Seminar 1 - A.G

Ex 1

a) $A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$

$\det A = ?$

$\det A = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \xrightarrow{L_1+L_2+L_3} \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} \frac{C_2-C_1}{C_3-C_1} (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix}$

$= (a+b+c) \begin{vmatrix} c-b & a-b \\ a-c & b-c \end{vmatrix} = (a+b+c)\left[-(b-c)^2 - (a-b)(a-c)\right]$

$= (a+b+c)\left[-b^2 + 2bc - c^2 - a^2 + ac + ab - bc\right]$

$= -(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$

$= -\frac{1}{2}(a+b+c)(2a^2+2b^2+2c^2-2ab-2ac-2bc)$

$= -\frac{1}{2}(a+b+c)(a^2-2ab+b^2+b^2-2bc+c^2+a^2-2ac+c^2)$

$\Delta = -\frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(a-c)^2\right]$

b) $A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$

$\det A = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \frac{C_2-C_1}{C_3-C_1} \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$

$= 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} b-a & c-a \\ (b-a)(b+a) & (c-a)(c+a) \end{vmatrix}$

$= (b-a)(c-a) \cdot \begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} = (b-a)(c-a)(c+\cancel{a}-b-\cancel{a})$

$= (b-a)(c-a)(c-b)$

EX 2 : $A = \begin{pmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{Z})$

a) $m = ?$ a.î $A^{-1} \in M_3(\mathbb{Z})$

b) $A^{-1} = ?$ $m = 0$

a) $A^{-1} = \frac{1}{\det A} \cdot A^* \in \mathcal{M}_3(\mathbb{Z}) \Leftrightarrow$

$A \cdot A^{-1} = A^{-1} \cdot A = I_3 \ | \det$

$\det A \cdot \det(A^{-1}) = 1 \Rightarrow \det A = \pm 1$

$\det(A), \det(A^{-1}) \in \mathbb{Z}$

$\det A = \begin{vmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{vmatrix} \underline{\underline{C_2 - C_1}} \begin{vmatrix} 2 & -3 & 3m+4 \\ 1 & m-1 & 1 \\ -1 & 0 & 0 \end{vmatrix}$

$= (-1) \cdot (-1)^{3+1} \cdot \begin{vmatrix} -3 & 3m+4 \\ m-1 & 1 \end{vmatrix} = -(-3 - 3m^2 + 3m - 4m + 4)$

$= 3m^2 + m - 1$

Caz 1 : $3m^2 + m - 1 = 1 \Rightarrow 3m^2 + m - 2 = 0$

$m_1 = -1 \in \mathbb{Z}$

$m_1 \cdot m_2 = -\frac{2}{3}$ (din Viete)

$m_2 = \frac{2}{3} \notin \mathbb{Z}$

Caz 2 : $3m^2 + m - 1 = -1 \Rightarrow m(3m+1) = 0$

$m_3 = 0 \in \mathbb{Z}$

$m_4 = -\frac{1}{3} \notin \mathbb{Z}$

$\Rightarrow m \in \{-1, 0\}$

b) $A = \begin{pmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{Z})$ $\quad m = 0$

$$A^t = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & -1 \\ 4 & 1 & 0 \end{pmatrix}$$

$$A^* = \begin{pmatrix} 1 & -4 & -1 \\ -1 & 4 & 2 \\ -1 & 3 & 1 \end{pmatrix}$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} -1 & -1 \\ 4 & 0 \end{vmatrix} = -4$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} -1 & 0 \\ 4 & 1 \end{vmatrix} = -1$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -1$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} = 2$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} = -1$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & -1 \\ -1 & -1 \end{vmatrix} = -(-2-1) = 3$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} = 1$$

$\det A = -1$

$$A^{-1} = \frac{1}{\det A} \cdot A^* = \begin{pmatrix} -1 & 4 & 1 \\ 1 & -4 & -2 \\ 1 & -3 & -1 \end{pmatrix}$$

OBS: $A \in \mathcal{M}_m(\mathbb{C})$

$$\det(\alpha A) = \alpha^m \det A$$

$$A A^{-1} = A^{-1} A = I_m \quad / \det$$

$$\det A \cdot \det(A^{-1}) = 1$$

$$A^{-1} = \frac{1}{\det A} \cdot A^* \quad / \det$$

$$A^{-1} = \frac{1}{\det A} \cdot A^* \quad | \det$$

$$\det(A^{-1}) = \left(\frac{1}{\det(A)}\right)^m \cdot \det(A^*) \iff$$

$$\iff \frac{1}{\det(A)} = \left(\frac{1}{\det(A)}\right)^m \cdot \det(A^*)$$

$$\det A^* = (\det A)^{m-1}, \quad m \geq 2$$

EX 3. $A = \begin{pmatrix} 1+a^2 & b \cdot a & c \cdot a \\ ba & 1+b^2 & cb \\ c \cdot a & cb & 1+c^2 \end{pmatrix}$

$\det A^* = ?$

$m = 3 \Rightarrow \det A^* = (\det A)^2$

$$\begin{array}{cccccc} 1 & 1' & 2 & 2' & 3 & 3' \end{array}$$
$$\begin{vmatrix} 1 + a^2 & 0 + ba & 0 + ca \\ 0 + ab & 1 + b^2 & 0 + cb \\ 0 + ac & 0 + bc & 1 + c^2 \end{vmatrix}$$

$$
\begin{aligned}
&\cdot\,|1\;\;2\;\;3| \;+\\
&\cdot\,|1\;\;2\;\;3'| \;+\\
&\cdot\,|1\;\;2'\;\;3| \;+\\
&|1\;\;2'\;\;3'| \;+ \;=0\\
&\cdot\,|1'\;\;2\;\;3| \;+\\
&|1'\;\;2\;\;3'| \;+ \;=0\\
&|1'\;\;2'\;\;3| \;+ \;=0\\
&|1'\;\;2'\;\;3'| \;+ \;=0
\end{aligned}
$$

$$
= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & ca \\ 0 & 1 & cb \\ 0 & 0 & c^2 \end{vmatrix} + \begin{vmatrix} 1 & ba & 0 \\ 0 & b^2 & 0 \\ 0 & bc & 1 \end{vmatrix} + \begin{vmatrix} a^2 & 0 & 0 \\ ab & 1 & 0 \\ ac & 0 & 1 \end{vmatrix} =
$$

$$
= 1 + 1\cdot(-1)^{1+1}\cdot\begin{vmatrix} 1 & cb \\ 0 & c^2 \end{vmatrix} + 1\cdot(-1)^{2+2}\cdot\begin{vmatrix} b^2 & 0 \\ bc & 1 \end{vmatrix} + 1\cdot(-1)^{3+3}\cdot\begin{vmatrix} a^2 & 0 \\ ab & 1 \end{vmatrix} =
$$

$$= 1 + c^2 + b^2 + a^2$$

$$\Rightarrow \det A^t = (a^2+b^2+c^2+1)^2$$

Ex 10 Fie $A \in \mathcal{M}_2(\mathbb{C})$.

Dacă $\exists\; k \geq 2$ a.î. $A^k = O_2$, atunci $A^2 = O_2$

Hamilton Cayley: $A^2 - Tr(A)\cdot A + I_2 \det A = O_2$

$(\forall)\; A \in \mathcal{M}_2(\mathbb{C})$

$\Rightarrow A_k^2 = O_2$

$\det(A^k) = \det O_2 \Rightarrow \det A = 0$

$A^2 = Tr(A)\cdot A$

__OBS__: $A^2 = \alpha A \Rightarrow A^n = \alpha^{n-1} A \quad (\forall)\; n \geq 2$

$A^3 = \alpha A^2 = \alpha^2 A$

$A^2 = Tr(A)\cdot A \Rightarrow A^k = (Tr A)^{k-1}\cdot A$

$\Rightarrow O_2 = (Tr A)^{k-1}\cdot A$

! $Tr A$ = suma elem. pe diagonală

$0 = (Tr\, A)^{k-1}\, Tr\, A$

$0 = (Tr\, A)^{k} \Rightarrow Tr\, A = 0$

$\Rightarrow A^2 = O_2$

EX11: $f: \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$, $f(x) = x^n$ nu e inj; nu e surj pt niciun $n \geq 2$

Fie $X_1 = O_2 \Rightarrow f(X_1) = O_2$
Fie $X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow f(X_2) = O_2$
$\Rightarrow f(x_1) = f(X_2)$ dar $x_1 \neq X_2$

$X_2^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O_2$

$\Rightarrow f$ nu este injectivă

Demonstrăm că X_2 nu aparține Im f

p.p prin red. la absurd că $\exists\, X \in \mathcal{M}_2(\mathbb{C})$ a.î. $f(X) = X_2 \Rightarrow X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow X^{2n} = O_2$

$\xrightarrow[\text{precedent}]{\text{ex.}} X^2 = O_2 \Longrightarrow \underbrace{X^n}_{=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = O_2, (\forall)\, n \geq 2$ (F)

$\Rightarrow$ presupunerea este falsă