Tutorial 4: $X33Y33Z33$ 3D problem.

Ana Paula Fernandes & Gilmar Guimarães

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Abstract

Tutorial 4 is about the $X33Y33Z33$ 3D-problem.

Equation

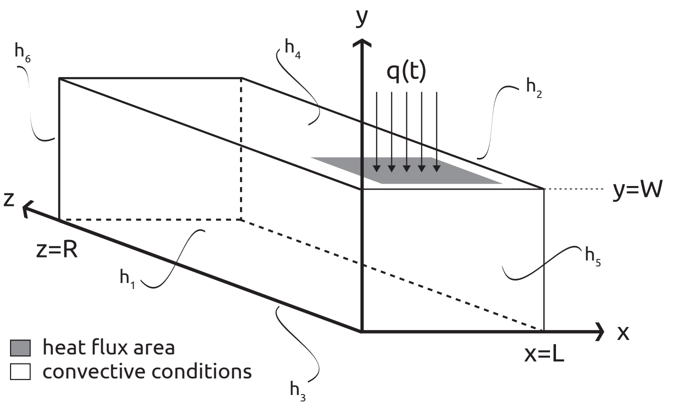

Figure 1: $X33Y33Z33$ problem

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1a}$$

with bondary conditions in x-direction

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = -h_1(T - T_\infty); \tag{1b}$$

$$-k\frac{\partial T}{\partial x}\bigg|_{x=L} = h_2(T - T_\infty) \tag{1c}$$

with bondary conditions in y-direction

$$-k\frac{\partial T}{\partial y}\bigg|_{y=0} = -h_3(T - T_\infty) \tag{1d}$$

$$-k\frac{\partial T}{\partial y}\bigg|_{y=W} = -q(t) + h_4(T - T_\infty) \tag{1e}$$

with bondary conditions in z-direction

$$-k\frac{\partial T}{\partial z}\bigg|_{z=0} = -h_5(T - T_\infty) \tag{1f}$$

$$-k\frac{\partial T}{\partial z}\bigg|_{z=R} = h_6(T - T_\infty) \tag{1g}$$

initial condition

$$T(x, y, z, 0) = F(x, y, z) - T_\infty \tag{1h}$$

If $\theta = T - T_\infty$,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{2a}$$

with bondary conditions

$$-k\frac{\partial\theta}{\partial x}\bigg|_{x=0} = -h_1\theta; -k\frac{\partial\theta}{\partial x}\bigg|_{x=L} = h_2\theta; \tag{2b}$$

$$-k\frac{\partial\theta}{\partial y}\bigg|_{y=0} = -h_3\theta;\ -k\frac{\partial\theta}{\partial y}\bigg|_{y=W} - h_4\theta = -q(t); \tag{2c}$$

$$-k\frac{\partial\theta}{\partial y}\bigg|_{z=0} = -h_5\theta;\ -k\frac{\partial\theta}{\partial y}\bigg|_{z=R} = h_6\theta \tag{2d}$$

initial temperature

$$\theta(x, y, z, 0) = F(x, y, z) - T_\infty \tag{2e}$$

By Green functions

$$\begin{aligned}\theta(x, y, z, t) = &\int_0^L \int_0^W \int_0^R G(x, y, z, t|x', y', z', 0)\theta(x, y, z, 0)dx'dy'dz' \\ &+ \frac{\alpha}{k}\int_0^t \int_{L_1}^{L_2} \int_{R_1}^{R_2} q(\tau)G(x, y, z, t|x', W, z', \tau)dx'dz'd\tau\end{aligned} \tag{3}$$

Where $G(x, y, z, t|x', y', z', \tau) = G_{X33}G_{Y33}G_{Z33}$. Green function in x-direction is given by (Cole et al., 2010)

$$\begin{aligned}G_{X33}(x, t|x', \tau) = &\frac{2}{L}\sum_{m=1}^{\infty} e^{-\alpha_m^2\alpha(t-\tau)/L^2}\left[\alpha_m\cos\left(\frac{\alpha_m x}{L}\right) + B_1\sin\left(\frac{\alpha_m x}{L}\right)\right] \\ &\times \frac{\left[\alpha_m\cos\left(\frac{\alpha_m x'}{L}\right) + B_1\sin\left(\frac{\alpha_m x'}{L}\right)\right]}{\left(\alpha_m^2 + B_1^2\right)\left[1 + \frac{B_2}{(\alpha_m^2 + B_2^2)}\right] + B_1}\end{aligned} \tag{4}$$

where $\tan\alpha_m = \frac{\alpha_m(B_1+B_2)}{\alpha_m^2 - B_1B_2}$ e $B_1 = \frac{h_1 L}{k}$ e $B_2 = \frac{h_2 L}{k}$.

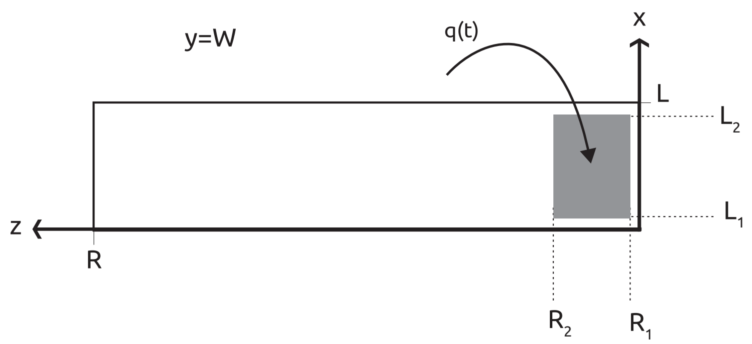

Figure 2: $X33Y33Z33$ problem - top wiew $(y = W)$

For y-direction,

$$
\begin{aligned}
G_{Y33}(y,t|y',\tau) = \frac{2}{W}\sum_{n=1}^{\infty} e^{-\beta_n^2\alpha(t-\tau)/W^2}\left[\beta_n \cos\left(\frac{\beta_n y}{W}\right) + B_3 \sin\left(\frac{\beta_n y}{W}\right)\right] \\
\times \frac{\left[\beta_n \cos\left(\frac{\beta_n y'}{W}\right) + B_3 \sin\left(\frac{\beta_n y'}{W}\right)\right]}{\left(\beta_n^2 + B_3^2\right)\left[1 + \frac{B_4}{(\beta_n^2+B_4^2)}\right] + B_3}
\end{aligned}
\tag{5}
$$

onde $\tan\beta_n = \frac{\beta_n(B_3+B_4)}{\beta_n^2 - B_3 B_4}$ e $B_3 = \frac{h_3 W}{k}$ e $B_4 = \frac{h_4 W}{k}$.

and for z-direction

$$
\begin{aligned}
G_{Z33}(z,t|z',\tau) = \frac{2}{R}\sum_{p=1}^{\infty} e^{-\gamma_p^2\alpha(t-\tau)/R^2}\left[\gamma_p \cos\left(\frac{\gamma_p z}{R}\right) + B_5 \sin\left(\frac{\gamma_p z}{R}\right)\right] \\
\times \frac{\left[\gamma_p \cos\left(\frac{\gamma_p z'}{R}\right) + B_5 \sin\left(\frac{\gamma_p z'}{R}\right)\right]}{\left(\gamma_p^2 + B_5^2\right)\left[1 + \frac{B_6}{(\gamma_p^2+B_6^2)}\right] + B_5}
\end{aligned}
\tag{6}
$$

onde $\tan\gamma_p = \frac{\gamma_p(B_5+B_6)}{\gamma_p^2 - B_5 B_6}$ e $B_5 = \frac{h_5 R}{k}$ e $B_6 = \frac{h_6 R}{k}$.

The eigenvalues α_m, β_n and γ_p are the roots of the transcendental equation and $m = 1, ...M$, $n = 1, ...N$ e $p = 1, ..., P$ and B_i is the Biot number, $i = 1, ..., 6$.

Fernandes et al. (2015),

$$
\begin{aligned}
&G_{X33}G_{Y33}G_{Z33} = G(x, y, z, t|x', y', z', \tau) \\
&= \frac{8}{LWR} \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} e^{-\left(\frac{\alpha_m^2}{L^2} + \frac{\beta_n^2}{W^2} + \frac{\gamma_p^2}{R^2}\right)\alpha(t-\tau)} \\
&\times \frac{\left[\alpha_m \cos\left(\frac{\alpha_m x'}{L}\right) + B_1 \sin\left(\frac{\alpha_m x'}{L}\right)\right]}{(\alpha_m^2 + B_1^2)\left[1 + \frac{B_2}{(\alpha_m^2 + B_2^2)}\right] + B_1} \\
&\times \frac{\left[\beta_n \cos\left(\frac{\beta_n y'}{W}\right) + B_3 \sin\left(\frac{\beta_n y'}{W}\right)\right]}{(\beta_n^2 + B_3^2)\left[1 + \frac{B_4}{(\beta_n^2 + B_4^2)}\right] + B_3} \\
&\times \frac{\left[\gamma_p \cos\left(\frac{\gamma_p z'}{R}\right) + B_5 \sin\left(\frac{\gamma_p z'}{R}\right)\right]}{(\gamma_p^2 + B_5^2)\left[1 + \frac{B_6}{(\gamma_p^2 + B_6^2)}\right] + B_5} \\
&\times \left[\alpha_m \cos\left(\frac{\alpha_m x}{L}\right) + B_1 \sin\left(\frac{\alpha_m x}{L}\right)\right] \\
&\times \left[\beta_n \cos\left(\frac{\beta_n y}{W}\right) + B_3 \sin\left(\frac{\beta_n y}{W}\right)\right] \\
&\times \left[\gamma_p \cos\left(\frac{\gamma_p z}{R}\right) + B_5 \sin\left(\frac{\gamma_p z}{R}\right)\right]
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&\theta(x,y,z,t) = \\
&8\theta_0 \sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-\left(\frac{\alpha_m^2}{L^2}+\frac{\beta_n^2}{W^2}+\frac{\gamma_p^2}{R^2}\right)\alpha t} \\
&\times \frac{\left[\alpha_m \cos\left(\frac{\alpha_m x}{L}\right) + B_1 \sin\left(\frac{\alpha_m x}{L}\right)\right]}{(\alpha_m^2 + B_1^2)\left[1 + \frac{B_2}{(\alpha_m^2+B_2^2)}\right] + B_1} \times \frac{\left[\beta_n \cos\left(\frac{\beta_n y}{W}\right) + B_3 \sin\left(\frac{\beta_n y}{W}\right)\right]}{(\beta_n^2 + B_3^2)\left[1 + \frac{B_4}{(\beta_n^2+B_4^2)}\right] + B_3} \\
&\times \frac{\left[\gamma_p \cos\left(\frac{\gamma_p z}{R}\right) + B_5 \sin\left(\frac{\gamma_p z}{R}\right)\right]}{(\gamma_p^2 + B_5^2)\left[1 + \frac{B_6}{(\gamma_p^2+B_6^2)}\right] + B_5} \times \frac{1}{\alpha_m \beta_n \gamma_p} \\
&\times \left[\alpha_m \sin\alpha_m - B_1(\cos\alpha_m - 1)\right]\left[\beta_n \sin\beta_n - B_3(\cos\beta_n - 1)\right]\left[\gamma_p \sin\gamma_p - B_5(\cos\gamma_p - 1)\right] \\
&+ \frac{\alpha}{k}\frac{8}{LWR} \sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-\left(\frac{\alpha_m^2}{L^2}+\frac{\beta_n^2}{W^2}+\frac{\gamma_p^2}{R^2}\right)\alpha t} \\
&\times \frac{\left[\alpha_m \cos\left(\frac{\alpha_m x}{L}\right) + B_1 \sin\left(\frac{\alpha_m x}{L}\right)\right]}{(\alpha_m^2 + B_1^2)\left[1 + \frac{B_2}{(\alpha_m^2+B_2^2)}\right] + B_1} \\
&\times \frac{\left[\beta_n \cos\left(\frac{\beta_n y}{W}\right) + B_3 \sin\left(\frac{\beta_n y}{W}\right)\right]}{(\beta_n^2 + B_3^2)\left[1 + \frac{B_4}{(\beta_n^2+B_4^2)}\right] + B_3}\left[\beta_n \cos(\beta_n) + B_3 \sin(\beta_n)\right] \\
&\times \frac{\left[\gamma_p \cos\left(\frac{\gamma_p z}{R}\right) + B_5 \sin\left(\frac{\gamma_p z}{R}\right)\right]}{(\gamma_p^2 + B_5^2)\left[1 + \frac{B_6}{(\gamma_p^2+B_6^2)}\right] + B_5} \\
&\times \left\{L\left[\sin\left(\frac{\alpha_m L_2}{L}\right) - \sin\left(\frac{\alpha_m L_1}{L}\right)\right] - \frac{B_1 L}{\alpha_m}\left[\cos\left(\frac{\alpha_m L_2}{L}\right) - \cos\left(\frac{\alpha_m L_1}{L}\right)\right]\right\} \\
&\times \left\{R\left[\sin\left(\frac{\gamma_p R_2}{R}\right) - \sin\left(\frac{\gamma_p R_1}{R}\right)\right] - \frac{B_5 R}{\gamma_p}\left[\cos\left(\frac{\gamma_p R_2}{R}\right) - \cos\left(\frac{\gamma_p R_1}{R}\right)\right]\right\} \\
&\times \int_0^t \left[q(\tau) e^{\left(\frac{\alpha_m^2}{L^2}+\frac{\beta_n^2}{W^2}+\frac{\gamma_p^2}{R^2}\right)\alpha\tau}\right] d\tau
\end{aligned}
\tag{8}
$$

Matlab Code Snippet

```
for d=1:length(t)-1
 somatorioexp1=0;
 somatorioexp2=0;
 for k=1:p
  for j=1:n
   for i=1:m
    % exponential term
    Amnp=((alfam(i)/L)^2+(betan(j)/W)^2+(gamap(k)/R )^2)*alfa;
    % temperature initial term
    parcelaexp1 = ...
    exp(-Amnp*t(d)) ...
    *((alfam(i)*cos(alfam(i)*x/L)+B1*sin(alfam(i)*x/L))/(((alfam(i)^2+B1^2)*(1+(B2/(alfam(i)^2+B2^2))))+B1))...
    *((betan(j)*cos(betan(j)*y/W)+B3*sin(betan(j)*y/W))/(((betan(j)^2+B3^2)*(1+(B4/(betan(j)^2+B4^2))))+B3))...
    *((gamap(k)*cos(gamap(k)*z/R)+B5*sin(gamap(k)*z/R))/(((gamap(k)^2+B5^2)*(1+(B6/(gamap(k)^2+B6^2))))+B5))...
    *(1/(alfam(i)*betan(j)*gamap(k)))...
    *(alfam(i)*sin(alfam(i))-B1*(cos(alfam(i))-1))...
    *(betan(j)*sin(betan(j))-B3*(cos(betan(j))-1))...
    *(gamap(k)*sin(gamap(k))-B5*(cos(gamap(k))-1));
    somatorioexp1=somatorioexp1+parcelaexp1;
    % heat flux term
    somatorioint=0;
    for f=1:d
     arg1 = ( t(d+1) - t(f+1) );
     arg2 = ( t(d+1) - t(f) );
     parcelaint = q(f)*(exp(-Amnp*arg1)-exp(-Amnp*arg2));
     somatorioint = somatorioint + parcelaint;
    end

    parcelaexp2 = ...
    (((alfam(i)*cos(alfam(i)*x/L)+B1*sin(alfam(i)*x/L))/((alfam(i)^2+B1^2)*(1+(B2/(alfam(i)^2+B2^2)))+B1)))...
   *(((betan(j)*cos(betan(j)*y/W)+B3*sin(betan(j)*y/W))/((betan(j)^2+B3^2)*(1+(B4/(betan(j)^2+B4^2)))+B3)))...
   *((betan(j)*cos(betan(j))+B3*sin(betan(j)))))...
   *(((gamap(k)*cos(gamap(k)*z/R)+B5*sin(gamap(k)*z/R))/((gamap(k)^2+B5^2)*(1+(B6/(gamap(k)^2+B6^2)))+B5)))...
   *(sin(alfam(i)*L2/L)-sin(alfam(i)*L1/L)-(B1/alfam(i))*(cos(alfam(i)*L2/L))+(B1/alfam(i))*(cos(alfam(i)*L1/L)))...
   *(sin(gamap(k)*R2/R)-sin(gamap(k)*R1/R)-(B5/gamap(k))*(cos(gamap(k)*R2/R))+(B5/gamap(k))*(cos(gamap(k)*R1/R)))...
   * (1/Amnp) * somatorioint;
   somatorioexp2 = somatorioexp2 + parcelaexp2;
  end
 end
end
 T(d)=...
 (8*Teta0) ...
 *  somatorioexp1   ...
```

```
+ (alfa/kCOND)*(8/W) ...
* somatorioexp2 + Tinf;
end
```

References

K. Cole, J. Beck, A. Haji-Sheikh, and B. Litkouhi. *Heat Conduction Using Green?s Functions, 2nd Edition*. Series in Computational Methods and Physical Processes in Mechanics and Thermal Sciences. CRC Press, 2010. ISBN 9781439895214.

A. P. Fernandes, M. B. dos Santos, and G. Guimarães. An analytical transfer function method to solve inverse heat conduction problems. *Applied Mathematical Modelling*, 2015. ISSN 0307-904X. doi: http://dx.doi.org/10.1016/j.apm.2015.02.012.