Seminar 3

Să se rezolve sist

(1)

$$\begin{cases} x + \alpha y + z = 1 \\ \alpha x - y + z = 1 \\ x + y - z = 2 \end{cases}$$

$\alpha \in \mathbb{R}$.
Discuție după $\mathbb{R}$

$$A = \left(\begin{array}{ccc|c} 1 & \alpha & 1 & 1 \\ \alpha & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 \end{array}\right)$$

$$\det A = \begin{vmatrix} 1 & \alpha & 1 \\ \alpha & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 1 + \alpha + \alpha + \cancel{1} - \cancel{1} + \alpha^2$$

$$\begin{matrix} 1 & \alpha & 1 \\ \alpha & -1 & 1 \end{matrix} \qquad = \alpha^2 + 2\alpha + 1 = (\alpha+1)^2$$

caz I $\Delta \neq 0 \Rightarrow \alpha \in \mathbb{R} - \{-1\}$ SCD $\Rightarrow$ S. Cramer
$\Rightarrow r_A = r_{\bar{A}} = 3$

$$d_x = \begin{vmatrix} 1 & \alpha & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 1 + \cancel{1} + 2\alpha + 2 - \cancel{1} + \alpha$$

$$\begin{matrix} 1 & \alpha & 1 \\ 1 & -1 & 1 \end{matrix} \qquad = 3\alpha + 3 = 3(\alpha+1)$$

$$x = \frac{d_x}{d} = \frac{3\cancel{(\alpha+1)}}{(\alpha+1)^{\cancel{2}}} = \frac{3}{\alpha+1}$$

$$d_y = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -\cancel{1} + 2\alpha \cancel{+1} - \cancel{1} - \cancel{2} + \alpha$$

$$\begin{matrix} 1 & 1 & 1 \\ \alpha & 1 & 1 \end{matrix} \qquad = 3\alpha - 3 = 3(\alpha-1)$$

$$y = \frac{d_y}{d} = \frac{3(\alpha-1)}{(\alpha+1)^2}$$

$$d_z = \begin{vmatrix} 1 & \alpha & 1 \\ \alpha & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2 + \alpha + \alpha + \cancel{1} - \cancel{1} - 2\alpha^2$$

$$\begin{matrix} 1 & \alpha & 1 \\ \alpha & -1 & 1 \end{matrix} \qquad = -2\alpha^2 + 2\alpha - 2$$

$$-2(\alpha^2 - \alpha + 1)$$

$$z = \frac{d_z}{d} = \frac{-2(\alpha^2 - \alpha + 1)}{(\alpha+1)^2}$$

$$S = \left\{ \left(\frac{3}{\alpha+1}, \frac{3(\alpha-1)}{(\alpha+1)^2}, \frac{-2(\alpha^2-\alpha+1)}{(\alpha+1)^2} \right) \right\}$$

CAZ II: $\Delta = 0 \Rightarrow \alpha = -1$

$$\bar{A} = \left(\underbrace{\begin{matrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{matrix}}_{A}\left|\begin{matrix} 1 \\ 1 \\ 2 \end{matrix}\right.\right)$$

$$d_p = \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} = -1 - 1 = -2 \neq 0$$

$$d_{car} = \begin{vmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2 - 1 - 1 \cancel{+1} \cancel{-1} - 2$$

$$\begin{matrix} 1 & -1 & 1 \\ -1 & -1 & 1 \end{matrix} \qquad = -6 \neq 0 \Rightarrow SI$$

(2) $\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ x + \alpha^2 z = 0 \end{cases}$ $\alpha \in \mathbb{R}$ $\bar{A} = \left(\underbrace{\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & \alpha^2 \end{matrix}}_{A}\left|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right.\right)$

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & \alpha^2 \end{vmatrix} = 5\alpha^2 + 0 + 12 - 15 + 0 - 8\alpha^2$$

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{matrix} \qquad = -3\alpha^2 - 3 = -3(\underbrace{\alpha^2 + 1}_{>0}) \neq 0$$

$\Rightarrow rA = r\bar{A} = 3 \Rightarrow SCD \Rightarrow \{(0,0,0)\}$

(4) $\triangle ABC$ și a, b, c lungimile lat.

$\begin{cases} ay + bx = c \\ cx + az = b \\ bz + cy = a \end{cases}$ Arătați că pt $\forall \triangle ABC \Rightarrow SCD$ și sol. unică (x_0, y_0, z_0) verifică $x_0, y_0, z_0 \in (-1,$

$$\det A = \begin{vmatrix} b & a & 0 \\ c & 0 & A \\ 0 & c & b \end{vmatrix} = 0 + 0 + 0 - abc - abc = -2abc \neq 0 \Rightarrow$$

$$\begin{matrix} b & a & 0 \\ c & 0 & a \end{matrix} \qquad \Rightarrow SCD \Rightarrow S\ Cramer$$

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$$\Delta x = \begin{vmatrix} b & 0 & a \\ a & c & b \\ c & a & 0 \end{vmatrix} \begin{matrix} b & 0 & a \\ a & c & b \end{matrix} = 0+0+a^3-0-c^2a-b^2a = -a(c^2+b^2-a)$$

$$x = \frac{\Delta x}{\Delta} = \frac{-a(c^2+b^2-a^2)}{-2abc} = \frac{c^2+b^2-a^2}{2bc}$$

Th cos : $a^2 = b^2+c^2-2bc\cos A$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$A \in (0,\pi) \Rightarrow x = \cos A \in (-1,1)$

analog : $y = \cos B$

$z = \cos C$

(5) $\begin{cases} x+y+z=0 \\ (b+c)x+(a+c)y+(a+b)z=0 \\ bcx+acy+abz=0 \end{cases}$ $a,b,c \in \mathbb{R}$, distincte două câte două

$$\bar{A} = \left(\underbrace{\begin{matrix} 1 & 1 & 1 \\ b+c & a+c & a+b \\ bc & ac & ab \end{matrix}}_{A} \middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right)$$

$$\det A \overset{C_2-C_1}{\underset{C_3-C_1}{=}} \begin{vmatrix} 1 & 0 & 0 \\ b+c & a-b & a-c \\ bc & c(a-b) & b(a-c) \end{vmatrix}$$

$$= (-1)^{1+1}\cdot 1 \cdot \begin{vmatrix} a-b & a-c \\ c(a-b) & b(a-c) \end{vmatrix} = (a-b)(a-c)\begin{vmatrix} 1 & 1 \\ c & b \end{vmatrix}$$

$$= \underbrace{(a-b)}_{\neq 0}\underbrace{(a-c)}_{\neq 0}\underbrace{(b-c)}_{\neq 0} \neq 0 \Rightarrow S.C.D \Rightarrow \exists \text{ sol } \{(0,0,0)\}$$

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⑥ $\begin{cases} x + 2y = m+1 \\ 2x + 3y = m-1 \\ mx + y = 3 \end{cases}$ $m = ? \Rightarrow S.i$

$$\bar{A} = \left(\begin{array}{cc|c} 1 & 2 & m+1 \\ 2 & 3 & m-1 \\ m & 1 & 3 \end{array}\right)$$

A

$d_p = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0 \Rightarrow r_A = 2$

$d_{car} \neq 0$

$$d_{car} = \begin{vmatrix} 1 & 2 & m+1 \\ 2 & 3 & m-1 \\ m & 1 & 3 \end{vmatrix} = 9 + \cancel{2m} + 2 + 2m^2 - \cancel{2m} - 3m(m+1) - m + 1 - 12$$

$$\begin{array}{ccc} 1 & 2 & m+1 \\ 2 & 3 & m-1 \end{array}$$

$= \cancel{1} + 2m^2 - 3m^2 - 3m - m - \cancel{1}$

$= -m^2 - 4m = -m(m+4)$

$d_{car} \neq 0 \Rightarrow -m(m+4) \neq 0$

$\Rightarrow m \in \mathbb{R} - \{-4, 0\}$

⑦ $\sum_{i=1}^{k} (1+i) x_i + \sum_{i=1}^{4-k} i \, x_{i+k} = 0, (\forall)\ k = \overline{1,3}$

pt. $k=1 \Rightarrow \sum_{i=1}^{1} (1+i) x_i + \sum_{i=1}^{3} i \, x_{i+1} = 0$

$2x_1 + x_2 + 2x_3 + 3x_4 = 0$

pt $k=2 \Rightarrow \sum_{i=1}^{2} (1+i) x_i + \sum_{i=1}^{2} i \, x_{i+2} = 0$

$2x_1 + 3x_2 + x_3 + 2x_4 = 0$

pt $k=3 \Rightarrow \sum_{i=1}^{3} (1+i) x_i + \sum_{i=1}^{1} i \, x_{i+3} = 0$

$2x_1 + 3x_2 + 4x_3 + x_4 = 0$

$$\begin{cases} 2x_1 + x_2 + 2x_3 + 3x_4 = 0 \\ 2x_1 + 3x_2 + x_3 + 2x_4 = 0 \\ 2x_1 + 3x_2 + 4x_3 + x_4 = 0 \end{cases} \quad \bar{A} = \left(\begin{array}{cccc|c} 2 & 1 & 2 & 3 & 0 \\ 2 & 3 & 1 & 2 & 0 \\ 2 & 3 & 4 & 1 & 0 \end{array}\right)$$

A

$$d_2 = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 6 - 2 = 4 \neq 0$$

$$d_3 = \begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 2 & 3 & 4 \end{vmatrix} = 24 + \cancel{12} + 2 - \cancel{12} - 6 - 8$$

$$\begin{matrix} 2 & 1 & 2 \\ 2 & 3 & 1 \end{matrix} \quad = 20 - 8 = 12 \neq 0$$

$\Rightarrow r_A = 3 = rg\,\bar{A} \Rightarrow$ SC simplu N.

x_1, x_2, x_3 variabile principale

x_4 var. secundare

$x_4 = \alpha$

$$\begin{cases} 2x_1 + x_2 + 2x_3 = -3\alpha \\ 2x_1 + 3x_2 + x_3 = -2\alpha \\ 2x_1 + 3x_2 + 4x_3 = -\alpha \end{cases} \xRightarrow{ec3 - ec2} \quad 3x_3 = \alpha, \quad x_3 = \frac{\alpha}{3}$$

$$2x_1 + x_2 = -3\alpha - \frac{2\alpha}{3} = \frac{-11\alpha}{3}$$

$$2x_1 + 3x_2 = -2\alpha - \frac{\alpha}{3} = -\frac{7\alpha}{3} \quad (-)$$

$$2x_2 = -\frac{7\alpha}{3} + \frac{11\alpha}{3} = \frac{4\alpha}{3}$$

$$x_2 = \frac{2\alpha}{3}$$

$$2x_1 = -\frac{11\alpha}{3} - \frac{2\alpha}{3} = -\frac{13\alpha}{3} \Rightarrow x_1 = \frac{-13\alpha}{6}$$

$$S = \left\{\left(-\frac{13\alpha}{6}, \frac{2\alpha}{3}, \frac{\alpha}{3}, \alpha\right)\right\}$$

(8) $\sum_{j=1}^{4} a_{ij} x_j = 4^{i-1}$, $(\forall)\ i = \overline{1,4}$, unde $a_{ij} = j^{i-1}$, $(\forall)\ i,j = \overline{1,4}$

$$\bar{A} = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 4 \\ 1^2 & 2^2 & 3^2 & 4^2 & 4^2 \\ 1^3 & 2^3 & 3^3 & 4^3 & 4^3 \end{array}\right)$$

A

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = (d-c)(d-c)(d-a)(c-b)(c\ldots$$
$$(b-a)$$

$$\Delta_A = \underbrace{(4-3)(4-2)(4-1)}_{3!}\underbrace{(3-2)(3-1)}_{2!}\underbrace{(2-1)}_{1!} \neq 0 \Rightarrow S.C.$$

$$\Delta_{x_1} = \Delta_{x_2} = \Delta_{x_3} = 0 \quad (2 \text{ col egale})$$

$$\Delta_{x_4} = \Delta_A = 3! \cdot 2! \cdot 1!$$

$$S = \{(0, 0, 0, 1)\}$$

Spații vectoriale

"$+$" $: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ (lege internă)

"$\cdot$" $: \underline{\mathbb{R}} \times \mathbb{R}^2 \to \mathbb{R}^2$ (lege externă)

1 ex: a) $(x, y) + (x', y') = (x + x', 0)$

$\alpha(x, y) = (\alpha x, \alpha y)$

$(\mathbb{R}^2, +, \cdot)\big|_{\mathbb{R}}$ este spațiu vectorial?

"$+$" nu are elem. neutru $\Rightarrow (\mathbb{R}^2, +)$ nu e grup abelian $\Rightarrow (\mathbb{R}, +, \cdot)\big|_{\mathbb{R}}$ nu e spațiu vectorial

ex: b) $(x, y) + (x', y') = (x + x', y')$

$\alpha(x, y) = (\alpha x, \alpha y)$

$\Rightarrow$ "$+$" nu e comutativă $\Rightarrow (\mathbb{R}^2, +)$ nu e gr. abelian

$\Rightarrow (\mathbb{R}, +, \cdot)$ nu e sp. vect.

c) $(x, y) + (x', y') = (x + x', y + y')$

$\alpha(x, y) = (0, \alpha y)$

axioma 5: $1_K \cdot v = v \Rightarrow 1 \cdot (x, y) = (x, y)$ (F)

$\Rightarrow$ axioma nu este satisfăcută

$\Rightarrow$ nu este spațiu vectorial

d) $(x,y)+(x',y')=(x+x', y+y')$

$\alpha(x,y)=(\alpha x, \alpha y)$

$\Rightarrow (\mathbb{R}^2,+,\cdot)|_{\mathbb{R}}$ spațiu vect

(2) $"+": \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^2$

$"\cdot": \mathbb{C}\times\mathbb{R}^2 \to \mathbb{R}^2$

$(x,y)+(x',y')=(x+x', y+y')$

$(a+ib)\overset{(x+iy)}{(x,y)}=(ax-by, ay+bx)$

ax. 2 $(a+ib)[(a'+ib')\cdot(x,y)]=[(a+ib)(a'+ib')](x,y)$

$(\mathbb{R}^2,+,\cdot)|_{\mathbb{C}}$ e sp. vect ?

OBS: $K=(\{a_0,a_1,a_2,a_3\},+)$ grup abelian

$((\forall)\ x\in K \Rightarrow x+x=e=a_0)$

$\mathbb{Z}_2=\{\hat{0},\hat{1}\}$

$"\cdot": \mathbb{Z}_2\times K \to K$

$\hat{0}\cdot a_i=a_0$

$\hat{1}\cdot a_i=a_i$, $\forall$ $i=\overline{0,3}$

$(K,+,\cdot)|_{\mathbb{Z}_2}$