UNIT-2

BOOLEAN ALGEBRA

Boolian algebra is the branch of algebra in which the values of the variables are the truth values true and false, usually denoted by 1 and 0 respecievely.

Definition

A Boolian algebra is a set A with elements 0 and 1, two binary operations $\vee$ (+ / $\oplus$ / join) and $\wedge$ ($\cdot$ / * / meet) and a unary operation (') such that the following properties hold for all $a, b, c \in A$

1) Identity laws

$a+0=a$ (0 is the identity for '+')

$a\cdot 1=a$ (1 is the identity for '$\cdot$')

2) complement laws

$a+\bar{a}=1$

$a\cdot\bar{a}=0$

3) Associative law

$a+(b+c)=(a+b)+c$

$a\cdot(b\cdot c)=(a\cdot b)\cdot c$

4) commutative law

$a+b=b+a$

$a\cdot b=b\cdot a$

5) Disteributive law

$a+(b\cdot c)=(a+b)\cdot(a+c)$

$a\cdot(b+c)=(a\cdot b)+(a\cdot c)$

The general notation of a boolean algebra is $(B, +, \cdot, -, 0, 1)$

Note

1). The elements 0 and 1 are in B, 0 is the identity element for the operation '+'. Thus 0 is said to be the zero elements

2) 1 is the identity element for the operatⁿ '$\cdot$' - thus 1 is said to be unit element.

Question

1) let L be the set of all logical statement for statements p and q, The '$\vee$' is denoted by $p\vee q$ and $\wedge$ denoted by $p\vee q$ define the operation '+' and '$\cdot$' by $p+q=p\vee q$

$p\cdot q=p\wedge q$, $\bar{p}=\neg p$

S.T $(L, +, \cdot, -, c, t)$ is a boolean algebra where c is stt that never hold and t is the stt that inversly hold

Solⁿ :- 1) identity $p+0=p\vee c=p$

$p\cdot 1=p\wedge t=p$

2) complement

$p+\bar{p}=p\vee\neg p=1$ (T)

$p\cdot\bar{p}=p\wedge\neg p=0$ (F)

3) Associative law

$p+(q+r) = p \vee (q \vee r) = (p \vee q) \vee r = (p+q)+r$

$p \cdot (q \cdot r) = p \wedge (q \wedge r) = (p \wedge q) \wedge r = (p \cdot q) \cdot r$

4) commutative law

$p+q = p \vee q = q \vee p = q+p$

$p \cdot q = p \wedge q = q \wedge p = q \cdot p$

5) Distributive law

$p \cdot (q+r) = p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) = (p \cdot q) + (p \cdot r)$

$p+(q \cdot r) = p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r) = (p+q) \cdot (q+r)$

Hence $(L, +, \cdot, -, 0, 1)$ is a boolean algebra.

2) Let $B = \{0, 1\}$ be the bits (binary digits) with binary operation '+' and '·' and unary operation '−' as shown in the below table

+	1	0
1	1	1
0	1	0

·	1	0
1	1	0
0	0	0

−	1	0
	0	1

S.T $(B, +, \cdot, -, 0, 1)$ is a boolean algebra

sol^n:-

1) identity law $1+0=1$

$1 \cdot 1 = 1$

2) complement law

$1+0=1$ $((T)=0)$

$1\cdot 0=0$ $((T)=0)$

3) Associative law

$1+(1+0) = (1+1) = 1 = (1+1)+0 = 1+0 = 1$

$1\cdot(1\cdot 0) = 1\cdot 0 = 0 = (1\cdot 1)\cdot 0 = 1\cdot 0 = 0$

4) commutative law

$1+0=0+1=1$

$1\cdot 0 = 0\cdot 1 = 0$

5) Distributive law

$1+(1\cdot 0) = (1+1)\cdot(1+0) = 1\cdot 1 = 1$

$1\cdot(1+0) = (1\cdot 1)+(1\cdot 0) = 1+0 = 1$

$\therefore (B, +, \cdot, -, 0, 1)$ is a boolean algebra

properties of boolean algebra

$(B, +, \cdot, -, 0, 1)$ is a boolean algebra, Then the following properties hold

1) idempotant law

$a+a=a$

$a\cdot a=a \quad \forall\, a\in B$

proof :- $a+a=a$

$a = a+0$ (by identity law)

$= a+(a\cdot\bar{a})$ (by complement law)

$(a+a)\cdot(a+\bar{a})$ (by distributive law)

$(a+a)\cdot 1$ (by identity law)

$\therefore a = a+a$

proof :- $a\cdot a = a$

$a = a\cdot 1$ (by identity law)

$= a\cdot(a+\bar{a})$ (by complement law)

$= (a\cdot a)+(a\cdot\bar{a})$ (by distributive law)

$= (a\cdot a)+0$ (by complement law)

$= a\cdot a$ (by identity law)

$\therefore a = a\cdot a$

2) Dominance law

$a+1=1$

$a\cdot 0 = 0 \quad \forall\, a\in B$

proof :- $a+1 = (a+1)\cdot 1$ (by identity law)

$= (a+1)\cdot(a+\bar{a})$ (by complement law)

$= a+(1\cdot\bar{a})$ (by distributive law)

$= a+(\bar{a}\cdot 1)$ (by commutative)

$= a+\bar{a}$ (by identity)

$= 1$

$\therefore a+1=1$

proof: $a\cdot 0=0$

$a\cdot 0=(a\cdot 0)+0$ (by identity law)

$=(a\cdot 0)+(a\cdot\bar{a})$ (by complement law)

$=a\cdot(0+\bar{a})$ (by distributive)

$=a\cdot(\bar{a}+0)$ (by commutati)

$=a\cdot\bar{a}$ (by identity law)

$=0$ (by complement law)

$\therefore a\cdot 0=0$

3) Absorption law

- $a\cdot(a+b)=a$
- $a+(a\cdot b)=a \quad \forall\, a,b\in B$

proof :- $a\cdot(a+b)=(a+0)\cdot(a+b)$ (by identity law)

$=a+(0\cdot b)$ (by distributive law)

$=a+(0\cdot b)$ (by commutative law)

$=a+0$ (by dominance law)

$=a$ (by identity law)

$\therefore a\cdot(a+b)=a$

proof :- $a+(a\cdot b)=a$

$a+(a\cdot b)=(a\cdot 1)+(a\cdot b)$ identity

$=a\cdot(1+b)$ distributive

$=a\cdot(b+1)$ commutati

$=a\cdot 1$ dominance

$=a$ identity

$a+(a\cdot b)=a$

Note :- Let B be the non-empty set with two binary operations $\wedge$, $\vee$ and a unary operation $'$ we define operations '+', '.' by $a+b=a\vee b$, $a\cdot b=a\wedge b$, then,

1) Idempotent law

$a\vee a=a$

$a\wedge a=a \quad \forall\, a\in B$

2) Dominance law

$a\vee 1=1$

$a\wedge 0=0 \quad \forall\, a\in B$

3) Absorption law

$a\wedge(a\vee b)=a$

$a\vee(a\wedge b)=a \quad \forall\, a,b\in B$

4) Demorgans law.

$(a+b)' = a'.b'$ | $(a\vee b)' = a'\wedge b'$

$(a.b)' = a'+b'$ | $(a\wedge b)' = a'\vee b'$ $\forall a,b \in B$

proof :- $(a\vee b)' = a'\wedge b'$

$(a\vee b)\vee(a'\wedge b') = [(a\vee b)\vee a']\wedge[(a\vee b)\vee b']$

distributive

$= [(a\vee a')\vee b]\wedge[a\vee(b\vee b')]$

$= (1\vee b)\wedge(a\vee 1)$ (complement)

$= 1\wedge 1$ (by dominance)

$= 1$ (by identity)

$(a\vee b)\wedge(a'\wedge b') = [a\wedge(a'\wedge b')]\vee[b\wedge(a'\wedge b')]$

$= [(a\wedge a')\wedge b']\vee[(b\wedge b')\wedge a']$

complement

$= (0\wedge b')\vee(0\wedge a')$ dominance

$= 0\vee 0$

$= 0$ complement law

thus $a'\wedge b'$ is the complement of $a\vee b$

ie $(a\vee b)' = a'\wedge b'$

ie $(a\wedge b)' = a'\vee b'$

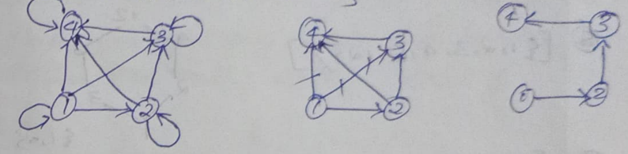
Hence the proof

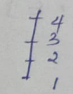
Hasse diagram / poset diagram

(partial order set diagram)

① $A = \{1,2,3,4\}$

$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$

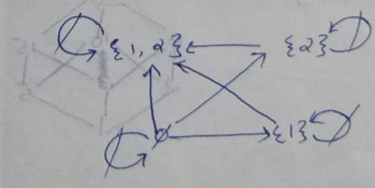


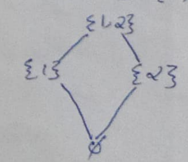


② $A = \{1,2\}$

$P(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$R = \{(\emptyset,\{1\}), (\emptyset,\{2\}), (\{1\},\{1\}), (\{1\},\{1,2\}), (\{2\},\{2\}), (\{2\},\{1,2\}), (\{1,2\},\{1,2\})\}$

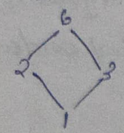




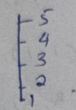
③ $A = \{1,2,3,6\}$

$R = \{(1,1),(1,2),(1,3),(1,6),(2,2),(2,6),(3,3),$

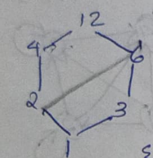
(3,6),(6,6)}



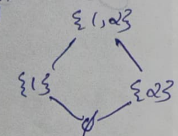
④ [{1,2,3,4,5}, ≤]



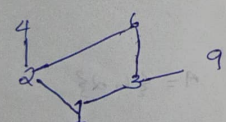
⑤ [{1,2,3,4,6,12}, /]



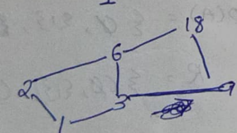
⑥ [{∅, {1}, {2}, {1,2}}, ⊆]



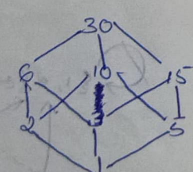
⑦ [{1,2,3,4,6,9}, /]



⑧ [{1,2,3,6,9,18}, /]



⑨ [{1,2,3,5,6,10,15,30}, /]

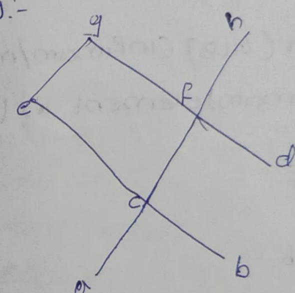


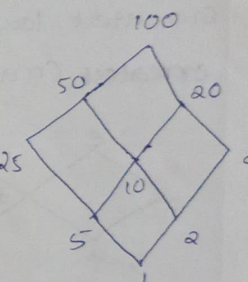
upper bound and lower bound

upper bound :- let B be a subset of a set A. An element $x \in A$ is upper bound of B, if $(y,x) \in$ poset $\forall y \in B$

Lower bound :- Let B be a subset of a set A, An element $x \in A$ is in lower bound of B, if $(x,y) \in$ poset $\forall y \in B$

Eg:-

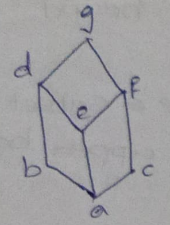




B = {5,10}
LB(B) = {1,5}
UB(B) = {50,100,20,10}

B = {5,10,2,4}
LB(B) = {1}
UB(B) = {20,100}

B = {e,c}	B = {c,f,d}
LB(B) = {a,b,c}	LB(B) = ∅
UB(B) = {g,e}	UB(B) = {g,h,f}



$B = \{d, g\}$

$LB(B) = \{a, b, e, d\}$

$UB(B) = \{g\}$

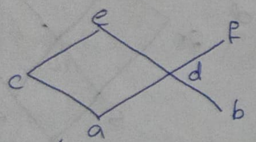
$B = \{e, f\}$

$LB(B) = \{a, e\}$

$UB(B) = \{g, f\}$

Least upper bound (LUB) (supremum) (Join/∨) :- least element in upper bound

Greatest lower (GLB) (infimum/meet/∧) :- Greatest (maximum) element in lower bound



$B = \{c, d\}$

$UB(B) = \{e\}$

$LUB(B) = \{e\}$

$LB(B) = \{a\}$

$GLB(B) = \{a\}$

$B = \{a, b\}$

$UB(B) = \{e, f, d\}$

$LUB(B) = \{d\}$

$LB(B) = \emptyset$

$GLB(B) = \emptyset$

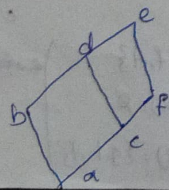
$B = \{e, f\}$

$UB(B) = \emptyset$

$LUB(B) = \emptyset$

$LB(B) = \{a, d, b\}$

$GLB(B) = \{d\}$



$B = \{a, c, f\}$

$UB(B) = \{e, f\}$

$LUB(B) = \{f\}$

$LB(B) = \{a\}$

$GLB(B) = \{a\}$

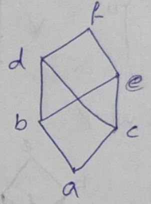
$B = \{d, c\}$

$UB(B) = \{d, e\}$

$LUB(B) = \{d\}$

$LB(B) = \{a, c\}$

$GLB(B) = \{c\}$



$B = \{d, e\}$

$UB(B) = \{f\}$

$LUB(B) = \{f\}$

$LB(B) = \{b, c, e\}$

$GLB(B) = \emptyset$

$B = \{b, c\}$

$UB(B) = \{d, e, f\}$

$LUB(B) = \emptyset$

$LB(B) = \{a\}$

$GLB(B) = \{a\}$

Upper bound and Lower bound

$A = \{a, b, c, d, e, f, g, h\}$

$R = \{(a,), (a, c), (a, e), (a, f), (a, g), (a, h), (b, b), (b, c), (b, e), (b, f), (b, g), (b, h), (c, c), (c, e), (c, g), (c, f), (c, h), (d, d), (d, f), (d, g), (d, h), (e, e), (e, g), (f, f), (f, h), (g, g), (h, h)\}$

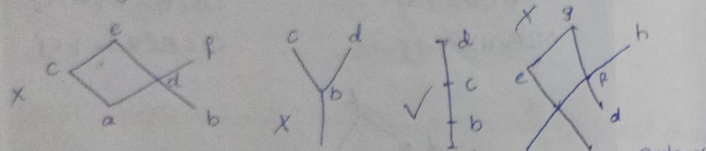
$B = \{e, c\}$

$LB(B) = \{a, b, c\}$

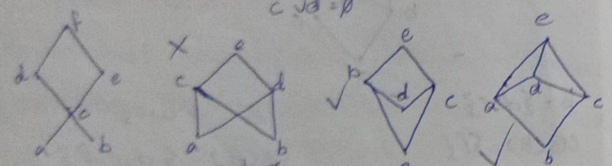
$UB(B) = \{g, e\}$

Join Semi lattice single top

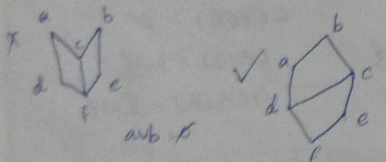
In a poset if LUB / Join / supremum / $\vee$ exist for every pair of elements, then poset is called Join-semi lattice



$e \vee f = \phi$ $\quad c \vee d = \phi$ $\quad g \vee h = \phi$



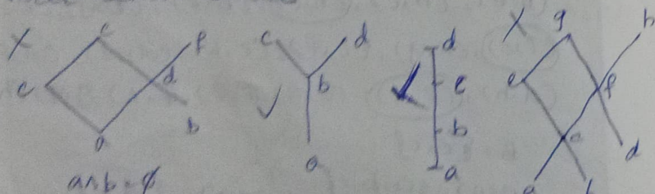
$a \vee b = \phi$



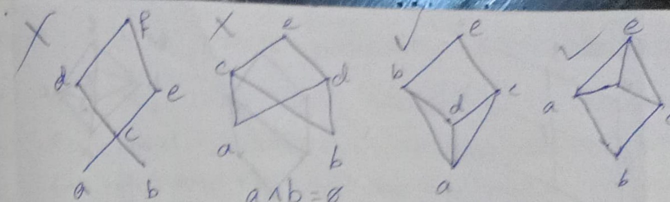
$a \vee b$ ϕ

Meet Semi lattice single bottom

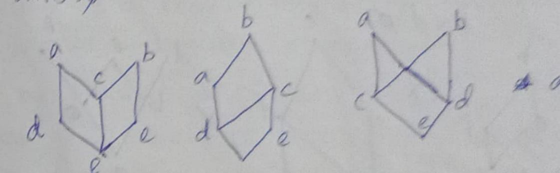
In a poset if GLB / meet / infimum / $\wedge$ exist for every pair of element, then the poset is called meet-semi-lattice



$a \wedge b = \phi$ $\quad a \wedge b = \phi$



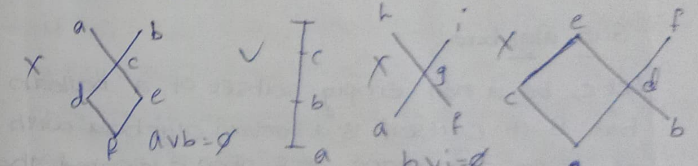
$a \wedge b = \phi$ $\quad a \wedge b = \phi$ $\quad b \wedge d = \phi$



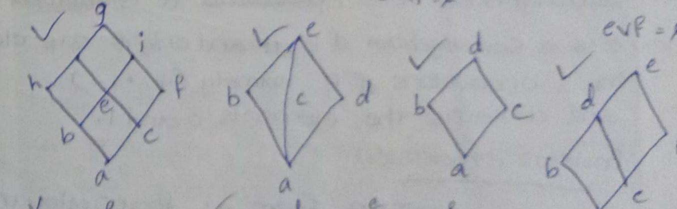
$a \wedge b = \phi$

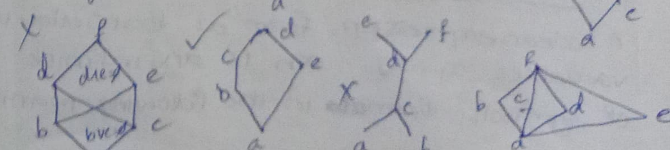
Lattice

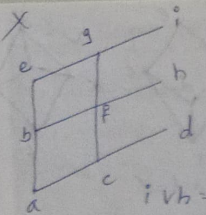
A poset is called lattice if it is both MSL & JSL ($\wedge \vee$)

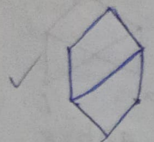


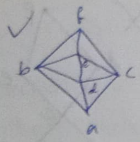
$a \vee b = \phi$ $\quad b \vee i = \phi$ $\quad e \wedge f = \phi$ $\quad a \wedge b = \phi$ $\quad e \vee f = \phi$





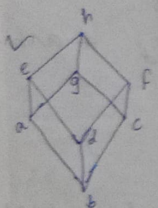


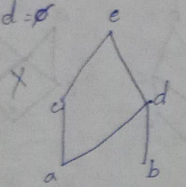


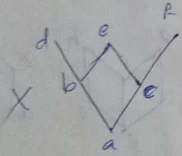


$i \vee h = \emptyset$

$h \vee d = \emptyset$







$d \vee e = \emptyset$

$e \vee f = \emptyset$

$a \wedge b = \emptyset$

Sub-algebra

Let C, be a non-empty subset of a boolean algebra B if C itself is a boolean algebra with respect to the operations on B, then C is called the subalgebra of B. ~~operations~~ It is obvious that C is a subalgebra of B, if and only if C is closed the 3 operations of B namely $(+, \cdot, -)$ and contains the elements 0 and 1.

Boolean expression

A boolean expression, form or formula in n variables $x_1, x_2, \dots x_n$ is any infinite string of symbols formed in the following manner.

1) 0 and 1 are boolean expressions
2) $x_1, x_2 \dots x_n$ are boolean expression.
3) if x_1, and x_2 are boolean expressions, then $(\alpha_1 \cdot \alpha_2)$ and $(\alpha_1 + \alpha_2)$ are also boolean expressions.
4) If α is a boolean expression, Then ~~$(\alpha_1 \cdot \alpha_2)$~~ α' is also a boolean expression
5) No strings of symbols expecept those formed in accordance with rule 1 to 4 are boolean expressions. ~~or more~~ we shall generally denote a boolean expression by $\alpha, \beta, \gamma \dots$ or more explicitly as $\alpha(x_1, x_2, \dots x_n)$

Boolean function

if $x_1, x_2 \dots x_n$ are boolean variables, a function from $B^n = \{x_1, x_2, \dots x_n\}$ to B $= \{0, 1\}$ is called a boolean fn of degree n'. Each boolean expression represent a boolean fn, which is evaluated by substituting the values 0 or 1 for each variable

Minterm (SOP)

A Boolean expression in n variables $x_1, x_2 \dots x_n$ consisting of the product of n terms such as

$$x_1^{a_1} * x_2^{a_2} * \dots x_n^{a_n} = \underset{i=1}{\overset{n}{*}} x_i^{a'}$$

in which a_i is either 0 or 1, x_i^0 stands for x_i and x_i^- stands for x_i for i = 1,2, ... n is called minterm ~~or~~ complete product, or a fundamental product of the n variables

Maxterm (pos)

A boolean expression in n variables $x_1, x_2, \dots x_n$ consisting of the product of n terms such as

$$x_1^{a_1} + x_2^{a_2} + \dots + x_n^{a_n} = \prod_{i=1}^{n} x_i^{a_i}$$

~~in which a_i is either 0 or 1, x_i^0 stands for x_i and x_i^- stands for x_i~~ for i = 1,2, ... n is called ~~minterms~~ complete product, or a ~~fundamental product of~~ the n variables.

in which a_i is either 0 or 1, x_i^0 stands for x_i and x_i' stands for x_i' for i = 1,2, ... n is called maxterm

Variables A	B	minterm	maxterm
0	0	$\bar A \bar B \to m_0$	$AB \to m_0$
0	1	$\bar A B \to m_1$	$A\bar B \to m_1$
1	0	$A \bar B \to m_2$	$A'B \to m_2$
1	1	$AB \to m_3$	$\bar A \bar B \to m_3$

Disjunctive normal forms

when a boolean fn is expressed as a sum of minterms it is called it's sum of products or disjunctive normal form.

Conjunctive Normal form

when a boolean fn is expressed as a product of maxterms it is called its product of sum or conjunctive normal forms

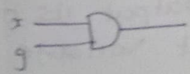
Note

A boolean fn expressed in the disjunctive normal form or conjunctive normal are said to be cannonical form

Design of Digital circuit

the boolean expression can be graphically represents by using logical circuit

AND gate

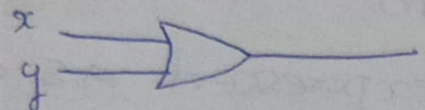


x	y	output
L	L	L
L	H	L
H	L	L
H	H	H

0	0	0
0	1	0
1	0	0
1	1	1

An AND gate is circuit that has two inputs and one output. The output voltage of an AND gate is high if both of the input voltage are high and output voltage is low if either one or both of the input voltage is low.

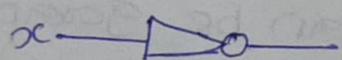
OR gate



x	y	output
L	L	L
L	H	H
H	L	H
H	H	H

the output voltage of an OR gate is high if either one or both of the input voltage is high, and output voltage are low.

NOT gate

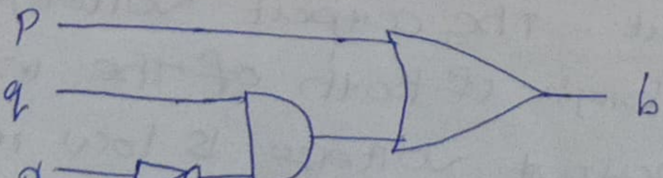


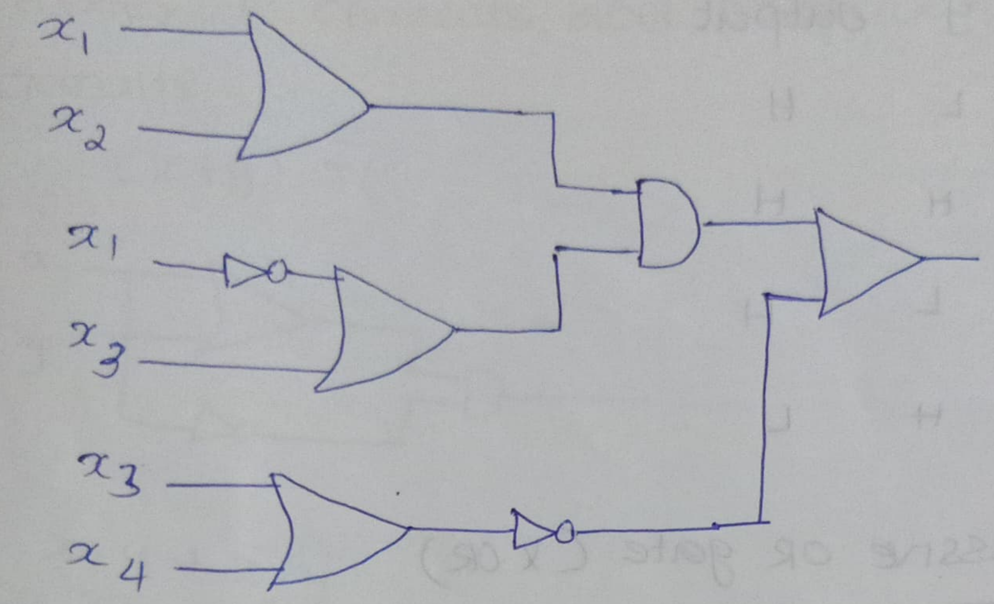
x	output
L	H
H	L

A NOT gate or an inverter is a circuit that has one input and one output. its output voltage is high if the input voltage is low, and its output voltage is low if the input voltage is high

Ex:-

$p \vee (q \wedge \bar{r})$





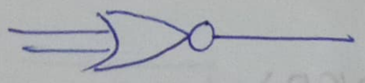
$(x_1 \vee x_2)(\bar{x}_1 \vee x_3) \quad (\overline{x_3 \vee x_4})$

$[(x_1 \vee x_2) \wedge (\bar{x}_1 \vee x_3)] \vee (\overline{x_3 \vee x_4})$

Advanced Gates

NOR Gate

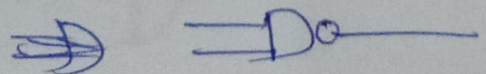
it's the complement of OR gate



x	y	output
L	L	L
L	H	L
H	L	L
H	H	H

NAND gate

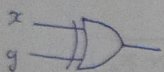
The complement of AND gate



x	y	outpat
L	L	H
L	H	H
H	L	H
H	H	L

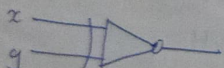
Exclussive OR gate (X OR)

It's written as $a \oplus b$



x	y	outpat
L	L	L
L	H	H
H	L	H
H	H	L

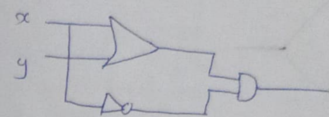
Exclussive NOR gate (X NOR)



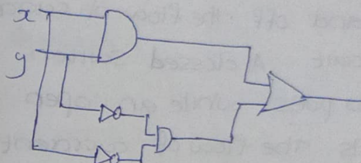
x	y	outpat
L	L	H
L	H	L
H	L	L
H	H	H

1- Constract circuits and produce the following circuits

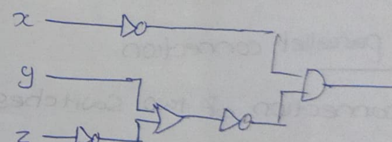
$(x+y)\cdot\bar{x}$



2) $x\cdot y+\bar{x}\cdot\bar{y}$

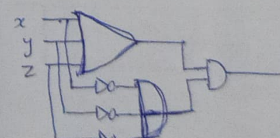


3) $\bar{x}\cdot(\overline{y+\bar{z}})$

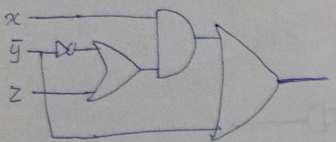


4) $(x+y+z)\cdot(\bar{x}\cdot\bar{y}\cdot\bar{z})$

Ans:-



5) $x \cdot (\bar{y}+z)+y$



Boolean Algebra Switching theory

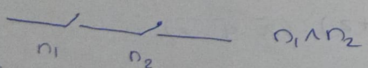
An electronic circuit is ~~an~~ two state device used for turning on and off the flow of current in an electric circuit. A closed switch allows the current to pass while an open (off) switch prevents the flow of current.

closed switch ————

open switch ——/——

Series and parallel connection.

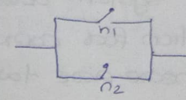
1) Here the connection of two switches n_1 and n_2 denoted $n_1 \wedge n_2$



Switches		$n_1 \wedge n_2$
n_1	n_2	state
closed	closed	on
closed	open	off
open	closed	off
open	open	on

2) parallel connection

This connection of two switches n_1 and n_2 denoted by $n_1 \vee n_2$

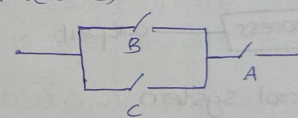


Switches		$n_1 \vee n_2$
n_1	n_2	state
closed	open	on
open	closed	on
closed	closed	on
open	open	off.

Questions

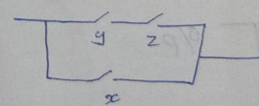
1) construct the circuit, represented by boolean expression

a) $A \wedge (B \vee C)$



An:-

b) $x \vee (y \wedge z)$

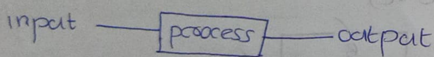


control system

It is a device or a set of device that manage commands directs regulates the behaviour of system and also control system are used in ~~indus~~ industrial production for controlling equipments or machine. there are two classes of control system.

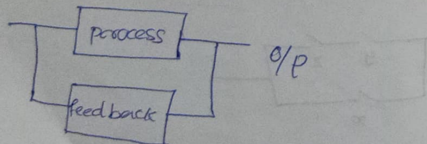
1) open loop control system.

An open loop control system acts completely on the basis of I/p and the O/p has no effects on the control action. This is also called non - feed back control system



2) closed loop control system

The system record the O/p instead I/p and modify ~~it~~ it according to the need . it is also reffered as feed back control system



Questions

1) Given an example of relation which is both Symmetric and antisymmetric

An:- $R = \{(1,1)\}$

$R = \{(2,2)(3,3),(4,4)\}$

2) Define greatest lower bond in poset

The infimum of a subset S of a partially ordered set T is the greatest element in T that is $\leq$ all elements of S, if such an element exist.

3) write the least upperbound & glb of the set $\{3,5\}$ in the poset $\langle Z - \{0\}, \leq \rangle$ whose $\leq$ is the divides relation

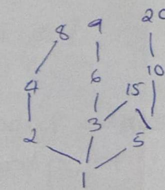
An:- $\{(1,1)\ (1,2),(1,3),(1,4) \dots$

$(2,2),(2,4),(2,6),(2,8) \dots$

$(3,3),(3,6),(3,9),(3,12),(3,15)$

$(4,4),(4,8),(4,12),(4,16) \dots$

$(5,5),(5,10) \dots \}$



glb = 1

lub = 15

6) is relation $\{\langle x, y \rangle / x \geq y\}$ antisymmetric Justify

yes

$x \leq y$ and $y \leq x$, then $x = y$

7) write the boolean expression $x_1 \oplus x_2$ in an equivalent sum of products canonical form in the three variables x_1, x_2, x_3.

$$(x_1 * x_2 * x_3) + (x_1' * x_2 * x_3) + (x_1 * x_2' * x_3) + (x_1 * x_2 * x_3')$$

$$+ (x_1' * x_2' * x_3) + (x_1 * x_2' * x_3') + (x_1' * x_2 * x_3')$$