CHAPTER

4

Conservation Laws

OUTLINE

CHAPTER OBJECTIVES

- To present a derivation of the governing equations for moving fluids starting from the principles of mass, momentum, and energy conservation for a material volume.
- To illustrate the application of the integral forms of the mass and momentum conservation equations to stationary, steadily moving, and accelerating control volumes.
- To develop the constitutive equation for a Newtonian fluid and provide the Navier-Stokes differential momentum equation.
- To show how the differential momentum equation is modified in noninertial frames of reference.
- To develop the differential energy equation and highlight its internal coupling between mechanical and thermal energies.

- To present several common extensions and simplified forms of the equations of motion.
- To derive and describe the dimensionless numbers that appear naturally when the equations of motion are put in dimensionless form.

4.1. INTRODUCTION

The governing principles in fluid mechanics are the conservation laws for mass, momentum, and energy. These laws are presented in this order in this chapter and can be stated in *integral* form, applicable to an extended region, or in *differential* form, applicable at a point or to a fluid particle. Both forms are equally valid and may be derived from each other. The integral forms of the equations of motion are stated in terms of the evolution of a control volume and the fluxes of mass, momentum, and energy that cross its control surface. The integral forms are typically useful when the spatial extent of potentially complicated flow details are small enough for them to be neglected and an average or integral flow property, such as a mass flux, a surface pressure force, or an overall velocity or acceleration, is sought. The integral forms are commonly taught in first courses on fluid mechanics where they are specialized to a variety of different control volume conditions (stationary, steadily moving, accelerating, deforming, etc.). Nevertheless, the integral forms of the equations are developed here for completeness and to unify the various control volume concepts.

The differential forms of the equations of motion are coupled nonlinear partial differential equations for the dependent flow-field variables of density, velocity, pressure, temperature, etc. Thus, the differential forms are often more appropriate for detailed analysis when field information is needed instead of average or integrated quantities. However, both approaches can be used for either scenario when appropriately refined for the task at hand. In the development of the differential equations of fluid motion, attention is given to determining when a solvable system of equations has been found by comparing the number of equations with the number of unknown dependent field variables. At the outset of this monitoring effort, the fluid's thermodynamic characteristics are assumed to provide as many as two equations, the thermal and caloric equations of state (1.12).

The development of the integral and differential equations of fluid motion presented in this chapter is not unique, and alternatives are readily found in other references. The version presented here is primarily based on that in Thompson (1972).

4.2. CONSERVATION OF MASS

Setting aside nuclear reactions and relativistic effects, mass is neither created nor destroyed. Thus, individual mass elements—molecules, grains, fluid particles, etc.—may be tracked within a flow field because they will not disappear and new elements will not spontaneously appear. The equations representing conservation of mass in a flowing fluid are based on the principle that the mass of a specific collection of neighboring fluid particles is constant. The volume occupied by a specific collection of fluid particles is called a *material volume* $V(t)$. Such a volume moves and deforms within a fluid flow so that it always contains the same mass elements; none enter the volume and none leave it. This implies that a material volume's surface $A(t)$, a material surface, must move at the local fluid velocity $\mathbf{u}$ so that fluid particles inside $V(t)$ remain inside and fluid particles outside $V(t)$ remain outside. Thus, a statement of conservation of mass for a material volume in a flowing fluid is:

$$\frac{d}{dt}\int\limits_{V(t)} \rho(\mathbf{x},t)dV = 0. \tag{4.1}$$

where ρ is the fluid density. Figure 3.18 depicts a material volume when the control surface velocity $\mathbf{b}$ is equal to $\mathbf{u}$. The primary concept here is equivalent to an infinitely flexible, perfectly sealed thin-walled balloon containing fluid. The balloon's contents play the role of the material volume $V(t)$ with the balloon itself defining the material surface $A(t)$. And, because the balloon is sealed, the total mass of fluid inside the balloon remains constant as the balloon moves, expands, contracts, or deforms.

Based on (4.1), the principle of mass conservation clearly constrains the fluid density. The implications of (4.1) for the fluid velocity field may be better displayed by using Reynolds transport theorem (3.35) with $F = \rho$ and $\mathbf{b} = \mathbf{u}$ to expand the time derivative in (4.1):

$$\int\limits_{V(t)} \frac{\partial\rho(\mathbf{x},t)}{\partial t}dV + \int\limits_{A(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n}dA = 0. \tag{4.2}$$

This is a mass-balance statement between integrated density changes within $V(t)$ and integrated motion of its surface $A(t)$. Although general and correct, (4.2) may be hard to utilize in practice because the motion and evolution of $V(t)$ and $A(t)$ are determined by the flow, which may be unknown.

To develop the integral equation that represents mass conservation for an *arbitrarily moving* control volume $V^*(t)$ with surface $A^*(t)$, (4.2) must be modified to involve integrations over $V^*(t)$ and $A^*(t)$. This modification is motivated by the frequent need to conserve mass within a volume that is not a material volume, for example a stationary control volume. The first step in this modification is to set $F = \rho$ in (3.35) to obtain

$$\frac{d}{dt}\int\limits_{V^*(t)} \rho(\mathbf{x},t)dV - \int\limits_{V^*(t)} \frac{\partial\rho(\mathbf{x},t)}{\partial t}dV - \int\limits_{A^*(t)} \rho(\mathbf{x},t)\mathbf{b}\cdot\mathbf{n}dA = 0. \tag{4.3}$$

The second step is to choose the arbitrary control volume $V^*(t)$ to be instantaneously coincident with material volume $V(t)$ so that *at the moment of interest* $V(t) = V^*(t)$ and $A(t) = A^*(t)$. At this coincidence moment, the $(d/dt)\int\rho dV$-terms in (4.1) and (4.3) are not equal; however, the volume integration of $\partial\rho/\partial t$ in (4.2) is equal to that in (4.3) and the surface integral of $\rho\mathbf{u}\cdot\mathbf{n}$ over $A(t)$ is equal to that over $A^*(t)$:

$$\int\limits_{V^*(t)} \frac{\partial\rho(\mathbf{x},t)}{\partial t}dV = \int\limits_{V(t)} \frac{\partial\rho(\mathbf{x},t)}{\partial t}dV = -\int\limits_{A(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n}dA = -\int\limits_{A^*(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n}dA \tag{4.4}$$

where the middle equality follows from (4.2). The two ends of (4.4) allow the central volume-integral term in (4.3) to be replaced by a surface integral to find:

$$\frac{d}{dt}\int_{V^*(t)} \rho(\mathbf{x},t)dV + \int_{A^*(t)} \rho(\mathbf{x},t)(\mathbf{u}(\mathbf{x},t)-\mathbf{b})\cdot\mathbf{n}dA = 0, \tag{4.5}$$

where $\mathbf{u}$ and $\mathbf{b}$ must both be observed in the same frame of reference; they are not otherwise restricted. This is the general integral statement of conservation of mass for an arbitrarily moving control volume. It can be specialized to stationary, steadily moving, accelerating, or deforming control volumes by appropriate choice of $\mathbf{b}$. In particular, when $\mathbf{b} = \mathbf{u}$, the arbitrary control volume becomes a material volume and (4.5) reduces to (4.1).

The differential equation that represents mass conservation is obtained by applying Gauss' divergence theorem (2.30) to the surface integration in (4.2):

$$\int_{V(t)} \frac{\partial\rho(\mathbf{x},t)}{\partial t}dV + \int_{A(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n}dA = \int_{V(t)} \left\{\frac{\partial\rho(\mathbf{x},t)}{\partial t} + \nabla\cdot\Big(\rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\Big)\right\}dV = 0. \tag{4.6}$$

The final equality can only be possible if the integrand vanishes at every point in space. If the integrand did not vanish at every point in space, then integrating (4.6) in a small volume around a point where the integrand is nonzero would produce a nonzero integral. Thus, (4.6) requires:

$$\frac{\partial\rho(\mathbf{x},t)}{\partial t} + \nabla\cdot(\rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)) = 0, \text{ or, in index notation } \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0. \tag{4.7}$$

This relationship is called the *continuity equation*. It expresses the principle of conservation of mass in differential form, but is insufficient for fully determining flow fields because it is a single equation that involves two field quantities, ρ and $\mathbf{u}$, and $\mathbf{u}$ is a vector with three components.

The second term in (4.7) is the divergence of the mass-density flux $\rho\mathbf{u}$. Such *flux divergence* terms frequently arise in conservation statements and can be interpreted as the net loss at a point due to divergence of a flux. For example, the local ρ will increase with time if $\nabla\cdot(\rho\mathbf{u})$ is negative. Flux divergence terms are also called *transport* terms because they transfer quantities from one region to another without making a net contribution over the entire field. When integrated over the entire domain of interest, their contribution vanishes if there are no sources at the boundaries.

The continuity equation may alternatively be written using (3.5) the definition of D/Dt and $\partial(\rho u_i)/\partial x_i = u_i\partial\rho/\partial x_i + \rho\partial u_i/\partial x_i$ [see (B3.6)]:

$$\frac{1}{\rho(\mathbf{x},t)}\frac{D}{Dt}\rho(\mathbf{x},t) + \nabla\cdot\mathbf{u}(\mathbf{x},t) = 0. \tag{4.8}$$

The derivative $D\rho/Dt$ is the time rate of change of fluid density following a fluid particle. It will be zero for *constant density* flow where $\rho =$ constant throughout the flow field, and for *incompressible* flow where the density of fluid particles does not change but different fluid particles may have different density:

$$\frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0. \tag{4.9}$$

Taken together, (4.8) and (4.9) imply:

$$\nabla \cdot \mathbf{u} = 0 \tag{4.10}$$

for incompressible flows. Constant density flows are a subset of incompressible flows; $\rho =$ constant is a solution of (4.9) but it is not a general solution. A fluid is usually called *incompressible* if its density does not change with *pressure*. Liquids are almost incompressible. Gases are compressible, but for flow speeds less than ~100 m/s (that is, for Mach numbers < 0.3) the fractional change of absolute pressure in an air flow is small. In this and several other situations, density changes in the flow are also small and (4.9) and (4.10) are valid.

The general form of the continuity equation (4.7) is typically required when the derivative $D\rho/Dt$ is nonzero because of changes in the pressure, temperature, or molecular composition of fluid particles.

4.3. STREAM FUNCTIONS

Consider the steady form of the continuity equation (4.7),

$$\nabla \cdot (\rho \mathbf{u}) = 0. \tag{4.11}$$

The divergence of the curl of any vector field is identically zero (see Exercise 2.19), so $\rho\mathbf{u}$ will satisfy (4.11) when written as the curl of a vector potential $\mathbf{\Psi}$,

$$\rho \mathbf{u} = \nabla \times \mathbf{\Psi}, \tag{4.12}$$

which can be specified in terms of two scalar functions: $\mathbf{\Psi} = \chi \nabla \psi$. Putting this specification for $\mathbf{\Psi}$ into (4.12) produces $\rho\mathbf{u} = \nabla\chi \times \nabla\psi$, because the curl of any gradient is identically zero (see Exercise 2.20). Furthermore, $\nabla\chi$ is perpendicular to surfaces of constant χ, and $\nabla\psi$ is perpendicular to surfaces of constant ψ, so the mass flux $\rho\mathbf{u} = \nabla\chi \times \nabla\psi$ will be parallel to surfaces of constant χ and constant ψ. Therefore, three-dimensional streamlines are the intersections of the two stream surfaces, or stream functions in a three-dimensional flow.

The situation is illustrated in Figure 4.1. Consider two members of each of the families of the two stream functions $\chi = a$, $\chi = b$, $\psi = c$, $\psi = d$. The intersections shown as darkened lines in Figure 4.1 are the streamlines. The mass flux $\dot{m}$ through the surface A bounded by the four stream functions (shown in gray in Figure 4.1) is calculated with area element dA having $\mathbf{n}$ as shown and Stokes' theorem.

Defining the mass flux $\dot{m}$ through A, and using Stokes' theorem produces

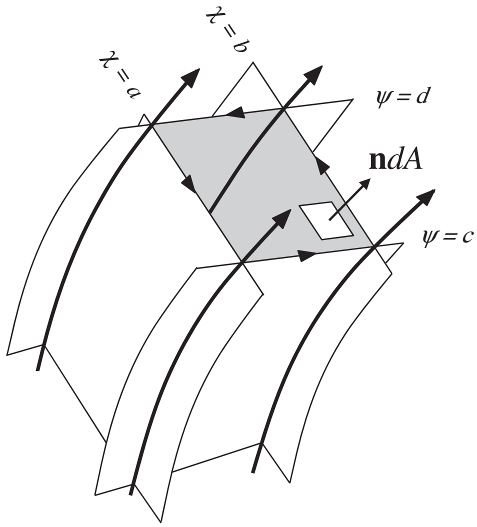


FIGURE 4.1 Isometric view of two members from each family of stream surfaces. The solid curves are streamlines and these lie at the intersections of the surfaces. The unit vector $\mathbf{n}$ points in the stream direction and is perpendicular to the gray surface that is bordered by the nearly rectangular curve C made up of segments defined by $\chi = a$, $\chi = b$, $\psi = c$, and $\psi = d$. The arrows on this border indicate the integration direction for Stokes' theorem.

$$\dot{m} = \int_A \rho\mathbf{u}\cdot\mathbf{n}dA = \int_A (\nabla\times\mathbf{\Psi})\cdot\mathbf{n}dA = \int_C \mathbf{\Psi}\cdot d\mathbf{s} = \int_C \chi\nabla\psi\cdot d\mathbf{s} = \int_C \chi d\psi$$
$$= b(d-c) + a(c-d) = (b-a)(d-c).$$

Here we have used the vector identity $\nabla\psi\cdot d\mathbf{s} = d\psi$. The mass flow rate of the stream tube defined by adjacent members of the two families of stream functions is just the product of the differences of the numerical values of the respective stream functions.

As a special case, consider two-dimensional flow in (x, y)-Cartesian coordinates where all the streamlines lie in z = constant planes. In this situation, z is one of the three-dimensional stream functions, so we can set $\chi = -z$, where the sign is chosen to obey the usual convention. This produces $\nabla\chi = -\mathbf{e}_z$, so $\rho\mathbf{u} = -\mathbf{e}_z \times \nabla\psi$, or

$$\rho u = \partial\psi/\partial y, \quad \text{and} \quad \rho v = -\partial\psi/\partial x$$

in conformity with Exercise 4.7.

Similarly, for axisymmetric three-dimensional flow in cylindrical polar coordinates (Figure 3.3c), all the streamlines lie in φ = constant planes that contain the z-axis so $\chi = -\varphi$ is one of the stream functions. This produces $\nabla\chi = -R^{-1}\mathbf{e}_\varphi$ and $\rho\mathbf{u} = \rho(u_R, u_z) = -R^{-1}\,\mathbf{e}_\varphi \times \nabla\psi$, or

$$\rho u_R = -R^{-1}\left(\partial\psi/\partial z\right), \quad \text{and} \quad \rho u_z = R^{-1}\left(\partial\psi/\partial R\right).$$

We note here that if the density is constant, mass conservation reduces to $\nabla\cdot\mathbf{u} = 0$ (steady or not) and the entire preceding discussion follows for $\mathbf{u}$ rather than $\rho\mathbf{u}$ with the interpretation of stream function values in terms of volumetric flux rather than mass flux.

4.4. CONSERVATION OF MOMENTUM

In this section, the momentum-conservation equivalent of (4.5) is developed from Newton's second law, the fundamental principle governing fluid momentum. When applied to a material volume $V(t)$ with surface area $A(t)$, Newton's second law can be stated directly as:

$$\frac{d}{dt}\int_{V(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)dV = \int_{V(t)} \rho(\mathbf{x},t)\mathbf{g}dV + \int_{A(t)} \mathbf{f}(\mathbf{n},\mathbf{x},t)dA, \tag{4.13}$$

where $\rho\mathbf{u}$ is the momentum per unit volume of the flowing fluid, $\mathbf{g}$ is the body force per unit mass acting on the fluid within $V(t)$, $\mathbf{f}$ is the surface force per unit area acting on $A(t)$, and $\mathbf{n}$ is the outward normal on $A(t)$. The implications of (4.13) are better displayed when the time derivative is expanded using Reynolds transport theorem (3.35) with $F = \rho\mathbf{u}$ and $\mathbf{b} = \mathbf{u}$:

$$\begin{aligned}\int_{V(t)} \frac{\partial}{\partial t}(\rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t))dV + \int_{A(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)(\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n})dA \\ = \int_{V(t)} \rho(\mathbf{x},t)\mathbf{g}dV + \int_{A(t)} \mathbf{f}(\mathbf{n},\mathbf{x},t)dA.\end{aligned} \tag{4.14}$$

This is a momentum-balance statement between integrated momentum changes within $V(t)$, integrated momentum contributions from the motion of $A(t)$, and integrated volume and surface forces. It is the momentum conservation equivalent of (4.2).

To develop an integral equation that represents momentum conservation for an arbitrarily moving control volume $V^*(t)$ with surface $A^*(t)$, (4.14) must be modified to involve integrations over $V^*(t)$ and $A^*(t)$. The steps in this process are entirely analogous to those taken between (4.2) and (4.5) for conservation of mass. First set $F = \rho\boldsymbol{u}$ in (3.35) and rearrange it to obtain:

$$\int_{V^*(t)} \frac{\partial}{\partial t}(\rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t))dV = \frac{d}{dt}\int_{V^*(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)dV - \int_{A^*(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\mathbf{b}\cdot\mathbf{n}dA = 0. \tag{4.15}$$

Then choose $V^*(t)$ to be instantaneously coincident with $V(t)$ so that at the moment of interest:

$$\int_{V(t)} \frac{\partial}{\partial t}(\rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t))dV = \int_{V^*(t)} \frac{\partial}{\partial t}(\rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t))dV,$$

$$\int_{A(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)(\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n})dA = \int_{A^*(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)(\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n})dA,$$

$$\int_{V(t)} \rho(\mathbf{x},t)\mathbf{g}dV = \int_{V^*(t)} \rho(\mathbf{x},t)\mathbf{g}dV, \quad \text{and} \int_{A(t)} \mathbf{f}(\mathbf{n},\mathbf{x},t)dA = \int_{A^*(t)} \mathbf{f}(\mathbf{n},\mathbf{x},t)dA.$$

(4.16a, 4.16b, 4.16c, 4.16d)

Now substitute (4.16a) into (4.15) and use this result plus (4.16b, 4.16c, 4.16d) to convert (4.14) to:

$$\frac{d}{dt}\int\limits_{V^*(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)dV + \int\limits_{A^*(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)(\mathbf{u}(\mathbf{x},t)-\mathbf{b})\cdot\mathbf{n}dA$$
$$= \int\limits_{V^*(t)} \rho(\mathbf{x},t)\mathbf{g}dV + \int\limits_{A^*(t)} \mathbf{f}(\mathbf{n},\mathbf{x},t)dA. \tag{4.17}$$

This is the general integral statement of momentum conservation for an arbitrarily moving control volume. Just like (4.5), it can be specialized to stationary, steadily moving, accelerating, or deforming control volumes by appropriate choice of **b**. For example, when $\mathbf{b} = \mathbf{u}$, the arbitrary control volume becomes a material volume and (4.17) reduces to (4.13).

At this point, the forces in (4.13), (4.14), and (4.17) merit some additional description that facilitates the derivation of the differential equation representing momentum conservation and allows its simplification under certain circumstances.

The body force, $\rho\mathbf{g}dV$, acting on the fluid element dV does so without physical contact. Body forces commonly arise from gravitational, magnetic, electrostatic, or electromagnetic *force fields*. In addition, in accelerating or rotating frames of reference, *fictitious* body forces arise from the frame's noninertial motion (see Section 4.7). By definition body forces are distributed through the fluid and are proportional to mass (or electric charge, electric current, etc.). In this book, body forces are specified per unit mass and carry the units of acceleration.

Body forces may be conservative or nonconservative. *Conservative body forces* are those that can be expressed as the gradient of a potential function:

$$\mathbf{g} = -\nabla\Phi \quad \text{or} \quad g_j = -\partial\Phi/\partial x_j, \tag{4.18}$$

where Φ is called the *force potential*; it has units of energy per unit mass. When the z-axis points vertically upward, the force potential for gravity is $\Phi = gz$, where g is the acceleration of gravity, and (4.18) produces $\mathbf{g} = -g\mathbf{e}_z$. Forces satisfying (4.18) are called *conservative* because the work done by conservative forces is independent of the path, and the sum of fluid-particle kinetic and potential energies is conserved when friction is absent.

Surface forces, **f**, act on fluid elements through direct contact with the surface of the element. They are proportional to the contact area and carry units of stress (force per unit area). Surface forces are commonly resolved into components normal and tangential to the contact area. Consider an arbitrarily oriented element of area dA in a fluid (Figure 2.5). If **n** is the surface normal with components n_i, then from (2.15) the components f_j of the surface force per unit area $\mathbf{f}(\mathbf{n}, \mathbf{x}, t)$ on this element are $f_j = n_i\tau_{ij}$, where τ_{ij} is the stress tensor. Thus, the normal component of **f** is $\mathbf{n}\cdot\mathbf{f} = n_i f_i$, while the tangential component is the vector $\mathbf{f} - (\mathbf{n}\cdot\mathbf{f})\mathbf{n}$ which has components $f_k - (n_i f_i)n_k$.

Other forces that influence fluid motion are surface- and interfacial-tension forces that act on lines or curves embedded within interfaces between liquids and gases or between immiscible liquids (see Figure 1.4). Although these forces are commonly important in flows with such interfaces, they do not appear directly in the equations of motion, entering instead through the boundary conditions.

Before proceeding to the differential equation representing momentum conservation, the use of (4.5) and (4.17) for stationary, moving, and accelerating control volumes having a variety of sizes and shapes is illustrated through a few examples. In all four examples, equations representing mass and momentum conservation must be solved simultaneously.

EXAMPLE 4.1

A long bar with constant cross section is held perpendicular to a uniform horizontal flow of speed U_∞, as shown in Figure 4.2. The flowing fluid has density ρ and viscosity μ (both constant). The bar's cross section has characteristic transverse dimension d, and the span of the bar is l with $l \gg d$. The average horizontal velocity profile measured downstream of the bar is $U(y)$, which is less than U_∞ due to the presence of the bar. Determine the required force per unit span, $-F_D/l$, applied to the ends of the bar to hold it in place. Assume the flow is steady and two dimensional in the plane shown. Ignore body forces.

Solution

Before beginning, it is important to explain the sign convention for fluid dynamic drag forces. The drag force on an inanimate object is *the force applied to the object by the fluid*. Thus, for stationary objects, drag forces are positive in the downstream direction, the direction the object would accelerate if released. However, the control volume laws are written for *forces applied to the contents of the volume*. Thus, from Newton's third law, a positive drag force on an object implies a negative force on the fluid. Therefore, the F_D appearing in Figure 4.2 is a positive number and this will be borne out by the final results. Here we also note that since the horizontal velocity downstream of the bar, the wake velocity $U(y)$, is less than U_∞, the fluid has been decelerated inside the control volume and this is consistent with a force from the body opposing the motion of the fluid as shown.

The basic strategy is to select a stationary control volume, and then use (4.5) and (4.17) to determine the force F_D that the body exerts on the fluid per unit span. The first quantitative step in the solution is to select a rectangular control volume with flat control surfaces aligned with the coordinate directions. The inlet, outlet, and top and bottom sides of such a control volume are shown in Figure 4.2. The vertical sides parallel to the x-y plane are not shown. However, the flow does not vary in the third direction and is everywhere parallel to these surfaces so these merely need be selected a comfortable distance l apart. The inlet control surface should be far enough upstream of the bar so that the inlet fluid velocity is $U_\infty \mathbf{e}_x$, the pressure is p_∞, and both are uniform.

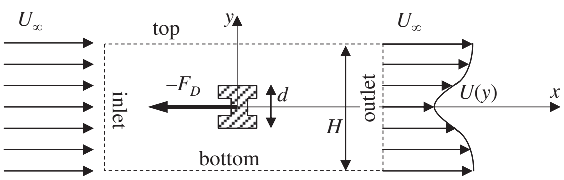


FIGURE 4.2 Momentum and mass balance for flow past long bar of constant cross section placed perpendicular to the flow. The intersection of the recommended stationary control volume with the x-y plane is shown with dashed lines. The force $-F_D$ holds the bar in place and slows the fluid that enters the control volume.

The top and bottom control surfaces should be separated by a distance H that is large enough so that these boundaries are free from shear stresses, and the horizontal velocity and pressure are so close to U_∞ and p_∞ that any difference can be ignored. And finally, the outlet surface should be far enough downstream so that streamlines are nearly horizontal there and the pressure can again be treated as equal to p_∞.

For steady flow and the chosen stationary volume, the control surface velocity is $\mathbf{b} = 0$ and the time derivative terms in (4.5) and (4.17) are both zero. In addition, the surface force integral contributes $-F_D\mathbf{e}_x$ where the beam crosses the control volume's vertical sides parallel to the x-y plane. The remainder of the surface force integral contains only pressure terms since the shear stress is zero on the control surface boundaries. After setting the pressure to p_∞ on all control surfaces, (4.5) and (4.17) simplify to:

$$\int_{A^*(t)} \rho\mathbf{u}(\mathbf{x})\cdot\mathbf{n}dA = 0, \quad \text{and} \quad \int_{A^*} \rho\mathbf{u}(\mathbf{x})\mathbf{u}(\mathbf{x})\cdot\mathbf{n}dA = -\int_{A^*} p_\infty\mathbf{n}dA - F_D\mathbf{e}_x.$$

In this case the pressure integral may be evaluated immediately by using Gauss' divergence theorem:

$$\int_{A^*} p_\infty\mathbf{n}dA = \int_{V^*} \nabla p_\infty dV = 0,$$

with the final value (zero) occurring because p_∞ is a constant. After this simplification, denote the fluid velocity components by $(u,v) = \mathbf{u}$, and evaluate the mass and x-momentum conservation equations:

$$-\int_{inlet} \rho U_\infty l dy + \int_{top} \rho v l dx - \int_{bottom} \rho v l dx + \int_{outlet} \rho U(y) l dy = 0, \text{ and}$$

$$-\int_{inlet} \rho U_\infty^2 l dy + \int_{top} \rho U_\infty v l dx - \int_{bottom} \rho U_\infty v l dx + \int_{outlet} \rho U^2(y) l dy = -F_D$$

where $\mathbf{u}\cdot\mathbf{n}dA$ is: $-U_\infty l dy$ on the inlet surface, $+vldx$ on the top surface, $-vldx$ on the bottom surface, and $+U(y)ldy$ on the outlet surface where l is the span of the flow into the page. Dividing both equations by ρl, and combining like integrals produces:

$$\int_{top} v dx - \int_{bottom} v dx = \int_{-H/2}^{+H/2} (U_\infty - U(y))dy, \text{ and}$$

$$U_\infty\left(\int_{top} v dx - \int_{bottom} v dx\right) + \int_{-H/2}^{+H/2} (U^2(y) - U_\infty^2)dy = -F_D/\rho l.$$

Eliminating the top and bottom control surface integrals between these two equations leads to:

$$F_D/l = \rho \int_{-H/2}^{+H/2} U(y)(U_\infty - U(y))dy,$$

which produces a positive value of F_D when $U(y)$ is less than U_∞. An essential feature of this analysis is that there are nonzero mass fluxes through the top and bottom control surfaces. The final formula here is genuinely useful in experimental fluid mechanics since it allows F_D/l to be determined from single-component velocity measurements made in the wake of an object.

EXAMPLE 4.2

Using a stream-tube control volume of differential length ds, derive the Bernoulli equation, $(½)\rho U^2 + gz + p/\rho =$ constant along a streamline, for steady, inviscid, constant density flow where U is the local flow speed.

Solution

The basic strategy is to use a stationary stream-tube-element control volume, (4.5), and (4.17) to determine a simple differential relationship that can be integrated along a streamline. The geometry is shown in Figure 4.3. For steady inviscid flow and a stationary control volume, the control surface velocity $\mathbf{b} = 0$, the surface friction forces are zero, and the time derivative terms in (4.5) and (4.17) are both zero. Thus, these two equations simplify to:

$$\int_{A^*(t)} \rho \mathbf{u}(\mathbf{x}) \cdot \mathbf{n} dA = 0 \quad \text{and} \quad \int_{A^*(t)} \rho \mathbf{u}(\mathbf{x}) \mathbf{u}(\mathbf{x}) \cdot \mathbf{n} dA = \int_{V^*(t)} \rho \mathbf{g} dV - \int_{A^*(t)} p \mathbf{n} dA.$$

The geometry of the volume plays an important role here. The nearly conical curved surface is tangent to the velocity while the inlet and outlet areas are perpendicular to it. Thus, $\mathbf{u} \cdot \mathbf{n} dA$ is: $-UdA$ on the inlet surface, zero on the nearly conical curved surface, and $+[U + (\partial U/\partial s)ds]dA$ on the outlet surface. Therefore, conservation of mass with constant density leads to

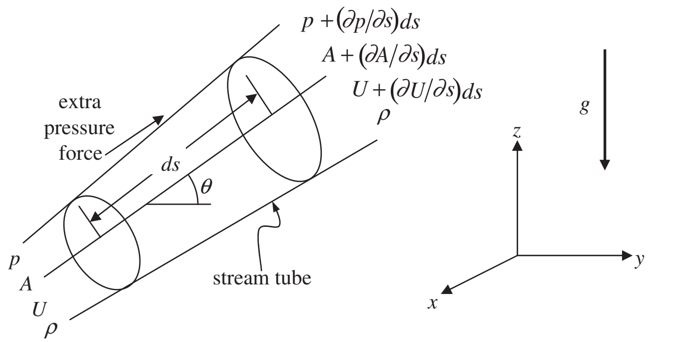


FIGURE 4.3 Momentum and mass balance for a short segment of a stream tube in steady inviscid constant density flow. Here, the inlet and outlet areas are perpendicular to the flow direction, and they are small enough so that only first-order corrections in the stream direction need to be considered. The alignment of gravity and stream tube leads to a vertical change of $\sin\theta\, ds = dz$ between its two ends. The area difference between the two ends of the stream tube leads to an extra pressure force.

$$-\rho UA + \rho\left(U + \frac{\partial U}{\partial s}ds\right)\left(A + \frac{\partial A}{\partial s}ds\right) = 0,$$

where first-order variations in U and A in the stream-wise direction are accounted for. Now consider the stream-wise component of the momentum equation recalling that $\mathbf{u} = U\mathbf{e}_u$ and setting $\mathbf{g} = -g\mathbf{e}_z$. For inviscid flow, the only surface force is pressure, so the simplified version of (4.17) becomes

$$\begin{aligned}&-\rho U^2 A + \rho\left(U + \frac{\partial U}{\partial s}ds\right)^2\left(A + \frac{\partial A}{\partial s}ds\right)\\ &= -\rho g \sin\theta\left(A + \frac{\partial A}{\partial s}\frac{ds}{2}\right)ds + pA + \left(p + \frac{\partial p}{\partial s}\frac{ds}{2}\right)\frac{\partial A}{\partial s}ds - \left(p + \frac{\partial p}{\partial s}ds\right)\left(A + \frac{\partial A}{\partial s}ds\right).\end{aligned}$$

Here, the middle pressure term comes from the extra pressure force on the nearly conical surface of the stream tube.

To reach the final equation, use the conservation of mass result to simplify the flux terms on the left side of the stream-wise momentum equation. Then, simplify the pressure contributions by canceling common terms, and note that $\sin\theta\, ds = dz$ to find

$$\begin{aligned}-\rho U^2 A + \rho U\left(U + \frac{\partial U}{\partial s}ds\right)A &= \rho UA\frac{\partial U}{\partial s}ds\\ &= -\rho g\left(A + \frac{\partial A}{\partial s}\frac{ds}{2}\right)dz + \frac{\partial p}{\partial s}\frac{\partial A}{\partial s}\frac{(ds)^2}{2} - A\frac{\partial p}{\partial s}ds - \frac{\partial p}{\partial s}\frac{\partial A}{\partial s}(ds)^2.\end{aligned}$$

Continue by dropping the second-order terms that contain $(ds)^2$ or $dsdz$, and divide by ρA to reach:

$$U\frac{\partial U}{\partial s}ds = -gdz - \frac{1}{\rho}\frac{\partial p}{\partial s}ds, \quad \text{or} \quad \left[d(U^2/2) + gdz + (1/\rho)dp = 0\right]_{\text{along a streamline}}.$$

Integrate the final differential expression along the streamline to find:

$$\frac{1}{2}U^2 + gz + p/\rho = \text{a constant along a streamline.} \tag{4.19}$$

EXAMPLE 4.3

Consider a small solitary wave that moves from right to left on the surface of a water channel of undisturbed depth h (Figure 4.4). Denote the acceleration of gravity by g. Assuming a small change in the surface elevation across the wave, derive an expression for its propagation speed, U, when the channel bed is flat and frictionless.

Solution

Before starting the control volume part of this problem, a little dimensional analysis goes a *long* way toward determining the final solution. The statement of the problem has only three parameters,

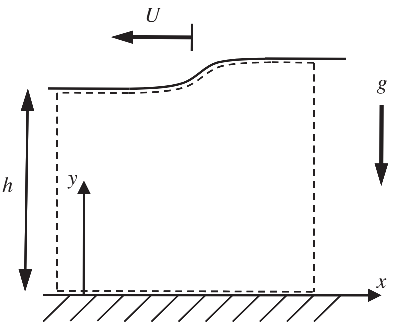


FIGURE 4.4 Momentum and mass balance for a small amplitude water wave moving into quiescent water of depth h. The recommended moving control volume is shown with dashed lines. The wave is driven by the imbalance of static pressure forces on the vertical inlet (left) and outlet (right) control surfaces.

U, g, and h, and there are two independent units (length and time). Thus, there is only one dimensionless group, U^2/gh, so it must be a constant. Therefore, the final answer must be in the form: $U = const \cdot \sqrt{gh}$, so the value of the following control volume analysis lies merely in determining the constant.

Choose the control volume shown and assume it is moving at speed $\mathbf{b} = -U\mathbf{e}_x$. Here we assume that the upper and lower control surfaces coincide with the water surface and the channel's frictionless bed. They are shown close to these boundaries in Figure 4.4 for clarity. Apply the integral conservation laws for mass and momentum.

$$\frac{d}{dt}\int\limits_{V^*(t)} \rho dV + \int\limits_{A^*(t)} \rho(\mathbf{u}-\mathbf{b})\cdot\mathbf{n}\, dA = 0, \quad \frac{d}{dt}\int\limits_{V^*(t)} \rho\mathbf{u} dV + \int\limits_{A^*(t)} \rho\mathbf{u}(\mathbf{u}-\mathbf{b})\cdot\mathbf{n} dA$$
$$= \int\limits_{V^*(t)} \rho\mathbf{g} dV + \int\limits_{A^*(t)} \mathbf{f} dA.$$

With this choice of a moving control volume, its contents are constant so both the d/dt terms are zero; thus,

$$\int\limits_{A^*} \rho(\mathbf{u}+U\mathbf{e}_x)\cdot\mathbf{n} dA = 0, \quad \text{and} \quad \int\limits_{A^*(t)} \rho\mathbf{u}(\mathbf{u}+U\mathbf{e}_x)\cdot\mathbf{n} dA = \int\limits_{V^*(t)} \rho\mathbf{g} dV + \int\limits_{A^*(t)} \mathbf{f} dA.$$

Here, all velocities are referred to a stationary coordinate frame, so that $\mathbf{u} = 0$ on the inlet side of the control volume in the undisturbed fluid layer. In addition, label the inlet (left) and outlet (right) water depths as h_{in} and h_{out}, respectively, and save consideration of the simplifications that occur when $(h_{out} - h_{in}) \ll (h_{out} + h_{in})/2$ for the end of the analysis. Let U_{out} be the horizontal flow speed on the outlet side of the control volume and assume its profile is uniform. Therefore $(\mathbf{u}+U\mathbf{e}_x)\cdot\mathbf{n}dA$ is $-Uldy$ on the inlet surface, and $+(U_{\text{out}} + U)ldy$ on the outlet surface, where l is (again) the width of the flow into the page. With these replacements, the conservation of mass equation becomes:

$$-\rho U h_{in} l + \rho(U_{out} + U) h_{out} l = 0, \quad \text{or} \quad U h_{in} = (U_{out} + U) h_{out},$$

and the horizontal momentum equation becomes:

$$-\rho(0)(0+U)h_{in}l + \rho U_{out}(U_{out}+U)h_{out}l = -\int\limits_{inlet} p\mathbf{n}\cdot\mathbf{e}_x dA - \int\limits_{outlet} p\mathbf{n}\cdot\mathbf{e}_x dA - \int\limits_{top} p\mathbf{n}\cdot\mathbf{e}_x dA.$$

Here, no friction terms are included, and the body force term does not appear because it has no horizontal component. First consider the pressure integral on the top of the control volume, and let $y = h(x)$ define the shape of the water surface:

$$\begin{aligned}
-p_o\int \mathbf{n}\cdot\mathbf{e}_x dA &= -p_o\int \frac{(-\,dh/dx, 1)}{\sqrt{1+(dh/dx)^2}}\cdot(1,0)l\sqrt{1+(dh/dx)^2}dx \\
&= -p_o\int\left(-\frac{dh}{dx}\right)ldx = p_o\int\limits_{h_{in}}^{h_{out}} ldh = p_o l(h_{out}-h_{in})
\end{aligned}$$

where the various square-root factors arise from the surface geometry; p_o is the (constant) atmospheric pressure on the water surface. The pressure on the inlet and outlet sides of the control volume is hydrostatic. Using the coordinate system shown, integrating (1.8), and evaluating the constant on the water surface produces $p = p_o + \rho g(h - y)$. Thus, the integrated inlet and outlet pressure forces are:

$$\begin{aligned}
&\int\limits_{inlet} pdA - \int\limits_{outlet} pdA - \int\limits_{top} p_o\mathbf{n}\cdot\mathbf{e}_x dA \\
&= \int\limits_0^{h_{in}} (p_o + \rho g(h_{in}-y))ldy - \int\limits_0^{h_{out}} (p_o + \rho g(h_{out}-y))ldy + p_o(h_{out}-h_{in})l \\
&= \int\limits_0^{h_{in}} \rho g(h_{in}-y)ldy - \int\limits_0^{h_{out}} \rho g(h_{out}-y)ldy = \rho g\left(\frac{h_{in}^2}{2} - \frac{h_{out}^2}{2}\right)l
\end{aligned}$$

where the signs of the inlet and outlet integrals have been determined by evaluating the dot products and we again note that the constant reference pressure p_o does not contribute to the net pressure force. Substituting this pressure force result into the horizontal momentum equation produces:

$$-\rho\big(0\big)\big(0+U\big)h_{in}l + \rho U_{out}\big(U_{out}+U\big)h_{out}l = \frac{\rho g}{2}\big(h_{in}^2 - h_{out}^2\big)l.$$

Dividing by the common factors of ρ and l,

$$U_{out}\big(U_{out}+U\big)h_{out} = \frac{g}{2}\big(h_{in}^2 - h_{out}^2\big),$$

and eliminating U_{out} via the conservation of mass relationship, $U_{out} = (h_{in} - h_{out})U/h_{out}$, leads to:

$$U\frac{(h_{in}-h_{out})}{h_{out}}\left(U\frac{(h_{in}-h_{out})}{h_{out}} + U\right)h_{out} = \frac{g}{2}\big(h_{in}^2 - h_{out}^2\big).$$

Dividing by the common factor of $(h_{in} - h_{out})$ and simplifying the left side of the equation produces:

$$U^2 \frac{h_{in}}{h_{out}} = \frac{g}{2}(h_{in} + h_{out}), \quad \text{or} \quad U = \sqrt{\frac{gh_{out}}{2h_{in}}(h_{in} + h_{out})} \approx \sqrt{gh},$$

where the final approximate equality holds when the inlet and outlet heights differ by only a small amount with both nearly equal to h.

EXAMPLE 4.4

Derive the differential equation for the vertical motion for a simple rocket having nozzle area A_e that points downward, exhaust discharge speed V_e, and exhaust density ρ_e, without considering the internal flow within the rocket (Figure 4.5). Denote the mass of the rocket by $M(t)$ and assume the discharge flow is uniform.

Solution

Select a control volume (not shown) that contains the rocket and travels with it. This will be an accelerating control volume and its velocity $\mathbf{b} = b(t)\mathbf{e}_z$ will be the rocket's vertical velocity. In addition, the discharge velocity is specified with respect to the rocket, so in a stationary frame of reference, the absolute velocity of the rocket's exhaust is $\mathbf{u} = u_z\mathbf{e}_z = (-V_e + b)\mathbf{e}_z$.

The conservation of mass and vertical-momentum equations are:

$$\frac{d}{dt}\int_{V^*(t)} \rho dV + \int_{A^*(t)} \rho(\mathbf{u}-\mathbf{b})\cdot\mathbf{n}\, dA = 0, \quad \frac{d}{dt}\int_{V^*(t)} \rho u_z dV + \int_{A^*(t)} \rho u_z(\mathbf{u}-\mathbf{b})\cdot\mathbf{n}\, dA$$
$$= -g\int_{V^*(t)} \rho dV + \int_{A^*(t)} f_z dA.$$

Here we recognize the first term in each equation as the time derivative of the rocket's mass M, and the rocket's vertical momentum Mb, respectively. (The second of these identifications is altered when the rocket's internal flows are considered; see Thompson, 1972, pp. 43–47.) For ordinary rocketry, the

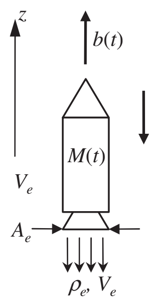


FIGURE 4.5 Geometry and parameters for a simple rocket having mass $M(t)$ that is moving vertically at speed $b(t)$. The rocket's exhaust area, density, and velocity (or specific impulse) are A_e, ρ_e, and V_e, respectively.

rocket exhaust exit will be the only place that mass and momentum cross the control volume boundary and here $\mathbf{n} = -\mathbf{e}_z$; thus $(\mathbf{u} - \mathbf{b})\cdot\mathbf{n}\, dA = (-V_e\mathbf{e}_z)\cdot(-\mathbf{e}_z)dA = V_e dA$ over the nozzle exit. In addition, we will denote the integral of vertical surface stresses by F_S, a force that includes the aerodynamic drag on the rocket and the pressure thrust produced when the rocket nozzle's outlet pressure exceeds the local ambient pressure. With these replacements, the above equations become:

$$\frac{dM}{dt} + \rho_e V_e A_e = 0, \quad \frac{d}{dt}(Mb) + \rho_e(-V_e + b)V_e A_e = -Mg + F_S.$$

Eliminating $\rho_e V_e A_e$ between the two equations produces:

$$\frac{d}{dt}(Mb) + (-V_e + b)\left(-\frac{dM}{dt}\right) = -Mg + F_S,$$

which reduces to:

$$M\frac{d^2 z_R}{dt^2} = -V_e\frac{dM}{dt} - Mg + F_S,$$

where z_R is the rocket's vertical location and $dz_R/dt = b$. From this equation it is clear that negative dM/dt (mass loss) may produce upward acceleration of the rocket when its exhaust discharge velocity V_e is high enough. In fact, V_e is the crucial figure of merit in rocket propulsion and is commonly referred to as the *specific impulse*, the thrust produced per unit rate of mass discharged.

Returning now to the development of the equations of motion, the differential equation that represents momentum conservation is obtained from (4.14) after collecting all four terms into the same volume integration. The first step is to convert the two surface integrals in (4.14) to volume integrals using Gauss' theorem (2.30):

$$\int_{A(t)} \rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)(\mathbf{u}(\mathbf{x},t)\cdot\mathbf{n})dA = \int_{V(t)} \nabla\cdot(\rho(\mathbf{x},t)\mathbf{u}(\mathbf{x},t)\mathbf{u}(\mathbf{x},t))dV = \int_{V(t)} \frac{\partial}{\partial x_i}\left(\rho u_i u_j\right)dV, \quad \text{and}$$

$$\int_{A(t)} \mathbf{f}(\mathbf{n},\mathbf{x},t)dA = \int_{A(t)} n_i\tau_{ij}dA = \int_{V(t)} \frac{\partial}{\partial x_i}\left(\tau_{ij}\right)dV, \tag{4.20a, 4.20b}$$

where the explicit listing of the independent variables has been dropped upon moving to index notation. Substituting (4.20a, 4.20b) into (4.14) and collecting all the terms on one side of the equation into the same volume integration produces:

$$\int_{V(t)} \left\{\frac{\partial}{\partial t}\left(\rho u_j\right) + \frac{\partial}{\partial x_i}\left(\rho u_i u_j\right) - \rho g_j - \frac{\partial}{\partial x_i}\left(\tau_{ij}\right)\right\}dV = 0. \tag{4.21}$$

Similarly to (4.6), the integral in (4.21) can only be zero for any material volume if the integrand vanishes at every point in space; thus (4.21) requires:

$$\frac{\partial}{\partial t}\left(\rho u_j\right) + \frac{\partial}{\partial x_i}\left(\rho u_i u_j\right) = \rho g_j + \frac{\partial}{\partial x_i}\left(\tau_{ij}\right). \tag{4.22}$$

This equation can be put into a more standard form by expanding the leading two terms,

$$\frac{\partial}{\partial t}\left(\rho u_j\right)+\frac{\partial}{\partial x_i}\left(\rho u_i u_j\right) = \rho\frac{\partial u_j}{\partial t}+u_j\left[\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho u_i)\right]+\rho u_i\frac{\partial u_j}{\partial x_i} = \rho\frac{Du_j}{Dt}, \tag{4.23}$$

recognizing that the contents of the [,]-brackets are zero because of (4.7), and using the definition of D/Dt from (3.5). The final result is:

$$\rho\frac{Du_j}{Dt} = \rho g_j+\frac{\partial}{\partial x_i}\left(\tau_{ij}\right), \tag{4.24}$$

which is sometimes called *Cauchy's equation of motion*. It relates fluid-particle acceleration to the net body (ρg_i) and surface force ($\partial\tau_{ij}/\partial x_j$) on the particle. It is true in any continuum, solid or fluid, no matter how the stress tensor τ_{ij} is related to the velocity field. However, (4.24) does not provide a complete description of fluid dynamics, even when combined with (4.7) because the number of dependent field variables is greater than the number of equations. Taken together, (4.7), (4.24), and two thermodynamic equations provide at most $1+3+2=6$ scalar equations but (4.7) and (4.24) contain ρ, u_j, and τ_{ij} for a total of $1+3+9=13$ unknowns. Thus, the number of unknowns must be decreased to produce a solvable system. The fluid's stress-strain rate relationship(s) or constitutive equation provides much of the requisite reduction.

4.5. CONSTITUTIVE EQUATION FOR A NEWTONIAN FLUID

As previously described in Section 2.4, the stress at a point can be completely specified by the nine components of the stress tensor τ; these components are illustrated in Figures 2.4 and 2.5, which show the directions of *positive* stresses on the various faces of small cubical and tetrahedral fluid elements. The first index of τ_{ij} indicates the direction of the normal to the surface on which the stress is considered, and the second index indicates the direction in which the stress acts. The diagonal elements τ_{11}, τ_{22}, and τ_{33} of the stress matrix are the normal stresses, and the off-diagonal elements are the tangential or shear stresses. Although finite size elements are shown in these figures, the stresses apply on the various planes when the elements shrink to a point and the elements have vanishingly small mass. Denoting the cubical volume in Figure 2.4 by $dV = dx_1dx_2dx_3$ and considering the torque produced on it by the various stresses' components, it can be shown that the stress tensor is symmetric,

$$\tau_{ij} = \tau_{ji}, \tag{4.25}$$

by considering the element's rotational dynamics in the limit $dV\rightarrow 0$ (see Exercise 4.30). Therefore, the stress tensor has only six independent components. However, this symmetry is violated if there are body-force couples proportional to the mass of the fluid element, such as those exerted by an electric field on polarized fluid molecules. Antisymmetric stresses must be included in such circumstances.

The relationship between the stress and deformation in a continuum is called a *constitutive equation*, and a linear constitutive equation between stress τ_{ij} and $\partial u_i/\partial x_j$ is examined here. A fluid that follows the simplest possible linear constitutive equation is known as a *Newtonian* fluid.

In a fluid at rest, there are only normal components of stress on a surface, and the stress does not depend on the orientation of the surface; the stress is *isotropic*. The only second-order isotropic tensor is the Kronecker delta, δ_{ij}, from (2.16). Therefore, the stress in a static fluid must be of the form

$$\tau_{ij} = -p\delta_{ij}, \tag{4.26}$$

where p is the *thermodynamic pressure* related to ρ and T by an equation of state such as that for a perfect gas $p = \rho RT$ (1.22). The negative sign in (4.26) occurs because the normal components of $\boldsymbol{\tau}$ are regarded as positive if they indicate tension rather than compression (see Figure 2.4).

A moving fluid develops additional stress components, σ_{ij}, because of viscosity, and these stress components appear as both diagonal and off-diagonal components within $\boldsymbol{\tau}$. A simple extension of (4.26) that captures this phenomenon and reduces to (4.26) when fluid motion ceases is:

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}. \tag{4.27}$$

This decomposition of the stress into fluid-static (p) and fluid-dynamic (σ_{ij}) contributions is approximate, because p is only well defined for equilibrium conditions. However, molecular densities, speeds, and collision rates are typically high enough, so that fluid particles (as defined in Section 1.8) reach local thermodynamic equilibrium conditions in nearly all fluid flows so that p in (4.27) is still the thermodynamic pressure.

The fluid-dynamic contribution, σ_{ij}, to the stress tensor is called the *deviatoric stress tensor*. For it to be invariant under Galilean transformations, it cannot depend on the absolute fluid velocity so it must depend on the velocity gradient tensor $\partial u_i/\partial x_j$. However, by definition, stresses only develop in fluid elements that change shape. Therefore, only the symmetric part of $\partial u_i/\partial x_j$, S_{ij} from (3.12), should be considered in the fluid constitutive equation because the antisymmetric part of $\partial u_i/\partial x_j$, R_{ij} from (3.13), corresponds to pure rotation of fluid elements. The most general linear relationship between σ_{ij} and S_{ij} that produces $\sigma_{ij} = 0$ when $S_{ij} = 0$ is

$$\sigma_{ij} = K_{ijmn}S_{mn}, \tag{4.28}$$

where K_{ijmn} is a fourth-order tensor having 81 components that may depend on the local thermodynamic state of the fluid. Equation (4.28) allows *each* of the nine components of σ_{ij} to be linearly related to *all* nine components of S_{ij}. However, this level of generality is unnecessary when the stress tensor is symmetric, and the fluid is isotropic.

In an isotropic fluid medium, the stress–strain rate relationship is independent of the orientation of the coordinate system. This is only possible if K_{ijmn} is an isotropic tensor. All fourth-order isotropic tensors must be of the form:

$$K_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu\delta_{im}\delta_{jn} + \gamma\delta_{in}\delta_{jm} \tag{4.29}$$

(see Aris, 1962, pp. 30–33), where λ, μ, and γ are scalars that depend on the local thermodynamic state. In addition, σ_{ij} is symmetric in i and j, so (4.28) requires that K_{ijmn} also be symmetric in i and j, too. This requirement is consistent with (4.29) only if

$$\gamma = \mu. \tag{4.30}$$

Therefore, only two constants, μ and λ, of the original 81, remain after the imposition of material-isotropy and stress-symmetry restrictions. Substitution of (4.29) into the constitutive equation (4.28) yields

$$\sigma_{ij} = 2\mu S_{ij} + \lambda S_{mm}\,\delta_{ij},$$

where $S_{mm} = \nabla\cdot\mathbf{u}$ is the volumetric strain rate (see Section 3.6). The complete stress tensor (4.27) then becomes

$$\tau_{ij} = -p\delta_{ij} + 2\mu S_{ij} + \lambda S_{mm}\,\delta_{ij}, \tag{4.31}$$

and this is the appropriate multi-dimensional extension of (1.3).

The two scalar constants μ and λ can be further related as follows. Setting $i = j$, summing over the repeated index, and noting that $\delta_{ii} = 3$, we obtain

$$\tau_{ii} = -3p + (2\mu + 3\lambda)S_{mm},$$

from which the pressure is found to be

$$p = -\frac{1}{3}\tau_{ii} + \left(\frac{2}{3}\mu + \lambda\right)\nabla\cdot\mathbf{u}. \tag{4.32}$$

The diagonal terms of S_{ij} in a flow may be unequal. In such a case the stress tensor τ_{ij} can have unequal diagonal terms because of the presence of the term proportional to μ in (4.31). We can therefore take the average of the diagonal terms of $\boldsymbol{\tau}$ and define a *mean pressure* (as opposed to thermodynamic pressure p) as

$$\overline{p} \equiv -\frac{1}{3}\tau_{ii}. \tag{4.33}$$

Substitution into (4.32) gives

$$p - \overline{p} = \left(\frac{2}{3}\mu + \lambda\right)\nabla\cdot\mathbf{u}. \tag{4.34}$$

For a completely incompressible fluid we can only define a mechanical or mean pressure, because there is no equation of state to determine a thermodynamic pressure. (In fact, the *absolute pressure in an incompressible fluid is indeterminate*, and only its *gradients* can be determined from the equations of motion.) The λ-term in the constitutive equation (4.31) drops out when $S_{mm} = \nabla\cdot\mathbf{u} = 0$, and no consideration of (4.34) is necessary. So, for *incompressible fluids*, the constitutive equation (4.31) takes the simple form:

$$\tau_{ij} = -p\delta_{ij} + 2\mu S_{ij} \quad \text{(incompressible)}, \tag{4.35}$$

where p can only be interpreted as the mean pressure experienced by a fluid particle. For a compressible fluid, on the other hand, a thermodynamic pressure can be defined, and it seems that p and $\overline{p}$ can be different. In fact, equation (4.34) relates this difference to the rate of expansion through the proportionality constant $\mu_v = \lambda + 2\mu/3$, which is called the *coefficient of bulk viscosity*. It has an appreciable effect on sound absorption and shock-wave structure. It is generally found to be nonzero in polyatomic gases because of relaxation effects associated with molecular rotation. However, the *Stokes assumption*,

$$\lambda + \frac{2}{3}\mu = 0, \tag{4.36}$$

is found to be accurate in many situations because either the fluid's μ_v or the flow's dilatation rate is small. Interesting historical aspects of the Stokes assumption can be found in Truesdell (1952).

Without using (4.36), the stress tensor (4.31) is:

$$\tau_{ij} = -p\delta_{ij} + 2\mu\left(S_{ij} - \frac{1}{3}S_{mm}\delta_{ij}\right) + \mu_v S_{mm}\,\delta_{ij}. \tag{4.37}$$

This linear relation between $\boldsymbol{\tau}$ and $\mathbf{S}$ is consistent with Newton's definition of the viscosity coefficient μ in a simple parallel flow $u(y)$, for which (4.37) gives a shear stress of $\tau = \mu(du/dy)$. Consequently, a fluid obeying equation (4.37) is called a *Newtonian fluid* where μ and μ_v may only depend on the local thermodynamic state. The off-diagonal terms of (4.37) are of the type

$$\tau_{12} = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right),$$

and directly relate the shear stress to shear strain rate via the viscosity μ. The diagonal terms of (4.37) combine pressure and viscous effects. For example, the first diagonal component of (4.37) is

$$\tau_{11} = -p + 2\mu\left(\frac{\partial u_1}{\partial x_1}\right) + \left(\mu_v - \frac{2}{3}\mu\right)\frac{\partial u_m}{\partial x_m},$$

which means that the normal viscous stress on a plane normal to the x_1-axis is proportional to the extension rate in the x_1 direction and the average expansion rate at the point.

The linear Newtonian friction law (4.37) might only be expected to hold for small strain rates since it is essentially a first-order expansion of the stress in terms of S_{ij} around $\tau_{ij} = 0$. However, the linear relationship is surprisingly accurate for many common fluids such as air, water, gasoline, and oils. Yet, other liquids display non-Newtonian behavior at moderate rates of strain. These include solutions containing long-chain polymer molecules, concentrated soaps, melted plastics, emulsions and slurries containing suspended particles, and many liquids of biological origin. These liquids may violate Newtonian behavior in several ways. For example, shear stress may be a *nonlinear* function of the local strain rate, which is the case for many liquid plastics that are *shear thinning*; their viscosity drops with increasing strain rate. Alternatively, the stress on a non-Newtonian fluid particle may depend on the local strain rate and on its *history*. Such memory effects give the fluid some elastic properties that may allow it to mimic solid behavior over short periods of time. In fact there is a whole class of *viscoelastic* substances that are neither fully fluid nor fully solid. Non-Newtonian fluid mechanics is beyond the scope of this text but its fundamentals are well covered elsewhere (see Bird et al., 1987).

4.6. NAVIER-STOKES MOMENTUM EQUATION

The momentum conservation equation for a Newtonian fluid is obtained by substituting (4.37) into Cauchy's equation (4.24) to obtain:

$$\rho\left(\frac{\partial u_j}{\partial t}+u_i\frac{\partial u_j}{\partial x_i}\right) = -\frac{\partial p}{\partial x_j}+\rho g_j+\frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)+\left(\mu_v-\frac{2}{3}\mu\right)\frac{\partial u_m}{\partial x_m}\delta_{ij}\right], \tag{4.38}$$

where we have used $(\partial p/\partial x_i)\delta_{ij}=\partial p/\partial x_j$, (3.5) with $F=u_j$, and (3.12). This is the *Navier-Stokes momentum equation*. The viscosities, μ and μ_v, in this equation can depend on the thermodynamic state and indeed μ, for most fluids, displays a rather strong dependence on temperature, decreasing with T for liquids and increasing with T for gases. Together, (4.7) and (4.38) provide $1+3=4$ scalar equations, and they contain ρ, p, and u_j for $1+1+3=5$ dependent variables. Therefore, when combined with suitable boundary conditions, (4.7) and (4.38) provide a complete description of fluid dynamics when ρ is constant or when a single (known) relationship exists between p and ρ. In the later case, the fluid or the flow is said to be *barotropic*. When the relationship between p and ρ also includes the temperature T, the internal (or thermal) energy e of the fluid must also be considered. These additions allow a caloric equation of state to be added to the equation listing, but introduces two more dependent variables, T and e. Thus, in general, a third field equation representing conservation of energy is needed to fully describe fluid dynamics.

When temperature differences are small within the flow, μ and μ_v can be taken outside the spatial derivative operating on the contents of the [,]-brackets in (4.38), which then reduces to

$$\rho\frac{Du_j}{Dt} = -\frac{\partial p}{\partial x_j}+\rho g_j+\mu\frac{\partial^2 u_j}{\partial x_i^2}+\left(\mu_v+\frac{1}{3}\mu\right)\frac{\partial}{\partial x_j}\frac{\partial u_m}{\partial x_m}\ \text{(compressible)}. \tag{4.39a}$$

For incompressible fluids $\nabla\cdot\mathbf{u}=\partial u_m/\partial x_m=0$, so (4.39a) in vector notation reduces to:

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p+\rho\mathbf{g}+\mu\nabla^2\mathbf{u}\ \text{(incompressible)}. \tag{4.39b}$$

Interestingly, the net viscous force per unit volume in incompressible flow, the last term on the right in this equation, can be obtained from the divergence of the strain rate tensor or from the curl of the vorticity (see Exercise 4.38):

$$\left(\mu\nabla^2\mathbf{u}\right)_j=\mu\frac{\partial^2 u_j}{\partial x_i^2} = 2\mu\frac{\partial S_{ij}}{\partial x_i} = \mu\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right) = -\mu\varepsilon_{jik}\frac{\partial \omega_k}{\partial x_i} = -\mu(\nabla\times\boldsymbol{\omega})_j. \tag{4.40}$$

This result would seem to pose a paradox since it shows that the net viscous force depends on the vorticity even though rotation of fluid elements was explicitly excluded from entering (4.37), the precursor of (4.40). This paradox is resolved by realizing that the net viscous force is given by either a spatial *derivative* of the vorticity or a spatial *derivative* of the deformation rate. The net viscous force vanishes when $\boldsymbol{\omega}$ is uniform in space (as in solid-body rotation), in which case the incompressibility condition requires that the deformation rate is zero everywhere as well.

If viscous effects are negligible, which is commonly true away from the boundaries of the flow field, (4.39) further simplifies to the *Euler equation*

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p+\rho\mathbf{g}. \tag{4.41}$$

4.7. NONINERTIAL FRAME OF REFERENCE

The equations of fluid motion in a noninertial frame of reference are developed in this section. The equations of motion given in Sections 4.4 through 4.6 are valid in an inertial frame of reference, one that is stationary or that is moving at a constant speed with respect to a stationary frame of reference. Although a stationary frame of reference cannot be defined precisely, a frame of reference that is stationary with respect to distant stars is adequate for our purposes. Thus, noninertial-frame effects may be found in other frames of reference known to undergo nonuniform translation and rotation. For example, the fluid mechanics of rotating machinery is often best analyzed in a rotating frame of reference, and modern life commonly places us in the noninertial frame of reference of a moving and maneuvering vehicle. Fortunately, in many laboratory situations, the relevant distances and time scales are short enough so that a frame of reference attached to the earth (sometimes referred to as the *laboratory frame* of reference) is a suitable inertial frame of reference. However, in atmospheric, oceanic, or geophysical studies where time and length scales are much larger, the earth's rotation may play an important role, so an earth-fixed frame of reference must often be treated as a noninertial frame of reference.

In a noninertial frame of reference, the continuity equation (4.7) is unchanged but the momentum equation (4.38) must be modified. Consider a frame of reference O′1′2′3′ that translates at velocity $d\mathbf{X}(t)/dt = \mathbf{U}(t)$ and rotates at angular velocity $\boldsymbol{\Omega}(t)$ with respect to a stationary frame of reference O123 (see Figure 4.6). The vectors $\mathbf{U}$ and $\boldsymbol{\Omega}$ may be resolved in either frame. The same clock is used in both frames so $t = t'$. A fluid particle P can be located in the rotating frame $\mathbf{x}' = (x'_1, x'_2, x'_3)$ or in the stationary frame $\mathbf{x} = (x_1, x_2, x_3)$, and these distances are simply related via vector addition: $\mathbf{x} = \mathbf{X} + \mathbf{x}'$. The velocity $\mathbf{u}$ of the fluid particle is obtained by time differentiation:

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} = \frac{d\mathbf{X}}{dt} + \frac{d\mathbf{x}'}{dt} = \mathbf{U} + \frac{d}{dt}(x'_1\mathbf{e}'_1 + x'_2\mathbf{e}'_2 + x'_3\mathbf{e}'_3)$$
$$= \mathbf{U} + \frac{dx'_1}{dt}\mathbf{e}'_1 + \frac{dx'_2}{dt}\mathbf{e}'_2 + \frac{dx'_3}{dt}\mathbf{e}'_3 + x'_1\frac{d\mathbf{e}'_1}{dt} + x'_2\frac{d\mathbf{e}'_2}{dt} + x'_3\frac{d\mathbf{e}'_3}{dt} = \mathbf{U} + \mathbf{u}' + \boldsymbol{\Omega} \times \mathbf{x}', \quad (4.42)$$

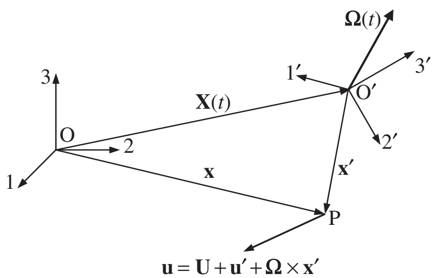


FIGURE 4.6 Geometry showing the relationship between a stationary coordinate system O123 and a noninertial coordinate system O′1′2′3′ that is moving, accelerating, and rotating with respect to O123. In particular, the vector connecting O and O′ is $\mathbf{X}$(t) and the rotational velocity of O′1′2′3′ is $\boldsymbol{\Omega}(t)$. The vector velocity $\mathbf{u}$ at point P in O123 is shown. The vector velocity $\mathbf{u}'$ at point P in O′1′2′3′ differs from u because of the motion of O′1′2′3′.

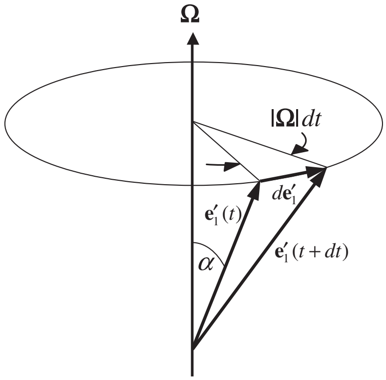


FIGURE 4.7 Geometry showing the relationship between $\boldsymbol{\Omega}$, the rotational velocity vector of O′1′2′3′, and the first coordinate unit vector $\mathbf{e}_1'$ in O′1′2′3′. Here, the increment $d\mathbf{e}_1'$ is perpendicular to $\boldsymbol{\Omega}$ and $\mathbf{e}_1'$.

where the final equality is based on the geometric construction of the cross product shown in Figure 4.7 for $\mathbf{e}_1'$, one of the unit vectors in the rotating frame. In a small time dt, the rotation of O′1′2′3′ causes $\mathbf{e}_1'$ to trace a small portion of a cone with radius $\sin\alpha$ as shown. The magnitude of the change in $\mathbf{e}_1'$ is $|\mathbf{e}_1'| = (\sin\alpha)|\boldsymbol{\Omega}|dt$, so $d|\mathbf{e}_1'|/dt = (\sin\alpha)|\boldsymbol{\Omega}|$, which is equal to the magnitude of $\boldsymbol{\Omega} \times \mathbf{e}_1'$. The direction of the rate of change of $\mathbf{e}_1'$ is perpendicular to $\boldsymbol{\Omega}$ and $\mathbf{e}_1'$, which is the direction of $\boldsymbol{\Omega} \times \mathbf{e}_1'$. Thus, by geometric construction, $d\mathbf{e}_1'/dt = \boldsymbol{\Omega} \times \mathbf{e}_1'$, and by direct extension to the other unit vectors, $d\mathbf{e}_i'/dt = \boldsymbol{\Omega} \times \mathbf{e}_i'$ (in mixed notation).

To find the acceleration $\mathbf{a}$ of a fluid particle at P, take the time derivative of the final version of (4.42) to find:

$$\mathbf{a} = \frac{d\mathbf{u}}{dt} = \frac{d}{dt}(\mathbf{U} + \mathbf{u}' + \boldsymbol{\Omega} \times \mathbf{x}') = \frac{d\mathbf{U}}{dt} + \mathbf{a}' + 2\boldsymbol{\Omega} \times \mathbf{u}' + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{x}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}'). \tag{4.43}$$

(see Exercise 4.42) where $d\mathbf{U}/dt$ is the acceleration of O′ with respect to O, $\mathbf{a}'$ is the fluid particle acceleration viewed in the noninertial frame, $2\boldsymbol{\Omega}\times\mathbf{u}'$ is the *Coriolis* acceleration, $(d\boldsymbol{\Omega}/dt)\times\mathbf{x}'$ is the acceleration caused by angular acceleration of the noninertial frame, and the final term is the *centripetal* acceleration.

In fluid mechanics, the acceleration $\mathbf{a}$ of fluid particles is denoted $D\mathbf{u}/Dt$, so (4.43) is rewritten:

$$\left(\frac{D\mathbf{u}}{Dt}\right)_{O123} = \left(\frac{D'\mathbf{u}'}{Dt}\right)_{O'1'2'3'} + \frac{d\mathbf{U}}{dt} + 2\boldsymbol{\Omega} \times \mathbf{u}' + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{x}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}'). \tag{4.44}$$

This equation states that fluid particle acceleration in an inertial frame is equal to the sum of: the particle's acceleration in the noninertial frame, the acceleration of the noninertial frame, the Coriolis acceleration, the particle's apparent acceleration from the noninertial frame's angular acceleration, and the particle's centripetal acceleration. Substituting (4.44) into (4.39), produces:

$$\rho\left(\frac{D'\mathbf{u}'}{Dt}\right)_{O'1'2'3'} = -\nabla' p + \rho\left[\mathbf{g} - \frac{d\mathbf{U}}{dt} - 2\boldsymbol{\Omega} \times \mathbf{u}' - \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{x}' - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}')\right] + \mu\nabla'^2\mathbf{u}' \tag{4.45}$$

as the incompressible-flow momentum conservation equation in a noninertial frame of reference where the primes denote differentiation, velocity, and position in the noninertial frame. Thermodynamic variables and the net viscous stress are independent of the frame of reference. Equation (4.45) makes it clear that the primary effect of a noninertial frame is the addition of extra body force terms that arise from the motion of the noninertial frame. The terms in [,]-brackets reduce to $\mathbf{g}$ alone when $O'1'2'3'$ is an inertial frame ($\mathbf{U}$ = constant and $\mathbf{\Omega} = 0$).

The four new terms in (4.45) may each be significant. The first new term $d\mathbf{U}/dt$ accounts for the acceleration of O' relative to O. It provides the apparent force that pushes occupants back into their seats or makes them tighten their grip on a handrail when a vehicle accelerates. An aircraft that is flown on a parabolic trajectory produces weightlessness in its interior when its acceleration $d\mathbf{U}/dt$ equals $\mathbf{g}$.

The second new term, the Coriolis term, depends on the fluid particle's velocity, not on its position. Thus, even at the earth's rotation rate of one cycle per day, it has important consequences for the accuracy of artillery and for navigation during air and sea travel. The earth's angular velocity vector $\mathbf{\Omega}$ points out of the ground in the northern hemisphere. The Coriolis acceleration $-2\mathbf{\Omega} \times \mathbf{u}$ therefore tends to deflect a particle to the right of its direction of travel in the northern hemisphere and to the left in the southern hemisphere. Imagine a low-drag projectile shot horizontally from the north pole with speed u (Figure 4.8). The Coriolis acceleration $2\Omega u$ constantly acts perpendicular to its path and therefore does not change the speed u of the projectile. The forward distance traveled in time t is ut, and the deflection is Ωut^2. The angular deflection is $\Omega ut^2/ut = \Omega t$, which is the earth's rotation in time t. This demonstrates that the projectile in fact travels in a straight line if observed from outer space (an inertial frame); its apparent deflection is merely due to the rotation of the earth underneath it. Observers on earth need an imaginary force to account for this deflection. A clear physical explanation of the Coriolis force, with applications to mechanics, is given by Stommel and Moore (1989).

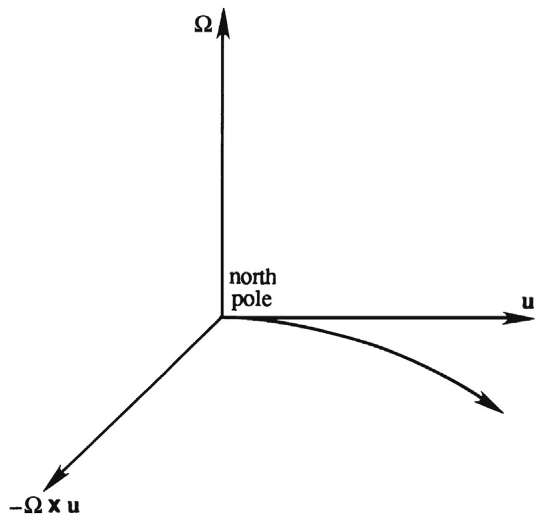


FIGURE 4.8 Particle trajectory deflection caused by the Coriolis acceleration when observed in a rotating frame of reference. If observed from a stationary frame of reference, the particle trajectory would be straight.

In the atmosphere, the Coriolis acceleration is responsible for wind circulation patterns around centers of high and low pressure in the earth's atmosphere. In an inertial frame, a nonzero pressure gradient accelerates fluid from regions of higher pressure to regions of lower pressure, as the first term on the right of (4.38) and (4.45) indicates. Imagine a cylindrical polar coordinate system (Figure 3.3c), with the z-axis normal to the earth's surface and the origin at the center of a high- or low-pressure region in the atmosphere. If it is a high pressure zone, u_R would be outward (positive) away from the z-axis in the absence of rotation since fluid will leave a center of high pressure. In this situation when there is rotation, the Coriolis acceleration $-2\mathbf{\Omega} \times \mathbf{u} = -2\Omega_z u_R \mathbf{e}_\varphi$ is in the $-\varphi$ direction (in the Northern hemisphere), or clockwise as viewed from above. On the other hand, if the flow is inward toward the center of a low-pressure zone, which reverses the direction of u_R, the Coriolis acceleration is counterclockwise. In the southern hemisphere, the direction of Ω_z is reversed so that the circulation patterns described above are reversed. Although the effects of a rotating frame will be commented on occasionally in this and subsequent chapters, most of the discussions involving Coriolis forces are given in Chapter 13, which covers geophysical fluid dynamics.

The third new acceleration term in [,]-brackets in (4.45) is caused by changes in the rotation rate of the frame of reference so it is of little importance for geophysical flows or for flows in machinery that rotate at a constant rate about a fixed axis. However, it does play a role when rotation speed or the direction of rotation vary with time.

The final new acceleration term in (4.45), the centrifugal acceleration, depends strongly on the rotation rate and the distance of the fluid particle from the axis of rotation. If the rotation rate is steady and the axis of rotation coincides with the z-axis of a cylindrical polar

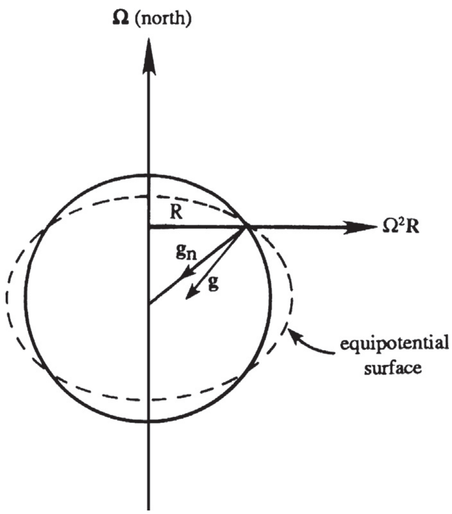


FIGURE 4.9 The earth's rotation causes it to budge near the equator and this leads to a mild distortion of equipotential surfaces from perfect spherical symmetry. The total gravitational acceleration is a sum of a centrally directed acceleration $\mathbf{g_n}$ (the Newtonian gravitation) and a rotational correction Ω^2R that points away from the axis of rotation.

coordinate system so that $\mathbf{\Omega} = (0, 0, \Omega)$ and $\mathbf{x}' = (R, \varphi, z)$, then $-\mathbf{\Omega}\times(\mathbf{\Omega} \times \mathbf{x}') = +\Omega^2 R\mathbf{e}_R$. This additional apparent acceleration can be added to the gravitational acceleration $\mathbf{g}$ to define an *effective gravity* $\mathbf{g}_e = \mathbf{g} + \Omega^2 R\mathbf{e}_R$ (Figure 4.9). Interestingly, a body-force potential for the new term can be found, but its impact might only be felt for relatively large atmospheric- or oceanic-scale flows (Exercise 4.43). The effective gravity is not precisely directed at the center of the earth and its acceleration value varies slightly over the surface of the earth. The equipotential surfaces (shown by the dashed lines in Figure 4.9) are perpendicular to the effective gravity, and the average sea level is one of these equipotential surfaces. Thus, at least locally on the earth's surface, we can write $\Phi_e = gz$, where z is measured perpendicular to an equipotential surface, and g is the local acceleration caused by the effective gravity. Use of the locally correct acceleration and direction for the earth's gravitational body force in the equations of fluid motion accounts for the centrifugal acceleration and the fact that the earth is really an ellipsoid with equatorial diameter 42 km larger than the polar diameter.

EXAMPLE 4.5

Find the radial, angular, and axial fluid momentum equations for viscous flow in the gaps between plates of a von Karman viscous impeller pump (see Figure 4.10) that rotates at a constant angular speed Ω_z. Assume steady constant-density constant-viscosity flow, neglect the body force for simplicity, and use cylindrical coordinates (Figure 3.3c).

Solution

First a little background: A von Karman viscous impeller pump uses rotating plates to pump viscous fluids via a combination of viscous and centrifugal forces. Although such pumps may be inefficient, they are wear-tolerant and may be used to pump abrasive fluids that would damage the

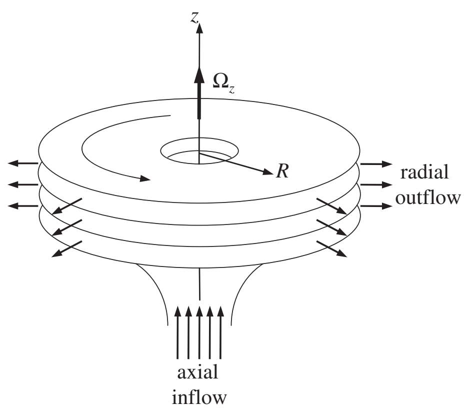


FIGURE 4.10 Schematic drawing of the impeller of a von Karman pump (Example 4.5).

vanes or blades of other pumps. Plus, their pumping action is entirely steady so they are exceptionally quiet, a feature occasionally exploited for air-moving applications in interior spaces occupied by human beings.

For steady, constant-density, constant-viscosity flow without a body force in a steadily rotating frame of reference, the momentum equation is a simplified version of (4.45):

$$\rho(\mathbf{u}'\cdot\nabla')\mathbf{u}' = -\nabla' p + \rho[-2\boldsymbol{\Omega}\times\mathbf{u}' - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x}')] + \mu\nabla'^2\mathbf{u}'.$$

Here we are not concerned with the axial inflow or the flow beyond the outer edges of the disks. Now choose the z-axis of the coordinate system to be coincident with the axis of rotation. For this choice, the flow between the disks should be axisymmetric, so we can presume that u'_R, u'_φ, u'_z, and p only depend on R and z. To further simplify the momentum equation, drop the primes, evaluate the cross products,

$$\boldsymbol{\Omega}\times\mathbf{u} = \Omega_z\mathbf{e}_z\times(u_R\mathbf{e}_R + u_\varphi\mathbf{e}_\varphi) = +\Omega_z u_R\mathbf{e}_\varphi - \Omega_z u_\varphi\mathbf{e}_R, \quad \text{and} \quad \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x}') = -\Omega_z^2 R\mathbf{e}_R,$$

and separate the radial, angular, and axial components to find:

$$\rho\left(u_R\frac{\partial u_R}{\partial R} + u_z\frac{\partial u_R}{\partial z} - \frac{u_\varphi^2}{R}\right) = -\frac{\partial p}{\partial R} + \rho\left[2\Omega_z u_\varphi + \Omega_z^2 R\right] + \mu\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u_R}{\partial R}\right) + \frac{\partial^2 u_R}{\partial z^2} - \frac{u_R}{R^2}\right)$$

$$\rho\left(u_R\frac{\partial u_\varphi}{\partial R} + u_z\frac{\partial u_\varphi}{\partial z} + \frac{u_R u_\varphi}{R}\right) = \rho[-2\Omega_z u_R] + \mu\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u_\varphi}{\partial R}\right) + \frac{\partial^2 u_\varphi}{\partial z^2} - \frac{u_\varphi}{R^2}\right)$$

$$\rho\left(u_R\frac{\partial u_z}{\partial R} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u_z}{\partial R}\right) + \frac{\partial^2 u_z}{\partial z^2}\right).$$

Here we have used the results found in the Appendix B for cylindrical coordinates. In the first two momentum equations, the terms in [,]-brackets result from rotation of the coordinate system.

4.8. CONSERVATION OF ENERGY

In this section, the integral energy-conservation equivalent of (4.5) and (4.17) is developed from a mathematical statement of conservation of energy for a fluid particle in an inertial frame of reference. The subsequent steps that lead to a differential energy-conservation equivalent of (4.7) and (4.24) follow the pattern set in Sections 4.2 and 4.5. For clarity and conciseness, the explicit listing of independent variables is dropped from the equations in this section.

When applied to a material volume $V(t)$ with surface area $A(t)$, conservation of internal energy per unit mass e and the kinetic energy per unit mass (½)$|\mathbf{u}|^2$ can be stated:

$$\frac{d}{dt}\int_{V(t)}\rho\left(e + \frac{1}{2}|\mathbf{u}|^2\right)dV = \int_{V(t)}\rho\mathbf{g}\cdot\mathbf{u}\,dV + \int_{A(t)}\mathbf{f}\cdot\mathbf{u}\,dA - \int_{A(t)}\mathbf{q}\cdot\mathbf{n}\,dA, \tag{4.46}$$

where the terms on the right are: work done on the fluid in $V(t)$ by body forces, work done on the fluid in $V(t)$ by surface forces, and heat transferred out of $V(t)$. Here, $\mathbf{q}$ is the heat flux vector and in general includes thermal conduction and radiation. The final term in (4.46) has a negative sign because the energy in $V(t)$ decreases when heat leaves $V(t)$ and this occurs when $\mathbf{q}\cdot\mathbf{n}$ is positive. Again, the implications of (4.46) are better displayed when the time derivative is expanded using Reynolds transport theorem (3.35),

$$\begin{aligned}\int_{V(t)} \frac{\partial}{\partial t}\left(\rho e + \frac{\rho}{2}|\mathbf{u}|^2\right) dV + \int_{A(t)} \left(\rho e + \frac{\rho}{2}|\mathbf{u}|^2\right)(\mathbf{u}\cdot\mathbf{n}) dA \\ = \int_{V(t)} \rho\mathbf{g}\cdot\mathbf{u} dV + \int_{A(t)} \mathbf{f}\cdot\mathbf{u} dA - \int_{A(t)} \mathbf{q}\cdot\mathbf{n} dA.\end{aligned} \tag{4.47}$$

Similar to the prior developments for mass and momentum conservation, this result can be generalized to an arbitrarily moving control volume $V^*(t)$ with surface $A^*(t)$:

$$\begin{aligned}\frac{d}{dt}\int_{V^*(t)} \rho\left(e + \frac{1}{2}|\mathbf{u}|^2\right) dV + \int_{A^*(t)} \left(\rho e + \frac{\rho}{2}|\mathbf{u}|^2\right)(\mathbf{u}-\mathbf{b})\cdot\mathbf{n} dA \\ = \int_{V^*(t)} \rho\mathbf{g}\cdot\mathbf{u} dV + \int_{A^*(t)} \mathbf{f}\cdot\mathbf{u} dA - \int_{A^*(t)} \mathbf{q}\cdot\mathbf{n} dA,\end{aligned} \tag{4.48}$$

when $V^*(t)$ is instantaneously coincident with $V(t)$. And, just like (4.5) and (4.17), (4.48) can be specialized to stationary, steadily moving, accelerating, or deforming control volumes by appropriate choice of the control surface velocity $\mathbf{b}$.

The differential equation that represents energy conservation is obtained from (4.47) after collecting all four terms under the same volume integration. The first step is to convert the three surface integrals in (4.47) to volume integrals using Gauss' theorem (2.30):

$$\begin{aligned}\int_{A(t)} \left(\rho e + \frac{\rho}{2}|\mathbf{u}|^2\right)(\mathbf{u}\cdot\mathbf{n}) dA &= \int_{V(t)} \nabla\cdot\left(\rho e\mathbf{u} + \frac{\rho}{2}|\mathbf{u}|^2\mathbf{u}\right) dV \\ &= \int_{V(t)} \frac{\partial}{\partial x_i}\left(\rho\left(e + \frac{1}{2}u_j^2\right)u_i\right) dV,\end{aligned} \tag{4.49}$$

$$\int_{A(t)} \mathbf{f}\cdot\mathbf{u} dA = \int_{A(t)} n_i\tau_{ij}u_j dA = \int_{V(t)} \frac{\partial}{\partial x_i}(\tau_{ij}u_j) dV, \tag{4.50}$$

and

$$\int_{A(t)} \mathbf{q}\cdot\mathbf{n} dA = \int_{A(t)} q_i n_i dA = \int_{V(t)} \nabla\cdot\mathbf{q} dA = \int_{V(t)} \frac{\partial}{\partial x_i} q_i dA, \tag{4.51}$$

where in (4.49) $u_j^2 = u_1^2 + u_2^2 + u_3^2$ because the summation index j is implicitly repeated. Substituting (4.49) through (4.51) into (4.47) and putting all the terms together into the same volume integration produces:

$$\int_{V(t)} \left\{ \frac{\partial}{\partial t}\left(\rho\left[e + \frac{1}{2}u_j^2\right]\right) + \frac{\partial}{\partial x_i}\left(\rho\left[e + \frac{1}{2}u_j^2\right]u_i\right) - \rho g_i u_i - \frac{\partial}{\partial x_i}\left(\tau_{ij}u_j\right) + \frac{\partial q_i}{\partial x_i}\right\} dV = 0. \tag{4.52}$$

Similar to (4.6) and (4.21), the integral in (4.52) can only be zero for any material volume if its integrand vanishes at every point in space; thus (4.52) requires:

$$\frac{\partial}{\partial t}\left(\rho\left[e + \frac{1}{2}u_j^2\right]\right) + \frac{\partial}{\partial x_i}\left(\rho\left[e + \frac{1}{2}u_j^2\right]u_i\right) = \rho g_i u_i + \frac{\partial}{\partial x_i}\left(\tau_{ij}u_j\right) - \frac{\partial q_i}{\partial x_i}. \tag{4.53}$$

This differential equation is a statement of conservation of energy containing terms for fluid particle internal energy, fluid particle kinetic energy, work, energy exchange, and heat transfer. It is commonly revised and simplified so that its terms are more readily interpreted. The second term on the right side of (4.53) represents the total rate of work done on a fluid particle by surface stresses. By performing the differentiation, and then using (4.27) to separate out pressure and viscous surface-stress terms, it can be decomposed as follows:

$$\frac{\partial}{\partial x_i}\left(\tau_{ij}u_j\right) = \tau_{ij}\frac{\partial u_j}{\partial x_i} + u_j\frac{\partial \tau_{ij}}{\partial x_i} = \left(-p\frac{\partial u_j}{\partial x_j} + \sigma_{ij}\frac{\partial u_j}{\partial x_i}\right) + \left(-u_j\frac{\partial p}{\partial x_j} + u_j\frac{\partial \sigma_{ij}}{\partial x_i}\right). \tag{4.54}$$

In the final equality, the terms in the first set of (,)-parentheses are the pressure and viscous-stress work terms that lead to the deformation of fluid particles while the terms in the second set of (,)-parentheses are the product of the local fluid velocity with the net pressure force and the net viscous force that lead to either an increase or decrease in the fluid particle's kinetic energy. (Recall from (4.24) that $\partial\tau_{ij}/\partial x_j$ represents the net surface force.) Substituting (4.54) into (4.53), expanding the differentiations on the left in (4.53), and using the continuity equation (4.7) to drop terms produces:

$$\rho\frac{D}{Dt}\left(e + \frac{1}{2}u_j^2\right) = \rho g_i u_i + \left(-p\frac{\partial u_j}{\partial x_j} + \sigma_{ij}\frac{\partial u_j}{\partial x_i}\right) + \left(-u_j\frac{\partial p}{\partial x_j} + u_j\frac{\partial \sigma_{ij}}{\partial x_i}\right) - \frac{\partial q_i}{\partial x_i} \tag{4.55}$$

(see Exercise 4.45). This equation contains both mechanical and thermal energy terms. A separate equation for the mechanical energy can be constructed by multiplying (4.22) by u_j and summing over j. After some manipulation, the result is:

$$\rho\frac{D}{Dt}\left(\frac{1}{2}u_j^2\right) = \rho g_j u_j - u_j\frac{\partial p}{\partial x_j} + u_j\frac{\partial}{\partial x_i}\left(\sigma_{ij}\right) \tag{4.56}$$

(see Exercise 4.46), where (4.27) has been used for τ_{ij}. Subtracting (4.56) from (4.55), dividing by $\rho = 1/v$, and using (4.8) produces:

$$\frac{De}{Dt} = -p\frac{Dv}{Dt} + \frac{1}{\rho}\sigma_{ij}S_{ij} - \frac{1}{\rho}\frac{\partial q_i}{\partial x_i}, \tag{4.57}$$

where the fact that σ_{ij} is symmetric has been exploited so $\sigma_{ij}(\partial u_j/\partial x_i) = \sigma_{ij}(S_{ji} + R_{ji}) = \sigma_{ij}S_{ij}$ with S_{ij} given by (3.12). Equation (4.57) is entirely equivalent to the first law of thermodynamics (1.10)—the change in energy of a system equals the work put into the system minus the heat lost by the system. The difference is that in (4.57), all the terms have units of power per unit mass instead of energy. The first two terms on the right in (4.57) are the pressure and viscous work done on a fluid particle while the final term represents heat transfer from the fluid particle. The pressure work and heat transfer terms may have either sign.

The viscous work term in (4.57) is the kinetic energy dissipation rate per unit mass, and it is commonly denoted by $\varepsilon = (1/\rho)\sigma_{ij}S_{ij}$. It is the product of the viscous stress acting on a fluid element and the deformation rate of a fluid element, and represents the viscous work put into fluid element deformation. This work is irreversible because deformed fluid elements do not return to their prior shape when a viscous stress is relieved. Thus, ε represents the irreversible conversion of mechanical energy to thermal energy through the action of viscosity. It is always positive and can be written in terms of the viscosities and squares of velocity field derivatives (see Exercise 4.47):

$$\varepsilon \equiv \frac{1}{\rho}\sigma_{ij}S_{ij} = \frac{1}{\rho}\left(2\mu S_{ij} + \left(\mu_v - \frac{2}{3}\mu\right)\frac{\partial u_m}{\partial x_m}\delta_{ij}\right)S_{ij} = 2\nu\left(S_{ij} - \frac{1}{3}\frac{\partial u_m}{\partial x_m}\delta_{ij}\right)^2 + \frac{\mu_v}{\rho}\left(\frac{\partial u_m}{\partial x_m}\right)^2, \quad (4.58)$$

where $\nu \equiv \mu/\rho$ is the kinematic viscosity, (1.4), and

$$\sigma_{ij} = +\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(\mu_v - \frac{2}{3}\mu\right)\frac{\partial u_m}{\partial x_m}\delta_{ij} \quad (4.59)$$

for a Newtonian fluid. Here we note that only shear deformations contribute to ε when $\mu_v = 0$ or when the flow is in incompressible. As described in Chapter 12, ε plays an important role in the physics and description of turbulent flow. It is proportional to μ (and μ_v) and the square of velocity gradients, so it is more important in regions of high shear. The internal energy increase resulting from high ε could appear as a hot lubricant in a bearing, or as burning of the surface of a spacecraft on reentry into the atmosphere.

The final energy-equation manipulation is to express q_i in terms of the other dependent field variables. For nearly all the circumstances considered in this text, heat transfer is caused by thermal conduction alone, so using (4.58) and Fourier's law of heat conduction (1.2), (4.57) can be rewritten:

$$\rho\frac{De}{Dt} = -p\frac{\partial u_m}{\partial x_m} + 2\mu\left(S_{ij} - \frac{1}{3}\frac{\partial u_m}{\partial x_m}\delta_{ij}\right)^2 + \mu_v\left(\frac{\partial u_m}{\partial x_m}\right)^2 + \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right), \quad (4.60)$$

where k is the fluid's thermal conductivity. It is presumed to only depend on thermodynamic conditions, as is the case for μ and μ_v.

At this point the development of the differential equations of fluid motion is complete. The field equations (4.7), (4.38), and (4.60) are general for a Newtonian fluid that follows Fourier's law of heat conduction. These field equations and two thermodynamic equations provide: $1 + 3 + 1 + 2 = 7$ scalar equations. The dependent variables in these equations are ρ, e, p, T, and u_j, a total of $1 + 1 + 1 + 1 + 3 = 7$ unknowns. The number of equations is equal to the number of unknown field variables; therefore, solutions are in principle possible for suitable boundary conditions.

Interestingly, the evolution of the entropy s in fluid flows can be deduced from (4.57) by using Gibb's property relation (1.18) for the internal energy $de = Tds - pd(1/\rho)$. When made specific to time variations following a fluid particle, it becomes:

$$\frac{De}{Dt} = T\frac{Ds}{Dt} - p\frac{D(1/\rho)}{Dt}. \tag{4.61}$$

Combining (4.57), (4.58), and (4.61) produces:

$$\frac{Ds}{Dt} = -\frac{1}{\rho T}\frac{\partial q_i}{\partial x_i} + \frac{\varepsilon}{T} = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(\frac{q_i}{T}\right) - \frac{q_i}{\rho T^2}\left(\frac{\partial T}{\partial x_i}\right) + \frac{\varepsilon}{T}, \tag{4.62}$$

and using Fourier's law of heat conduction, this becomes:

$$\frac{Ds}{Dt} = +\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(\frac{k}{T}\frac{\partial T}{\partial x_i}\right) + \frac{k}{\rho T^2}\left(\frac{\partial T}{\partial x_i}\right)^2 + \frac{\varepsilon}{T}. \tag{4.63}$$

The first term on the right side is the entropy gain or loss from heat conduction. The last two terms, which are proportional to the square of temperature and velocity gradients (see (4.58)), represent the *entropy production* caused by heat conduction and viscous generation of heat. The second law of thermodynamics requires that the entropy production due to irreversible phenomena should be positive, so that $\mu, \kappa, k > 0$. Thus, explicit appeal to the second law of thermodynamics is not required in most analyses of fluid flows because it has already been satisfied by taking positive values for the viscosities and the thermal conductivity. In addition (4.63) requires that fluid particle entropy be preserved along particle trajectories when the flow is inviscid and non-heat-conducting, i.e., when $Ds/Dt = 0$.

4.9. SPECIAL FORMS OF THE EQUATIONS

The general equations of motion for a fluid may be put into a variety of special forms when certain symmetries or approximations are valid. Several special forms are presented in this section. The first applies to the integral form of the momentum equation and corresponds to the classical mechanics principle of conservation of angular momentum. The second through fifth special forms arise from manipulations of the differential equations to generate Bernoulli equations. The sixth special form applies when the flow has constant density and the gravitational body force and hydrostatic pressure cancel. The final special form for the equations of motion presented here, known as the Boussinesq approximation, is for low-speed incompressible flows with constant transport coefficients and small changes in density.

Angular Momentum Principle for a Stationary Control Volume

In the mechanics of solids bodies it is shown that

$$d\mathbf{H}/dt = \mathbf{M}, \tag{4.64}$$

where $\mathbf{M}$ is the torque of all external forces on the body about any chosen axis, and $d\mathbf{H}/dt$ is the rate of change of angular momentum of the body about the same axis. For the fluid in a material control volume, the angular momentum is

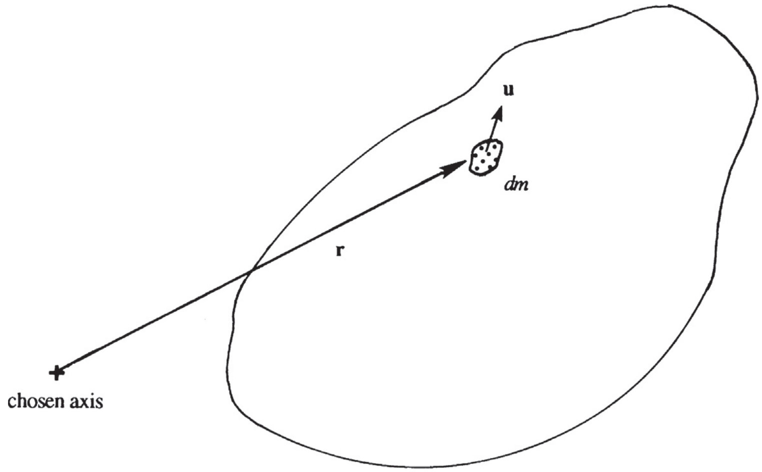


FIGURE 4.11 Definition sketch for the angular momentum theorem where $dm = \rho dV$. Here the chosen axis points out of the page, and elemental contributions to the angular momentum about this axis are $\mathbf{r} \times \rho \mathbf{u} dV$.

$$\mathbf{H} = \int_{V(t)} (\mathbf{r} \times \rho\mathbf{u}) dV,$$

where $\mathbf{r}$ is the position vector from the chosen axis (Figure 4.11). Inserting this in (4.64) produces:

$$\frac{d}{dt}\int_{V(t)} (\mathbf{r} \times \rho\mathbf{u}) dV = \int_{V(t)} (\mathbf{r} \times \rho\mathbf{g}) dV + \int_{A(t)} (\mathbf{r} \times \mathbf{f}) dA,$$

where the two terms on the right are the torque produced by body forces and surface stresses, respectively. As before, the left-hand term can be expanded via Reynolds transport theorem to find:

$$\begin{aligned}\frac{d}{dt}\int_{V(t)} (\mathbf{r} \times \rho\mathbf{u}) dV &= \int_{V(t)} \frac{\partial}{\partial t}(\mathbf{r} \times \rho\mathbf{u}) dV + \int_{A(t)} (\mathbf{r} \times \rho\mathbf{u})(\mathbf{u}\cdot\mathbf{n}) dA \\ &= \int_{V_o} \frac{\partial}{\partial t}(\mathbf{r} \times \rho\mathbf{u}) dV + \int_{A_o} (\mathbf{r} \times \rho\mathbf{u})(\mathbf{u}\cdot\mathbf{n}) dA \\ &= \frac{d}{dt}\int_{V_o} (\mathbf{r} \times \rho\mathbf{u}) dV + \int_{A_o} (\mathbf{r} \times \rho\mathbf{u})(\mathbf{u}\cdot\mathbf{n}) dA,\end{aligned}$$

where V_o and A_o are the volume and surface of a stationary control volume that is instantaneously coincident with the material volume, and the final equality holds because V_o does not vary with time. Thus, the stationary volume angular momentum principle is:

$$\frac{d}{dt}\int_{V_o} (\mathbf{r} \times \rho\mathbf{u}) dV + \int_{A_o} (\mathbf{r} \times \rho\mathbf{u})(\mathbf{u}\cdot\mathbf{n}) dA = \int_{V_o} (\mathbf{r} \times \rho\mathbf{g}) dV + \int_{A_o} (\mathbf{r} \times \mathbf{f}) dA. \quad (4.65)$$

The angular momentum principle (4.65) is analogous to the linear momentum principle (4.17) when $\mathbf{b} = 0$, and is very useful in investigating rotating fluid systems such as turbomachines, fluid couplings, dishwashing-machine spray rotors, and even lawn sprinklers.

EXAMPLE 4.6

Consider a lawn sprinkler as shown in Figure 4.12. The area of each nozzle exit is A, and the jet velocity is U. Find the torque required to hold the rotor stationary.

Solution

Select a stationary volume V_o with area A_o as shown by the dashed lines. Pressure everywhere on the control surface is atmospheric, and there is no net moment due to the pressure forces. The control surface cuts through the vertical support and the torque M exerted by the support on the sprinkler arm is the only torque acting on V_o. Apply the angular momentum balance

$$\int_{A_o} (\mathbf{r} \times \rho \mathbf{u})(\mathbf{u} \cdot \mathbf{n}) dA = \int_{A_o} (\mathbf{r} \times \mathbf{f}) dA = M,$$

where the time derivative term must be zero for a stationary rotor. Evaluating the surface flux terms produces:

$$\int_{A_o} (\mathbf{r} \times \rho \mathbf{u})(\mathbf{u} \cdot \mathbf{n}) dA = (a\rho U \cos\alpha) UA + (a\rho U \cos\alpha) UA = 2a\rho AU^2 \cos\alpha.$$

Therefore, the torque required to hold the rotor stationary is $M = 2a\rho AU^2 \cos\alpha$. When the sprinkler is rotating at a steady state, this torque is balanced by both air resistance and mechanical friction.

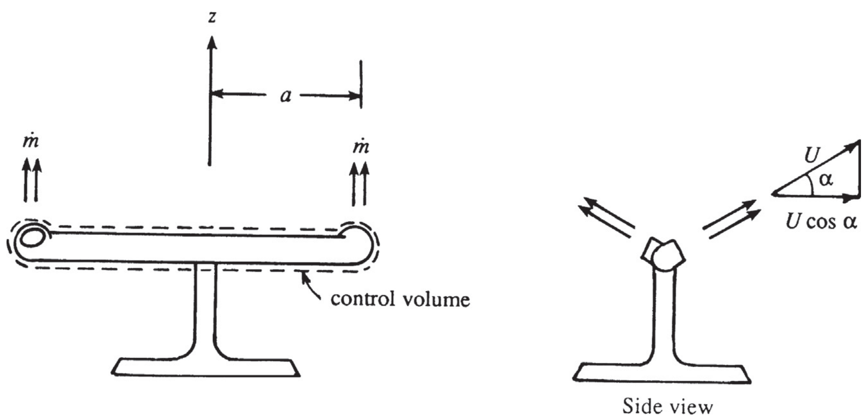


FIGURE 4.12 Lawn sprinkler.

Bernoulli Equations

Various conservation laws for mass, momentum, energy, and entropy were presented in the preceding sections. Bernoulli equations are not separate laws, but are instead derived from the Navier-Stokes momentum equation (4.38) and the energy equation (4.60) under various sets of conditions.

First consider inviscid flow ($\mu = \mu_v = 0$) where gravity is the only body force so that (4.38) reduces to the Euler equation (4.41):

$$\frac{\partial u_j}{\partial t} + u_i \frac{\partial u_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} - \frac{\partial}{\partial x_j}\Phi, \tag{4.66}$$

where $\Phi = gz$ is the body force potential, g is the acceleration of gravity, and the z-axis is vertical. If the flow is also barotropic, then $\rho = \rho(p)$, and

$$\frac{1}{\rho}\frac{\partial p}{\partial x_j} = \frac{\partial}{\partial x_j}\int_{p_o}^{p} \frac{dp'}{\rho(p')}, \tag{4.67}$$

where dp/ρ is a perfect differential, p_o is a reference pressure, and p' is the integration variable. In this case the integral depends only on its endpoints, and not on the path of integration. Constant density, isothermal, and isentropic flows are barotropic. In addition, the advective acceleration in (4.66) may be rewritten in terms of the velocity-vorticity cross product, and the gradient of the kinetic energy per unit mass:

$$u_i \frac{\partial u_j}{\partial x_i} = -(\mathbf{u} \times \boldsymbol{\omega})_j + \frac{\partial}{\partial x_j}\left(\frac{1}{2}u_i^2\right) \tag{4.68}$$

(see Exercise 4.50). Substituting (4.67) and (4.68) into (4.66) produces:

$$\frac{\partial u_j}{\partial t} + \frac{\partial}{\partial x_j}\left[\frac{1}{2}u_i^2 + \int_{p_o}^{p} \frac{dp'}{\rho(p')} + gz\right] = (\mathbf{u} \times \boldsymbol{\omega})_j, \tag{4.69}$$

where all the gradient terms have been collected together to form the Bernoulli function B = the contents of the [,]-brackets.

Equation (4.69) can be used to deduce the evolution of the Bernoulli function in inviscid barotropic flow. First consider steady flow ($\partial u_j/\partial t = 0$) so that (4.69) reduces to

$$\nabla B = \mathbf{u} \times \boldsymbol{\omega}. \tag{4.70}$$

The left-hand side is a vector normal to the surface B = constant whereas the right-hand side is a vector perpendicular to both $\mathbf{u}$ and $\boldsymbol{\omega}$ (Figure 4.13). It follows that surfaces of constant B must contain the streamlines and vortex lines. Thus, an inviscid, steady, barotropic flow satisfies

$$\frac{1}{2}u_i^2 + \int_{p_o}^{p} \frac{dp'}{\rho(p')} + gz = \text{constant along streamlines and vortex lines.} \tag{4.71}$$

This is the first of several possible *Bernoulli equations*. If, in addition, the flow is irrotational ($\boldsymbol{\omega} = 0$), then (4.70) implies that

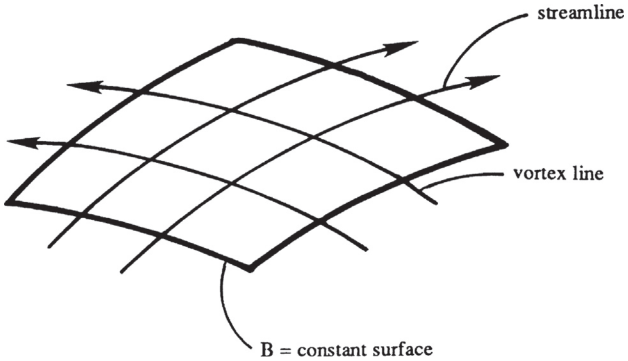


FIGURE 4.13 A surface defined by streamlines and vortex lines. Within this surface the Bernoulli function defined as the contents of the [,]-brackets in (4.69) is constant in steady flow. Note that the streamlines and vortex lines can be at an arbitrary angle.

$$\frac{1}{2}u_i^2 + \int_{p_o}^{p} \frac{dp'}{\rho(p')} + gz = \text{constant everywhere.} \tag{4.72}$$

It may be shown that a sufficient condition for the existence of the surfaces containing streamlines and vortex lines is that the flow be barotropic. Incidentally, these are called *Lamb surfaces* in honor of the distinguished English applied mathematician and hydrodynamicist, Horace Lamb. In a general nonbarotropic flow, a path composed of streamline and vortex line segments can be drawn between any two points in a flow field. Then (4.71) is valid with the proviso that the integral be evaluated on the specific path chosen. As written, (4.71) requires that the flow be steady, inviscid, and have only gravity (or other conservative) body forces acting upon it. Irrotational flow is presented in Chapter 6. We shall note only the important point here that, in a nonrotating frame of reference, barotropic irrotational flows remain irrotational if viscous effects are negligible. Consider the flow around a solid object, say an airfoil (Figure 4.14). The flow is irrotational at all points outside the thin viscous layer close to the surface of the body. This is because a particle P on a streamline outside the viscous layer started from some point S, where the flow is uniform and consequently irrotational. The Bernoulli equation (4.72) is therefore satisfied everywhere outside the viscous layer in this example.

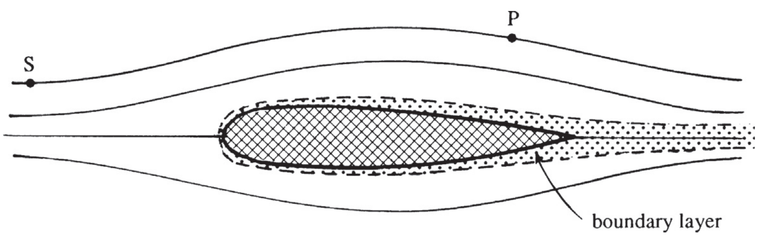


FIGURE 4.14 Flow over a solid object. Viscous shear stresses are usually confined to a thin layer near the body called a *boundary layer*. Flow outside the boundary layer is irrotational, so if a fluid particle at S is initially irrotational it will remain irrotational at P because the streamline it travels on does not enter the boundary layer.

An unsteady form of Bernoulli's equation can be derived only if the flow is irrotational. In this case, the velocity vector can be written as the gradient of a scalar potential ϕ (called the *velocity potential*):

$$\mathbf{u} \equiv \nabla\phi. \tag{4.73}$$

Putting (4.73) into (4.69) with $\boldsymbol{\omega} = 0$ produces:

$$\nabla\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \int_{p_o}^{p} \frac{dp'}{\rho(p')} + gz\right] = 0, \quad \text{or} \quad \frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \int_{p_o}^{p} \frac{dp'}{\rho(p')} + gz = B(t), \tag{4.74}$$

where the integration function $B(t)$ is independent of location. Here ϕ can be redefined to include B,

$$\phi \rightarrow \phi + \int_{t_o}^{t} B(t')dt',$$

without changing its use in (4.73); then the second part of (4.74) provides a second Bernoulli equation for unsteady, inviscid, irrotational, barotropic flow:

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \int_{p_o}^{p} \frac{dp'}{\rho(p')} + gz = \text{constant}. \tag{4.75}$$

This form of the Bernoulli equation will be used in studying irrotational wave motions in Chapter 7.

A third Bernoulli equation can be obtained for steady flow ($\partial/\partial t = 0$) from the energy equation (4.55) in the absence of viscous stresses and heat transfer ($\sigma_{ij} = q_i = 0$):

$$\rho u_i \frac{\partial}{\partial x_i}\left(e + \frac{1}{2}u_j^2\right) = \rho u_i g_i - \frac{\partial}{\partial x_j}\left(\rho u_j p/\rho\right). \tag{4.76}$$

When the body force is conservative with potential gz, and the steady continuity equation, $\partial(\rho u_i)/\partial x_i = 0$, is used to simplify (4.76), it becomes:

$$\rho u_i \frac{\partial}{\partial x_i}\left(e + \frac{p}{\rho} + \frac{1}{2}u_j^2 + gz\right) = 0. \tag{4.77}$$

From (1.13) $h = e + p/\rho$, so (4.77) states that gradients of the sum $h + |\mathbf{u}|^2/2 + gz$ must be normal to the local streamline direction u_i. Therefore, a third Bernoulli equation is:

$$h + \frac{1}{2}|\mathbf{u}|^2 + gz = \text{constant on streamlines}. \tag{4.78}$$

Equation (4.63) requires that inviscid, non-heat-conducting flows are isentropic (s does not change along particle paths), and (1.18) implies $dp/\rho = dh$ when $s =$ constant. Thus the path integral $\int dp/\rho$ becomes a function h of the endpoints only if both heat conduction and viscous stresses may be neglected in the momentum Bernoulli equations (4.71), (4.72), and (4.75). Equation (4.78) is very useful for high-speed gas flows where there is significant interplay between kinetic and thermal energies along a streamline. It is nearly the same as

(4.71), but does not include the other barotropic and vortex-line-evaluation possibilities allowed by (4.71).

Interestingly, there is also a Bernoulli equation for constant-viscosity constant-density irrotational flow. It can be obtained by starting from (4.39), using (4.68) for the advective acceleration, and noting from (4.40) that $\nabla^2\mathbf{u} = -\nabla\times\boldsymbol{\omega}$ in incompressible flow:

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\frac{\partial\mathbf{u}}{\partial t} + \rho\nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) - \rho\mathbf{u}\times\boldsymbol{\omega} = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{u} = -\nabla p + \rho\mathbf{g} - \mu\nabla\times\boldsymbol{\omega}. \quad (4.79)$$

When $\rho =$ constant, $\mathbf{g} = -\nabla(gz)$, and $\boldsymbol{\omega} = 0$, the second and final parts of this extended equality require:

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho gz + p\right) = 0. \quad (4.80)$$

Now, form the dot product of this equation with the arc-length element $\mathbf{e}_u ds = d\mathbf{s}$ directed along a chosen streamline, integrate from location 1 to location 2 along this streamline, and recognize that $\mathbf{e}_u\cdot\nabla = \partial/\partial s$ to find:

$$\rho\int_1^2\frac{\partial\mathbf{u}}{\partial t}\cdot d\mathbf{s} + \int_1^2\mathbf{e}_u\cdot\nabla\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho gz + p\right)ds = \rho\int_1^2\frac{\partial\mathbf{u}}{\partial t}\cdot d\mathbf{s} + \int_1^2\frac{\partial}{\partial s}\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho gz + p\right)ds = 0. \quad (4.81)$$

The integration in the second term is elementary, so a fourth Bernoulli equation for constant-viscosity constant-density irrotational flow is:

$$\int_1^2\frac{\partial\mathbf{u}}{\partial t}\cdot d\mathbf{s} + \left(\frac{1}{2}|\mathbf{u}|^2 + gz + \frac{p}{\rho}\right)_2 = \left(\frac{1}{2}|\mathbf{u}|^2 + gz + \frac{p}{\rho}\right)_1, \quad (4.82)$$

where 1 and 2 denote upstream and downstream locations on the same streamline at a single instant in time. Alternatively, (4.80) can be written using (4.73) as

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + gz + \frac{p}{\rho} = \text{constant}. \quad (4.83)$$

To summarize, there are (at least) four Bernoulli equations: (4.71) is for inviscid, steady, barotropic flow; (4.75) is for inviscid, irrotational, unsteady, barotropic flow; (4.78) is for inviscid, isentropic, steady flow; and (4.82) or (4.83) is for constant-viscosity, irrotational, unsteady, constant density flow. Perhaps the simplest form of these is (4.19).

There are many useful and important applications of Bernoulli equations. A few of these are described in the following paragraphs.

Consider first a simple device to measure the local velocity in a fluid stream by inserting a narrow bent tube (Figure 4.15), called a *pitot tube* after the French mathematician Henri Pitot (1695–1771), who used a bent glass tube to measure the velocity of the river Seine. Consider two points (1 and 2) at the same level, point 1 being away from the tube and point 2 being immediately in front of the open end where the fluid velocity $\mathbf{u}_2$ is zero. If the flow is steady

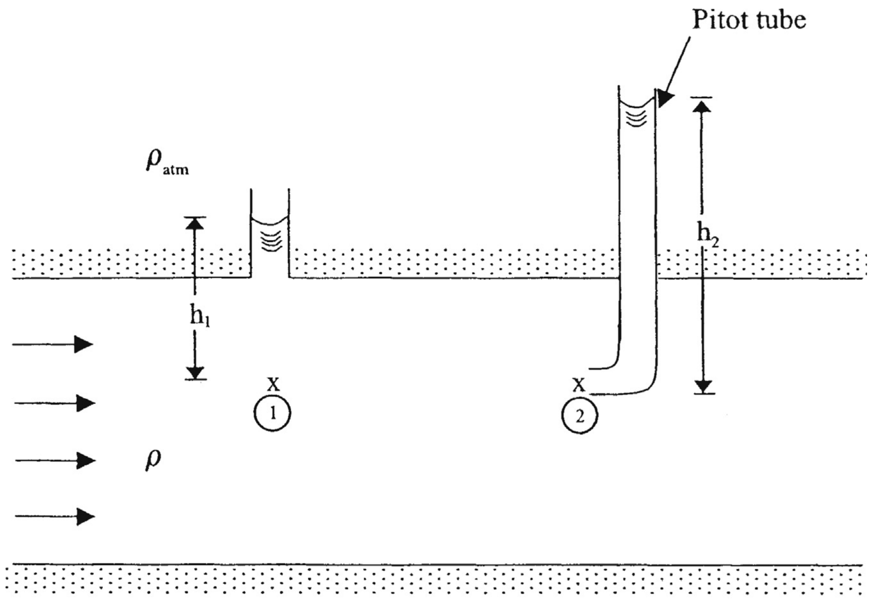


FIGURE 4.15 Pitot tube for measuring velocity in a duct. The first port measures the static pressure while the second port measures the static and dynamic pressure. Using the steady Bernoulli equation for incompressible flow, the height difference $h_2 - h_1$ can be related to the flow speed.

and irrotational with constant density along the streamline that connects 1 and 2, then (4.19) gives

$$\frac{p_1}{\rho} + \frac{1}{2}|\mathbf{u}|_1^2 = \frac{p_2}{\rho} + \frac{1}{2}|\mathbf{u}|_2^2 = \frac{p_2}{\rho},$$

from which the magnitude of $\mathbf{u}_1$ is found to be

$$|\mathbf{u}|_1 = \sqrt{2(p_2 - p_1)/\rho}.$$

Pressures at the two points are found from the hydrostatic balance

$$p_1 = \rho g h_1 \quad \text{and} \quad p_2 = \rho g h_2,$$

so that the magnitude of $\mathbf{u}_1$ can be found from

$$|\mathbf{u}|_1 = \sqrt{2g(h_2 - h_1)}.$$

Because it is assumed that the fluid density is very much greater than that of the atmosphere to which the tubes are exposed, the pressures at the tops of the two fluid columns are assumed to be the same. They will actually differ by $\rho_{atm} g(h_2 - h_1)$. Use of the hydrostatic

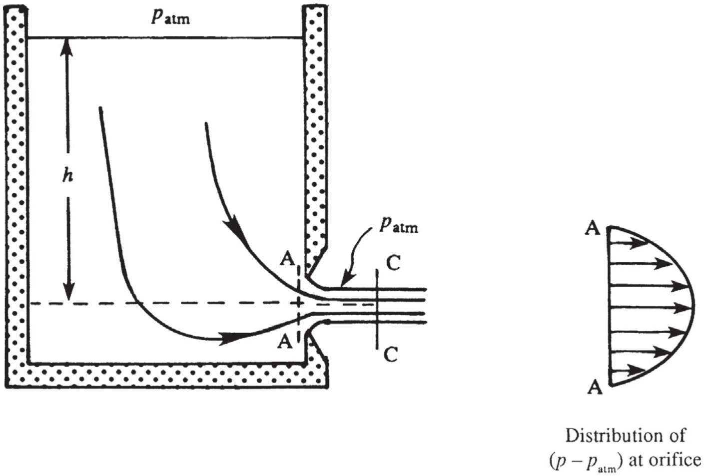


FIGURE 4.16 Flow through a sharp-edged orifice. Pressure has the atmospheric value everywhere across section CC; its distribution across orifice AA is indicated. The basic finding here is that the width of the fluid jet that emerges from the tank at AA is larger than the width of the jet that crosses CC.

approximation above station 1 is valid when the streamlines are straight and parallel between station 1 and the upper wall.

The pressure p_2 measured by a pitot tube is called *stagnation pressure* or *total pressure*, which is larger than the local static pressure. Even when there is no pitot tube to measure the stagnation pressure, it is customary to refer to the local value of the quantity $(p + \rho|\boldsymbol{u}|^2/2)$ as the local *stagnation pressure*, defined as the pressure that would be reached if the local flow is *imagined* to slow down to zero velocity frictionlessly. The quantity $\rho u^2/2$ is sometimes called the *dynamic pressure*; stagnation pressure is the sum of static and dynamic pressures.

As another application of Bernoulli's equation, consider the flow through an orifice or opening in a tank (Figure 4.16). The flow is slightly unsteady due to lowering of the water level in the tank, but this effect is small if the tank area is large compared to the orifice area. Viscous effects are negligible everywhere away from the walls of the tank. All streamlines can be traced back to the free surface in the tank, where they have the same value of the Bernoulli constant $B = |\mathbf{u}|^2/2 + p/\rho + gz$. It follows that the flow is irrotational, and B is constant *throughout* the flow.

We want to apply a Bernoulli equation between a point at the free surface in the tank and a point in the jet. However, the conditions right at the opening (section A in Figure 4.16) are not simple because the pressure is *not* uniform across the jet. Although pressure has the atmospheric value everywhere on the free surface of the jet (neglecting small surface tension effects), it is not equal to the atmospheric pressure *inside* the jet at this section. The streamlines at the orifice are curved, which requires that pressure must vary across the width of the jet in order to balance the centrifugal force. The pressure distribution across the orifice (section A) is shown in Figure 4.16. However, the streamlines in the jet become parallel a short distance away from the orifice (section C in Figure 4.16), where the jet area is smaller than the orifice area. The pressure across section C is uniform and equal to the atmospheric value (p_{atm}) because it has that value at the surface of the jet.

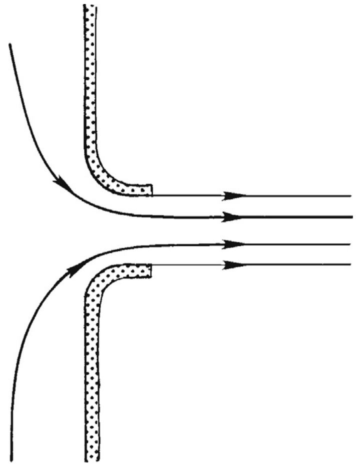


FIGURE 4.17 Flow through a rounded orifice. Here the pressure and velocity can achieve parallel outflow inside the tank, so the width of the jet does not change outside the tank.

Application of the Bernoulli equation (4.19) for steady constant-density flow between a point on the free surface in the tank and a point at C gives

$$\frac{p_{atm}}{\rho} + gh = \frac{p_{atm}}{\rho} + \frac{u^2}{2},$$

from which the average jet velocity magnitude u is found as

$$u = \sqrt{2gh},$$

which simply states that the loss of potential energy equals the gain of kinetic energy. The mass flow rate is approximately

$$\dot{m} = \rho A_c u = \rho A_c \sqrt{2gh},$$

where A_c is the area of the jet at C. For orifices having a sharp edge, A_c has been found to be $\approx$ 62% of the orifice area because the jet contracts downstream of the orifice opening.

If the orifice has a well-rounded opening (Figure 4.17), then the jet does not contract, the streamlines right at the exit are then parallel, and the pressure at the exit is uniform and equal to the atmospheric pressure. Consequently the mass flow rate is simply $\rho A\sqrt{2gh}$, where A equals the orifice area.

Neglect of Gravity in Constant Density Flows

When the flow velocity is zero, the Navier-Stokes momentum equation for incompressible flow (4.39b) reduces to a balance between the hydrostatic pressure p_s, and the steady body force acting on the hydrostatic density ρ_s,

$$0 = -\nabla p_s + \rho_s \mathbf{g},$$

which is equivalent to (1.8). When this hydrostatic balance is subtracted from (4.39b), the pressure difference from hydrostatic, $p' = p - p_s$, and the density difference from hydrostatic, $\rho' = \rho - \rho_s$, appear:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p' + \rho' \mathbf{g} + \mu \nabla^2 \mathbf{u}. \tag{4.84}$$

When the fluid density is constant, $\rho' = 0$ and the gravitational-body-force term disappears leaving:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p' + \mu \nabla^2 \mathbf{u}. \tag{4.85}$$

Because of this, steady body forces (like gravity) in constant density flow are commonly omitted from the momentum equation, and pressure is measured relative to its local hydrostatic value. Furthermore, the prime on p in (4.85) is typically dropped in this situation. However, when the flow includes a free surface, a fluid-fluid interface across which the density changes, or other variations in density, the gravitational-body-force term should reappear.

The Boussinesq Approximation

For flows satisfying certain conditions, Boussinesq in 1903 suggested that density changes in the fluid can be neglected except where ρ is multiplied by g. This approximation also treats the other properties of the fluid (such as μ, k, C_p) as constants. It is commonly useful for analyzing oceanic and atmospheric flows. Here we shall discuss the basis of the approximation in a somewhat intuitive manner and examine the resulting simplifications of the equations of motion. A formal justification, and the conditions under which the Boussinesq approximation holds, is given in Spiegel and Veronis (1960).

The Boussinesq approximation replaces the full continuity equation (4.7) by its incompressible form (4.10), $\nabla \cdot \mathbf{u} = 0$, to indicate that the relative density changes following a fluid particle, $\rho^{-1}(D\rho/Dt)$, are small compared to the velocity gradients that compose $\nabla \cdot \mathbf{u}$. Thus, the Boussinesq approximation cannot be applied to high-speed gas flows where density variations induced by velocity divergence cannot be neglected (see Section 4.11). Similarly, it cannot be applied when the vertical scale of the flow is so large that hydrostatic pressure variations cause significant changes in density. In a hydrostatic field, the vertical distance over which the density changes become important is of order $c^2/g \sim 10$ km for air where c is the speed of sound. (This vertical distance estimate is consistent with the scale height of the atmosphere; see Section 1.10.) The Boussinesq approximation therefore requires that the vertical scale of the flow be $L \ll c^2/g$.

In both cases just mentioned, density variations are caused by pressure variations. Now suppose that such pressure-compressibility effects are small and that density changes are caused by temperature variations alone, as in a thermal convection problem. In this case, the Boussinesq approximation applies when the temperature variations in the flow are small. Assume that ρ changes with T according to $\delta\rho/\rho = -\alpha\delta T$, where $\alpha = -\rho^{-1}(\partial\rho/\partial T)_p$ is the thermal expansion coefficient (1.20). For a perfect gas at room temperature $\alpha = 1/T \sim 3 \times 10^{-3}\,\mathrm{K}^{-1}$ but for typical liquids $\alpha \sim 5 \times 10^{-4}\,\mathrm{K}^{-1}$. Thus, for a temperature difference in the fluid of 10°C, density variations can be at most a few percent and it turns out that $\rho^{-1}(D\rho/Dt)$ can also be no larger than a few percent of the velocity gradients in $\nabla\cdot\mathbf{u}$. To see this, assume that the flow field is characterized by a length scale L, a velocity scale U, and a temperature scale δT. By this we mean that the velocity varies by U and the temperature varies by δT between locations separated by a distance of order L. The ratio of the magnitudes of the two terms in the continuity equation is

$$\frac{(1/\rho)(D\rho/Dt)}{\nabla\cdot\mathbf{u}} \sim \frac{(1/\rho)u(\partial\rho/\partial x)}{\partial u/\partial x} \sim \frac{(U/\rho)(\delta\rho/L)}{U/L} = \frac{\delta\rho}{\rho} = \alpha\delta T \ll 1,$$

which allows (4.7) to be replaced by its incompressible form (4.10).

The Boussinesq approximation for the momentum equation is based on its form for incompressible flow, and proceeds from (4.84) divided by ρ_s:

$$\frac{\rho}{\rho_s}\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_s}\nabla p' + \frac{\rho'}{\rho_s}\mathbf{g} + \frac{\mu}{\rho_s}\nabla^2\mathbf{u}.$$

When the density fluctuations are small $\rho/\rho_s \cong 1$ and $\mu/\rho_s \cong \nu$ (= the kinematic viscosity), so this equation implies:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_0}\nabla p' + \frac{\rho'}{\rho_0}\mathbf{g} + \nu\nabla^2\mathbf{u}, \tag{4.86}$$

where ρ_0 is a constant reference value of ρ_s. This equation states that density changes are negligible when conserving momentum, except when ρ' is multiplied by g. In flows involving buoyant convection, the magnitude of $\rho' g/\rho_s$ is of the same order as the vertical acceleration $\partial w/\partial t$ or the viscous term $\nu\nabla^2 w$.

The Boussinesq approximation to the energy equation starts from (4.60), written in vector notation,

$$\rho\frac{De}{Dt} = -p\nabla\cdot\mathbf{u} + \rho\varepsilon - \nabla\cdot\mathbf{q}, \tag{4.87}$$

where (4.58) has been used to insert ε, the kinetic energy dissipation rate per unit mass. Although the continuity equation is approximately $\nabla\cdot\mathbf{u} = 0$, an important point is that the volume expansion term $p(\nabla\cdot\mathbf{u})$ is *not* negligible compared to other dominant terms of equation (4.87); only for incompressible liquids is $p(\nabla\cdot\mathbf{u})$ negligible in (4.87). We have

$$-p\nabla\cdot\mathbf{u} = \frac{p}{\rho}\frac{D\rho}{Dt}; \frac{p}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{\mathrm{p}}\frac{DT}{Dt} = -p\alpha\frac{DT}{Dt}.$$

Assuming a perfect gas, for which $p = \rho RT$, $C_{\mathrm{p}} - C_{\mathrm{v}} = R$, and $\alpha = 1/T$, the foregoing estimate becomes

$$-p\nabla\cdot\mathbf{u} = -\rho RT\alpha\frac{DT}{Dt} = -\rho(C_{\mathrm{p}} - C_{\mathrm{v}})\frac{DT}{Dt}.$$

Equation (4.87) then becomes

$$\rho C_p \frac{DT}{Dt} = \rho\varepsilon - \nabla \cdot \mathbf{q}, \tag{4.88}$$

where $e = C_v T$ for a perfect gas. Note that C_v (instead of C_p) would have appeared on the left side of (4.88) if $\nabla \cdot \mathbf{u}$ had been dropped from (4.87).

The heating due to viscous dissipation of energy is negligible under the restrictions underlying the Boussinesq approximation. Comparing the magnitude of $\rho\varepsilon$ with the left-hand side of (4.88), we obtain

$$\frac{\rho\varepsilon}{\rho C_p (DT/Dt)} \sim \frac{2\mu S_{ij} S_{ij}}{\rho C_p u_i (\partial T/\partial x_i)} \sim \frac{\mu U^2/L^2}{\rho C_p U(\delta T/L)} = \frac{\nu U}{(C_p \delta T)L}.$$

In typical situations this is extremely small ($\sim 10^{-7}$). Neglecting $\rho\varepsilon$, and assuming Fourier's law of heat conduction (1.2) with constant k, (4.88) finally reduces to

$$\frac{DT}{Dt} = \kappa \nabla^2 T, \tag{4.89}$$

where $\kappa \equiv k/\rho C_p$ is the thermal diffusivity.

Summary

The Boussinesq approximation applies if the Mach number of the flow is small, propagation of sound or shock waves is not considered, the vertical scale of the flow is not too large, and the temperature differences in the fluid are small. Then the density can be treated as a constant in both the continuity and the momentum equations, except in the gravity term. Properties of the fluid such as μ, k, and C_p are also assumed constant. Omitting Coriolis accelerations, the set of equations corresponding to the Boussinesq approximation is: (4.9) and/or (4.10), (4.86) with $\mathbf{g} = -g\mathbf{e}_z$, (4.89), and $\rho = \rho_0[1 - \alpha(T - T_0)]$, where the z-axis points upward. The constant ρ_0 is a reference density corresponding to a reference temperature T_0, which can be taken to be the mean temperature in the flow or the temperature at an appropriate boundary. Applications of the Boussinesq set can be found in several places in this book, for example, in the analysis of wave propagation in a density-stratified medium, thermal instability, turbulence in a stratified medium, and geophysical fluid dynamics.

4.10. BOUNDARY CONDITIONS

The differential equations for the conservation laws require boundary conditions for proper solution. Specifically, the Navier-Stokes momentum equation (4.38) requires the specification of the velocity vector on all surfaces bounding the flow domain. For an external flow, one that is not contained by walls or surfaces at specified locations, the fluid's velocity vector and the thermodynamic state must be specified on a closed distant surface.

On a solid boundary or at the interface between two immiscible fluids, conditions may be derived from the three basic conservation laws as follows. In Figure 4.18, a small cylindrical control volume is drawn through the interface separating medium 1 (fluid) from medium 2 (solid or liquid immiscible with fluid 1). Here $+\mathbf{n}dA$ and $-\mathbf{n}dA$ are the end face—directed area

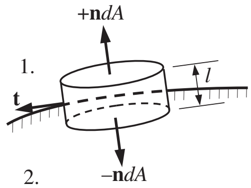


FIGURE 4.18 Interface between two media for evaluation of boundary conditions. Here medium 1 is a fluid, and medium 2 is a solid or a second fluid that is immiscible with the first fluid. Boundary conditions can be determined by evaluating the equations of motion in the small cylindrical control volume shown and then letting l go to zero with the volume straddling the interface.

elements in medium 1 and medium 2, respectively. The circular surfaces are locally tangent to the interface, and separated from each other by a distance l. Now apply the conservation laws to the volume defined by the cylindrical volume. Next, let $l \to 0$, keeping the two round area elements in the two different media. As $l \to 0$, all volume integrals $\to 0$ and the integral over the side area, which is proportional to l, tends to zero as well. The unit vector **n** is normal to the interface and points into medium 1. Mass conservation gives $\rho_1 \mathbf{u}_1 \cdot \mathbf{n} = \rho_2 \mathbf{u}_2 \cdot \mathbf{n}$ at each point on the interface as the end face area becomes small. (Here we assume that the coordinates are fixed to the interface, that is, the interface is at rest. Later in this section we show the modifications necessary when the interface is moving.)

If medium 2 is a solid, then $\mathbf{u}_2 = 0$ there. If medium 1 and medium 2 are immiscible liquids, no mass flows across the boundary surface. In either case, $\mathbf{u}_1 \cdot \mathbf{n} = 0$ on the boundary. The same procedure applied to the integral form of the momentum equation (4.17) gives the result that the force/area on the surface, $n_i \tau_{ij}$, is continuous across the interface if surface tension is neglected. If surface tension is included, a jump in pressure in the direction normal to the interface must be added; see Section 1.6 and the discussion later in this section.

Applying the integral form of energy conservation (4.48) to a small cylindrical control volume of infinitesimal height l that straddles the interface gives the result that $n_i q_i$ is continuous across the interface, or explicity, $k_1(\partial T_1/\partial n) = k_2(\partial T_2/\partial n)$ at the interface surface. The heat flux must be continuous at the interface; it cannot store heat.

Two more boundary conditions are required to completely specify a problem and these are not consequences of any conservation law. These boundary conditions are: no slip of a viscous fluid is permitted at a solid boundary $\mathbf{u}_1 \cdot \mathbf{t} = 0$; and no temperature jump is permitted at the boundary $T_1 = T_2$. Here **t** is a unit vector tangent to the boundary.

Known violations of the no-slip boundary condition occur for superfluid helium at or below 2.17 K, which has an immeasurably small (essentially zero) viscosity. On the other hand, the *appearance* of slip is created when water or water-based fluids flow over finely textured *super-hydrophobic* (strongly water repellent) coated surfaces. This is described by Gogte et al. (2005). Surface textures must be much smaller than the capillary length for water and were typically about 10μm in this case. The fluid did not slip on the protrusions but did not penetrate the valleys because of the surface tension, giving the appearance of slip. Both slip and temperature jump are known to occur in highly rarefied gases, where the mean distances between intermolecular collisions become of the order of the length scales of interest in the problem. The details are closely related to the manner of gas-surface interaction of momentum and energy. A review of these topics is provided by McCormick (2005).

Moving and Deforming Boundaries

Consider a surface in space that may be moving or deforming in some arbitrary way. Examples may be flexible solid boundaries, the interface between two immiscible liquids, or a moving shock wave, which is described in Chapter 15. The first two examples do not permit mass flow across the interface, whereas the third does. Such a surface can be defined and its motion described in inertial coordinates by $\eta(x, y, z, t) = 0$. We often must treat problems in which boundary conditions must be satisfied on such a moving, deforming interface. Let the velocity of a point that remains on the interface, where $\eta = 0$, be $\mathbf{u}_s$. An observer riding on that point sees:

$$d\eta/dt = \partial\eta/\partial t + (\mathbf{u}_s \cdot \nabla)\eta = 0 \quad \text{on} \quad \eta = 0. \tag{4.90}$$

A fluid particle has velocity $\mathbf{u}$. If no fluid flows across $\eta = 0$, then $\mathbf{u} \cdot \nabla\eta = \mathbf{u}_s \cdot \nabla\eta = -\partial\eta/\partial t$. Thus the condition that there be no mass flow across the surface becomes

$$\partial\eta/\partial t + (\mathbf{u} \cdot \nabla)\eta \equiv D\eta/Dt = 0 \quad \text{on} \quad \eta = 0. \tag{4.91}$$

If there is mass flow across the surface, it is proportional to the relative velocity between the fluid and the surface, $(u_{rel})_n = \mathbf{u} \cdot \mathbf{n} - \mathbf{u}_s \cdot \mathbf{n}$, where $\mathbf{n} = \nabla\eta/|\nabla\eta|$.

$$(u_{rel})_n = (\mathbf{u} \cdot \nabla\eta + \partial\eta/\partial t)/|\nabla\eta| = (1/|\nabla\eta|)D\eta/Dt. \tag{4.92}$$

Thus the mass flow rate across the surface (per unit surface area) is represented by

$$(\rho/|\nabla\eta|)D\eta/Dt \quad \text{on} \quad \eta = 0. \tag{4.93}$$

Again, if no mass flows across the surface, the requirement is $D\eta/Dt = 0$ on $\eta = 0$.

Surface Tension Revisited

As discussed in Section 1.6, attractive intermolecular forces dominate in a liquid, whereas in a gas repulsive forces are larger. However, as a liquid-gas phase boundary is approached from the liquid side, these attractive forces are not felt equally because there are many fewer liquid-phase molecules near the phase boundary. Thus there tends to be an unbalanced attraction to the interior of the liquid of the molecules on the phase boundary. This unbalanced attraction is called *surface tension* and its manifestation is a pressure increment across a curved interface. A somewhat more detailed description is provided in texts on physicochemical hydrodynamics. Two excellent sources are Probstein (1994, Chapter 10) and Levich (1962, Chapter VII).

Lamb, in *Hydrodynamics* (1945, 6th Edition, p. 456), writes, "Since the condition of stable equilibrium is that the free energy be a minimum, the surface tends to contract as much as is consistent with the other conditions of the problem." Thus we are led to introduce the Helmholtz free energy (per unit mass) f via

$$f = e - Ts, \tag{4.94}$$

where the notation is consistent with that used in Section 1.8. If the free energy is a minimum, then the system is in a state of stable equilibrium, and F is called the *thermodynamic potential* at constant volume (Fermi, 1956, *Thermodynamics*, p. 80). For a reversible, isothermal change, the work done on the system is the gain in total free energy F,

$$df = de - Tds - sdT, \tag{4.95}$$

where the last term is zero for an isothermal change. Then, from (1.18), $dF = -pdv$ = work done on the system. (These relations suggest that surface tension decreases with increasing temperature.)

For an interface of area $= A$, separating two fluids of densities ρ_1 and ρ_2, with volumes V_1 and V_2, respectively, and with a surface tension coefficient σ (corresponding to free energy per unit area), the total (Helmholtz) free energy F of the system can be written as

$$F = \rho_1 V_1 f_1 + \rho_2 V_2 f_2 + A\sigma. \tag{4.96}$$

If $\sigma > 0$, then the two media (fluids) are immiscible and A will reach a local minimum value at equilibrium. On the other hand, if $\sigma < 0$, corresponding to surface compression, then the two fluids mix freely since the minimum free energy will occur when A has expanded to the point that the spacing between its folds reaches molecular dimensions and the two-fluid system has uniform composition.

When $\sigma > 0$, it and the curvature of the fluid interface determine the pressure difference across the interface. Here, we shall assume that σ = const. Flows driven by surface tension gradients are called *Marangoni flows* and are not discussed here. Consider the situation depicted in Figure 4.19 where the pressure above a curved interface is higher than that below it by an increment Δp, and the shape of the fluid interface is given by $\eta(x, y, z) = z - h(x,y) = 0$. The origin of coordinates and the direction of the z-axis are chosen so that h, $\partial h/\partial x$, and $\partial h/\partial y$ are all zero at $\mathbf{x} = (0, 0, 0)$. Plus, the directions of the x- and y-axes are chosen so that the surface's principal radii of curvature, R_1 and R_2, are found in the x-z and y-z planes, respectively. Thus, the surface's shape is given by

$$\eta(x,y,z) = z - \left(x^2/2R_1\right) - \left(y^2/2R_2\right) = 0$$

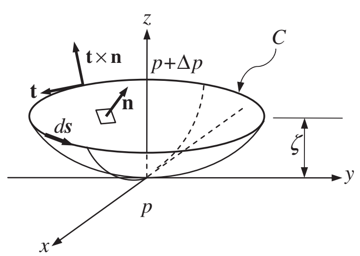


FIGURE 4.19 The curved surface shown is tangent to the x-y plane at the origin of coordinates. The pressure above the surface is Δp higher than the pressure below the surface, creating a downward force. Surface tension forces pull in the local direction of $\mathbf{t} \times \mathbf{n}$, which is slightly upward, all around the curve C and thereby balances the downward pressure force.

in the vicinity of the origin. A closed curve C is defined by the intersection of the curved surface and the plane $z = \zeta$. The goal here is to determine how the pressure increment Δp depends on R_1 and R_2 when pressure and surface tension forces are balanced as the area enclosed by C approaches zero.

First determine the net pressure force $\mathbf{F}_p$ on the surface A bounded by C. The unit normal $\mathbf{n}$ to the surface η is

$$\mathbf{n} = \frac{\nabla\eta}{|\nabla\eta|} = \frac{(-\,x/R_1, -y/R_2, 1)}{\sqrt{(x/R_1)^2 + (y/R_2)^2 + 1}},$$

and the area element is

$$dA = \sqrt{1 + (\partial\eta/\partial x)^2 + (\partial\eta/\partial y)^2}\, dxdy = \sqrt{1 + (x/R_1)^2 + (y/R_2)^2}\, dxdy,$$

so

$$\mathbf{F}_p = -\iint_A \Delta p \mathbf{n} dA = -\Delta p \int_{-\sqrt{2R_1\zeta}}^{+\sqrt{2R_1\zeta}} \left[\int_{-\sqrt{2R_2\zeta - x^2R_2/R_1}}^{+\sqrt{2R_2\zeta - x^2R_2/R_1}} (-x/R_1, -y/R_2, 1) dy \right] dx. \tag{4.97}$$

The minus sign appears here because greater pressure above the surface (positive Δp) must lead to a downward force and the vertical component of $\mathbf{n}$ is positive. The x- and y-components of $\mathbf{F}_p$ are zero because of the symmetry of the situation (odd integrand with even limits). The remaining double integration for the z-component of $\mathbf{F}_p$ produces:

$$\left(\mathbf{F}_p\right)_z = \mathbf{e}_z \cdot \mathbf{F}_p = -\pi \Delta p \sqrt{2R_1\zeta}\sqrt{2R_2\zeta}.$$

The net surface tension force $\mathbf{F}_{st}$ on C can be determined from the integral,

$$\mathbf{F}_{st} = \sigma \oint_C \mathbf{t} \times \mathbf{n}\, ds, \tag{4.98}$$

where $ds = dx\sqrt{1 + (dy/dx)^2}$ is an arc length element of curve C, and $\mathbf{t}$ is the unit tangent to C so

$$\mathbf{t} = -\frac{(1, dy/dx, 0)}{\sqrt{1 + (dy/dx)^2}} = \frac{(-\,y/R_2, x/R_1, 0)}{\sqrt{(y/R_2)^2 + (x/R_1)^2}},$$

and dy/dx is found by differentiating the equation for C, $\zeta = (x^2/2R_1) - (y^2/2R_2)$, with ζ regarded as constant. On each element of C, the surface tension force acts perpendicular to $\mathbf{t}$ and tangent to A. This direction is given by $\mathbf{t} \times \mathbf{n}$ so the integrand in (4.98) is

$$\mathbf{t} \times \mathbf{n} ds = \frac{(R_2/y)dx}{\sqrt{1 + (x/R_1)^2 + (y/R_2)^2}} \left(\frac{x}{R_1}, \frac{y}{R_2}, \frac{x^2}{R_1^2} + \frac{y^2}{R_2^2} \right) \cong \frac{R_2}{y} \left(\frac{x}{R_1}, \frac{y}{R_2}, \frac{x^2}{R_1^2} + \frac{y^2}{R_2^2} \right) dx,$$

where the approximate equality holds when x/R_1 and $y/R_2 \ll 1$ and the area enclosed by C approaches zero. The symmetry of the integration path will cause the x- and y-components of $\mathbf{F}_{st}$ to be zero, leaving

$$(\mathbf{F}_{st})_z = \mathbf{e}_z \cdot \mathbf{F}_{st} = 4\sigma \int_0^{\sqrt{2R_1\zeta}} \frac{R_2}{\sqrt{2R_2\zeta - (R_2/R_1)x^2}} \left[\frac{2\zeta}{R_2} + \frac{x^2}{R_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right] dx,$$

where y has been eliminated from the integrand using the equation for C, and the factor of four appears because the integral shown only covers one-quarter of the path defined by C. An integration variable substitution in the form $\sin\xi = x/\sqrt{2R_1\zeta}$ allows the integral to be evaluated:

$$(\mathbf{F}_{st})_z = \mathbf{e}_z \cdot \mathbf{F}_{st} = \pi\sigma\sqrt{2R_1\zeta}\sqrt{2R_2\zeta}\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

For static equilibrium, $\mathbf{F}_p + \mathbf{F}_{st} = 0$, so the evaluated results of (4.97) and (4.98) require:

$$\Delta p = \sigma(1/R_1 + 1/R_2), \tag{1.5}$$

where the pressure is greater on the side of the surface with the centers of curvature of the interface. Batchelor (1967, p. 64) writes,

> *An unbounded surface with a constant sum of the principal curvatures is spherical, and this must be the equilibrium shape of the surface. This result also follows from the fact that in a state of (stable) equilibrium the energy of the surface must be a minimum consistent with a given value of the volume of the drop or bubble, and the sphere is the shape which has the least surface area for a given volume.*

The original source of this analysis is Lord Rayleigh's (1890) "On the Theory of Surface Forces."

For an air bubble in water, gravity is an important factor for bubbles of millimeter size, as we shall see here. The hydrostatic pressure in a liquid is obtained from $p_L + \rho g z = \text{const.}$, where z is measured positively upwards from the free surface and gravity acts downwards. Thus for a gas bubble beneath the free surface,

$$p_G = p_L + \sigma\left(1/R_1 + 1/R_2\right) = \text{const.} - \rho g z + \sigma\left(1/R_1 + 1/R_2\right).$$

Gravity and surface tension forces are of the same order over a length scale $(\sigma/\rho g)^{1/2}$. For an air bubble in water at 288 K, this scale $= [7.35 \times 10^{-2}\ \text{N/m}/(9.81\ \text{m/s}^2 \times 10^3\ \text{kg/m}^3)]^{1/2} = 2.74 \times 10^{-3}$ m.

EXAMPLE 4.7

Calculation of the shape of the free surface of a liquid adjoining an infinite vertical plane wall. Here let $z = \zeta(x)$ define the free surface shape. With reference to Figure 4.20 where the y-axis points into the page, $1/R_1 = [\partial^2\zeta/\partial x^2][1 + (\partial\zeta/\partial x)^2]^{-3/2}$, and $1/R_2 = [\partial^2\zeta/\partial y^2][1 + (\partial\zeta/\partial y)^2]^{-3/2} = 0$. At the free surface, $\rho g\zeta - \sigma/R_1 = const.$ As $x \to \infty$, $\zeta \to 0$, and $R_2 \to \infty$, so $const. = 0$. Then $\rho g\zeta/\sigma - \zeta''/(1 + \zeta'^2)^{3/2} = 0$.

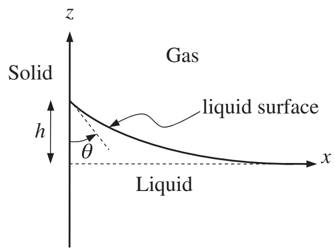


FIGURE 4.20 Free surface of a liquid adjoining a vertical plane wall. Here the contact angle is θ and the liquid rises to $z = h$ at the solid wall.

Multiply by the integrating factor ζ' and integrate. We obtain $(\rho g/2\sigma)\zeta^2 + (1 + \zeta'^2)^{-1/2} = C$. Evaluate C as $x \to \infty$, $\zeta \to 0$, $\zeta' \to 0$. Then $C = 1$. We look at $x = 0$, $z = \zeta(0) = h$ to find h. The slope at the wall, $\zeta' = \tan(\theta + \pi/2) = -\cot\theta$. Then $1 + \zeta'^2 = 1 + \cot^2\theta = \csc^2\theta$. Thus we now have $(\rho g/2\sigma)h^2 = 1 - 1/\csc\theta = 1 - \sin\theta$, so that $h^2 = (2\sigma/\rho g)(1 - \sin\theta)$. Finally we seek to integrate to obtain the shape of the interface. Squaring and rearranging the result above, the differential equation we must solve may be written as $1 + (d\zeta/dx)^2 = [1 - (\rho g/2\sigma)\zeta^2]^{-2}$. Solving for the slope and taking the negative square root (since the slope is negative for positive x),

$$d\zeta/dx = -\left\{1 - \left[1 - \left(\rho g/2\sigma\right)\zeta^2\right]^2\right\}^{1/2}\left[1 - \left(\rho g/2\sigma\right)\zeta^2\right]^{-1}.$$

Define $\sigma/\rho g = \delta^2$, $\zeta/\delta = \gamma$. Rewriting the equation in terms of x/δ and γ, and separating variables:

$$2(1 - \gamma^2/2)\gamma^{-1}\left(4 - \gamma^2\right)^{-1/2} d\gamma = d\left(x/\delta\right).$$

The integrand on the left is simplified by partial fractions and the constant of integration is evaluated at $x = 0$ when $\eta = h/\delta$. Finally,

$$\cosh^{-1}(2\delta/\zeta) - \left(4 - \zeta^2/\delta^2\right)^{1/2} - \cosh^{-1}(2\delta/h) + (4 - h^2/\delta^2)^{1/2} = x/\delta$$

gives the shape of the interface in terms of $x(\zeta)$.

Analysis of surface tension effects results in the appearance of additional dimensionless parameters in which surface tension is compared with other effects such as viscous stresses, body forces such as gravity, and inertia. These are defined in the next section.

4.11. DIMENSIONLESS FORMS OF THE EQUATIONS AND DYNAMIC SIMILARITY

For a properly specified fluid flow problem or situation, the differential equations of fluid motion, the fluid's constitutive and thermodynamic properties, and the boundary conditions may be used to determine the dimensionless parameters that govern the situation of interest even before a solution of the equations is attempted. The dimensionless parameters so determined set the importance of the various terms in the governing differential equations, and thereby indicate which phenomena will be important in the resulting flow. This section describes and presents the primary dimensionless parameters or numbers required in the remainder of the text. Many others not mentioned here are defined and used in the broad realm of fluid mechanics.

The dimensionless parameters for any particular problem can be determined in two ways. They can be deduced directly from the governing differential equations if these equations are known; this method is illustrated here. However, if the governing differential equations are unknown or the parameter of interest does not appear in the known equations, dimensionless parameters can be determined from dimensional analysis (see Section 1.11). The advantage of adopting the former strategy is that dimensionless parameters determined from the equations of motion are more readily interpreted and linked to the physical phenomena occurring in the flow. Thus, knowledge of the relevant dimensionless parameters frequently aids the solution process, especially when assumptions and approximations are necessary to reach a solution.

In addition, the dimensionless parameters obtained from the equations of fluid motion set the conditions under which scale model testing with small models will prove useful for predicting the performance of larger devices. In particular, two flow fields are considered to be dynamically similar when their dimensionless parameters match, and their geometries are scale similar; that is, any length scale in the first flow field may be mapped to its counterpart in the second flow field by multiplication with a single scale ratio. When two flows are dynamically similar, analysis, simulations, or measurements from one flow field are directly applicable to the other when the scale ratio is accounted for. Moreover, use of standard dimensionless parameters typically reduces the parameters that must be varied in an experiment or calculation, and greatly facilitates the comparison of measured or computed results with prior work conducted under potentially different conditions.

To illustrate these advantages, consider the drag force F_D on a sphere, of diameter d moving at a speed U through a fluid of density ρ and viscosity μ. Dimensional analysis (Section 1.11) using these five parameters produces the following possible dimensionless scaling laws:

$$\frac{F_D}{\rho U^2 d^2} = \Psi\left(\frac{\rho U d}{\mu}\right), \quad \text{or} \quad \frac{F_D \rho}{\mu^2} = \Phi\left(\frac{\mu}{\rho U d}\right). \tag{4.99}$$

Both are valid, but the first is preferred because it contains dimensionless groups that either come from the equations of motion or are traditionally defined in the study of fluid dynamic drag. If dimensionless groups were not used, experiments would have to be conducted to determine F_D as a function of d, keeping U, ρ, and μ fixed. Then, experiments would have to be conducted to determine F_D as a function of U, keeping d, ρ, and μ fixed, and so on. However, such a duplication of effort is unnecessary when dimensionless groups are used. In fact, use of the first dimensionless law above allows experimental results from a wide range of conditions to be simply plotted with two axes (see Figure 4.21) even though the full complement of experiments may have involved variations in all five dimensional parameters.

The idea behind dimensional analysis is intimately associated with the concept of similarity. In fact, a collapse of all the data on a single graph such as the one in Figure 4.21 is

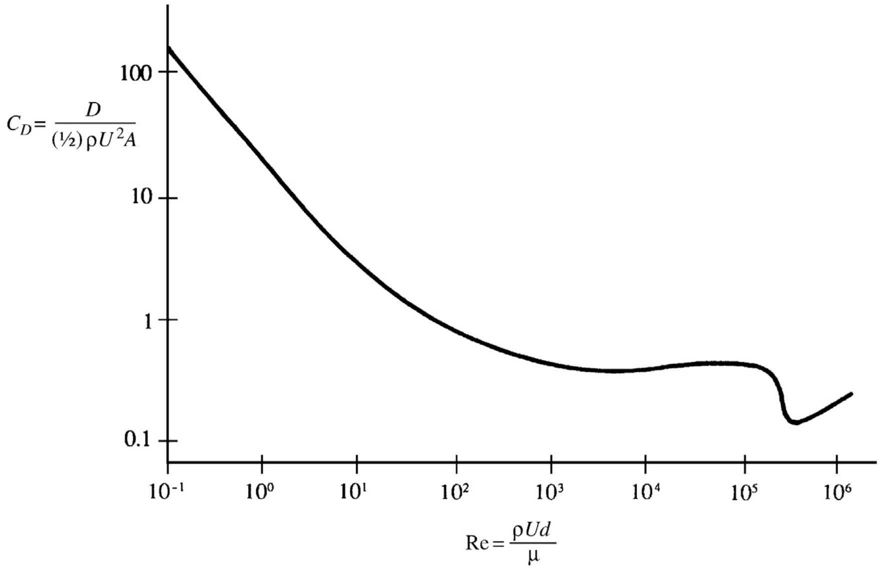


FIGURE 4.21 Coefficient of drag C_D for a sphere vs. the Reynolds number Re based on sphere diameter. At low Reynolds number $C_D \sim 1/\mathrm{Re}$, and above Re $\sim 10^3$, $C_D \sim$ constant (except for the dip between $\mathrm{Re} = 10^5$ and 10^6). These behaviors (except for the dip) can be explained by simple dimensional reasoning. The reason for the dip is the transition of the laminar boundary layer to a turbulent one, as explained in Chapter 9.

possible only because in this problem all flows having the same value of the dimensionless group known as the *Reynolds number* $\mathrm{Re} = \rho U d/\mu$ are dynamically similar. This dynamic similarity is assured because the Reynolds number appears when the equations of motion are cast in dimensionless form.

The use of dimensionless parameters pervades fluid mechanics to such a degree that this chapter and this text would be considered incomplete without this section, even though this topic is typically covered in first-course fluid mechanics texts. For clarity, the discussion begins with the momentum equation, and then proceeds to the continuity and energy equations.

Consider the flow of a fluid having nominal density ρ and viscosity μ through a flow field characterized by a length scale l, a velocity scale U, and a rotation or oscillation frequency Ω. The situation here is intended to be general so that the dimensional parameters obtained from this effort will be broadly applicable. Particular situations that would involve all five parameters include pulsating flow through a tube, flow past an undulating self-propelled body, or flow through a turbomachine.

The starting point is the Navier-Stokes momentum equation (4.39) simplified for incompressible flow. (The effect of compressibility is deduced from the continuity equation in the next subsection.)

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{u} \tag{4.39b}$$

This equation can be rendered dimensionless by defining dimensionless variables:

$$x_i^* = x_i/l, \quad t^* = \Omega t, \quad u_j^* = u_j/U, \quad p^* = (p - p_\infty)/\rho U^2, \quad \text{and} \quad g_j^* = g_j/g, \tag{4.100}$$

where g is the acceleration of gravity. It is clear that the boundary conditions in terms of the dimensionless variables (4.100) are independent of l, U, and Ω. For example, consider the viscous flow over a circular cylinder of radius R. When the velocity scale U is the free-stream velocity and the length scale is the radius R, then, in terms of the dimensionless velocity $u^* = u/U$ and the dimensionless coordinate $r^* = r/R$, the boundary condition at infinity is $u^* \to 1$ as $r^* \to \infty$, and the condition at the surface of the cylinder is $u^* = 0$ at $r^* = 1$. In addition, because pressure enters (4.39b) only as a gradient, the pressure itself is not of consequence; only pressure differences are important. The conventional practice is to render $p - p_\infty$ dimensionless, where p_∞ is a suitably chosen reference pressure. Depending on the nature of the flow, $p - p_\infty$ could be made dimensionless in terms of a generic viscous stress $\mu U/l$, a hydrostatic pressure ρgl, or, as in (4.100), a dynamic pressure ρU^2. Substitution of (4.100) into (4.39) produces:

$$\left[\frac{\Omega l}{U}\right]\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^* = -\nabla^* p^* + \left[\frac{gl}{U^2}\right]\mathbf{g}^* + \left[\frac{\mu}{\rho Ul}\right]\nabla^{*2}\mathbf{u}^*, \tag{4.101}$$

where $\nabla^* = l\nabla$. It is apparent that two flows (having different values of Ω, U, l, g, or μ) will obey the same dimensionless differential equation if the values of the dimensionless groups $\Omega l/U$, gl/U^2, and $\mu/\rho Ul$ are identical. Because the dimensionless boundary conditions are also identical in the two flows, it follows that *they will have the same dimensionless solutions*. Products of these dimensionless groups appear as coefficients in front of different terms when the pressure is presumed to have alternative scalings (see Exercise 4.59).

The parameter groupings shown in [,]-brackets in (4.100) have the following names and interpretations:

$$\mathrm{St} = \text{Strouhal number} \equiv \frac{\text{unsteady acceleration}}{\text{advective acceleration}} \propto \frac{\partial u/\partial t}{u(\partial u/\partial x)} \propto \frac{\Omega U}{U^2/l} = \frac{\Omega l}{U}, \tag{4.102}$$

$$\mathrm{Re} = \text{Reynolds number} \equiv \frac{\text{inertia force}}{\text{viscous force}} \propto \frac{\rho u(\partial u/\partial x)}{\mu(\partial^2 u/\partial x^2)} \propto \frac{\rho U^2/l}{\mu U/l^2} = \frac{\rho Ul}{\mu}, \text{ and} \tag{4.103}$$

$$\mathrm{Fr} = \text{Froude number} \equiv \left[\frac{\text{inertia force}}{\text{gravity force}}\right]^{1/2} \propto \left[\frac{\rho u(\partial u/\partial x)}{\rho g}\right]^{1/2} \propto \left[\frac{\rho U^2/l}{\rho g}\right]^{1/2} = \frac{U}{\sqrt{gl}}. \tag{4.104}$$

The Strouhal number sets the importance of unsteady fluid acceleration in flows with oscillations. It is relevant when flow unsteadiness arises naturally or because of an imposed frequency. The Reynolds number is the most common dimensionless number in fluid mechanics. Low Re flows involve small sizes, low speeds, and high kinematic viscosity such as bacteria swimming through mucous. High Re flows involve large sizes, high speeds, and low kinematic viscosity such as an ocean liner steaming at full speed.

St, Re, and Fr have to be equal for dynamic similarity of two flows in which unsteadiness and viscous and gravitational effects are important. Note that the mere presence of gravity does not make the gravitational effects dynamically important. For flow around an object in a homogeneous fluid, gravity is important only if surface waves are generated. Otherwise, the effect of gravity is simply to add a hydrostatic pressure to the entire system that can be combined with the pressure gradient (see "Neglect of Gravity in Constant Density Flows" earlier in this chapter).

Interestingly, in a density-stratified fluid, gravity can play a significant role without the presence of a free surface. The effective gravity force in a two-layer situation is the buoyancy force per unit volume $(\rho_2 - \rho_1)g$, where ρ_1 and ρ_2 are fluid densities in the two layers. In such a case, an internal Froude number is defined as:

$$\mathrm{Fr}' \equiv \left[\frac{\text{inertia force}}{\text{buoyancy force}}\right]^{1/2} \propto \left[\frac{\rho_1 U^2/l}{(\rho_2 - \rho_1)g}\right]^{1/2} = \frac{U}{\sqrt{g'l}}, \tag{4.105}$$

where $g' \equiv g\,(\rho_2 - \rho_1)/\rho_1$ is the *reduced gravity*. For a continuously stratified fluid having a maximum buoyancy frequency N (see 1.29), the equivalent of (4.104) is $\mathrm{Fr}' \equiv U/Nl$. Alternatively, the internal Froude number may be replaced by the Richardson Number $= \mathrm{Ri} \equiv 1/\mathrm{Fr}'^2 = g'l/U^2$, which can also be refined to a gradient Richardson number $\equiv N^2(z)/(dU/dz)^2$ that is important in studies of instability and turbulence in stratified fluids.

Under dynamic similarity, all the dimensionless numbers match and there is one dimensionless solution. The dimensional consistency of the equations of motion ensures that all flow quantities may be set in dimensionless form. For example, the local pressure at point $\mathbf{x} = (x, y, z)$ can be made dimensionless in the form

$$\frac{p(\mathbf{x}, t) - p_\infty}{(1/2)\rho U^2} \equiv C_p = \Psi\left(\mathrm{St}, \mathrm{Fr}, \mathrm{Re}; \frac{\mathbf{x}}{l}, \Omega t\right), \tag{4.106}$$

where $C_p = (p - p_\infty)/(1/2)\rho U^2$ is called the *pressure coefficient* (or the Euler number), and Ψ represents the dimensionless solution for the pressure coefficient in terms of dimensionless parameters and variables. The factor of ½ in (4.106) is conventional but not necessary. Similar relations also hold for any other dimensionless flow variables such as velocity $\mathbf{u}/U$. It follows that in dynamically similar flows, dimensionless flow variables are identical at *corresponding points and times* (that is, for identical values of $\mathbf{x}/l$, and Ωt). Of course there are many instances where the flow geometry may require two or more length scales: $l, l_1, l_2, \ldots l_n$. When this is the case, the aspect ratios $l_1/l, l_2/l, \ldots l_n/l$ provide a dimensionless description of the geometry, and would also appear as arguments of the function Ψ in a relationship like (4.106). Here a difference between relations (4.99) and (4.106) should be noted. Equation (4.99) is a relation between *overall* flow parameters, whereas (4.106) holds *locally* at a point.

In the foregoing analysis we have assumed that the imposed unsteadiness in boundary conditions is important. However, time may also be made dimensionless via $t^* = Ut/l$, as would be appropriate for a flow with steady boundary conditions. In this case, the time derivative in (4.39) should still be retained because the resulting flow may still be naturally unsteady since flow oscillations can arise spontaneously even if the boundary conditions are steady. But, we know from dimensional considerations, such unsteadiness must have a time scale proportional to l/U.

In the foregoing analysis we have also assumed that an imposed velocity U is relevant. Consider now a situation in which the imposed boundary conditions are purely unsteady.

To be specific, consider an object having a characteristic length scale l oscillating with a frequency Ω in a fluid at rest at infinity. This is a problem having an imposed length scale and an *imposed time scale* $1/\Omega$. In such a case a velocity scale $U = l\Omega$ can be constructed. The preceding analysis then goes through, leading to the conclusion that $\mathrm{St} = 1$, $\mathrm{Re} = Ul/\nu = \Omega l^2/\nu$, and $\mathrm{Fr} = U/(gl)^{1/2} = \Omega(l/g)^{1/2}$ have to be duplicated for dynamic similarity.

All dimensionless quantities are identical in dynamically similar flows. For flow around an immersed body, like a sphere, we can define the (dimensionless) drag and lift coefficients,

$$C_D \equiv \frac{F_D}{(1/2)\rho U^2 A} \quad \text{and} \quad C_L \equiv \frac{F_L}{(1/2)\rho U^2 A}, \tag{4.107, 4.108}$$

where A is a reference area, and F_D and F_L are the drag and lift forces, respectively, experienced by the body; as in (4.106) the factor of 1/2 in (4.107) and (4.108) is conventional but not necessary. For blunt bodies such as spheres and cylinders, A is taken to be the maximum cross section perpendicular to the flow. Therefore, $A = \pi d^2/4$ for a sphere of diameter d, and $A = bd$ for a cylinder of diameter d and length b, with its axis perpendicular to the flow. For flows over flat plates, and airfoils, on the other hand, A is taken to be the *planform area*, that is, $A = sl$; here, l is the average length of the plate or chord of the airfoil in the direction of flow and s is the width perpendicular to the flow, sometimes called the *span*.

The values of the drag and lift coefficients are identical for dynamically similar flows. For flow about a steadily moving ship, the drag is caused both by gravitational and viscous effects so we must have a functional relation of the form $C_D = C_D(\mathrm{Fr}, \mathrm{Re})$. However, in many flows gravitational effects are unimportant. An example is flow around a body that is far from a free surface and does not generate gravity waves. In this case, Fr is irrelevant, so $C_D = C_D(\mathrm{Re})$ is all that is needed when the effects of compressibility are unimportant. This is the situation portrayed by the first member of (4.99) and illustrated in Figure 4.21.

A dimensionless form of the continuity equation should indicate when flow-induced pressure differences induce significant departures from incompressible flow. However, the simplest possible scaling fails to provide any insights because the continuity equation itself does not contain the pressure. Thus, a more fruitful starting point for determining the relative size of $\nabla\cdot\mathbf{u}$ is (4.9),

$$\nabla\cdot\mathbf{u} = -\frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{1}{\rho c^2}\frac{Dp}{Dt}, \tag{4.9}$$

along with the assumption that pressure-induced density changes will be isentropic, $dp = c^2 d\rho$ where c is the sound speed (1.19). Using the following dimensionless variables,

$$x_i^* = x_i/l, \quad t^* = Ut/l, \quad u_j^* = u_j/U, \quad p^* = \left(p - p_\infty\right)/\rho_o U^2, \quad \text{and} \quad \rho^* = \rho/\rho_o, \tag{4.109}$$

where ρ_o is a reference density, the outside members of (4.9) can be rewritten:

$$\nabla^*\cdot\mathbf{u}^* = -\left[\frac{U^2}{c^2}\right]\frac{1}{\rho^*}\frac{Dp^*}{Dt^*}, \tag{4.110}$$

which specifically shows that the square of

$$M = \text{Mach number} \equiv \left[\frac{\text{inertia force}}{\text{compressibility force}}\right]^{1/2} \propto \left[\frac{\rho U^2/l}{\rho c^2/l}\right]^{1/2} = \frac{U}{c} \tag{4.111}$$

sets the size of isentropic departures from incompressible flow. In engineering practice, gas flows are considered incompressible when $M < 0.3$, and from (4.110) this corresponds to ~10% departure from ideal incompressible behavior when $(1/\rho^*)(Dp^*/Dt^*)$ is unity. Of course, there may be nonisentropic changes in density too and these are considered in Thompson (1972, pp. 137–146). Flows in which $M < 1$ are called *subsonic*, whereas flows in which $M > 1$ are called *supersonic*. At high subsonic and supersonic speeds, matching Mach number between flows is required for dynamic similarity.

There are many possible thermal boundary conditions for the energy equation, so a fully general scaling of (4.60) is not possible. Instead, a simple scaling is provided here based on constant specific heats, neglect of μ_v, and constant free-stream and wall temperatures, T_o and T_w, respectively. In addition, for simplicity, an imposed flow oscillation frequency is not considered. The starting point of the scaling provided here is a mild revision of (4.60) that involves the enthalpy h per unit mass,

$$\rho\frac{Dh}{Dt} = \frac{Dp}{Dt} + \rho\varepsilon + \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right), \tag{4.112}$$

where ε is given by (4.58). Using $dh \cong C_p dT$, the following dimensionless variables:

$$\varepsilon^* = \rho_o l^2 \varepsilon/\mu_o U^2, \quad \mu^* = \mu/\mu_o, \quad k^* = k/k_o, \quad T^* = (T - T_o)/(T_w - T_o), \tag{4.113}$$

and those defined in (4.106), (4.107) becomes:

$$\rho^*\frac{DT^*}{Dt^*} = \left[\frac{U^2}{C_p(T_w - T_o)}\right]\frac{Dp^*}{Dt^*} + \left[\frac{U^2}{C_p(T_w - T_o)}\frac{\mu_o}{\rho_o Ul}\right]\rho^*\varepsilon^* + \left[\frac{k_o}{C_p\mu_o}\frac{\mu_o}{\rho_o Ul}\right]\nabla^*(k^*\nabla^* T^*). \tag{4.114}$$

Here the relevant dimensionless parameters are:

$$\text{Ec} = \text{Eckert number} \equiv \frac{\text{kinetic energy}}{\text{thermal energy}} = \frac{U^2}{C_p(T_w - T_o)}, \tag{4.115}$$

$$\text{Pr} = \text{Prandtl number} \equiv \frac{\text{momentum diffusivity}}{\text{thermal diffusivity}} = \frac{\nu}{\kappa} = \frac{\mu_o/\rho_o}{k_o/\rho_o C_p} = \frac{\mu_o C_p}{k_o}, \tag{4.116}$$

and we recognize $\rho_o Ul/\mu_o$ as the Reynolds number in (4.114) as well. In low-speed flows, where the Eckert number is small the middle terms drop out of (4.114), and the full energy equation (4.112) reduces to (4.89). Thus, low Ec is needed for the Boussinesq approximation.

The Prandtl number is a ratio of two molecular transport properties. It is therefore a fluid property and independent of flow geometry. For air at ordinary temperatures and pressures, Pr = 0.72, which is close to the value of 0.67 predicted from a simplified kinetic theory model assuming hard spheres and monatomic molecules (Hirschfelder, Curtiss, & Bird, 1954, pp. 9–16). For water at 20 °C, Pr = 7.1. Dynamic similarity between flows involving thermal effects requires equality of the Eckert, Prandtl, and Reynolds numbers.

And finally, for flows involving surface tension σ, there are several relevant dimensionless numbers:

$$\mathrm{We} = \text{Weber number} \equiv \frac{\text{inertia force}}{\text{surface tension force}} \propto \frac{\rho U^2 l^2}{\sigma l} = \frac{\rho U^2 l}{\sigma}, \tag{4.117}$$

$$\mathrm{Bo} = \text{Bond number} \equiv \frac{\text{gravity force}}{\text{surface tension force}} \propto \frac{\rho l^3 g}{\sigma l} = \frac{\rho l^2 g}{\sigma}, \tag{4.118}$$

$$\mathrm{Ca} = \text{Capillary number} \equiv \frac{\text{viscous stress}}{\text{surface tension stress}} \propto \frac{\mu U/l}{\sigma/l} = \frac{\mu U}{\sigma}. \tag{4.119}$$

Here, for the Weber and Bond numbers, the ratio is constructed based on a ratio of forces as in (4.107) and (4.108), and not forces per unit volume as in (4.103), (4.104), and (4.111). At high Weber number, droplets and bubbles are easily deformed by fluid acceleration or deceleration, for example during impact with a solid surface. At high Bond numbers surface tension effects are relatively unimportant compared to gravity, as is the case for long-wavelength, ocean surface waves. At high capillary numbers viscous forces dominate those from surface tension; this is commonly the case in machinery lubrication flows. However, for slow bubbly flow through porous media or narrow tubes (low Ca) the opposite is true.

EXAMPLE 4.8

A ship 100 m long is expected to cruise at 10 m/s. It has a submerged surface of 300 m^2. Find the model speed for a 1/25 scale model, neglecting frictional effects. The drag is measured to be 60 N when the model is tested in a towing tank at the model speed. Estimate the full scale drag when the skin-friction drag coefficient for the model is 0.003 and that for the full-scale prototype is half of that.

Solution

Estimate the model speed neglecting frictional effects. Then the nondimensional drag force depends only on the Froude number:

$$D/\rho U^2 l^2 = f\left(U/\sqrt{gl}\right).$$

Equating Froude numbers for the model (denoted by subscript "m") and full-size prototype (denoted by subscript "p"), we get

$$U_\mathrm{m} = U_\mathrm{p}\sqrt{g_\mathrm{m} l_\mathrm{m}/g_\mathrm{p} l_\mathrm{p}} = 10\sqrt{1/25} = 2 \text{ m/s}.$$

The total drag on the model was measured to be 60 N at this model speed. Of the total measured drag, a part was due to frictional effects. The frictional drag can be estimated by treating the surface of the hull as a flat plate, for which the drag coefficient C_D is a function of the Reynolds number. Using a value of $v = 10^{-6}$ m^2/s for water, we get

$$Ul/\nu \ (\text{model}) = [2(100/25)]/10^{-6} = 8 \times 10^6,$$
$$Ul/\nu \ (\text{prototype}) = 10\ (100)/10^{-6} = 10^9.$$

The problem statement sets the frictional drag coefficients as

$$C_D(\text{model}) = 0.003,$$
$$C_D(\text{prototype}) = 0.0015.$$

and these are consistent with Figure 9.11. Using a value of $\rho = 1000\ \text{kg/m}^3$ for water, we estimate:

$$\text{Frictional drag on model} = \frac{1}{2}C_D\rho U^2 A = (0.5)\ (0.003)\ (1000)\ (2)^2\ (300/25^2) = 2.88\ \text{N}.$$

Out of the total model drag of 60 N, the wave drag is therefore $60 - 2.88 = 57.12$ N.

Now the *wave drag* still obeys the scaling law above, which means that $D/\rho U^2 l^2$ for the two flows are identical, where D represents wave drag alone. Therefore,

$$\text{Wave drag on prototype} = (\text{Wave drag on model})\ (\rho_p/\rho_m)\ (l_p/l_m)^2\ (U_p/U_m)^2$$
$$= 57.12\ (1)\ (25)^2\ (10/2)^2 = 8.92 \times 10^5\ \text{N}.$$

Having estimated the wave drag on the prototype, we proceed to determine its frictional drag.

$$\text{Frictional drag on prototype} = \frac{1}{2}C_D\rho U^2 A = (0.5)\ (0.0015)\ (1000)\ (10)^2\ (300) = 0.225 \times 10^5\ \text{N}.$$

Therefore, total drag on prototype $= (8.92 + 0.225) \times 10^5 = 9.14 \times 10^5$ N.

If we did not correct for the frictional effects, and assumed that the measured model drag was all due to wave effects, then we would have found a prototype drag of

$$D_p = D_m(\rho_p/\rho_m)\ (l_p/l_m)^2\ (U_p/U_m)^2 = 60\ (1)\ (25)^2\ (10/2)^2 = 9.37 \times 10^5\ \text{N}.$$

EXERCISES

4.1. Let a one-dimensional velocity field be $u = u(x, t)$, with $v = 0$ and $w = 0$. The density varies as $\rho = \rho_0(2 - \cos \omega t)$. Find an expression for $u(x, t)$ if $u(0, t) = U$.

4.2. Consider the one-dimensional Cartesian velocity field: $\mathbf{u} = (\alpha x/t, 0, 0)$ where α is a constant. Find a spatially uniform, time-dependent density field, $\rho = \rho(t)$, that renders this flow field mass conserving when $\rho = \rho_o$ at $t = t_o$.

4.3. Find a nonzero density field $\rho(x,y,z,t)$ that renders the following Cartesian velocity fields mass conserving. Comment on the physical significance and uniqueness of your solutions.

a) $\mathbf{u} = (U \sin(\Omega t - kx), 0, 0)$ where U, ω, k are positive constants
[*Hint*: exchange the independent variables x,t for a single independent variable $\xi = \omega t - kx$]

b) $\mathbf{u} = (-\Omega y, +\Omega x, 0)$ with $\Omega = \text{constant}$ [*Hint*: switch to cylindrical coordinates]
c) $\mathbf{u} = (A/x, B/y, C/z)$ where A, B, C are constants

4.4. A proposed conservation law for ξ, a new fluid property, takes the following form: $\frac{d}{dt}\int_{V(t)} \rho\xi dV + \int_{A(t)} \boldsymbol{\Theta}\cdot\mathbf{n}dS = 0$, where $V(t)$ is a material volume that moves with the fluid velocity $\mathbf{u}$, $A(t)$ is the surface of $V(t)$, ρ is the fluid density, and $\boldsymbol{\Theta} = -\rho\gamma\nabla\xi$.
a) What partial differential equation is implied by the above conservation statement?
b) Use the part a) result and the continuity equation to show: $\frac{\partial\xi}{\partial t} + \mathbf{u}\cdot\nabla\xi = \frac{1}{\rho}\nabla\cdot(\rho\gamma\nabla\xi)$.

4.5. The components of a mass flow vector $\rho\mathbf{u}$ are $\rho u = 4x^2y$, $\rho v = xyz$, $\rho w = yz^2$.
a) Compute the net mass outflow through the closed surface formed by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.
b) Compute $\nabla\cdot(\rho\mathbf{u})$ and integrate over the volume bounded by the surface defined in part a).
c) Explain why the results for parts a) and b) should be equal or unequal.

4.6. Consider a simple fluid mechanical model for the atmosphere of an ideal spherical star that has a surface gas density of ρ_o and a radius r_o. The escape velocity from the surface of the star is v_e. Assume that a tenuous gas leaves the star's surface radially at speed v_o uniformly over the star's surface. Use the steady continuity equation for the gas density ρ and fluid velocity $\mathbf{u} = (u_r, u_\theta, u_\varphi)$ in spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho u_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(\rho u_\varphi) = 0$$

to determine the following.
a) Determine ρ when $v_o \geq v_e$ so that $\mathbf{u} = (u_r, u_\theta, u_\varphi) = \left(v_o\sqrt{1-(v_e^2/v_o^2)(1-(r_o/r))}, 0, 0\right)$.
b) Simplify the result from part a) when $v_o \gg v_e$ so that: $\mathbf{u} = (u_r, u_\theta, u_\varphi) = (v_o, 0, 0)$.
c) Simplify the result from part a) when $v_o = v_e$.
d) Use words, sketches, or equations to describe what happens when $v_o < v_e$. State any assumptions that you make.

4.7. The definition of the stream function for two-dimensional, constant-density flow in the x-y plane is: $\mathbf{u} = -\mathbf{e}_z \times \nabla\psi$, where $\mathbf{e}_z$ is the unit vector perpendicular to the x-y plane that determines a right-handed coordinate system.
a) Verify that this vector definition is equivalent to $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$ in Cartesian coordinates.
b) Determine the velocity components in r-θ polar coordinates in terms of r-θ derivatives of ψ.
c) Determine an equation for the z-component of the vorticity in terms of ψ.

4.8. A curve of $\psi(x,y) = C_1$ (= a constant) specifies a streamline in steady two-dimensional, constant-density flow. If a neighboring streamline is specified by $\psi(x,y) = C_2$, show that the volume flux per unit depth into the page between the streamlines equals $C_2 - C_1$ when $C_2 > C_1$.

4.9. The well-known undergraduate fluid mechanics textbook by Fox et al. (2009) provides the following statement of conservation of momentum for a constant-shape (nonrotating) control volume moving at a non-constant velocity $\mathbf{U} = \mathbf{U}(t)$:

$$\frac{d}{dt}\int\limits_{V^*(t)} \rho\mathbf{u}_{rel}dV + \int\limits_{A^*(t)} \rho\mathbf{u}_{rel}(\mathbf{u}_{rel}\cdot\mathbf{n})dA = \int\limits_{V^*(t)} \rho\mathbf{g}dV + \int\limits_{A^*(t)} \mathbf{f}dA - \int\limits_{V^*(t)} \rho\frac{d\mathbf{U}}{dt}dV.$$

Here $\mathbf{u}_{rel} = \mathbf{u} - \mathbf{U}(t)$ is the fluid velocity observed in a frame of reference moving with the control volume while $\mathbf{u}$ and $\mathbf{U}$ are observed in a nonmoving frame. Meanwhile, equation (4.17) states this law as

$$\frac{d}{dt}\int\limits_{V^*(t)} \rho\mathbf{u}dV + \int\limits_{A^*(t)} \rho\mathbf{u}(\mathbf{u}-\mathbf{U})\cdot\mathbf{n}dA = \int\limits_{V^*(t)} \rho\mathbf{g}dV + \int\limits_{A^*(t)} \mathbf{f}dA$$

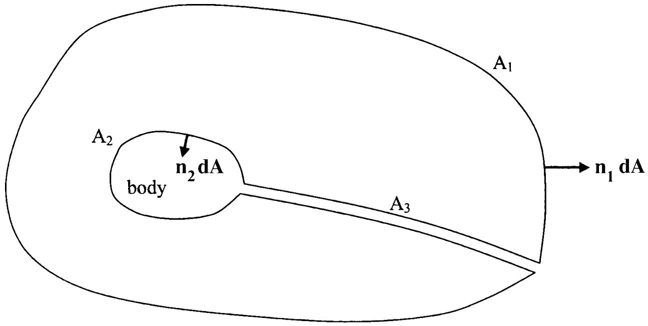
where the replacement $\mathbf{b} = \mathbf{U}$ has been made for the velocity of the accelerating control surface $A^*(t)$. Given that the two equations above are not identical, determine if these two statements of conservation of fluid momentum are contradictory or consistent.

4.10. A jet of water with a diameter of 8 cm and a speed of 25 m/s impinges normally on a large stationary flat plate. Find the force required to hold the plate stationary. Compare the average pressure on the plate with the stagnation pressure if the plate is 20 times the area of the jet.

4.11. Show that the thrust developed by a stationary rocket motor is $F = \rho AU^2 + A(p - p_{\text{atm}})$, where p_{atm} is the atmospheric pressure, and p, ρ, A, and U are, respectively, the pressure, density, area, and velocity of the fluid at the nozzle exit.

4.12. Consider the propeller of an airplane moving with a velocity U_1. Take a reference frame in which the air is moving and the propeller [disk] is stationary. Then the effect of the propeller is to accelerate the fluid from the upstream value U_1 to the downstream value $U_2 > U_1$. Assuming incompressibility, show that the thrust developed by the propeller is given by $F = \rho A(U_2^2 - U_1^2)/2$, where A is the projected area of the propeller and ρ is the density (assumed constant). Show also that the velocity of the fluid at the plane of the propeller is the average value $U = (U_1 + U_2)/2$. [*Hint*: The flow can be idealized by a pressure jump of magnitude $\Delta p = F/A$ right at the location of the propeller. Also apply Bernoulli's equation between a section far upstream and a section immediately upstream of the propeller. Also apply the Bernoulli equation between a section immediately downstream of the propeller and a section far downstream. This will show that $\Delta p = \rho(U_2^2 - U_1^2)/2$.]

4.13. Generalize the control volume analysis of Example 4.1 by considering the control volume geometry shown for steady two-dimensional flow past an arbitrary body in the absence of body forces. Show that the force the fluid exerts on the body is given by the Euler momentum integral: $F_j = -\int\limits_{A_1}(\rho u_i u_j - \tau_{ij})n_i dA$, and $0 = \int\limits_{A_1}\rho u_i n_i dA$.



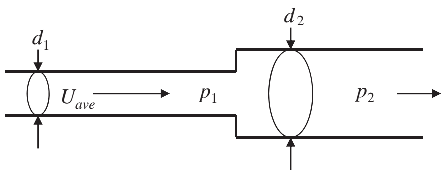
4.14. The pressure rise $\Delta p = p_2 - p_1$ that occurs for flow through a sudden pipe-cross-sectional-area expansion can depend on the average upstream flow speed U_{ave}, the upstream pipe diameter d_1, the downstream pipe diameter d_2, and the fluid density ρ and viscosity μ. Here p_2 is the pressure downstream of the expansion where the flow is first fully adjusted to the larger pipe diameter.

a) Find a dimensionless scaling law for Δp in terms of U_{ave}, d_1, d_2, ρ, and μ.

b) Simplify the result of part a) for high-Reynolds-number turbulent flow where μ does not matter.

c) Use a control volume analysis to determine Δp in terms of U_{ave}, d_1, d_2, and ρ for the high Reynolds number limit. [*Hints*: 1) a streamline drawing might help in determining or estimating the pressure on the vertical surfaces of the area transition, and 2) assume uniform flow profiles wherever possible.]

d) Compute the ideal flow value for Δp using the Bernoulli equation (4.19) and compare this to the result from part c) for a diameter ratio of $d_1/d_2 = ½$. What fraction of the maximum possible pressure rise does the sudden expansion achieve?



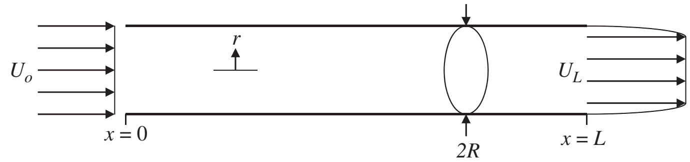
4.15. Consider how pressure gradients and skin friction develop in an empty wind tunnel or water tunnel test section when the flow is incompressible. Here the fluid has viscosity μ and density ρ, and flows into a horizontal cylindrical pipe of length L with radius R at a uniform horizontal velocity U_o. The inlet of the pipe lies at $x = 0$. Boundary layer growth on the pipe's walls induces the horizontal velocity on the pipe's centerline to be U_L at $x = L$; however, the pipe-wall boundary layer thickness remains much smaller than R. Here, L/R is of order 10, and $\rho U_o R/\mu \gg 1$. The radial coordinate from the pipe centerline is r.

a) Determine the displacement thickness, δ_L^*, of the boundary layer at $x = L$ in terms of U_o, U_L, and R. Assume that the boundary layer displacement thickness is zero at $x = 0$. [The boundary layer displacement thickness, δ^*, is the thickness of the zero-flow-speed layer that displaces the outer flow by the same amount as the actual boundary layer. For a boundary layer velocity profile $u(y)$ with $y =$ wall-normal coordinate and $U =$ outer flow velocity, δ^* is defined by: $\delta^* = \int_0^\infty (1 - (u/U))dy$.]

b) Determine the pressure difference, $\Delta P = P_L - P_o$, between the ends of the pipe in terms of ρ, U_o, and U_L.

c) Assume the horizontal velocity profile at the outlet of the pipe can be approximated by: $u(r, x = L) = U_L(1 - (r/R)^n)$ and estimate average skin friction, $\overline{\tau}_w$, on the inside of the pipe between $x = 0$ and $x = L$ in terms of ρ, U_o, U_L, R, L, and n.

d) Calculate the skin friction coefficient, $c_f = \overline{\tau}_w/(1/2)\rho U_o^2$, when $U_o = 20.0$ m/s, $U_L = 20.5$ m/s, $R = 1.5$ m, $L = 12$ m, $n = 80$, and the fluid is water, i.e., $\rho = 10^3$ kg/m^3.

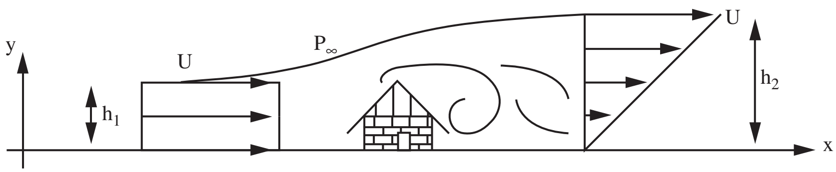


4.16. Consider the situation depicted below. Wind strikes the side of a simple residential structure and is deflected up over the top of the structure. Assume the following: two-dimensional steady inviscid constant-density flow, uniform upstream velocity profile, linear gradient in the downstream velocity profile (velocity U at the upper boundary and zero velocity at the lower boundary as shown), no flow through the upper boundary of the control volume, and constant pressure on the upper boundary of the control volume. Using the control volume shown:

a) Determine h_2 in terms of U and h_1.

b) Determine the *direction* and *magnitude* of the horizontal force on the house per unit depth into the page in terms of the fluid density ρ, the upstream velocity U, and the height of the house h_1.

c) Evaluate the magnitude of the force for a house that is 10 m tall and 20 m long in wind of 22 m/sec (approximately 50 miles per hour).

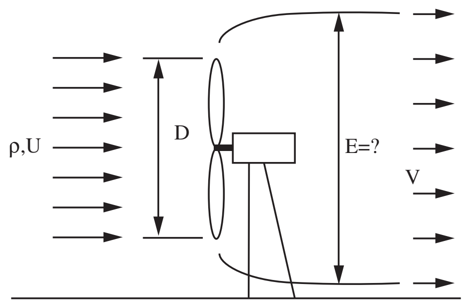


4.17. A large wind turbine with diameter D extracts a fraction η of the kinetic energy from the airstream (density $= \rho =$ constant) that impinges on it with velocity U.

a) What is the diameter of the wake zone, E, downstream of the windmill?

b) Determine the magnitude and direction of the force on the windmill in terms of ρ, U, D, and η.

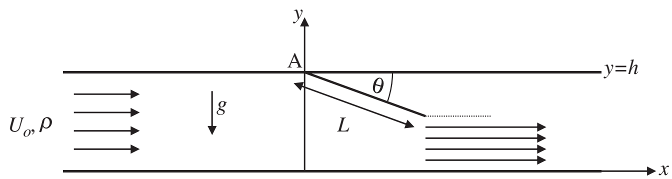
c) Does your answer approach reasonable limits as $\eta \to 0$ and $\eta \to 1$?



4.18. An incompressible fluid of density ρ flows through a horizontal rectangular duct of height h and width b. A uniform flat plate of length L and width b attached to the top of the duct at point A is deflected to an angle θ as shown.

a) Estimate the pressure difference between the upper and lower sides of the plate in terms of x, ρ, U_o, h, L, and θ when the flow separates cleanly from the tip of the plate.

b) If the plate has mass M and can rotate freely about the hinge at A, determine a formula for the angle θ in terms of the other parameters. You may leave your answer in terms of an integral.



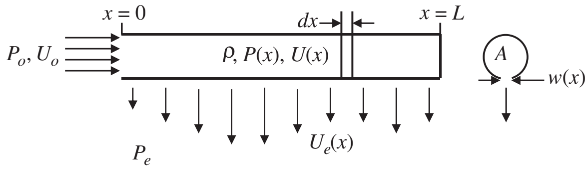
4.19. A pipe of length L and cross sectional area A is to be used as a fluid-distribution manifold that expels a steady uniform volume flux per unit length of an incompressible liquid from $x = 0$ to $x = L$. The liquid has density ρ, and is to be expelled from the pipe through a slot of varying width, $w(x)$. The goal of this problem is to determine $w(x)$ in terms of the other parameters of the problem. The pipe-inlet pressure and liquid velocity at $x = 0$ are P_o and U_o, respectively, and the pressure outside the pipe is P_e. If $P(x)$ denotes the pressure on the inside of the pipe, then the liquid velocity through the slot U_e is determined from: $P(x) - P_e = (1/2)\rho U_e^2$. For this problem assume that the expelled liquid exits the pipe perpendicular to the pipe's axis, and note that $wU_e = \text{const.} = U_o A/L$, even though w and U_e both depend on x.

a) Formulate a dimensionless scaling law for w in terms of x, L, A, ρ, U_o, P_o, and P_e.

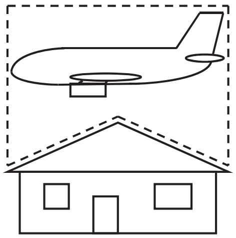
b) Ignore the effects of viscosity, assume all profiles through the cross section of the pipe are uniform, and use a suitable differential-control-volume analysis to show that:

$$A\frac{dU}{dx} + wU_e = 0, \quad \text{and} \quad \rho\frac{d}{dx}U^2 = -\frac{dP}{dx}.$$

c) Use these equations and the relationships stated above to determine $w(x)$ in terms of x, L, A, ρ, U_o, P_o, and P_e. Is the slot wider at $x = 0$ or at $x = L$?



4.20. The take-off mass of a Boeing 747-400 may be as high as 400,000 kg. An Airbus A380 may be even heavier. Using a control volume (CV) that comfortably encloses the aircraft, explain why such large aircraft do not crush houses or people when they fly low overhead. Of course, the aircraft's wings generate lift but they are entirely contained within the CV and do not coincide with any of the CV's surfaces; thus merely stating the lift balances weight is not a satisfactory explanation. Given that the CV's vertical body-force term, $-g \int_{CV} \rho dV$, will exceed 4×10^6 N when the airplane and air in the CV's interior are included, your answer should instead specify which of the CV's surface forces or surface fluxes carries the signature of a large aircraft's impressive weight.

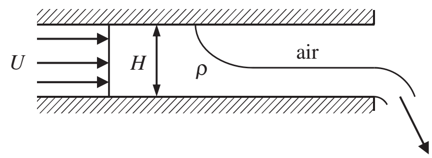


4.21. [1]An inviscid incompressible liquid with density ρ flows in a wide conduit of height H and width B into the page. The inlet stream travels at a uniform speed U and fills the conduit. The depth of the outlet stream is less than H. Air with negligible density fills the gap above the outlet stream. Gravity acts downward with acceleration g. Assume the flow is steady for the following items.

a) Find a dimensionless scaling law for U in terms of ρ, H, and g.

b) Denote the outlet stream depth and speed by h and u, respectively, and write down a set of equations that will allow U, u, and h to be determined in terms of ρ, H, and g.

c) Solve for U, u, and h in terms of ρ, H, and g. [*Hint*: solve for h first.]



[1]Based on a lecture example of Professor P. E. Dimotakis

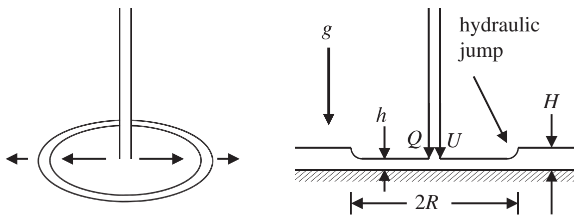
4.22. A hydraulic jump is the shallow-water-wave equivalent of a gas-dynamic shock wave. A steady radial hydraulic jump can be observed safely in one's kitchen, bathroom, or backyard where a falling stream of water impacts a shallow pool of water on a flat surface. The radial location R of the jump will depend on gravity g, the depth of the water behind the jump H, the volume flow rate of the falling stream Q, and the stream's speed, U, where it impacts the plate. In your work, assume $\sqrt{2gh} \ll U$ where r is the radial coordinate from the point where the falling stream impacts the surface.

a) Formulate a dimensionless law for R in terms of the other parameters.

b) Use the Bernoulli equation and a control volume with narrow angular and negligible radial extents that contains the hydraulic jump to show that:

$$R \cong \frac{Q}{2\pi U H^2}\left(\frac{2U^2}{g} - H\right).$$

c) Rewrite the result of part b) in terms of the dimensionless parameters found for part a).

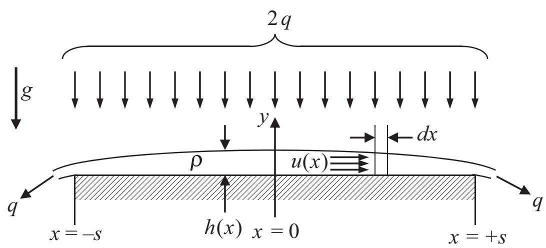


4.23. A fine uniform mist of an inviscid incompressible fluid with density ρ settles steadily at a total volume flow rate (per unit depth into the page) of $2q$ onto a flat horizontal surface of width $2s$ to form a liquid layer of thickness $h(x)$ as shown. The geometry is two dimensional.

a) Formulate a dimensionless scaling law for h in terms of x, s, q, ρ, and g.

b) Use a suitable control volume analysis, assuming $u(x)$ does not depend on y, to find a single cubic equation for $h(x)$ in terms of $h(0)$, s, q, x, and g.

c) Determine $h(0)$.



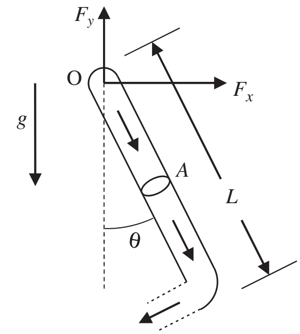
4.24. A thin-walled pipe of mass m_o, length L, and cross-sectional area A is free to swing in the x-y plane from a frictionless pivot at point O. Water with density ρ fills the pipe, flows into it at O perpendicular to the x-y plane, and is expelled at a right angle from the pipe's end as shown. The pipe's opening also has area A and gravity g acts downward. For a steady mass flow rate of $\dot{m}$, the pipe hangs at angle θ with respect to the vertical as shown. Ignore fluid viscosity.

a) Develop a dimensionless scaling law for θ in terms of m_o, L, A, ρ, $\dot{m}$, and g.

b) Use a control volume analysis to determine the force components, F_x and F_y, applied to the pipe at the pivot point in terms of θ, m_o, L, A, ρ, $\dot{m}$, and g.

c) Determine θ in terms of m_o, L, A, ρ, $\dot{m}$, and g.

d) Above what value of $\dot{m}$ will the pipe rotate without stopping?



4.25. Construct a house of cards, or light a candle. Get the cardboard tube from the center of a roll of paper towels and back away from the cards or candle a meter or two so that by blowing you cannot knock down the cards or blow out the candle unaided. Now use the tube in two slightly different configurations. First, place the tube snugly against your face encircling your mouth, and try to blow down the house of cards or blow out the candle. Repeat the experiment while moving closer until the cards are knocked down or the candle is blown out (you may need to get closer to your target than might be expected; do not hyperventilate; do not start the cardboard tube on fire). Note the distance between the end of the tube and the card house or candle at this point. Rebuild the card house or relight the candle and repeat the experiment, except this time hold the tube a few centimeters away from your face and mouth, and blow through it. Again, determine the greatest distance from which you can knock down the cards or blow out the candle.

a) Which configuration is more effective at knocking the cards down or blowing the candle out?

b) Explain your findings with a suitable control-volume analysis.

c) List some practical applications where this effect might be useful.

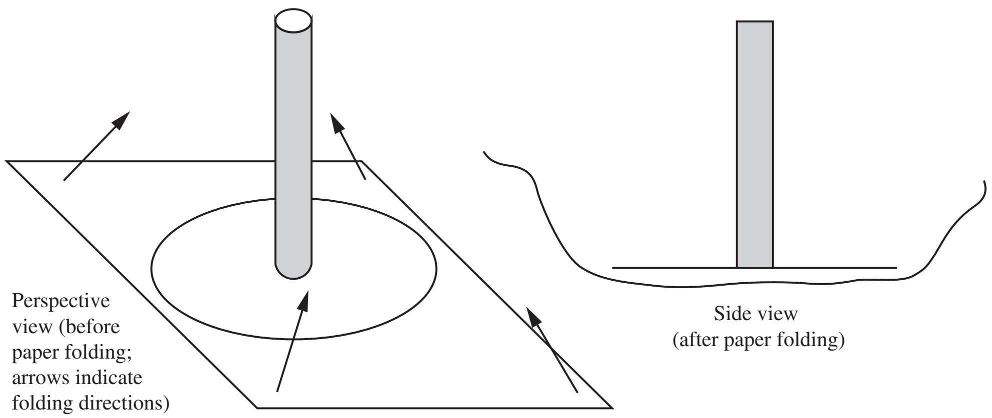
4.26. [2]Attach a drinking straw to a 15-cm-diameter cardboard disk with a hole at the center using tape or glue. Loosely fold the corners of a standard piece of paper upward so that

[2]Based on a demonstration done for the 3rd author by Professor G. Tryggvason

the paper mildly cups the cardboard disk (see drawing). Place the cardboard disk in the central section of the folded paper. Attempt to lift the loosely folded paper off a flat surface by blowing or sucking air through the straw.

a) Experimentally determine if blowing or suction is more effective in lifting the folded paper.

b) Explain your findings with a control volume analysis.



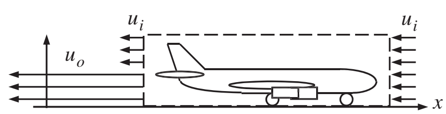
4.27. A rectangular tank is placed on wheels and is given a constant horizontal acceleration a. Show that, at steady state, the angle made by the free surface with the horizontal is given by $\tan\theta = a/g$.

4.28. Starting from rest at $t = 0$, an airliner of mass M accelerates at a constant rate $\mathbf{a} = a\mathbf{e}_x$ into a headwind, $\mathbf{u} = -u_i\mathbf{e}_x$. For the following items, assume that: 1) the x-component of the fluid velocity is $-u_\mathrm{i}$ on the front, sides, and back upper half of the control volume (CV), 2) the x-component of the fluid velocity is $-u_\mathrm{o}$ on the back lower half of the CV, 3) changes in M can be neglected, 4) changes of air momentum inside the CV can be neglected, and 5) the airliner has frictionless wheels. In addition, assume constant air density ρ and uniform flow conditions exist on the various control surfaces. In your work, denote the CV's front and back area by A. (This approximate model is appropriate for real commercial airliners that have the engines hung under the wings.)

a) Find a dimensionless scaling law for u_o at $t = 0$ in terms of u_i, ρ, a, M, and A.

b) Using a CV that encloses the airliner (as shown) determine a formula for $u_o(t)$, the time-dependent air velocity on the lower half of the CV's back surface.

c) Evaluate u_o at $t = 0$, when $M = 4 \times 10^5$ kg, $a = 2.0$ m/s^2, $u_i = 5$ m/s, $\rho = 1.2$ kg/m^3, and $A = 1200$ m^2. Would you be able to walk comfortably behind the airliner?

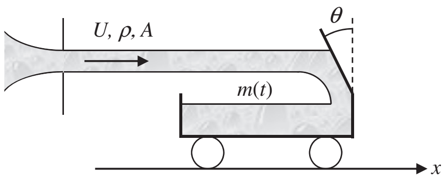


4.29. [3]A cart that can roll freely in the x direction deflects a horizontal water jet into its tank with a vane inclined to the vertical at an angle θ. The jet issues steadily at velocity U with density ρ, and has cross-sectional area A. The cart is initially at rest with a mass of m_0. Ignore the effects of surface tension, the cart's rolling friction, and wind resistance in your answers.

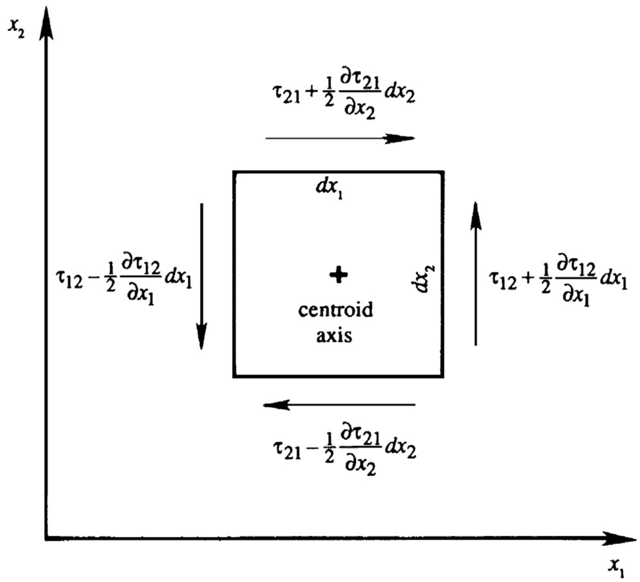
a) Formulate dimensionless law for the mass, $m(t)$, in the cart at time t in terms of t, θ, U, ρ, A, and m_0.

b) Formulate a differential equation for $m(t)$.

c) Find a solution for $m(t)$ and put it in dimensionless form.



4.30. Prove that the stress tensor is symmetric by considering first-order changes in surface forces on a vanishingly small cube in rotational equilibrium. Work with rotation about the number 3 coordinate axis to show $\tau_{12} = \tau_{21}$. Cyclic permutation of the indices will suffice for showing the symmetry of the other two shear stresses.



4.31. Obtain an empty plastic milk jug with a cap *that seals tightly*, and a frying pan. Fill both the pan and jug with water to a depth of approximately 1 cm. Place the jug in the pan with the cap off. Place the pan on a stove and turn up the heat until the water in the

[3]Similar to problem 4.170 on page 157 in Fox et al. (2009)

frying pan boils vigorously for a few minutes. Turn the stove off, and quickly put the cap tightly on the jug. *Avoid spilling or splashing hot water on yourself*. Remove the capped jug from the frying pan and let it cool to room temperature. Does anything interesting happen? If something does happen, explain your observations in terms of surface forces. What is the origin of these surface forces? Can you make any quantitative predictions about what happens?

4.32. In cylindrical coordinates (R,φ,z), two components of a steady incompressible viscous flow field are known: $u_\varphi = 0$, and $u_z = -Az$ where A is a constant, and body force is zero.

a) Determine u_R so that the flow field is smooth and conserves mass.

b) If the pressure, p, at the origin of coordinates is P_o, determine $p(R,\varphi,z)$ when the density is constant.

4.33. Solid body rotation with a constant angular velocity, Ω, is described by the following Cartesian velocity field: $\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{x}$. For this velocity field:

a) Compute the components of:

$$\tau_{ij} = -p\delta_{ij} + \mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right] + \mu_v\delta_{ij}\frac{\partial u_k}{\partial x_k}.$$

b) Consider the case of $\Omega_1 = \Omega_2 = 0$, $\Omega_3 \neq 0$, with $p = p_o$ at $x_1 = x_2 = 0$. Use the differential momentum equation in Cartesian coordinates to determine $p(r)$, where $r^2 = x_1^2 + x_2^2$, when there is no body force and $\rho =$ constant. Does your answer make sense? Can you check it with a simple experiment?

4.34. Using only (4.7), (4.22), (4.37), and (3.12) show that $\rho\frac{D\mathbf{u}}{Dt} + \nabla p = \rho\mathbf{g} + \mu\nabla^2\mathbf{u} + \left(\mu_v + \frac{1}{3}\mu\right)\nabla(\nabla\cdot\mathbf{u})$ when the dynamic (μ) and bulk (μ_v) viscosities are constants.

4.35. [4]Air, water, and petroleum products are important engineering fluids and can usually be treated as Newtonian fluids. Consider the following materials and try to classify them as: Newtonian fluid, non-Newtonian fluid, or solid. State the reasons for your choices and note the temperature range where you believe your answers are correct. Simple impact, tensile, and shear experiments in your kitchen or bathroom are recommended. Test and discuss at least five items.

a) toothpaste
b) peanut butter
c) shampoo
d) glass
e) honey
f) mozzarella cheese
g) hot oatmeal
h) creamy salad dressing
i) ice cream
j) silly putty

[4]Based on a suggestion from Professor W. W. Schultz

4.36. The equations for conservation of mass and momentum for a viscous Newtonian fluid are (4.7) and (4.39a) when the viscosities are constant.

a) Simplify these equations and write them out in primitive form for steady constant-density flow in two dimensions where $u_i = (u_1(x_1, x_2), u_2(x_1, x_2), 0)$, $p = p(x_1, x_2)$, and $g_j = 0$.

b) Determine $p = p(x_1, x_2)$ when $u_1 = Cx_1$ and $u_2 = -Cx_2$, where C is a positive constant.

4.37. Starting from (4.7) and (4.39b), derive a Poisson equation for the pressure, p, by taking the divergence of the constant-density momentum equation. [In other words, find an equation where $\partial^2 p/\partial x_j^2$ appears by itself on the left side and other terms not involving p appear on the right side.] What role does the viscosity μ play in determining the pressure in constant density flow?

4.38. Prove the equality of the two ends of (4.40) without leaving index notation or using vector identities.

4.39. The viscous compressible fluid conservation equations for mass and momentum are (4.7) and (4.38). Simplify these equations for constant-density, constant-viscosity flow and where the body force has a potential, $g_j = -\partial\Phi/\partial x_j$. Assume the velocity field can be found from $u_j = \partial\phi/\partial x_j$, where the scalar function ϕ depends on space and time. What are the simplified conservation of mass and momentum equations for ϕ?

4.40. The viscous compressible fluid conservation equations for mass and momentum are (4.7) and (4.38).

a) In Cartesian coordinates (x,y,z) with $\mathbf{g} = (g_x, 0, 0)$, simplify these equations for unsteady one-dimensional unidirectional flow where: $\rho = \rho(x,t)$ and $\mathbf{u} = (u(x,t), 0, 0)$.

b) If the flow is also incompressible, show that the fluid velocity depends only on time, i.e., $u(x,t) = U(t)$, and show that the equations found for part a) reduce to

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} = 0, \quad \text{and} \quad \rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \rho g_x.$$

c) If $\rho = \rho_o(x)$ at $t = 0$, and $u = U(0) = U_o$ at $t = 0$, determine implicit or explicit solutions for $\rho = \rho(x,t)$ and $U(t)$ in terms of x, t, $\rho_o(x)$, U_o, $\partial p/\partial x$, and g_x.

4.41. [5]**a)** Derive the following equation for the velocity potential for irrotational inviscid compressible flow in the absence of a body force:

$$\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}\left(|\nabla\phi|^2\right) + \frac{1}{2}\nabla\phi\cdot\nabla\left(|\nabla\phi|^2\right) - c^2\nabla^2\phi = 0$$

where $\nabla\phi = \mathbf{u}$. Start from the Euler equation (4.41), use the continuity equation, assume that the flow is isentropic so that p depends only on ρ, and denote $(\partial p/\partial\rho)_s = c^2$.

b) What limit does $c \to \infty$ imply?

c) What limit does $|\nabla\phi| \to 0$ imply?

[5]Obtained from Professor Paul Dimotakis

4.42. Derive (4.43) from (4.42).

4.43. For steady constant-density inviscid flow with body force per unit mass $\mathbf{g} = -\nabla\Phi$, it is possible to derive the following Bernoulli equation: $p + \frac{1}{2}\rho|\mathbf{u}|^2 + \rho\Phi = $ constant along a streamline.

a) What is the equivalent form of the Bernoulli equation for constant-density inviscid flow that appears steady when viewed in a frame of reference that rotates at a constant rate about the z-axis, i.e., when $\mathbf{\Omega} = (0, 0, \Omega_z)$ with Ω_z constant?

b) If the extra term found in the Bernoulli equation is considered a pressure correction: Where on the surface of the earth (i.e., at what latitude) will this pressure correction be the largest? What is the absolute size of the maximum pressure correction when changes in R on a streamline are 1 m, 1 km, and 10^3 km?

4.44. For many atmospheric flows, rotation of the earth is important. The momentum equation for inviscid flow in a frame of reference rotating at a constant rate $\mathbf{\Omega}$ is:

$$\partial\mathbf{u}/\partial t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla\Phi - (1/\rho)\nabla p - 2\mathbf{\Omega}\times\mathbf{u} - \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{x})$$

For steady two-dimensional horizontal flow, $\mathbf{u} = (u, v, 0)$, with $\Phi = gz$ and $\rho = \rho(z)$, show that the streamlines are parallel to constant pressure lines when the fluid particle acceleration is dominated by the Coriolis acceleration $|(\mathbf{u}\cdot\nabla)\mathbf{u}| \ll |2\mathbf{\Omega}\times\mathbf{u}|$, and when the local pressure gradient dominates the centripetal acceleration $|\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{x})| \ll |\nabla p|/\rho$. [This seemingly strange result governs just about all large-scale weather phenomena like hurricanes and other storms, and it allows weather forecasts to be made based on surface pressure measurements alone.]

Hints:

1. If $Y(x)$ defines a streamline contour, then $dY/dx = v/u$ is the streamline slope.
2. Write out all three components of the momentum equation and build the ratio v/u.
3. Using hint 1, the pressure increment along a streamline is: $dp = (\partial p/\partial x)dx + (\partial p/\partial y)dY$.

4.45. Show that (4.55) can be derived from (4.7), (4.53), and (4.54).

4.46. Multiply (4.22) by u_j and sum over j to derive (4.56).

4.47. Starting from $\varepsilon = (1/\rho)\sigma_{ij}S_{ij}$, derive the rightmost expression in (4.58).

4.48. For many gases and liquids (and solids too!), the following equations are valid:
$\mathbf{q} = -k\nabla T$ (Fourier's law of heat conduction, k = thermal conductivity, T = temperature),
$e = e_o + C_vT$ (e = internal energy per unit mass, C_v = specific heat at constant volume), and
$h = h_o + C_pT$ (h = enthalpy per unit mass, C_p = specific heat at constant pressure),
where e_o and h_o are constants, and C_v and C_p are also constants. Start with the energy equation

$$\rho\frac{\partial e}{\partial t} + \rho u_i\frac{\partial e}{\partial x_i} = -p\frac{\partial u_i}{\partial x_i} + \sigma_{ij}S_{ij} - \frac{\partial q_i}{\partial x_i}$$

for each of the following items.

a) Derive an equation for T involving u_j, k, ρ, and C_v for incompressible flow when $\sigma_{ij} = 0$.

b) Derive an equation for T involving u_j, k, ρ, and C_p for flow with $p = const.$ and $\sigma_{ij} = 0$.

c) Provide a physical explanation as to why the answers to parts a) and b) are different.

4.49. Derive the following alternative form of (4.60): $\rho C_p \frac{DT}{Dt} = \alpha T \frac{Dp}{Dt} + \rho\varepsilon + \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)$, where ε is given by (4.58) and α is the thermal expansion coefficient defined in (1.20). [*Hint*: $dh = (\partial h/\partial T)_p dT + (\partial h/\partial p)_T dp$]

4.50. Show that (4.68) is true without abandoning index notation or using vector identities.

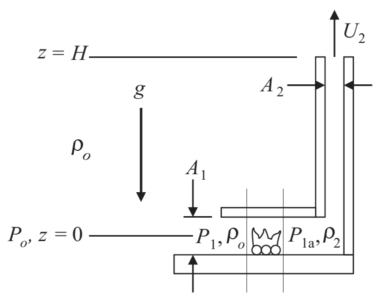
4.51. Consider an incompressible planar Couette flow, which is the flow between two parallel plates separated by a distance b. The upper plate is moving parallel to itself at speed U, and the lower plate is stationary. Let the x-axis lie on the lower plate. The pressure and velocity fields are independent of x, and the fluid is uniform with constant viscosity.

a) Show that the pressure distribution is hydrostatic and that the solution of the Navier-Stokes equation is $u(y) = Uy/b$.

b) Write the expressions for the stress and strain rate tensors, and show that the viscous kinetic-energy dissipation per unit volume is $\mu U^2/b^2$.

c) Using a rectangular control volume for which the two horizontal surfaces coincide with the walls and the two vertical surfaces are perpendicular to the flow: evaluate the kinetic energy equation (4.56) within this control volume, and show that the balance is between the viscous dissipation and the work done in moving the upper surface.

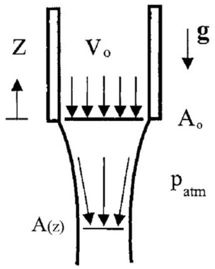
4.52. Determine the outlet speed, U_2, of a chimney in terms of ρ_o, ρ_2, g, H, A_1, and A_2. For simplicity, assume the fire merely decreases the density of the air from ρ_o to ρ_2 ($\rho_o > \rho_2$) and does not add any mass to the airflow. (This mass flow assumption isn't true, but it serves to keep the algebra under control in this problem.) The relevant parameters are shown in the figure. Use the steady Bernoulli equation into the inlet and from the outlet of the fire, but perform a control volume analysis across the fire. Ignore the vertical extent of A_1 compared to H and the effects of viscosity.



4.53. A hemispherical vessel of radius R containing an inviscid constant-density liquid has a small rounded orifice of area A at the bottom. Show that the time required to lower the level from h_1 to h_2 is given by

$$t = \frac{2\pi}{A\sqrt{2g}}\left[\frac{2}{3}R\left(h_1^{3/2} - h_2^{3/2}\right) - \frac{1}{5}\left(h_1^{5/2} - h_2^{5/2}\right)\right].$$

4.54. Water flows through a pipe in a gravitational field as shown in the accompanying figure. Neglect the effects of viscosity and surface tension. Solve the appropriate conservation equations for the variation of the cross-sectional area of the fluid column $A(z)$ after the water has left the pipe at $z = 0$. The velocity of the fluid at $z = 0$ is uniform at V_0 and the cross-sectional area is A_0.



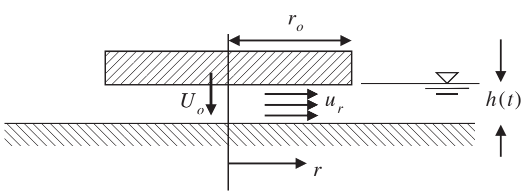
4.55. Redo the solution for the orifice-in-a-tank problem allowing for the fact that in Figure 4.16, $h = h(t)$, but ignoring fluid acceleration. Estimate how long it takes for the tank to empty.

4.56. A circular plate is forced down at a steady velocity U_o against a flat surface. Frictionless incompressible fluid of density ρ fills the gap $h(t)$. Assume that $h \ll r_o$, the plate radius, and that the radial velocity $u_r(r,t)$ is constant across the gap.

a) Obtain a formula for $u_r(r,t)$ in terms of r, U_o, and h.

b) Determine $\partial u_r(r,t)/\partial t$.

c) Calculate the pressure distribution under the plate assuming that $p(r = r_o) = 0$.



4.57. A frictionless, incompressible fluid with density ρ resides in a horizontal nozzle of length L having a cross-sectional area that varies smoothly between A_i and A_o via: $A(x) = A_i + (A_o - A_i)f(x/L)$, where f is a function that goes from 0 to 1 as x/L goes from 0 to 1. Here the x-axis lies on the nozzle's centerline, and $x = 0$ and $x = L$ are the horizontal locations of the nozzle's inlet and outlet, respectively. At $t = 0$, the pressure at the inlet of the nozzle is raised to $p_i > p_o$, where p_o is the (atmospheric) outlet pressure of the nozzle, and the fluid begins to flow horizontally through the nozzle.

a) Derive the following equation for the time-dependent volume flow rate $Q(t)$ through the nozzle from the unsteady Bernoulli equation and an appropriate conservation-of-mass relationship.

$$\frac{\dot{Q}(t)}{A_i}\int_{x=0}^{x=L}\frac{A_i}{A(x)}\,dx + \frac{Q^2(t)}{2}\left(\frac{1}{A_o^2}-\frac{1}{A_i^2}\right) = \left(\frac{p_i - p_o}{\rho}\right)$$

b) Solve the equation of part a) when $f(x/L) = x/L$.

c) If $\rho = 10^3$ kg/m^3, $L = 25$ cm, $A_i = 100$ cm^2, $A_o = 30$ cm^2, and $p_i - p_o = 100$ kPa for $t \geq 0$, how long does it take for the flow rate to reach 99% of its steady-state value?

4.58. Using the small slope version of the surface curvature $1/R_1 \approx d^2\zeta/dx^2$, redo Example 4.7 to find h and $\zeta(x)$ in terms of x, σ, ρ, g, and θ. Show that the two answers are consistent when θ approaches $\pi/2$.

4.59. Redo the dimensionless scaling leading to (4.101) by choosing a generic viscous stress, $\mu U/l$, and then a generic hydrostatic pressure, $\rho g l$, to make $p - p_\infty$ dimensionless. Interpret the revised dimensionless coefficients that appear in the scaled momentum equation, and relate them to St, Re, and Fr.

4.60. From Figure 4.21, it can be seen that $C_D \propto 1/\text{Re}$ at small Reynolds numbers and that C_D is approximately constant at large Reynolds numbers. Redo the dimensional analysis leading to (4.99) to verify these observations when:

a) Re is low and fluid inertia is unimportant so ρ is no longer a parameter.

b) Re is high and the drag force is dominated by fore-aft pressure differences on the sphere and μ is no longer a parameter.

4.61. Suppose that the power to drive a propeller of an airplane depends on d (diameter of the propeller), U (free-stream velocity), ω (angular velocity of the propeller), c (velocity of sound), ρ (density of fluid), and μ (viscosity). Find the dimensionless groups. In your opinion, which of these are the most important and should be duplicated in model testing?

4.62. A 1/25 scale model of a submarine is being tested in a wind tunnel in which $p = 200$ kPa and $T = 300$ K. If the prototype speed is 30 km/hr, what should be the free-stream velocity in the wind tunnel? What is the drag ratio? Assume that the submarine would not operate near the free surface of the ocean.

4.63. A set of small-scale tank-draining experiments are performed to predict the liquid depth, h, as a function of time t for the draining process of a large cylindrical tank that is geometrically similar to the small-experiment tanks. The relevant parameters are gravity g, initial tank depth H, tank diameter D, orifice diameter d, and the density and viscosity of the liquid, ρ and μ, respectively.

a) Determine a general relationship between h and the other parameters.

b) Using the following small-scale experiment results, determine whether or not the liquid's viscosity is an important parameter.

$H = 8$ cm, $D = 24$ cm, $d = 8$ mm		$H = 16$ cm, $D = 48$ cm, $d = 1.6$ cm	
h (cm)	t (s)	h (cm)	t (s)
8.0	0.00	16.0	0.00
6.8	1.00	13.3	1.50
5.0	2.00	9.5	3.00
3.0	3.00	5.3	4.50
1.2	4.00	1.8	6.00
0.0	5.30	0.0	7.50

c) Using the small-scale-experiment results above, predict how long it takes to completely drain the liquid from a large tank having $H = 10$ m, $D = 30$ m, and $d = 1.0$ m.

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Supplemental Reading

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