Seminar 7

Proiecții și simetrii

1) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1+x_2-x_3, -x_1-x_2+x_3, x_1+x_2+x_3)$

a) $[f]_{R,R}$, $f(x_1, x_2, x_3)$

$R = \{e_1' = e_1+e_2+e_3, e_2' = e_1+e_3, e_3' = e_1+e_2\}$

lucrare

b) $\mathbb{R}^3 = \operatorname{Im} f \oplus W$

$s: \mathbb{R}^3 \to \mathbb{R}^3$ simetric față de W

t 0 $s(v_1+v_2) = -v_1+v_2$

$s(0,1,1) = ?$

c) $R^3 = f(V) + U$

$V' = \{x \in \mathbb{R}^3 \mid x_1+2x_2+x_3=0, \ -x_1+x_2+2x_3=0\}$

$p: \mathbb{R}^3 \to \mathbb{R}^3$, proiecția pe $f(V')$

$p(u_1+u_2) = u_1$ $\quad p(2,-1,3)$

b) I $y \in \operatorname{Im} f \Rightarrow x \in \mathbb{R}^3 \mid f(x) = y$

$\operatorname{Im} f = \{y \in \mathbb{R}^3 \mid y_1+y_2=0\}$

$$\begin{cases} x_1+x_2-x_3 = y_1 \\ -x_1-x_2+x_3 = y_2 \\ x_1+x_2+x_3 = y_3 \end{cases} \qquad A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & y_1 \\ -1 & -1 & 1 & y_2 \\ 1 & 1 & 1 & y_3 \end{array}\right)$$

$$\Delta c = \begin{vmatrix} 1 & -1 & y_1 \\ -1 & 1 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = 0 \overset{L_1+L_2}{=} \begin{vmatrix} 0 & 0 & y_1+y_2 \\ -1 & 1 & y_2 \\ 1 & 1 & y_3 \end{vmatrix}$$

$= y_1+y_2 = 0$

$(y_1, -y_1, y_3) = y_1\overset{f_1}{(1,-1,0)} + y_3\overset{f_2}{(0,0,1)}$

$$A' = \begin{vmatrix} 1 \\ 2 \\ -2 \end{vmatrix}$$

$R_1'' = \{e_1'', e_2''\}$ reper in $\mathrm{Im}\, f$

$$rg\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 3 = max \overset{SLi'}{\Rightarrow} SLi$$

$$W = <\{e_3''\}>$$

Det. comp lui $(0,1,1)$ în rap cu reperul $\{e_1'', e_2'', e_3''\}$

$$(0,1,1) = \underbrace{\alpha(1,-1,0) + \beta(0,0,1)}_{v_1} + \gamma\underbrace{(1,0,0)}_{v_2}$$

$$\begin{cases} \alpha + \gamma = 0 & \Rightarrow \gamma = 1 \\ -\alpha = 1 & \Rightarrow \alpha = -1 \\ \beta = 1 \end{cases}$$

$v_1 = (-1,1,0) + (0,0,1) = (-1,1,1)$

$v_2 = (1,0,0)$

$s(0,1,1) = (1,-1,-1) + (1,0,0) = (2,-1,-1)$

! vectorul complementar e cu -

c) $V' = \begin{cases} x_1 + 2x_2 + x_3 = 0 \\ -x_1 + x_2 + 2x_3 = 0 \end{cases}$ $\left(\begin{array}{cc|c|c} 1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right)$

$$\begin{array}{l} x_1 + 2x_2 = -x_3 \\ -x_1 + x_2 = -2x_3 \\ \hline 3x_2 = -3x_3 \end{array} \quad (+)$$

$\Rightarrow x_2 = -x_3$

$x_1 = -x_3 + 2x_3 = x_3$

$V' = \{(x_3, -x_3, x_3)\} = x_3\{\underbrace{(1,-1,1)}_{=z \quad e_1'''}\}$

$f(z) = f(1,-1,1) = (-1,1,1) = e_1'''$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$e_2''' = (1,0,0)$ $\quad U = <\{e_2''', e_3'''\}>$

$e_3''' = (0,1,0)$

$$(2,-1,3) = \alpha'(-1,1,1) + \beta'(1,0,0) + \gamma'(0,1,0)$$

$$\begin{cases} -\alpha' + \beta' = 2 & \Rightarrow \beta' = 5 \\ \alpha' + \gamma' = -1 & \Rightarrow \gamma' = -4 \\ \alpha' = 3 \end{cases}$$

$p(2,-1,3) = u_1 = (-3, 3, 3)$

② $f: \mathbb{R}_2[x] \to \mathbb{R}_1[x]$, $f(p) = p'$

a) $[f]_{R,R'}$ $\quad R = \{x^2, 1+x, 2-x\}$ reper în $\mathbb{R}_2[x]$

$R' = \{x, 1+3x\}$ reper în $\mathbb{R}_1[x]$

b) $\mathbb{R}_2[x] = \mathrm{Ker}(f) \oplus W$, $p_1, p_2 : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$

p_1 pr pe $\mathrm{Ker}(f)$, $p_1(1-x+3x^2)$

p_2 pr pe W, $p_2(2x+x^2)$

a) $f(x^2) = 2x = a(x) + b(1+3x)$

$a = 2$
$b = 0$

$$\Rightarrow A = \begin{pmatrix} 2 & -3 & 3 \\ 0 & 1 & -1 \end{pmatrix} = [f]_{R,R'}$$

$f(1+x) = 1 = a'x + b'(1+3x)$

$b' = 1$
$a' = -3$

$f(2-x) = -1 = a''x + b''(1+3x)$

$b'' = -1$
$a'' = 3$

b) $\mathrm{Ker}(f) = \{p \in \mathbb{R}_2[x] \mid \underset{\text{derivat}}{p'} = 0\} = \langle\{1\}\rangle$

$W = \langle\{x, x^2\}\rangle$

$1 - x + 3x^2 = 1 \cdot 1 + (-1)x + 3 \cdot x^2$

$\Rightarrow p_1(1 - x - 3x^2) = 1$

$2x + x^2 = \underbrace{0 \cdot 1}_{\in \mathrm{Ker}} + \underbrace{2 \cdot x + 1 \cdot x^2}_{\in W} \Rightarrow$

$\Rightarrow p_2(2x + x^2) = 2x + x^2$

$\cdot 7$

4) $f: \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2^{s}(\mathbb{R})$

$f(A) = A + A^t$

a) $[f]_{R_0, R_0'}$ $\quad R_0 = \{E_{11}, E_{12}, E_{21}, E_{22}\}$

reper canonic in $\mathcal{M}_2 \mathbb{R}$

$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$R_0' = \{E_{11}, E_{12}+E_{21}, E_{22}\}$ reper in $\mathcal{M}_2^{s}$

b) $\operatorname{Ker} f$, $\operatorname{Im} f$

c) $f(V)$ $\quad, V = \left\{ \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}, c, d \in \mathbb{R} \right\}$

a) $f(E_{11}) = E_{11} + E_{11}^t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 2E_{11} + 0(E_{12}+E_{21}) + 0 \cdot E_{22}$ $\quad [f]_{R_0, R_0'} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

$f(E_{12}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = E_{12} + E_{21}$

$f(E_{21}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = E_{12} + E_{21}$

$f(E_{22}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = 0 \cdot E_{11} + 0(E_{12}+E_{21}) + 2E_{22}$

b) $\operatorname{Ker}(f) = \{ A \in \mathcal{M}_2 \mathbb{R} \mid A + A^t = O_2 \}$

Deci $A = -A^t$ $\quad A \in \mathcal{M}_2^{a}(\mathbb{R})$

$A = \begin{pmatrix} a & b \\ -c & d \end{pmatrix} = -\begin{pmatrix} a & c \\ b & d \end{pmatrix} \Rightarrow \begin{matrix} a = -a \\ b = -c \\ c = -b \\ d = -d \end{matrix}$

$= b \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

$\operatorname{Ker} f = \left\langle \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\} \right\rangle$

Th dim: $\underbrace{\dim \mathcal{M}_2 \mathbb{R}}_{=4} = \underbrace{\dim \operatorname{Ker} f}_{=1} + \underbrace{\dim \operatorname{Im} f}_{\Rightarrow = 3}$

$\left.\begin{matrix} \operatorname{Im} f \subset \mathcal{M}_2^{s}(\mathbb{R}) \text{ subsp vect} \\ \dim \operatorname{Im} f = \dim \mathcal{M}_2^{s}(\mathbb{R}) = 3 \end{matrix}\right\} \Rightarrow \operatorname{Im} f = \mathcal{M}_2^{s}(\mathbb{R})$

c) $V = \left\{ c\underbrace{\begin{pmatrix}0&0\\1&0\end{pmatrix}}_{E_{21}} + d\underbrace{\begin{pmatrix}0&0\\0&1\end{pmatrix}}_{E_{22}} \;\middle|\; c,d\in\mathbb{R} \right\}$

$V = \langle\{E_{21}, E_{22}\}\rangle$

$f(E_{21}) = E_{12} + E_{21}$

$f(E_{22}) = 2E_{22}$

$\Rightarrow f(V) = \langle\{E_{12}+E_{21}, E_{22}\}\rangle$

__Obs__: $A^t = -A$ (Matrice antisimetrice