PROJECT ALED

Cernei Ana, Vonica Andreea, Cernei Tudor

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1 Link Google Colab

This is the link for the Graphic representations and interpretations in Google Colab.

2 Falling Chain Description

Problem Statement

This problem refers to a chain falling freely under the influence of gravity. The objective is to determine the speed of the chain as a function of its position, modeling the vertical motion of the chain. The phenomenon is influenced by factors such as the length of the chain, its position, and speed at various points.

Description of the Equations Used

The mathematical model of the problem is represented by a first-order differential equation, where x is the independent variable (position), and v is the dependent variable (the speed of the chain). The initial equation is:

$$xv\frac{dv}{dx} + v^2 = 32x$$

This equation is rewritten to allow for the application of an integrating factor. By multiplying with the factor $\mu(x) = x$, the equation becomes solvable, and the obtained solution reflects the behavior of the chain's speed as a function of its position.

3 Falling Chain Problem

a) Given Equation

$$xv\frac{dv}{dx} + v^2 = 32x$$

where x is independent and v is dependent.

Rewrite in Differential Form

$$xv\,dv + v^2\,dx = 32x\,dx$$

$$xv\,dv = (32x - v^2)\,dx$$

$$xv\,dv + (-32x + v^2)\,dx = 0$$

Verify if it's an Exact Equation

Define:

$$M(x,v) = xv \quad \text{and} \quad N(x,v) = v^2 - 32x$$

Check:

$$\frac{\partial M}{\partial v} = 2v, \quad \frac{\partial N}{\partial x} = v$$

Since these partial derivatives are not equal, this is not an exact equation.

Find the Integrating Factor

The integrating factor is given by:

$$\mu(x) = e^{\int \frac{M_v - N_x}{N}\,dx}$$

Calculate:

$$\frac{M_v - N_x}{N} = \frac{v}{vx} = \frac{1}{x}$$

Thus,

$$\mu(x) = e^{\int \frac{x}{xv}\,dx} = e^{\ln x} = x$$

Multiply Integrating Factor and Rewrite Equation

Multiply by the integrating factor $\mu(x) = x$:

$$x^2 v\,dv + x(v^2 - 32x)\,dx = 0$$

Solve for $g(x)$

Find the integral of N:

$$\int N\,dv = x^2\frac{v^2}{2} + g(x)$$

Then,the derivative of the function with respect to x is

$$M = v^2x - 32x^2 = xv^2 + g'(x)$$

Now , we resolve the equation

$$v^2x - 32x^2 = xv^2 + g'(x) \quad | - v^2x$$

Since we need to find $g(x)$, we first compute $g'(x)$ by matching terms:

$$g'(x) = -32x^2$$

Integrating $g'(x)$ with respect to x:

$$g(x) = \int -32x^2\,dx = -\frac{32x^3}{3}$$

Solution Becomes

$$\int N\,dv = x^2\frac{v^2}{2} + g(x) = C$$

$$\frac{x^2v^2}{2} - \frac{32x^3}{3} = C$$

where C is a constant of integration. Using initial conditions $x = 3$ ft, $v(0) = 3$, we find:

$$\frac{3^2 0^2}{2} - \frac{32x^3}{3} = C$$

$$C = -288$$

Explicit Solution of $v(x)$

$$\frac{x^2v^2}{2} - \frac{32x^3}{3} = -288$$

Solving for $v(x)$:

$$x^2v^2 = 2(-288 + \frac{32x^3}{3})$$

$$x^2v^2 = 2(-288 + \frac{32x^3}{3})$$

$$v(x) = \sqrt{\frac{2(-288) + 32x^3}{3x^2}}$$

$$v(x) = \sqrt{2 * 32\frac{(-27) + x^3}{3x^2}}$$

$$v(x) = 8\sqrt{\frac{x^3 - 27}{3x^2}}$$

b) Solve for $v(x)$ Using $x = 8$

$$v(x) = 8\sqrt{\frac{x^3}{3x^2} - \frac{27}{3x^2}}$$

$$v(x) = 8\sqrt{\frac{x}{3} - \frac{9}{x^2}}$$

$$v(8) = 8\sqrt{\frac{8}{3} - \frac{9}{8^2}} \approx 12.7$$

4 Streamlines Description

Problem Statement

The differential equation models streamlines, which are paths that fluid particles follow in a steady flow around a circular object. The circular boundary of the object is given by

$$x^2 + y^2 = 1$$

Description of the Equations Used

The differential equation modeling the streamlines is:

$$\frac{2xy}{(x^2+y^2)^2}\,dx + \left[1 + \frac{y^2 - x^2}{(x^2+y^2)^2}\right]\,dy = 0$$

This can be written in the form $M(x,y)\,dx + N(x,y)\,dy = 0$, where $M(x,y) = \frac{2xy}{(x^2+y^2)^2}$ and $N(x,y) = 1 + \frac{y^2-x^2}{(x^2+y^2)^2}$. To verify the exactness of the equation, the partial derivatives of each term are calculated. Once the equation is confirmed to be exact, the integration method is applied to obtain the solution that describes the streamline.

5 Streamlines Problem

The differential equation we are examining is:

$$\frac{2xy}{(x^2+y^2)^2}\,dx + \left[1 + \frac{y^2 - x^2}{(x^2+y^2)^2}\right]\,dy = 0.$$

This can be written in the form:

$$M(x,y)\,dx + N(x,y)\,dy = 0,$$

where

$$M(x,y) = \frac{2xy}{(x^2+y^2)^2}, \quad N(x,y) = 1 + \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Condition for Exactness

The condition for exactness is:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Finding the Partial Derivatives Step-by-Step

To calculate $\frac{\partial M(x,y)}{\partial y}$, we use the quotient rule for differentiation of a function $f(y) = \frac{u}{v}$, where:

$$f'(y) = \frac{u'v - uv'}{v^2}.$$

Here, $u = 2xy$ and $v = (x^2 + y^2)^2$.

Step 1: Differentiate $u = 2xy$ with respect to y

$$u = 2xy$$

The derivative of u with respect to y is:

$$u' = \frac{d}{dy}(2xy) = 2x.$$

Step 2: Differentiate $v = (x^2 + y^2)^2$ with respect to y

Using the chain rule to differentiate v:

$$v = (x^2 + y^2)^2$$

The derivative of v with respect to y is:

$$v' = 2(x^2 + y^2) \cdot \frac{d}{dy}(x^2 + y^2) = 2(x^2 + y^2) \cdot 2y = 4y(x^2 + y^2).$$

Step 3: Apply the Quotient Rule

Now we substitute into the quotient rule:

$$\frac{\partial M}{\partial y} = \frac{u'v - uv'}{v^2}.$$

Thus,

$$\frac{\partial M}{\partial y} = \frac{(2x) \cdot (x^2 + y^2)^2 - (2xy) \cdot 4y(x^2 + y^2)}{(x^2 + y^2)^4}.$$

Step 4: Simplify the Expression

Expanding and simplifying the terms, we get:

$$\frac{\partial M}{\partial y} = \frac{2x(x^2 + y^2) - 8xy^2}{(x^2 + y^2)^3} = \frac{2x(x^2 - 3y^2)}{(x^2 + y^2)^3}.$$

Calculating $\frac{\partial N(x,y)}{\partial x}$ Step-by-Step

Now we calculate $\frac{\partial N(x,y)}{\partial x}$.

$$N(x,y) = 1 + \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

Step 1: Differentiate the Constant Term

The constant term 1 has a derivative of 0.

Step 2: Differentiate $\frac{y^2 - x^2}{(x^2+y^2)^2}$ with respect to x

Using the quotient rule again, let: - $u = y^2 - x^2$, so $u' = -2x$. - $v = (x^2 + y^2)^2$, which has a derivative $v' = 4x(x^2 + y^2)$ (as previously calculated).

Substitute into the quotient rule:

$$\frac{\partial N}{\partial x} = \frac{u'v - uv'}{v^2}.$$

Thus,

$$\frac{\partial N}{\partial x} = \frac{(-2x)(x^2 + y^2)^2 - (y^2 - x^2) \cdot 4x(x^2 + y^2)}{(x^2 + y^2)^4}.$$

Step 3: Simplify the Expression

Expanding and simplifying, we obtain:

$$\frac{\partial N}{\partial x} = \frac{2x(x^2 - 3y^2)}{(x^2 + y^2)^3}.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the differential equation is exact.

Finding the Solution

To find the function f, where $\frac{\partial f}{\partial x} = M(x,y)$, we proceed as follows:

$$f = \int \frac{2xy}{(x^2 + y^2)^2} \, dx$$

Let $u = x^2 + y^2$, so $du = 2x \, dx$:

$$f = \int \frac{y}{u^2} \, du = -\frac{y}{x^2 + y^2} + h(y),$$

where $h(y)$ is a function of y alone.

Now, we solve for the unknown function $h(y)$ by taking the derivative of f with respect to y and comparing it to $N(x,y)$.

$$\frac{\partial f}{\partial y} = N(x, y)$$

$$\frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2} + h(y)\right) = 1 + \frac{y^2 - x^2}{(x^2+y^2)^2}$$

Calculating the derivative, we get:

$$-\frac{(x^2+y^2) - 2y \cdot y}{(x^2+y^2)^2} + h'(y) = 1 + \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\frac{y^2 - x^2}{(x^2+y^2)^2} + h'(y) = 1 + \frac{y^2 - x^2}{(x^2+y^2)^2}$$

Thus, we find that $h'(y) = 1$.

Integrate both sides to solve for $h(y)$:

$$\int h'(y)\, dy = \int 1\, dy$$

$$h(y) = y + c$$

Therefore, the solution is:

$$f(x, y) = -\frac{y}{x^2+y^2} + y = c$$

If we consider the case when $c = 0$, the solution is:

$$-\frac{y}{x^2+y^2} + y = 0$$

$$y = \frac{x^2+y^2}{y}$$

$$1 = x^2 + y^2$$

Hence, the solution for $c = 0$ is $1 = x^2 + y^2$.

6 Conclusion

Exact differential equations are powerful and versatile tools that help us model and solve complex problems across mathematics, physics, engineering, and other scientific fields. They provide a clear method to find solutions for dynamic phenomena by transforming the problem into the integration of a potential function, simplifying the analysis and interpretation of the system. This process enables precise and applicable solutions, which are fundamental for accurately understanding and simulating the behavior of physical systems.