Seminar 8 < Vectori. Valori Proprii. Diagonalizare

① Fie $f \in End(\mathbb{R}^3)$, $R_0 = \{e_1, e_2, e_3\}$ rep. canonic

a) $f(e_1) = e_2$

$f(e_2) = e_1 + e_2 + e_3$

$f(e_3) = e_2$

Det R reper în $\mathbb{R}^3$ a.î. $[f]_{R,R} = A'$ diagonală

SOL: $f(e_1) = \underline{0} \cdot e_1 + \underline{1} \cdot e_2 + \underline{0} \cdot e_3$

$f(e_2) = 1 \cdot e_1 + 1 \cdot e_2 + 1 \cdot e_3$

$f(e_3) = 0 \cdot e_1 + 1 \cdot e_2 + 0 \cdot e_3$

$A = [f]_{R_0, R_0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Polinomul caracteristic:

$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} \xrightarrow{C_1 = C_1 - C_3}$

$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ \lambda & 1 & -\lambda \end{vmatrix} = \lambda \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \xrightarrow{L_1 = L_1 + L_3} \lambda \begin{vmatrix} 0 & 2 & -\lambda \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$

$= \lambda \cdot (-1)^{1+3} \cdot 1 \cdot \begin{vmatrix} 2 & -\lambda \\ 1-\lambda & 1 \end{vmatrix} = \lambda(2 + \lambda - \lambda^2)$

$= -\lambda(\lambda^2 - \lambda - 2) = -\lambda(\lambda - 2)(\lambda + 1) = 0$

OBS: $\lambda^3 - \sigma_1 \lambda^2 + \sigma_2 \lambda - \sigma_3 = 0 \Rightarrow \lambda^3 - \lambda^2 - 2\lambda = 0$

$\sigma_1 = Tr\, A$

$\sigma_2 = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 + 0 - 1 = -2$

$\sigma_3 = \det A = 0$

$\lambda_1 = 0$, $m_1 = 1$ $\quad$ m (multiplicitatea rădăcinilor)

$\lambda_2 = 2$, $m_2 = 1$

$\lambda_3 = -1$, $m_3 = 1$

$\sigma(f) = \{0, 2, -1\} = Spec(f)$ [nu am multiplicit.

$V_{\lambda_1} = \{x \in \mathbb{R}^3 / f(x) = 0 \cdot x\}$

$AX = 0 \cdot X$

$(A - 0 I_3) X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow AX = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = Ker f$

$\dim V_{\lambda_1} = 3 - rg A = 3 - 2 = 1 = m_1$ (A)

$\Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_2 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_2 = 0 \end{cases}$

$x_1 = -x_3$

$V_{\lambda_1} = \{(-x_3, 0, x_3),\ x_3 \in \mathbb{R}\}$

$\{x_3(-1, 0, 1)\}$

$<\{(-1, 0, 1)\}> \quad = e_1'$

$R_1 = \{e_1'\}$

Analog pt V_{λ_2} și V_{λ_3} $\Rightarrow$ dim = multiplicitățile fiecărui, dacă vrem ca teorema să meargă

$V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_2 \cdot x = 2 \cdot x\}$

$AX = 2X \Rightarrow AX - 2X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$X(A - 2I_3) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\det(A - 2I_3) = 0 \Rightarrow rg = 2$

$\dim V_{\lambda_2} = 3 - rg A = 3 - 2 = 1 = m_2$

$\begin{cases} -2x_1 + x_2 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_2 - 2x_3 = \end{cases} \Rightarrow \begin{cases} x_1 - x_2 = -x_3 \\ x_2 = 2x_3 \end{cases}$

$x_1 = x_3$

$V_{\lambda_2} = \{(x_3, 2x_3, x_3)\}\ x_3 \in \mathbb{R}$

$= \{x_3(1, 2, 1)\} \quad = <\{(1, 2, 1)\}> = R_2$

e_2'

$V_{\lambda_3} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_3 \cdot x = 2 \cdot x\}$

$AX = -X \Rightarrow (A + I_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \underline{\underline{L_2 - L_1}}$

ASA CEVA PICĂ LA LUCRARE.

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$\det(A + I_3) = 0$ $(rg = 2)$
$L_2 - L_1 = L_3$

$\Rightarrow \begin{cases} x_1 + 2x_2 = -x_3 \\ x_2 = -x_3 \end{cases} \Rightarrow x_1 = -x_3 - 2x_2 = -x_3 + 2x_3 = x_3$

$V_{\lambda_3} = \{(x_3, -x_3, x_3), x_3 \in \mathbb{R}\} = \{x_3(1,-1,1)\}$

$R_3' = \langle\{(1,-1,1)\}\rangle$

$R = R_1' \cup R_2' \cup R_3'$ rep în $\mathbb{R}^3$

a.î. $[f]_{R,R} = A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

2) $f \in End(\mathbb{R}^3)$, $R_0 = \{e_1, e_2, e_3\}$ rep canonic

a) $[f]_{R_0,R_0} = A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

- Det R rep în $\mathbb{R}^3$ a.î. $[f]_{R,R}$ diagonală
- $A^n = ?$

$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 \xrightarrow{L_1+L_2+L_3} (2-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$

$\xrightarrow[C_3 - C_1]{C_2 - C_1} (2-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 1 & -(\lambda+1) & 0 \\ 1 & 0 & -(\lambda+1) \end{vmatrix} = (2-\lambda)(\lambda+1)^2 = 0$

$\Rightarrow \lambda_2 = 2$, $m_2 = 1$
$\lambda_A = -1$, $m_A = 2$

$\sigma(f) = \{2, -1\}$
$Spec(f) = \{\underset{-1}{\lambda_1} = \underset{-1}{\lambda_1} < \underset{2}{\lambda_2}\}$

$V_{\lambda_1} = \{x \in \mathbb{R}_3 \mid f(x) = -x\}$
$AX = -X$
$(A + I_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$r(A + I_3) = 1 \Rightarrow \dim V_{\lambda_1} = 3 - 1 = 2 = m_1$

$\{ x_1 + x_2 + x_3 = 0 \Rightarrow x_3 = -x_1 - x_2$

$V_{\lambda_1} = \{ (x_1, x_2, -x_1 - x_2) \}\ x_1, x_2 \in \mathbb{R}$

$= \{ x_1 \underbrace{(1, 0, -1)}_{e_1'} + x_2 \underbrace{(0, 1, -1)}_{e_2'} \}$

$R_1 = \{ e_1', e_2' \}$ (reper în V_{λ_1})

$V_{\lambda_2} = \{ X \in \mathbb{R}^3 \mid f(x) = 2X \}$

$AX = 2X$

$\Rightarrow (A - 2I_3) X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\det(A - 2I_3) = \begin{vmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 0 \Rightarrow rg = 2$

$\dim V_{\lambda_2} = 3 - 2 = 1 = m_2$

$$\begin{cases} x_1 - 2x_2 = -x_3 \\ x_1 + x_2 = 2x_3 \end{cases} \quad (-)$$

$-3x_2 = -3x_3 \Rightarrow x_2 = x_3$

$x_1 = -x_3 + 2x_3 = x_3$

$V_{\lambda_2} = \{ (x_3, x_3, x_3),\ x_3 \in \mathbb{R} \}$

$\{ x_3 (1, 1, 1) \}$

$R_2 = \langle \{ \underbrace{(1, 1, 1)}_{e_3'} \} \rangle = \{ e_3' \}$ reper în V_{λ_2}

$R = R_1 \cup R_2$ reper în $\mathbb{R}_3$ a.î. $[g]_R = A'$

$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ matrice diagonală

- $A^n = ?$

$R_0 \xrightarrow{C} R$

$e_1' = (1, 0, -1) = e_1 - e_3$

$e_2' = (0, 1, -1) = e_2 - e_3$

$e_3' = (1, 1, 1) = e_1 + e_2 + e_3$

$C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$

$A' = C^{-1} \cdot AC \Rightarrow A = CA'C^{-1}$

$A^n = \underbrace{CA'C^{-1} \cdot CA'C^{-1} \cdot \ldots \cdot CA'C^{-1}}_{\text{de } n \text{ ori}} = C \cdot A'^n \cdot C^{-1}$

$\det C = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 + 1 + 1 = 3 \neq 0$

$C^t = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ $\quad C^* = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

$C^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

$A^n = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-1)^n \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

$\ldots \Rightarrow$ se înmulțesc

Obs: pt a calcula A^n avem nevoie de A', matr diag.

(5) $f \in End(\mathbb{R}^3)$

Dacă $\lambda_1 = 3$, $\lambda_2 = -2$, $\lambda_3 = 1$ sunt valorile proprii și $v_1 = (-3, 2, 1)$ $v_2 = (-2, 1, 0)$ $v_3 = (-6, 3, 1)$ sunt vect proprii coresp aflați care este $A = [f]_{R_0, R_0}$?

$R' = \{v_1, v_2, v_3\}$ reper în $\mathbb{R}^3$ R' - SLi

vect proprii coresp la val proprii distincte

$$A' = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A' = C^{-1} A C \Rightarrow A = C A' C^{-1}$$

$R_0 \xrightarrow{C} R'$

$v_1 = -3e_1 + 2e_2 + e_3$

$v_2 = -2e_1 + e_2$

$v_3 = -6e_1 + 3e_2 + e_3$

$$C = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\det C = \begin{vmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \\ -3 & -2 & -6 \\ 2 & 1 & 3 \end{vmatrix} = -3 - \cancel{6} + \cancel{6} + 4 = 1$$

$$C^t = \begin{pmatrix} -3 & 2 & 1 \\ -2 & 1 & 0 \\ -6 & 3 & 1 \end{pmatrix}$$

. . . . se calculează