1.36)$$

$$Q = a_e \frac{\rho_{\infty} A}{m V \pi h_{\infty}} k (T_{eq} - T_w) \left[ e^{-S_\theta^2} + \sqrt{\pi} S_\theta \operatorname{erf} S_\theta \right]. \quad (1.37)$$

Here, X is the drag of the plate, Y its lift, Q the total heat flux, A the area of one side of the plate,  $\theta$  the angle of attack, and  $T_{eq}$  the temperature of the thermally insulated plate ( $Q = 0$ ):

$$T_{eq} = T_\infty \left\{ 1 + \frac{1}{4} \left[ 2S^2 + 1 - \frac{1}{1 + \sqrt{\pi} S_\theta e^{-S_\theta^2} \operatorname{erf} S_\theta} \right] \right\}. \quad (1.38)$$

For a sphere of diameter D (monatomic gas):

$$X = \frac{\pi D^2 \rho_{\infty} V^2}{8} \left\{ \frac{(2-a_n+a_r)}{2S^3} \left[ \frac{4S^4 + 4S^2 - 1}{2S} \operatorname{erf} S + \frac{2S^2 + 1}{\sqrt{\pi}} e^{-S^2} \right] + \frac{2a_n}{3S} \sqrt{\frac{\pi T_w}{T_\infty}} \right\}, \quad (1.39)$$

\*See H. S. Tsien, J. Aeron. Sci., Vol. 13, No. 12 (1946); J. Stalder, G. Goodwin, and M. Creager, NASA Rept. Nos. 1032 (1951) and 1093 (1952); L. Talbot, J. Aerospace Sci., Vol. 26, No. 1 (1959); S.A. Schaaf and L. Talbot, "Handbook of Supersonic Aerodynamics," Washington, 1959, Sec. 16; R. G. Barantsev and U. Ischen-goi, Vestn. Lenin-grad Univ., No. 13 (1961); R. Monti, L'Aerotecnica, Vol. 41, No. 6 (1961); M. D. Sirinan, Missil, Vol. 3, No. 5 (1961); S. A. Schaaf, Mechanics Raref. Gases, Handbuch der Physik 8/2 (1963); G. N. Patterson, Molecular Flow of Gases, Wiley, 1956, V. P. Shidlovskii, "Introduction to Rarefied Gas Dynamics," Nauka Press, 1965.

$$Q = a_e \frac{\pi D^2 \rho_{\infty}}{m V \pi h_{\infty}} \frac{k (T_{eq} - T_w)}{S} \left[ S^2 + S \operatorname{ierfc} S + \frac{1}{2} \operatorname{erf} S \right], \quad (1.40)$$

$$T_{eq} = T_\infty \left\{ 1 + \frac{S^2}{4} \frac{2S(2S^2+1)(S+\operatorname{ierfc} S)+(2S^2-1)\operatorname{erf} S}{2S^3(S+\operatorname{ierfc} S)+\operatorname{erf} S} \right\}. \quad (1.41)$$

Here,

$$\operatorname{ierfc} x = \frac{2}{\sqrt{\pi}} \int_0^x du \int_u^\infty e^{-t^2} dt.$$

For many applications we are interested in the flow of a free-molecule stream over an infinite cylinder, at right angles to the cylinder axis. Under laboratory conditions it is very difficult to obtain free-molecule streams of large dimensions. Therefore, to obtain streams with a large ratio of mean free path to characteristic model dimension we need to find ways of reducing the latter. In this respect, thin wires — cylinders, of diameter up to some hundredths of a millimeter — are ideal objects of investigation. Knowing the aerodynamic characteristics of thin wires (hot wire anemometers) in free-molecule flow, we may use them to determine the parameters of streams of rarefied gases.\*

Assuming a scheme of completely diffuse reflection ( $a_r = 1$ ), we have, for a cylinder perpendicular to the flow:

$$X = \frac{\rho_{\infty} V^2}{2} D L (C_{xl} + C_{xr}), \quad (1.42)$$

where

$$C_{xl} = \frac{\sqrt{\pi} e^{-S^{3/2}}}{S} \left\{ I_0 \left( \frac{S^2}{2} \right) + \left( \frac{1+2S^2}{2} \right) \left[ I_0 \left( \frac{S^2}{2} \right) + I_1 \left( \frac{S^2}{2} \right) \right] \right\},$$

$$C_{xr} = \frac{\pi^{3/2}}{4 V \sqrt{h_r}},$$

$$Q = a_e \frac{k n_{\infty} T_{\infty}}{\pi^{3/2} V h_{\infty}} \left\{ -\frac{T_w}{T_{\infty}} (z_1 + z_2) + \frac{1}{2} \left[ (S^2 + 2) z_1 + \left( S^2 + \frac{5}{2} \right) z_2 \right] \right\}, \quad (1.43a)$$

for a monatomic gas, and

\*H. S. Tsien, J. Aerospace Sci., Vol. 15, No. 10 (1948).

$$Q = \alpha_e \frac{k n_\infty T_\infty}{\pi^{3/2} V h_\infty} \left\{ -\frac{3}{2} \frac{T_w}{T_\infty} (z_1 + z_2) + \frac{1}{2} \left[ (S^2 + 3) z_1 + \left( S^2 + \frac{7}{2} \right) z_2 \right] \right\} \quad (1.43b)$$

for a diatomic gas.

Here,  $D$  is the cylinder diameter,  $L$  is its length,  $I_n(x)$  is the modified Bessel function of order  $n$ ,

$$z_1 = \pi e^{-S^2/2} \text{ and } z_2 = \pi S^2 e^{-S^2/2} \left[ I_0 \left( \frac{S^2}{2} \right) + I_1 \left( \frac{S^2}{2} \right) \right].$$

In obtaining formulas (1.42) and (1.43), we assumed\* that  $T_w = \text{const}$  and  $T_r = \text{const}$  over the whole surface of the cylinder. The quantity  $T_r$  was determined in terms of the accommodation coefficient from formula (1.34) for the total fluxes to the cylinder. We also took into account that the internal energy of a diatomic gas is equal to  $kT_\infty$  and accommodates with the same accommodation coefficient as does the translational energy, i.e.,  $\alpha_e = \alpha_e^B$ .

It follows from (1.43) that, under the assumptions made, the temperature of a thermally insulated cylinder (i.e., with  $Q = 0$ ) does not depend on the accommodation coefficient. Since the total energy of the reflected molecules  $E_r$  must be equal to the energy of the incident molecules, then, according to (1.34),  $E_i = E_w$ , i.e.,  $T_r = T_w = T_{eq}$ . But  $T_r$  is the single unknown parameter in expression (1.42) for the drag. Therefore, in this case, the drag of the cylinder is also independent of the accommodation coefficient. This permits an experimental test of this model; since the accommodation coefficient does not affect the result, the molecules of a monatomic gas should reflect diffusely and the temperature  $T_r$  should be constant over the surface. In Figs. 49 and 50, calculated values for the drag and equilibrium temperature of a cylinder are compared with experimental values. Taking account of possible experimental errors, the agreement may be considered satisfactory.

We note that the equilibrium temperature of the cylinder is higher than the stagnation temperature (Fig. 49). That interesting property is not typical of a cylinder only.<sup>†</sup>

\*See the paper by J. Stalder, C. Goodwin, and M. Creager, NASA Rept. Nos. 1032 (1951) and 1093 (1952).

<sup>†</sup>See W. D. Hayes and R. F. Probstein, "Hypersonic Flow Theory," 2nd ed., Academic Press, New York, 1966; see also J. Stalder, G. Goodwin, and M. Creager, cited above.

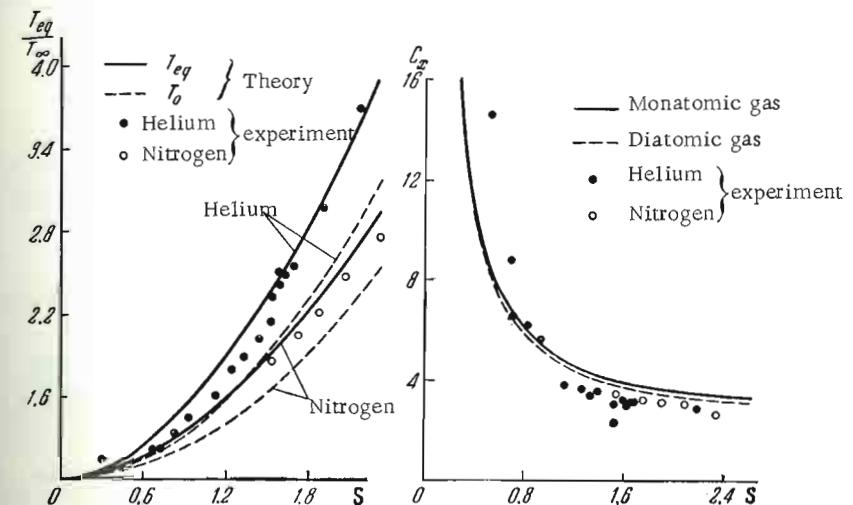


Fig. 49

In fact, let us examine a thermally insulated surface element. For a monatomic gas, according to (1.25), the temperature does not depend on the accommodation coefficient, and is equal to

$$T_{eq} = T_\infty \left\{ 1 + \frac{S^2}{2} + \frac{1}{4} \left[ 1 - \frac{1}{1 + V \pi S_\theta e^{-S_\theta^2} (1 + \operatorname{erf} S_\theta)} \right] \right\}.$$

The square bracket is always positive and, therefore,

$$T_{eq} \geq T_\infty \left( 1 + \frac{1}{2} S^2 \right).$$

The adiabatic stagnation temperature of the gas is equal to

$$T_0 = T_\infty \left( 1 + \frac{\gamma - 1}{2} M_\infty^2 \right) = T_\infty \left( 1 + \frac{2}{5} S^2 \right) < T_{eq},$$

i.e., the equilibrium temperature of any ( $\theta \leq \pi/2$ ) thermally insulated surface element in a free-molecule stream is higher than the stagnation temperature.

This fact is obvious when  $S \rightarrow \infty$ . In fact, in that case, the whole energy of the oncoming stream reaches the wall. Each incident molecule brings an energy  $\frac{1}{2} m V^2$ . On the average, each re-

flected molecule must carry away the same energy. For this to be true, the wall temperature, according to (1.22), must be equal to

$$T_w = T_r = T_{eq} = \frac{mV^2}{4k}.$$

For adiabatic stagnation, a fraction of the energy of the incident molecules is expended in working against the pressure forces and

$$T_0 = \frac{mV^2}{5k}.$$

When  $S$  decreases, not all the energy of the incident stream reaches the wall, since molecules traveling upstream with large thermal velocities do not reach the wall. Therefore, the difference between  $T_{eq}$  and  $T_0$  must decrease with decreasing  $S$ .

Those conclusions are fully in agreement with the experimental data presented in Fig. 49.

An experimental test of the applicability of our model is difficult for heat transfer, since an unknown accommodation coefficient appears in formula (1.43) for the heat flux. Measurements carried out at  $M = 1.6-2.0$  on wires of platinum, nickel, and several other materials, are in reasonably good agreement with theory for  $\alpha_e = 0.9$ .\*

To illustrate the influence of the accommodation coefficient, the aerodynamic coefficients for a plate and a rocket-shaped body are presented in Figs. 51 and 52. The calculation was performed for completely diffuse reflection at  $S = 7$ .

The results for  $S \rightarrow \infty$ , i.e., either when  $V \rightarrow \infty$ , or  $T_\infty \rightarrow 0$ , are particularly graphic. In that case, we may argue that a monoenergetic beam of molecules of velocity  $V$  strikes the body. If the body is cold (i.e.,  $V\sqrt{h_w} \rightarrow \infty$ ), and  $\alpha_e \sim 1$ , the velocity of the reflected molecules is small and may be neglected. In that limiting case,

$$p = \rho_\infty V^2 \sin^2 \theta, \quad \tau = \rho_\infty V^2 \sin \theta \cos \theta, \quad q = \frac{1}{2} \rho_\infty V^3 \sin \theta. \quad (1.44)$$

For an arbitrary body the drag and the heat-flux referred to the frontal area  $A$  of the body are equal to

\*See the works by Stalder et al. cited above.

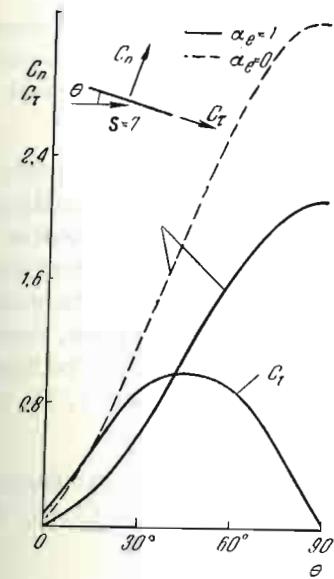


Fig. 51

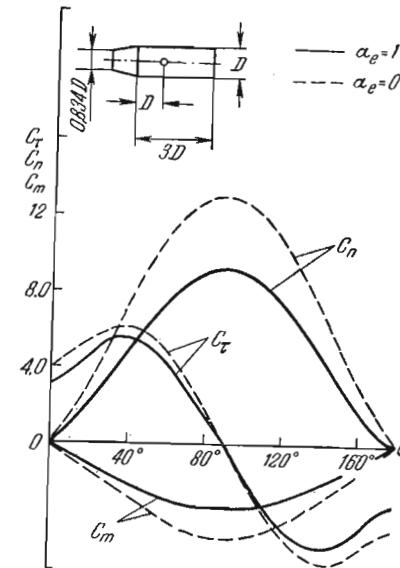


Fig. 52

$$C_x = \frac{2X}{\rho_\infty V^2 A} = 2, \quad \frac{2Q}{\rho_\infty V^3 A} = 1. \quad (1.45)$$

The lift vanishes. In a free-molecule stream the lift results from the reaction of the reflected molecules and the thermal motion of the molecules of the incident stream. Therefore, when  $S \gg 1$ , the value of the lift force is very sensitive to a change in accommodation coefficient. As  $\alpha_e$  decreases, the momentum of the reflected particles increases and, therefore, so does the lift force (Fig. 51).

Let us compare the properties of a cold flat plate, in a hypersonic free-molecule stream with  $\alpha_e \sim 1$ , and in a continuous flow, as calculated for simplicity by the Newtonian method. In Newtonian flow it is assumed\* that after reaching the body the gas moves along its surface, and transfers its normal momentum to the body. In the limiting case of hypersonic free-molecule flow

\*See, for example, G. G. Chernyi, "Introduction to Supersonic Flow," (English translation by R. F. Probstein), Academic Press, New York, 1961; W. D. Hayes and R. F. Probstein, "Hypersonic Flow Theory," 2nd ed., Academic Press, New York, 1966.

examined above, the plate receives not only the normal but also the tangential momentum of the gas. Therefore, the force normal to the plate is the same in both flows, but in the free-molecule flow there is also a tangential force.

Newtonian theory does not take into account the viscosity of the fluid, while all effects are included in the kinetic calculation. Therefore, we must add, to the forces calculated by Newtonian theory, the friction drag, calculated, for example, from boundary layer theory. It is clear, however, that only part of the tangential momentum is absorbed in the boundary layer. Therefore, even allowing for friction, the drag in free-molecule flow must be higher than in continuous flow, and the lift-to-drag ratio (i.e.,  $Y/X$ ), is correspondingly less.

If we assume that the molecules are reflected specularly, it is clear that the normal force is double the Newtonian force, while the tangential force vanishes as in Newtonian theory.

For hypersonic free-molecule flows, just as for continuous hypersonic flows, aerodynamic characteristics are independent of Mach number (of  $S$ ). More exactly, when  $S \gg 1$  and  $S_0 \gg 1$  the flow depends only on the velocity of the incident stream  $V$ , and its density  $\rho_\infty$ , but does not depend on its temperature. To illustrate that behavior of the aerodynamic characteristics, we present values of the drag coefficient and of the Stanton number for a flat plate, a cylinder, and a sphere, in Figs. 53 and 54. The Stanton number is defined by the relation

$$St = \frac{2\alpha}{\kappa + 1} \frac{Q}{A\rho_\infty V (T_{eq} - T_w) \alpha_e}.$$

The surface  $A$  of the body has been chosen as the characteristic area in Fig. 54. In Fig. 53, the drag coefficient of the cylinder and sphere is referred to the normal section, while for the plate it is the area of one side. It is assumed that  $T_r = T_w = T_\infty$  or, equivalently,  $\alpha_r = \alpha_n = \alpha_e = 1$ . In practice, for a cold body, with  $S \geq 5$  the drag is equal to its asymptotic value as  $S \rightarrow \infty$ .

For slender bodies, i.e., with  $S_0 < 1$ , the drag coefficient does not approach a constant value.

The temperature of the body is determined by a balance of the heat fluxes to the body surface. In addition to the heat trans-

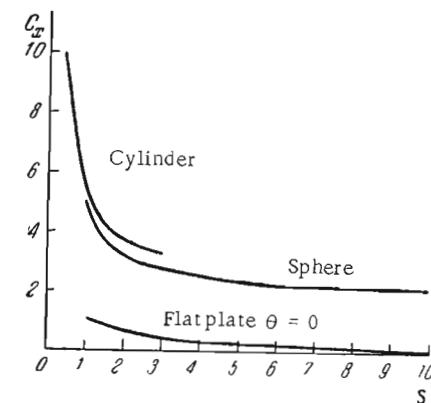


Fig. 53

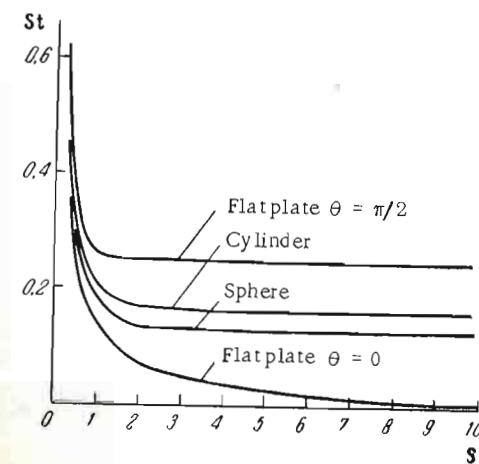


Fig. 54

mitted to the surface by the incident molecules, the surface may receive heat from sources in the body, from heat conduction in the body, and from radiation. Under flight conditions close to orbital, a considerable part of the heat is lost by radiation, and usually  $T_w/T_\infty \leq 1$ , i.e., the body may be considered cold.\*

\*S. Abarbanel, J. Aerospace Sci., Vol. 28, No. 4 (1961).

The expressions presented above for aerodynamic characteristics apply only to the simplest bodies. In general, the calculation reduces to very laborious quadratures, especially if one does not make the assumption that  $T_r$  is constant over the body surface.

A number of approximate methods\* have been proposed to simplify the calculations when  $S \gg 1$ , and have been used to calculate the aerodynamic characteristics of a number of bodies. If the fraction of the body surface where  $S_\theta \leq 1$  is small, then, by neglecting terms of order greater than  $S_\theta^{-2}$  in (1.23) and (1.24), we obtain the approximation

$$\left. \begin{aligned} p &= \frac{\rho_\infty V^2}{2S^2} \left\{ 2(2 - a_n) S_\theta^2 + a_n S_\theta \sqrt{\pi} \sqrt{\frac{T_w}{T_\infty}} + 2 - a_n \right\}, \\ \tau &= a_n \rho_\infty V^2 \sin \theta \cos \theta. \end{aligned} \right\} \quad (1.46)$$

In that approximation, the aerodynamic characteristics of an axisymmetric body, at an angle of attack  $\alpha$ , after integration of  $p$  and  $\tau$  over the surface, take the form

$$\begin{aligned} C_x &= \frac{2\pi}{A} \int_0^l \left[ (2 - a_n - a_r) \frac{\xi^3 \sin^3 \alpha}{1 + \xi^2 \operatorname{tg}^2 \alpha} F \right. \\ &\quad \left. + \frac{a_n V \pi}{S} \sqrt{\frac{T_w}{T_\infty}} \frac{\xi^2 \sin^2 \alpha}{\sqrt{1 + \xi^2 \operatorname{tg}^2 \alpha}} G + \left( \frac{2 - a_n}{S^2} + 2a_r \right) \xi H \sin \alpha \right] r dx, \end{aligned} \quad (1.47)$$

$$\begin{aligned} C_y &= C_x \operatorname{ctg} \alpha - \frac{2\pi}{A \cos \alpha} \int_0^l \left[ 2(2 - a_n - a_r) \frac{\xi^3 \sin^2 \alpha}{1 + \xi^2 \operatorname{tg}^2 \alpha} G \right. \\ &\quad \left. + \frac{2 - a_n}{S^2} \xi M + \left( \frac{a_n V \pi}{S} \sqrt{\frac{T_w}{T_\infty}} \frac{\xi^2 \sin \alpha}{\sqrt{1 + \xi^2 \operatorname{tg}^2 \alpha}} + 2a_r \cos^2 \alpha \right) H \right] r dx, \end{aligned} \quad (1.48)$$

$$\begin{aligned} C_m &= -\frac{2\pi}{At} \int_0^l \left\{ \left[ 2(2 - a_n - a_r) \frac{\xi^3 \sin^2 \alpha}{1 + \xi^2 \operatorname{tg}^2 \alpha} (F - G) \right. \right. \\ &\quad \left. \left. + \frac{a_n V \pi}{S} \sqrt{\frac{T_w}{T_\infty}} \frac{\xi^2 \sin \alpha}{\sqrt{1 + \xi^2 \operatorname{tg}^2 \alpha}} (G - H) + \frac{2 - a_n}{S^2} \xi (H - M) \right] (r \xi \operatorname{tg} \alpha + x) \right. \\ &\quad \left. + 2a_r \xi \sin^2 \alpha \left[ \left( x - \frac{r \xi}{\operatorname{tg} \alpha} \right) H + \frac{r \xi}{\operatorname{tg} \alpha} G \right] \right\} r dx. \end{aligned} \quad (1.49)$$

\* W. A. Gustafson, A.R.S. J., Vol. 29, No. 9 (1959); B. M. Schrello, A.R.S. J., Vol. 30, No. 8 (1960). The method of the latter paper has been extended by V. S. Galkin to the case of rotating bodies [Inzh. Zh., No. 5 (1965)].

Here a cylindrical system of coordinates is used; the  $x$  axis is directed along the body axis; the origin of coordinates is chosen to be the leading edge of the body; the moment

$$M_z = \frac{1}{2} C_m V^2 A l$$

is calculated about the origin of coordinates;  $r(x)$  is the equation of the body;  $\alpha$  is the angle of attack,  $\xi = (dr/dx) \operatorname{ctg} \alpha$ ; and

$$F = \frac{1}{\pi} \left[ \left( 1 + \frac{3}{2\xi^2} \right) \arccos(-\xi) + \frac{1}{\xi} \left( \frac{11}{6} + \frac{2}{3\xi^2} \right) \sqrt{1 - \xi^2} \right] \text{ for } \xi \leq 1,$$

$$F = 1 + \frac{3}{2\xi^2} \text{ for } \xi \geq 1,$$

$$G = \frac{1}{\pi} \left[ \left( 1 + \frac{1}{2\xi^2} \right) \arccos(-\xi) + \frac{3}{2\xi} \sqrt{1 - \xi^2} \right] \text{ for } \xi \leq 1,$$

$$G = 1 + \frac{1}{2\xi^2} \text{ for } \xi \geq 1,$$

$$H = \frac{1}{\pi} \left[ \arccos(-\xi) + \frac{1}{\xi} \sqrt{1 - \xi^2} \right] \text{ for } \xi \leq 1,$$

$$H = 1 \text{ for } \xi \geq 1,$$

$$M = \frac{1}{\pi} \arccos(-\xi) \text{ for } \xi \leq 1,$$

$$M = 1 \text{ for } \xi \geq 1.$$

Thus, for example, for the drag of a cone of vertex half-angle  $\delta$ , and of a sphere, we have, respectively,

$$\begin{aligned} C_x &= 2(2 - a_n - a_r) F \sin^2 \delta \cos^3 \alpha + \frac{a_n V \pi}{S} \sqrt{\frac{T_w}{T_\infty}} G \sin \delta \cos^2 \alpha \\ &\quad + \left( \frac{2 - a_n}{S^2} + 2a_r \right) H \cos \alpha, \end{aligned} \quad (1.50)$$

$$C_x = (2 + a_r - a_n) + \frac{2 - a_n}{S^2} + \frac{2}{3} \frac{a_n V \pi}{S} \sqrt{\frac{T_w}{T_\infty}}. \quad (1.51)$$

For a sphere with  $S \gg 1$ , formula (1.51) gives results in good agreement with the exact value. However, this approximation is not sufficiently accurate for bodies elongated in the stream direction.

It was assumed above that the body is immersed in a uniform equilibrium stream. In a number of cases we are also interested in the flow over bodies with different boundary conditions at infinity. As we have noted, thin wires (hot-wire anemometers) may

serve as instruments for the measurement of flow properties. In particular, hot-wire anemometers may be used to determine the properties of a stream with velocity or temperature gradients. For flows near equilibrium, the distribution function of the Navier-Stokes approximation may be taken as the distribution function of the free stream  $f_\infty$ . The flow over a cylinder of such a nonuniform stream has been examined by Bell and Schaaf.\* Their analysis shows that the effect of the nonuniformity is appreciable only for very low stream velocities and for very strong gradients.

### §6.2. Free-Molecule Flows. Flow Over Concave Bodies

Free-molecule flow over convex bodies has been examined in the preceding section. The molecular number flux on each surface element of a convex body is independent of other parts of the body. When we examine free-molecule flow over bodies with concavities (and also over a group of bodies), we must take account of the shading of some parts of the body by other parts, as well as of the incidence on some parts of the body of molecules reflected from other parts.

Let us consider a point  $M_1$  on the surface of a body. We designate by  $\Omega(M_1)$  the solid angle subtended at the point  $M_1$  by the rest of the body or by other bodies (Fig. 55). We assume the molecular reflection law (9.6) of Chapter II:

$$f_r(x_1, \xi_1) = \int_{\xi_1 \cdot n < 0} K(x_1, \xi_i, \xi_r) f_i(x_1, \xi) d\xi, \quad (2.1)$$

where  $K(\xi_i, \xi_r)$  is a given function which depends on the properties of the surface,  $n$  is the outward normal, and  $x_1$  is the coordinate of point  $M_1$ . Then the distribution function of the molecules incident at point  $M_1$  may be represented in the form

$$f_i(x_1, \xi_1) = \int_{\xi_2 \cdot n_2 < 0} K(x_2, \xi_2, \xi_1) f(x_2, \xi_2) d\xi_2, \quad n_1 \cdot \xi_1 < 0, \quad (2.2a)$$

if the velocity  $\xi_1$  lies within  $\Omega(M_1)$ , and

$$f_i(x_1, \xi_1) = f_\infty(\xi_1), \quad (2.2b)$$

\*S. Bell and S. A. Schaaf, A.R.S. J., 23: 314 (1953); Jet Propulsion, 25: 168 (1955).

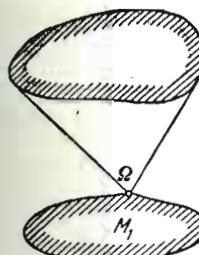


Fig. 55

if  $\xi_1$  lies in the solid angle  $\bar{\Omega} = 2\pi - \Omega(M_1)$ . The points  $M_2$  are points on the body surface which we may reach from the point  $M_1$  along the vector  $(-\xi_1)$ . Equation (2.2a) relates the distribution functions at different points of the body surface. An analytical solution of that equation is extremely complicated in the general case.

In practical applications, diffuse reflection ( $\alpha_r = 1$ ), with a Maxwellian distribution, is very frequently assumed, namely [see (10.3) and (10.3a) of Chapter II]:

$$K(x_2, \xi_2, \xi_1) = -\frac{2}{\pi} h_r^2(x_2) e^{-h_r(x_2) \xi_1^2} (\xi_2 \cdot n_2). \quad (2.3)$$

In that case, inside  $\Omega$ , instead of (2.2a), for the molecules incident at the point  $M_1$ , we have

$$f_i(x_1, \xi_1) = N(x_2) \frac{2}{\pi} h_r^2(x_2) e^{-h_r(x_2) \xi_1^2}, \quad \xi_1 \cdot n_1 < 0, \quad (2.4)$$

where  $N(x_2)$  is the number of molecules incident per unit time on unit area near the point  $M_2(x_2)$ .

Similarly, for the reflected molecules ( $\xi_1 \cdot n_1 > 0$ ) at the point  $M_1$ , we have

$$f(x_1, \xi_1) = N(x_1) \frac{2}{\pi} h_r^2(x_1) e^{-h_r(x_1) \xi_1^2}. \quad (2.5)$$

In general, if the distribution function for the reflected molecules does not depend on the specific form of the distribution function for the incident molecules, we may write

$$K(x_2, \xi_2, \xi_1) = -F(x_2, \xi_1)(\xi_2 \cdot n_2). \quad (2.6)$$

The function  $F$  may depend on one or several of the macroscopic parameters of the incident molecules (on their total energy, momentum, etc.). For such a kernel,

$$\left. \begin{aligned} f(x_1, \xi_1) &= N(x_2) F(x_2, \xi_1) && \text{when } \xi_1 \cdot n_1 < 0, \xi_1 \text{ inside } \Omega, \\ f(x_1, \xi_1) &= N(x_1) F(x_1, \xi_1) && \text{when } \xi_1 \cdot n_1 > 0. \end{aligned} \right\} \quad (2.7)$$

The number of molecules incident on the surface at point  $M_1$  is equal to

$$N(\mathbf{x}_1) = N_\infty(\mathbf{x}_1) - \int_{\Omega} N(\mathbf{x}_2) F(\mathbf{x}_2, \xi_1) (\xi_1 \cdot \mathbf{n}_1) d\xi_1, \quad (2.8)$$

$$\xi_1 \cdot \mathbf{n}_1 < 0,$$

where  $N_\infty(\mathbf{x}_1)$  is the number of molecules of the undisturbed stream incident on the surface element,

$$N_\infty(\mathbf{x}_1) = - \int_{2\pi - \Omega} f_\infty(\xi_1) (\xi_1 \cdot \mathbf{n}_1) d\xi_1, \quad \xi_1 \cdot \mathbf{n}_1 < 0. \quad (2.9)$$

We introduce the polar coordinates  $\xi$ ,  $\theta$ , and  $\varphi$ , where  $\theta$  is the polar angle measured from the normal  $\mathbf{n}$ , and  $\varphi$  is the azimuthal angle. Then,

$$N(\mathbf{x}_1) = N_\infty(\mathbf{x}_1) + \int_{\Omega} N(\mathbf{x}_2) \bar{F}(\mathbf{x}_2) \sin \theta_1 \cos \theta_1 d\theta_1 d\varphi, \quad (2.10)$$

where

$$\bar{F}(\mathbf{x}_2) = \int_0^\infty F(\mathbf{x}_2, \xi_1) \xi_1^3 d\xi_1.$$

The integral equation (2.10) may be rewritten in the form

$$N(\mathbf{x}_1) = N_\infty(\mathbf{x}_1) + \left. \begin{aligned} &\int_{A_2} N(\mathbf{x}_2) G(\mathbf{x}_2, \mathbf{x}_1) dA_2, \\ &G(\mathbf{x}_2, \mathbf{x}_1) = F(\mathbf{x}_2) \frac{\cos \theta_1 \cos \theta_2}{r_{12}^2}. \end{aligned} \right\} \quad (2.11)$$

Here,  $dA_2$  is a surface element,  $\theta_1$  and  $\theta_2$  are the angles between lines joining the points  $M_1$  and  $M_2$ , and the normals  $\mathbf{n}_1$  and  $\mathbf{n}_2$ , respectively,  $r_{12}$  is the distance between the points  $M_1$  and  $M_2$ , and the integration is carried out over the entire surface visible from point  $M_1$ .

In particular, for the kernel (2.3), the integral  $\bar{F} = \pi^{-1}$ , and

$$G(\mathbf{x}_2, \mathbf{x}_1) = \frac{\cos \theta_1 \cos \theta_2}{\pi r_{12}^2} = G(\mathbf{x}_1, \mathbf{x}_2). \quad (2.12)$$

In that case, Eq. (2.11) is a Fredholm equation with a symmetrical kernel.

We note that under these assumptions, Eq. (2.11) does not contain either the surface temperature or the accommodation coefficient, i.e., the number of molecules incident on various surface elements of the body depends only on the distribution function of the incident stream and on the body geometry. We also note that the kernel of the integral equation (2.11) has the form (2.12) for any function  $F$ , which depends only on the modulus of  $\xi$ , or, more generally, if  $\int F \xi_1^3 d\xi_1$  does not depend on  $\theta$  and  $\varphi$ .

In fact, according to (2.7),

$$N(\mathbf{x}_1) = + \int_{\xi_1 \cdot \mathbf{n}_1 > 0} f_r(\mathbf{x}_1, \xi_1) (\xi_1 \cdot \mathbf{n}_1) d\xi_1 = N(\mathbf{x}_1) \int F(\mathbf{x}_1, \xi_1) (\xi_1 \cdot \mathbf{n}_1) d\xi_1,$$

and, therefore,

$$\begin{aligned} 1 &= \int_{\xi_1 \cdot \mathbf{n}_1 > 0} F(\mathbf{x}_1, \xi_1) (\xi_1 \cdot \mathbf{n}_1) d\xi_1 = \int_0^\infty F \xi_1^3 d\xi_1 \int_0^{\pi/2} \int_0^{2\pi} \sin \theta \cos \theta d\theta d\varphi \\ &= \pi \int_0^\infty F \xi_1^3 d\xi_1 = \pi \bar{F}. \end{aligned}$$

The reflection law which satisfies these conditions is called the cosine law, since the number of molecules leaving a unit area in a direction  $\theta$  is proportional to  $\cos \theta$ .

This discussion indicates that in the case of cosine reflection, the number of molecules reflected from a surface element  $dA_2$  and incident on the surface element  $dA_1$  is equal to

$$N(\mathbf{x}_2) G(\mathbf{x}_2, \mathbf{x}_1) dA_2 dA_1.$$

Similarly, the number of molecules reflected from element  $dA_1$  and incident on the element  $dA_2$  is equal to

$$N(\mathbf{x}_1) G(\mathbf{x}_1, \mathbf{x}_2) dA_1 dA_2.$$

Then,

$$G(\mathbf{x}_1, \mathbf{x}_2) = G(\mathbf{x}_2, \mathbf{x}_1).$$

The function  $G(\mathbf{x}_1, \mathbf{x}_2)$  may be regarded as the influence function of element  $dA_1$  on element  $dA_2$ , or  $G dA_1 dA_2$  as the probability that a molecule reflected from the element  $dA_1$  will strike element  $dA_2$ .

We note that this model of free-molecule flow is similar to the scattering of light at surfaces according to Lambert's law.\* That analogy may be used for the simulation of free-molecule flows.

Let the energy accommodation coefficient be specified as

$$\alpha_e = \frac{E_l - E_r}{E_l - E_w}. \quad (2.13)$$

The energy of the molecules incident per unit area is equal to

$$E_i(\mathbf{x}_1) = E_{i\infty}(\mathbf{x}_1) - \int_{\Omega} e_r(\xi_1) N(\mathbf{x}_2) F(\mathbf{x}_2, \xi_1) (\xi_1 \cdot \mathbf{n}_1) d\xi_1, \quad (2.14)$$

$$\xi_1 \cdot \mathbf{n}_1 < 0,$$

where  $E_{i\infty}$  is the energy brought by molecules arriving from infinity, and  $e_r(\xi_1)$  is the energy carried away from the surface by a molecule of velocity  $\xi_1$ .

Similarly,

$$E_r(\mathbf{x}_1) = \int_{\Omega} e_r(\xi_1) N(\mathbf{x}_1) F(\mathbf{x}_1, \xi_1) (\xi_1 \cdot \mathbf{n}_1) d\xi_1. \quad (2.15)$$

For  $E_w$ , as in the preceding section, we obtain

$$E_w(\mathbf{x}_1) = N(\mathbf{x}_1) \frac{2}{\pi} h_w^2 \int_{\Omega} e_r(\xi_1) e^{-h_w(x_1)} \xi_1^2 (\xi_1 \cdot \mathbf{n}_1) d\xi_1. \quad (2.16)$$

If  $F$  and  $e_r$  depend only on the modulus of the velocity  $\xi$ , we can carry out the integration with respect to  $\xi$  in (2.14)–(2.16):

$$E_i(\mathbf{x}_1) = E_{i\infty}(\mathbf{x}_1) + \int_{A_2} N(\mathbf{x}_2) \frac{\tilde{F}(\mathbf{x}_2) \cos \theta_1 \cos \theta_2}{r_{12}^2} dA_2, \quad (2.17)$$

$$E_r = \pi N(\mathbf{x}_1) \tilde{F}(\mathbf{x}_1),$$

where

$$\tilde{F}(\mathbf{x}_1) = \int_0^\infty e_r(\xi_1) F(\mathbf{x}_1, \xi_1) \xi_1^3 d\xi_1. \quad (2.18)$$

Substituting that expression into (2.13), we obtain an integral equation for  $\tilde{F}$ :

$$\pi N(\mathbf{x}_1) \tilde{F}(\mathbf{x}_1) = \alpha_e N(\mathbf{x}_1) \frac{2}{\pi} h_w^2(\mathbf{x}_1) \int_{\Omega} e_r(\xi_1) e^{-h_w(x_1)} \xi_1^2 d\xi_1$$

$$+ (1 - \alpha_e) E_{i\infty}(\mathbf{x}_1) + (1 - \alpha_e) \int_{A_2} N(\mathbf{x}_2) \tilde{F}(\mathbf{x}_2) \frac{\cos \theta_1 \cos \theta_2}{r_{12}^2} dA_2. \quad (2.19)$$

By noting that  $E_r(\mathbf{x}_2) = \pi \tilde{F}(\mathbf{x}_2) N(\mathbf{x}_2)$ , Eq. (2.19) may be rewritten in the form

$$E_r(\mathbf{x}_1) = \alpha_e E_w + (1 - \alpha_e) E_{i\infty} + (1 - \alpha_e) \int_{A_2} E_r(\mathbf{x}_2) G(\mathbf{x}_2, \mathbf{x}_1) dA_2. \quad (2.20)$$

Therefore, if the function  $\tilde{F}$  does not depend on  $\varphi$  and  $\theta$ , then  $\pi \tilde{F}(\mathbf{x}_2)$  is the average energy brought by a single molecule reflected from the element  $dA_2$ , and  $E_r(\mathbf{x}_2) G(\mathbf{x}_1, \mathbf{x}_2) dA_1 dA_2$  is the fraction of the energy of the molecules reflected from the element  $dA_2$ , brought to the element  $dA_1$ . In other words, instead of the actual scheme of reflection, we consider all molecules reflected uniformly in all directions with the same energy. If the molecules do not have internal energy and are reflected with a Maxwellian distribution, then the energy brought per molecule is equal to  $m/h_r$ . In that case, we consider all molecules to travel with a single velocity equal to  $(2/h_r)^{1/2}$ . Thus, the problem reduces to the solution of the integral equations (2.11) and (2.19) or (2.20). If the kernel  $K$  is given in the form (2.3), then, having a solution for  $\tilde{F}$ , we can find  $h_r(\mathbf{x}_1)$  by means of (2.18). Having  $N(\mathbf{x}_1)$  and  $h_r(\mathbf{x}_1)$ , we can calculate all the aerodynamic characteristics by means of straightforward quadratures.

Let us examine an adiabatic wall, i.e., we assume that at each point of the body  $E_i = E_r$ . In that case, it follows from the definition of  $\alpha_e$  that  $E_i = E_w = E_r$ . By replacing the energy  $E_w$  by  $E_r$  in Eq. (2.20), we obtain

\*E. Larish, Izv. Akad. Nauk SSSR, Mekhan. i Mashinostr., No. 3 (1960).

$$E_r(x_1) = E_{i\infty}(x_1) + \int_{A_2} E_r(x_2) G(x_2, x_1) dA_2.$$

It is easy to see that this equation is analogous to Eq. (2.11). Therefore, if  $E_{i\infty}(x_1) = \text{const } N_\infty(x_1)$ , then

$$\frac{E_r(x_1)}{N(x_1)} = \frac{E_w(x_1)}{N(x_1)} = \frac{E_{i\infty}(x_1)}{N_\infty(x_1)}. \quad (2.21)$$

If we assume that the reflected molecules possess both translational and internal energy corresponding to the wall temperature  $T_w$ , we then obtain (see § 6.1)

$$\frac{E_w(x_1)}{N(x_1)} = \frac{1}{2} \frac{\kappa+1}{\kappa-1} kT_{eq} = \frac{E_{i\infty}}{N_\infty}.$$

Thus, the equilibrium temperature of the wall of a concave body is equal to

$$T_{eq} = \frac{2}{k} \frac{\kappa-1}{\kappa+1} \frac{E_{i\infty}}{N_\infty}. \quad (2.22)$$

This expression takes a very simple form for a hypersonic stream ( $S \rightarrow \infty$ ); in that case,  $E_{i\infty}$  is proportional to  $N_\infty$ , and

$$T_{eq} = \frac{\kappa-1}{\kappa+1} \frac{m}{k} V^2 = V^2(c_p + c_v). \quad (2.23)$$

It is interesting to note that in this case the equilibrium temperature is the same for the surface elements which molecules reach directly from infinity, and for the shaded elements which receive only molecules reflected by other parts of the body. That result

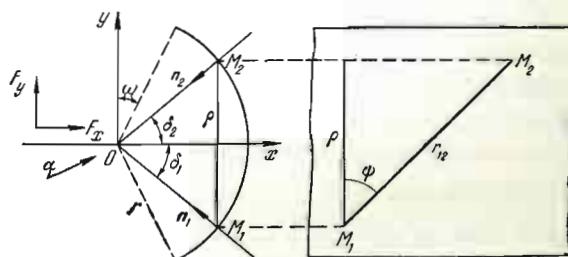


Fig. 56

depends neither on the accommodation coefficient nor on the shape of the body.

As a very simple example we consider a hypersonic gas flow over an infinite cylindrical surface (Fig. 56). We neglect the thermal velocities of the incident molecules. We consider  $\alpha < \omega$ , so that there is no shading of some parts by others.

Using the notation of Fig. 56, we have \*

$$\left. \begin{aligned} \cos \theta_1 = \cos \theta_2 &= \frac{\rho}{2r} \cos \psi, & r_{12} &= \frac{\rho}{\cos \psi}, & dA_2 &= \frac{r\rho d\delta_1 d\psi}{\cos^2 \psi}, \\ \rho &= 2r \sin \left| \frac{\delta_1 - \delta_2}{2} \right|, & G(x_1, x_2) &= \frac{\cos \theta_1 \cos \theta_2}{\pi r_{12}^2} = \frac{\cos^4 \psi}{4\pi r^2}. \end{aligned} \right\} \quad (2.24)$$

In that notation, Eq. (2.11) takes the form

$$N(x_1) = Vn_\infty \cos(\delta_1 - \alpha) + \frac{1}{2\pi} \int N(x_2) \cos^2 \psi \sin \left| \frac{\delta_1 - \delta_2}{2} \right| d\delta_2 d\psi. \quad (2.25)$$

Since  $N(x_2)$  does not depend on  $\psi$ , we may carry out the integration with respect to  $\psi$ :

$$N(\delta_1) = Vn_\infty \cos(\delta_1 - \alpha) + \frac{1}{4} \int_{-\omega}^{+\omega} N(\delta_2) \sin \left| \frac{\delta_1 - \delta_2}{2} \right| d\delta_2. \quad (2.26)$$

We differentiate that expression twice with respect to  $\delta_1$ ; we have

$$\frac{d^2N(\delta_1)}{d\delta_1^2} = -Vn_\infty \cos(\delta_1 - \alpha) - \frac{1}{16} \int_{-\omega}^{+\omega} N(\delta_2) \sin \left| \frac{\delta_1 - \delta_2}{2} \right| d\delta_2 + \frac{N(\delta_1)}{4}. \quad (2.27)$$

By multiplying equations (2.26) by  $1/4$  and combining them with (2.27), we obtain

$$\frac{d^2N(\delta_1)}{d\delta_1^2} = -\frac{3}{4} Vn_\infty \cos(\delta_1 - \alpha). \quad (2.28)$$

\*M. T. Chahine, "Rarefied Gas Dynamics," Second Symposium, Academic Press, New York, 1961; M. T. Pratt, A.I.A.A.J., Vol. 1, No. 7 (1963); E. W. Sparrow, V. K. Johnson, T. S. Lundgreen, and T. S. Chen, J. of Heat Transfer, Vol. 86, No. 1 (1964).

The general solution of that equation,

$$N(\delta_1) = A + B\delta_1 + \frac{3}{4} Vn_\infty \cos(\delta_1 - \alpha), \quad (2.29)$$

contains two arbitrary constants, which depend on the flow geometry, i.e., on  $\alpha$  and  $\omega$ . To determine them we substitute (2.29) into the integral equation (2.26) and equate coefficients of the identical functions of  $\delta$ . We obtain

$$A = \frac{Vn_\infty}{4} \cos \alpha \left( 1 + 2 \sin^2 \frac{\omega}{2} \right), \quad B = \frac{Vn_\infty}{2} \sin \alpha \frac{\cos^3 \left( \frac{\omega}{2} \right)}{2 \cos \left( \frac{\omega}{2} \right) + \omega \sin \left( \frac{\omega}{2} \right)}.$$

Thus, the number of molecules  $N(x_1)$  striking the surface is completely determined. We now calculate the force acting on the body, assuming for simplicity that  $\alpha_e = 1$ , i.e.,  $h_r = h_w$ .

The normal and tangential moment transmitted to the body by the molecules per unit area may be represented in the form

$$p(\delta) = p_{n\infty} + p_{nr} + p_{nb}, \quad \tau(\delta) = p_{t\infty} + p_{tb}, \quad (2.30)$$

where  $p_{nb}$  and  $p_{tb}$  are, respectively, the normal and tangential moment transferred to a surface element by molecules reflected from other parts of the body, and  $p_{nr}$  is the momentum transferred by the re-emitted molecules.

For  $p_{n\infty}$ ,  $p_{nr}$ , and  $p_{t\infty}$ , we evidently have

$$\left. \begin{aligned} p_{n\infty} &= mn_\infty V^2 \cos^2(\alpha - \delta), \\ p_{nr} &= \frac{1}{2} mN(\delta) \sqrt{\frac{\pi}{h_w}}, \\ p_{t\infty} &= mn_\infty V^2 \sin(\alpha - \delta) \cos(\alpha - \delta). \end{aligned} \right\} \quad (2.31)$$

The calculation of  $p_{nb}$  and  $p_{tb}$  is somewhat more complicated. In analogy with (2.10) we write

$$\begin{aligned} p_{nb} &= m \int N(x_2) F_1(x_2, \xi_1) \xi_1^2 \cos^2 \theta_1 d\xi_1, \\ p_{tb} &= -m \int N(x_2) F_2(x_2, \xi_1) \xi_1^2 \cos \psi \cos \delta_1 \cos \theta_1 d\xi_1. \end{aligned}$$

Substituting  $F = 2\pi^{-1}h_w^2 \exp(-h_w^2 \xi^2)$ , and integrating with respect to  $\xi_1$ , we obtain

$$p_{nb} = \frac{3}{4} \frac{m}{\sqrt{h_w}} \int_{A_2} N(x_2) G(x_1, x_2) \cos \theta_1 dA_2,$$

$$p_{tb} = -\frac{3}{4} \frac{m}{\sqrt{h_w}} \int_{A_2} G(x_1, x_2) N(x_2) \cos \psi \sin \delta_1 dA_2.$$

In calculating the momentum we assume that all the molecules move with the same velocity, equal to  $\sqrt{\frac{3}{4}h_w^{-1}}$ . Using the variables of (2.24) and integrating with respect to  $\psi$ , we finally obtain

$$p_{nb} = \frac{m}{2\sqrt{\pi h_w}} \int_{-\omega}^{+\omega} N(\delta_2) \sin^2 \frac{1}{2}(\delta_1 - \delta_2) d\delta_2, \quad (2.32)$$

$$p_{tb} = -\frac{m}{2\sqrt{\pi h_w}} \int_{-\omega}^{+\omega} N(\delta_2) \sin \frac{1}{2}(\delta_1 - \delta_2) \cos \frac{1}{2}(\delta_1 - \delta_2) d\delta_2. \quad (2.33)$$

Hence substituting the expression found earlier for  $N(\delta)$ , and carrying out the quadratures, we can calculate the distribution of normal and tangential forces on the body surface.

If we only need the overall aerodynamic characteristics, we can avoid the calculation of those integrals, since the total forces acting on the body are due only to the momentum transferred to it by molecules arriving from infinity and departing to infinity. The forces due to molecules which move from one part of the body to another are internal. In exactly the same way we can calculate the total energy flux. The momentum and energy carried away by molecules departing for infinity may be written in the form

$$\left. \begin{aligned} P_{r\infty} &= mN(x_1) \int_{\overline{\Omega}(x_1)} F(x_1, \xi_1) (\xi_1 \cdot n_1) \xi_1 d\xi_1, \\ E_{r\infty} &= mN(x_1) \int_{\overline{\Omega}(x_1)} F(x_1, \xi_1) e_r(\xi_1) (\xi_1 \cdot n_1) d\xi_1. \end{aligned} \right\} \quad (2.34)$$

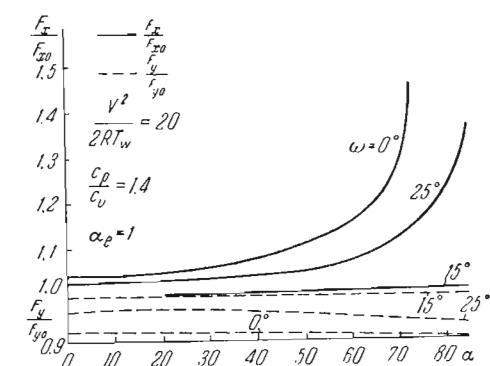


Fig. 57

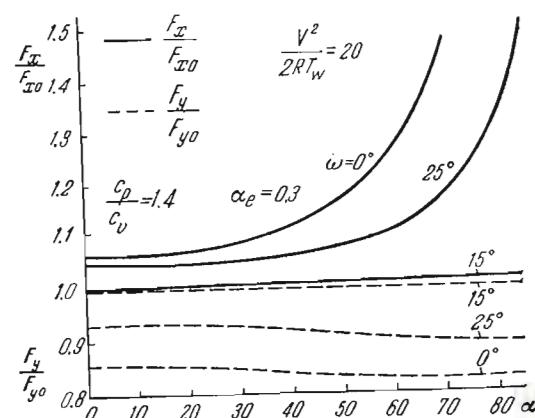


Fig. 58

The calculation of the aerodynamic forces acting on a concave cylinder has been carried out in a number of papers.\* In the paper by Sparrow et al., the calculations were carried out allowing for shading and for an arbitrary  $\alpha_e$ . In the papers by Larish, Pratt, and Chahine, the flow over a concave hemisphere was examined. Some of the results presented by Sparrow et al. are shown in Figs. 57-59.

\*See the papers cited above.

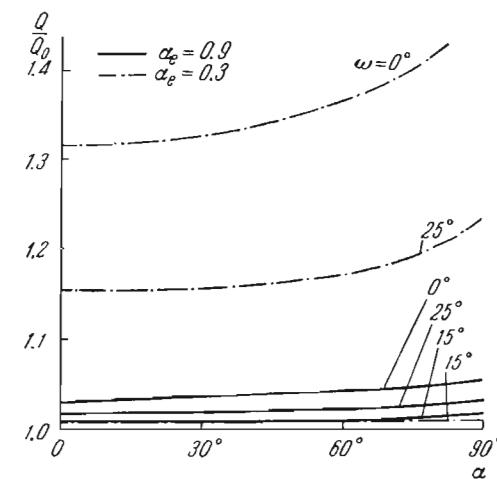


Fig. 59

Figures 57 and 58 show the axial and transverse forces acting on the inner surface of a cylinder (Fig. 56).  $F_{x0}$  and  $F_{y0}$  are the forces calculated without allowing for repeated collisions of reflected molecules with the body. Figure 59 shows the total heat flux on the isothermal internal surface of the body, referred to the flux calculated with no account of the energy brought by molecules reflected from other parts of the body.

### §6.3. Free-Molecule Flows in Tubes

Let us now examine the flow of gas in a tube which joins two containers. Historically that is one of the oldest problems of rarefied gasdynamics.\*

**1.** We first examine two containers separated by an infinitely thin diaphragm with an aperture whose diameter is much smaller than the mean free path of the molecules in both containers. Let the gas in each container be in equilibrium, with pressures  $p_1$  and  $p_2$ , and temperatures  $T_1$  and  $T_2$ , respectively. We shall consider

\*See, for example, Knudsen's book (M. Knudsen, "Kinetic Theory of Gases," 1934), and also M. Knudsen, Ann. Physik, Vol. 28 (1909); M. Smoluchowski, Ann. Physik, Vol. 33 (1910).

the containers to be large enough so that the molecules passing from one container to the other through the small aperture do not change the state of the gas in the containers. After a molecule has passed through the aperture into either container, it collides with other molecules or with the wall at a large distance from the aperture, and can only return near the aperture after experiencing many collisions and coming to equilibrium with the gas in that container.

For a Maxwellian distribution of molecules in the containers, the number of molecules passing from container 1 to container 2 through unit area of aperture per unit time is evidently equal to

$$N_{12} = \int (\xi \cdot n) f_1 d\xi = \frac{n_1}{2V\pi h_1}, \quad (3.1)$$

where  $n$  is the normal to the plane of the aperture. Similarly, the number of molecules passing through the aperture from container 2 to container 1 is equal to

$$N_{21} = \frac{n_2}{2V\pi h_2}. \quad (3.2)$$

Therefore, the total flux of particles from container 1 to container 2 is equal to

$$N = N_{12} - N_{21} = \frac{1}{2V\pi} \left( \frac{n_1}{Vh_1} - \frac{n_2}{Vh_2} \right). \quad (3.3)$$

We know that the condition for no gas flow from one container to the other ( $N = 0$ ), in classical gasdynamics (i.e., when the mean free path is much less than the aperture diameter) is that the pressure be the same in both containers. It may be seen from (3.3) that in a Knudsen gas the mass flow through the aperture is equal to zero if

$$n_1 \sqrt{h_2} = n_2 \sqrt{h_1}$$

or

$$\frac{p_1}{VT_1} = \frac{p_2}{VT_2}, \quad (3.4)$$

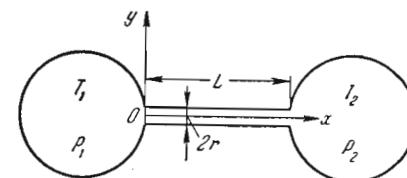


Fig. 60

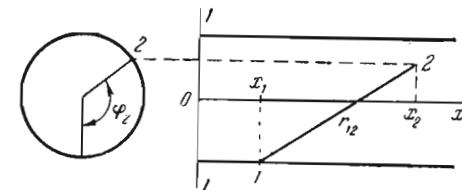


Fig. 61

since

$$p_i = kn_i T_i \quad \text{and} \quad h_i = \frac{m}{2kT_i}.$$

Equation (3.3) has been verified experimentally by Knudsen,\* who investigated the flow of gas through a porous membrane. The discrepancy between theory and experiment proved to be within the range of experimental error.

**2.** Now let the containers be joined by a cylindrical tube of diameter  $2r = d$  and length  $L$  (Fig. 60), where the tube diameter is much smaller than the mean free path. If the molecules were reflected specularly by the wall of the tube, it is clear that each molecule entering the tube at one end would leave at the other end. Therefore, the gas flow would be the same as in the preceding case of flow through an orifice. If the reflection were diffuse, as a result of one or several collisions with the wall, some of the molecules would leave the tube by the end which they entered. Then the gas flow rate would clearly be reduced.

\*See Knudsen's previously cited monograph.

Since the molecules do not collide with each other, the flux of molecules may be divided into two independent streams: a flux from the reservoir on the left, and a flux from the reservoir on the right. Those fluxes may be examined separately.

For instance, we examine the discharge of gas from reservoir 1.\*

Let  $\mathbf{x}_1$  be a point on the inner surface of the tube (Fig. 61). We assume that the molecules are scattered diffusely by the tube surface according to the cosine law. Then, for the number flux of molecules  $N(\mathbf{x}_1)$ , per unit area, we obtain the integral equation (2.11) again; but now  $N_\infty(\mathbf{x}_1)$  is the number of molecules reaching unit area directly from reservoir 1 without collisions:

$$N_1(\mathbf{x}_1) = N_\infty(\mathbf{x}_1) + \int_{A_2} N(\mathbf{x}_2) G(\mathbf{x}_2, \mathbf{x}_1) dA_2, \quad (3.5)$$

where  $A_2$  is the inner surface of the tube and  $dA_2 = r d\varphi dx$ . Since  $N(x)$  does not depend on  $\varphi$ ,

$$\int_{A_2} N(\mathbf{x}_2) G(\mathbf{x}_2, \mathbf{x}_1) r d\varphi dx = r \int_0^L N(\mathbf{x}_2) dx \int_0^{2\pi} \frac{\cos \theta_1 \cos \theta_2}{\pi r_{12}} d\varphi_2.$$

Without restriction of generality, the angle  $\varphi$  may be measured from the point  $\mathbf{x}_1$  so that  $\varphi_1 = 0$ . It is easy to see from Fig. 61 that

$$\frac{\cos \theta_1 \cos \theta_2}{\pi r_{12}} = \frac{4r^2 \sin^4 \frac{\varphi_2}{2}}{\pi \left[ (x_2 - x_1)^2 + 4r^2 \sin^2 \frac{\varphi_2}{2} \right]^2} = \frac{\sin^4 \frac{\varphi}{2}}{4\pi r^2 \left[ (X_2 - X_1)^2 + \sin^2 \frac{\varphi_2}{2} \right]^2},$$

where  $X = x/2r$ . Then

$$G^*(\mathbf{x}_1, \mathbf{x}_2) = 2r^2 \int_0^{2\pi} G(\mathbf{x}_2, \mathbf{x}_1) d\varphi_2 =$$

\*D. Clausing, Ann. Physik, Vol. 12 (1932); G. N. Patterson, UTIA Rev., No. 18 (1962); E. M. Sparrow and V. K. Johnson, J. of Heat Transfer, No. 2 (1963).

$$= \frac{1}{\pi} \int_0^\pi \frac{\sin^4 \frac{\varphi_2}{2} d\varphi_2}{\left[ (X_2 - X_1)^2 + \sin^2 \frac{\varphi_2}{2} \right]^2} = 1 - |X_2 - X_1| \frac{2(X_2 - X_1)^2 + 3}{2[1 + (X_2 - X_1)^2]^{3/2}}. \quad (3.6)$$

We now calculate  $N_\infty(\mathbf{x}_1)$ . Let  $dA_2$  be a surface element of the inlet. Then the number of molecules passing through the element  $dA_2$  and falling on the element of tube surface  $dA_1$  is equal to

$$\frac{n_1}{2\sqrt{\pi h_1}} G(\mathbf{x}_2, \mathbf{x}_1) dA_2 dA_1, \quad \left( h_1 = \frac{m}{2\pi T_1} \right), \quad (3.7)$$

where  $n_1$  is the number density of molecules in reservoir 1. Integrating that expression over the entrance cross section  $\pi r^2$ , following manipulations which are unwieldy but not difficult, we obtain

$$N_\infty(\mathbf{x}_1) = \frac{1}{2} N_{01} \left[ \frac{1+2X_1^2}{\sqrt{1+X_1^2}} - 2X_1 \right], \quad \left( N_{01} = \frac{n_1}{2\sqrt{\pi h_1}} \right), \quad (3.8)$$

where  $N_{01}$  is the number of molecules entering the tube from container 1 per unit inlet area.

We note that if we replace  $N_{01}$  by  $N(\mathbf{x}_1)$  in (3.7), expression (3.7) will determine the number of molecules from element  $dA_1$  which leave the tube through element  $dA_2$ . Therefore, the number of molecules reflected from  $dA_1$  and leaving without collisions with the tube through the left end of the tube (into container 1) is equal to

$$N_{x_1 \rightarrow 1} dA_1 = N(\mathbf{x}_1) \frac{1}{2} \left[ \frac{1+2X_1^2}{\sqrt{1+X_1^2}} - 2X_1 \right] dA_1 = N(\mathbf{x}_1) \chi_{x_1 \rightarrow 1} dA_1. \quad (3.9)$$

Similarly, the number of molecules leaving through the right end of the tube is equal to

$$N_{x_1 \rightarrow 2} dA_1 = N(\mathbf{x}_1) \frac{1}{2} \left[ \frac{1+2(L-X_1)^2}{\sqrt{1+(L-X_1)^2}} - 2(L-X_1) \right] dA_1 \\ = N(\mathbf{x}_1) \chi_{x_1 \rightarrow 2} dA_1. \quad (3.10)$$

The functions  $\chi_{x_1 \rightarrow 1}$  and  $\chi_{x_1 \rightarrow 2}$  determine the probability of exit of a molecule which starts from the element  $dA_1$ , through sections 1 and 2, respectively, without collisions. Taking account of (3.6) and (3.8), and introducing the new variable  $\chi = N(\mathbf{x}_1)/N_{01}$ , we

write Eq. (3.5) in the form  $(\bar{L} = L/d)$

$$\begin{aligned}\chi(X_1) &= \frac{1}{2} \left[ \frac{1+2X_1^2}{\sqrt{X_1^2+1}} - 2X_1 \right] \\ &+ \int_0^{\bar{L}} \left\{ 1 - |X_2 - X_1| \frac{2(X_2 - X_1)^2 + 3}{2[1 + (X_2 - X_1)^2]^{3/2}} \right\} \chi(X_2) dX_2. \quad (3.11)\end{aligned}$$

This equation is called the Clausius equation. If Eq. (3.11) is solved, then the mass flow of gas from reservoir 1 through unit area of the entrance is determined by the expression

$$\begin{aligned}N_{12} &= N_{01} - \int_{A_1} N_{x_1 \rightarrow 1} dA_1 \\ &= N_{01} \left[ 1 - 2 \int_0^{\bar{L}} \chi(X_1) \left( \frac{1+2X_1^2}{\sqrt{1+X_1^2}} - 2X_1 \right) dX_1 \right]. \quad (3.12)\end{aligned}$$

The flux of particles from reservoir 2 has thus been calculated completely.

The function  $\chi(x)$  possesses an interesting symmetry property:

$$\chi(\bar{L} - X) = 1 - \chi(X). \quad (3.13)$$

Let us write Eq. (3.11) in the form

$$\chi(X_1) = \chi_{x_1 \rightarrow 1} + \int_0^{\bar{L}} \chi(X_2) G^*(X_1, X_2) dX_2. \quad (3.14)$$

Let  $g(X_1) = N(X_1)/N_{02}$  define the specific flux on element  $dA_1$ , due to the flux of molecules with density  $N_{02}$  from the right container. Then

$$g(X_1) = \chi_{x_1 \rightarrow 2} + \int_0^{\bar{L}} g(X_2) G^*(X_1, X_2) dX_2. \quad (3.15)$$

It is evident that

$$g(X_1) = \chi(\bar{L} - X_1). \quad (3.16)$$

Moreover,

$$\chi_{x_1 \rightarrow 1} + \chi_{x_1 \rightarrow 2} + \int_0^{\bar{L}} G^*(X_1, X_2) dX_2 = 1, \quad (3.17)$$

since the total probability that a molecule starting from the element  $dA_1$  either exits through the end, or strikes the tube wall, is clearly equal to unity. Combining Eqs. (3.14) and (3.15), and taking into account relations (3.16) and (3.17), we obtain

$$\chi(X_1) + \chi(\bar{L} - X_1) - 1 = \int_0^{\bar{L}} [\chi(X_2) + \chi(\bar{L} - X_2) - 1] G^* dX_2.$$

The unique solution of this integral equation is the trivial solution (3.13). The property (3.13) permits us to calculate the function  $\chi$  for half of the tube, and it is always true that

$$\chi\left(\frac{L}{2}\right) = \frac{1}{2}.$$

Calculations show that the dependence of  $\chi$  on  $X$  is nearly linear. Figure 62 shows the variation of gas flow rate between the two reservoirs, obtained by numerical integration of (3.11).\* For comparison, Clausius's solution, obtained by replacing  $\chi$  by the linear function  $\chi(x) = \alpha + (1 - 2\alpha)(x/L)$  is also shown.

The gas flow rate  $N$  is referred to the flow rate  $N_0$ , which would occur if the containers were joined directly ( $L = 0$ ). The quantity  $\frac{4}{3}(d/L)N_0$  gives the flow rate for very long tubes.†

If the function  $\chi$  is known, then simple quadratures give the variation of the gas flow rate over the channel cross section. Results of such calculations are shown in Fig. 63‡ (ρ is the distance from the tube axis). Note the slip at the wall shown in the figure.

For a tube of variable cross section it is also easy to construct an integral equation similar to (3.11).¶

\*The calculations are from the paper by Sparrow and Johnson cited above.

†That formula for very long tubes was obtained in the papers by Knudsen and Smoluchowski referred to at the beginning of the section.

‡E. M. Sparrow and A. Hajj-Sheikh, Phys. Fluids, Vol. 7, No. 8 (1964).

¶I. P. Otmakhova, Vestn. Mosk. Univ., No. 6 (1959); E. M. Sparrow and V. K. Johnson, A.I.A.A. J., Vol. 1, No. 5 (1963); R.P. Torkowski, J. Phys. Chem., Vol. 67, No. 2 (1963).

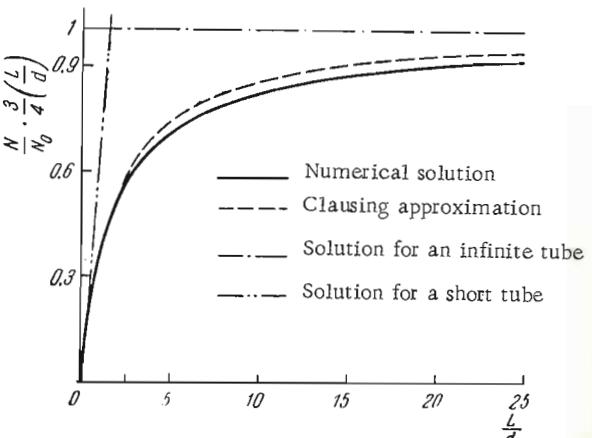


Fig. 62

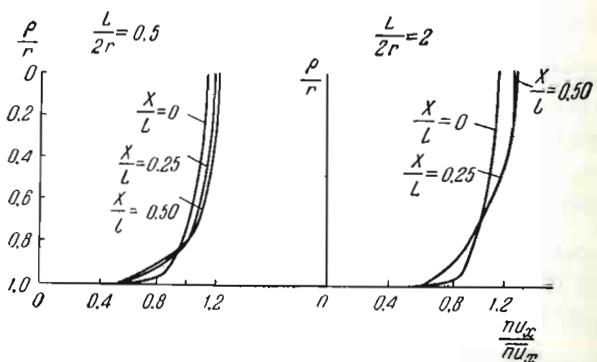


Fig. 63

Equations of the Clausing type are also obtained when we examine flows in complicated channels with diaphragms similar to the traps in vacuum equipment. It is important to note that the gas flow rate through such channels, when there is cosine reflection, depends only on the temperature and pressure of the gas at the entrance and exit of the channel, and does not depend on the channel wall temperature. Therefore, for example, cooling of the walls of traps, in high-vacuum pumps, can have no

effect unless their temperature is low enough to condense the gas (condensed molecules "stick" to the walls).

3. For tubes of moderate length, the Monte Carlo method may be applied effectively. The gas flow rate through a tube with an entrance section  $A_1$  and an exit section  $A_2$  is equal to

$$N = N_{01}P_{1 \rightarrow 2} - N_{02}P_{2 \rightarrow 1}, \quad (3.18)$$

where  $N_{0i}$  is the number of molecules entering the tube through section  $A_i$ ;  $P_{1 \rightarrow 2}$  is the probability that a molecule entering through section 1 leaves through section 2, and  $P_{2 \rightarrow 1}$  is the probability that a molecule entering through section 2 leaves through section 1.

If the distribution of molecules of the gas in both containers is Maxwellian, then

$$A_1P_{1 \rightarrow 2} = A_2P_{2 \rightarrow 1}. \quad (3.19)$$

In fact, when the pressures and temperatures of the gas in the two containers are equal, then  $N = 0$  and  $N_{01}$  and  $N_{02}$  are proportional to the cross-sectional areas  $A_1$  and  $A_2$ ; also, the probability of free flight is the same for similar entrance molecular distribution functions. Relation (3.19) allows us to restrict the calculation to the probability  $P_{1 \rightarrow 2}$ .

For an arbitrary entrance, molecular distribution function and arbitrary molecular reflection from the channel wall, the probability  $P_{1 \rightarrow 2}$  may be determined as follows. Two random numbers are drawn to determine a point  $(y, z)$  of the channel entrance section. Then, with a probability density corresponding to the distribution function of the particles entering the channel, three random numbers are drawn at  $(y, z)$  to determine the direction and velocity of the particle entering the channel. If the chosen direction allows the particle to leave through section 2, then unity is stored in the memory. Otherwise, there is an impact point on the wall. At that point a draw is made for the direction and velocity of the reflected particle, in accordance with the particle reflection law (probability density §6.2). Three results are possible:

1. The molecule leaves through section 2, and unity is stored in the memory.

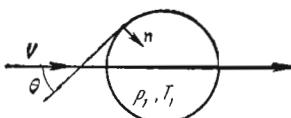


Fig. 64

tinued until the particle leaves through section 2 (result 1) or section 1 (result 0). Denoting by  $N_i$  the number of positive results (the sum of all the units) and by  $N$  the total number of draws, we evidently have

$$P_{1 \rightarrow 2} = \lim_{N \rightarrow \infty} \frac{N_i}{N}.$$

For special channel configurations and special laws of reflection, the procedure may be simplified. In particular, if the molecules are reflected according to a cosine law, there is no need to make a draw for the velocity of the reflected particle.\*

#### §6.4. Devices for Pressure Measurement in Free-Molecule Flow

The simplest device for pressure measurement in free-molecule flow takes the form of a reservoir with a small orifice (Fig. 64), whose diameter must be much less than the molecular mean free path both of the incident stream and in the enclosure.† If that device (reservoir) is located in a moving stream, its dimensions also must be smaller than the mean free path of molecules in the incident stream, since, otherwise, the flux of molecules incident on the aperture will be disturbed by the device. The pressure in the reservoir (pressure measuring device) is determined by the condition that the number of molecules of the incident stream which pass through the orifice into the reservoir equals the number of molecules leaving the reservoir. If the normal to the entrance plane of the reservoir makes an angle  $\frac{1}{2}\pi - \theta$  with the velocity vector  $\mathbf{V}$  of a uniform equilibrium stream, then, according to (1.11), the number of molecules passing through unit

\*See, for example, D. H. Davis, L. L. Levenson, and N. Milleron, "Rarefied Gas Dynamics," Second Symposium, Academic Press, New York, 1961. See also the previous section.

†See, for example, G. N. Patterson, UTIA Rev., No. 18 (1962).

2. The molecule leaves through section 1, and zero is stored in the memory.

3. The molecule again strikes the channel wall. In that case, a draw is made for the new direction and velocity of the particle. The process is con-

area of the orifice per unit time is equal to

$$N_t = \frac{n_\infty}{2\sqrt{\pi h_\infty}} [e^{-S_\theta^2} + \sqrt{\pi} S_\theta (1 + \operatorname{erf} S_\theta)]. \quad (4.1)$$

The number of molecules leaving through unit area of the orifice is clearly equal to

$$N_r = \frac{n_1}{2\sqrt{\pi h_1}} = \frac{p_1}{\sqrt{2\pi m k T_1}}, \quad (4.2)$$

where the subscript 1 denotes the parameters of the gas in the reservoir.

By equating the number of entering and leaving molecules, we obtain the following expression for the pressure in the reservoir:

$$p_1 = p_\infty \sqrt{\frac{T_1}{T_\infty}} [e^{-S_\theta^2} + \sqrt{\pi} S_\theta (1 + \operatorname{erf} S_\theta)]. \quad (4.3)$$

If we measure  $p_1$  for three values of  $\theta$ , we can determine the velocity ratio  $S$  without knowing the temperature and pressure of the oncoming stream or the reservoir temperature. For example, we denote the total pressure reading (i.e.,  $\theta = \pi/2$ ) by  $p'_0$ , the reading of the device with  $\theta = -\pi/2$  by  $p_3$ , and the static pressure reading ( $\theta = 0$ ) by  $p_s$ . Then,

$$\left. \begin{aligned} p'_0 &= p_\infty \sqrt{\frac{T_1}{T_\infty}} [e^{-S^2} + \sqrt{\pi} S (1 + \operatorname{erf} S)], \\ p_3 &= p_\infty \sqrt{\frac{T_1}{T_\infty}} [e^{-S^2} - \sqrt{\pi} S (1 - \operatorname{erf} S)], \end{aligned} \right\} \quad (4.4)$$

whence

$$S = \frac{p'_0 - p_3}{2\sqrt{\pi} p_s}. \quad (4.5)$$

For large velocities (large  $S$ ) the pressure  $p_3$  may be neglected in comparison with the pressure  $p'_0$ , and to determine  $S$  we require only two measurements:

$$S = \frac{p'_0}{2\sqrt{\pi} p_s}. \quad (4.5a)$$

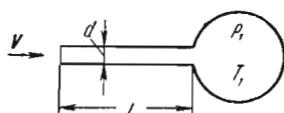


Fig. 65

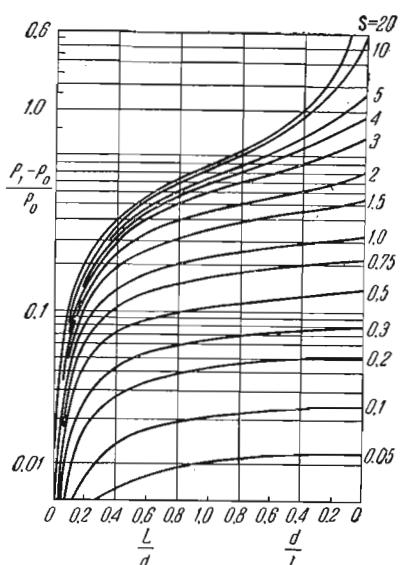


Fig. 66

We note that the pressure indicated by the device is not equal to the pressure on a surface element located at the same place as the orifice.

This device takes a very simple form when the orifice is located directly at the pressure gauge; it is convenient for measurements in free-molecule streams of large dimensions (for example, in free flight). A manometer, for example, a Pirani-type manometer, can serve simultaneously as a reservoir. However, its range of application is limited by the requirement that the mean free path of the incident stream be larger than both the orifice and the device.

We therefore examine total pressure gauges of the traditional type, namely of a tube of diameter much smaller than the mean free path (Fig. 65). The length of the tube must be chosen so that the reservoir induces no disturbance of the stream near the open end of the tube.

In steady flow, the pressure in the reservoir must be such that the flux of molecules from the reservoir through the tube is equal to the flux of molecules through the tube into the reservoir. Since the molecules do not collide among themselves in the tube, the two fluxes can be calculated independently.

The discharge from a container through a tube was examined in § 6.3. The motion of incident molecules along the tube differs from the discharge from a container only through the distribution function of molecules at the entrance of the tube. Therefore, without going into details, we present the final curve after Rotte and Leew\* (Fig. 66), which is of interest for applications. Here,  $p_0$  is the pressure which would occur in the reservoir for  $L = 0$ . These results were obtained in the Clausing approximation, i.e., with a linear approximation function  $\chi(X)$ .

### §6.5. Nearly Free-Molecule Flows

Let us examine high Knudsen number flows with collisions. We write the Boltzmann equation for steady flow in the absence of external forces in the dimensionless form (see § 2.11):

$$\xi_i \frac{\partial f}{\partial x_i} = \epsilon J(f, f), \quad (\epsilon = \frac{L}{\lambda} = \frac{1}{Kn} = \frac{n_0 \sigma L G}{U} \ll 1). \quad (5.1)$$

Here the coordinates of physical space are referred to a characteristic dimension of the flow or of the immersed body  $L$ , and the

\*D.E. Rotte and J.H. Leew, A.I.A.A.J., Vol. 1, No. 1 (1963); see also D.L. Chambre, S.A. Schaaf, J. Aerospace Sci., Vol. 15, No. 12 (1948); A.I. Ivanovskii, Tr. Tsentr. Aerolog. Obser., No. 56 (1964); J.C. Whang, A.I.A.A.J., Vol. 1, No. 8 (1963); S.J. Townsend, G. N. Patterson, and S. R. M. Sinclair, "Rarefied Gas Dynamics, Fourth Symposium, Academic Press, New York, 1965. Conical gauges are examined in the last two papers. Free-molecule flow in a gauge, mounted at an angle of attack with respect to a stream, was examined in the paper by P. C. Hughes and J. H. Leew, in: "Rarefied Gas Dynamics," Fourth Symposium, Academic Press, New York, 1965.

velocity of the molecules to the characteristic velocity  $U$ . We will show below that, in general, at any point  $\mathbf{x}$  the ratio of  $U$  to the characteristic velocity of the molecules  $G$  may be of different order for different groups of molecules. Therefore, the flow cannot be characterized by a single parameter or by a single Knudsen number. However, for simplicity, we characterize the flow in this section by a single parameter  $\epsilon$  or, in other words, assume that all the molecules have the same mean free path  $\lambda$ .

It is natural to seek a solution of (5.1) either as a series in  $\epsilon$ , or by the method of successive approximations, starting from the solution for free-molecule flow ( $\epsilon \rightarrow 0$  or  $Kn \rightarrow \infty$ ).

We represent the function  $f$  as a series (see § 3.6):

$$f(\mathbf{x}, \xi) = f^{(0)}(\mathbf{x}, \xi) + \epsilon f^{(1)}(\mathbf{x}, \xi) + \dots \quad (5.2)$$

Substituting this expansion into Eq. (5.1) and equating terms with equal powers of  $\epsilon$ , we obtain

$$\xi_l \frac{\partial f^{(0)}}{\partial x_l} = 0, \quad \xi_l \frac{\partial f^{(1)}}{\partial x_l} = J(f^{(0)}, f^{(0)}), \quad \xi_l \frac{\partial f^{(2)}}{\partial x_l} = 2J(f^{(0)}, f^{(1)}), \dots \quad (5.3)$$

The first of these equations is already familiar to us as the equation for free-molecule flow. Thus, the solution of the nonlinear integro-differential Boltzmann equation has been reduced to the solution of a recurrent system of linear differential equations.

It is easy to see that the method of successive approximations leads to a system of equations equivalent to (5.3), if we take the free-molecule solution as the first approximation and, to obtain each successive approximation, we substitute the preceding solution into the right side of the Boltzmann equation.

In fact, let us denote the  $N$ -th approximation to the distribution function by  $F^{(N)}$ . Let  $F^{(0)}$  be the free-molecule solution of the problem. Then,

$$\xi_l \frac{\partial F^{(0)}}{\partial x_l} = 0, \quad \xi_l \frac{\partial F^{(1)}}{\partial x_l} = \epsilon J(F^{(0)}, F^{(0)}), \quad \xi_l \frac{\partial F^{(2)}}{\partial x_l} = \epsilon J(F^{(1)}, F^{(1)}), \dots \quad (5.4)$$

By comparing Eqs. (5.3) and (5.4), it is easy to find that, up to higher-order terms,

$$F^{(0)} = f^{(0)}, \quad F^{(1)} = f^{(0)} + \epsilon f^{(1)}, \quad F^{(2)} = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \dots$$

The method of successive approximations may be modified somewhat by applying it to one of the integral forms of the Boltzmann equations.

We write the Boltzmann equation in the form [compare formula (7.1) of Chapter II]:

$$\xi \frac{\partial f}{\partial l} = \epsilon J = \epsilon [J_1(\mathbf{x}, \xi) - f(\mathbf{x}, \xi) J_2(\mathbf{x}, \xi)]. \quad (5.5)$$

Here the differentiation is taken along the trajectories of the molecules of velocity  $\xi$ . We write that equation in the integral forms [compare (7.4) and (7.5) of Chapter II]:

$$f(\mathbf{x}_2, \xi) = f(\mathbf{x}_1, \xi) + \frac{\epsilon}{\xi} \int_{l_1}^{l_2} [J_1(\tau, \xi) - f(\tau, \xi) J_2(\tau, \xi)] d\tau, \quad (5.6)$$

$$f(\mathbf{x}_2, \xi) = f(\mathbf{x}_1, \xi) \exp \left\{ -\frac{\epsilon}{\xi} \int_{l_1}^{l_2} J_2(s, \xi) ds \right\} \\ + \frac{\epsilon}{\xi} \int_{l_1}^{l_2} J_1(\tau, \xi) \exp \left\{ -\frac{\epsilon}{\xi} \int_{\tau}^{l_2} J_2(s, \xi) ds \right\} d\tau, \quad (5.7)$$

where  $\mathbf{x}_2$  is any point on the straight line through the point  $\mathbf{x}_1$  in the direction of the vector  $\xi$ , while  $l$ ,  $\tau$ , and  $s$  are coordinates calculated along this line.

The successive approximations are constructed, as above, by substituting the preceding approximation into the right side of the equation. There is a certain arbitrariness in the choice of the function  $f(\mathbf{x}_1, \xi)$ . In each approximation in the calculation of the function  $f(\mathbf{x}_2, \xi)$  we can integrate from the boundaries of the flow region, i.e., choose the points  $\mathbf{x}_1$  to belong to the boundary. However, we can proceed in smaller steps, by choosing, for example,  $\mathbf{x}_1 = \mathbf{x}_2 - \xi \Delta t$ , where  $\Delta t$  is some time interval. The smaller the interval  $\Delta t$ , the larger the number of approximations required to obtain the desired solution. On the other hand, we may expect that when  $\Delta t$  is reduced, the process will converge in cases where the process does not converge for larger  $\Delta t$  or when  $\mathbf{x}_1$  is chosen on the boundary.

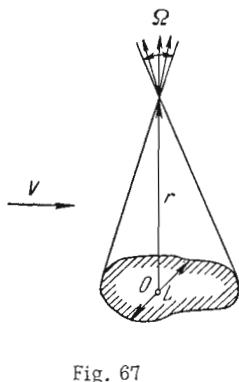


Fig. 67

Although the conditions for convergence have not been clearly established for any of the calculation schemes mentioned above, it appears that to calculate flows in a finite region of characteristic dimension of order  $L$ , they are all roughly equivalent.\*

However, it is easy to show that for an infinite region the method of successive approximations, based on Eqs. (5.3), (5.4), and (5.6) leads to the appearance of secular terms.

Let us examine the flow of an infinite uniform equilibrium stream of characteristic mean free path  $\lambda$  past a body of characteristic dimension  $L \gg \lambda$ . We assume, for definiteness, that the molecules are reflected diffusely from the body surface. In the zeroth (free-molecule) approximation the distribution function of the molecules at an arbitrary point of flow differs from the distribution function of the incident stream only for molecules reaching the body with the solid angle  $\Omega$  (Fig. 67). Therefore, the collision integral  $J(\mathbf{x}, \xi)$  decreases with increasing distance from the body as  $\Omega \sim 1/r^2$  for all velocities  $\xi$  lying outside  $\Omega$ . Along trajectories leading away from the body, the collision integral does not decrease. Therefore, the distribution function of the first approximation  $F^{(1)}$  or  $f^{(1)}$  increases without limit with increasing distance from the body along those trajectories.

Indeed, that result should have been expected, because we assumed that the solution of Eq. (5.1) was given by a series (5.2), in which the functions  $f^{(v)}$  depend only on the variable  $\mathbf{x} = \mathbf{x}^*/L$  and do not depend on  $\epsilon$ .† Actually, the functions  $f^{(v)}$  may depend on  $\mathbf{x}$  in the combinations  $\mathbf{x} = \mathbf{x}_0$ ,  $\epsilon\mathbf{x} = \mathbf{x}_1$ ,  $\epsilon^2\mathbf{x} = \mathbf{x}_2$ , and so on. Therefore, instead of (5.2), in the general case we should seek a solution in the form (see §§2.3 and 3.7)

$$\begin{aligned} f(\mathbf{x}, \epsilon\mathbf{x}, \epsilon^2\mathbf{x}, \dots, \xi, \epsilon) &\equiv f(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots, \xi, \epsilon) \\ &= f^{(0)}(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots, \xi) + \epsilon f^{(1)}(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots, \xi) + \dots \quad (5.2a) \end{aligned}$$

\*We have already used similar schemes when examining Couette flows in §4.2.

†Here,  $\mathbf{x}_i^*$  are dimensionless coordinates.

Then for the derivatives we have

$$\frac{\partial f}{\partial \mathbf{x}} = \frac{\partial f}{\partial \mathbf{x}_0} + \epsilon \frac{\partial f}{\partial \mathbf{x}_1} + \epsilon^2 \frac{\partial f}{\partial \mathbf{x}_2}.$$

By substituting that expression for the derivatives and the series (5.2a) into Eq. (5.1), and equating coefficients of equal powers of  $\epsilon$ , we obtain, instead of equations (5.3),

$$\left. \begin{aligned} \xi \cdot \frac{\partial f^{(0)}}{\partial \mathbf{x}_0} &= 0, \quad \xi \cdot \frac{\partial f^{(1)}}{\partial \mathbf{x}_0} + \xi \cdot \frac{\partial f^{(0)}}{\partial \mathbf{x}_1} = J(f^{(0)}, f^{(0)}), \\ \xi \cdot \frac{\partial f^{(2)}}{\partial \mathbf{x}_0} + \xi \cdot \frac{\partial f^{(1)}}{\partial \mathbf{x}_1} + \xi \cdot \frac{\partial f^{(0)}}{\partial \mathbf{x}_2} &= 2J(f^{(0)}, f^{(1)}) \dots \end{aligned} \right\} \quad (5.3a)$$

It is easy to see that  $\mathbf{x}_1 = \epsilon\mathbf{x} = \epsilon\mathbf{x}^*/L = \mathbf{x}^*/\lambda$ , i.e., the coordinates  $\mathbf{x}_1$  are coordinates on the  $\lambda$ -scale. It may be seen from the second equation of (5.3a) that  $f^{(1)}$  does not grow without limit as  $\mathbf{x}_0 \rightarrow \infty$  if, in the  $\lambda$ -scale, the function  $f^{(0)}$  satisfies the equation

$$\xi \cdot \frac{\partial f^{(0)}}{\partial \mathbf{x}_1} = J(f^{(0)}, f^{(0)}),$$

i.e., once again the Boltzmann equation. Now, by letting  $\mathbf{x}_1$  go to infinity in that equation, we see that the function  $f^{(0)}(\mathbf{x}_2)$  in the  $\lambda/\epsilon$ -scale satisfies the equation

$$J(f^{(0)}, f^{(0)}) = 0.$$

It follows from the third equation of (5.3a) that  $f^{(1)}$  in this scale satisfies the equation

$$\xi \cdot \frac{\partial f^{(0)}}{\partial \mathbf{x}_2} = 2J(f^{(0)}, f^{(1)}).$$

It is easy to see that the last two equations belong to the recurrent system of integral equations for the functions  $f^{(v)}$  in the Hilbert-Enskog theory.\* Therefore, at distances much greater than  $\lambda$ , i.e., when  $\mathbf{x}^* = O(\lambda/\epsilon)$ , the solution of the Boltzmann equation tends to the Hilbert solution, and if we restrict attention to two terms of the series (5.3a), then the flow may be described by the Navier-Stokes equations (see §3.8). Near the body, i.e., on

\*See §§3.7 and 3.8.

the L-scale, the solution of the Boltzmann equation for small  $\epsilon$  can also be simplified, since it converges to the solution of the recurrent system of differential equations (5.3a). However, in the intermediate region, i.e., at distances of the order  $\lambda$  from the body, the flow is described by integro-differential equations (the Boltzmann equation for  $f^{(0)}$  and linear equations for  $f^{(v)}$  with  $v \geq 1$ ) and, therefore, the investigation of the flow as a whole cannot be simplified.

We limit our problem to the determination only of the first correction to the free-molecule value for the distribution function on the body.

We examine the model equation, which we write in the same dimensionless coordinates as in Eqs. (5.1) and (5.2):

$$\xi \frac{df}{dl} = \epsilon n(f_0 - f). \quad (5.8)$$

We write the solution of that equation in the form

$$f(x_2, \xi) = f(x_1, \xi) \exp \left\{ -\frac{\epsilon}{\xi} \int_{l_1}^{l_2} n ds \right\} + \frac{\epsilon}{\xi} \int_{l_1}^{l_2} n f_0(\tau, \xi) \exp \left\{ -\frac{\epsilon}{\xi} \int_{\tau}^{l_2} n ds \right\} d\tau. \quad (5.9)$$

For example, we seek the distribution function at some point  $x_2 = x_w$  on the body.

We consider a trajectory which arrives at the point  $x_2$  from infinity. On that trajectory we choose a point  $x_1$  located at a sufficiently large distance  $R$  from the body. It is easy to see that far from the body the hydrodynamic quantities  $n$ ,  $u_i$ , and  $T$  differ from their values at infinity by quantities of order  $1/r^2$ . Therefore, at large distances

$$f_0 = f_\infty + \frac{\varphi_1(\xi)}{r^2} + \epsilon \frac{\varphi_2(\xi)}{r^2} + \dots \quad (5.10)$$

Let us assume that for  $r > R$  the expansion (5.10) and similar expansions for the hydrodynamic quantities are valid. At the same time, since we are interested in the asymptotic solution as  $\epsilon \rightarrow 0$ , we can always choose  $\lambda$  large enough so that  $R \ll \lambda$ .

The distribution function at the point  $x_2$  is determined by Eq. (5.9) with the integration carried out from infinity. Then the first term in (5.9) clearly vanishes, and we have

$$f(x_w, \xi) = \frac{\epsilon}{\xi} \int_{l_\infty}^{l_w} n f_0(\tau, \xi) \exp \left\{ -\frac{\epsilon}{\xi} \int_{\tau}^{l_w} n ds \right\} d\tau, \quad (5.11)$$

where the limit  $l_\infty$  indicates that the integration is started at infinity and quantities referring to point  $x_2$  on the body surface are denoted by the subscript w.

We write (5.11) in the form

$$f(x_w, \xi) = \frac{\epsilon}{\xi} \int_{l_\infty}^{l_w} n f_\infty(\xi) \exp \left\{ -\frac{\epsilon}{\xi} \int_{\tau}^{l_w} n ds \right\} d\tau + \frac{\epsilon}{\xi} \int_{l_\infty}^{l_w} n (f_0 - f_\infty) \exp \left\{ -\frac{\epsilon}{\xi} \int_{\tau}^{l_w} n ds \right\} d\tau = f_\infty + \frac{\epsilon}{\xi} \int_{l_\infty}^{l_w} n (f_0 - f_\infty) \exp \left\{ -\frac{\epsilon}{\xi} \int_{\tau}^{l_w} n ds \right\} d\tau. \quad (5.12)$$

We divide the last integral into three integrals:

$$\frac{\epsilon}{\xi} \int_{l_\infty}^{l_w} n (f_0 - f_\infty) \exp \left\{ -\frac{\epsilon}{\xi} \int_{\tau}^{l_w} n ds \right\} d\tau = \frac{\epsilon}{\xi} \int_{l_\infty}^{l_\lambda} [\ ] d\tau + \frac{\epsilon}{\xi} \int_{l_\lambda}^{l_1} [\ ] d\tau + \frac{\epsilon}{\xi} \int_{l_1}^{l_w} [\ ] d\tau, \quad (5.13)$$

where for brevity the integrand on the left has been replaced by square brackets;  $l_1$  is the value of the variable  $\tau$  at the intersection of the line of integration with the sphere  $r = R$ , and  $l_\lambda$  is the value of  $\tau$  at its intersection with the sphere of radius  $r = 1/\epsilon$ , i.e., with the sphere whose radius is equal to the mean free path  $\lambda$ .

In the first and second integrals, up to terms of order  $1/r^2$ , we can replace integration with respect to  $\tau$  by an integration with

respect to  $r$ . Since, when  $r > R$ , the difference  $f_0 - f_\infty$  may be represented in the form (5.10), we easily obtain the result that the first integral is of order  $\epsilon^2$ . In fact,

$$\begin{aligned} & \frac{\epsilon}{\xi} \int_{1/\epsilon}^{\infty} n \frac{\varphi_1}{r^2} \exp \left\{ -\frac{\epsilon}{\xi} \int_{r_w}^r n ds \right\} dr \\ &= \frac{\epsilon^2}{\xi} \int_1^{\infty} \frac{n \varphi_1}{x^2} \exp \left\{ -\frac{1}{\xi} \int_{r_w}^x n dy \right\} dx \sim O(\epsilon^2). \end{aligned} \quad (5.14)$$

In exactly the same way it is easy to see that the second integral is equal to

$$\begin{aligned} & \frac{\epsilon^2}{\xi} \int_{\epsilon R}^1 \frac{n \varphi_1}{x^2} \exp \left\{ -\frac{1}{\xi} \int_{r_w}^x n dy \right\} dx \\ &= \epsilon \frac{n(R) \varphi_1(\xi, R)}{\xi R} + O\left(\frac{\epsilon}{R^2}\right) + O(\epsilon^2 \ln \epsilon). \end{aligned} \quad (5.15)$$

We note that replacing the exponential under the integral sign by unity introduces an error of order  $\epsilon^2$ .

It is clear that in the last integral of (5.13) the exponential may also be replaced by unity, since the integration is carried out at distances of order unity. Thus, the distribution function on the body may be represented in the form

$$f(x_w, \xi) = f_\infty(\xi) + \frac{\epsilon}{\xi} \int_{l_\lambda}^{l_w} n(f_0 - f_\infty) d\tau + O(\epsilon^2 \ln \epsilon). \quad (5.16)$$

Therefore, the distribution function on the body,  $f(x_w, \xi)$ , differs from the free-molecule distribution function  $f_\infty(\xi)$  by a quantity of order  $\epsilon$ . Similarly, we can show that at any point  $x_2$  in a sphere of radius  $R$ ,

$$f(x_2, \xi) = f_\infty(\xi) + O(\epsilon) \text{ for } \xi \text{ outside } \Omega \quad (5.17)$$

and

$$f(x_2, \xi) = f_r^{(0)}(x_w, \xi) + O(\epsilon) \text{ for } \xi \text{ inside } \Omega \quad (5.18)$$

Here,  $f_r^{(0)}$  is the free-molecule value of the distribution function for the reflected molecules at the body.

Similarly, outside the sphere of radius  $R$ , taking account of (5.10), we have

$$f(x_2, \xi) = f_\infty(\xi) + O\left(\frac{\epsilon}{R}\right) \quad (5.19)$$

for  $\xi$  outside  $\Omega$ , and

$$f(x_2, \xi) = f_r^{(0)}(x_w, \xi) e^{-\frac{\epsilon}{\xi} n_\infty r^2} + f_\infty(\xi) \left( 1 - e^{-\frac{\epsilon}{\xi} n_\infty r^2} \right) + O\left(\frac{\epsilon}{R}\right) \quad (5.20)$$

for  $\xi$  inside  $\Omega$ . The exponential in the first term accounts for the decay of the stream of reflected molecules by the incident stream.

To find the distribution function on the body from formula (5.16), we must calculate the function  $f_0(\xi)$ . Using the estimates of (5.17)–(5.20), we easily find that, dropping terms of order  $\epsilon^2 \ln \epsilon$ , the function  $f_0$  may be replaced by its free-molecule value  $f_0^{(0)}$  if we neglect the attenuation of the reflected and incident streams. Since formula (5.16) only takes into account collisions of the molecules inside a sphere of radius equal to the mean free path, the probability of second collisions is small, and the errors made by neglecting them are of order  $\epsilon^2$ .

A similar order-of-magnitude analysis may easily be made for two-dimensional bodies. It is clear that the perturbations of the hydrodynamic quantities in the plane case fall off as  $1/r$  and, therefore, so does  $f_0 - f_\infty \sim \varphi/r$ . In terms of that estimate, an argument similar to the preceding indicates that the distribution function on the body contains correction terms both of order  $\epsilon \ln \epsilon$ , and of order  $\epsilon$ .\* The first correction may be obtained, as in the three-dimensional case, by calculating only the first collisions within a sphere of radius equal to the mean free path. However, to obtain the correction of order  $\epsilon$ , we must calculate both the attenuation of the reflected molecules and the collisions outside the sphere of radius  $r = \lambda$ , i.e., we must take multiple collisions into account.

\*R. S. Iroshnikov, Inzh. Zh., Vol. 1, No. 3 (1961).

Extending the results obtained for the model equation to the Boltzmann equation, we can assert that in the three-dimensional case the first correction to the distribution function at the body is of order  $\epsilon$  and can be obtained either from Eq. (5.6), or from Eq. (5.7), by substituting the free-molecule distribution function on the right side without allowance for attenuation. On the other hand, the first iteration of the free-molecule flow does not give the correct correction of order  $\epsilon$  at large distances from the body. The convergence is not uniform over space. For a fixed Knudsen number, evidently, the successive iterations also correct the solution there. However, the range of Knudsen numbers for which the iterations converge is not yet clear.

In the plane case, to obtain solutions by successive approximations, taking account of terms of order  $\epsilon$ , we can only use Eq. (5.7), which is strictly valid only for molecules of finite interaction radius. The procedure in this case may be represented as follows.\* The distribution function for the incident molecules  $f_i^{(0)}$  is calculated, allowing for attenuation by the body. It is clear that

$$f_i^{(0)}(\xi) = f_\infty(\xi)$$

for trajectories outside  $\Omega$ , and

$$f_i^{(0)}(\xi) = 0$$

inside  $\Omega$ . The distribution function for the reflected molecules at the body,  $f_r^{(0)}(\mathbf{x}_w, \xi)$ , is found from  $f_i^{(0)}(\xi)$  at the body and from the law of reflection. Then from formula (5.7) we find the distribution function for the reflected molecules,  $f_r^{(0)}(\mathbf{x}, \xi)$ . Here, in (5.7), the integrals  $J_1$  and  $J_2$  are calculated using the function  $f_i^{(0)}$ , and  $f(\mathbf{x}_1, \xi)$  is set equal to  $f_r^{(0)}(\mathbf{x}_w, \xi)$ . Then, from formula (5.7), the distribution function at the body is calculated afresh. Since the integration is carried out from infinity, the first term drops out. The integrals  $J_1$  and  $J_2$  are calculated using the function  $f_i^{(0)}$  outside  $\Omega$  and the function  $f_r^{(0)}$  inside  $\Omega$ .

In most cases the first collision theory is defined as a theory in which only collisions between incident and reflected molecules

\*See, for example, D. R. Willis, "Rarefied Gas Dynamics," First Symposium, Pergamon Press, New York, 1960.

are considered. It was shown above that in the three-dimensional case a calculation based on first collisions is sufficient to obtain the correction of order  $\epsilon$  at the body. However, in the general case, it is necessary to calculate both collisions of the reflected molecules with the incident stream, and collisions of the molecules of both groups among themselves. As the Mach number of the oncoming stream increases, the role of the last two types of collisions decreases. In the limiting case of hypersonic flow ( $M = \infty$ ) there are no collisions among incident molecules. If the velocities of the reflected molecules are much smaller than the velocity of the incident molecules, the momentum and energy brought to the body as a result of collisions among the reflected molecules is small compared to the momentum and energy brought to the body as a result of collisions of the reflected molecules with the incident molecules.

The problem of obtaining the first correction to the distribution function, either formula (5.6), or by formula (5.7), reduces to the calculation of multidimensional integrals with a complicated kernel. The calculations are simplified for a hypersonic incident stream, since the number of the integrations is reduced, and the calculation of the collisions is appreciably simplified.

The calculation of those multiple integrals may be conveniently accomplished by the Monte Carlo method. However, it is simpler to calculate one-collision flows directly by the Monte Carlo method, without writing out the integrals. The function  $f_i^{(0)}(\mathbf{x}, \xi)$  is calculated as before. Knowing  $f_i^{(0)}(\mathbf{x}_w, \xi)$  at the body, we find the function  $f_r^{(0)}(\mathbf{x}_w, \xi)$  from the law of reflection of the molecules. From random numbers uniformly distributed over the surface, we select two to determine the point  $\mathbf{x}_w$  of the surface. Then, by choosing three random numbers with a probability density corresponding to  $f_r^{(0)}(\mathbf{x}_w, \xi)$ , we select a reflected molecule, i.e., we determine its velocity and direction. Two random numbers are then drawn, to correspond to the probabilities of the mean free path of the reflected molecule and to its collision parameters; then we calculate the outcome of the collision of the reflected and incident molecules. If after collision one or both molecules fall into a given cell on the surface of the body, then we store their momentum and energy in that cell. We choose a new reflected molecule, and the calculation is repeated. Here, as above, the calculation is simplified appreciably for hypersonic flow. Examples

of calculations by the Monte Carlo method are presented in the next section.

### §6.6. Hypersonic Nearly Free-Molecule Flows. The Molecular Boundary Layer

It was assumed in the preceding section that the flow is characterized by a single mean free path  $\lambda$ . For moderate Mach number ( $M \sim 1$ ) flows past bodies, the single characteristic mean free path may be taken as the mean free path for molecules at infinity (see §2.11).

To have a free-molecule flow at moderate Mach numbers, it is sufficient to require that the Knudsen number  $\lambda_\infty/L \gg 1$ .<sup>\*</sup> Experiments show that in some cases the free-molecule regime is attained with  $\text{Kn} \approx 2 - 3$ .

We shall show below that, in hypersonic flow, we cannot select a single criterion to define a region of free-molecule or near-free-molecule flow. The criteria turn out to depend on the body shape and the laws of interaction of the molecules among themselves and with the body surface (accommodation coefficients).

Let us examine the special features of hypersonic flows.<sup>†</sup> To obtain qualitative results we restrict our examination to the framework of elementary kinetic theory (see §1.5). Let a body be immersed in an undisturbed equilibrium hypersonic flow of velocity  $\mathbf{V}$ , density  $n_\infty$ , and temperature  $T_\infty$ . The molecular mean free path in a system of coordinates fixed in the gas is equal to

$$\lambda_\infty = \frac{c}{n_\infty \sigma_\infty g}, \quad (6.1)$$

where  $c$  is the mean thermal velocity,  $g$  is the mean relative velocity, and  $\sigma_\infty$  is the collision cross section of the molecules. Since for a Maxwellian distribution the velocities  $c$  and  $g$  are of the same order, (6.1) may be rewritten in the form

$$\lambda_\infty \sim \frac{1}{n_\infty \sigma_\infty}. \quad (6.2)$$

<sup>\*</sup>H. S. Tsien, J. Aeron. Sci., Vol. 13, No. 12 (1946).

<sup>†</sup>In what follows, we follow the work of M. N. Kogan, Prikl. Mat. i Mekhan., Vol. 26, No. 3 (1962).

If the gas motion is considered in a fixed system of coordinates, for example in a coordinate system fixed with respect to a nozzle or a body, then the distance travelled by molecules between collisions in the flow direction is evidently equal to

$$\lambda_{11} \sim \frac{\lambda_\infty}{c} V \sim M \lambda_\infty \quad \text{for } M \gg 1. \quad (6.3)$$

If  $L$  is a characteristic flow dimension, then

$$\frac{\lambda_{11}}{L} \sim \frac{\lambda_\infty}{L} M \sim \text{Kn} M. \quad (6.4)$$

Therefore, between collisions in a hypersonic stream molecules travel a streamwise distance larger by a factor of  $M$  than in the transverse direction. Because of that anisotropy, the presence of comparatively small longitudinal gradients invalidates the continuum equations,<sup>\*</sup> while they remain valid in the presence of large transverse gradients (for example, in the hypersonic boundary layer).

The collision cross section of the molecules enters into the definition of mean free path. As we saw in Chapter I, in the general case the effective collision cross section depends on the relative velocity of the molecules. To estimate the variation in collision cross section, we use the relationship between the mean free path and the viscosity:

$$\mu \sim m n c \lambda_\infty \sim \rho a \lambda_\infty \sim \rho \sqrt{T} \lambda_\infty \sim \frac{\sqrt{T}}{\sigma}. \quad (6.5)$$

Assuming the Sutherland law of variation of viscosity with temperature, we obtain

$$\sigma = A \left( 1 + \frac{S}{T} \right), \quad (6.6)$$

where  $S$  is the Sutherland constant, and  $A$  is a constant. The constant  $S$  for gases such as nitrogen, oxygen, helium, and hydrogen lies in the range 80–140.

If  $\sigma_\infty$  is the collision cross section at temperature  $T_\infty$ , and the relative velocity is  $c \sim a_\infty \sim \sqrt{T_\infty}$ , then at a relative velocity

<sup>\*</sup>That effect appears, for example, in discharge of a gas into a vacuum. See §6.9.

$V$ , the collision cross section is equal to

$$\sigma = \sigma_\infty \frac{T_\infty}{T} \frac{T+S}{T_\infty+S}, \quad (6.7)$$

where  $T \sim T_\infty V^2/a_\infty^2 \sim T_\infty M^2$ .

Under conditions in air the relative velocity of the molecules is of order  $5 \cdot 10^4$  cm/sec (since  $T_\infty \sim 300^\circ\text{K}$ ). With an increase in the relative velocity of the molecules, the cross section may vary by one third at most (as  $T \rightarrow \infty$ ). The comparatively slight variation in collision cross section is due to the fact that even at room temperature the relative velocity of the molecules turns out to be so large that the interaction of the molecules is determined by the steep part of the potential curve. If the temperature  $T_\infty$  is so low that molecular collisions are determined by the remote gently sloping parts of the potential curve, an increase in the relative velocity of the molecules may appreciably change the effective molecular cross section. This phenomenon may be encountered, for example, in hypersonic wind tunnels operating with helium. The temperature of the stream in the working section of the tunnel may be equal to  $5-10^\circ\text{K}$ , while the velocity of the incident molecules relative to molecules reflected from a body placed in the stream may correspond to a temperature of  $T \geq 300^\circ\text{K}$ .

To illustrate the influence of the variation of collision cross section with relative velocity, we consider two limiting cases below: "hard" molecules with  $\sigma = \text{const}$  and "soft" molecules with  $\sigma$  inversely proportional to the relative velocity of the molecules. The latter case corresponds to a Maxwellian gas.

We assume their reflection to be diffuse with a Maxwellian distribution. The mean velocity  $V_2$  of the reflected molecules is determined by the wall temperature  $T_w$  and the accommodation coefficient  $\alpha_e$ .

When we examine the flow over a body we must consider several characteristic path lengths: the mean free path of the incident stream in the field of molecules reflected from the body,  $\lambda_{12}$ ; the mean free path of the reflected molecules among the incident molecules,  $\lambda_{21}$ ; the mean free path of the reflected molecules colliding with each other,  $\lambda_{22}$ ; as well as the path lengths  $\lambda_{11}$  and  $\lambda_\infty$  introduced above. We note, in the general case, that  $\lambda_{12} \neq \lambda_{21}$ .

We consider two typical cases: hypersonic flow past a plate perpendicular and parallel to the incident stream. We also briefly describe the flow over a cone and a wedge.

### 1. Flat Plate Perpendicular to the Stream.

Here the thermal velocity of the molecules of the incident stream is unimportant for a qualitative assessment (since the velocity of sound  $a \ll V$ ). We can therefore assume that a uniform beam of molecules strikes the body at a velocity  $V$ . The mean free path of the molecules  $\lambda_\infty$  in that case characterizes only the density of incident molecules.

The dimensionless parameter which determines the flow is the ratio  $M_2 = V/V_2$  since the velocity  $V_2$  determines the density of the reflected molecules  $n_2$ . From the impermeability condition we have

$$n_2 \sim \frac{n_\infty V}{V_2} \sim n_\infty M_2. \quad (6.8)$$

If the temperature  $T_w$  of the body is of the order of  $T_\infty$  ("cold" body), for example in the case of high-speed flight, and the accommodation coefficient  $\alpha_e \sim 1$ , then  $V_2 \sim a_\infty$  and  $M_2 \gg 1$ . If the accommodation coefficient is small (less than 0.5, let us say, in practice), then the velocity of the reflected molecules is of the same order as  $V$ , i.e.,  $M_2 \sim 1$ . If the body is "hot," i.e., its temperature is of the order of the gas stagnation temperature, then  $M_2 \sim 1$  even when  $\alpha_e \sim 1$ . We note that the hypersonic regime  $M_2 \gg 1$  may be realized even when  $M \sim 1$ . This occurs, for example, when the body is highly cooled when  $\alpha_e \sim 1$ .

**1.1.** Let  $M_2 \gg 1$ ,  $\sigma = \sigma_\infty = \text{const}$ , and  $M_2 = M$ .\* It follows from (6.8) that  $n_2 \sim n_\infty M_2 \gg n_\infty$  i.e., a "cushion" of reflected molecules forms around the body. Since, in that case, the relative velocities  $V_{12} \sim V_{21} \sim V$ , while  $V_{22} \sim V_2$ , for the characteristic path lengths we have the estimates

$$\lambda_{12} \sim \frac{V}{n_2 \sigma_\infty V} \sim \frac{\lambda_\infty}{M_2}, \quad \lambda_{21} \sim \frac{V_2}{n_\infty \sigma_\infty V} \sim \frac{\lambda_\infty}{M_2}, \quad \lambda_{22} \sim \frac{V_2}{n_2 \sigma_\infty V_2} \sim \frac{\lambda_\infty}{M_2}, \quad (6.9)$$

i.e., all the mean free paths are of the same order.

\*The case  $M_2 \neq M$ . is considered just as simply. In that case, however, all formulas have a complicating factor of the type  $\sqrt{T_\infty/T_w}$ , which only introduces a small correction.

For the flow to be near free-molecular, it is necessary that

$$\frac{\lambda_{12}}{L} \sim \frac{\lambda_{21}}{L} \sim \frac{\lambda_{22}}{L} \sim \frac{Kn}{M_2} \gg 1, \quad (6.10)$$

where L is the characteristic dimension of the plate.

Thus, in this flow, the characteristic rarefaction criterion is  $Kn/M_2$ , and not  $Kn$ .

In a free-molecule flow the number of molecules incident on the plate, and the momentum and energy brought by them, are, respectively,

$$N_0 \sim n_\infty V L^2, \quad P_0 \sim m N_0 V, \quad Q = m N_0 V^2. \quad (6.11)$$

A molecule approaching the body experiences a number of collisions of order  $n_2 \sigma_\infty L$ . Therefore, the number of molecules compared to a free-molecule flux, and the momentum and the energy lost by collisions, are, respectively,

$$N_- \sim N_0 n_2 \sigma_\infty L \sim N_0 \frac{M_2}{Kn}, \quad P_- \sim N_- m V, \quad Q \sim N_- m V^2. \quad (6.12)$$

In addition, because of collisions, the plate is also struck by molecules which would otherwise fly past it. It is easy to see that the additional number of molecules  $N_+$ , their momentum  $P_+$ , and energy  $Q_+$ , are of the same order as  $N_-$ ,  $P_-$ , and  $Q_-$ .

In fact, let us consider a hemispherical region of radius L and center at the center of the plate. The density of reflected molecules there is roughly of order  $n_2$ . At distances from the center of the plate  $r$  greater than L, the density of the reflected molecules falls off as  $n_2(L/r)^2$ . Inside the hemisphere,  $Vn_\infty n_2 \sigma L^3$  incident molecules collide with molecules reflected from the plate. Since the solid angle subtended by the plate, at points inside the sphere is of order unity, the number of molecules incident on the plate after collision is also of order  $Vn_\infty n_2 \sigma L^3$ , and the momentum and energy they bring is  $m n_\infty n_2 \sigma L^3 V^2$  and  $m n_\infty n_2 \sigma L^3 V^3$ , respectively. It is evident that  $n_\infty V n_2 (L/r)^2 \sigma_\infty r^2 dr$  collisions occur in a spherical shell of thickness  $dr$  outside the sphere. Since the number of molecules incident on the plate after a collision is proportional to the solid angle subtended by the plate, the number of such

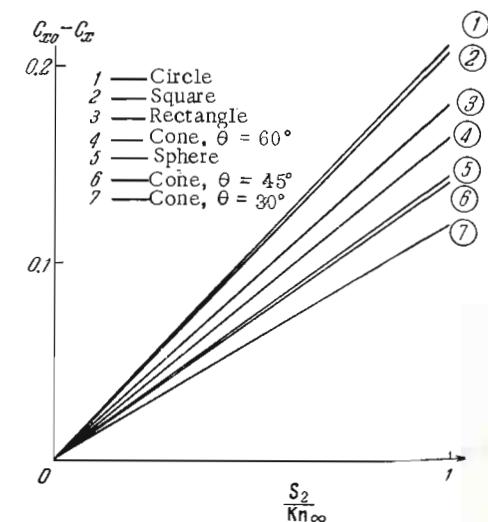


Fig. 68

particles is equal to  $n_\infty V n_2 (L/r)^4 \sigma_\infty r^2 dr$ . Integrating from L to  $\infty$ , we clearly obtain the result that the number of molecules reaching the plate from the space outside a sphere of radius  $\sim L$  after a collision, is of the same order as the number whose last collision was inside the sphere of radius L. Comparing this result with (6.12), we see that their number, and the momentum and energy which they bring, are of the same order as the amount lost through collisions.

The change in energy and momentum brought to the plate, due to collisions of the reflected molecules among themselves, is small, because of the smallness of their velocities.

Therefore,

$$C_x = 2 \frac{P_0 - P_- + P_+}{\rho_\infty V^2 L^2} \sim C_{x0} + \frac{AM}{Kn}, \quad (6.13)$$

$$q = 2 \frac{Q_0 - Q_- - Q_+}{\rho_\infty V^3 L^2} \sim q_0 + \frac{AM}{Kn}, \quad (6.14)$$

where the subscript 0 refers to free-molecule flow, and the A's are constants.

It may be seen from the flow geometry that most collisions occur near the body, and that most of the molecules that have collided strike the plate after the collision, bringing to it the momentum and energy which they would have brought without undergoing a collision. Therefore, for a plate perpendicular to the stream (and more generally for blunt bodies), the coefficient  $A$  must be small. Calculations show that it is negative, i.e., the drag and the heat flux are lower than in free-molecule flow.

Figure 68 shows the results of first-collision calculations of drag for a circular\* and rectangular† plate, and also for a sphere‡ and a cone.¶ The calculations were carried out for a hard-sphere gas under the assumption of completely diffuse reflection.

The graph shows the quantity

$$C_{x0} - C_x = \alpha \frac{S_2}{Kn_\infty}$$

$$\left( S_2 = V_\infty \sqrt{\frac{m}{2kT_2}}, Kn_\infty = \frac{1}{2\sqrt{2}\sigma n_\infty R_0}, R_0 = \sqrt{\frac{A}{\pi}} \right);$$

where  $\sigma = \pi d^2$  is the molecular collision cross section,  $T_2$  is the temperature of the reflected molecules, and  $A$  is the area of the center section;  $\theta$  is the half-angle of the cone; the aspect of the rectangle is 5. As is to be expected, the difference from the free-molecule case is small.

**1.2.** We now examine the same flow, under the assumption that the effective collision cross section varies in inverse proportion to the velocity of the colliding molecules. As above, we consider that  $M \sim M_2 \gg 1$  and  $T_w \sim T_\infty$ . Since  $V_{12} \sim V_{21} \sim V$ ,

$$\sigma_2 \sim \sigma_{12} \sim \sigma_{21} \sim \sigma_\infty \frac{a}{V} \sim \frac{\sigma_\infty}{M}.$$

Then,

$$\lambda_{12} \sim \frac{V}{n_2 \sigma_2 V} \sim \lambda_\infty, \quad \lambda_{21} \sim \frac{V_2}{n_\infty \sigma_2 V} \sim \lambda_\infty, \quad \lambda_{22} \sim \frac{V_2}{n_2 \sigma_2 V_2} \sim \frac{\lambda_\infty}{M}. \quad (6.15)$$

It is easy to see that if the shortest of the mean free paths  $\lambda_{22} \gg L$ , i.e., if  $Kn/M \gg 1$  the flow may be considered to be exactly as it was in case 1.1. However, here the number of collisions of reflected molecules among themselves is larger by a factor  $M$  than the number of collisions with incident molecules. Therefore, the momentum imparted to the plate as a result of collisions of the reflected molecules among themselves is of the same order as the momentum imparted as a result of collisions of reflected molecules with incident molecules. However, the energy imparted as a result of reflected-reflected collisions is lower by a factor  $M$  than the energy imparted as a result of collisions of reflected and incident molecules.

By means of estimates similar to those made above in case 1.1, we have

$$C_x = C_{x0} + \frac{A}{Kn}, \quad q = q_0 + \frac{A}{Kn}, \quad (Kn \gg M). \quad (6.16)$$

If  $\lambda_\infty \gg L$ , but  $\lambda_{22} \leq L$ , the flow is not nearly free-molecular, since the reflected molecules experience many collisions within distances of order  $L$ . However, the motion of the molecules of the incident stream in the field of reflected molecules may be considered as nearly free-molecular.

In fact, let us examine the limiting case when  $\lambda_{22} \ll L$  or  $1 \ll Kn \ll M$ . At a small distance of the order of several mean free paths  $\lambda_{22}$  from the wall, the flux of reflected molecules may already be regarded as the discharge of a continuous medium into vacuum. When a reflected molecule collides with an incoming molecule, it acquires a velocity of order  $V$ , i.e., its velocity increases roughly by a factor  $M$ . Therefore, the mean free path of the reflected molecules also increases by a factor  $M$ , i.e., it becomes of order  $\lambda_{22}M \sim \lambda_\infty$ . In such a collision a reflected molecule appears to transform into a molecule of different kind with a collision cross section smaller by a factor  $M$ . The oncoming molecule appears to detach a molecule from the continuous mean flux of reflected molecules, and thereafter both molecules have a long mean free path  $\lambda_\infty \gg L$ . Therefore, if the flow of the reflected molecules is known, the interaction of the reflected and incident streams may

\*V. A. Perepukhov, Zh. Vychisl. Mat. i Mat. Fiz., Vol. 1, No. 4 (1961).

†O. G. Fridlander, Zh. Prikl. Mekhan. i Tekhn. Fiz., No. 3 (1963).

‡V. A. Perepukhov, Zh. Vychisl. Mat. i Met. Fiz., No. 2 (1967).

¶M. N. Kogan and L. M. Degtyarev, Astronaut. Acta, No. 1 (1965).

be considered to be a nearly free-molecule flow. The main difficulty is in calculation of the flux of reflected molecules. The calculation of that flux is a problem interesting in itself.\*

**1.3.** Now let  $M_2 \sim 1$  and  $\sigma = \text{const} = \sigma_\infty$ . As noted, such a flow may occur when  $\alpha_e \ll 1$  and  $T_w \sim T_\infty$ , or with a strongly heated wall, whose temperature is of the order of the stagnation temperature. The latter case often arises in wind tunnel investigations.

It is easy to see in that case that  $n_2 \sim n_\infty$ , and

$$\lambda_{12} \sim \lambda_{21} \sim \lambda_{22} \sim \lambda_\infty. \quad (6.17)$$

The flow is analogous to that examined in the case 1.1, with the difference that now the flow is free-molecular or nearly so when  $Kn \gg 1$  and not when  $Kn/M \gg 1$ .

In a manner similar to that of case 1.1, we obtain

$$C_x = C_{x0} + \frac{A}{Kn}, \quad q = q_0 + \frac{A}{Kn} \quad (6.18)$$

with  $Kn \gg 1$ .

**1.4.** If  $M_2 \sim 1$  and  $\sigma$  varies inversely with the velocity, then  $\sigma_2 \sim \sigma_\infty/M$ . In that case,  $\sigma_{22}$  is also equal to  $\sigma_2$ . Since  $n_2 \sim n_\infty$ , we have

$$\lambda_{11} \sim \lambda_{12} \sim \lambda_{21} \sim \lambda_{22} \sim M\lambda_\infty. \quad (6.19)$$

Therefore, when  $MKn \gg 1$ ,

$$C_x = C_{x0} + \frac{A}{MKn}, \quad q = q_0 + \frac{A}{MKn}. \quad (6.20)$$

We note that for large enough Mach numbers, this flow may occur even when  $\lambda_\infty \ll L$ , or  $Kn \ll 1$ , i.e., in a dense medium. When  $MKn \gg 1$  the flow is nearly free-molecular even if the incident flow is considered to be a continuous medium.

The flow may be represented as follows. Assume that the plate absorbs all the molecules incident upon it, and then emits

\*A similar flow arises, for example, in evaporation in a vacuum. See also the end of §4.2.

molecules of a new kind which possess a cross section smaller by a factor  $M$  and, therefore, a mean free path larger by a factor  $M$ . Since the characteristic mean free path for interaction of the new molecules among themselves and with molecules of the incident flow is now of order  $\lambda_\infty M \gg L$ , collisions are rare, and the flow is nearly free-molecular. The reflected molecules collide with incident molecules and appear to transform them into molecules of a new kind. The removal of molecules from the incident stream causes a perturbation which is propagated in the stream at the speed of sound,  $a_\infty$ . But, while the gas travels a distance of order  $L$ , those disturbances are propagated to a distance  $L/M \ll \lambda_\infty$ . We can therefore neglect any gasdynamic rearrangement of the incident stream. This is especially clear in the limiting case of a sufficiently dense beam of molecules traveling in parallel straight lines.

It is interesting to note that, depending on the conditions, the similarity criterion varies by a factor of  $M^2$  from  $Kn/M$  in case 1.1 to  $MKn$  in case 1.4. If in the first case we increase the velocity without limit, when the density of the incident stream is arbitrarily small, the flow tends to that of a continuous medium. In the second case, with the density arbitrarily large, the flow tends to free molecule as the velocity is increased without limit.

Therefore, the boundary of free-molecule flows is displaced. It is evident, for example, that in the case 1.1, i.e., with  $\alpha_e \sim 1$ , free-molecule flow is achieved at much lower density (at much higher altitude), than in case 1.3, i.e., with  $\alpha_e \ll 1$ .

**2. Flat Plate Parallel to the Stream.** In this case the molecules fall on the plate either because of thermal agitation or as a result of collisions. In a free-molecule stream the number of molecules reaching the plate is  $N_0 \sim L^2 n_\infty a_\infty$ , where the speed of sound  $a_\infty$  characterizes the velocity of thermal motion.

We examine the same four cases as for the plate perpendicular to the stream.

**2.1.** Let  $M \sim M_2 \gg 1$  and  $\sigma = \text{const} = \sigma_\infty$ . In this case, it follows from the condition of impermeability that  $n_2 \sim n_\infty$ , since  $V_2 \sim a_\infty$ . The characteristic mean free paths are estimated as follows:

$$\lambda_{12} \sim \frac{V}{n_\infty \sigma_\infty V} \sim \lambda_\infty, \quad \lambda_{21} \sim \frac{V_2}{n_\infty \sigma_\infty V_\infty} \sim \frac{\lambda_\infty}{M}, \quad \lambda_{22} \sim \frac{V_2}{n_\infty \sigma_\infty V_2} \sim \lambda_\infty. \quad (6.21)$$

Therefore, in a hypersonic flow over a flat plate, the least mean free path is that of reflected molecules colliding with molecules of the incident stream.

Let  $\lambda_{21} \gg L$ , i.e.,  $Kn \gg M \gg 1$ . Each incoming molecule undergoes  $n_2 \sigma L$  collisions near the body or  $n_\infty \sigma L$  collisions in this case. Therefore, of the  $N_0 \sim L^2 n_\infty a_\infty$  incoming molecules, a number

$$N_- \sim N_0 n_\infty \sigma L \sim n_\infty^2 a_\infty \sigma L^3 \sim \frac{N_0}{Kn}$$

does not strike the body. Altogether,  $n_\infty^2 V \sigma L^3$  collisions occur around the body and, therefore, the number of molecules striking the plate as a result of collisions is

$$N_+ \sim n_\infty^2 V \sigma L^3 \sim N_0 \frac{M}{Kn},$$

i.e., as a result of collisions the number of molecules striking the plate is larger by a factor  $M$  than the number lost because of collisions. Since the molecules have an average velocity of order  $V$  after a collision, the momentum and energy imparted to the plate as a result of collisions in a steady parallel hypersonic stream is greater than the amount lost, i.e., the drag of the plate and the heat flux are larger than in free-molecule flow:

$$C_x = C_{x0} + \frac{A}{Kn} \sim C_{x0} \left(1 + A_1 \frac{M}{Kn}\right), \quad C_{x0} \sim \frac{A_0}{M}. \quad (6.22)$$

The normal momentum (pressure) increases particularly strongly. In a free-molecule stream each molecule brings a normal momentum of order  $ma_\infty$ , while each molecule striking the plate after a collision brings a momentum of order  $mV$ , i.e., larger by a factor  $M$ . In a free-molecule stream,  $p = p_\infty$ . In the case examined here

$$p = p_\infty + AN_+ mV_\infty \sim p_\infty \left(1 + A \frac{M^2}{Kn}\right). \quad (6.23)$$

The flow past a flat plate possesses an interesting property: it remains nearly free-molecular even in the case when  $\lambda_{21}$  is of the same order or even less than  $L$ . Indeed, when a reflected molecule collides with an incident molecule, the former acquires a velocity of order  $V$ , and, therefore, its mean free path increases by a factor  $M$ , so that it can experience a second collision only at a distance  $\lambda_\infty$ , which is much greater than  $L$ .

It is evident that as  $\lambda_{21}$  decreases, the number of reflected molecules which experience a collision far from the body increases and, therefore, the number which return to the plate also increases. Therefore, the number of molecules striking the plate,  $N_w$ , increasingly differs from  $N_0$  and, therefore, the density of the reflected molecules  $n_2$  also increases.

We examine the limiting case  $\lambda_{21} \ll L$  or  $1 \ll Kn \ll M$ . In that case, all the reflected molecules undergo collisions at distances of the order  $\lambda_{21} \sim \lambda_\infty/M \ll L$ . Since the mean free path of the molecules after collision is of the order  $\lambda_\infty$ , then, within a layer of thickness  $\delta \sim \lambda_\infty/M$  the molecules may experience only one collision each.

$N_w$  molecules leave the plate per unit time. They all collide inside that layer with  $N_w$  molecules of the incident stream. Since the velocity of the reflected molecules is of the order of the thermal velocity  $a_\infty \ll V$ , we may argue roughly that in a collision a free-stream molecule has no thermal velocity, while a reflected molecule is at rest. Therefore, on the average, in a collision half the molecules go to the outer edge of the layer. Thus, almost  $N_w$  molecules return to the wall. Molecules leave the layer only through its edges. The number of such molecules is proportional to the ratio of the area of the edges of the layer to the area of the plate, i.e., is equal to

$$N_w \frac{\delta}{L} \sim N_w \frac{Kn}{M}.$$

On the other hand, the number of particles which enters the layer is  $N_0 - N_- + N_0$ , where  $N_0$  is the number of particles entering the layer as a result of collisions outside the layer, and  $N_-$  is the number of molecules removed from the stream  $N_0$  as a result of collisions. As we noted above,  $N_w$  molecules leave the layer through its outer edge at a velocity  $V$ . Therefore, the density of these

molecules is  $n_2 \sim N_w / L^2 V$ . Therefore, the number of molecules entering the layer as a result of collisions with those molecules is of order

$$n_\infty V n_2 \delta L^3 \sim \frac{V n_2 L^2}{Kn} \sim \frac{N_w}{Kn}.$$

The loss of molecules  $N_-$  consists of the loss due to collisions outside the layer and within the layer. The first is equal to  $N_0 n_2 \sigma L \sim N_w / M Kn$ . The second is determined by the number of incoming molecules colliding in the layer  $n_2^* \sigma L$ , where  $n_2^* \sim N_w / \alpha_\infty L^2$  is the density of reflected molecules in the layer. For the flow to be nearly free-molecular, we must impose the condition

$$n_2^* \sigma L \ll 1. \quad (6.24)$$

Therefore,

$$N_- \sim N_0 n_2^* \sigma L \sim \frac{N_w}{Kn} \ll N_0. \quad (6.25)$$

Taking into account (6.25), the balance condition for molecules in the layer may be written in the form

$$N_0 \sim N_w \frac{Kn}{M}. \quad (6.26)$$

Therefore,

$$N_w \sim N_0 \frac{M}{Kn} \text{ and } n_2^* \sim n_\infty \frac{M}{Kn}. \quad (6.27)$$

The condition (6.24) may now be written in the form

$$Kn^2 \gg M \gg Kn \gg 1. \quad (6.28)$$

Conditions (6.28) are the conditions for existence of the layer. It is clear that those conditions can be fulfilled only for very large Knudsen and Mach numbers. Thus, when conditions (6.28) are observed in a highly rarefied hypersonic stream around a flat plate, there may be a thin layer of thickness  $\delta \sim L Kn/M$  with a particle density that is greater by a factor of  $M/Kn$  than in the free stream (Fig. 69). This layer is called the molecular boundary layer.\*

\*See the paper by Kogan cited above.

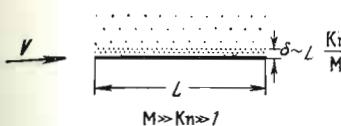


Fig. 69

The great majority of particles reflected from the plate experience their first collision with the incoming molecules inside the layer. After collision they either return to the plate or leave the

layer and experience a second collision far from the body (at distances of the order  $\lambda_\infty \gg L$ ). Since the contribution of those remote collisions is negligible, the molecular boundary layer may be calculated within the framework of the first collision theory, in spite of the fact that one of the mean free paths ( $\lambda_{21}$ ) is much shorter than the characteristic flow dimension  $L$ .

It is easy to verify that the drag and the pressure when such a layer is present are again determined by formulas (6.22) and (6.23). However, since in the case of a molecular layer  $M \gg Kn$ , the contribution of the collisions is much larger than the corresponding free-molecule values. For regimes intermediate between  $Kn \gg M$  and  $Kn^2 \gg M \gg Kn$  the drag and pressure are also determined by formulas (6.22) and (6.23).

This discussion may easily be generalized for the case when  $M_2 \gg 1$ , but is not equal to  $M$ . It is easy to verify, in that case, that

$$n_2 \sim n_\infty \frac{M_2}{M}, \quad \lambda_{12} \sim \lambda_\infty \frac{M}{M_2}, \quad \lambda_{21} \sim \frac{\lambda_\infty}{M_2}, \quad \lambda_{22} \sim \lambda_\infty \frac{M}{M_2}. \quad (6.29)$$

Correspondingly, instead of (6.22) and (6.23), we have

$$C_x = C_{x0} \left( 1 + A \frac{M_2}{Kn} \right) \text{ and } p = p_\infty \left( 1 + A \frac{M_2 M}{Kn} \right). \quad (6.30)$$

The conditions for the existence of the molecular boundary layer take the form

$$\frac{M}{M_2} Kn^2 \gg M_2 \gg Kn \gg 1. \quad (6.31)$$

These estimates are in good agreement with computations.

Figures 70-72 show examples of calculation of the flow past a plate by a first-collision theory, allowing for attenuation of the density of reflected molecules by collisions with incoming molecules. The molecules are regarded as hard spheres. The reflec-

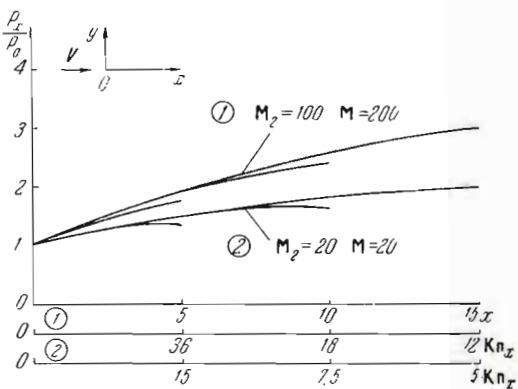


Fig. 70

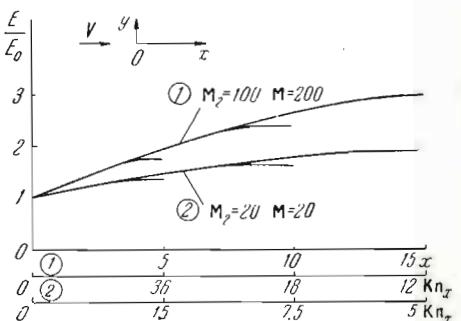


Fig. 71

tion from the wall is diffuse. The computation was made by the Monte Carlo method.\* The Knudsen number is  $Kn_x = \lambda/x = (\sigma n_\infty x)^{-1}$ , where  $x$  is the distance measured from the leading edge of the plate.

Consistent with the estimates made above, the energy and momentum imparted to the plate as a result of collisions increase with the ratio  $M_2/Kn_x$ . The pressure  $p = P_y$  increases especially rapidly. The pressure due to the collisions is several times

\* M. N. Kogan and L. M. Degtyarev, Astronaut. Acta, No. 1 (1965); see also V. A. Perepukhov, Zh. Vychisl. Mat. i Mat. Fiz., Vol. 3, No. 3 (1963).

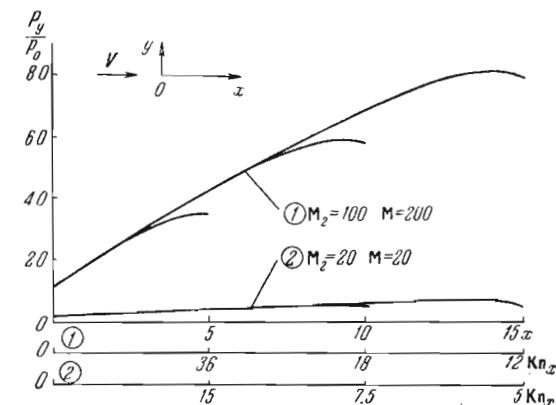


Fig. 72

greater than the corresponding free-molecule value. In the case of the molecular layer, the pressure increases by a factor of ten (Fig. 72). It is interesting to note that the flow past the plate has an essentially parabolic character, i.e., an increase in the length of the plate has only a slight effect on the flow upstream. However, it should be noted that the flow parameters at the plate leading edge are not equal to the corresponding free-molecule values. This indicates that the parabolic behavior is established only at some distance from the leading edge.\*

We also note that the friction, pressure, and heat flux increase along the plate whereas, according to boundary layer theory, allowing for interaction with the external flow, those quantities fall off along the plate. Boundary layer theory is asymptotic as  $x \rightarrow \infty$ . These calculations reflect to some extent the behavior of the stream in a small region around the plate leading edge,† owing to the parabolic feature mentioned.

It is evident that the friction, pressure, and heat transfer first increase with increasing  $x$ , attain a maximum, and then fall off, tending for large  $x$  to values determined by boundary layer theory.

\* To explain the process of transition to a parabolic curve, a more detailed calculation of the flow at the front of the plate is required.

† On this topic, see also § 5.2.

For that reason the drag of the plate for a given Mach number has a maximum at some Reynolds number. With decreasing Reynolds number the drag first increases in accordance with the predictions of boundary layer theory. At some Reynolds number it reaches a maximum and then decreases to the free-molecule value.

The molecular boundary layer phenomenon may be approached in a somewhat different manner. Let us examine the flow past a plate of a stream of particles traveling in parallel straight lines with velocity  $\mathbf{V}$  and density  $n_\infty$ .

Let that stream strike the plate at a small angle  $\theta$ . In addition, let the velocity of the reflected molecules  $V_2 \ll V$ , i.e.,  $M_2 \gg 1$ . In that case, a molecular boundary layer of thickness  $L \text{Kn}/M_2$  will be formed around the plate; the conditions for its existence are determined by the inequalities  $\text{Kn} \ll M_2$  and  $\text{Kn}^2 \gg \theta M_2^2$ . Those conditions may be met in various kinds of accelerators of the molecular beam type.

**2.2.** Let us now examine a flow which differs from the above only in that  $\sigma$  is proportional to the relative velocity, i.e., we shall put  $M \sim M_2 \gg 1$  and  $\sigma_2 \sim \sigma_\infty M^{-1}$

Since then,  $V_2 \sim a_\infty$  and  $n_2 \sim n_\infty$ , we have

$$\lambda_{12} \sim \frac{V}{n_\infty \sigma_2 V} \sim \lambda_\infty M, \quad \lambda_{21} \sim \frac{V_2}{n_\infty \sigma_2 V} \sim \lambda_\infty, \quad \lambda_{22} \sim \frac{V_2}{n_\infty \sigma_\infty V} \sim \lambda_\infty. \quad (6.32)$$

We consider three cases:  $\text{Kn} \gg 1$ ,  $\text{Kn} \sim 1$  and  $\text{Kn} \ll 1$ . If  $\lambda_\infty \gg L$ , i.e.,  $\text{Kn} \gg 1$  then near the plate there are  $n_\infty^2 V \sigma_\infty L^3 M^{-1}$  collisions of incident molecules with reflected molecules and, therefore,

$$C_x \sim \frac{A_0}{M} + \frac{A}{MKn}, \quad p \sim p_\infty \left(1 + A \frac{M}{Kn}\right). \quad (6.33)$$

The correction to the drag due to collision of the reflected molecules among themselves and to loss of momentum of incoming molecules to the reflected molecules is of order  $(Kn M^2)^{-1}$ . Here the drag, and particularly the pressure, are again greater than in the free-molecule flow. If  $Kn \geq M$ , we must take into account the momentum carried away by the reflected molecules, which is of order  $M^{-2}$ .

When  $\lambda_\infty \sim L$ , i.e., when  $Kn \sim 1$ , all the reflected molecules undergo collisions among themselves and with incoming molecules at distances of order  $L$  from the plate. The collisions of the reflected molecules among themselves do not change the order of their density or velocity. A collision of a reflected molecule with an incident molecule transforms the reflected molecule into a molecule of a different kind with a mean free path greater by a factor  $M$  and equal to  $\lambda_\infty M$ .

It is easy to see that in that case formulas (6.33) are valid for the drag and the pressure, and when  $Kn \sim 1$  they may be written in the form

$$C_x = (A_0 + A) \frac{1}{M}, \quad p = p_\infty (1 + AM). \quad (6.34)$$

In the limiting case when  $\lambda_\infty \ll L$ , i.e., when  $Kn \ll 1$ , all the reflected molecules collide with incoming molecules in a thin layer of thickness  $\lambda_\infty$ . Considerations analogous to those presented in case 2.1 indicate that  $N_w \sim N_0 Kn^{-1}$ ,  $n_2^* \sim n_\infty Kn^{-1}$ . in the molecular boundary layer. Therefore, the characteristic mean free paths in the molecular layer are

$$\lambda_{12}^* \sim \lambda_\infty Kn M, \quad \lambda_{21}^* \sim \lambda_\infty, \quad \lambda_{22}^* \sim \lambda_\infty Kn.$$

To avoid any appreciable attenuation of the incoming stream in the molecular layer, we require that  $\lambda_{12}^* \gg L$  or  $Kn^2 M \gg 1$ . Otherwise, the estimates given must be altered.

In this case, the expressions (6.33) are valid again, but, since  $Kn \ll 1$ , we may expect a considerable increase in drag, heat transfer, and particularly in the pressure, when allowance is made for the collisions.

As in case 2.1, because of the "regeneration" of the molecules, the reflected molecules in the molecular layer experience no more than one collision with incoming molecules. But, since the mean free path  $\lambda_{22}^*$  of the reflected molecules among themselves is much shorter than the layer thickness  $\lambda_\infty$ , the reflected molecules experience many collisions among themselves, before they collide with an incident molecule. Therefore, this flow cannot be discussed in terms of a first-collision theory.

**2.3.** We now examine the case of high-speed reflected molecules,  $V_2 \sim V$ , i.e.,  $M_2 \sim 1$ . We put  $\sigma = \sigma_\infty$ . In that case,  $n_2 \sim n_\infty M^{-1}$  and

$$\lambda_{12} \sim M\lambda_\infty, \quad \lambda_{21} \sim \lambda_\infty, \quad \lambda_{22} \sim M\lambda_\infty. \quad (6.35)$$

This case is similar to the case 2.1 in the sense that the shortest mean free path is  $\lambda_{21}$ . However, all the molecules here have the same order of velocity,  $\sim V$ , and no regeneration of molecules occurs. Therefore, the flow is nearly free-molecular only when  $\lambda_\infty \gg L$ , i.e., when  $Kn \gg 1$ . In that case,

$$C_x = C_{x0} + \frac{A}{KnM}, \quad p = p_\infty \left(1 + A \frac{M}{Kn}\right). \quad (6.36)$$

The constants  $A > 0$ .

**2.4.** If  $M_2 \sim 1$  and  $\sigma$  is inversely proportional to the relative velocity, then

$$\lambda_{12} \sim M^2\lambda_\infty, \quad \lambda_{21} \sim M\lambda_\infty, \quad \lambda_{22} \sim M^2\lambda_\infty.$$

This case is similar to the preceding one. The flow is nearly free-molecular if  $M\lambda_\infty \gg 1$ . In that case,

$$C_x = C_{x0} + \frac{A}{KnM^2}, \quad p = p_\infty \left(1 + A \frac{1}{Kn}\right). \quad (6.37)$$

The flow may also be nearly free-molecular when  $Kn \ll 1$  if  $KnM \gg 1$ .

**3. Flow over a Plate at an Angle of Attack.** Here, as in the theory of hypersonic flow of a continuous medium, we must distinguish between flow with  $M \sin \vartheta \gg 1$  and  $M \sin \vartheta \ll 1$  where  $\vartheta$  is the angle of attack. In the first case the flow is close to flow about a plate perpendicular to the stream, and in the second case to flow about a plate parallel to the stream.

Let  $M \sin \vartheta \gg 1$  and  $M = M_2 \gg 1$ . We examine only the case  $\sigma = \sigma_\infty = \text{const}$ . Since  $N_0 \sim Vn_\infty L^2 \sin \vartheta$  and  $n_2 \sim n_\infty M_2 \sin \vartheta$

$$\lambda_{12} \sim \frac{\lambda_\infty}{M_2 \sin \vartheta}, \quad \lambda_{21} \sim \frac{\lambda_\infty}{M_2}, \quad \lambda_{22} \sim \frac{\lambda_\infty}{M_2 \sin \vartheta}. \quad (6.38)$$

If  $\lambda_{21} \gg L$ , the flow is close to free-molecular. It is easy to see that

$$N_- \sim n_\infty^2 V M_2 \sigma_\infty L^3 \sin^2 \vartheta \quad \text{and} \quad N_+ \sim n_\infty^2 V_\infty M_2 \sigma_\infty L^3 \sin \vartheta. \quad (6.39)$$

Therefore the drag and the lift, referred to  $L^2$ , are equal to

$$\left. \begin{aligned} C_x &= A_0 \sin \vartheta - A_1 \frac{M_2}{Kn} \sin^2 \vartheta + A_2 \frac{M_2}{Kn} \sin \vartheta, \\ C_y &= \left( A_1 \frac{M_2}{Kn} + A_0 \frac{1}{M_2} \right) \sin \vartheta. \end{aligned} \right\} \quad (6.40)$$

The lift when  $M_2 \gg 1$  is determined mainly by the collisions. The lift-to-drag ratio  $K = C_y/C_x$  is much larger than in free-molecule flow. That situation is characteristic more generally than for a flat plate. We saw in § 6.1 that, in a hypersonic free-molecule flow, the lift on any body is due to the reactive momentum of the reflected molecules and, therefore,  $C_y \sim 1/M_2$ . Therefore, it is usual to assume that the lift-to-drag ratio is small when  $Kn \gg 1$ . However, consideration of collisions when  $M_2 \gg 1$  leads to a radical change in the lift and pitching moment characteristics of bodies, while the drag differs comparatively little from its free-molecule value. Those conclusions were obtained for near-free-molecule flows (with  $M_2/Kn \ll 1$ ). With further increase of  $M_2$  and a fixed Knudsen number, the flow tends to a continuous flow, and the lift-to-drag ratio continues to increase.

We saw above that  $A_1 > A_2 > 0$  with  $\vartheta = 1/2\pi$  and the plate drag is lower than in free-molecule flow. It may be seen from (6.40) that when  $\vartheta \rightarrow 0$  but  $M \sin \vartheta \gg 1$ , the plate drag becomes larger than in free-molecule flow. With further reduction in  $\vartheta$ , when  $M\vartheta \rightarrow 0$  we arrive in the limit to flow past a plate parallel to the stream.

**4. Flow Past a Cone.** In flow past a cone with vertex half-angle  $\theta$  we must also distinguish between the regimes  $M \sin \theta \gg 1$  and  $M \sin \theta \ll 1$ .

Considerations similar to those just presented for the flat plate, indicate that the drag referred to the normal section of the cone (when  $M \sin \theta \gg 1$ ,  $M = M_2 \gg 1$ , and  $\sigma = \sigma_\infty$ ) is given by the formula (for small  $\theta$ )

$$C_x \sim C_{x0} - A_1 \frac{M}{Kn} + A_2 \frac{M}{Kn \theta}, \quad (6.41)$$

where  $Kn = \lambda_\infty/R$  and  $R$  is the radius of the cone base.

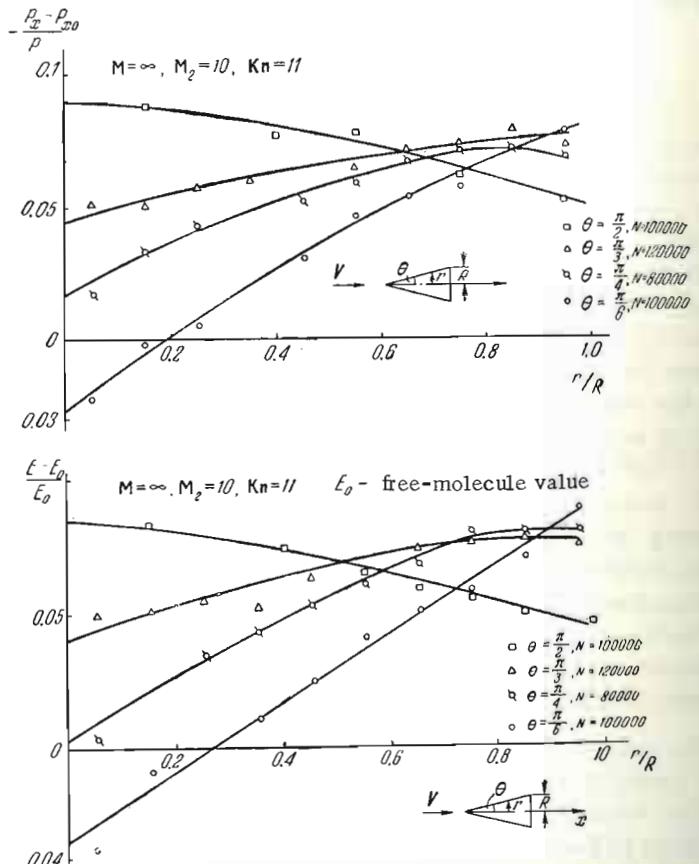


Fig. 73

When  $M\theta \ll 1$  the molecules strike the cone surface mainly as a result of thermal agitation. In that case,  $N_0 \sim LRn_\infty a_\infty$ , where  $L = R/\theta$  is the length of the cone and  $n_2 \sim n_\infty$ . It is easy to see that when  $\theta$  tends to zero for a fixed diameter  $2R$ , the cone drag increases without limit. However, for the flow to be considered nearly free-molecular, it is necessary that the cone length  $L \ll \lambda_\infty$ . When that condition is fulfilled, the drag, referred to the normal section, is

$$C_x \sim C_{x0} + A_1 \frac{1}{Kn \theta^2} = C_{x0} \left( 1 + \frac{AM}{Kn 0} \right), \quad \left( C_{x0} \sim \frac{A_0}{M\theta}, \quad Kn = \frac{\lambda_\infty}{R} \right). \quad (6.42)$$

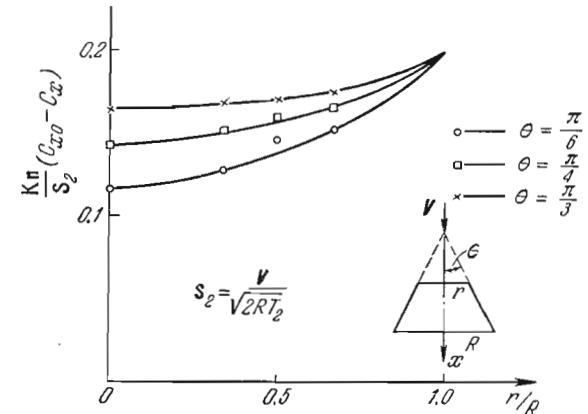


Fig. 74

It is interesting to note that when  $\theta$  tends to zero (or, analogously, the cone base diameter vanishes) for fixed  $L$  and  $\lambda_\infty$ , the flow past a cone at zero angle of attack does not tend to free-molecule flow. Yet the flow past a cone at an angle of attack  $\alpha$  for  $Ma \gg 1$  tends to a free-molecule flow as the cone diameter tends to zero.

Figure 73 presents the results of some calculations for a cone by the Monte Carlo method\* for a hypersonic flow ( $M = \infty$ ). These curves fully confirm our qualitative conclusions. With reduction of the angle  $\theta$ , the negative correction to the free-molecule value in formula (6.41) decreases, while the positive one increases. On the graph,  $N$  is the number of trials in the Monte Carlo method, corresponding to the results presented. We note that the forces acting on the body differ markedly from the free-molecule values, although the Knudsen number is

$$Kn = (2\sqrt{2}\sigma Re_\infty)^{-1} \approx 11.$$

Figure 74 shows the results of calculations for truncated cones. The calculation was also performed by the Monte Carlo method.†

\*M. N. Kogan and L. M. Degtyarev, Astronaut. Acta, No. 1 (1965).

†V. A. Perepukhov, Zh. Vychisl. Mat. i Mat. Fiz., No. 2 (1967).

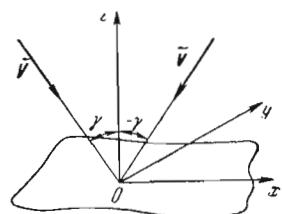


Fig. 75

### §6.7. The Reverse Flow Theorem for Nearly Free-Molecule Flows\*

Let a flat plate of arbitrary shape, located in the plane  $xy$ , be immersed in a stream of rarefied gas whose velocity vector  $\mathbf{V}$  lies in the plane  $xz$  and makes an angle  $\gamma$  with the  $z$  axis (Fig. 75). Let the characteristic dimension of the plate be  $L$ .

In free-molecule flow each surface element  $dS$  receives the same number of molecules, which transfer the same momentum and energy. As a result of the interaction of the molecules with the plate surface,<sup>†</sup> each element creates the same field of reflected molecules. The distribution function for molecules of the oncoming stream near the plate differs from the distribution function at infinity, since some of the molecules are screened by the plate. It is clear that all elements of the plate screen the flow in the same way.

Let us examine, in the first approximation, the influence of molecular collisions on the number, momentum, and energy flux brought to the plate by the molecules. It was shown in §6.5 that, in calculating the collisions in the first approximation, we should not take account of attenuation of the oncoming stream by the reflected molecules. Collisions of reflected molecules with reflected molecules are appreciable only within a sphere of radius  $R \sim L \ll \lambda$ . But, at distances much smaller than  $\lambda$ , the attenuation may be neglected. Thus, only the attenuation of the flux of reflected molecules by the incident molecules is appreciable. Therefore, the field of the incident and reflected molecules created by each element of the surface does not depend on the presence of other elements. It is clear that in that case the pattern of scattered molecules after the first collision is a superposition of molecular scattering by molecules reflected by separate elements of the plate.

\*M. N. Kogan, Doklady Akad. Nauk SSSR, Vol. 144, No. 6 (1962).

<sup>†</sup>The specific interaction law here does not matter. It is important only that all elements of the plate reflect the molecules identically.

We noted above that, in free-molecule flow, all elements of the plate are equivalent, so that the normal and tangential momentum and heat flux brought to the plate by the molecules are the same for arbitrary plates of a given area. When collisions occur, the number, momentum, and energy of particles falling on any element of the plate differ from the corresponding values in free-molecule flow. The change in the number of molecules and in the momentum and energy they carry to a given element evidently depend on the field of reflected molecules, which is the superposition of the fields of individual elements. Depending on the relative location of the elements, the aerodynamic properties of the elements of the plate vary and, therefore, the aerodynamic characteristics of an entire plate of specified area depend on its shape.

Let us denote by  $w_{ij}^k(x_i - x_j, y_i - y_j)$  the influence function of the  $i$ -th element on the  $j$ -th element. The quantity  $\vec{w}_{ij}^k dS_i dS_j$  is the change in the  $k$ -th property due to the element  $dS_j$ , as a result of collisions of the incident molecules and of molecules reflected from the  $j$ -th element, with molecules reflected from the  $i$ -th element.

The subscript  $k$  indicates the property which we are dealing with (normal or tangential momentum, heat flux, number of particles, and so on).

The total change in the  $k$ -th property as a result of collisions is evidently equal to

$$\int \int \int \int \vec{w}_{ij}^k dS_i dS_j, \quad (7.1)$$

where the integration is carried out over the whole plate.

The vector velocity of the incident stream lies in the plane  $xz$  and makes an angle  $\gamma$  with the normal to the plate (Fig. 75). In addition to the undisturbed stream we examine the flow whose vector velocity also lies in the plane  $xz$ , but which makes an angle  $-\gamma$  with the normal. That stream is labeled the reverse stream. The speed of the reverse stream  $|\vec{V}'|$  is equal to the speed of the direct stream  $|\vec{V}|$ .

For the flow examined here there exists a reverse-flow theorem, which asserts that on a plate of arbitrary shape in a direct and reversed stream the nor-

mal momentum, energy, and particle flux are the same, while the tangential momentum has its sign changed.

We introduce the influence function in the reverse stream  $\vec{w}_{ij}^k$ . It is easy to see that

$$\vec{w}_{ij}^k = \vec{w}_{ji}^k \quad (7.2)$$

for the particle flux, the normal momentum, and the energy, and

$$\vec{w}_{ij}^k = -\vec{w}_{ji}^k \quad (7.3)$$

for the components of tangential momentum.

The total change of particle flux, normal momentum, and energy in the reverse stream is equal to

$$\int \int \int \int \vec{w}_{ij}^k dS_i dS_j = \int \int \int \int \vec{w}_{ji}^k dS_i dS_j = \int \int \int \int \vec{w}_{ij}^k dS_i dS_j. \quad (7.4)$$

Similarly, for the tangential momentum, we have

$$\int \int \int \int \vec{w}_{ij}^k dS_i dS_j = - \int \int \int \int \vec{w}_{ji}^k dS_i dS_j = - \int \int \int \int \vec{w}_{ij}^k dS_i dS_j. \quad (7.5)$$

Thus, the aerodynamic characteristics of plates in the direct and reverse streams are the same.

It was assumed above that all elements reflect in the same way. However, the principle of superposition is valid also when different elements reflect differently. In general, for example, the different elements of a surface may have different temperatures.

The superposition principle allows us first to find the influence functions and then to determine the characteristics of plates of different shapes by simple quadratures. This result has been used in a number of papers,\* and also in calculations of flow over a plate at an angle of attack, the results of which were presented in the preceding section.

\* See, for example, the papers: O. G. Fridlender, *Zh. Prikl. Mekhan. i Tekhn. Fiz.*, No. 3 (1963); V. A. Perepukhov, *Inzh. Zh.*, No. 5 (1965).

### §6.8. Comparison of Theoretical and Experimental Data on Flows at Large Knudsen Numbers\*

1. At present, both theoretical and experimental data on flow over bodies at large Knudsen numbers are very sparse. The theoretical investigations are limited mainly to first-collision approximations for hard spheres.† A few calculations have also been carried out by means of the model equations.‡ Experimental data over a wide range of Knudsen numbers are available only for cylinders (fine wires) and strips perpendicular to the flow.

When the results of calculations are compared with experimental data, in addition to errors in the method of computation, we must consider errors due to our insufficient knowledge of the laws of interaction of molecules with the wall and between themselves.

If the body is thermally insulated, the energy carried away by the reflected molecules is equal to that brought by the incident molecules, and does not depend on the accommodation coefficients. If each point of the body surface is thermally insulated (the wall is absolutely nonconducting thermally), and if the momentum of the reflected molecules and the temperature of the wall are uniquely related to the energy carried away by the reflected molecules, it is evident that both the drag of each element of the body, and the temperature distribution over its surface, do not depend on the accommodation coefficients. In another limiting case — the case when the body is absolutely conducting — the drag and the body temperature are independent of the energy accommodation coefficient, if the latter is determined as an average over the whole body (compare § 6.1). Thus, by comparing the theoretical and experimental results for the drag and temperature of a thermally insulated body, we may eliminate the accommodation coefficient from examination.

\* See M. N. Kogan, in a paper presented at the 7th Conference on Problems and Methods in Hydrodynamics, Jurata, Poland, 1965.

† Among these we note the following: M. Heinemann, *Commun. Pure Appl. Math.*, Vol. 1, No. 3 (1948); M. Lunc and J. Lubonski, *Arch. Mech. Stosowanej*, 8: 597 (1956); R. M. L. Baker and A. F. Charwat, *Phys. Fluids*, 1: 73 (1958). See also the papers cited in the two preceding sections.

‡ See M. Rose, *Phys. Fluids*, Vol. 7, No. 8 (1964); G. J. Maslach, D. R. Willis, S. Tang, and D. Ko, "Rarefied Gas Dynamics," Fourth Symposium, Academic Press, New York, 1965.

As noted above, the available theoretical results were obtained either for hard-sphere molecules, or by means of the model equations. It is therefore necessary to find the relation between the properties of real molecules and the diameter of the spheres or the parameters of interaction of the molecules entering into the model equation. We recall that the model equation is the best approximation to the Boltzmann equation for a Maxwellian (more accurately, a pseudo-Maxwellian) gas.\*

We saw in § 6.5 that collisions of the incident and reflected molecules play an important role in free-molecular flows, especially at hypersonic velocities. Therefore, the diameters of hard spheres, or the interaction parameters of other theoretical models of molecules, must be so chosen that the interaction of the molecules is best approximated for those collisions.

Both experimental and theoretical results must be expressed, in the final analysis, in terms of parameters of the incident flow. When we pass from molecular interaction parameters corresponding to the incident flow to interaction parameters appropriate to molecular collisions, we must remember that, for a given mathematical model, they may change differently than in the interaction of real molecules. Thus, for example, if the hard-sphere model has been assumed in the calculations, then, by definition, the collision cross section remains constant in the incident stream, as well as in collisions of incident molecules with reflected molecules.

But, in a real gas, the collision cross sections decrease with increasing relative velocity of the molecules. It is evident that comparable data may be obtained only if the collision cross section of the model hard-sphere molecules is taken equal to the actual cross section for collisions of reflected and incident molecules, and the transition to free-stream parameters is made in accordance with the actual law of molecular interaction. We should bear in mind that for a given gas the transition to free-stream parameters in a wind tunnel experiment (especially in hypersonic tunnels) and under free-flight conditions, may prove to be different. As already noted in § 6.6, the temperature of the incident stream  $T_\infty$  in wind tunnels at large Mach numbers is often much lower than the temperature of the incident stream at free-flight

\*See § 2.8.

conditions at the same Mach number. Therefore, the relative velocities of incident molecules under wind tunnel conditions are much lower than in free flight. But for lower relative velocities the collision cross section varies much more rapidly with the relative velocity of the colliding molecules than for large relative velocities. The result, for example, may be that under wind tunnel conditions the molecules behave like Maxwellian molecules, whereas, in free flight, their cross section does not change much and, therefore, their behavior may be approximated satisfactorily by hard spheres. Therefore, a calculation carried out for hard spheres at fixed Mach and Knudsen numbers may agree with the results of free-flight investigations at the same Mach and Knudsen numbers, whereas the calculation corresponds to wind tunnel tests at different values of Knudsen number of the incident stream.

Let us crudely assume, for example, that in a wind tunnel the collision cross section of the molecules changes as for a Maxwellian gas, in inverse proportion to the relative velocity of the colliding molecules. Since the relative velocity of the molecules in the incident stream is of the order of the velocity of sound  $a$ , while the relative velocity of the molecules in the collision of an incident molecule with a reflected molecule is of the order of the velocity of the incident stream  $V$ , the collision cross section of incident molecules is larger by a factor  $M$  than the cross section of molecules when an incident molecule collides with a reflected molecule. We denote by  $\text{Kn}_s$  the Knudsen number for the calculation carried out with hard-sphere molecules, and by  $\text{Kn}_{\infty t}$  the Knudsen number determined from the parameters of the incident stream in a tunnel. From what was said above, it is clear that the results of calculation may be comparable with the results of experiment only if we choose

$$\text{Kn}_s = \text{Kn}_{\infty t} M. \quad (8.1)$$

On the other hand, a calculation carried out for Maxwellian molecules is comparable with the results of a wind tunnel experiment at identical Mach and Knudsen numbers of the incident stream. However, for comparison with the result of a free-flight experiment (for which we put  $\sigma = \text{const}$ ), the calculation must be carried out at a Knudsen number of

$$\text{Kn}_M = \frac{\text{Kn}_{\infty f}}{M}, \quad (8.2)$$

where  $Kn_{\infty f}$  is the Knudsen number of the free-flight experiment.

Note that, in going from the wind tunnel experiment to free flight, we must generally not compare data at identical incident Mach and Knudsen numbers. In essence, experiments conducted at substantially different incident flow temperatures may be regarded as experiments carried out in different gases, since, at low temperatures, the viscosity varies roughly in proportion to the temperature, while at high temperatures it is proportional to the square root of the temperature. In this example the wind-tunnel experiment is carried out in a Maxwellian gas, while the free-flight experiment takes place in a hard-sphere gas. Their data will be comparable if

$$Kn_{\infty t} = \frac{Kn_{\infty f}}{M}. \quad (8.3)$$

Formulas (8.1)-(8.3) are estimates only, of course. For an actual comparison of experimental and theoretical data, a more accurate calculation is necessary. Thus, the relative velocity of incident molecules was assumed equal to the speed of sound  $a = \sqrt{\pi k T_{\infty} / m}$ , while a more accurate mean relative molecular velocity in a Maxwell distribution is

$$\bar{\xi}_{\text{rel}} = 4 \sqrt{\frac{k T_{\infty}}{\pi m}}.$$

The mean velocity of reflected molecules may be taken, for example, as the most probable velocity or the velocity of molecules whose momentum is equal to the momentum re-emitted by molecules in a Maxwellian distribution.

If the molecules are reflected diffusely with a Maxwellian distribution corresponding to the temperature  $T_r$ , then the first method predicts a mean velocity,

$$v_2 = \sqrt{\frac{2kT_r}{m}} = v_r,$$

and the second,

$$v_2 = \sqrt{\frac{\pi k T_r}{2m}} = \frac{\sqrt{\pi}}{2} v_r.$$

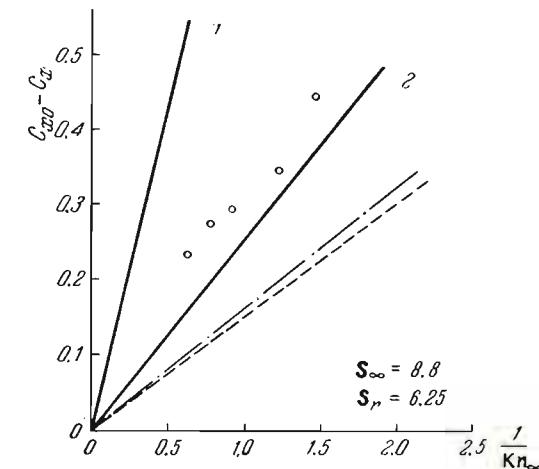


Fig. 76

For a cold body  $v_r \ll V$ , and the difference between these definitions of mean velocity is negligible.

The actual variation of collision cross section may be found, for example, from the variation of viscosity with temperature.

Let us, for instance, compare experimental and theoretical data for spheres. Qualitatively, the flow past a sphere is similar to the flow past a plate perpendicular to the stream discussed in § 6.6. For a given Mach number, the quantity  $C_x - C_{x0}$  varies as  $Kn^{-1}$  regardless of the interaction law of molecules with the surface or each other. Such a comparison is shown in Fig. 76.\* Curve 1 was obtained from a first-collision theory for hard-sphere molecules.† Curve 2 was obtained by means of the modified model equation‡

$$\frac{df}{dt} = A(n_i^2 \Phi_{ii} + 2n_r n_i \Phi_{ri} + n_r^2 \Phi_{rr}) - Anf. \quad (8.4)$$

\*The graph presents experimental data from the paper by M. Kinslow and J. L. Potter, A.I.A.A.J., 1:2467 (1963). The experiments were performed at Knudsen numbers for which the theories compared are already, strictly speaking, unsuitable.

†V. A. Perepukhov, Zh. Vychisl. Mat. i Mat. Fiz., Vol. 6, No. 2 (1967).

‡See G. J. Maslach, D. R. Willis, S. Tang, and D. Ko, "Rarefied Gas Dynamics," Fourth Symposium, Academic Press, New York, 1965.

Here,

$$\Phi_{kl} = \left( \frac{m}{2\pi k T_{kl}} \right)^{3/2} \exp \left\{ -\frac{m}{2kT_{kl}} (\xi - u_{kl})^2 \right\}, \quad (k, l = r, i),$$

$$n_k = \int_{\Omega_k} f d\xi, \quad u_{kl} = \frac{1}{2n_k} \int_{\Omega_k} f \xi d\xi + \frac{1}{2n_l} \int_{\Omega_l} f \xi d\xi,$$

$$T_{kl} = \frac{m}{6kn_k} \int_{\Omega_k} f (\xi - u_{kl})^2 d\xi + \frac{m}{6kn_l} \int_{\Omega_l} f (\xi - u_{kl})^2 d\xi.$$

The subscript *r* refers to properties of molecules whose velocities lie in the solid angle  $\Omega_r$ , subtended by the body at a given point (molecules leaving the body). The subscript *i* refers to all other directions of molecular velocities. That form of the collision term permits a more complete allowance for the interaction of individual groups of molecules [reflected molecules with incident molecules (*ri*), incident molecules with each other (*ii*), and reflected molecules with each other (*rr*)].

All the calculations were carried out for diffuse reflection. The calculations for the modified model equation were made by an iteration method. In examining Couette flow (see §4.2), we saw that in the linear approximation the results of calculations for the model equation coincided with the corresponding solutions of the Boltzmann equation, if the coefficient *A* is expressed in terms of the viscosity:

$$A = \frac{kT_\infty}{\mu(T_\infty)}. \quad (8.5)$$

The diameter of hard-sphere molecules may also be expressed in terms of the viscosity (see §3.8):

$$\sigma = \pi d^2 = \left( \frac{mkT}{\pi} \right)^{1/2} \frac{1}{\mu(T_\infty)}. \quad (8.6)$$

The Knudsen number  $Kn_\infty$  expressed in terms of  $M_\infty$  and  $Re_\infty$  of the incident flow, is equal to

$$Kn_\infty = \sqrt{\frac{1}{2} \pi \frac{M_\infty}{Re_\infty}}.$$

We see on the graph that, plotted in terms of that parameter, the results of calculations obtained from the model equation and from

the first-collision hard-sphere theory disagree appreciably.\* However, that comparison is invalid, since it is between results of calculations carried out not only by two different methods, but also for different gases. For the hard-sphere data to become comparable with the results obtained from the model equation, i.e., for Maxwellian molecules, the data for spheres must be plotted in terms of a Knudsen number, altered in accordance with formula (8.1). With allowance for the corrections indicated above, it may be rewritten in the form

$$Kn_M = Kn_{\infty t} = Kn_s \frac{\bar{\xi}_{\text{rel}}}{V + v_r} = \frac{4 Kn_s}{V^{2\pi} S_\infty \left( 1 + \frac{1}{S_r} \right)} \quad (8.1a)$$

$$\left( S_\infty = V \sqrt{h_\infty}, S_r = \frac{V}{v_r} \right).$$

Curve 1 recalculated in that way is shown by a dotted line in Fig. 76. If we bear in mind the approximate nature of these calculations, the agreement of results obtained by different methods after replotting may be considered satisfactory. The fact that even the experimental results are now close to the theoretical calculations indicates that the behavior of real molecules under the conditions of the experiment is closer to the behavior of Maxwellian molecules than to hard spheres. The same conclusion was reached from the comparison made of experimental data with results obtained from the modified model equation for a cylinder and a strip.<sup>†</sup> See Figs. 77 and 78.

To correlate theoretical and experimental results in a viscous gas, use is often made of a Reynolds number in which the viscosity corresponds to the temperature behind the shock or to the stagnation temperature. In the language of kinetic theory, this is equivalent to choosing the stagnation conditions to define a characteristic relative molecular velocity. As we have seen, the relative velocity of molecules in the first collisions is somewhat larger than *V*. It is easy to see that the relative velocity of the molecules  $\xi_{\text{rel}}(T_0)$  is somewhat less than *V* for a monatomic gas and larger for a diatomic gas, for which at large Mach numbers

\* We note that the results given here of Perepukhov's calculations give a correction to the free-molecule drag of a sphere, which is less by a figure of about two than in the paper by Baker and Charwat [R.M.L. Baker and A. F. Charwat, Phys. Fluids, 1: 73 (1958)].

<sup>†</sup> See Masiach et al., cited above.

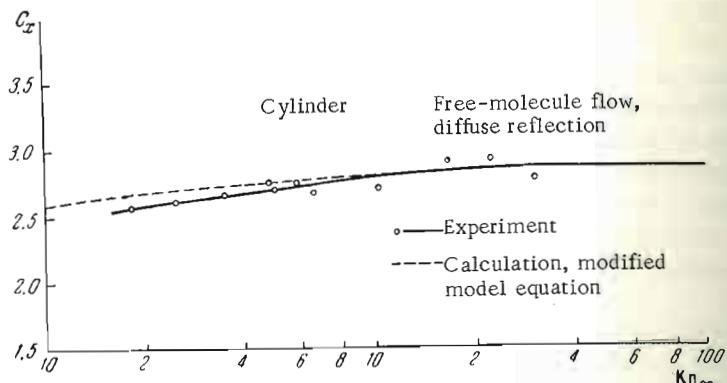


Fig. 77

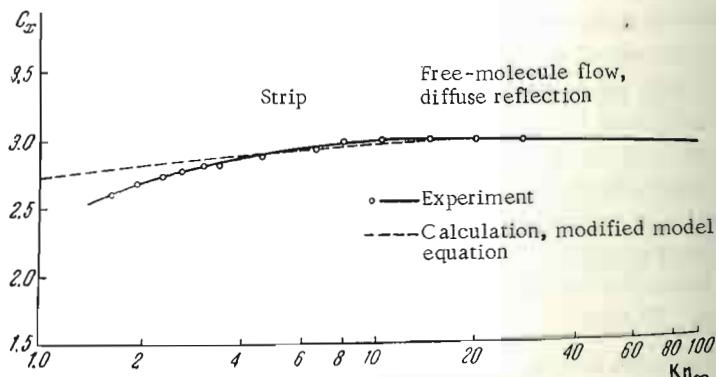


Fig. 78

the estimate

$$\frac{V}{\xi_{\text{rel}}(T_0)} \sim \frac{1}{4} \sqrt{2\pi \frac{\kappa}{\kappa-1}}$$

is valid. Here, instead of formula (8.1a), we evidently have

$$\text{Kn}_M = \text{Kn}_{\text{out}} = \text{Kn}_s \sqrt{\frac{T_\infty}{T_0}}. \quad (8.1b)$$

The data for hard spheres recalculated according to that formula are shown by dot-dash lines on Fig. 76.

Thus, when we use a Reynolds number based on stagnation parameters to correlate the data, we decrease somewhat the relative velocities of the molecules during first collisions. However, as the Knudsen number decreases, the role of collisions of incident molecules with reflected molecules decreases, and the correlation of data in terms of stagnation temperature becomes more correct. Therefore, the data can be correlated over a wide range of Knudsen numbers, for example, in terms of the parameter\*

$$\alpha = \frac{1}{\text{Kn}} \left( \frac{T_0}{T_\infty} \right) \frac{\mu(T_\infty)}{\mu(T_0)}, \quad (8.7)$$

which goes over to  $1/\text{Kn}$  for Maxwellian molecules, while for hard spheres it corresponds to formula (8.1b).

Data for a "cold" body ( $S_r \gg 1$ ) were correlated above. If the body is "hot" or  $\alpha_e$  is small, the velocity of the reflected molecules is of order  $V$  and, therefore, the relative velocity of the molecules in first collisions is more than twice the velocity  $\xi_{\text{rel}}(T_0)$ . In that case, a recalculation based on the stagnation temperature [according to formula (8.1b)] is considerably less correct than a recalculation according to formula (8.1a). That situation often arises in continuous wind tunnels where the temperature of the body is comparable to the stagnation temperature, and in nearly free-molecular flows even higher, as noted in § 6.1.

2. For comparison of theory with experiment at large Knudsen numbers, a very interesting topic of research is the discharge of a gas through a small orifice into vacuum. The characteristics of the flow do not depend on the interaction of the molecules with the walls, and thereby we eliminate the element of uncertainty associated with lack of knowledge of the reflection of molecules from solid surfaces. On the other hand, the discharge of a gas through an orifice at moderate pressures has been very accurately measured.

\*Cf. the analogous parameter in: F. S. Sherman, D. R. Willis, and G. J. Maslach, in a paper presented at the 11th International Congress on Applied Mechanics, Munich, 1964.

In § 6.3, we have examined the free-molecule discharge of a gas through an orifice. The effect of collisions may be taken into account by the method of iterations, using, for example, the integral form of the Boltzmann equation (see §§ 2.7 and 6.5)

$$f^{(n+1)}(x, \xi) = f(x_0, \xi) \exp \left\{ - \int_{x_0}^x \frac{J_2^{(n)}(l', \xi)}{\xi} dl' \right\}$$

$$+ \int_{x_0}^x J_1^{(n)}(l, \xi) \exp \left\{ - \int_l^x \frac{J_2^{(n)}(l', \xi)}{\xi} dl' \right\} \frac{dl}{\xi}. \quad (8.8)$$

We recall that this integral equation is valid only for molecules of finite interaction radius.

Since the measured quantity is the flow rate of the gas, it is sufficient to determine the distribution function in the plane of the orifice. For the trajectories of molecules which arrive at the orifice plane from the high-pressure vessel, the function  $f(x_0, \xi)$  in formula (8.8) must be put equal to the equilibrium Maxwellian distribution function of the molecules in the vessel, since it is assumed that the dimensions of the vessel are so large that the distribution function at a sufficiently large distance from the orifice has not been perturbed by the discharge. For the trajectories originating in the low-pressure vessel (theoretically from the vacuum chamber) the function  $f(x_0, \xi)$  must evidently be put equal to zero. As the zeroth approximation for the distribution function we may take, for example, the distribution function of a free-molecule discharge. It is easy to see that, sufficiently far from the orifice, the distribution function will differ substantially from free-molecule functions even at an arbitrarily low pressure. This evidently leads to nonuniform convergence of the successive approximations, just as in the calculation of the flow past bodies in a nearly free-molecular stream (see § 6.5). Yet, we may hope that the first iteration gives a correct result near the orifice, as it did in calculating distribution function on the body. In fact, even the first iteration for the total pressure (8.8) has not yet been carried out, even for the simplest molecular models.

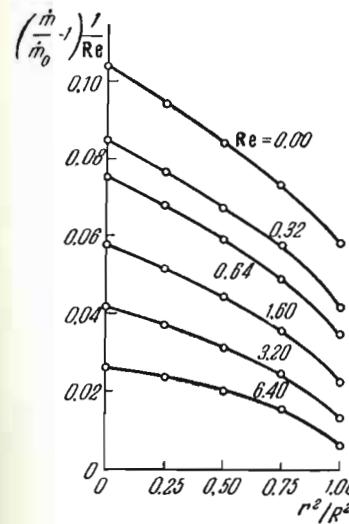


Fig. 79

Willis\* computed the first iteration for the model equation, i.e., for Eq. (8.8) in which the integral  $J_2$  was replaced by  $A_n$ , and the integral  $J_1$  by  $A_n f_0$ , where  $f_0$  is the local Maxwellian distribution, and  $A$  is a coefficient which defines the interaction of the molecules. In the calculations he assumed that the portion of molecules returning to the high-pressure vessel as a result of collisions was small, and neglected it. Therefore, the local mass flow rate is determined by the expression

$$\dot{m} = \int_{\xi_x > 0} \xi_x f d\xi. \quad (8.9)$$

where  $\xi_x$  is the velocity component along the flow axis; here,  $\xi_x > 0$  for molecules leaving the high-pressure vessel.

For a circular orifice at sufficiently large Knudsen numbers or sufficiently small Reynolds numbers, the mass flow rate may be represented in the form

$$\dot{m} = \dot{m}_0 (1 + a \text{Re} + \dots), \quad (\text{Re} = 2R\rho \left( \frac{kT}{m} \right)^{1/2} \frac{1}{\mu}), \quad (8.10)$$

where  $R$  is the orifice radius,  $m_0$  is the mass flow rate in a free-molecule stream, and  $T$  is the temperature of the gas in the vessel.

Generally speaking, the first iteration gives not only the value of the coefficient  $a$ , but also the values of some of the higher order terms, whose calculation is beyond the accuracy of the approximation. Therefore, only the quantity

$$\lim_{\text{Re} \rightarrow 0} \left( \frac{\dot{m}}{\dot{m}_0} - 1 \right) = a \quad (8.11)$$

is justified by the analysis.

\*D. R. Willis, J. Fluid Mech., 21:1 (1965).

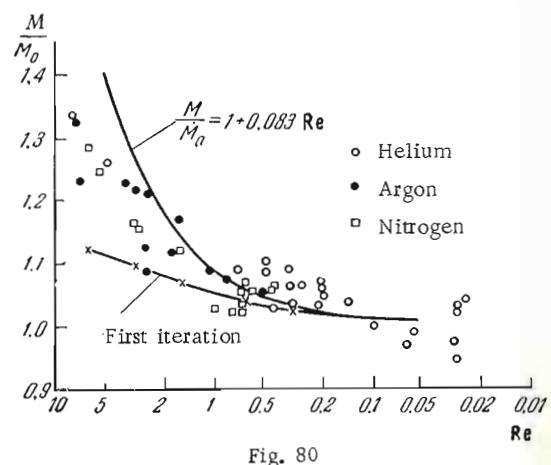


Fig. 80

Figure 79 shows the variation of local mass flow over the orifice for various pressures or various  $Re$  numbers (the origin of coordinates is chosen at the center of the orifice). The curve corresponding to  $Re=0$  determines the local value of the coefficient  $a$ . All the other curves take into account higher-order terms and serve only to estimate the region of applicability of the first approximation.

The total mass flow rate through the orifice,  $\dot{M}$ , may also be represented in the form (8.10):

$$\dot{M} = \dot{M}_0(1 + A Re + \dots). \quad (8.12)$$

A limiting process analogous to (8.11) gives for  $A$  the value 0.083. A comparison is made in Fig. 80 of the curve

$$\dot{M} = \dot{M}_0(1 + 0.083 Re) \quad (8.13)$$

with the results of measurements of flow rate made by Liepmann\* for helium, nitrogen, and argon. The figure also shows a curve corresponding to the full first approximation. The results of the experiment are in qualitative agreement with the curve (8.13), which is the only one justified in the framework of the first approximation.

\*H. W. Liepmann, J. Fluid Mech., 10: 65 (1961).

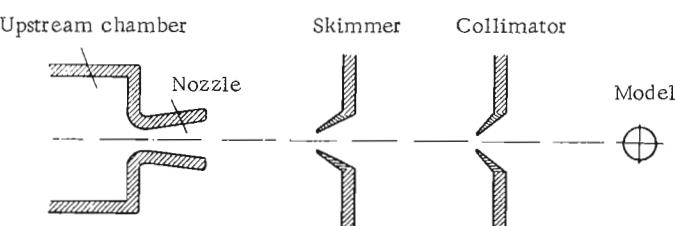


Fig. 81

Unfortunately, the scatter of the experimental points is somewhat too large at  $Re < 1$  where we may expect satisfactory accuracy from the first iteration.

### §6.9. Discharge into Vacuum

A number of practical problems require the creation of high-speed streams of a highly rarefied gas. The need for such streams arises, for example, in the design of jets for the control of space equipment or in simulating conditions of flight at high altitudes. The latter problem is especially complex, since, in order to carry out aerodynamic tests, it is not sufficient to discharge the gas with a low specific impulse; one needs streams which can be precisely controlled (capable of being calculated reliably or of being easily measured experimentally) with a uniform core of sufficient dimensions.

1. One of the simplest ways of obtaining easily calculated streams of highly rarefied gas (molecular beams) has already been examined in §§ 6.3 and 6.8. It is the free-molecule discharge of a gas into vacuum through an orifice of diameter much smaller than the mean free path of the molecules in the vessel. However, that method has two substantial drawbacks: the stream velocity is low and the intensity of the stream is small. In fact, the molecules are discharged from the vessel at thermal velocities, proportional to the square root of the gas temperature in the vessel, and inversely proportional to the mass of the molecules. For example, for nitrogen, at standard temperature in the vessel, the mean thermal velocity of the molecules is  $v_m < 10^5$  cm/sec. Therefore, to obtain molecules with velocities, let us say, commensurate with

the velocity of an Earth satellite, i.e.,  $\sim 10^6$  cm/sec, we need to heat the gas to several thousand degrees. To obtain a gas under such conditions, and to keep it for any length of time, presents a problem in itself. However, even if we succeed in obtaining such a high-temperature gas, its molecules would possess completely different properties, since they would undergo dissociation and ionization.

The intensity of a stream, i.e., the number of molecules crossing a unit area of the orifice per unit time, is proportional to the density and the velocity of the molecules and, therefore, to the square root of the temperature. An increase of temperature is associated with difficulties, which were mentioned above. The opportunity for increase in the gas density is limited, since its mean free path must be much larger than the diameter of the orifice. Otherwise, the flow becomes difficult to calculate and the determination of the flow parameters becomes very complicated. However, we cannot realize a high beam intensity even within those limits. Since the discharging molecules are distributed according to the cosine law, the intensity of the beam falls off in inverse proportion to the square of the distance from the aperture. Therefore, at the location of the model, the intensity of the beam is still far too low (by several orders of magnitude).

To increase the intensity of the stream, an interesting scheme was proposed by Kantrowitz and Grey.\* Their idea is to draw off a free-molecule stream, not from a gas at rest, but from a gas expanded in a nozzle (Fig. 81). It is known that the velocity of a gas expanded to a high Mach number in a nozzle is close to its maximum velocity, and larger by a factor of roughly two than the velocity of molecules in the upstream chamber.

At large Mach numbers the velocity of directed motion of the molecules is much greater than the thermal velocity, i.e., the molecules travel almost as a parallel beam.<sup>†</sup>

Therefore, if the gas expands to a high Mach number (in practice,  $M = 5-8$ ) ahead of the skimmer, then with the same skimmer aperture diameter as for the simple source described above, and for the same gas density ahead of the skimmer, the gas

\*A. Kantrowitz and J. Grey, Rev. Sci. Instr., 22: 328 (1951).

<sup>†</sup>For more details regarding the structure of a gasdynamic discharge into a vacuum, see below.

flow rate will be several times larger, not only because of increased flow velocity, but also because of the directed motion of the molecules. But the main effect of the directionality of the stream is that the intensity of the beam between the skimmer and the test model varies comparatively little. However, this ideal scheme may be carried out only if all the molecules which strike the skimmer surface are absorbed by it (as on cryogenic panels). Otherwise, the molecules reflected from the skimmer distort the flow at the entrance to the skimmer orifice, and disturb its directionality. To increase the beam intensity one tries to have the maximum possible gas density ahead of the aperture and, therefore, the minimum possible mean free path of the molecules. It is usually chosen to be comparable to the diameter of the orifice. Therefore, the mean free path turns out to be small with respect to the characteristic dimensions of the skimmer. In front of the skimmer there is a compression zone (or a shock); the flow differs appreciably from the ideal scheme; its directionality is disturbed, and the source intensity is appreciably decreased. Sometimes this decrease is explained by the bluntness of the skimmer leading edges. However, in fact, it is associated with the general flow pattern of the gas through the skimmer which occurs for any wall thickness.\* The flow near the orifice of the skimmer becomes very complex, passing through all regimes from continuum almost to free-molecule. As we saw in the previous chapters, the methods of calculating transition flows have been applied only in one-dimensional problems. The complex transition two-dimensional flow near the skimmer orifice has only been studied in detail experimentally.<sup>†</sup>

Generally speaking, the calculation and investigation of flow near the skimmer orifice is a problem just as complex as the problems which are studied in that type of equipment.

2. As we saw above, the intensity of the discharge from an orifice increases with increasing gas density in the vessel. In the limit we arrive at conditions when the molecular mean free path becomes much smaller than the diameter of the orifice and we have a gasdynamic efflux into a vacuum.

\*We saw in § 5.2 that even at the leading edge of an infinitely thin plate, parallel to the stream, with  $\lambda < L$  the flow is not free-molecule.

<sup>†</sup>A large number of papers devoted to experimental investigation of flows in equipment of a similar type may be found in the Proceedings of all the Symposia on the Dynamics of Rarefied Gases.

In principle (without account of dissipative processes), an ideal gasdynamic expansion into vacuum through an orifice or a nozzle, accelerates the gas to any Mach number and to an arbitrarily low density. However, in fact, when we try to obtain a rarefied gas stream by means of a nozzle, the rapid growth of the boundary layer in the diverging part of the nozzle prevents the establishment of the flow conditions calculated in the ideal scheme. In order to avoid that difficulty, a great deal of attention has been devoted in recent years to the study of freely expanding jets.\* In that flow the nozzle has no walls and, therefore, there is no boundary layer to disturb the establishment of the regime. However, it turns out that in that flow the presence of dissipative processes does not allow us to obtain arbitrarily high Mach numbers.

Let us examine this interesting phenomenon.

The discharge of a jet into vacuum is a complex two-dimensional flow, in which we find all the regimes from the continuum to the nearly free-molecular. The construction of a solution of the Boltzmann equation for this problem is too complex at the present time. However, the problem may be simplified, since the flow from a cylindrical or spherical source to some extent simulates the flow along the axis of a plane or an axisymmetric jet, respectively.† Thus, the investigation reduces to the solution of a one-dimensional problem for the Boltzmann equation. Even so, an exact solution of the Boltzmann equation corresponding to a point or line source has not been found.

We may hope to construct an approximate solution of the problem by considering the flow from a sonic line to some sufficiently large Mach number by means of the usual inviscid hydrodynamic source flow, and further downstream by means of the Boltzmann equation (Fig. 82). The lines where the solutions are matched must be located in the region where the dissipative processes may still be neglected, and where the solution of the Boltzmann

\*See, for example, F. S. Sherman, Arch. Mech. Stosowanej, 2: 16 (1964); V. N. Gusev, Mekhan. Zhdostki i Gaza, No. 1 (1967).

†See M. D. Ladyzhenskii, Doklady Akad. Nauk SSSR, Vol. 134, No. 2 (1960); M. D. Ladyzhenskii, Prikl. Mat. i Mekhan., Vol. 26, No. 4 (1962); H. Ashkenas and F. S. Sherman, "Rarefied Gas Dynamics," Fourth Symposium, Academic Press, New York, 1965.

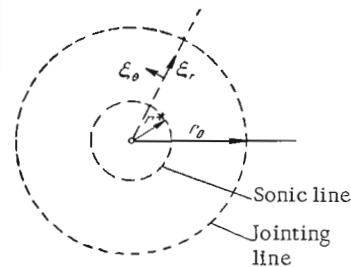


Fig. 82

In fact, in accordance with the formulation of problems of the Boltzmann equation (see § 2.9) on the matching line we may assign a distribution function only for molecules whose velocity is positive.

The distribution function for molecules with velocities  $\xi_r < 0$  must be obtained from the solution of the Boltzmann equation under the condition that the pressure vanishes at infinity.

If in the internal (hydrodynamic) region the solution were an exact solution of the kinetic equation, then the values of the distribution function for  $\xi_r < 0$  found from solution of the external problem would coincide with the corresponding values obtained for the internal problem. However, the gasdynamic solution corresponds to the approximation involving the replacement of the distribution function by a local Maxwellian distribution. Therefore, it is impossible to carry out a rigorous matching.

In their paper (see above), Brook and Oman circumvented that difficulty by further coarsening the flow scheme. Since the matching line is located in the high Mach-number region, the majority of the molecules have velocities  $\xi_r > 0$ , and only a small fraction return upstream. Therefore, molecules traveling upstream may be neglected in general, by considering the distribution function to differ from zero only for  $\xi_r > 0$ . Later on, by making a rough estimate of the terms of the model equation, and dropping most of them, they obtain the equation

\*J. W. Brook and R. A. Oman, "Rarefied Gas Dynamics," Fourth Symposium, Academic Press, New York, 1965.

equation at some section agrees with the solution of the Euler equations. The problem in that formulation has been examined by Brook and Oman\* for the model Boltzmann equation, which, as we repeatedly stated above, allows us to obtain satisfactory qualitative results.

Strictly speaking, the solution of the Euler equations and of the kinetic equations cannot be matched.

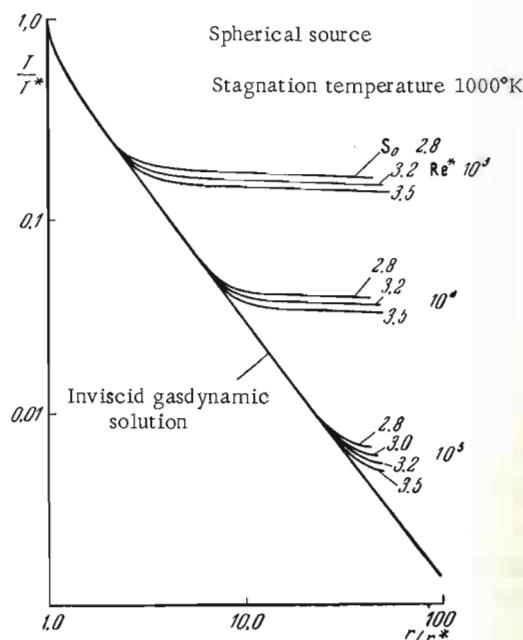


Fig. 83

$$\xi_r \frac{d}{dr} (r^\nu f) = Anr^\nu (f_0 - f), \quad (9.1)$$

where  $\nu = 2$  for a spherical source, and  $\nu = 1$  for a cylindrical source.

That equation, rewritten in integral form, was solved numerically by the method of successive approximations.

The calculations show that the velocity and density of the flow, both for the cylindrical and the spherical source, differ only slightly from the velocity and density of an ideal source. The variation in temperature proves to be of greatest interest. In the expansion of the gas, the temperature first drops, as for an ideal source, and then its pattern sharply changes and it becomes nearly constant (results of calculations for a spherical source are presented in Fig. 83; the picture is similar for a cylindrical source).

The location of the temperature knee depends on the Reynolds number  $Re^*$  or the Knudsen number at the critical section (on the line  $M=1$ ). The higher  $Re^*$ , the longer does the actual flow follow the ideal source flow and the higher the Mach number at which stabilization begins.<sup>†</sup>

Since the velocity tends to a limiting value close to the maximum discharge velocity, and the temperature becomes stabilized, stabilization of the Mach number also occurs.<sup>‡</sup>

Thus, dissipative processes prevent the attainment of an arbitrarily high Mach number in a free jet. It may be said that the growth of the boundary layer on the nozzle walls which prevents the increase of Mach number is due to the action of "transverse viscosity" (dissipative processes due to the gradients of the flow parameters, perpendicular to the stream direction). Then the above-described phenomenon of temperature and Mach-number freezing may be regarded as due to the influence of "longitudinal viscosity" (longitudinal gradients).

Because of the extremely simplified formulation of the problem, this analysis may claim only qualitative validity.

A more careful analysis has been given by Edwards and Cheng<sup>¶</sup> and by Hamel and Willis.<sup>§</sup> In those papers, the moment equations were used. For a spherical source, the qualitative results agree with those described above. In the paper by Hamel and Willis, in which a rigorous matching of the asymptotic solutions of the moment equations in the external and internal regions was performed, it was shown that in the external region the same

<sup>†</sup>Figure 83 shows several curves corresponding to the same value of  $Re^*$ . Each curve corresponds to a different matching location ( $r_0$ ) or, analogously, to a different Mach number or velocity ratio  $S_0$  at the beginning of the kinetic calculation. It is interesting to note that a shift in  $r_0$  is accompanied by a corresponding shift in the frozen temperature, in spite of the fact that the temperature obtained from the kinetic calculation agrees to a large extent with the temperature for an ideal source. The reason for such a "good memory" in the solution up to the end has not been explained.

<sup>‡</sup>Stabilization of temperature and of Mach number have been observed experimentally. See J. B. Anderson, R. P. Anders, J. B. Fenn, and G. Maise, "Rarefied Gas Dynamics," Fourth Symposium, Academic Press, New York, 1965.

<sup>¶</sup>R. H. Edwards and H. K. Cheng, A.I.A.A. J., Vol. 4, No. 3 (1966).

<sup>§</sup>B. Hamel and D. R. Willis, Phys. Fluids, Vol. 9, No. 5 (1966).

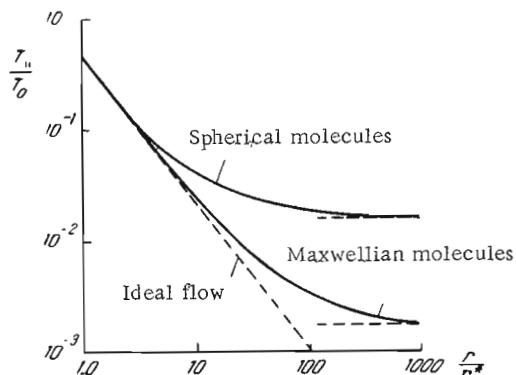


Fig. 84

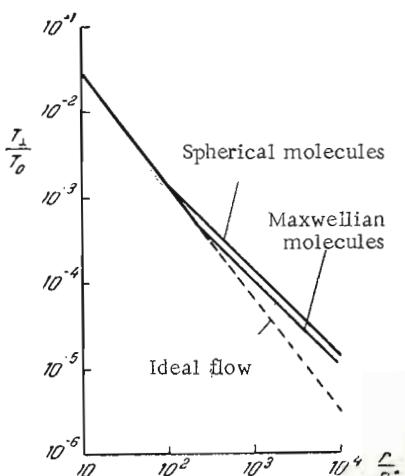


Fig. 85

results can be obtained if the distribution function is represented in the form

$$f = n \left[ \left( \frac{m}{2\pi k T_{\perp}} \right) \exp \left( -\frac{m \xi_{\perp}^2}{2k T_{\perp}} \right) \right] \left[ \left( \frac{m}{2\pi k T_{\parallel}} \right)^{1/2} \exp \left( -\frac{m (\xi_{\parallel} - u)^2}{2\pi T_{\parallel}} \right) \right]. \quad (9.2)$$

This is an ellipsoidal distribution, similar to that introduced by Holway for the investigation of the structure of a shock wave (see § 4.4). The distribution contains two temperatures:  $T_{\parallel}$  is the parallel temperature, which characterizes the scatter with respect to the velocities  $\xi_{\parallel} = \xi_r$ , and  $T_{\perp}$  is the transverse temperature, which characterizes the scatter with respect to transverse velocities  $\xi_{\perp}$ . The transverse temperature is determined by molecular collisions. As the flow expands, collisions become less frequent. The flow increasingly approximates a radial expansion. Therefore, the parallel temperature is stabilized (Fig. 84), while the transverse temperature falls almost in the same way as for an ideal source (Fig. 85). However, this fall is slower than might be expected from geometric considerations (i.e., proportional to  $r^{-2}$ ). This is apparently due to collisions. Although the collisions are rare, at each collision the transverse temperature obtains a substantial "addition" (in relation to  $T_{\perp}$ ). Since  $T_{\parallel} \gg T_{\perp}$ , that "transfer" of energy has no effect on the parallel temperature.

In contrast with Brook and Oman, the papers of Edwards and Cheng, and Hamel and Willis show no freezing of temperature in the cylindrical source.

The stabilization of temperature in a cylindrical source had already been observed earlier by Ladyzhenskii,\* who investigated the flow in a plane and a spherical source by means of the Navier-Stokes equations. Assuming a power law variation of viscosity with temperature  $\mu \sim T^n$  ( $n = 1$  for Maxwellian molecules,  $n = 0.5$  for spherical molecules), Ladyzhenskii showed that when  $n \geq n_0$  ( $n_0 = 1$  for the plane case and  $n_0 > 1$  for the spherical case), the viscosity does not distort the flow of the ideal source. When  $n < n_0$  in the plane case, temperature stabilization occurs, and in the spherical case there is stabilization of the temperature, and the velocity falls to zero.

The kinetic analysis in the paper by Hamel and Willis also shows that when  $n > n_0$ , the real flow does not differ from the ideal source flow.

However, when  $n < n_0$ , i.e., in the region where dissipative processes appear, the results of a kinetic analysis and of the

\* M. D. Ladyzhenskii, Prikl. Mat. i Mekhan., Vol. 26, No. 4 (1962).

Navier-Stokes equations differ. The Navier-Stokes analysis leads to a higher influence of dissipation.

For the problem examined, the Navier-Stokes equations permit correct determination of the boundary of the region in which it is necessary to take into account dissipative processes, but they lead to qualitatively incorrect results in evaluating the actual influence of those processes. As has been noted already by Ladyzhenskii,\* the dissipative processes become appreciable in the region where, strictly speaking, the Navier-Stokes equations have already become invalid.

## APPENDIX 1

The Boltzmann equation in curvilinear coordinates\* is

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{\xi_1}{H_1} \frac{\partial f}{\partial q_1} + \frac{\xi_2}{H_2} \frac{\partial f}{\partial q_2} + \frac{\xi_3}{H_3} \frac{\partial f}{\partial q_3} + & \left( -\frac{\xi_1 \xi_2}{H_1 H_2} \frac{\partial H_1}{\partial q_2} - \frac{\xi_1 \xi_3}{H_1 H_3} \frac{\partial H_1}{\partial q_3} \right. \\ & + \frac{\xi_2^2}{H_2 H_1} \frac{\partial H_2}{\partial q_1} + \frac{\xi_3^2}{H_3 H_1} \frac{\partial H_3}{\partial q_1} \Big) \frac{\partial f}{\partial \xi_1} + \left( \frac{\xi_1^2}{H_1 H_2} \frac{\partial H_1}{\partial q_2} - \frac{\xi_2 \xi_3}{H_2 H_3} \frac{\partial H_2}{\partial q_3} \right. \\ & - \frac{\xi_2 \xi_1}{H_2 H_1} \frac{\partial H_2}{\partial q_1} + \frac{\xi_3^2}{H_3 H_2} \frac{\partial H_3}{\partial q_2} \Big) \frac{\partial f}{\partial \xi_2} + \left( \frac{\xi_1^2}{H_1 H_3} \frac{\partial H_1}{\partial q_3} + \frac{\xi_2^2}{H_2 H_3} \frac{\partial H_2}{\partial q_3} \right. \\ & \left. - \frac{\xi_3 \xi_1}{H_3 H_1} \frac{\partial H_3}{\partial q_1} - \frac{\xi_3 \xi_2}{H_3 H_2} \frac{\partial H_3}{\partial q_2} \right) \frac{\partial f}{\partial \xi_3} = J; \end{aligned}$$

where  $q_1$ ,  $q_2$ , and  $q_3$  are curvilinear orthogonal coordinates,  $\xi_1$ ,  $\xi_2$ ,  $\xi_3$  are projections of the vector velocity of the molecules on the curvilinear coordinate axes,  $H_1$ ,  $H_2$ ,  $H_3$  are the Lamé coefficients.

The Boltzmann equation in cylindrical coordinates is

$$\begin{aligned} q_1 = r, & \quad q_2 = \varphi, \quad q_3 = z, \\ H_1 = 1, & \quad H_2 = r, \quad H_3 = 1, \\ \xi_1 = \xi_r, & \quad \xi_2 = \xi_\varphi, \quad \xi_3 = \xi_z, \\ \frac{\partial f}{\partial t} + \xi_r \frac{\partial f}{\partial r} + \frac{\xi_\varphi}{r} \frac{\partial f}{\partial \varphi} + \xi_z \frac{\partial f}{\partial z} + \frac{\xi_\varphi^2}{r} \frac{\partial f}{\partial \xi_r} - \frac{\xi_r \xi_\varphi}{r} \frac{\partial f}{\partial \xi_\varphi} = J. \end{aligned}$$

The Boltzmann equation in spherical coordinates is

$$\begin{aligned} q_1 = r, & \quad q_2 = \varphi, \quad q_3 = \theta, \\ H_1 = 1, & \quad H_2 = r \sin \theta, \quad H_3 = r, \\ \xi_1 = \xi_r, & \quad \xi_2 = \xi_\varphi, \quad \xi_3 = \xi_\theta, \end{aligned}$$

\*The derivation of the Boltzmann equation in curvilinear coordinates may be found, for example, in the monograph by Vlasov (A. A. Vlasov, "Statistical Distribution Functions," Nauka Press, 1966), or in the paper by Shakhov [E. M. Shakhov, Mekhan. Zhidkosti i Gaza, No. 2 (1967)].

\*See the paper cited above.

$$\frac{\partial f}{\partial t} + \xi_r \frac{\partial f}{\partial r} + \frac{\xi_\varphi}{r \sin \theta} \frac{\partial f}{\partial \varphi} + \frac{\xi_\theta}{r} \frac{\partial f}{\partial \theta} + \frac{\xi_\varphi^2 + \xi_\theta^2}{r} \frac{\partial f}{\partial \xi_r}$$

$$- \left( \frac{\xi_\varphi \xi_\theta}{r} \operatorname{ctg} \theta + \frac{\xi_r \xi_\varphi}{r} \right) \frac{\partial f}{\partial \xi_\varphi} + \left( \frac{\xi_\varphi^2}{r} \operatorname{ctg} \theta - \frac{\xi_r \xi_\theta}{r} \right) \frac{\partial f}{\partial \xi_\theta} = J.$$

**APPENDIX 2****Some Frequently Encountered Integrals**

$$1. \int_{-\infty}^{+\infty} e^{-\beta x^2} dx = \sqrt{\frac{\pi}{\beta}}, \quad \int_{-\infty}^{+\infty} x^2 e^{-\beta x^2} dx = \frac{1}{2\beta} \sqrt{\frac{\pi}{\beta}},$$

$$\int_{-\infty}^{+\infty} x^{2n} e^{-\beta x^2} dx = \frac{(2n-1)(2n-3)\dots5\cdot3\cdot1}{2^n} \sqrt{\frac{\pi}{\beta^{2n+1}}}.$$

$$2. \int_0^{\infty} x e^{-\beta x^2} dx = \frac{1}{2\beta}, \quad \int_0^{\infty} x^3 e^{-\beta x^2} dx = \frac{1}{2\beta^2},$$

$$\int_0^{\infty} x^5 e^{-\beta x^2} dx = \frac{1}{\beta^3}, \quad \int_0^{\infty} x^{2n+1} e^{-\beta x^2} dx = \frac{n!}{2\beta^{n+1}},$$

$$\int_0^{\infty} x^n e^{-\beta x^2} dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\beta^{\frac{n+1}{2}}}.$$

$$3. \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_i^2 x^2 e^{-\beta x^2} dx_1 dx_2 dx_3 = \frac{5}{4} \frac{\pi^{3/2}}{\beta^{7/2}}$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_i^2 x^4 e^{-\beta x^2} dx_1 dx_2 dx_3 = \frac{35}{8} \frac{\pi^{3/2}}{\beta^{9/2}},$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^4 e^{-\beta x^2} dx_1 dx_2 dx_3 = \frac{15}{4} \frac{\pi^{3/2}}{\beta^{7/2}}.$$

$$(x^2 = x_1^2 + x_2^2 + x_3^2),$$

4.  $F(x)$  an arbitrary function,  $x^2 = x_1^2 + x_2^2 + x_3^2$ .

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_i^4 F(x) dx_1 dx_2 dx_3 = \frac{1}{5} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^4 F(x) dx_1 dx_2 dx_3,$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_i^2 x_j^2 F(x) dx_1 dx_2 dx_3 = \frac{1}{3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^4 F(x) dx_1 dx_2 dx_3,$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_i^2 x_j^2 F(x) dx_1 dx_2 dx_3 = \frac{1}{15} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^4 F(x) dx_1 dx_2 dx_3.$$

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