C6 AG

Aplicații liniare

$(V_i, +, \cdot)_{/\mathbb{K}}$ sp. vect, $i=\overline{1,2}$

$f: V_1 \to V_2$ apl. liniară $\Leftrightarrow \begin{cases} f(x+y) = f(x)+f(y) \\ f(ax) = a f(x), \ \forall x,y \in V_1 \\ \qquad \forall a \in \mathbb{K} \end{cases}$

$\Leftrightarrow f(ax+by) = a f(x) + b f(y), \ \forall x,y \in V_1, \forall a,b \in \mathbb{K}.$

$\operatorname{Ker} f = \{ x \in V_1 \mid f(x) = 0_{V_2} \} \subseteq V_1$

$\operatorname{Im} f = \{ y \in V_2 \mid \exists x \in V_1 \text{ aî } f(x) = y \} \subseteq V_2$

Prop a) f inj $\Leftrightarrow \operatorname{Ker} f = \{0_{V_1}\}$

b) f surj $\Leftrightarrow \dim \operatorname{Im} f = \dim V_2$.

Teorema dimensiunii

$f: V_1 \to V_2$ liniară

$\dim V_1 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$

Dem

Fie $\mathcal{B}_0 = \{e_1, \dots, e_k\}$ bază în $\operatorname{Ker} f \subseteq V_1$

Extindem la $\mathcal{B}_1 = \{e_1, \dots, e_k, \underline{e_{k+1}, \dots, e_n}\}$ bază în V_1

Dem că $\mathcal{B} = \{ f(e_{k+1}), \dots, f(e_n)\}$ bază în $\operatorname{Im} f$

① $\mathcal{B}$ este SLI

Fie $a_i \in \mathbb{K}$, $i = \overline{k+1,n}$ aî $\sum_{i=k+1}^{n} a_i f(e_i) = 0_{V_2}$

$\xRightarrow{f \text{ lin}} f\Big(\underbrace{\sum_{i=k+1}^{n} a_i e_i}_{\in \operatorname{Ker} f = \langle \mathcal{B}_0 \rangle}\Big) = 0_{V_2} \Rightarrow \sum_{i=k+1}^{n} a_i e_i = \sum_{j=1}^{k} a_j e_j \Rightarrow$

$\sum_{i=k+1}^{n} a_i e_i - \sum_{j=1}^{k} a_j e_j = 0_{V_1} \xRightarrow{\mathcal{B}_1 \text{ SLI}} \begin{matrix} a_i = 0, \ i = \overline{k+1,n} \\ a_j = 0, \ j = \overline{1,k} \end{matrix}$

② $\operatorname{Im} f = \langle \mathcal{B} \rangle$

$\forall y \in \operatorname{Im} f$, $\exists x \in V_1 = \langle \mathcal{B}_1 \rangle$ aî $f(x) = y$

$$f\Big(\underbrace{\sum_{j=1}^{k} a_j e_j}_{\ker f} + \sum_{i=k+1}^{n} a_i e_i\Big) = y \Rightarrow f\Big(\sum_{i=k+1}^{n} a_i e_i\Big) = y$$

$$f\Big(\sum_{i=k+1}^{n} a_i e_i\Big) = \sum_{i=k+1}^{n} a_i f(e_i)$$

Deci $\mathcal{B} = \{f(e_{k+1}), \dots, f(e_n)\}$ bază în $\operatorname{Im} f$.

$$\dim V_1 = n = \underbrace{k}_{\dim \ker(f)} + \underbrace{n-k}_{\dim \operatorname{Im} f}$$

Prop

a) f injectivă $\Leftrightarrow \ker f = \{0_{V_1}\} \Leftrightarrow \dim V_1 = \dim \operatorname{Im} f$.

b) f surjectivă $\Leftrightarrow \dim \operatorname{Im} f = \dim V_2$.

$\Leftrightarrow \dim V_1 = \dim \ker f + \dim V_2$

c) f bijectivă $\Leftrightarrow \dim V_1 = \dim V_2$.

Teoremă

$V_1 \simeq V_2$ (sp. vect. izomorfe) $\Leftrightarrow \dim V_1 = \dim V_2$

Dem

"$\Rightarrow$" $\exists f : V_1 \to V_2$ izomorfism de sp vect $\Leftrightarrow \begin{cases} f \text{ lin} \\ f \text{ bij} \end{cases}$

Cf. Prop c) $\dim V_1 = \dim V_2$.

"$\Leftarrow$" $\dim V_1 = \dim V_2 = n$.

$R_1 = \{e_1, \dots, e_n\}$ reper în V_1, $R_2 = \{e_1', \dots, e_n'\}$ reper în V_2.

Construim $f : V_1 \to V_2$, $f(e_i) = e_i'$, $\forall i = \overline{1,n}$

$f(x) = f\Big(\sum_{i=1}^{n} x_i e_i\Big) = \sum_{i=1}^{n} x_i \underbrace{f(e_i)}_{e_i'} = x'$ (am extins prin liniaritate)

$\forall x' \in V_2$, $x' = \sum_{i=1}^{n} x_i e_i'$, $\exists! x \in V_1$, $x = \sum_{i=1}^{n} x_i e_i$

aî $f(x) = x' \Rightarrow f$ bij

Deci f izomorfism de $V_1 \simeq V_2$

EX $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1, x_1+x_2+x_3, x_1+x_2+x_3)$

$\text{Im} f = ?$

SOL $\text{Ker} f = \{x \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3}\}$ $\begin{cases} x_1 = 0 \\ x_1+x_2+x_3 = 0 \\ x_1+x_2+x_3=0 \end{cases} \Rightarrow \begin{matrix} x_1 = 0 \\ x_2 = -x_3 \end{matrix}$

$\text{Ker} f = \{(0, -x_3, x_3), x_3 \in \mathbb{R}\} = \langle\{(0,-1,1)\}\rangle$

$\mathcal{B}_0 = \{(0,-1,1)\}$ bază în Ker f

Extindem la $\mathcal{B}_1 = \mathcal{B}_0 \cup \underline{\{(0,1,0), (1,0,0)\}}$ bază în $\mathbb{R}^3$

$\text{rg}\begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = 3 = \max$

$\mathcal{B} = \{f(0,1,0), f(1,0,0)\}$ bază în Im f

$f(0,1,0) = (0,1,1)$, $f(1,0,0) = (1,1,1)$

$\text{Im} f = \langle\{(0,1,1), (1,1,1)\}\rangle = \{a(0,1,1) + b(1,1,1) \mid a, b \in \mathbb{R}\}$

$y \in \text{Im} f \Leftrightarrow \exists a, b \in \mathbb{R}$ a.î. $y = (b, a+b, a+b)$

$\begin{cases} b = y_1 \\ a+b = y_2 \\ a+b = y_3 \end{cases}$ $A = \left(\begin{array}{cc|c} 0 & 1 & y_1 \\ 1 & 1 & y_2 \\ 1 & 1 & y_3 \end{array}\right)$

$\begin{vmatrix} 0 & 1 & y_1 \\ 1 & 1 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} 0 & 1 & y_1 \\ 0 & 0 & y_2 - y_3 \\ 1 & 1 & y_3 \end{vmatrix} = 0$

$y_2 - y_3 = 0 \Rightarrow \text{Im} f = \{y \in \mathbb{R}^3 \mid y_2 - y_3 = 0\}$

Prop $f: V_1 \to V_2$ liniară

a) f inj $\Leftrightarrow$ f transformă $\forall$ SLI din V_1 într-un SLI din V_2

b) f surj $\Leftrightarrow$ — " — SG — " — SG — " —

c) f bij $\Leftrightarrow$ f transf. $\forall$ reper din V_1 într-un reper din V_2

Dem

a) "$\Rightarrow$" Ip: f inj

$S = \{v_1, \dots, v_k\}$ SLI în $V_1 \Rightarrow f(S) = \{f(v_1), \dots, f(v_k)\}$ SLI în V_2

$$\sum_{i=1}^{k} a_i f(v_i) = 0_{V_2} \overset{f\ lin}{\Longrightarrow} f\left(\sum_{i=1}^{k} a_i v_i\right) = 0_{V_2}$$

$$\sum_{i=1}^{k} a_i v_i = 0 \underset{\text{SLI în } V_1}{\overset{S}{\Rightarrow}} a_i = 0,\ \forall i = \overline{1,k} \quad \ker f = \{0_{V_1}\} \text{ (ip: } f \text{ inj)} \Rightarrow f(S) \text{ este SLI}$$

"$\Leftarrow$" Ip: $\forall S$ SLI în $V_1 \Rightarrow f(S)$ SLI în V_2.

Pp abs $\exists\, x \in \ker f \Rightarrow f(x) = 0_{V_2}$, $x \neq 0_{V_1}$

$\{x\}$ SLI $\Rightarrow \{f(x) = 0_{V_2}\}$ SLI Contrad. Pp. e falsă

$\Rightarrow f$ inj

b) "$\Rightarrow$" Ip f = surj : $\operatorname{Im} f = V_2$.

Fie $S = \{v_1, \dots, v_k\}$ SG pt V_1 i.e $V_1 = \langle S \rangle$

Dem că $V_2 = \langle f(S) \rangle$. "$\supset$"

Dem $V_2 \overset{"}{\subset} \langle f(S) \rangle$

Fie $y \in V_2 \overset{f\ surj}{\Longrightarrow} \exists\, x \in V_1$ a.î. $f(x) = y$

$$y = f\left(\sum_{i=1}^{k} a_i v_i\right) = \sum_{i=1}^{k} a_i f(v_i) \Rightarrow V_2 = \langle f(S) \rangle$$

"$\Leftarrow$" $V_1 = \langle S \rangle \Rightarrow V_2 = \langle f(S) \rangle$

Dem că f e surj

$\forall\, y \in V_2,\ \exists\, x \in V_1$ a.î. $f(x) = y$.

$y = \sum_{i=1}^{k} a_i f(v_i)$, $x = \sum_{i=1}^{n} a_i v_i$

$$f(x) = f\left(\sum_{i=1}^{k} a_i v_i\right) = \sum_{i=1}^{k} a_i f(v_i) = y$$

Matricea asociată unei aplicații liniare.

$f : V_1 \to V_2$ liniară

$\mathcal{R}_1 = \{e_1, \dots, e_n\} \xrightarrow{f} \mathcal{R}_2 = \{\overline{e}_1, \dots, \overline{e}_m\}$

reper în V_1 reper în V_2.

$$f(e_i) = \sum_{j=1}^{m} a_{ji}\overline{e}_j \quad , \forall i = \overline{1,n} \qquad A \in \mathcal{M}_{m,n}(\mathbb{K})$$

$$f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i f(e_i) = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{m} a_{ji}\overline{e}_j\right) =$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ji}x_i\right)\overline{e}_j$$

$$f(x) = y = \sum_{j=1}^{m} y_j \overline{e}_j$$

$$\Rightarrow y_j = \sum_{i=1}^{n} a_{ji} x_i \quad \forall j = \overline{1,m}$$

$$Y = AX \Leftrightarrow \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$[f]_{\mathcal{R}_1, \mathcal{R}_2} = A$$

Teorema de caract. a apl. liniare

$f : V_1 \to V_2$ liniară $\Leftrightarrow \exists A \in \mathcal{M}_{m,n}(\mathbb{K})$ aî coord. lui x în rap. cu reperul $\mathcal{R}_1 = \{e_1, \dots, e_n\}$ din V_1 și coord. lui $y = f(x)$ în rap. cu reperul $\mathcal{R}_2 = \{\overline{e}_1, \dots, \overline{e}_m\}$ din V_2. verifică $Y = AX \Leftrightarrow \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$y = \sum_{j=1}^{m} y_j \overline{e}_j \quad , \quad x = \sum_{i=1}^{n} x_i e_i$$

OBS

$$\begin{array}{ccc} \mathcal{R}_1 = \{e_1, \dots, e_n\} & \xrightarrow{A} & \mathcal{R}_2 = \{\overline{e}_1, \dots, \overline{e}_m\} \\ C \downarrow & & \downarrow D \\ \mathcal{R}_1' = \{e_1', \dots, e_n'\} & \xrightarrow{A'} & \mathcal{R}_2' = \{\overline{e}_1', \dots, \overline{e}_m'\} \end{array}$$

$A' = D^{-1}AC$

$C \in GL(n, \mathbb{K})$

$D \in GL(m, \mathbb{K})$

$A = [f]_{\mathcal{R}_1, \mathcal{R}_2}$; $A' = [f]_{\mathcal{R}_1', \mathcal{R}_2'}$ $\quad$ $rg A' = rg A$ = invariant

OBS $f \in End(V)$ $\quad f: V \to V$ liniară

$$\begin{array}{ccc} \mathcal{R}_1=\{e_1,..,e_n\} & \xrightarrow{A} & \mathcal{R}_1=\{e_1,..,e_n\} \\ C\downarrow & & \downarrow C \\ \mathcal{R}_1'=\{e_1',..,e_n'\} & \xrightarrow{A'} & \mathcal{R}_1'=\{e_1',..,e_n'\} \end{array}$$

$\boxed{A'=C^{-1}AC}$

$A=[f]_{\mathcal{R}_1,\mathcal{R}_2}$

$A'=[f]_{\mathcal{R}_1',\mathcal{R}_2'}$

EX

$f:\mathbb{R}^2\to\mathbb{R}^2,\ f(x)=(x_1+x_2,\ 2x_2)$

$\mathcal{R}_0=\{e_1=(1,0),\ e_2=(0,1)\}$ rep canonic

$\mathcal{R}'=\{e_1'=e_1-2e_2,\ e_2'=e_1+e_2\}$.

a) $[f]_{\mathcal{R}_0,\mathcal{R}_0}$; b) $[f]_{\mathcal{R}',\mathcal{R}'}$

Sol

a) $\mathcal{R}_0 \xrightarrow{f} \mathcal{R}_0$

$f(e_1)=f(1,0)=(1,0)=e_1+0\cdot e_2$ $\quad A=\begin{pmatrix}1&1\\0&2\end{pmatrix}$

$f(e_2)=f(0,1)=(1,2)=e_1+2e_2$

sau $f(x)=y \Leftrightarrow \begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}1&1\\0&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$ $\quad A=[f]_{\mathcal{R}_0,\mathcal{R}_0}$

$\qquad\qquad\qquad\ \ \begin{pmatrix}x_1+x_2\\2x_2\end{pmatrix}$

b) $f(e_1')=f(e_1-2e_2)=f(e_1)-2f(e_2)=(1,0)-2(1,2)=(-1,-4)$

$=ae_1'+be_2'=a(1,-2)+b(1,1)=(a+b,-2a+b)$

$\begin{cases}a+b=-1\\-2a+b=-4\end{cases}$ $\ominus$

$3a\quad /=3$

$a=1$

$b=-1-1=-2$

$A'=\begin{pmatrix}1&0\\-2&2\end{pmatrix}=[f]_{\mathcal{R}',\mathcal{R}'}$

$f(e_2')=f(e_1+e_2)=f(e_1)+f(e_2)=(1,0)+(1,2)=(2,2)=ce_1'+de_2'$

$=(c+d,-2c+d)$

$\begin{cases}c+d=2\\-2c+d=2\end{cases}$

$c=0,\ d=2$

$$\begin{array}{ccc} \mathcal{R}_0 & \xrightarrow{A} & \mathcal{R}_0 \\ C\downarrow & & \downarrow C \\ \mathcal{R}' & \xrightarrow{A'} & \mathcal{R}' \end{array}$$

$A'=C^{-1}AC$

$e_1'=e_1-2e_2$

$e_2'=e_1+e_2$

$C=\begin{pmatrix}1&1\\-2&1\end{pmatrix}$

Prop $f: V_1 \to V_2$ liniară

a) f injectivă $\Leftrightarrow \dim V_1 = rg\, A$

b) f surjectivă $\Leftrightarrow \dim V_2 = rg\, A$

c) f bijectivă $\Leftrightarrow A \in GL(n, \mathbb{K})$

Dem

a) f inj $\Leftrightarrow \operatorname{Ker} f = \{0_{V_1}\}$

$\operatorname{Ker} f = \{x \in V_1 \mid AX = 0\} = S(A)$, $\underbrace{\dim \operatorname{Ker} f}_{0} = \dim V_1 - rg\, A$

$\Leftrightarrow \dim V_1 = rg\, A$

b) f surjectivă $\Leftrightarrow \dim \operatorname{Im} f = \dim V_2$.

T dim: $\dim V_1 = \underbrace{\dim \operatorname{Ker} f}_{\dim V_1 - rg\, A} + \dim \operatorname{Im} f$ $\Rightarrow \dim V_2 = rg\, A$

c) f bij $\Leftrightarrow \dim V_1 = \dim V_2 = rg\, A \Leftrightarrow A \in GL(n, \mathbb{K})$

OBS

a) $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$ $\quad h = g \circ f$

$x \longrightarrow f(x) = y \longrightarrow g(y) = g(f(x)) = z$

$Z = A_g Y = A_g A_f X$

$Z = A_h X$ $\quad \Rightarrow \boxed{A_{g \circ f} = A_g \cdot A_f}$

b) $V \xrightarrow{f} V \xrightarrow{f^{-1}} V$ $\quad$ (id_V)

$I_n = A_{f^{-1}} \cdot A_f$

Analog $I_n = A_f \cdot A_{f^{-1}}$ $\quad \Rightarrow A_f^{-1} = A_{f^{-1}}$

c) $GL(V) = \{f: V \to V \mid f \text{ lin} + \text{bij}\}$

$\varphi: (GL(V), \circ) \longrightarrow (GL(n, \mathbb{K}), \cdot)$ izomorfism de grupuri

$\varphi(f) = A_f$ $\quad$ 1) $\varphi(f \circ g) = \varphi(f) \cdot \varphi(g)$

2) φ bij

Ex $f:\mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (2x_1+2x_2, x_1+x_3, x_1+3x_2-2x_3)$

a) f nu este izom. sp v

b) $f|_{V'} : V' \xrightarrow{\alpha} V''$ izom sp vect, α

$V' = \{x \in \mathbb{R}^3 \mid x_1+x_2-x_3=0\}$

$V'' = \{x \in \mathbb{R}^3 \mid 3x_1-4x_2-2x_3=0\}$

c) $f(V' \cap V'') = ?$

SOL a) $[f]_{R_0,R_0} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix} = A$

$f(x)=y \iff \begin{pmatrix} 2x_1+2x_2 \\ x_1+x_3 \\ x_1+3x_2-2x_3 \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\det A = \begin{vmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & -2 \end{vmatrix} = 0 \Rightarrow A \notin GL(3,\mathbb{R})$

$\Rightarrow f$ nu e inj

$\operatorname{Ker} f = \{x \in \mathbb{R}^3 \mid AX = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\}$

$\dim \operatorname{Ker} f = 3 - \operatorname{rg} A = 3-2 = 1$

$\dim \operatorname{Im} f = 3 - 1 = 2$

b) $\dim V' = 3-1 = 2$

$V' = \{(x_1, x_2, x_1+x_2) \mid x_1, x_2 \in \mathbb{R}\}$, $R' = \{\underset{e_1'}{(1,0,1)}, \underset{e_2'}{(0,1,1)}\}$

$(x_1, x_2, x_1+x_2) = x_1(1,0,1) + x_2(0,1,1)$

R' SG și $\dim V' = |R'| = 2 \to R'$ reper în V'

$f(e_1') = f(1,0,1) = (2\cdot1+2\cdot0, 1+1, 1+3\cdot0-2\cdot1) = (2,2,-1) = e_1''$

$3\cdot2-4\cdot2-2(-1) = 6-8+2 = 0 \Rightarrow e_1'' \in V''$

$f(e_2') = f(0,1,1) = (2\cdot0+2\cdot1, 0+1, 0+3\cdot1-2\cdot1) = (2,1,1) = e_2''$

$3\cdot2-4\cdot1-2\cdot1 = 6-6 = 0 \Rightarrow e_2'' \in V''$

Dem că $R'' = \{e_1'', e_2''\}$ reper în V'' $\operatorname{rg}\begin{pmatrix} 2 & 2 \\ 2 & 1 \\ -1 & 1 \end{pmatrix} = 2 = \max$

$\dim V'' = 3-1 = 2$

R'' este SLI $\Big| \Rightarrow R''$ reper în V''

c) $f(V' \cap V'')$, $V' \cap V'' = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{cases} x_1 + x_2 - x_3 = 0 \\ 3x_1 - 4x_2 - 2x_3 = 0 \end{cases} \right\}$

$\dim V' \cap V'' = 3 - rg B = 3 - 2 = 1$ $\quad B = \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 3 & -4 & -2 \end{array}\right) \begin{array}{|c} 0 \\ 0 \end{array}$

$\begin{cases} x_1 + x_2 = x_3 \\ 3x_1 - 4x_2 = 2x_3 \end{cases} \Big| \cdot 4$

$7x_1 \quad / \quad = 6x_3$

$x_1 = \frac{6}{7} x_3$

$x_2 = x_3 - \frac{6}{7} x_3 = \frac{1}{7} x_3$

$V' \cap V'' = \left\{ \left(\frac{6}{7} x_3, \frac{1}{7} x_3, x_3 \right) = \frac{x_3}{7}(6,1,7),\ x_3 \in \mathbb{R} \right\} = \langle \{(6,1,7)\} \rangle$

$f|_{V' \cap V''} : V' \cap V'' \longrightarrow \mathbb{R}^3$

$\underset{\underset{1}{\parallel}}{\dim (V' \cap V'')} = \dim \left(Ker\left(f|_{V' \cap V''}\right) \right) + \underbrace{\dim f(V' \cap V'')}$

$f(6,1,7) = (2 \cdot 6 + 2 \cdot 1,\ 6 + 7,\ 6 + 3 \cdot 1 - 2 \cdot 7)$

$= (14, 13, -5) \neq 0_{\mathbb{R}^3}$

$f(V' \cap V'') = \langle \{(14, 13, -5)\} \rangle$, $\dim (f(V' \cap V'')) = 1$.

<u>Def</u> $(V^* = \{ f : V \to \mathbb{K},\ f \text{ liniară} \}, +, \cdot)/_{\mathbb{K}}$ <u>sp. vectorial dual</u>

$+ : V^* \times V^* \longrightarrow V^*$ $\quad (f+g)(x) := f(x) + g(x)$

$\cdot : \mathbb{K} \times V \longrightarrow V$ $\quad (\alpha f)(x) := \alpha f(x)$, $\forall f, g \in V^*$

$\forall x \in V, \alpha \in \mathbb{K}$

<u>Teoremă</u> $\quad V \simeq V^*$

<u>Dem</u> $\mathcal{R} = \{e_1, \dots, e_n\}$ reper în V

Construim $\mathcal{R}^* = \{e_1^*, \dots, e_n^*\}$, $e_i^* : V \to \mathbb{K}$ care

verifică $e_i^*(e_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$ $\quad e_1^*(e_1) = 1, \dots, e_1^*(e_n) = 0$

$e_n^*(e_1) = 0, \dots, e_n^*(e_n) = 1.$

Extindem prin liniaritate: $e_i^*(x) = e_i^*\left(\sum_{j=1}^{n} x_j e_j \right) =$

$= \sum_{j=1}^{n} x_j e_i^*(e_j) = x_i$, $\forall i = \overline{1,n}$

Dem. $\mathcal{R}^* = \{e_1^*, \dots, e_n^*\}$ reper în V^*

① SLI $\quad \sum_{i=1}^{n} a_i e_i^* = 0 \Rightarrow \sum_{i=1}^{n} a_i \overbrace{e_i^*(e_j)}^{\delta_{ij}} = 0 \Rightarrow a_j = 0, \ \forall j = \overline{1,n}$

② SG. $\quad \forall f \in V^*, \ f = \sum_{i=1}^{n} f_i e_i^*$

$$f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i f(e_i) = \sum_{i=1}^{n} f(e_i) e_i^*(x), \forall x \in V$$

$$f = \sum_{i=1}^{n} f(e_i) e_i^*, \quad f_i = f(e_i), \ i = \overline{1,n}$$

Exemple de endomorfisme

Def $\quad p : V = V_1 \oplus V_2 \longrightarrow V = V_1 \oplus V_2$ lin.

p s.n. proiecție pe V_1, de-a lungul lui $V_2 \Leftrightarrow$

$$p(v) = p(\underset{\substack{\cap \\ V_1}}{v_1} + \underset{\substack{\cap \\ V_2}}{v_2}) = v_1$$

Prop $\quad p \in \mathrm{End}(V)$

p = proiecție $\Leftrightarrow p \circ p = p$.

Dem "$\Rightarrow$" $p : V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2 \quad p(v) = v_1$

$p \circ p(v) = p(v_1) = p(v_1 + 0) = v_1 = p(v), \forall v \Rightarrow p \circ p = p$

"$\Leftarrow$" $p \in \mathrm{End}(V)$ a.î. $p \circ p = p$.

Fie $V_1 = \mathrm{Im}\, p$, $V_2 = \mathrm{Ker}\, p$. Dem că $V \overset{?}{=} \mathrm{Im}\, p \oplus \mathrm{Ker}\, p$.

"$\subseteq$" Fie $\underset{\substack{\cap \\ V}}{v} = \underset{\substack{\cap \\ \mathrm{Im}\, v}}{p(v)} + \underset{\substack{\cap \\ \mathrm{Ker}\, p}}{(v - p(v))}$.

$$p(v - p(v)) = p(v) - p(p(v)) = p(v) - p(v) = 0$$

$\oplus$ Fie $v \in \mathrm{Im}\, p \cap \mathrm{Ker}\, p \Rightarrow v = p(w) \Rightarrow \underset{\substack{\| \\ 0}}{p(v)} = p(\underset{\substack{\| \\ p(w) \\ \| \\ v}}{p(w)})$

$p(v) = 0$

$\boxed{v = 0}$

$V = \mathrm{Im}\, p \oplus \mathrm{Ker}\, p \qquad p(v) = p(v_1 + v_2) = p(\underset{\substack{\| \\ p(u)}}{v_1}) = p(p(u)) = p(u) = v_1$.

Ex. $V_1 = <\{(1,2,3)\}>$, $\mathbb{R}^3 = V_1 \oplus V_2$

$p: V_1 \oplus V_2 \to V_1$ pr. pe V_1

$p(1,5,0) = ?$

$\mathcal{R} = \underset{\substack{\| \\ \{(1,2,3)\}}}{\mathcal{R}_1} \cup \{ \underset{\substack{\| \\ e_1}}{(1,0,0)}, \underset{\substack{\| \\ e_2}}{(0,1,0)} \}$ reper în $\mathbb{R}^3$, $V_2 = <\{e_1, e_2\}>$

$(1,5,0) = a(1,2,3) + b(1,0,0) + c(0,1,0) = (a+b, 2a+c, 3a)$

$$\begin{cases} a+b = 1 \\ 2a+c = 5 \\ 3a = 0 \end{cases} \qquad \begin{aligned} a &= 0 \\ b &= 1 \\ c &= 5 \end{aligned}$$

$$(1,5,0) = \underbrace{0\cdot(1,2,3)}_{\substack{v_1 \\ \cap \\ V_1}} + \underbrace{(1,0,0) + 5(0,1,0)}_{\substack{v_2 \\ \cap \\ V_2}} \Rightarrow p(1,5,0) = (0,0,0)$$

T3C

1. $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1 - x_2 + x_3,\ x_1 - x_2 + x_3,\ x_3)$

a) $[f]_{R_0, R_0} = A = ?$

b) Să se det $\operatorname{Ker} f$, $\operatorname{Im} f$. Precizați câte un reper în fiecare

c) $\mathbb{R}^3 = \operatorname{Ker} f \oplus W$, $W = ?$.

Fie $p : \operatorname{Ker} f \oplus W \to \operatorname{Ker} f$ proiecția pe $\operatorname{Ker} f$, de-a lungul lui W și $s : \mathbb{R}^3 \to \mathbb{R}^3$ simetria față de $\operatorname{Ker} f$.

Calculați $p(1,0,3)$, $s(1,0,3)$

2. $S : V \to W$ liniară

$S^* : W^* \to V^*$, $S^*(f) = f \circ S$, $\forall f \in W^*$

(pull-back)

a) S^* este liniară

b) S surj $\Rightarrow$ S^* inj.