CHAPTER

# 5

# Vorticity Dynamics

OUTLINE



### CHAPTER OBJECTIVES

- To introduce the basic concepts and phenomena associated with vortex lines, tubes, and sheets in viscous and inviscid flows.
- To derive and state classical theorems and equations for vorticity production and transport in inertial and rotating frames of reference.
- To develop the relationship that describes how vorticity at one location induces fluid velocity at another.
- To present some of the intriguing phenomena of vortex dynamics.

## 5.1. INTRODUCTION

Vorticity is a vector field that is twice the angular velocity of a fluid particle. A concentration of codirectional or nearly codirectional vorticity is called a *vortex*. Fluid motion leading to

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circular or nearly circular streamlines is called *vortex motion*. In two dimensions $(r,\theta)$, a uniform distribution of plane-normal vorticity with magnitude $\omega$ produces solid body rotation,

$$u_\theta = \omega r/2, \tag{5.1}$$

while a perfect concentration of plane-normal vorticity located at $r = 0$ with circulation $\Gamma$ produces irrotational flow for $r > 0$,

$$u_\theta = \Gamma/2\pi r. \tag{5.2}$$

Both of these flow fields are steady and both produce closed (circular) streamlines. However, in the first, fluid particles rotate, but in the second, for $r \neq 0$, they do not. In the second flow field, the vorticity is infinite on a line perpendicular to the $r$-$\theta$ plane that intersects it at $r = 0$, but is zero elsewhere. Thus, such an *ideal line vortex* is also known as an *irrotational vortex*. It is a useful idealization that will be exploited in this chapter, in Chapter 6, and in Chapter 14.

In general, vorticity in a flowing fluid is neither unidirectional nor steady. In fact, we can commonly think of vorticity as being embedded in fluid elements so that an element's vorticity may be reoriented or concentrated or diffused depending on the motion and deformation of that fluid element and on the torques applied to it by surrounding fluid elements. This conjecture is based on the fact that the dynamics of three-dimensional time-dependent vorticity fields can often be interpreted in terms of a few fundamental principles. This chapter presents these principles and some aspects of flows with vorticity, starting with fundamental vortex concepts.

A *vortex line* is a curve in the fluid that is everywhere tangent to the local vorticity vector. Here, of course, we recognize that a vortex line is not strictly linear; it may be curved just as a streamline may be curved. A vortex line is related to the vorticity vector the same way a streamline is related to the velocity vector. Thus, if $\omega_x$, $\omega_y$, and $\omega_z$ are the Cartesian components of the vorticity vector $\boldsymbol{\omega}$, then the components of an element $d\mathbf{s} = (dx, dy, dz)$ of a vortex line satisfy

$$dx/\omega_x = dy/\omega_y = dz/\omega_z, \tag{5.3}$$

which is analogous to (3.7) for a streamline. As a further similarity, vortex lines do not exist in irrotational flow just as streamlines do not exist in stationary fluid. Elementary examples of vortex lines are supplied by the flow fields (5.1) and (5.2). For solid-body rotation (5.1), all lines perpendicular to the $r$-$\theta$ plane are vortex lines, while in the flow field of an irrotational vortex (5.2) the lone vortex line is perpendicular to the $r$-$\theta$ plane and passes through it at $r = 0$.

In a region of flow with nontrivial vorticity, the vortex lines passing through any closed curve form a tubular surface called a *vortex tube* (Figure 5.1), which is akin to a stream tube (Figure 3.6). The circulation around a narrow vortex tube is $d\Gamma = \boldsymbol{\omega}\cdot\mathbf{n}dA$ just as the volume flow rate in a narrow stream tube is $dQ = \mathbf{u}\cdot\mathbf{n}dA$. The *strength of a vortex tube* is defined as the circulation computed on a closed circuit lying on the surface of the tube that encircles it just once. From Stokes' theorem it follows that the strength of a vortex tube, $\Gamma$, is equal to the vorticity in the tube integrated over its cross-sectional area. Thus, when Gauss' theorem is applied to the volume $V$ defined by a section of a vortex tube, such as that shown in Figure 5.1, we find that

$$\int_V \nabla \cdot \omega dV = \int_A \omega \cdot \mathbf{n} dA = \left\{ \int_{lower\ end} + \int_{curved\ side} + \int_{upper\ end} \right\} \omega \cdot \mathbf{n} dA$$
$$= -\Gamma_{lower\ end} + \Gamma_{upper\ end} = 0, \tag{5.4}$$

where $\boldsymbol{\omega} \cdot \mathbf{n}$ is zero on the curved sides of the tube, and the final equality follows from $\nabla \cdot \boldsymbol{\omega} = \nabla \cdot (\nabla \times \mathbf{u}) = 0$. Equation (5.4) states that a vortex tube's strength is independent of where it is measured, $\Gamma_{lower\ end} = \Gamma_{upper\ end}$, and this implies that vortex tubes cannot end within the fluid, a concept that can be extended to vortex lines in the limit as a vortex tube's cross-sectional area goes to zero. However, vortex lines and tubes can terminate on solid surfaces or free surfaces, or they can form loops. This kinematic constraint is often useful for determining the topology of vortical flows.

As we will see in this and other chapters, fluid viscosity plays an essential role in the diffusion of vorticity, and in the reconnection of vortex lines. However, before considering these effects, the role of viscosity in the two basic vortex flows (5.1) and (5.2) is examined. Assuming incompressible flow, we shall see that in one of these flows the viscous terms in the momentum equation drop out, although the viscous stress and dissipation of energy are nonzero.

As discussed in Chapter 3, fluid elements undergoing solid-body rotation (5.1) do not deform ($S_{ij} = 0$), so the Newtonian viscous stress tensor (4.37) reduces to $\tau_{ij} = -p\delta_{ij}$, and Cauchy's equation (4.24) reduces to Euler's equation (4.41). When the solid-body rotation field, $u_r = 0$ and $u_\theta = \omega r/2$, is substituted into (4.41), it simplifies to:

$$-\rho u_\theta^2/r = -\partial p/\partial r, \text{ and } 0 = -\partial p/\partial z - \rho g. \tag{5.5a, 5.5b}$$

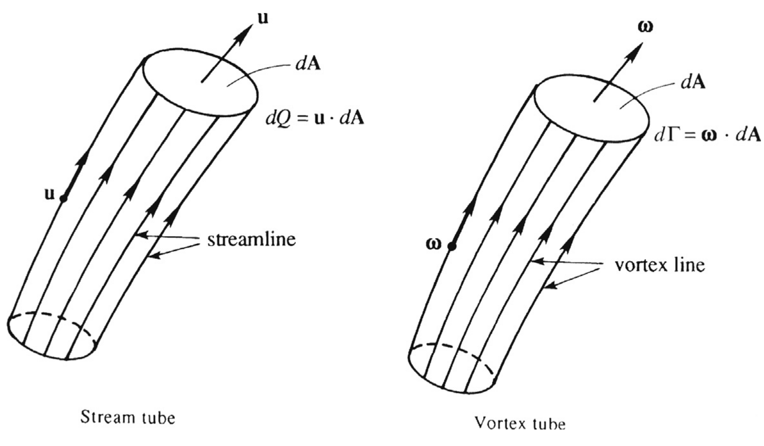


**FIGURE 5.1** Analogy between stream tubes and vortex tubes. The lateral sides of stream and vortex tubes are locally tangent to the flow's velocity and vorticity fields, respectively. Stream and vortex tubes with cross-sectional area $d\mathbf{A}$ carry constant volume flux $\mathbf{u} \cdot d\mathbf{A}$ and constant circulation $\boldsymbol{\omega} \cdot d\mathbf{A}$, respectively.

Integrating (5.5a) produces $p(r,z) = \rho\omega^2 r^2/8 + f(z)$, where $f$ is an undetermined function. Integrating (5.5b) produces $p(r,z) = -\rho g z + g(r)$, where $g$ is an undetermined function. These two equations are consistent when:

$$p(r,z) - p_o = \frac{1}{8}\rho\omega^2 r^2 - \rho g z \tag{5.6}$$

where $p_o$ is the pressure at $r = 0$ and $z = 0$. To determine the shape of constant pressure surfaces, solve (5.6) for $z$ to find:

$$z = \frac{\omega^2 r^2}{8g} - \frac{p(r,z) - p_o}{\rho g}.$$

Hence, surfaces of constant pressure are paraboloids of revolution (Figure 5.2).

The important point to note is that viscous stresses are absent in steady solid-body rotation. (The viscous stresses, however, are important during the transient period of *initiating* solid body rotation, say by steadily rotating a tank containing a viscous fluid initially at rest.) In terms of velocity, (5.6) can be written as

$$-\frac{1}{2}u_\theta^2 + gz + \frac{p(r,z)}{\rho} = const.,$$

and, when compared to (4.19), this shows that the Bernoulli function $B = u_\theta^2/2 + gz + p/\rho$ is *not* constant for points on different streamlines. This outcome is expected because the flow is rotational.

For the flow induced by an irrotational vortex (5.2), the viscous stress is

$$\sigma_{r\theta} = \mu\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right] = -\frac{\mu\Gamma}{\pi r^2},$$

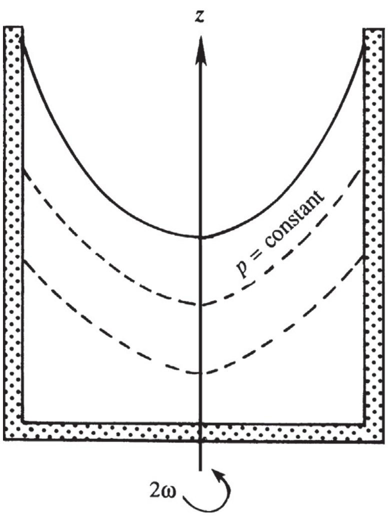


**FIGURE 5.2** The steady flow field of a viscous liquid in a steadily rotating tank is solid body rotation. When the axis of rotation is parallel to the (downward) gravitational acceleration, surfaces of constant pressure in the liquid are paraboloids of revolution.

which is nonzero everywhere because fluid elements deform (see Figure 3.16). However, the interesting point is that the *net viscous force* on an element is zero for $r > 0$ (see Exercise 5.4) because the viscous forces on the surfaces of an element cancel out, leaving a zero resultant. Thus, momentum conservation is again represented by the Euler equation. Substitution of (5.2) into (5.5), followed by integration, yields

$$p(r,z) - p_\infty = -\frac{\rho\Gamma^2}{8\pi^2 r^2} - \rho g z, \tag{5.7}$$

where $p_\infty$ is the pressure far from the line vortex at $z = 0$. This can be rewritten:

$$z = -\frac{\Gamma^2}{8\pi^2 r^2 g} - \frac{p(r,z) - p_\infty}{\rho g},$$

which shows that surfaces of constant pressure are hyperboloids of revolution of the second degree (Figure 5.3). Equation (5.7) can also be rewritten:

$$\frac{1}{2}u_\theta^2 + gz + \frac{p(r,z)}{\rho} = const.,$$

which shows that Bernoulli's equation is applicable between any two points in the flow field, as is expected for irrotational flow.

One way of generating the flow field from an irrotational vortex is by rotating a solid circular cylinder with radius $a$ in an infinite viscous fluid (see Figure 8.7). It is shown in Section 8.2 that the steady solution of the Navier-Stokes equations satisfying the no-slip boundary condition ($u_\theta = \omega a/2$ at $r = a$) is

$$u_\theta = \omega a^2/2r \text{ for } r \geq a,$$

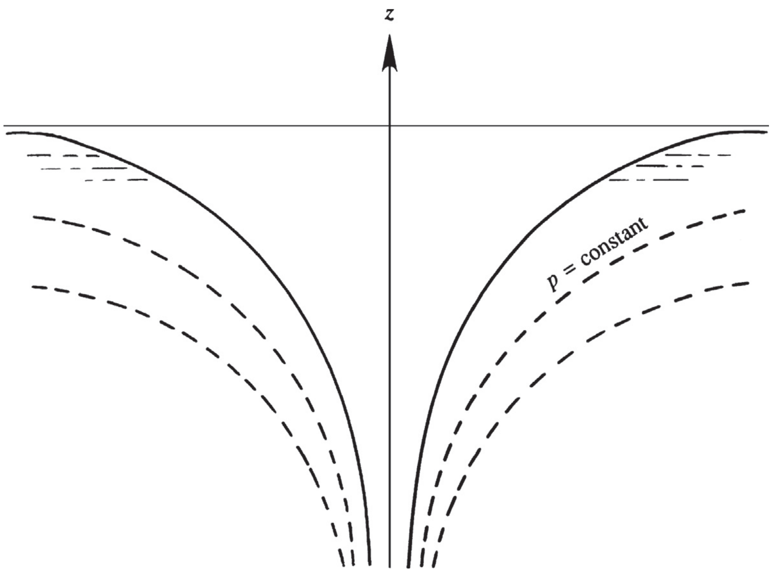


**FIGURE 5.3** Surfaces of constant pressure in the flow induced by an ideal linear vortex that coincides with the $z$-axis and is parallel to the (downward) gravitational acceleration.

where $\omega/2$ is the cylinder's constant rotation rate; see (8.11). When the motions inside and outside the cylinder are considered, this flow field precisely matches that of a Rankine vortex with core size $a$; see (3.28) with $\Gamma = \pi a^2 \omega$. The presence of the nonzero-radius cylinder leads to a flow field without a singularity that is irrotational for $r > a$. Viscous stresses are present, and the resulting viscous dissipation of kinetic energy is exactly compensated by the work done at the surface of the cylinder. However, there is no *net* viscous force at any point in the steady state. Interestingly, the application of the moment of momentum principle (see Section 4.9) to a large-radius cylindrical control volume centered on the rotating solid cylinder shows that the torque that rotates the solid cylinder is transmitted to an arbitrarily large distance from the axis of rotation. Thus, any attempt to produce this flow in a stationary container would require the application of counteracting torque on the container.

These examples suggest that *irrotationality does not imply the absence of viscous stresses*. Instead, it implies the *absence of net viscous forces*. Viscous stresses will be present whenever fluid elements deform. Yet, when $\boldsymbol{\omega}$ is uniform and nonzero (solid body rotation), there is no viscous stress at all. However, solid-body rotation is unique in this regard, and this uniqueness is built into the Newtonian-fluid viscous stress tensor (4.59). In general, fluid element rotation is accomplished and accompanied by viscous effects. Indeed, viscosity is a primary agent for vorticity generation and diffusion.

## 5.2. KELVIN'S CIRCULATION THEOREM

By considering the analogy with electrodynamics, Helmholtz published several theorems for vortex motion in an inviscid fluid in 1858. Ten years later, Kelvin introduced the idea of circulation and proved the following theorem: *In an inviscid, barotropic flow with conservative body forces, the circulation around a closed curve moving with the fluid remains constant with time*, if the motion is observed from a nonrotating frame. This theorem can be stated mathematically as

$$D\Gamma/Dt = 0 \tag{5.8}$$

where $D/Dt$ is defined by (3.5) and represents the total time rate of change following the fluid elements that define the closed curve, $C$ (a material contour), used to compute the circulation $\Gamma$. Such a material contour is shown in Figure 5.4.

Kelvin's theorem can be proved by time differentiating the definition of the circulation (3.18):

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\int_C u_i dx_i = \int_C \frac{Du_i}{Dt} dx_i + \int_C u_i \frac{D}{Dt}(dx_i), \tag{5.9}$$

where $dx_i$ are the components of the arc length element $d\mathbf{x}$ of $C$. Using (4.39) and (4.59), the first term on the right side of (5.9) may be rewritten:

$$\int_C \frac{Du_i}{Dt} dx_i = \int_C \left(-\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i + \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}\right) dx_i = -\int_C \frac{1}{\rho} dp - \int_C d\Phi + \int_C \left(\frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}\right) dx_i, \tag{5.10}$$

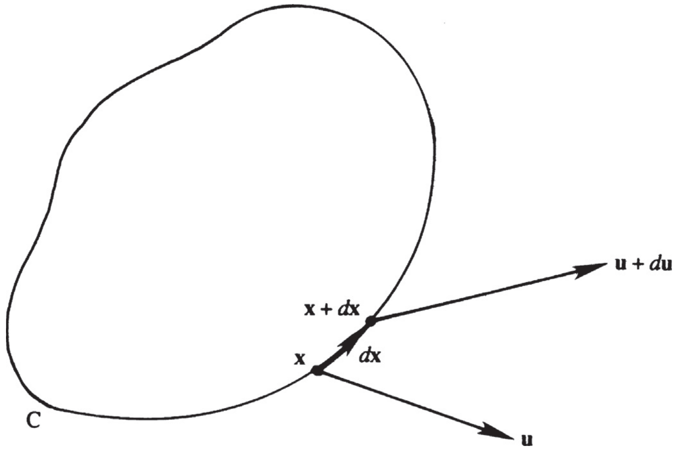


**FIGURE 5.4** Contour geometry for the Proof of Kelvin's circulation theorem. Here the short segment $d\mathbf{x}$ of the contour C moves with the fluid so that $D(d\mathbf{x})/Dt = d\mathbf{u}$.

where the replacements $(\partial p/\partial x_i)dx_i = dp$ and $(\partial\Phi/\partial x_i)dx_i = d\Phi$ have been made, and $\Phi$ is the body force potential (4.18). For a barotropic fluid, the first term on the right side of (5.10) is zero because $C$ is a closed contour, and $\rho$ and $p$ are single valued at each point in space. Similarly, the second integral on the right side of (5.10) is zero since $\Phi$ is also single valued at each point in space.

Now consider the second term on the right side of (5.9). The velocity at point $\mathbf{x} + d\mathbf{x}$ on $C$ is:

$$\mathbf{u} + d\mathbf{u} = \frac{D}{Dt}(\mathbf{x} + d\mathbf{x}) = \frac{D\mathbf{x}}{Dt} + \frac{D}{Dt}(d\mathbf{x}), \text{ so } du_i = \frac{D}{Dt}(dx_i).$$

Thus, the last term in (5.9) then becomes

$$\int_C u_i \frac{D}{Dt}(dx_i) = \int_C u_i du_i = \int_C d\left(\frac{1}{2}u_i^2\right) = 0,$$

where the final equality again follows because $C$ is a closed contour and $\mathbf{u}$ is a single-valued vector function. Hence, (5.9) simplifies to:

$$\frac{D\Gamma}{Dt} = \int_C \left(\frac{1}{\rho}\frac{\partial\sigma_{ij}}{\partial x_j}\right) dx_i, \tag{5.11}$$

and Kelvin's theorem (5.8) is proved when the fluid is inviscid ($\mu = \mu_v = 0$) or when the net viscous force ($\partial\sigma_{ij}/\partial x_j$) is zero along $C$. This latter condition occurs when $C$ lies entirely in irrotational fluid.

From this short proof we see that the three ways to create or destroy vorticity in a flow are: nonconservative body forces, a nonbarotropic pressure-density relationship, and nonzero net viscous forces. Examples of each follow. The Coriolis acceleration is a nonconservative body force that occurs in rotating frames of reference, and it generates a drain or bathtub

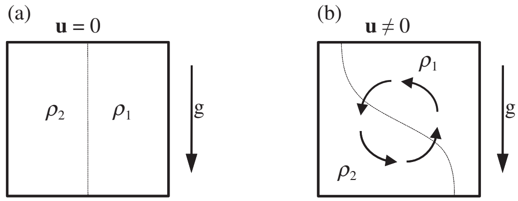


**FIGURE 5.5** Schematic drawings of two fluids with differing density that are initially stationary and separated within a rectangular container. Gravity acts downward as shown. Here the density difference is baroclinic because it depends on fluid composition and pressure, not on pressure alone. (a) This drawing shows the initial condition immediately before the barrier between the two fluids is removed. (b) This drawing shows the resulting fluid motion a short time after barrier removal. The deflection of the fluid interface clearly indicates that vorticity has been created.

vortex when a filled tank, initially at rest on the earth's surface, is drained. Nonbarotropic effects can lead to vorticity generation when a vertical barrier is removed between two side-by-side initially motionless fluids having different densities in the same container and subject to a gravitational field. The two fluids will tumble as the heavier one slumps to the container's bottom and the lighter one surges to the container's top (see Figure 5.5 and Exercise 5.5). Nonzero net viscous forces create vorticity at solid boundaries where the no-slip condition is maintained. A short distance away from a solid boundary, the velocity parallel to the boundary may be large. Vorticity is created when such near-wall velocity gradients arise.

Kelvin's theorem implies that irrotational flow will remain irrotational if the following four restrictions are satisfied.

**(1)** There are no net viscous forces along $C$. If $C$ moves into regions where there are net viscous forces such as within a boundary layer that forms on a solid surface, then the circulation changes. The presence of viscous effects causes *diffusion* of vorticity into or out of a fluid circuit and consequently changes the circulation.

**(2)** The body forces are conservative. Conservative body forces such as gravity act through the center of mass of a fluid particle and therefore do not generate torques that cause fluid particle rotation.

**(3)** The fluid density must depend on pressure only (barotropic flow). A flow will be barotropic if the fluid is homogeneous and one of the two independent thermodynamic variables is constant. Isentropic, isothermal, and constant density conditions lead to barotropic flow. Flows that are not barotropic are called *baroclinic*. Here fluid density depends on the pressure *and* the temperature, composition, salinity, and/or concentration of dissolved constituents. Consider fluid elements in barotropic and baroclinic flows (Figure 5.6). For the barotropic element, lines of constant $p$ are parallel to lines of constant $\rho$, which implies that the resultant pressure forces pass through the center of mass of the element. For the baroclinic element, the lines of constant $p$ and $\rho$ are not parallel. The net pressure force does not pass through the center of mass, and the resulting torque changes the vorticity and circulation. As described above, Figure 5.6 depicts a situation where vorticity is generated in a baroclinic flow.

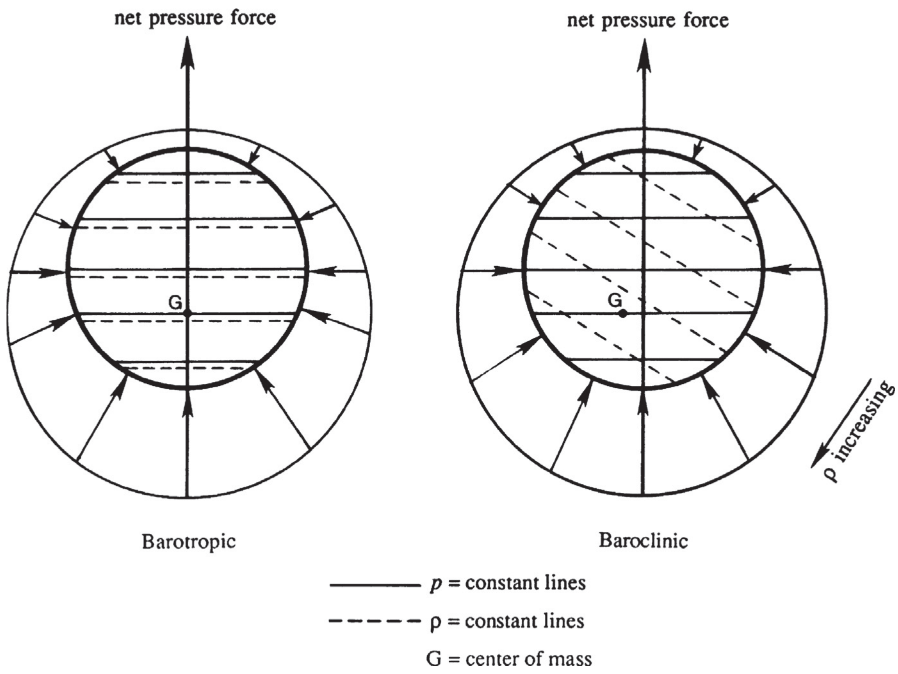


**FIGURE 5.6** Mechanism of vorticity generation in baroclinic flow, showing that the net pressure force does not pass through the center of mass G of the fluid element. The radially inward arrows indicate pressure forces on an element.

**(4)** The frame of reference must be an inertial frame. As described in Section 4.7, the conservation of momentum equation includes extra terms when the frame of reference rotates and accelerates, and these extra terms were not considered in the short proof given above.

## 5.3. HELMHOLTZ'S VORTEX THEOREMS

Under the same four restrictions, Helmholtz proved the following theorems for vortex motion:

**(1)** Vortex lines move with the fluid.
**(2)** The strength of a vortex tube (its circulation) is constant along its length.
**(3)** A vortex tube cannot end within the fluid. It must either end at a solid boundary or form a closed loop—a *vortex ring* or *loop*.
**(4)** The strength of a vortex tube remains constant in time.

Here, we only highlight the proof of the first theorem, which essentially says that fluid particles that at any time are part of a vortex line always belong to the same vortex line. To prove this result, consider an area $S$, bounded by a curve, lying on the surface of a vortex

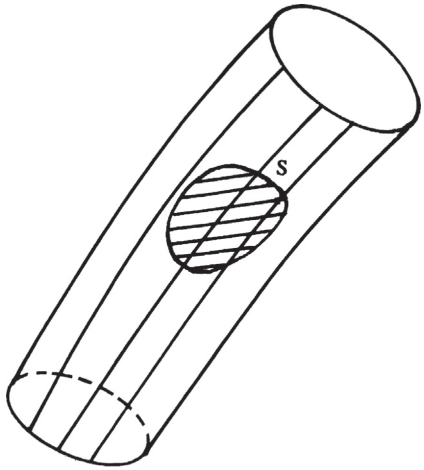


**FIGURE 5.7** Vortex tube and surface geometry for Helmholtz's first vortex theorem. The surface $S$ lies within a closed contour on the surface of a vortex tube.

tube without embracing it (Figure 5.7). Since the vorticity vectors are everywhere lying parallel to $S$ (none are normal to $S$), it follows that the circulation around the edge of $S$ is zero. After an interval of time, the same fluid particles form a new surface, $S'$. According to Kelvin's theorem, the circulation around $S'$ must also be zero. As this is true for any $S$, the component of vorticity normal to every element of $S'$ must vanish, demonstrating that $S'$ must lie on the surface of the vortex tube. Thus, vortex tubes move with the fluid, a result we will also be able to attain from the field equation for vorticity. Applying this result to an infinitesimally thin vortex tube, we get the Helmholtz vortex theorem that vortex lines move with the fluid. A different proof may be found in Sommerfeld (1964, pp. 130–132).

## 5.4. VORTICITY EQUATION IN A NONROTATING FRAME

An equation governing the vorticity in an inertial frame of reference is derived in this section. The *fluid density is assumed to be constant*, so that the flow is barotropic. Viscous effects are retained but the viscosity is assumed to be constant. Baroclinic effects and a rotating frame of reference are considered in Section 5.6. The derivation given here uses vector notation and several vector identities. In Section 5.6, the derivation is completed in tensor notation.

Vorticity $\boldsymbol{\omega}$ is the curl of the velocity, so, as previously noted, $\nabla\cdot\boldsymbol{\omega} = \nabla\cdot(\nabla\times\mathbf{u}) = 0$. An equation for the vorticity can be obtained from the curl of the momentum conservation equation (4.39b)

$$\nabla\times\left\{\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \nu\nabla^2\mathbf{u}\right\}. \tag{5.12}$$

When $\mathbf{g}$ is conservative and (4.18) applies, the curl of the first two terms on the right side of (5.12) will be zero because they are gradients of scalar functions. The acceleration term on the left side of (5.12) becomes:

$$\nabla \times \left\{ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right\} = \frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times \{(\mathbf{u} \cdot \nabla)\mathbf{u}\} = \frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times \{\nabla(\mathbf{u} \cdot \mathbf{u}) + \boldsymbol{\omega} \times \mathbf{u}\}$$
$$= \frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}),$$

so (5.12) reduces to

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = \nu \nabla^2 \boldsymbol{\omega},$$

where we have also used the identity $\nabla \times \nabla^2 \mathbf{u} = \nabla^2 (\nabla \times \mathbf{u})$ in rewriting the viscous term. The second term in the above equation can be written as

$$\nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u},$$

based on the vector identity (B.3.10), and the fact that $\nabla \cdot \mathbf{u} = 0$ and $\nabla \cdot \boldsymbol{\omega} = 0$. Thus, (5.12) becomes

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}. \tag{5.13}$$

This is the field equation governing vorticity in a fluid with constant $\rho$ and conservative body forces. The term $\nu \nabla^2 \boldsymbol{\omega}$ represents the rate of change of $\boldsymbol{\omega}$ caused by diffusion of vorticity in the same way that $\nu \nabla^2 \mathbf{u}$ represents acceleration caused by diffusion of momentum. The term $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$ represents the rate of change of vorticity caused by the stretching and tilting of vortex lines. This important mechanism of vorticity alteration is discussed further in Section 5.6. Note that pressure and gravity terms do not appear in (5.13) since these forces act through the center of mass of an element and therefore generate no torque. In addition, note that (5.13) might appear upon first glance to be a linear equation for $\boldsymbol{\omega}$. However, the vorticity is the curl of the velocity so both the advective part of the $D\boldsymbol{\omega}/Dt$ term and the $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$ term represent nonlinearities.

## 5.5. VELOCITY INDUCED BY A VORTEX FILAMENT: LAW OF BIOT AND SAVART

For a variety of applications in aero- and hydrodynamics, the flow induced by a concentrated distribution of vorticity (a vortex) with arbitrary orientation must be calculated. Here we consider the simple case of incompressible flow where $\nabla \cdot \mathbf{u} = 0$. Taking the curl of the vorticity produces:

$$\nabla \times \omega = \nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u} = -\nabla^2 \mathbf{u},$$

where the second equality follows from an identity of vector calculus (B.3.13). The two ends of this extended equality form a Poisson equation, and its solution is the vorticity-induced portion of the fluid velocity:

$$\mathbf{u}(\mathbf{x}, t) = -\frac{1}{4\pi} \int_{V'} \frac{1}{|\mathbf{x} - \mathbf{x}'|} (\nabla' \times \omega(\mathbf{x}', t)) d^3 x', \tag{5.14}$$

where $V'$ encloses the vorticity of interest and $\nabla'$ operates on the $\mathbf{x}'$ coordinates (see Exercise 5.8). This result can be further simplified by rewriting the integrand in (5.14):

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}(\nabla' \times \omega(\mathbf{x}',t)) = \nabla' \times \left(\frac{\omega(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\right) - \nabla'\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) \times \omega(\mathbf{x}',t)$$
$$= \nabla' \times \left(\frac{\omega(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\right) + \left(\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right) \times \omega(\mathbf{x}',t),$$

to obtain:

$$\mathbf{u}(\mathbf{x},t) = -\frac{1}{4\pi}\int_{V'} \nabla' \times \left(\frac{\omega(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\right) d^3x' + \frac{1}{4\pi}\int_{V'} \frac{\omega(\mathbf{x}',t) \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'.$$

Here the first integral is zero when $V'$ is chosen to capture a segment of the vortex, but it takes several steps to deduce this. First, rewrite the curl operation in index notation and apply Gauss' divergence theorem:

$$\int_{V'} \nabla' \times \left(\frac{\omega(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\right) d^3x' = \int_{V'} \varepsilon_{kij}\frac{\partial}{\partial x_i'}\left(\frac{\omega_j(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\right) d^3x' = \int_{A'} \varepsilon_{kij}\left(\frac{\omega_j(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\right) n_i d^2x'$$
$$= \int_{A'} \frac{\mathbf{n} \times \omega(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} d^2x', \tag{5.15}$$

where $A'$ is the surface of $V'$ and $\mathbf{n}$ is the outward normal on $A'$. Now choose $V'$ to be a volume aligned so that its end surfaces are locally normal to $\boldsymbol{\omega}(\mathbf{x}',t)$ while its curved lateral surface lies outside the concentration of vorticity as shown in Figure 5.8. For this volume, the final integral in (5.15) is zero because $\mathbf{n} \times \boldsymbol{\omega} = 0$ on its end surfaces since $\boldsymbol{\omega}(\mathbf{x}',t)$ and $\mathbf{n}$ are parallel there, and because $\boldsymbol{\omega}(\mathbf{x}',t) = 0$ on its lateral surface. Thus, (5.14) reduces to:

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{4\pi}\int_{V'} \frac{\boldsymbol{\omega}(\mathbf{x}',t) \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'. \tag{5.16}$$

If an elemental vortex segment of length $dl$ is considered so that $V' = \Delta A' dl$, and the observation location, $\mathbf{x}$, is sufficiently distant from the vorticity concentration location $\mathbf{x}'$ so that $(\mathbf{x}-\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|^3$ is effectively constant over the vorticity concentration, then (5.16) may be simplified to:

$$d\mathbf{u}(\mathbf{x},t) \cong \frac{1}{4\pi}\int_{\Delta A'} |\boldsymbol{\omega}(\mathbf{x}',t)|\,\mathbf{e}_\omega d^2x' \times \frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} dl = \frac{\Gamma dl}{4\pi}\mathbf{e}_\omega \times \frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}, \tag{5.17}$$

where $d\mathbf{u}$ is the velocity induced by the vortex segment, and $\Gamma$ and $\mathbf{e}_\omega$ are the strength and direction of the vortex segment at $\mathbf{x}'$, respectively. This is an expression of the Biot-Savart vortex induction law.

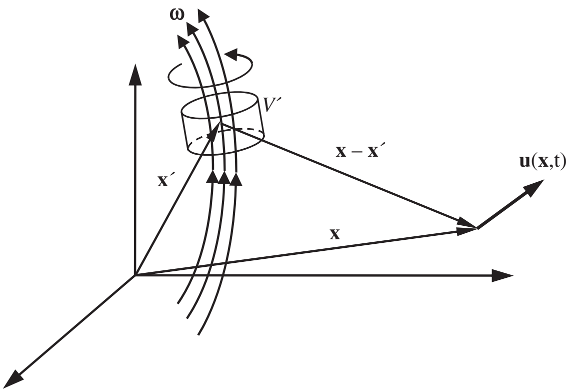


**FIGURE 5.8** Geometry for derivation of Law of Biot and Savart. The location of the vorticity concentration or vortex is $\mathbf{x}'$. The location of the vortex-induced velocity $\mathbf{u}$ is $\mathbf{x}$. The volume $V'$ contains a segment of the vortex. Its flat ends are perpendicular to the vorticity in the vortex, while its curved lateral sides lie outside the vortex.

## 5.6. VORTICITY EQUATION IN A ROTATING FRAME

A vorticity equation was derived in Section 5.4 for a fluid of uniform density observed from an inertial frame of reference. Here, this equation is generalized to a rotating frame of reference and a nonbarotropic fluid. The flow, however, will be assumed nearly incompressible in the Boussinesq sense, so that the continuity equation is approximately $\nabla \cdot \mathbf{u} = 0$. And, for conciseness, the comma notation for spatial derivatives (Section 2.14) is adopted.

The first step is to show that $\nabla \cdot \boldsymbol{\omega} = \omega_{i,i}$ is zero. From the definition $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, we obtain

$$\omega_{i,i} = (\varepsilon_{inq} u_{q,n})_{,i} = \varepsilon_{inq} u_{q,ni}.$$

In the last term, $\varepsilon_{inq}$ is antisymmetric in $i$ and $n$, whereas the derivative $u_{q,ni}$ is symmetric in $i$ and $n$. As the contracted product of a symmetric and an antisymmetric tensor is zero, it follows that

$$\omega_{i,i} = 0 \quad \text{or} \quad \nabla \cdot \boldsymbol{\omega} = 0 \, . \tag{5.18}$$

Hence, the vorticity field is divergence free (solenoidal), even for compressible and unsteady flows.

The continuity and momentum equations for a nearly incompressible flow in a steadily rotating coordinate system are

$$u_{i,i} = 0, \text{ and } \frac{\partial u_i}{\partial t} + u_j u_{i,j} + 2\varepsilon_{ijk}\Omega_j u_k = -\frac{1}{\rho} p_{,i} + g_i + \nu u_{i,jj}, \tag{5.19, 5.20}$$

where $\boldsymbol{\Omega}$ is the angular velocity of the coordinate system and $g_i$ is the effective gravity (including centrifugal acceleration); see Section 4.7. The advective acceleration can be written as

$$u_j u_{i,j} = u_j(u_{i,j} - u_{j,i}) + u_j u_{j,i} = -u_j \varepsilon_{ijk}\omega_k + \frac{1}{2}(u_j u_j)_{,i} = -(\mathbf{u}\times\boldsymbol{\omega})_i + \frac{1}{2}(u_j^2)_{,i}, \tag{5.21}$$

where we have used the relation

$$\varepsilon_{ijk}\omega_k = \varepsilon_{ijk}\varepsilon_{kmn}u_{n,m} = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})u_{n,m} = u_{j,i} - u_{i,j}. \tag{5.22}$$

The viscous diffusion term can be written as

$$\nu u_{i,jj} = \nu(u_{i,j} - u_{j,i})_{,j} + \nu u_{j,ij} = -\nu\varepsilon_{ijk}\omega_{k,j}, \tag{5.23}$$

where we have used (5.22) and the fact that $u_{j,ij} = 0$ because of (5.19). Equation (5.23) says that $\nu\nabla^2\mathbf{u} = -\nu\nabla\times\boldsymbol{\omega}$, which we have used several times before (e.g., see (4.40)). Because $\boldsymbol{\Omega}\times\mathbf{u} = -\mathbf{u}\times\boldsymbol{\Omega}$, the Coriolis acceleration term in (5.20) can be rewritten

$$2\varepsilon_{ijk}\Omega_j u_k = -2\varepsilon_{ijk}\Omega_k u_j. \tag{5.24}$$

Substituting (5.21), (5.23), and (5.24) into (5.20), we obtain

$$\partial u_i/\partial t + \left(\frac{1}{2}u_j^2 + \Phi\right)_{,i} - \varepsilon_{ijk}u_j(\omega_k + 2\Omega_k) = -(1/\rho)p_{,i} - \nu\varepsilon_{ijk}\omega_{k,j}, \tag{5.25}$$

where we have also set $\mathbf{g} = -\nabla\Phi$; see (4.18).

Equation (5.25) is another form of the Navier-Stokes momentum equation, so the rotating-frame-of-reference vorticity equation is obtained by taking its curl. Since $\omega_n = \varepsilon_{nqi}u_{i,q}$, we need to operate on (5.25) by $\varepsilon_{nqi}(\ )_{,q}$ which produces:

$$\frac{\partial}{\partial t}\left(\varepsilon_{nqi}u_{i,q}\right) + \varepsilon_{nqi}\left(\frac{1}{2}u_j^2 + \Phi\right)_{,iq} - \varepsilon_{nqi}\varepsilon_{ijk}\left[u_j(\omega_k + 2\Omega_k)\right]_{,q} = -\varepsilon_{nqi}\left(\frac{1}{\rho}p_{,i}\right)_{,q} - \nu\varepsilon_{nqi}\varepsilon_{ijk}\omega_{k,jq}. \tag{5.26}$$

The second term on the left side vanishes on noticing that $\varepsilon_{nqi}$ is antisymmetric in $q$ and $i$, whereas the derivative $(u_j^2/2 + \Pi)_{,iq}$ is symmetric in $q$ and $i$. The third term on the left side of (5.26) can be written as

$$\begin{aligned}
-\varepsilon_{nqi}\varepsilon_{ijk}\left[u_j\left(\omega_k + 2\Omega_k\right)\right]_{,q} &= -(\delta_{nj}\delta_{qk} - \delta_{nk}\delta_{qj})\left[u_j(\omega_k + 2\Omega_k)\right]_{,q} \\
&= -\left[u_n(\omega_k + 2\Omega_k)\right]_{,k} + \left[u_j(\omega_n + 2\Omega_n)\right]_{,j} \\
&= -u_n(\omega_{k,k} + 2\Omega_{k,k}) - u_{n,k}(\omega_k + 2\Omega_k) + u_j(\omega_n + 2\Omega_n)_{,j} \\
&= -u_n(0+0) - u_{n,k}(\omega_k + 2\Omega_k) + u_j(\omega_n + 2\Omega_n)_{,j} \\
&= -u_{n,j}(\omega_j + 2\Omega_j) + u_j\,\omega_{n,j},
\end{aligned} \tag{5.27}$$

where we have used $u_{i,i} = 0$, $\omega_{i,i} = 0$ and the fact that the derivatives of $\Omega$ are zero.

The first term on the right-hand side of (5.26) can be written as

$$-\varepsilon_{nqi}\left(\frac{1}{\rho}p_{,i}\right)_{,q} = -\frac{1}{\rho}\varepsilon_{nqi}\,p_{,iq} + \frac{1}{\rho^2}\varepsilon_{nqi}\rho_{,q}p_{,i}$$
$$= 0 + \frac{1}{\rho^2}[\nabla\rho \times \nabla p]_n, \tag{5.28}$$

which involves the $n$-component of the vector $\nabla\rho \times \nabla p$. The viscous term in (5.26) can be written as

$$-\nu\varepsilon_{nqi}\varepsilon_{ijk}\omega_{k,jq} = -\nu(\delta_{nj}\delta_{qk} - \delta_{nk}\delta_{qj})\omega_{k,jq}$$
$$= -\nu\omega_{k,nk} + \nu\omega_{n,jj} = \nu\omega_{n,jj}. \tag{5.29}$$

If we use (5.27) through (5.29), then (5.26) becomes

$$\frac{\partial\omega_n}{\partial t} = u_{n,j}(\omega_j + 2\Omega_j) - u_j\omega_{n,j} + \frac{1}{\rho^2}[\nabla\rho \times \nabla p]_n + \nu\omega_{n,jj}.$$

Changing the free index from $n$ to $i$ produces

$$\frac{D\omega_i}{Dt} = (\omega_j + 2\Omega_j)u_{i,j} + \frac{1}{\rho^2}[\nabla\rho \times \nabla p]_i + \nu\omega_{i,jj}.$$

In vector notation this can be written:

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} + 2\Omega)\cdot\nabla\mathbf{u} + \frac{1}{\rho^2}\nabla\rho \times \nabla p + \nu\nabla^2\boldsymbol{\omega}. \tag{5.30}$$

This is the *vorticity equation* for a nearly incompressible (i.e., Boussinesq) fluid observed from a frame of reference rotating at a constant rate $\boldsymbol{\Omega}$. Here $\mathbf{u}$ and $\boldsymbol{\omega}$ are, respectively, the velocity and vorticity observed in this rotating frame of reference. As vorticity is defined as twice the angular velocity, $2\boldsymbol{\Omega}$ is the *planetary vorticity* and $(\boldsymbol{\omega} + 2\boldsymbol{\Omega})$ is the *absolute vorticity* of the fluid, measured in an inertial frame. In a nonrotating frame, the vorticity equation is obtained from (5.30) by setting $\boldsymbol{\Omega}$ to zero and interpreting $\mathbf{u}$ and $\boldsymbol{\omega}$ as the absolute velocity and vorticity, respectively.

The left side of (5.30) represents the rate of change of vorticity following a fluid particle. The last term $\nu\nabla^2\boldsymbol{\omega}$ represents the rate of change of $\boldsymbol{\omega}$ due to molecular diffusion of vorticity, in the same way that $\nu\nabla^2\mathbf{u}$ represents acceleration due to diffusion of velocity. The second term on the right-hand side is the rate of generation of vorticity due to baroclinicity of the flow, as discussed in Section 5.2. In a barotropic flow, density is a function of pressure alone, so $\nabla\rho$ and $\nabla p$ are parallel vectors. The first term on the right side of (5.30) represents vortex stretching and plays a crucial role in the dynamics of vorticity even when $\boldsymbol{\Omega} = 0$.

To better understand the vortex-stretching term, consider the natural coordinate system where $s$ is the arc length along a vortex line, $n$ points away from the center of vortex-line curvature, and $m$ lies along the second normal to $s$ (Figure 5.9). Then,

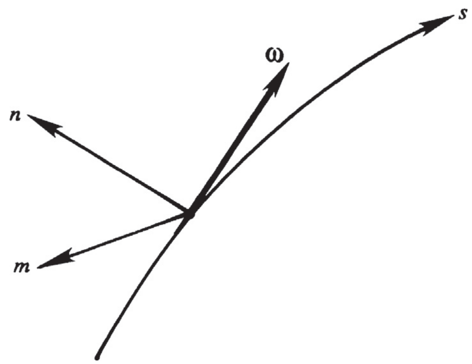


**FIGURE 5.9** Coordinate system aligned with the vorticity vector.

$$(\omega\cdot\nabla)\mathbf{u} = \left[\omega\cdot\left(\mathbf{e}_s\frac{\partial}{\partial s}+\mathbf{e}_n\frac{\partial}{\partial n}+\mathbf{e}_m\frac{\partial}{\partial m}\right)\right]\mathbf{u} = \omega\frac{\partial \mathbf{u}}{\partial s}, \tag{5.31}$$

where we have used $\boldsymbol{\omega}\cdot\mathbf{e}_n = \boldsymbol{\omega}\cdot\mathbf{e}_m = 0$, and $\boldsymbol{\omega}\cdot\mathbf{e}_s = \omega = |\boldsymbol{\omega}|$. Equation (5.31) shows that $(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}$ equals the magnitude of $\boldsymbol{\omega}$ times the derivative of $\mathbf{u}$ in the direction of $\boldsymbol{\omega}$. The quantity $\omega(\partial\mathbf{u}/\partial s)$ is a vector and has the components $\omega(\partial u_s/\partial s)$, $\omega(\partial u_n/\partial s)$, and $\omega(\partial u_m/\partial s)$. Among these, $\partial u_s/\partial s$ represents the increase of $u_s$ along the vortex line $s$, that is, the stretching of vortex lines. On the other hand, $\partial u_n/\partial s$ and $\partial u_m/\partial s$ represent the change of the normal velocity components along $s$ and, therefore, the rate of turning or tilting of vortex lines about the $m$ and $n$ axes, respectively.

To see the effect of these terms more clearly, write out the components of (5.30) for barotropic inviscid flow observed in an inertial frame of reference:

$$\frac{D\omega_s}{Dt} = \omega\frac{\partial u_s}{\partial s},\quad \frac{D\omega_n}{Dt} = \omega\frac{\partial u_n}{\partial s},\quad\text{and}\quad \frac{D\omega_m}{Dt} = \omega\frac{\partial u_m}{\partial s}. \tag{5.32}$$

The first equation of (5.32) shows that the vorticity along $s$ changes due to stretching of vortex lines, reflecting the principle of conservation of angular momentum. Stretching decreases the moment of inertia of fluid elements that constitute a vortex line, resulting in an increase of their angular rotation speed. Vortex stretching plays an especially crucial role in the dynamics of turbulent and geophysical flows. The second and third equations of (5.32) show how vorticity along $n$ and $m$ is created by the tilting of vortex lines. For example, in Figure 5.9, the turning of the vorticity vector $\boldsymbol{\omega}$ toward the $n$-axis will generate a vorticity component along $n$. *The vortex stretching and tilting term* $(\boldsymbol{\omega}\cdot\nabla)\,\mathbf{u}$ *is absent in two-dimensional flows, in which* $\boldsymbol{\omega}$ *is perpendicular to the plane of flow.*

To better understand how frame rotation influences vorticity, consider $\boldsymbol{\Omega} = \Omega\mathbf{e}_z$ so that $2(\boldsymbol{\Omega}\cdot\nabla)\mathbf{u} = 2\Omega(\partial\mathbf{u}/\partial z)$ and suppress all other terms on the right side of (5.30) to obtain the component equations:

$$\frac{D\omega_z}{Dt} = 2\Omega\frac{\partial w}{\partial z},\quad \frac{D\omega_x}{Dt} = 2\Omega\frac{\partial u}{\partial z},\quad\text{and}\quad \frac{D\omega_y}{Dt} = 2\Omega\frac{\partial v}{\partial z}.$$

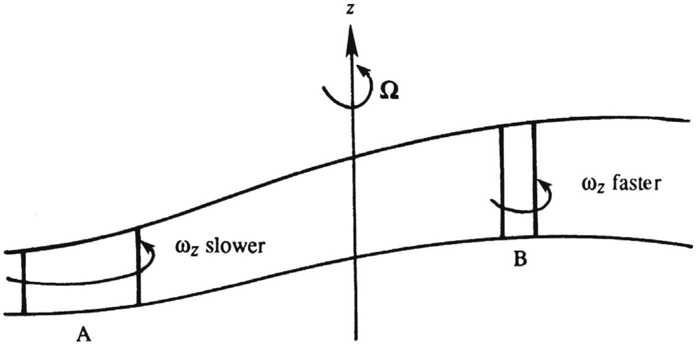


**FIGURE 5.10** Generation of relative vorticity due to stretching of fluid columns parallel to the planetary vorticity $2\boldsymbol{\Omega}$. A fluid column acquires $\boldsymbol{\omega}_z$ (in the same sense as $\boldsymbol{\Omega}$) by moving from location A to location B.

This shows that stretching of fluid lines in the $z$ direction increases $\omega_z$, whereas a tilting of vertical lines changes the relative vorticity along the $x$ and $y$ directions. Note that merely stretching or turning of vertical *fluid lines* is required for this mechanism to operate, in contrast to $(\boldsymbol{\omega} \cdot \nabla)\,\mathbf{u}$ where a stretching or turning of *vortex lines* is needed. This is because vertical fluid lines contain the planetary vorticity $2\boldsymbol{\Omega}$. A vertically stretching fluid column tends to acquire positive $\omega_z$, and a vertically shrinking fluid column tends to acquire negative $\omega_z$ (Figure 5.10). For this reason large-scale geophysical flows are almost always full of vorticity, and the change of $\boldsymbol{\omega}$ due to the presence of planetary vorticity $2\boldsymbol{\Omega}$ is a central feature of geophysical fluid dynamics.

Kelvin's circulation theorem for inviscid flow in a rotating frame of reference is modified to

$$\frac{D\Gamma_a}{Dt} = 0 \quad \text{where} \quad \Gamma_a \equiv \int_A (\boldsymbol{\omega} + 2\boldsymbol{\Omega})\cdot\mathbf{n}dA = \Gamma + 2\int_A \boldsymbol{\Omega}\cdot\mathbf{n}dA \tag{5.33}$$

(see Exercise 5.11). Here, $\Gamma_a$ is circulation due to the absolute vorticity $(\boldsymbol{\omega} + 2\boldsymbol{\Omega})$ and differs from $\Gamma$ by the amount of planetary vorticity intersected by the area $A$.

## 5.7. INTERACTION OF VORTICES

Vortices placed close to one another can mutually interact through their induced velocities and generate interesting motions. To examine such interactions, consider ideal concentrated-line vortices. A real vortex, with a core within which vorticity is distributed, can be idealized by a concentrated vortex line with circulation equal to the average vorticity in the core times the core area. Motion outside the vortex core is assumed irrotational, and therefore inviscid. It will be shown in the next chapter that irrotational motion of a constant density fluid is governed by the linear Laplace equation so the principle of superposition applies, and the velocity at a point can be obtained by adding the contribution of all vortices in the field. To determine the mutual interaction of line vortices, the important principle to keep in mind is the first Helmholtz vortex theorem—vortex lines move with the flow.

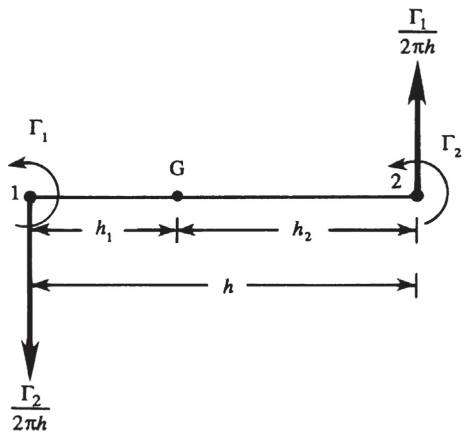


**FIGURE 5.11** Interaction of two line vortices of the same sign. Here the induced velocities are in opposite directions and perpendicular to the line connecting the vortices. Thus, if free to move, the two vortices will travel on circular paths centered on the point G where the combined velocity induced by the two vortices is zero.

Consider the interaction of two ideal line vortices of strengths $\Gamma_1$ and $\Gamma_2$, where both $\Gamma_1$ and $\Gamma_2$ are positive (i.e., counterclockwise vorticity). Let $h = h_1 + h_2$ be the distance between the vortices (Figure 5.11). Then the velocity at point 2 due to vortex $\Gamma_1$ is directed upward and equals

$$V_1 = \Gamma_1/2\pi h.$$

Similarly, the velocity at point 1 due to vortex $\Gamma_2$ is downward and equals

$$V_2 = \Gamma_2/2\pi h.$$

The vortex pair therefore rotates counterclockwise around this center of vorticity G, which remains stationary.

Now suppose that the two vortices have the same circulation of magnitude $\Gamma$, but an opposite sense of rotation (Figure 5.12). Then the velocity of each vortex at the location of the other is $\Gamma/(2\pi h)$ so the dual-vortex system translates at a speed $\Gamma/(2\pi h)$ relative to the fluid. A pair of counter-rotating vortices can be set up by stroking the paddle of a boat, or by briefly moving the blade of a knife in a bucket of water (Figure 5.13). After the paddle or knife is withdrawn, the vortices do not remain stationary but continue to move.

The behavior of a single vortex near a wall can be found by superposing two vortices of equal and opposite strength. The technique involved is called the *method of images* and has wide application in irrotational flow, heat conduction, acoustics, and electromagnetism. It is clear that the inviscid flow pattern due to vortex A at distance $h$ from a wall can be obtained by eliminating the wall and introducing instead a vortex of equal and opposite strength at the image point B (Figure 5.14). The velocity at any point P on the wall, made up of $V_A$ due to the real vortex and $V_B$ due to the image vortex, is parallel to the wall. The wall is therefore a streamline, and the inviscid boundary condition of zero normal velocity across a solid wall is satisfied. Because of the flow induced by the image vortex, vortex A moves with speed

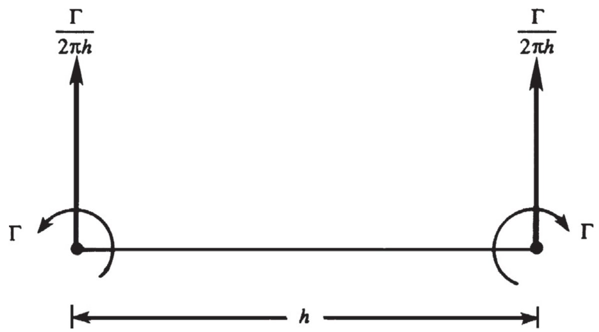


**FIGURE 5.12** Interaction of line vortices of opposite spin, but of the same magnitude. Here $\Gamma$ refers to the *magnitude* of circulation, and the induced velocities are in same direction and perpendicular to the line connecting the vortices. Thus, if free to move, the two vortices will travel along straight lines in the direction shown at speed $\Gamma/2\pi h$.

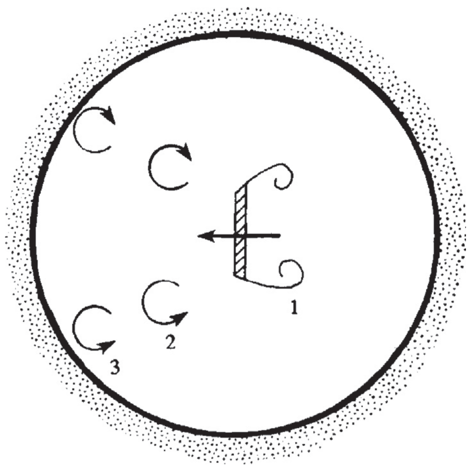


**FIGURE 5.13** Top view of a vortex pair generated by moving the blade of a knife in a bucket of water. Positions at three instances of time 1, 2, and 3 are shown. *(After Lighthill, 1986.)*

$\Gamma/(4\pi h)$ parallel to the wall. For this reason, vortices in the example of Figure 5.13 move apart along the boundary on reaching the side of the vessel.

Now consider the interaction of two doughnut-shaped vortex rings (such as smoke rings) of equal and opposite circulation (Figure 5.15a). According to the method of images, the flow field for a single ring near a wall is identical to the flow of two rings of opposite circulations. The translational motion of each element of the ring is caused by the induced velocity from each element of the same ring, plus the induced velocity from each element of the other vortex ring. In the figure, the motion at A is the resultant of $V_B$, $V_C$, and $V_D$, and this resultant has components parallel to and toward the wall. Consequently, the vortex ring increases in diameter and moves toward the wall with a speed that decreases monotonically (Figure 5.15b).

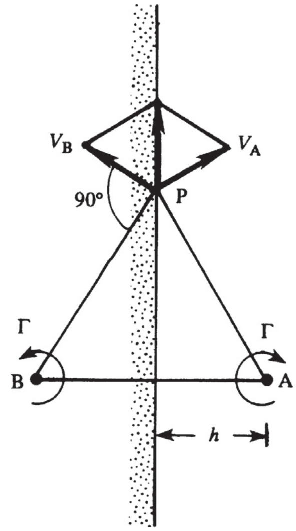


**FIGURE 5.14** Line vortex A near a wall and its image B. The sum of the induced velocities is parallel to the wall at all points P on the wall when the two vortices have equal and opposite strengths and they are equidistant from the wall.

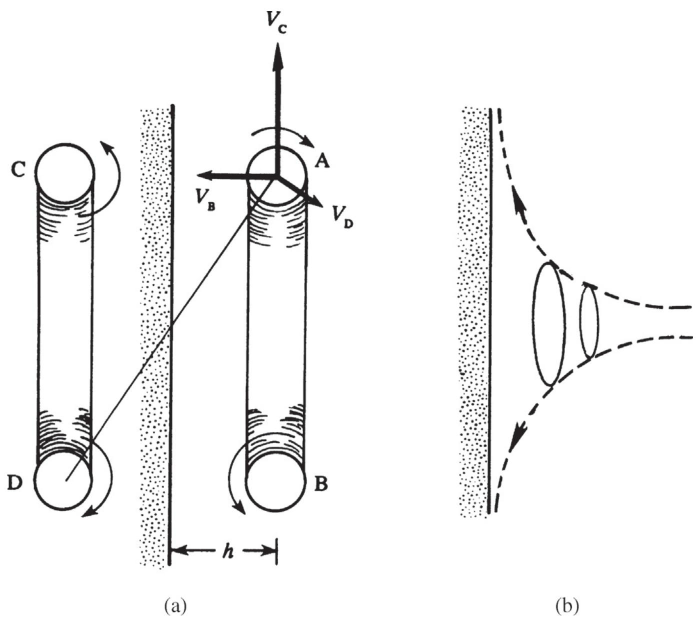


**FIGURE 5.15** (a) Torus or doughnut-shaped vortex ring near a wall and its image. A section through the middle of the ring is shown along with primary induced velocities at A from the vortex segments located at B, C, and D. (b) Trajectory of a vortex ring, showing that it widens while its translational velocity toward the wall decreases.

Finally, consider the interaction of two vortex rings of equal magnitude and similar sense of rotation. It is left to the reader (Exercise 5.15) to show that they should both translate in the same direction, but the one in front increases in radius and therefore slows down in its translational speed, while the rear vortex contracts and translates faster. This continues until the smaller ring passes through the larger one, at which point the roles of the two vortices are reversed. The two vortices can pass through each other forever in an ideal fluid. Further discussion of this intriguing problem can be found in Sommerfeld (1964, p. 161).

## 5.8. VORTEX SHEET

Consider an infinite number of ideal line vortices, placed side by side on a surface AB (Figure 5.16). Such a surface is called a *vortex sheet*. If the vortex filaments all rotate clockwise, then the tangential velocity immediately above AB is to the right, while that immediately below AB is to the left. Thus, a discontinuity of tangential velocity exists across a vortex sheet. If the vortex filaments are not infinitesimally thin, then the vortex sheet has a finite thickness, and the velocity change is spread out.

In Figure 5.16, consider the circulation around a circuit of dimensions $dn$ and $ds$. The normal velocity component $v$ is continuous across the sheet ($v = 0$ if the sheet does not move normal to itself), while the tangential component $u$ experiences a sudden jump. If $u_1$ and $u_2$ are the tangential velocities on the two sides, then

$$d\Gamma = u_2\, ds + v\, dn - u_1\, ds - v\, dn = (u_2 - u_1)\, ds.$$

Therefore the circulation per unit length, called the *strength of a vortex sheet*, equals the jump in tangential velocity:

$$\Gamma \equiv \frac{d\Gamma}{ds} = u_2 - u_1.$$

The concept of a vortex sheet is especially useful in discussing the flow over aircraft wings (Chapter 14).

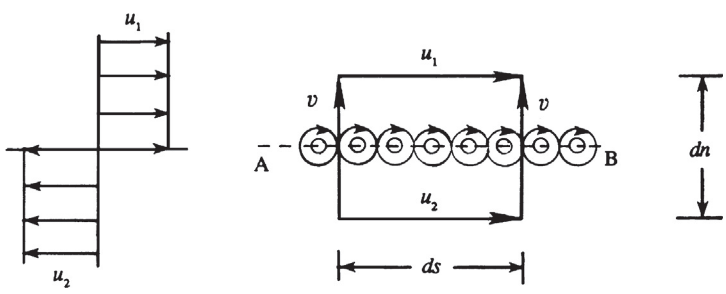


**FIGURE 5.16** A vortex sheet produces a change in the velocity that is tangent to it. Vortex sheets may be formed by placing many parallel ideal line vortices next to each other. The strength of a vortex sheet, $d\Gamma/ds = u_1 - u_2$, can be determined by computing the circulation on the rectangular contour shown and this strength may depend on the sheet-tangent coordinate.

## EXERCISES

**5.1.** A closed cylindrical tank 4 m high and 2 m in diameter contains water to a depth of 3 m. When the cylinder is rotated at a constant angular velocity of 40 rad/s, show that nearly 0.71 $m^2$ of the bottom surface of the tank is uncovered. [*Hint*: The free surface is in the form of a paraboloid. For a point on the free surface $p = p_o$, let $h$ be the height above the (imaginary) vertex of the paraboloid and $r$ be the local radius of the paraboloid. From Section 5.1 we have $h = \omega_0^2 r^2/2g$, where $\omega_0$ is the angular velocity of the tank. Apply this equation to the two points where the paraboloid cuts the top and bottom surfaces of the tank.]

**5.2.** A tornado can be idealized as a Rankine vortex with a core of diameter 30 m. The gauge pressure at a radius of 15 m is $-2000$ N/$m^2$ (i.e., the absolute pressure is 2000 N/$m^2$ below atmospheric).

**(a)** Show that the circulation around any circuit surrounding the core is 5485 $m^2$/s. [*Hint*: Apply the Bernoulli equation between infinity and the edge of the core.]

**(b)** Such a tornado is moving at a linear speed of 25 m/s relative to the ground. Find the time required for the gauge pressure to drop from $-500$ to $-2000$ N/$m^2$. Neglect compressibility effects and assume an air temperature of 25°C. (Note that the tornado causes a sudden decrease of the local atmospheric pressure. The damage to structures is often caused by the resulting excess pressure on the inside of the walls, which can cause a house to explode.)

**5.3.** The velocity field of a flow in cylindrical coordinates $(R, \varphi, z)$ is $\mathbf{u} = (u_R, u_\varphi, u_z) = (0, aRz, 0)$ where $a$ is a constant.

**(a)** Show that the vorticity components are $\boldsymbol{\omega} = (\omega_R, \omega_\varphi, \omega_z) = (-aR, 0, 2az)$.

**(b)** Verify that $\nabla \cdot \boldsymbol{\omega} = 0$.

**(c)** Sketch the streamlines and vortex lines in an $Rz$ plane. Show that the vortex lines are given by $zR^2 =$ constant.

**5.4.** Starting from the flow field of an ideal vortex (5.2), compute the viscous stresses $\sigma_{rr}$, $\sigma_{r\theta}$, and $\sigma_{\theta\theta}$, and show that the net viscous force on a fluid element, $(\partial\sigma_{ij}/\partial x_i)$, is zero.

**5.5.** Consider the situation depicted in Figure 5.5. Use a Cartesian coordinate system with a horizontal $x$-axis that puts the barrier at $x = 0$, a vertical $y$-axis that puts the bottom of the container at $y = 0$ and the top of the container at $y = H$, and a $z$-axis that points out of the page. Show that, at the instant the barrier is removed, the rate of baroclinic vorticity production at the interface between the two fluids is:

$$\frac{D\omega_z}{Dt} = \frac{2(\rho_2 - \rho_1)g}{(\rho_2 + \rho_1)\delta},$$

where the thickness of the density transition layer just after barrier removal is $\delta \ll H$, and the density in this thin interface layer is assumed to be $(\rho_1 + \rho_2)/2$. If necessary, also assume that fluid pressures match at $y = H/2$ just after barrier removal, and that the width of the container into the page is $b$. State any additional assumptions that you make.

**5.6.** At $t = 0$ a constant-strength $z$-directed vortex sheet is created in an $x$-$z$ plane ($y = 0$) in an infinite pool of a fluid with kinematic viscosity $\nu$, that is, $\boldsymbol{\omega}(y,0) = \mathbf{e}_z\gamma\delta(y)$. The symmetry of the initial condition suggests that $\boldsymbol{\omega} = \omega_z\mathbf{e}_z$ and that $\omega_z$ will only depend on $y$ and $t$. Determine $\boldsymbol{\omega}(y,t)$ for $t > 0$ via the following steps.

**a)** Determine a dimensionless scaling law for $\omega_z$ in terms of $\gamma$, $\nu$, $y$, and $t$.

**b)** Simplify the general vorticity equation (5.13) to a linear field equation for $\omega_z$ for this situation.

**c)** Based on the fact that the field equation is linear, simplify the result of part a) by requiring $\omega_z$ to be proportional to $\gamma$, plug the simplified dimensionless scaling law into the equation determined for part b), and solve this equation to find the undetermined function to reach:

$$\omega_z(y,t) = \frac{\gamma}{2\sqrt{\pi \nu t}} \exp\left\{-\frac{y^2}{4\nu t}\right\}$$

**5.7.** [1]**a)** Starting from the continuity and Euler equations for an inviscid compressible fluid, $\partial\rho/\partial t + \nabla \cdot (\rho \mathbf{u}) = 0$ and $\rho(D\mathbf{u}/Dt) = -\nabla p + \rho \mathbf{g}$, derive the Vazsonyi equation:

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\right) \cdot \nabla u + \frac{1}{\rho^3} \nabla \rho \times \nabla p,$$

when the body force is conservative: $\mathbf{g} = -\nabla\Phi$. This equation shows that $\boldsymbol{\omega}/\rho$ in a compressible flow plays nearly the same dynamic role as $\boldsymbol{\omega}$ in an incompressible flow [see (5.30) with $\boldsymbol{\Omega} = 0$ and $\nu = 0$].

**b)** Show that the final term in the Vazsonyi equation may also be written: $(1/\rho)\nabla T \times \nabla s$.

**c)** Simplify the Vazsonyi equation for barotropic flow.

**5.8.** Starting from the unsteady momentum equation for a compressible fluid with constant viscosities:

$$\rho \frac{D\mathbf{u}}{Dt} + \nabla p = \rho \mathbf{g} + \mu \nabla^2 \mathbf{u} + \left(\mu_v + \frac{1}{3}\mu\right) \nabla(\nabla \cdot \mathbf{u}),$$

show that

$$\frac{\partial \mathbf{u}}{\partial t} + \omega \times \mathbf{u} = T\nabla s - \nabla\left(h + \frac{1}{2}|\mathbf{u}|^2 + \Phi\right) - \frac{\mu}{\rho}\nabla \times \omega + \frac{1}{\rho}\left(\mu_v + \frac{4}{3}\mu\right)\nabla(\nabla \cdot \mathbf{u})$$

where $T$ = temperature, $h$ = enthalpy per unit mass, $s$ = entropy per unit mass, and the body force is conservative: $\mathbf{g} = -\nabla\Phi$. This is the viscous Crocco-Vazsonyi equation. Simplify this equation for steady inviscid non-heat-conducting flow to find the Bernoulli equation (4.78), $h + \frac{1}{2}|\mathbf{u}|^2 + \Phi$ = constant along a streamline, which is valid when the flow is rotational and nonisothermal.

**5.9.** **a)** Solve $\nabla^2 G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$ for $G(\mathbf{x},\mathbf{x}')$ in a uniform, unbounded three-dimensional domain, where $\delta(\mathbf{x} - \mathbf{x}') = \delta(x - x')\delta(y - y')\delta(z - z')$ is the three-dimensional Dirac delta function.

**b)** Use the result of part a) to show that: $\phi(\mathbf{x}) = -\frac{1}{4\pi} \int_{all\ x'} \frac{q(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'$ is the solution of the Poisson equation $\nabla^2\phi(\mathbf{x}) = q(\mathbf{x})$ in a uniform, unbounded three-dimensional domain.

[1]Obtained from Professor Paul Dimotakis

**5.10.** Start with the equations of motion in the rotating steadily coordinates, and prove Kelvin's circulation theorem $\frac{D}{Dt}(\Gamma_a) = 0$ where $\Gamma_a = \int (\omega + 2\boldsymbol{\Omega})\cdot d\mathbf{A}$. Assume that $\boldsymbol{\Omega}$ is constant, the flow is inviscid and barotropic and that the body forces are conservative. Explain the result physically.

**5.11.** In $(R,\varphi,z)$ cylindrical coordinates, consider the radial velocity $u_R = -R^{-1}(\partial\psi/\partial z)$, and the axial velocity $u_z = R^{-1}(\partial\psi/\partial R)$ determined from the axisymmetric stream function $\psi(R,z) = \frac{Aa^4}{10}\left(\frac{R^2}{a^2}\right)\left(1 - \frac{R^2}{a^2} - \frac{z^2}{a^2}\right)$ where $A$ is a constant. This flow is known as *Hill's spherical vortex*.

**a)** For $R^2 + z^2 \leq a^2$, sketch the streamlines of this flow in a plane that contains the $z$-axis. What does $a$ represent?

**b)** Determine $\mathbf{u} = u_R(R,z)\mathbf{e}_R + u_z(R,z)\mathbf{e}_z$.

**c)** Given $\omega_\varphi = (\partial u_R/\partial z) - (\partial u_z/\partial R)$, show that $\boldsymbol{\omega} = AR\mathbf{e}_\varphi$ in this flow and that this vorticity field is a solution of the vorticity equation (5.13).

**d)** Does this flow include stretching of vortex lines?

**5.12.** In $(R,\varphi,z)$ cylindrical coordinates, consider the flow field $u_R = -\alpha R/2$, $u_\phi = 0$, and $u_z = \alpha z$.

**a)** Compute the strain rate components $S_{RR}$, $S_{zz}$, and $S_{Rz}$. What sign of $\alpha$ causes fluid elements to elongate in the $z$ direction? Is this flow incompressible?

**b)** Show that it is possible for a steady vortex (a Burgers' vortex) to exist in this flow field by adding $u_\varphi = (\Gamma/2\pi R)[1 - \exp(-\alpha R^2/4\nu)]$ to $u_R$ and $u_z$ from part a) and then determining a pressure field $p(R,z)$ that together with $\mathbf{u} = (u_R, u_\varphi, u_z)$ solves the Navier-Stokes momentum equation for a fluid with constant density $\rho$ and kinematic viscosity $\nu$.

**c)** Determine the vorticity in the Burgers' vortex flow of part b).

**d)** Explain how the vorticity distribution can be steady when $\alpha \neq 0$ and fluid elements are stretched or compressed.

**e)** Interpret what is happening in this flow when $\alpha > 0$ and when $\alpha < 0$.

**5.13.** An ideal line vortex parallel to the $z$-axis of strength $\Gamma$ intersects the $x$-$y$ plane at $x = 0$ and $y = h$. Two solid walls are located at $y = 0$ and $y = H > 0$. Use the method of images for the following.

**a)** Based on symmetry arguments, determine the horizontal velocity $u$ of the vortex when $h = H/2$.

**b)** Show that for $0 < h < H$ the horizontal velocity of the vortex is:

$$u(0,h) = \frac{\Gamma}{4\pi h}\left(1 - 2\sum_{n=1}^{\infty}\frac{1}{(nH/h)^2 - 1}\right),$$

and evaluate the sum when $h = H/2$ to verify your answer to part a).

**5.14.** The axis of an infinite solid circular cylinder with radius $a$ coincides with the $z$-axis. The cylinder is stationary and immersed in an incompressible inviscid fluid, and the net circulation around it is zero. An ideal line vortex parallel to the cylinder with circulation $\Gamma$ passes through the $x$-$y$ plane at $x = L > a$ and $y = 0$. Here two image vortices are needed to satisfy the boundary condition on the cylinder's surface. If one of these is located at $x = y = 0$ and has strength $\Gamma$, determine the strength and location of the second image vortex.

**5.15.** Consider the interaction of two vortex rings of equal strength and similar sense of rotation. Argue that they go through each other, as described near the end of Section 5.7.

**5.16.** A constant-density irrotational flow in a rectangular torus has a circulation $\Gamma$ and volumetric flow rate $Q$. The inner radius is $r_1$, the outer radius is $r_2$, and the height is $h$. Compute the total kinetic energy of this flow in terms of only $\rho$, $\Gamma$, and $Q$.

**5.17.** Consider a cylindrical tank of radius $R$ filled with a viscous fluid spinning steadily about its axis with constant angular velocity $\mathbf{\Omega}$. Assume that the flow is in a steady state.

**(a)** Find $\int_A \boldsymbol{\omega} \cdot dA$ where A is a horizontal plane surface through the fluid normal to the axis of rotation and bounded by the wall of the tank.

**(b)** The tank then stops spinning. Find again the value of $\int_A \omega \cdot d\mathrm{A}$.

**5.18.** In Figure 5.11, locate point G.

**5.19.** Consider two-dimensional steady flow in the $x$-$y$ plane outside of a long circular cylinder of radius $a$ that is centered on and rotating about the $z$-axis at a constant angular rate of $\Omega_z$. Show that the fluid velocity on the $x$-axis is $\mathbf{u}(x,0) = (\Omega_z a^2/x)\mathbf{e}_y$ for $x > a$ when the cylinder is replaced by:

**a)** A circular vortex sheet of radius $a$ with strength $\gamma = \Omega_z a$

**b)** A circular region of uniform vorticity $\boldsymbol{\omega} = 2\Omega_z \mathbf{e}_z$ with radius $a$.

**c)** Describe the flow for $x^2 + y^2 < a^2$ for parts a) and b).

**5.20.** An ideal line vortex in a half space filled with an inviscid constant-density fluid has circulation $\Gamma$, lies parallel to the $z$-axis, and passes through the $x$-$y$ plane at $x = 0$ and $y = h$. The plane defined by $y = 0$ is a solid surface.

**a)** Use the method of images to find $\mathbf{u}(x,y)$ for $y > 0$ and show that the fluid velocity on $y = 0$ is $\mathbf{u}(x,0) = \Gamma h \mathbf{e}_x/[\pi(x^2 + h^2)]$.

**b)** Show that $\mathbf{u}(0,y)$ is unchanged for $y > 0$ if the image vortex is replaced by a vortex sheet of strength $\gamma(x) = -u(x,0)\mathbf{e}_z$ on $y = 0$.

**c)** (If you have the patience) Repeat part b) for $\mathbf{u}(x,y)$ when $y > 0$.

## Literature Cited

Lighthill, M. J. (1986). *An Informal Introduction to Theoretical Fluid Mechanics*. Oxford, England: Clarendon Press.
Sommerfeld, A. (1964). *Mechanics of Deformable Bodies*. New York: Academic Press.

## Supplemental Reading

Batchelor, G. K. (1967). *An Introduction to Fluid Dynamics*. London: Cambridge University Press.

Pedlosky, J. (1987). *Geophysical Fluid Dynamics*. New York: Springer-Verlag. (This book discusses the vorticity dynamics in rotating coordinates, with application to geophysical systems.)

Prandtl, L., & Tietjens, O. G. (1934). *Fundamentals of Hydro- and Aeromechanics*. New York: Dover Publications. (This book contains a good discussion of the interaction of vortices.)

Saffman, P. G. (1992). *Vortex Dynamics*. Cambridge: Cambridge University Press. (This book presents a wide variety of vortex topics from an applied mathematics perspective.)