C5 AG.

Teorema (Grassmann)

$(V,+,\cdot)|\mathbb{K}$ sp. vect. finit generat, $V_1, V_2 \subset V$ subsp vect

$\Rightarrow \dim_{\mathbb{K}}(V_1+V_2) = \dim_{\mathbb{K}} V_1 + \dim_{\mathbb{K}} V_2 - \dim_{\mathbb{K}}(V_1 \cap V_2)$

Dem

$\langle V_1 \cup V_2 \rangle = V_1 + V_2$

Not. $\dim_{\mathbb{K}} V = n$, $\dim_{\mathbb{K}} V_i = n_i$, $i=\overline{1,2}$, $\dim_{\mathbb{K}}(V_1 \cap V_2) = p$.

$n_i < n$, $i=\overline{1,2}$, $p < n$.

Fie $\mathcal{B}_0 = \{e_1, \dots, e_p\}$ bază în $V_1 \cap V_2$

Completăm la $\mathcal{B}_1 = \{e_1, \dots, e_p, \underline{f_{p+1}, \dots, f_{n_1}}\}$ bază în V_1

$\mathcal{B}_2 = \{e_1, \dots, e_p, g_{p+1}, \dots, g_{n_2}\}$ bază în V_2

Fie $\mathcal{B} = \{e_1, \dots, e_p, f_{p+1}, \dots, f_{n_1}, g_{p+1}, \dots, g_{n_2}\}$. Dem că $\mathcal{B}$ este bază în $V_1 + V_2$.

$\operatorname{card}\mathcal{B} = \cancel{p} + n_1 - \cancel{p} + n_2 - p = n_1 + n_2 - p$.

① $\mathcal{B}$ este SLI. Fie $a_1, \dots, a_p, b_{p+1}, \dots, b_{n_1}, c_{p+1}, \dots, c_{n_2} \in \mathbb{K}$ a.î.

$$\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j + \sum_{k=p+1}^{n_2} c_k g_k = 0$$

$$x = \underbrace{\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j}_{\in V_1} = \underbrace{-\sum_{k=p+1}^{n_2} c_k g_k}_{\in V_2} \in V_1 \cap V_2 = \langle \mathcal{B}_0 \rangle$$

$\exists a'_1, \dots, a'_p$ a.î. $x = \sum_{i=1}^{p} a'_i e_i$

1) $\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j = \sum_{i=1}^{p} a'_i e_i \Rightarrow \sum_{i=1}^{p} (a_i - a'_i) e_i + \sum_{j=p+1}^{n_1} b_j f_j = 0$

$\xRightarrow{\mathcal{B}_1 \text{ e SLI}}$ $a_i - a'_i = 0$, $\forall i = \overline{1,p}$

$b_j = 0$, $\forall j = \overline{p+1, n_1}$

$$-\sum_{k=p+1}^{n_2} c_k g_k = \sum_{i=1}^{p} a_i' e_i \Rightarrow \sum_{i=1}^{p} a_i' e_i + \sum_{k=p+1}^{n_2} c_k g_k = 0 \overset{B_2 \, S_1}{\underset{SLI}{\Longrightarrow}}$$

$$\Rightarrow a_i' = 0, \ \forall i = \overline{1,p} \qquad \Longrightarrow a_i = 0, \ \forall i = \overline{1,p}$$

$$c_k = 0, \ \forall k = \overline{p+1, n_2}$$

Deci $\mathcal{B}$ este SLI.

(2) $\mathcal{B}$ este SG. i.e $V_1 + V_2 = \langle \mathcal{B} \rangle$.

$\forall x \in V_1 + V_2 \Rightarrow \exists \, x_1 \in V_1, \ x_2 \in V_2$ aî $x = x_1 + x_2$

$$x = \left(\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j \right) + \left(\sum_{i=1}^{p} a_i' e_i + \sum_{k=p+1}^{n_2} c_k g_k \right)$$

$$= \sum_{i=1}^{p} (a_i + a_i') e_i + \sum_{j=p+1}^{n_1} b_j f_j + \sum_{k=p+1}^{n_2} c_k g_k \Rightarrow \mathcal{B} \text{ este SG}$$

Prin urmare, $\mathcal{B}$ este bază

$$\dim_{\mathbb{K}} (V_1 + V_2) = \operatorname{card} \mathcal{B} = n_1 + n_2 - p =$$

$$= \dim_{\mathbb{K}} V_1 + \dim_{\mathbb{K}} V_2 - \dim_{\mathbb{K}} (V_1 \cap V_2).$$

<u>OBS</u> $\dim_{\mathbb{K}} (V_1 \oplus V_2) = \dim_{\mathbb{K}} V_1 + \dim_{\mathbb{K}} V_2 \qquad (V_1 \cap V_2 = \{0_V\})$

<u>Teoremă</u> $A \in \mathcal{M}_{m,n}(\mathbb{R})$

$S(A) = \{ x \in \mathbb{R}^n \mid AX = 0 \} \subset \mathbb{R}^n$ subsp. vect

$\dim S(A) = n - rg(A)$

<u>Prop</u> $(V, +, \cdot)|_{\mathbb{K}}$ sp vect, $V' \subset V$ subsp. vect.
Coordonatele vect din V', în raport cu $\forall$ reper, sunt soluțiile unui SLO

i.e. $\exists \, A$ aî $V' = S(A)$

Aplicaţie

Fie $(\mathbb{R}^4, +, \cdot)_{/\mathbb{R}}$, $V' = \langle \{ \overset{u}{(1,1,0,0)}, \overset{v}{(1,0,1,-1)} \} \rangle$

a) Să se descrie V' printr-un sistem de ec. liniare.

b) $\mathbb{R}^4 = V' \oplus V''$, $V'' = ?$
(subsp. vect. complementar lui V')

$\mathcal{R}' = \{u, v\}$ este SG pt V'

$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$ $\quad rg A = 2 = max \overset{crit}{\underset{LI}{\Longrightarrow}} \mathcal{R}'$ este SLI

Deci $\mathcal{R}'$ este bază în V'

$\forall x \in V', \exists a, b \in \mathbb{R}$ a.î. $x = au + bv$

$(x_1, x_2, x_3, x_4) = a(1,1,0,0) + b(1,0,1,-1)$
$= (a+b, a, b, -b)$

$$\circledast \begin{cases} a+b = x_1 \\ a = x_2 \\ b = x_3 \\ -b = x_4 \end{cases} \quad \left(\begin{array}{cc|c} \boxed{\begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}} & & x_1 \\ & & x_2 \\ 0 & 1 & x_3 \\ 0 & -1 & x_4 \end{array}\right)$$

$\circledast$ este compatibil $\Leftrightarrow rg A = rg \bar{A} = 2$

$\Delta_p = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0$

$\Delta_{c_1} = \begin{vmatrix} 1 & 1 & x_1 \\ 1 & 0 & x_2 \\ 0 & 1 & x_3 \end{vmatrix} = \begin{vmatrix} 1 & \cancel{1} & \cancel{x_1} \\ 0 & -1 & x_2 - x_1 \\ 0 & 1 & x_3 \end{vmatrix} = -x_3 - x_2 + x_1 = 0$

$\Delta_{c_2} = \begin{vmatrix} 1 & 1 & x_1 \\ 1 & 0 & x_2 \\ 0 & -1 & x_4 \end{vmatrix} = \begin{vmatrix} 1 & \cancel{1} & \cancel{x_1} \\ 0 & -1 & x_2 - x_1 \\ 0 & -1 & x_4 \end{vmatrix} = -x_4 + x_2 - x_1 = 0$

$V' = \left\{ x \in \mathbb{R}^4 \,\middle|\, \begin{cases} x_1 - x_2 - x_3 = 0 \\ -x_1 + x_2 - x_4 = 0 \end{cases} \right\} = S(A')$

$A' = \left(\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \end{array}\right)$ $\quad \begin{cases} -x_3 = -x_1 + x_2 \\ -x_4 = x_1 - x_2 \end{cases}$

OBS

$\begin{cases} x_3 = x_1 - x_2 \\ x_4 = -x_1 + x_2 \end{cases}$ $\quad \left. \begin{array}{l} \mathcal{R}_1 = \{(1,0,1,-1), (0,1,-1,1)\} \text{ SG} \\ card\, \mathcal{R}_1 = dim\, V' = 2 \end{array} \right| \Rightarrow \mathcal{R}_1$ bază în V'

$V' = \{ (x_1, x_2, x_1 - x_2, -x_1 + x_2) \mid x_1, x_2 \in \mathbb{R} \} = \{ x_1(1,0,1,-1) + x_2(0,1,-1,1) \mid x_1, x_2 \in \mathbb{R} \}$

b) $\mathbb{R}^4 = V' \oplus V''$

$\mathcal{R}' = \{u, v\}$ bază în V'. Extindem la o bază în $\mathbb{R}^4$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad rg\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = 4 = max$$

nu convine

$\mathcal{R}'' = \{e_1, e_4\}$ $\qquad V'' = \langle \mathcal{R}'' \rangle$

$\mathcal{R}' \cup \mathcal{R}''$ bază în $\mathbb{R}^4$.

<u>Aplicație</u> $(\mathbb{R}^4, +, \cdot)/\mathbb{R}$, $V' = \{(x, y, z, t) \in \mathbb{R}^4 \mid x + y - z - 3t = 0\}$

$V'' = \{(x, y, z, t) \in \mathbb{R}^4 \mid x + y + z + 2t = 0\}$

$\mathbb{R}^4 = V' + V''$, dar suma <u>nu</u> e directă.

$\dim V' = \dim V'' = 4 - 1 = 3$ (V', V'' = hiperplane care trec prin origine)

$\dim(V' + V'') = 3 + 3 - \dim(V' \cap V'') = 6 - 2 = 4$

$$V' \cap V'' = \left\{ (x, y, z, t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} x + y - z - 3t = 0 \\ x + y + z + 2t = 0 \end{cases} \right\}$$

$\dim V' \cap V'' = 4 - 2 = 2$ $\qquad A = \left(\begin{array}{cc|cc} 1 & 1 & -1 & -3 \\ 1 & 1 & 1 & 2 \end{array}\right) \begin{matrix} 0 \\ 0 \end{matrix}$

$$\left.\begin{matrix} V' + V'' \subset \mathbb{R}^4 \\ \text{dar } \dim(V' + V'') = \dim \mathbb{R}^4 = 4 \end{matrix}\right\} \Rightarrow \mathbb{R}^4 = V' + V''$$

$V' \cap V'' \neq \{0_{\mathbb{R}^4}\}$ $\qquad$ suma nu e directă.

<u>Morfisme de spații vectoriale</u>

<u>Def</u> $(V_i, +, \cdot)/K_i$, $i = \overline{1,2}$ sp. vectoriale.

$f: V_1 \to V_2$ s.n. <u>aplicație semi-liniară</u> $\Leftrightarrow$

1) $f(x+y) = f(x) + f(y), \ \forall x, y \in V_1$

2) $\exists\, \theta : \mathbb{K}_1 \longrightarrow \mathbb{K}_2$ izomorfism de corpuri ai

$$f(\alpha x) = \theta(\alpha) f(x), \ \forall x \in V_1, \forall \alpha \in \mathbb{K}_1.$$

Dacă $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{K}$ şi $\theta = id_{\mathbb{K}}$, at f s.n. aplicaţie liniară sau morfism de spaţii vectoriale.

Exemple

1) $(V_i, +, \cdot)/\mathbb{R}$, $i = \overline{1,2}$

$\theta : \mathbb{R} \to \mathbb{R}$ automorfism de corpuri $\Rightarrow \theta = id_{\mathbb{R}}$

$f : V_1 \to V_2$ apl. semi-liniară $\Rightarrow$ f este apl. liniară.

2) $(V_i, +, \cdot)/\mathbb{C}$, $i = \overline{1,2}$ $\quad V_i = \mathbb{C}^n$, $i = \overline{1,2}$

$\theta : \mathbb{C} \to \mathbb{C}$ automorfism de corpuri $\theta(\alpha) = \overline{\alpha}$

$f : V_1 \longrightarrow V_2$, $f(z) = \overline{z}$; $f(z+u) = f(z) + f(u)$

$\quad \overset{\shortparallel}{\mathbb{C}^n} \quad \overset{\shortparallel}{\mathbb{C}^n}$ $\quad f(z_1, \dots, z_n) = (\overline{z_1}, \dots, \overline{z_n})$, $z = (z_1, \dots, z_n)$.

$f(\alpha z) = \overline{\alpha z} = \overline{\alpha}\, \overline{z} = \theta(\alpha) f(z)$

f este apl. semi-liniară şi nu e liniară.

Aplicaţii liniare

Def: $f : V_1 \to V_2$ apl. liniară

f s.n. izomorfism de sp vect dacă este şi bijectivă.

- $(V, +, \cdot)/\mathbb{K}$

$End(V) = \{ f : V \to V \mid f \text{ liniară} \}$

$Aut(V) = \{ f \in End(V) \mid f \text{ bijecție} \}$

OBS a) $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$ $\quad f, g$ liniare $\Rightarrow h = g \circ f$ liniară.

b) $f : (V_1, +) \to (V_2, +)$

f liniară $\Rightarrow$ f morf. grupuri şi $f(0_{V_1}) = 0_{V_2}$.

Exemple de aplicații liniare

① $f: V \longrightarrow V$, $f(x) = 0_V$, $f(x) = x$, $\forall x \in V$.

② $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $f(x) = y$ $\quad Y = AX$

$$\begin{pmatrix} y_1 \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} \dots & a_{1n} \\ a_{n1} \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

③ $f: \mathcal{M}_n(\mathbb{R}) \longrightarrow \mathbb{R}$, $f(A) = Tr(A)$, $\forall A \in \mathcal{M}_n(\mathbb{R})$

$f(A+B) = Tr(A+B) = Tr(A) + Tr(B) = f(A) + f(B)$

$f(\alpha A) = Tr(\alpha A) = \alpha Tr(A) = \alpha f(A)$, $\forall A, B \in \mathcal{M}_n(\mathbb{R})$, $\alpha \in \mathbb{R}$.

④ $f(A) = \det A$ nu este apl liniară

$f: \mathcal{M}_n(\mathbb{R}) \longrightarrow \mathbb{R}$.

Prop de caracterizare

$f: V_1 \longrightarrow V_2$ liniară $\Leftrightarrow$ $\forall x, y \in V_1$, $\forall a, b \in \mathbb{K}$: $f(ax+by) = af(x) + bf(y)$

$\Leftrightarrow f\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i f(x_i)$

$\forall x_i \in V_1$, $\forall a_i \in \mathbb{K}$, $i = \overline{1,n}$

Dem

„$\Rightarrow$" f lin: $f(x+y) = f(x) + f(y)$ și $f(ax) = af(x)$

$a \in \mathbb{K}$, $x \in V_1 \Rightarrow ax \in V_1$

$b \in \mathbb{K}$, $y \in V_1 \Rightarrow by \in V_1$ $\Rightarrow f(ax+by) = f(ax) + f(by)$

$= af(x) + bf(y)$

„$\Leftarrow$" $f(ax+by) = af(x) + bf(y)$, $\forall x, y \in V_1$, $\forall a, b \in \mathbb{K}$

Fie $a = b = 1_{\mathbb{K}}$

$f(1_{\mathbb{K}} x + 1_{\mathbb{K}} y) = 1_{\mathbb{K}} f(x) + 1_{\mathbb{K}} f(y) \Rightarrow f(x+y) = f(x) + f(y)$

Fie $b = 0_{\mathbb{K}}$

$f(ax + 0_{\mathbb{K}} y) = a f(x) + 0_{\mathbb{K}} f(y) \Rightarrow f(ax) = af(x)$

OBS

$f: V_1 \to V_2$ apl liniară

Dacă $V' \subset V_1$ ssp vect, at $f(V') \subset V_2$ ssp. vect.

$$\left.\begin{array}{l}\forall\, y_1, y_2 \in f(V') \\ \forall\, a, b \in \mathbb{K}\end{array}\right\} \overset{?}{\Longrightarrow} a y_1 + b y_2 \in f(V')$$

$\exists\, x_1, x_2 \in V'$ aî $\begin{cases} y_1 = f(x_1) \\ y_2 = f(x_2)\end{cases}$

$$a y_1 + b y_2 = a f(x_1) + b f(x_2) = f(\underbrace{a x_1 + b x_2}_{\in V' \ (V' \text{ ssp v})}) \in f(V')$$

Def $f: V_1 \to V_2$ apl. lin.

$\operatorname{Ker}(f) = \{ x \in V_1 \mid f(x) = 0_{V_2} \}$ nucleul lui f.

$\operatorname{Im} f = \{ y \in V_2 \mid \exists\, x \in V_1 \text{ aî } f(x) = y \}$ imaginea lui f

Prop $f: V_1 \to V_2$ apl liniară

a) $\operatorname{Ker} f \subseteq V_1$, $\operatorname{Im} f \subseteq V_2$ subsp. vect

b) f injectivă $\Leftrightarrow$ $\operatorname{Ker} f = \{0_{V_1}\}$

c) f surjectivă $\Leftrightarrow$ $\dim_{\mathbb{K}} \operatorname{Im} f = \dim_{\mathbb{K}} V_2$

Dem

a) $\operatorname{Ker} f \subseteq V_1$ subsp vect.

$$\left.\begin{array}{l}\forall\, x_1, x_2 \in \operatorname{Ker} f \\ \forall\, a, b \in \mathbb{K}\end{array}\right\} \Rightarrow a x_1 + b x_2 \in \operatorname{Ker} f$$

$$f(a x_1 + b x_2) = a \underset{0_{V_2}}{\underset{\parallel}{f(x_1)}} + b \underset{0_{V_2}}{\underset{\parallel}{f(x_2)}} = 0_{V_2}$$

$f(V_1) = \operatorname{Im} f \subseteq V_2$ ssp (cf. obs).

b) „$\Rightarrow$" f inj. Dem că $\operatorname{Ker} f = \{0_{V_1}\}$.

Fie $x \in \operatorname{Ker} f \Rightarrow \left.\begin{array}{l} f(x) = 0_{V_2} \\ \text{dar } f(0_{V_1}) = 0_{V_2}\end{array}\right\} \overset{inj}{\Longrightarrow} x = 0_{V_1} \Rightarrow \operatorname{Ker} f = \{0_{V_1}\}$.

„$\Leftarrow$" $\operatorname{Ker} f = \{0_{V_1}\} \Rightarrow f$ inj

Fie $x_1, x_2 \in V_1$ aî $f(x_1) = f(x_2) \Rightarrow f(x_1 - x_2) = 0_{V_2} \underset{\operatorname{Ker} f = \{0_{V_1}\}}{\Rightarrow} x_1 = x_2 \Rightarrow f$ inj

c) "$\Rightarrow$" Ip: f surj $\Rightarrow \operatorname{Im} f = V_2 \Rightarrow \dim_{\mathbb{K}} \operatorname{Im} f = \dim_{\mathbb{K}} V_2$

"$\Leftarrow$" Ip: $\left.\begin{array}{l}\dim_{\mathbb{K}} \operatorname{Im} f = \dim_{\mathbb{K}} V_2 \\ \text{dar } \operatorname{Im} f \subseteq V_2 \text{ sspv}\end{array}\right\} \Rightarrow \operatorname{Im} f = V_2 \Rightarrow f$ surj.

Consecință

$f: V_1 \to V_2$ liniară

f izomorfism. $\Leftrightarrow \begin{cases} \operatorname{Ker} f = \{0_V\} \\ \dim \operatorname{Im} f = \dim V_2. \end{cases}$

Teorema dimensiunii

$f: V_1 \to V_2$ apl. liniară

$\Rightarrow \dim V_1 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$

Ex1. $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (x_1 + x_2 - x_3, x_1 + x_2, x_1 + x_2 + x_3)$

a) f liniară

b) $\operatorname{Ker} f$, $\operatorname{Im} f = ?$

Precizați câte un reper în fiecare.

SOL

a) 1) $f(x+y) = f(x) + f(y)$

$x + y = (x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$f(x+y) = (x_1 + y_1 + x_2 + y_2 - x_3 - y_3, x_1 + y_1 + x_2 + y_2, x_1 + y_1 + x_2 + y_2 + x_3 + y_3)$

$= (x_1 + x_2 - x_3, x_1 + x_2, x_1 + x_2 + x_3) + (y_1 + y_2 - y_3, y_1 + y_2, y_1 + y_2 + y_3)$

$= f(x) + f(y)$

2) $f(\alpha x) = \alpha f(x)$

$f(\alpha x) = f(\alpha x_1, \alpha x_2, \alpha x_3) = (\alpha x_1 + \alpha x_2 - \alpha x_3, \alpha x_1 + \alpha x_2, \alpha x_1 + \alpha x_2 + \alpha x_3)$

$= \alpha (x_1 + x_2 - x_3, x_1 + x_2, x_1 + x_2 + x_3) = \alpha f(x)$

Deci f apl. liniară

b) $\operatorname{Ker} f = \{x \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3}\} = \left\{x \in \mathbb{R}^3 \,\middle|\, \begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 + x_2 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases}\right\}$

$A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right)$

$\operatorname{Ker} f = S(A)$

$\dim \mathrm{Ker} f = 3 - rg A = 3-2=1 \Rightarrow f$ nu e inj

$\begin{cases} x_2 - x_3 = -x_1 \\ x_2 = -x_1 \end{cases} \Rightarrow \begin{matrix} x_3 = 0 \\ x_2 = -x_1 \end{matrix}$

$\mathrm{Ker} f = \{ (x_1, -x_1, 0) \mid x_1 \in \mathbb{R} \} = \{ x_1(1,-1,0), x \in \mathbb{R} \}$

$\mathcal{R}_1 = \{ (1,-1,0) \}$ bază în Ker f

$\mathrm{Im} f = \{ y \in \mathbb{R}^3 \mid x \in \mathbb{R}^3$ ai $f(x) = y \}$.

$(*) \begin{cases} x_1 + x_2 - x_3 = y_1 \\ x_1 + x_2 = y_2 \\ x_1 + x_2 + x_3 = y_3 \end{cases}$ $(*)$ este sistem compatibil

$rg A = rg \bar{A}$

$A = \left(\begin{matrix} 1 & \boxed{1 & -1} \\ 1 & \boxed{1 & 0} \\ 1 & 1 & 1 \end{matrix} \middle| \begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix} \right)$ $\Delta_c = \begin{vmatrix} 1 & -1 & y_1 \\ 1 & 0 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = 0$

$\begin{vmatrix} 1 & -1 & y_1 \\ 1 & 0 & y_2 \\ 2 & 0 & y_1 + y_3 \end{vmatrix} = 0$ $\begin{vmatrix} 1 & y_2 \\ 2 & y_1 + y_3 \end{vmatrix} = 0 \Rightarrow y_1 - 2y_2 + y_3 = 0$

$\mathrm{Im} f = \{ y \in \mathbb{R}^3 \mid y_1 - 2y_2 + y_3 = 0 \}$ $\dim \mathrm{Im} f = 3 - 1 = 2$

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

<u>Th. dim</u>: $\underset{\overset{\parallel}{3}}{\dim \mathbb{R}^3} = \underset{\overset{\parallel}{1}}{\dim \mathrm{Ker} f} + \dim \mathrm{Im} f \Rightarrow \dim \mathrm{Im} f = 2$

$y_1 = 2y_2 - y_3$

$\mathrm{Im} f = \{ (2y_2 - y_3, y_2, y_3), y_2, y_3 \in \mathbb{R} \} = \{ y_2(2,1,0) + y_3(-1,0,1), y_2, y_3 \in \mathbb{R} \}$

$\left. \begin{matrix} \mathcal{R}_2 = \{ (2,1,0), (-1,0,1) \} \text{ SG pt Im } f \\ \text{card } \mathcal{R}_2 = \dim \mathrm{Im} f = 2 \end{matrix} \right\} \Rightarrow \mathcal{R}_2$ bază a Im f.

<u>OBS!</u> $\mathcal{R}_1 = \{ (1,-1,0) \}$ bază în Ker f

Extindem la o bază în $\mathbb{R}^3$

$rg \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3 = max$ $\mathcal{R}_1 \cup \{ e_1, e_3 \}$ bază în $\mathbb{R}^3$

$\{ f(e_1), f(e_3) \}$ bază în Im f

$f(e_1) = f(1,0,0) = (1,1,1)$

$f(e_3) = f(0,0,1) = (-1,0,1)$

$\mathcal{R}_2' = \{(1,1,1), (-1,0,1)\}$ bază în $\operatorname{Im} f$

$y \in \operatorname{Im} f \Leftrightarrow \exists\, a, b \in \mathbb{R}$ al $(y_1, y_2, y_3) = a(1,1,1) + b(-1,0,1)$

$$\begin{cases} a-b = y_1 \\ a = y_2 \\ a+b = y_3 \end{cases} \qquad \left(\begin{array}{cc|c} 1 & -1 & y_1 \\ 1 & 0 & y_2 \\ 1 & 1 & y_3 \end{array}\right)$$

$$\Delta_c = 0 \Rightarrow \begin{vmatrix} 1 & -1 & y_1 \\ 1 & 0 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = 0$$

<u>Ex</u>. $f : \mathbb{R}_4[X] \longrightarrow \mathbb{R}_2[X]$, $f(P) = P''$ $\quad \widetilde{P} = P$ (funcția polinomială asociată)

a) f liniară

b) $\operatorname{Ker} f = ?$ $\quad \operatorname{Im} f = ?$

$P = a_0 + a_1X + a_2X^2 + a_3X^3 + a_4X^4$

$\widetilde{P} : \mathbb{R} \to \mathbb{R}$

$\widetilde{P}(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$.

$f(aP + bQ) = (aP + bQ)'' = aP'' + bQ'' = af(P) + bf(Q)$
$\forall P, Q \in \mathbb{R}_4[X], \forall a, b \in \mathbb{R}$

b) $\operatorname{Ker} f = \{P \in \mathbb{R}_4[X] \,/\, f(P) = 0\} = \{P = a_0 + a_1X,\ a_0, a_1 \in \mathbb{R}\}$

$P'' = 0 \Rightarrow P = a_0 + a_1X$ $\qquad = \mathbb{R}_1[X]$

$\mathcal{R}_1 = \{1, X\}$ bază în $\operatorname{Ker} f$.

T. dim: $\underbrace{\dim \mathbb{R}_4[X]}_{5} = \underbrace{\dim \operatorname{Ker} f}_{2} + \dim \operatorname{Im} f \Rightarrow$

$\dim \operatorname{Im} f = 3 = \dim \mathbb{R}_2[X] \Rightarrow f$ surjectivă