CURS 3 - AG

20.10.2022

Spații vectoriale. Exemple.
Sisteme liniar independente / dependente
Sisteme de generatori. Baze

Exemple de spații vectoriale

① $(V_1, \oplus, \odot)_{/\mathbb{K}}, (V_2, \boxplus, \boxdot)_{/\mathbb{K}}$ spații vectoriale

$\Rightarrow (V_1 \times V_2, +, \cdot)_{/\mathbb{K}}$ sp. vect

$+ : (V_1 \times V_2) \times (V_1 \times V_2) \longrightarrow V_1 \times V_2$

$(x_1, y_1) + (x_2, y_2) = (x_1 \oplus x_2, y_1 \boxplus y_2), \forall (x_1,y_1),(x_2,y_2) \in V_1 \times V_2$

$\cdot : \mathbb{K} \times (V_1 \times V_2) \longrightarrow V_1 \times V_2$

$\alpha \cdot (x_1, y_1) = (\alpha \odot x_1, \alpha \boxdot y_1), \forall \alpha \in \mathbb{K}, \forall (x_1,y_1) \in V_1 \times V_2.$

Caz particular

$(\mathbb{R}, +, \cdot)_{/\mathbb{R}} \Rightarrow (\mathbb{R}^n, +, \cdot)_{/\mathbb{R}}$ sp. vect.

$(x_1, .., x_n) + (y_1, .., y_n) = (x_1 + y_1, x_2 + y_2, .., x_n + y_n)$

$\alpha (x_1, .., x_n) = (\alpha x_1, \alpha x_2, .., \alpha x_n), \forall \alpha \in \mathbb{R}, \forall (x_1, .., x_n), (y_1, .., y_n) \in \mathbb{R}^n.$

② $(\mathcal{M}_{m,n}(\mathbb{K}), +, \cdot)_{/\mathbb{K}}$ sp. vect.

$(a_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}} \longrightarrow (a_{11}, .., a_{1n}, a_{21}, .., a_{2n}, .., a_{m1}, .., a_{mn}) \in \mathbb{R}^{mn}.$

③ $(\mathbb{K}[X], +, \cdot)_{/\mathbb{K}}$ sp. vect

$P = a_0 + a_1 X + \ldots + a_n X^n$, $a_n \neq 0$, grad $P = n$

$(\mathbb{K}_n[X] = \{P \in \mathbb{K}[X] \mid \text{grad } P \leq n\}, +, \cdot)_{/\mathbb{K}}$ sp. vect

$P = a_0 + a_1 X + .. + a_n X^n \longrightarrow (a_0, a_1, .., a_n) \in \mathbb{K}^{n+1}$

④ $I = [a, b]$, $a < b$.

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a) $(\mathcal{C}(I)=\{f:I\to\mathbb{R} \mid f \text{ cont}\},+,\cdot)_{/\mathbb{R}}$ sp. vect

$(\mathcal{D}(I)=\{f:I\to\mathbb{R} \mid f \text{ deriv}\},+,\cdot)_{/\mathbb{R}}$ sp vect

$(\mathcal{P}(I)=\{f:I\to\mathbb{R} \mid f \text{ primitivabilă}\},+,\cdot)_{/\mathbb{R}}$ sp. vect

$(\mathcal{I}(I)=\{f:I\to\mathbb{R} \mid f \text{ integrabile}\},+,\cdot)_{/\mathbb{R}}$ sp. vect

__Def__ $(V,+,\cdot)_{/\mathbb{K}}$ sp. vect, $V' \subseteq V$ subm. nevidă

V' s.n. __subspațiu vectorială__ $\Leftrightarrow$ subm. închisă la adunarea vect și la „$\cdot$" cu scalari

i.e. $\forall x,y\in V' \Rightarrow x+y\in V'$

$\forall \alpha\in\mathbb{K}, \forall x\in V' \Rightarrow \alpha\cdot x\in V'$

__OBS__ $V'\subset V$ subsp. vect $\Rightarrow (V',+,\cdot)_{/\mathbb{K}}$ sp vect (cu operațiile induse)

__Prop__ $(V,+,\cdot)_{/\mathbb{K}}$ sp vect, $V'\subseteq V$ subm. nevidă

$V'\subset V$ subsp. vect $\Leftrightarrow [\forall a,b\in\mathbb{K}, \forall x,y\in V' \Rightarrow ax+by\in V']$

$\Leftrightarrow [\forall a_1,\dots,a_n\in\mathbb{K}, \forall x_1,\dots,x_n\in V' \Rightarrow a_1x_1+\dots+a_nx_n\in V']$

__Dem__

„$\Rightarrow$" $V'\subset V$ ssp vect (ipoteza)

$\forall a\in\mathbb{K}, \forall x\in V' \Rightarrow ax\in V'$

$\forall b\in\mathbb{K}, \forall y\in V' \Rightarrow by\in V'$ $\Rightarrow ax+by\in V'$

„$\Leftarrow$" $\forall a,b\in\mathbb{K}, \forall x,y\in V' \Rightarrow ax+by\in V'$ (ipoteza)

1) $a=1_{\mathbb{K}}, b=1_{\mathbb{K}}$.

$1_{\mathbb{K}}x+1_{\mathbb{K}}y=x+y\in V'$

2) $b=0_{\mathbb{K}}$

$ax+0_{\mathbb{K}}y=ax\in V'$

Exemple de subspații vectoriale

1) $(V,+,\cdot)/_{\mathbb{K}}$, $\{0_V\}, V \subseteq V$ subsp. vect.

2) $n < m$, $m \geq 2$ $\quad \mathbb{R}^n \subset \mathbb{R}^m$ subsp. vect

3) $(\mathcal{M}_n(\mathbb{R}),+,\cdot)/_{\mathbb{R}}$.

a) $V' = \{A = diag(a_1,\dots,a_n) = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \in \mathcal{M}_n(\mathbb{R})\}$.

b) $V'' = \{A \in \mathcal{M}_n(\mathbb{R}) \mid A = A^T\} = \mathcal{M}_n^s(\mathbb{R})$

c) $V''' = \{A \in \mathcal{M}_n(\mathbb{R}) \mid A = -A^T\} = \mathcal{M}_n^a(\mathbb{R})$

d) $W = \{A \in \mathcal{M}_n(\mathbb{R}) \mid Tr(A) = 0\}$

ssp vect.

<u>OBS</u> $GL(n,\mathbb{R})$, $O(n)$, $SO(n)$, $SL(n)$ $\subset \mathcal{M}_n(\mathbb{R})$ <u>nu</u> sunt subsp. vect.

4) $W = \{(x,y) \in \mathbb{R}^2 \mid ax+by=0,\ a^2+b^2>0\} \subset \mathbb{R}^2$ ssp vect.
(dreaptă care trece prin origine)

$W' = \{(x,y,z) \in \mathbb{R}^3 \mid ax+by+cz=0,\ a^2+b^2+c^2>0\} \subset \mathbb{R}^2$ ssp v
(plan care trece prin origine)

$W'' = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid a_1x_1+\dots+a_nx_n=0,\ a_1^2+\dots+a_n^2>0\} \subset \mathbb{R}^n$ ssp v
(hiperplan care trece prin origine)

$U = S(A) = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid \underset{(m,n)}{A}\underset{(n,1)}{X} = \underset{(m,1)}{0}\} \subset \mathbb{R}^n$ ssp vect
(mulț. sol unui SLO)
(este $\cap$ a m hiperplane).

Subspațiul vectorial generat de o mulțime

<u>Def</u> $(V,+,\cdot)/_{\mathbb{K}}$, $S \subseteq V$ subm. nevidă
$\langle S \rangle = \{x \in V \mid x = a_1x_1+\dots+a_nx_n$, unde $a_1,\dots,a_n \in \mathbb{K},\ x_1,\dots,x_n \in S\}$
spațiul vect. generat de S.

OBS Dacă $V = \langle S \rangle$, atunci S s.n. sistem de generatori (SG)

V s.n. spațiu vectorial finit generat dacă $\exists\ S$ = m. finită a.î. $V = \langle S \rangle$

OBS

a) $S \subset \langle S \rangle$

b) $\langle S \rangle$ = cel mai mic subsp. vect. al lui V, care conține S

c) $\langle \phi \rangle = \{0_V\}$ (Convenție)

Def $(V,+,\cdot)/\mathbb{K}$, $S \subset V$ subm. nevidă (SLI)

1) S s.n. sistem liniar independent $\Leftrightarrow$

$$\left[\begin{array}{l}\forall a_1,\dots,a_n \in \mathbb{K} \\ \forall x_1,\dots,x_n \in S\end{array} \text{ a.î. } \underbrace{\sum_{i=1}^{n} a_i x_i}_{a_1x_1+\dots+a_nx_n} = 0 \Rightarrow a_1 = \dots = a_n = 0_{\mathbb{K}}\right]$$

2) S s.n. sistem liniar dependent (SLD) $\Leftrightarrow$

$\exists\, x_1,\dots,x_n \in S$

$\exists\, a_1,\dots,a_n \in \mathbb{K}$, nu toți nuli a.î. $a_1x_1+\dots+a_nx_n = 0$.

Prop $(V,+,\cdot)/\mathbb{K}$, $x \in V$, $x \neq 0_V \Rightarrow \{x\}$ este SLI

Dem

Fie $a \in \mathbb{K}$ a.î. $a \cdot x = 0_V$

Pp abs $a \neq 0_{\mathbb{K}} \Rightarrow \exists\, a^{-1}$ (în corpul $(\mathbb{K},+,\cdot)$)

$\underbrace{a^{-1} \cdot a}_{1_{\mathbb{K}}} \cdot x = a^{-1} \cdot 0_V \Rightarrow x = 0_V$ Contrad. $\Rightarrow$ Pp. este falsă

$\Rightarrow a = 0_{\mathbb{K}} \Rightarrow \{x\}$ este SLI, $x \neq 0_V$

Def $(V,+,\cdot)/\mathbb{K}$, $S \subseteq V$ subm. nevidă

S s.n. bază $\Leftrightarrow \begin{cases} 1) & S \text{ este SLI} \\ 2) & S \text{ este SG} \end{cases}$

Exemple de baze

1) $(\mathbb{R},+,\cdot)/\mathbb{R}$ $\quad B_0=\{1\}$ este baza canonică

$\{1\}$ SLI $\quad (1\neq 0$ cf prop$)$

$\forall x\in\mathbb{R} \quad x=x\cdot 1\in\langle\{1\}\rangle \Rightarrow \mathbb{R}=\langle\{1\}\rangle \Rightarrow \{1\}$ SG

Deci $\{1\}$ SLI + SG $\Rightarrow$ bază

$B'=\{a\}$ bază

$a\neq 0_{\mathbb{R}}$

2) $(\mathbb{R}^2,+,\cdot)/\mathbb{R}$, $\quad B_0=\{e_1=(1,0), e_2=(0,1)\}$ baza canonică

a) SLI $\quad$ Fie $a,b\in\mathbb{R}$ aî $ae_1+be_2=0_{\mathbb{R}^2}$

$$a(1,0)+b(0,1)=(0,0) \Rightarrow \begin{cases} a=0 \\ b=0 \end{cases}$$

$$(a,0)+(0,b)$$

$$(a,b)$$

b) SG $\quad \forall x=(x_1,x_2)\in\mathbb{R}^2 \Rightarrow x=x_1e_1+x_2e_2$

$$(x_1,0)+(0,x_2)$$

$$x_1(1,0)+x_2(0,1)$$

$\Rightarrow \mathbb{R}^2=\langle\{e_1,e_2\}\rangle \Rightarrow \{e_1,e_2\}$ este SG

SLI + SG $\Rightarrow$ bază

3) $(\mathbb{R}[X],+,\cdot)/\mathbb{R}$, $\quad B_0=\{1,X,X^2,\dots\}$ bază

sp. vect care <u>NU</u> este finit generat.

$(\mathbb{R}_n[X]=\{P\in\mathbb{R}[X] \,,\, \text{grad}\,P\leq n\},+,\cdot)$

$B_0=\{1,X,\dots,X^n\}$ baza

a) SLI

$$a_0\cdot 1+a_1X+\dots+a_nX^n=0 \Leftrightarrow \begin{cases} a_0=0 \\ \vdots \\ a_n=0 \end{cases}$$

b) SG, $\forall P=a_0+a_1X+\dots+a_nX^n\in\langle B_0\rangle$

$\quad\ P\in\mathbb{R}_n[X]$

4) $(\mathcal{M}_{m,n}(\mathbb{R}),+,\cdot)/\mathbb{R}$;

$$B_0 = \left\{ E_{ij} = \begin{array}{c} \\ i \end{array}\begin{pmatrix} 0 & & 0 \\ - & 1 & - \\ 0 & & 0 \end{pmatrix}, \ \forall\, i=\overline{1,m},\ j=\overline{1,n} \right\}.$$

$\operatorname{card} B_0 = mn$

OBS a) $\forall$ subm $\neq \emptyset$ a unui SLI este un SLI

$S = \{x_1, \dots, x_n\}$ SLI $\Rightarrow S' = \{x_1, \dots, x_{n-1}\}$ SLI

$a_1 x_1 + \dots + a_{n-1} x_{n-1} = 0_V \Rightarrow a_1 x_1 + \dots + a_{n-1} x_{n-1} + 0_K x_n = 0$

$a_1, \dots, a_{n-1} \in \mathbb{K}$. $\Downarrow$ SLI

$a_1 = \dots = a_{n-1} = 0_K = 0_K$

b) $\forall$ supramulțime a unui SLD este un SLD.

$S = \{x_1, \dots, x_n\}$ SLD $\Rightarrow S' = S \cup \{x_{n+1}\}$ SLD.

Fie. $a_1 x_1 + \dots + a_n x_n + 0 \cdot x_{n+1} = 0_V. \xRightarrow{S\ SLD}$

$\ \parallel$

$a_1 x_1 + \dots + a_n x_n$.

nu toți scalari $a_1, \dots, a_n$ sunt nuli.

c) $\forall$ supramulțime a lui SG este un SG.

$S = \{x_1, \dots, x_n\}$, $S' = S \cup \{x_{n+1}\}$

$V = \langle S \rangle$

$\forall x \in V$, $x = a_1 x_1 + \dots + a_n x_n = a_1 x_1 + \dots + a_n x_n + 0 \cdot x_{n+1} \in \langle S' \rangle$.

d) $0_V \in S \Rightarrow S$ este SLD.

Teorema schimbului

Fie $(V,+,\cdot)/\mathbb{K}$ sp. vect finit generat

Fie $\{x_1, \dots, x_n\}$ SG
$\{y_1, \dots, y_n\}$ SLI $\Rightarrow \{y_1, \dots, y_n\}$ este SG

Dem

$V = \langle \{x_1, \dots, x_n\} \rangle \Rightarrow y_1 = a_1 x_1 + \dots + a_n x_n$, $a_1, \dots, a_n \in \mathbb{K}$

Ψ
y_1

Pp. prin abs $a_1 = .. = a_n = 0_{\mathbb{K}} \Rightarrow y_1 = 0_V \Rightarrow \{0_V, y_2, .., y_n\}$ SLD ↯.

$1_{\mathbb{K}} \cdot 0_V + 0_{\mathbb{K}} \cdot y_2 + .. + 0_{\mathbb{K}} y_n = 0_V$ Contrad.

Pp. este falsă $\exists\, a_1 \neq 0_{\mathbb{K}}$ (eventual renumerotez)

$y_1 = a_1 x_1 + ... + a_n x_n \Rightarrow x_1 = a_1^{-1}(y_1 - a_2 x_2 - ... - a_n x_n)$

$V = \langle \{x_1, x_2, .., x_n\} \rangle = \langle \{y_1, x_2, .., x_n\} \rangle$

$\ni y_2$

$y_2 = b_1 y_1 + a_2 x_2 + ... + a_n x_n$.

Pp. abs $a_2 = .. = a_n = 0_{\mathbb{K}} \Rightarrow y_2 = b_1 y_1 \Rightarrow$

$b_1 y_1 - 1_{\mathbb{K}} \cdot y_2 + 0 \cdot y_3 + ... + 0 y_n = 0_V \Rightarrow \{y_1, .., y_n\}$ SLD ↯.

$a_2 \neq 0_{\mathbb{K}}$

$x_2 = a_2^{-1}[y_2 - b_1 y_1 - x_3 a_3 - ... x_n a_n]$

$V = \langle \{x_1, .., x_n\} \rangle = \langle \{y_1, x_2, .., x_n\} \rangle = \langle \{y_1, y_2, x_3, .., x_n\} \rangle$

După un nr finit de pași

$V = \langle \{y_1, .., y_n\} \rangle \Rightarrow \{y_1, .., y_n\}$ SG.

Prop card $\forall$ SG (finit) $\geq$ card $\forall$ SLi (finit)

Dem. $\{x_1, .., x_n\}$ SG

Fie $\{y_1, .., y_{n+1}\} \subset V \Rightarrow \{y_1, .., y_{n+1}\}$ este SLD.

1) $\{y_1, .., y_n\}$ SLi $\xRightarrow{\text{Th sch}}$ $\{y_1, .., y_n\}$ SG

$V = \langle \{y_1, .., y_n\} \rangle \Rightarrow y_{n+1} = a_1 y_1 + ... + a_n y_n \Rightarrow$

$\ni y_{n+1}$

$a_1 y_1 + ... + a_n y_n - 1_{\mathbb{K}} y_{n+1} = 0_V \Rightarrow \{y_1, .., y_{n+1}\}$ SLD ($1_{\mathbb{K}} \neq 0_V$)

2) $\{y_1, .., y_n\}$ este SLD $\Rightarrow \{y_1, .., y_n, y_{n+1}\}$ SLD

($\forall$ supram a unui SLD e SLD)

Teoremă $(V,+,\cdot)/\mathbb{K}$ sp. vect. finit generat.

Dacă B_1, B_2 sunt baze, at. card B_1 = card $B_2 = n$

$n = \dim_{\mathbb{K}} V$ (dimensiunea sp. vect)

Dem

B_i baze $\Rightarrow$ B_i SLI + B_i SG $\forall i = \overline{1,2}$

1) B_1 SG, B_2 SLI $\xRightarrow{PROP}$ card $(B_1) \geq$ card (B_2)

2) B_2 SG, B_1 SLI $\xRightarrow{PROP}$ card $(B_2) \geq$ card (B_1)

$\Rightarrow$ card B_1 = card B_2 $= n$.

OBS $n = \dim_{\mathbb{K}} V$ = nr max de vect care form SLI

= nr min de vect care form SG.

OBS $(V,+,\cdot)/\mathbb{K}$ sp vect, $\dim_{\mathbb{K}} V = n$

$S = \{v_1, \dots, v_n\}$, card $S = n$

UAE 1) S bază

2) S e SLI

3) S este SG

Aplicații $(\mathbb{R}^3,+,\cdot)/\mathbb{R}$, $B_0 = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ baza canonică

$\dim_{\mathbb{R}} \mathbb{R}^3 = 3$

a) $S = \{(1,2,3), (-1,1,1), (0,2,3), (-1,0,0)\}$ SLD.

(M1) $n = 3$ nr. max de vect care form SLI

(M2) Fie $a, b, c, d \in \mathbb{R}$ a.î.

$a(1,2,3) + b(-1,1,1) + c(0,2,3) + d(-1,0,0) = (0,0,0)$

$$\begin{cases} a - b - d = 0 \\ 2a + b + 2c = 0 \\ 3a + b + 3c = 0 \end{cases} \qquad A = \left(\begin{array}{cccc|c} \boxed{1} & -1 & 0 & \boxed{-1} & 0 \\ 2 & 1 & 2 & 0 & 0 \\ 3 & 1 & 3 & 0 & 0 \end{array}\right)$$

$$\Delta_p = \begin{vmatrix} -1 & 0 & -1 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{vmatrix} \neq 0 \Rightarrow rg\,A = rg\,\bar{A} = 3 \quad SCsN$$

$a, b, c =$ var pr. $\quad d = \alpha$ var secundară

sist are și sol nenule. $\Rightarrow$ S este SLD.

b) $S' = \{(1,2,-1), (\alpha,1,1), (0,2,1)\}$.

Să se det $\alpha \in \mathbb{R}$ aî S' este SLI, resp SG.

Fie $a, b, c \in \mathbb{R}$ aî

$a(1,2,-1) + b(\alpha,1,1) + c(0,2,1) = 0_{\mathbb{R}^3}$

S' SLI $\Leftrightarrow a = b = c = 0$

$$\begin{cases} a + \alpha b = 0 \\ 2a + b + 2c = 0 \\ -a + b + c = 0 \end{cases} \qquad A = \left(\begin{matrix} 1 & \alpha & 0 \\ 2 & 1 & 2 \\ -1 & 1 & 1 \end{matrix}\right| \left.\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right)$$

$(a,b,c) = (0,0,0) \Leftrightarrow \det A \neq 0$

$$\Delta = \begin{vmatrix} 1 & \alpha+1 & 1 \\ 2 & 3 & 4 \\ -1 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} \alpha+1 & 1 \\ 3 & 4 \end{vmatrix} \neq 0$$

$-(4\alpha + 1) \neq 0 \Rightarrow \alpha \neq -\frac{1}{4}$

$\alpha \in \mathbb{R} \setminus \{-\frac{1}{4}\} \Leftrightarrow S'$ este SLI $\overset{OBS}{\Leftrightarrow} S'$ este SG

c) $S'' = \{(1,2,3), (-1,1,1), \cancel{(0,3,4)}\}$

- S'' este SLD
- Extrageți din S'' un SLI maximal și extindeți-l la o bază.

$(1,2,3) + (-1,1,1) = (0,3,4) \Rightarrow (1,2,3) + (-1,1,1) - (0,3,4) = 0_{\mathbb{R}^3}$

$\Rightarrow$ SLD.

$S''_1 = \{(1,2,3), (-1,1,1)\}$ este SLI ? (maximal)

$a(1,2,3) + b(-1,1,1) = (0,0,0)$

$$\begin{cases} a - b = 0 \\ 2a + b = 0 \\ 3a + b = 0 \end{cases} \quad A = \left(\begin{matrix} 1 & -1 \\ 2 & 1 \\ 3 & 1 \end{matrix}\right| \left.\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right) \qquad a = b = 0$$

$S_1'' \cup \{(1,0,0)\} = \{(1,2,3), (-1,1,1), (1,0,0)\}$.

$a(1,2,3) + b(-1,1,1) + c(1,0,0) = (0,0,0)$

$\begin{cases} a-b+c = 0 \\ 2a+b = 0 \\ 3a+b = 0 \end{cases} \qquad A = \left(\begin{array}{ccc} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & 0 \end{array}\right) \left| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right.$

$\det A \neq 0 \Rightarrow$ sol unică $(a,b,c) = (0,0,0)$.

EX $V' = \{(x,y) \in \mathbb{R}^2 \mid x-2y=0\} \subset \mathbb{R}^2$

a) $V' \subset \mathbb{R}^2$ ssp V

b) B bază în V'

(x'', y'')

$(ax+bx', ay+by')$

SOL a) Fie $(x,y), (x',y') \in V' \overset{?}{\Longrightarrow} a(x,y) + b(x',y') \in V'$ $\forall a,b \in \mathbb{R}$

$x-2y=0$

$x'-2y'=0$

$x''-2y'' = (ax+bx') - 2(ay+by')$

$= a\underbrace{(x-2y)}_{0} + b\underbrace{(x'-2y')}_{0} = 0 \Rightarrow (x'',y'') \in V'$

b) $x-2y=0 \Rightarrow x=2y$

$V' = \{(2y,y),\ y \in \mathbb{R}\} = \{y(2,1),\ y \in \mathbb{R}\} \Rightarrow V' = \langle\{(2,1)\}\rangle$

$\Downarrow$

$\{(2,1)\}$ SG.

$(2,1) \neq (0,0) \Rightarrow \{(2,1)\}$ SLI

$B = \{(2,1)\}$ bază în V'