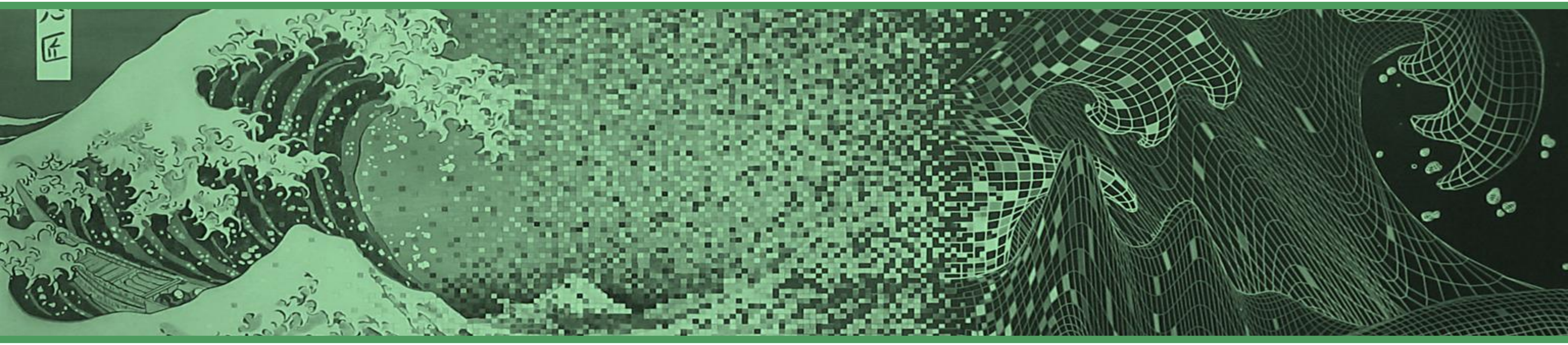
Bayesian learning

Probability theory and Bayesian models



Aprendizagem 2023

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Outline

- **Probability theory**
  - prior and posterior probability
  - maximum a posterior (MAP) and maximum likelihood (ML)
- **Bayes optimal classifier**
  - Bayesian learning in discrete spaces
  - Bayesian learning in numeric data spaces
- **Naïve Bayes classifier**
  - conditional independence
  - classification with naïve Bayes
  - estimating probabilities in small samples

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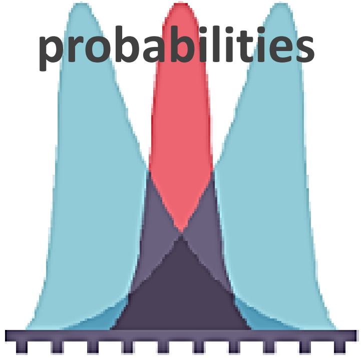
Uncertainty

- A key concept in the field in machine learning is ***uncertainty***
  - noise on measurements
  - finite size of data sets

- **Probability theory** provides a consistent framework to quantify and handle uncertainty
  - a central foundation in pattern recognition

versus



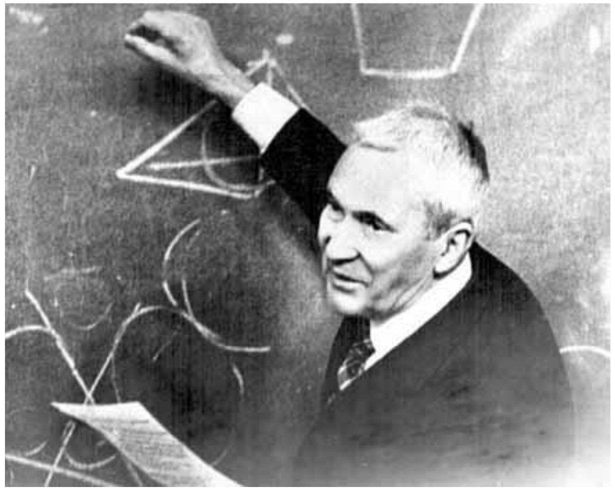
Kolmogorov’s axioms of probability (1933)

- To each sentence a, a numerical degree of belief between 0 and 1 is assigned

$$0 \leq p(a) \leq 1$$
$$p(\text{true}) = 1,\ p(\text{false}) = 0$$

- The probability of disjunction is given by

$$p(a \vee b) = p(a) + p(b) - p(a \wedge b)$$



Where do numerical degrees of belief come from?

- Humans *believe* in a subjective viewpoint from *experience*
  - this approach is called **Bayesian**
- [*subjectivist*] For a finite sample we can *estimate* the probability of a given phenomenon
  - count the *frequency* of the event in a *sample*
    - **frequentist** approach
  - do not know the true value because we cannot access the whole population of events
- [*objectivist*] From the true nature of the universe, e.g. the probability of heads in a fair coin is 0.5
  - **Platonic world** of ideas!
  - we can never verify whether a fair coin exists

Posterior probability

- If Ω is the set of all possible events, $p(\Omega) = 1$
  - $card(\Omega)$ is the number of elements of the set Ω
  - consider occurrences $a, b \subseteq \Omega$, then

$$p(a) = \frac{card(a)}{card(\Omega)} \qquad p(a \wedge b) = \frac{card(a \wedge b)}{card(\Omega)}$$

  - ... knowing

$$p(a|b) = \frac{card(a \wedge b)}{card(b)}$$

  - ... we get

$$p(a|b) = \frac{p(a \wedge b)}{p(b)}$$

Bayes' rule

- From…

$$p(a|b) = \frac{p(a \wedge b)}{p(b)} \qquad p(b|a) = \frac{p(a \wedge b)}{p(a)}$$

  - … we can infer the Bayes' rule

$$p(b|a) = \frac{p(a|b) \cdot p(b)}{p(a)}$$

Reverent Thomas Bayes (1702-1761)

He set down his findings on probability in "Essay Towards Solving a Problem in the Doctrine of Chances" (1763)

Law of total probability

- For mutually exclusive events $b_1, \ldots, bn$ with

$$\sum_{i=1}^{n} p(b_i) = 1$$

  - … the law of **total probability** is represented by

$$p(a) = \sum_{i=1}^{n} p(a) \wedge p(b_i) = \sum_{i=1}^{n} p(a, b_i)$$

$$p(a) = \sum_{i=1}^{n} p(a|b_i) \cdot p(b_i)$$

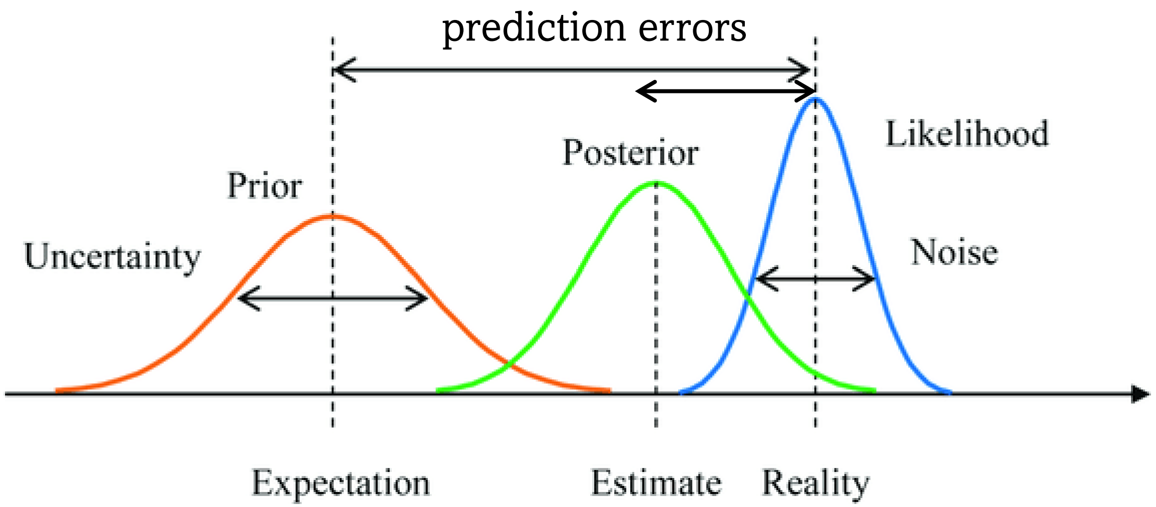
Rules of probability

sum rule $$p(X) = \sum_{Y} p(X, Y)$$

product rule $$p(X, Y) = p(Y|X)p(X)$$

Conditional probability: prior and posterior

- ***prior*** probability: before the evidence is obtained
  - $p(a)$ the prior probability that the proposition is true, e.g. $p(cavity) = 0.1$
- ***posterior*** probability: after the evidence is obtained
  - $P(a|b)$, the probability given that we know b, e.g. $P(cavity|toothache) = 0.8$



Bayes theorem

- Bayes rule can be used to determine the prior total probability $p(h)$ of hypothesis h given data D
  - *example*: what is the probability of infection given certain symptoms?

$$p(h|D) = \frac{p(D|h) \cdot p(h)}{p(D)}$$

  - $p(h)$ = prior probability of hypothesis h – 20% of patients are infected
  - $p(D)$ = prior probability of training data $D = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ – 40% of patients have identical symptoms
  - $p(D|h)$ = probability that the hypothesis h generates the data D
    - probability that infection generates the symptoms? 70% infected patients show identical symptoms
  - $p(h|D)$ = probability of h given D – $\frac{0.7 \times 0.2}{0.4}$ = 35% probability of being infected given my symptoms

Maximum a Posteriori (MAP)

- Given data D and Bayes rule assumption…
  - what is the most probable hypothesis h out of a set of possible hypothesis $h_1, h_2, \dots$?
  - to determine the maximum a posteriori hypothesis h_{MAP} we maximize

$$h_{MAP} = \underset{h}{\operatorname{argmax}}\ p(h|D)$$

$$h_{MAP} = \underset{h}{\operatorname{argmax}} \frac{p(D|h) \cdot p(h)}{p(D)}$$

  - $p(D)$ is the same for every hypothesis, hence…

$$h_{MAP} = \underset{h}{\operatorname{argmax}}\ p(D|h) \cdot p(h)$$

Maximum Likelihood (ML)

- Assuming every hypothesis has the same probability
  - same priors $p(h_1) = p(h_2) = \cdots$
  - useful to prevent biases towards specific hypotheses (e.g., having or not having a disease)
  - we can further simplify and choose the maximum likelihood (ML) hypothesis

$$h_{ML} = \underset{h}{\operatorname{argmax}}\ p(D|h)$$

  - *posterior* $\propto$ *likelihood* $\times$ *prior*
  - $p(D|h)$ is evaluated using the observed data D and is called ***likelihood function***

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Bayesian Learning

- For each hypothesis h in H, calculate the posterior probability

$$p(h|\mathbf{x}) = \frac{p(\mathbf{x}|h) \cdot p(h)}{p(\mathbf{x})}$$

- Return the hypothesis h_{MAP} with the highest posterior probability

$$h_{MAP} = \underset{h}{\operatorname{argmax}}\ p(h|\mathbf{x})$$

- **Exercise**: does patient have cancer or not?
  – *a patient takes a lab test and the result comes back positive*
  – *the test returns a correct positive result* $(+)$ *in only 98% of the cases in which the disease is present*
  … and a correct negative result $(-)$ *in only 97% of the cases in which the disease is not present*
  – *0.008 of the entire population have this cancer*

Suppose a *positive* result is returned...

$P(cancer) = 0.008 \qquad P(\neg cancer) = 0.992$

$P(+|cancer) = 0.98 \qquad P(-|cancer) = 0.02$

$P(+|\neg cancer) = 0.03 \qquad P(-|\neg cancer) = 0.97$

$$P(+|cancer) \cdot P(cancer) = 0.98 \cdot 0.008 = 0.0078$$

$$P(+|\neg cancer) \cdot P(\neg cancer) = 0.03 \cdot 0.992 = 0.0298$$

$$h_{MAP} = \neg cancer$$

$$P(cancer \mid +) = \frac{0.0078}{0.0078 + 0.0298} = 0.20745$$

$$P(\varnothing cancer \mid +) = \frac{0.0298}{0.0078 + 0.0298} = 0.79255$$

The result of Bayesian inference strongly depends on prior probabilities, which must be representative in order to apply the MAP

Estimating p(h)

- Let us draw some principles to estimate

$$p(h|\mathbf{x}) = \frac{p(\mathbf{x}|h) \cdot p(h)}{p(\mathbf{x})}$$

- Let us first start with $p(h)$
  - given no prior knowledge that *one hypothesis is more likely* than another
    - $p(h)$ can be assumed to be uniformly distributed

$$\forall_{h \in H} \; p(h) = \frac{1}{|H|}$$

  - otherwise, estimate the prior base on the observed frequency

Estimating $p(D|h)$

- If data is **discrete**:
  - probability of each possible occurrence using class-conditional probability mass function
    - we can use the frequentist approach introduced in the first lectures
      - e.g. I observe 2 out of 10 individuals with blue eyes and brown in shift A and 1 out of 8 in shift B, then $p(\mathbf{x} = [blue\ eyes, brown]|A) = 0.2$ and $p(\mathbf{x} = [blue\ eyes, brown]|B) = 0.125$
- If data is **real-valued**:
  - probability based on class-conditional probability density function
    - we can use empirical or theoretical distributions
      - e.g. assuming age and height to be independent and uniformly distributed in class A, where $\text{age}{\sim}U(20{,}30)$ and $\text{height}{\sim}U(150{,}180)$, then $p(\mathbf{x} = [22.3years, 154cm]|A) = \frac{1}{10} \times \frac{1}{30}$
- If data is **mixed**:
  - similarly to the above cases, the probability is drawn from the class-conditional joint distribution

Bayesian optimal classifier

- What is the most probable classification of the new instance given the training data?

$$h_{MAP} = \arg\max_h p(h|\mathbf{x}_{new}) = \arg\max_h \frac{p(\mathbf{x}_{new}|h)p(h)}{p(\mathbf{x}_{new})} = \arg\max_h p(\mathbf{x}_{new}|h)p(h)$$

… where the hypotheses correspond to our classes

– we ignore the denominator as it does not alter decision

- The Bayesian classifier has as many parameter as:
  – the number of priors minus 1
    – we can deduce one prior from the remaining ones
    – e.g. given h_1, h_2 and h_3, $p(h_3) = 1 - p(h_2) - p(h_1)$
  – the number of parameters associated with the class-conditional distributions, $p(\mathbf{x}|h)$

Bayesian optimal classifier: example

- Learning the Bayesian model given by priors and posteriors
  - priors $p(c=0)=\frac{4}{7}, p(c=1)=1-p(c=0)=\frac{3}{7}$
  - for each combination of possible values, learn the posteriors
    - $p(c=0 \mid v_1=0, v_2=\mathrm{A}, v_3=0)=\frac{p\,(c=0)p(v_1=0,v_2=\mathrm{A},v_3=0|c=0)}{p\,(v_1=0,v_2=\mathrm{A},v_3=0)}$
    - $p(c=0 \mid v_1=0, v_2=\mathrm{A}, v_3=1)=\frac{p\,(c=0)p(v_1=0,v_2=\mathrm{A},v_3=1|c=0)}{p\,(v_1=0,v_2=\mathrm{A},v_3=1)}$
    - …
    - $p(c=1 \mid v_1=1, v_2=\mathrm{C}, v_3=1)=\frac{p\,(c=1)p(v_1=1,v_2=\mathrm{C},v_3=1|c=1)}{p\,(v_1=1,v_2=\mathrm{C},v_3=1)}$
  - from data $p(v_1=0, v_2=\mathrm{A}, v_3=0|c=0)=1/3, \ldots, p(v_1=1, v_2=\mathrm{C}, v_3=1|c=1)=2/4$

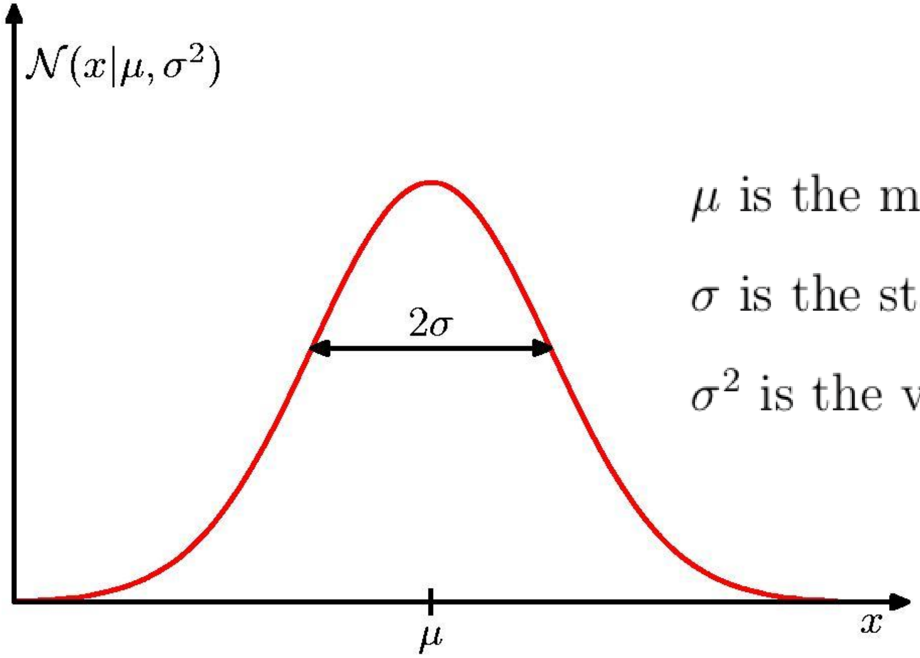
	y_1	y_2	y_3	$class$
$\mathbf{x}_1$	1	C	1	1
$\mathbf{x}_2$	1	C	1	0
$\mathbf{x}_3$	0	B	1	0
$\mathbf{x}_4$	0	A	0	0
$\mathbf{x}_5$	1	C	1	1
$\mathbf{x}_6$	0	B	1	1
$\mathbf{x}_7$	0	A	0	1

- Now we can classify new observations, e.g. $\mathbf{x}_{\text{new}}=[1, \mathrm{C}, 1]$
  - $p(c=1 \mid v_1=1, v_2=\mathrm{C}, v_3=1)=\frac{1}{p\,(v_1=1,v_2=\mathrm{C},v_3=1)} \times p\,(c=1)p(v_1=1, v_2=\mathrm{C}, v_3=1|c=1)=\frac{1}{p\,(v_1=1,v_2=\mathrm{C},v_3=1)} \times \frac{4}{7} \times \frac{2}{4}$
  - $p(c=1 \mid v_1=1, v_2=\mathrm{C}, v_3=1)=\frac{1}{p\,(v_1=1,v_2=\mathrm{C},v_3=1)} \times p\,(c=0)p(v_1=1, v_2=\mathrm{C}, v_3=1|c=0)=\frac{1}{p\,(v_1=1,v_2=\mathrm{C},v_3=1)} \times \frac{3}{7} \times \frac{1}{3}$
  - $\mathbf{x}_{\text{new}}$ is classified with class 1

Recall: Gaussian distribution

- Gaussian or normal distribution Defined by the probability

$$p(x|\mu,\sigma^2) = N(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2\sigma^2}(x-u)^2\right)$$



μ is the mean

σ is the standard deviation

σ^2 is the variance

$$\int_{-\infty}^{\infty} \mathcal{N}\left(x|\mu,\sigma^2\right)\,\mathrm{d}x = 1$$

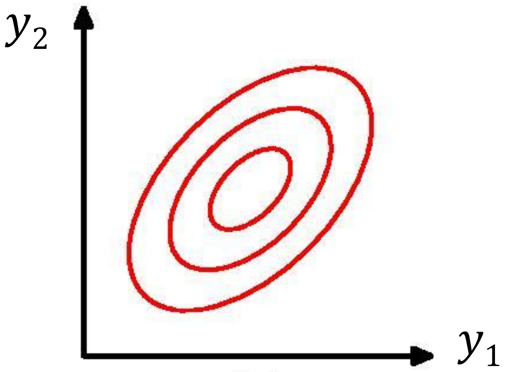
$$\mathcal{N}(x|\mu,\sigma^2) > 0$$

Multivariate Gaussian distribution in m-dimensional spaces

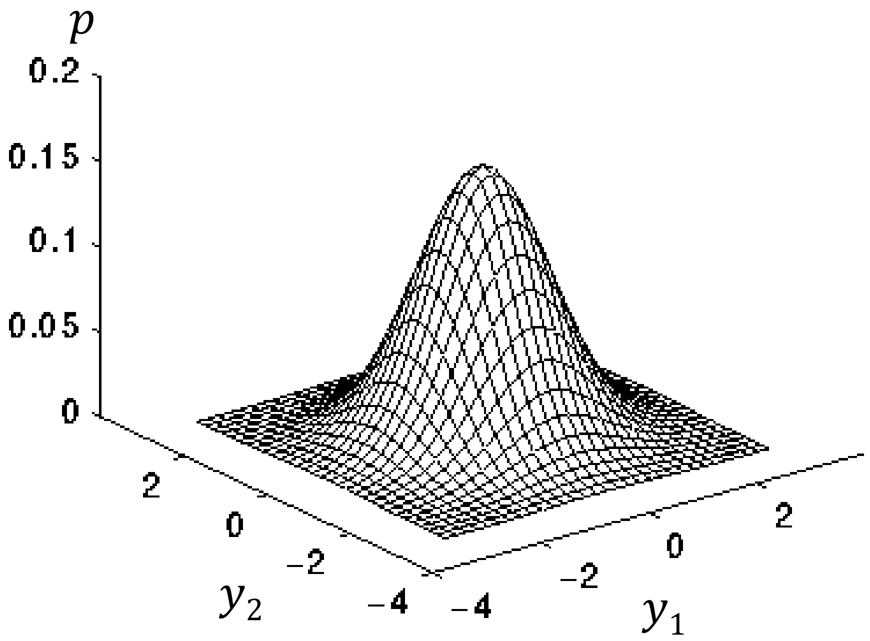
$$p(\mathbf{x}|\boldsymbol{\mu},\Sigma) = N(\mathbf{x}|\boldsymbol{\mu},\Sigma) = \frac{1}{(2\pi)^{m/2}\sqrt{|\Sigma|}}\exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{u})^T\cdot\Sigma^{-1}\cdot(\mathbf{x}-\mathbf{u})\right)$$

where…

- $\mathbf{u}$ is the m-dimensional mean vector
- Σ is a $m \times m$ covariance matrix
- $|\Sigma|$ is the determinant of Σ



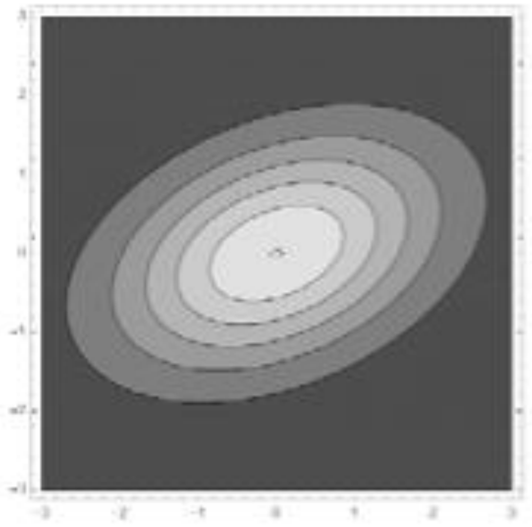
$m = 2$ dimensions



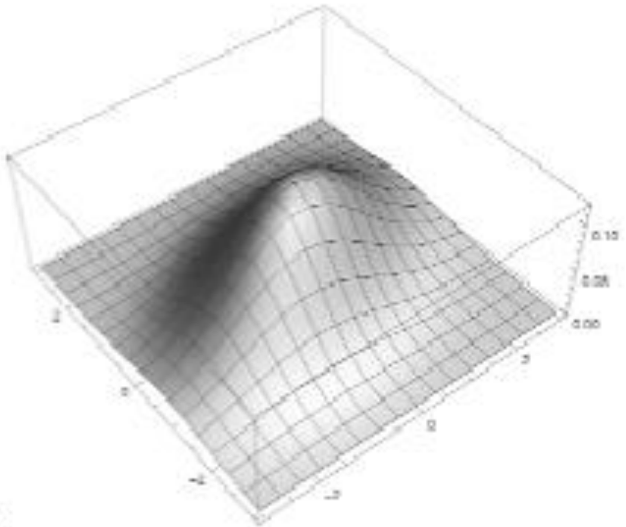
$m = 2$ dimensions

Multivariate Gaussian distribution in m-dimensional spaces

- Gaussian distribution over a $m = 2$ dimensional space with
  - average $\boldsymbol{\mu} = (0,0)^T$
  - covariance matrix $\Sigma = \begin{pmatrix} 2 & 0.5 \\ 0.5 & 1 \end{pmatrix}$
  - (a) two-dimensional and (b) three-dimensional plots of the Gaussian

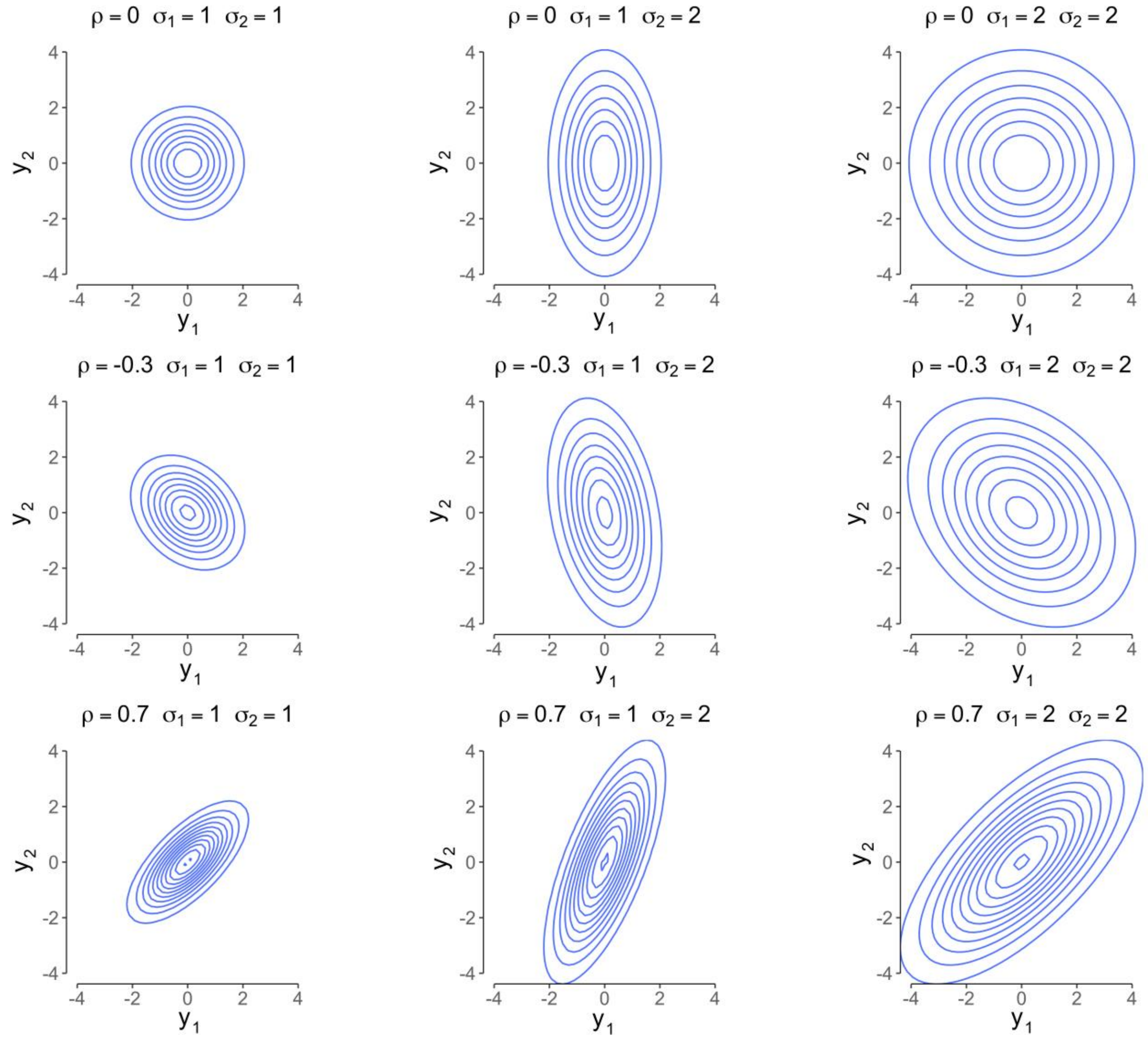


(a)



(b)

Multivariate Gaussian: covariances



$$\Sigma = \begin{pmatrix} cov(y_1, y_1) & cov(y_1, y_2) \\ cov(y_1, y_2) & cov(y_2, y_2) \end{pmatrix}$$

$$= \begin{pmatrix} {\sigma_1}^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & {\sigma_2}^2 \end{pmatrix}$$

$$cov(y_1, y_2) = \frac{\sum_{i=1}^{n}(x_{1i} - \overline{y_1}) \cdot (x_{2i} - \overline{y_2})}{n}$$

population n *versus*
sample $n - 1$

Multivariate Gaussian: example

- Approximate a multivariate Gaussian distribution using the following points:

$$X = \{(-2,2), (-1,3), (0,1), (-2,1)\}$$

  – maximum likelihood parameters

  – $\mu = \frac{1}{4}\left(\begin{bmatrix}-2\\2\end{bmatrix} + \begin{bmatrix}-1\\3\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}-2\\1\end{bmatrix}\right) = \begin{bmatrix}-1.25\\1.75\end{bmatrix}$

  – $\Sigma_{00} = \frac{1}{4-1}[(-2+1.25)(-2+1.25) + (-1+1.25)(-1+1.25) + (0+1.25)(0+1.25) + (-2+1.25)(-2+1.25)] \approx 0.92$

  – $\Sigma = \begin{pmatrix}\Sigma_{00} & \Sigma_{10}\\ \Sigma_{01} & \Sigma_{11}\end{pmatrix} = \begin{pmatrix}0.92 & -0.083\\ -0.083 & 0.92\end{pmatrix}$

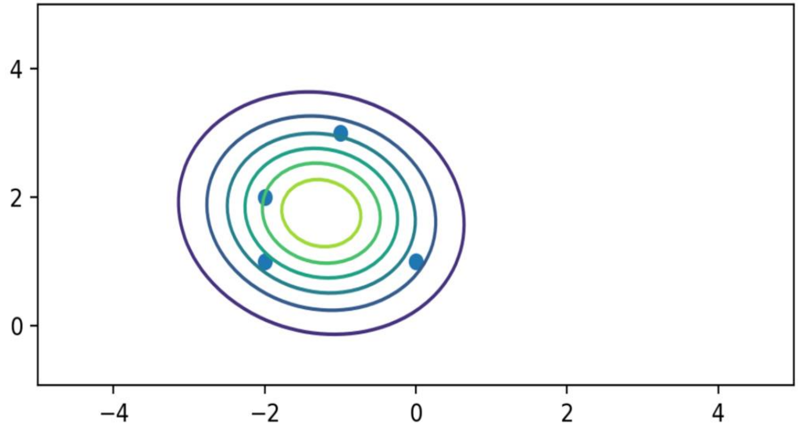
  – $\Sigma^{-1} = \begin{pmatrix}1.1 & 0.1\\ 0.1 & 1.1\end{pmatrix}$

  – we can write the Gaussian expression for a two-dimensional input $\mathbf{x} = [x_1 \;\; x_2]^T$ as follows

  – $N(\mathbf{x}|\boldsymbol{\mu}, \Sigma) = \frac{1}{(2\pi)^{2/2}\sqrt{0.083}} \exp\left(-\frac{1}{2}\left(\begin{bmatrix}x_1\\x_2\end{bmatrix} - \begin{bmatrix}-1.25\\1.75\end{bmatrix}\right)^T \begin{bmatrix}1.1 & 0.1\\0.1 & 1.1\end{bmatrix}\left(\begin{bmatrix}x_1\\x_2\end{bmatrix} - \begin{bmatrix}-1.25\\1.75\end{bmatrix}\right)\right)$

Multivariate Gaussian: example

- What is the shape of the previous 2-dimensional Gaussian?
  - fixing $\boldsymbol{\mu}$ and Σ inspection…



- What is the probability of observing $\mathbf{x} = (0,0)^T$?
  - $N\left(\begin{bmatrix}0\\0\end{bmatrix} \middle| \mu, \Sigma\right) = \frac{1}{2\pi\sqrt{0.083}} exp\left(-\frac{1}{2}\left(\begin{bmatrix}0\\0\end{bmatrix} - \begin{bmatrix}-1.25\\1.75\end{bmatrix}\right)^T \begin{bmatrix}1.1 & 0.1\\0.1 & 1.1\end{bmatrix}\left(\begin{bmatrix}0\\0\end{bmatrix} - \begin{bmatrix}-1.25\\1.75\end{bmatrix}\right)\right) = 0.0145$

Bayesian optimal classifier: example

- Consider a population of 100 individuals
  - 30 individuals have phenotype A, 30 have B, and remaining ones have C
  - the expression of three genes (variables) are characterized by the following 3-dimensional Gaussians

$$N_A\left(\boldsymbol{\mu}_A = \begin{bmatrix} 0.375 \\ 0.875 \\ 0.25 \end{bmatrix}, \Sigma_A = \begin{bmatrix} 3.41 & 1.34 & 2.6 \\ 1.34 & 2.125 & 1.18 \\ 2.6 & 1.18 & 2.8 \end{bmatrix}\right), N_B\left(\boldsymbol{\mu}_B = \begin{bmatrix} 0.5 \\ 0.125 \\ 0.875 \end{bmatrix}, \Sigma_B = \begin{bmatrix} 0.286 & 0.07 & -0.07 \\ 0.07 & 0.125 & 0.018 \\ -0.07 & 0.018 & 0.125 \end{bmatrix}\right), N_C\left(\boldsymbol{\mu}_C = \begin{bmatrix} 0 \\ -0.125 \\ 0.125 \end{bmatrix}, \Sigma_C = \begin{bmatrix} 1.7 & 1.14 & 1 \\ 1.14 & 1.55 & 0.73 \\ 1 & 0.73 & 0.98 \end{bmatrix}\right)$$

- **classify** observations $\mathbf{x}_1 = [0, 1.1, -0.8]$ and $\mathbf{x}_2 = [-0.3, 0.1, -0.3]$
  - to facilitate the calculus of class-conditional probabilities, $p(\mathbf{x}|N)$, we can use *scipy* or other package
  - $p(A|\mathbf{x}_1) = \frac{p(A)p(\mathbf{x}_1|A)}{p(\mathbf{x}_1)} = \frac{1}{p(\mathbf{x}_1)} \times \frac{30}{100} \times 0.019{=}0.0057, p(B|\mathbf{x}_1) = \frac{1}{p(\mathbf{x}_1)} \times \frac{30}{100} \times 5.4E - 14, p(C|\mathbf{x}_1) = \frac{1}{p(\mathbf{x}_1)} \times \frac{40}{100} \times 0.0088 = 0.0035$
  - $p(A|\mathbf{x}_2) = \frac{p(A)p(\mathbf{x}_2|A)}{p(\mathbf{x}_2)} = \frac{1}{p(\mathbf{x}_2)} \times \frac{30}{100} \times 0.0266 = 0.008, p(B|\mathbf{x}_2) = \frac{1}{p(\mathbf{x}_2)} \times \frac{30}{100} \times 6.8E - 6, p(C|\mathbf{x}_2) = \frac{1}{p(\mathbf{x}_2)} \times \frac{40}{100} \times 0.068 = 0.027$
  - $\mathbf{x}_1$ is classified with phenotype A and $\mathbf{x}_2$ is classified with phenotype C

Bayes optimal classifier

- **Advantages**
  - when data distributions are well-approximated, provides highly **accurate** results
  - priors can be easily neglected to not bias posteriors

- **Disadvantages**
  - requires a good amount of data to estimate joint distributions
    - impracticable in the presence of **high-dimensional data**
  - can be computationally **expensive**
    - discrete data: need to compute the posterior probability for every hypothesis
    - numeric data: need to approximate distributions
      - e.g. fitting multivariate Gaussians can be expensive due covariance matrix inversion

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Joint distribution

- A joint distribution for toothache, cavity, catch – *dentist's probe catches in my tooth* ☹
  - we need to know the conditional probabilities of the conjunction of toothache and cavity
  - what can a dentist conclude if the probe catches in the aching tooth?

$$P(cavity \mid toothache \wedge catch) = \frac{P(toothache \wedge catch | cavity) \cdot p(cavity)}{P(toothache \wedge catch)}$$

  - **Problem?**
    - For m possible variables there are 2^m possible combinations

	toothache		no toothache	
	catch	no catch	catch	no catch
cavity	0.108	0.012	0.072	0.008
no cavity	0.016	0.064	0.144	0.576

Conditional independence

- Once we know that the patient has cavity we do not expect the probability of the probe catching to depend on the presence of toothache
  - **independence**

$$P(catch|cavity \wedge toothache) = P(catch|cavity) \qquad P(a|b) = P(a)$$
$$P(toothache|cavity \wedge catch) = P(toothache|cavity) \qquad P(b|a) = P(b)$$

- The decomposition of large probabilistic domains into weakly connected subsets via conditional independence is one of the most important developments in the recent history of AI

$$P(a \wedge b) = P(a)P(b)$$
$$P(toothache, catch, cavity, Weather = cloudy) = P(Weather = cloudy)P(toothache, catch, cavity)$$

Naive Bayes Classifier

- In contrast with optimal Bayes, naïve Bayes places conditional assumption!
  - known to work well, even the assumption is not true!
    - a single cause directly influence a number of conditionally independent effects

$$P(\text{cause}, \text{effect}_1, \text{effect}_2, \ldots \text{effect}_n) = P(\text{cause}) \prod_{i=1}^{n} P(\text{effect}_i \mid \text{cause})$$

- Along with decision trees, neural networks, nearest neighbors, a widely used learning approaches
- **When** to use?
  - moderate or large training set available
  - variables describing instances are (class-conditionally) independent
- **Successful applications**
  - classifying text documents
  - diagnosis

Recall: Bayes classifier

- Assume target function $f\colon X \longrightarrow \Sigma$
  - where each instance $\mathbf{x}$ is described by features $x_1, x_2 \ldots x_m$
  - the most probable value of $f(\mathbf{x})$ is

$$h_{MAP} = \underset{h}{\operatorname{argmax}}\ p(h|\mathbf{x})$$

$$h_{MAP} = \underset{h}{\operatorname{argmax}} \frac{p(\mathbf{x}|h) \cdot p(h)}{p(\mathbf{x})}$$

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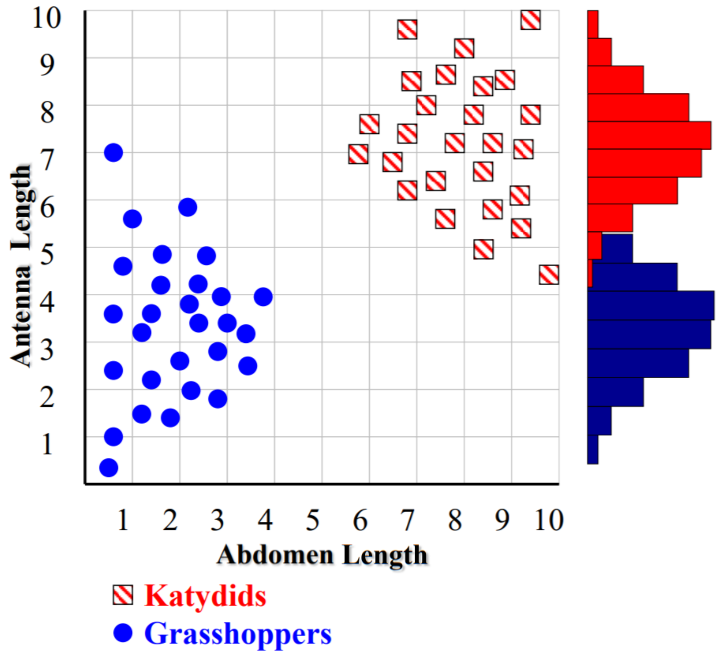

Naïve assumption

- Naive Bayes assumption:

$$p(h|\mathbf{x}) = p(h|y_1 = x_1, y_2 = x_2, \dots, y_m = x_m) = \prod_{j=1}^{m} p(h|y_j = x_j)$$

  – … yielding

    the naïve Bayes classifier:

$$h_{MAP} = \underset{h}{\operatorname{argmax}}\, p(h) \prod_{j=1}^{m} p(y_j = x_j|h)$$

  – for each target value h: estimate $p(h)$
  – for each attribute value x_j: estimate $p(x_j|h)$
  – are frequentist views the only possibility to estimate these probabilities?

$$h_{MAP} = \underset{h}{\operatorname{argmax}}\, \hat{p}(h) \prod_{j=1}^{m} \hat{p}(y_j = x_j|h)$$

Naïve Bayes: example

Goal

– learn NB classifier and classify $\mathbf{x}_{new} = (\mathrm{age} \le 30, \mathrm{income} = medium, \mathrm{student} = yes, \mathrm{rating} = fair)$

age	income	student	credit rating	buys computer
≤30	high	no	fair	no
≤30	high	no	excellent	no
31..40	high	no	fair	yes
>40	medium	no	fair	yes
>40	low	yes	fair	yes
>40	low	yes	excellent	no
31..40	low	yes	excellent	yes
≤30	medium	no	fair	no
≤30	low	yes	fair	yes
>40	medium	yes	fair	yes
≤30	medium	yes	excellent	yes
31..40	medium	no	excellent	yes
31..40	high	yes	fair	yes
>40	medium	no	excellent	no

Naïve Bayes: example

- p(buys_computer=*yes*)=9/14 and p(buys_computer=*no*)=5/14
- Class-conditional distributions $p(x|h)$
  - p(age= $\leq$ 30 | buys_computer=*yes*) = 2/9=0.22
  - p(age= $\leq$ 30 | buys_computer=*no*) = 3/5 =0.6
  - p(income=*medium* | buys_computer=*yes*) = 4/9 =0.44
  - p(income=*medium* | buys_computer=*no*) = 2/5 = 0.4
  - p(student=*yes* | buys_computer=*yes*) = 6/9 =0.67
  - p(student=*yes* | buys_computer=*no*) = 1/5 = 0.2
  - p(credit_rating=*fair* | buys_computer=yes) = 6/9 = 0.67
  - p(credit_rating=*fair* | buys_computer=*no*) = 2/5 = 0.4

- $\mathbf{x}_{new}$ = (age $\leq$ 30, income =medium, student=yes, rating=fair)
  - $p(\mathbf{x}|c_1)$: $p(\mathbf{x}|$buys_computer=*yes*)= 0.222 x 0.444 x 0.667 x 0.0.667 =0.044
  - $p(\mathbf{x}|c_2)$: $p(\mathbf{x}|$buys_computer=*no*)= 0.6 x 0.4 x 0.2 x 0.4 =0.019
  - $p(\mathbf{x}|c_1) \times p(c_1)$: $p(\mathbf{x}|$buys_computer=*yes*) $\times$ p(buys_computer=*yes*)=0.028
  - $p(\mathbf{x}|c_2) \times p(c_2)$: $p(\mathbf{x}|$buys_computer=*no*) $\times$ p(buys_computer=*no*)=0.007

- $\mathbf{x}$ belongs to class buys_computer=*yes*, $p(c_1|\mathbf{x})$ = 0.028/(0.028+0.007)

Estimating probabilities in small samples

- We have estimated probabilities based on the times an event occurs, n_a, over total opportunities, n
  – however n_a estimate can be poor when n is small
  – **problem**: what if none of the training instances with a target h have the value a?
    – n_a is 0 which will rewrite the joint probability $p(\mathbf{x}|h)$ as 0, irrespective of other features
- **Solution**?
  – when n_a is very small:

$$\text{from } p(y_j = a|h) = \frac{n_a}{n} \text{ to } \hat{p}(y_j = a|h) = \frac{n_a + d \times p(h)}{n + d}$$

    – n is number of training examples
    – n_a number of a occurrences from observations with target h
    – $p(h)$ is the prior estimate
    – d is the weight given to the prior: the number of "virtual" examples, generally $d \ll n$

Naïve Bayes: comments

- **Advantages**
  - easy to implement
  - good results obtained in most of the cases
- **Disadvantages**
  - class conditional independence *assumption*
    - loss of accuracy
    - dependencies exist among variables (e.g. symptoms)

- How to deal with these dependencies?
  - **Bayesian Belief Networks**
    - relaxes independence assumption towards sets of variables

Outline

- **Probability theory**
  - prior and posterior probability
  - maximum a posterior (MAP) and maximum likelihood (ML)
- **Bayes optimal classifier**
  - Bayesian learning in discrete spaces
  - Bayesian learning in numeric data spaces
- **Naïve Bayes classifier**
  - conditional independence
  - classification with naïve Bayes
  - estimating probabilities in small samples

Thank You

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