$R' \cup R''$ bază în $\mathbb{R}^4$

Seminar 6

examen

1 $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$

$S = \{\overbrace{(1,2,3)}^{u}, \overbrace{(-1,1,5)}^{v}\}_{V}$,

$S' = \{\overbrace{(1,5,11)}^{u'}, \overbrace{(2,1,-2)}^{v'}, (3,$

a) $<S> = <S'> + <V>$

b) Să se descrie V' în fct. de un sist de ec lin

c) V'' a.î. $\mathbb{R}^3 = V' \oplus V$

a) $rg\begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 5 \end{pmatrix} = 2 = max \xRightarrow{cli} S - SLI$

$w' = u' + v' \Rightarrow S'$ SLD

$rg\begin{pmatrix} 1 & 2 \\ 5 & 1 \\ 1 & -2 \end{pmatrix} = 2 = max \xRightarrow{cli} \{u', v'\} - SLI$ max p

$\Rightarrow \dim_{\mathbb{R}} <S> = 2$

$\dim_{\mathbb{R}} <S'> = 2$

$\exists\, a, b \in \mathbb{R}$ a.î. $u' = au + vb$

$(1,5,11) = a(1,2,3) + b(-1,1,5)$

$(1,5,11) = (a-b, 2a+b, 3a+5b)$

$$\begin{cases} a-b=1 \\ 2a+b=5 \\ 3a+5b=11 \end{cases} \quad A=\left(\begin{array}{cc|c} \overline{2} & \overline{11} & 5 \\ 3 & 5 & 11 \end{array}\right)$$

$$\det A = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 5 \\ 3 & 5 & 11 \end{vmatrix} = 11+10-15-3-25+22$$
$$\begin{matrix} 1 & -1 & 1 \\ 2 & 1 & 5 \end{matrix} \qquad = 21+22-43=0$$

$\Rightarrow rg\, A = rg\, \overline{A} \Rightarrow$ sist comp. det.

$\exists\, c, d$ a.î. $v' = cu + dv$

$(2,1,-2) = c(1,2,3) + d(-1,1,5)$

$$\begin{cases} c-d=2 \\ 2c+d=1 \\ 3c+5d=-2 \end{cases} \quad A=\left(\begin{array}{cc|c} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 3 & 5 & -2 \end{array}\right)$$

$$\begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 3 & 5 & -2 \end{vmatrix} = -2+20-3-6-5-4$$
$$\begin{matrix} 1 & -1 & 2 \\ 2 & 1 & 1 \end{matrix} \qquad = 20-12-8 = 20-20=0$$

$\Rightarrow$ S.C.D.

$\Rightarrow \dim_{\mathbb{R}} \langle S \rangle = \dim_{\mathbb{R}} \langle S' \rangle = 2$

$\Rightarrow \langle S \rangle = \langle S' \rangle$

b) Fie $x \in V'$

$\exists\, a, b \in \mathbb{R}$ a.î. $x = au + bv$

$$\begin{cases} a-b=x_1 \\ 2a+b=x_2 \\ 3a+5b=x_3 \end{cases} \quad \left(\begin{array}{cc|c} 1 & -1 & x_1 \\ 2 & 1 & x_2 \\ 3 & 5 & x_3 \end{array}\right)$$

$$\begin{vmatrix} 1 & -1 & x_1 \\ 2 & 1 & x_2 \\ 3 & 5 & x_3 \end{vmatrix} = x_3+10x_1-3x_2-3x_1-5x_2+2x_3$$
$$\begin{matrix} 1 & -1 & x_1 \\ 2 & 1 & x_2 \end{matrix} \qquad = 7x_1 - 8x_2 + 3x_3$$

$V' = \{ x \in \mathbb{R}_3 / 7x_1 - 8x_2 + 3x_3 = 0 \}$

c) $R' = \{u, v\}$ – reper în V'

extindem la un reper în $\mathbb{R}_3$

$R'' = R' \cup \{(0,0,1)\}$

$rg \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 3 & 5 & 1 \end{pmatrix} = 3 = max \Rightarrow SLi$ $\Bigg\} \Rightarrow R''$ spațiu generat

$\dim \mathbb{R}_3 = 3 = card\ R''$

$V'' = \langle \{0,0,1\} \rangle$

lucrare

2) $(\mathbb{R}^3, +, \cdot)/_{\mathbb{R}}$ $V' = \{ (x,y,z) \in \mathbb{R}^3 / \begin{cases} x - y + 2z = 0 \\ 2x + y + z = 0 \end{cases} \}$

Să se descompună $x = (-1, 3, 4)$ în rap cu

$\mathbb{R}^3 = V' \oplus V''$

$x = \underset{\in V'}{u'} + \underset{\in V''}{v'}$

$A = \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 2 & 1 & 1 & 0 \end{array} \right)$

$rg\ A = 2 \Rightarrow \dim \langle V' \rangle = \underbrace{3 - rg\ A}_{\text{TEORIA!}} = 1$

$\begin{cases} x - y = -2z \\ 2x + y = -z \end{cases} \Rightarrow x = -z$

$y = z$

$V' = \{ (-z, z, z);\ z \in \mathbb{R} \} = \langle V' \rangle = \langle \{(-1,1,1)\} \rangle$

$R' = \{(-1,1,1)\}$ este reper în V'

Extindem la un reper în $\mathbb{R}^3$

$R'' = \cancel{R' \cup} \{ \langle (1,0,0), (0,1,0) \}$

$rg \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = 3 = max \Rightarrow R = R \cup R'$ SLi

$\xRightarrow{prop} R$ reper în $\mathbb{R}^3$

$\Rightarrow V''$ sp. generat de R'' $V'' = \langle R'' \rangle$

$(-1, 3, 4) = \underbrace{a(-1,1,1)}_{u'} + \underbrace{b(1,0,0) + c(0,1,0)}_{v'}$

pică la examen !!

$$\begin{cases} -a+b=-1 & \Rightarrow b=3 \\ a+c=3 & \Rightarrow c=-1 \\ a=4 \end{cases}$$

$\Rightarrow u' = (-4, 4, 4)$

$v' = (3,0,0) + (0,-1,0) = (3,-1,0)$

$x = \underbrace{(-4,4,4)}_{\in V'} + \underbrace{(3,-1,0)}_{\in V''}$

Apl. liniare:

1. $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1 + x_2, -x_2)$

Ar că $f \in$ Automorfism $(\mathbb{R}^2)$

$f \in Aut(\mathbb{R}^2)$

a) f este liniară și bijectivă:

<u>liniaritate</u>: $f(ax+by) = a(f(x)) + b(f(y))$

$ax + by = a(x_1, x_2) + b(y_1, y_2)$

$= ax_1 + by_1, ax_2 + bx_2$

$f(ax+by) = f(ax_1 + by_1, ax_2 + by_2)$

$= (ax_1 + b y_2 + ax_2 + by_2, -ax_2 - by_2)$

$= (ax_1 + ax_2, -ax_2) + (bx_1 + bx_2, -bx_2)$

$= a(x_1 + x_2, -x_2) + b(x_1 + x_2, -x_2)$

$= af(x) + bf(y)$ $(\forall)\ x, y \in \mathbb{R}^2$

$(\forall)\ a, b \in \mathbb{R}$

<u>bijectivitate</u>:

<u>inj</u>: f inj $\Leftrightarrow$ $Ker(f) = \{0_{\mathbb{R}^2}\}$

$Ker(f) = \{x \in \mathbb{R}^2 / f(x) = 0_{\mathbb{R}^2}\}$

$$\begin{cases} x_1 + x_2 = 0 \\ -x_2 = 0 \end{cases} \Rightarrow x_1 = x_2 = 0$$

$\Rightarrow f$ este inj

surj: f surj $\Leftrightarrow \dim \operatorname{Im} f = \dim \mathbb{R}^2 = 2$

Folosim Th. dim.

$f: V_1 \to V_2$, liniară, $\dim V_1 = \dim \operatorname{Ker}(f) + \dim \operatorname{Im} f$.

$\dim \mathbb{R}^2 = \dim \operatorname{Ker}(f) + \dim \operatorname{Im} f$

$\Rightarrow 2 = 0 + \dim \operatorname{Im} f \Rightarrow f$. surj

$\Rightarrow f$ bij + f. liniară

$\Rightarrow f$ Automorfism

2) $f: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x_1, x_2, x_3) = f(x_1 + x_2 + x_3, -x_1 - 2x_2 - x_3, x_1 + x_2 + x_3)$

a) $[f]_{R_0 R_0} = A$, A^{-1}?

b) $\dim \operatorname{Ker}(f)$, $\dim \operatorname{Im} f$.

c) $V' = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_1 - x_2 + x_3 = 0 \\ x_1 + 2x_2 - x_3 = 0 \end{array} \right\}$

a) $R_0 = \{e_1, e_2, e_3\} \xrightarrow{A} R_0 = \{e_1, e_2, e_3\}$

MI $f(e_1) = a e_1 + b e_2 + c e_3$ $\quad A = \begin{pmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{pmatrix}$

$f(e_2) = a' e_1 + b' e_2 + c' e_3$

$f(e_3) = a'' e_1 + b'' e_2 + b''' e_3$

$f(x) = y \Leftrightarrow Y = AX \Leftrightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

MII $\begin{pmatrix} x_1 + 2x_2 + x_3 \\ -x_1 - 2x_2 - x_3 \\ x_1 + x_2 + x_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

MI $f(e_1) = f(1, 0, 0) = (1, -1, 1) = e - e_2 + e_3$

$f(e_2) = f(0, 1, 0) = (2, -2, 1)$

$f(e_3) = f(0, 0, 1) = (1, -1, 1)$

Scanned with CamScanner

GRUPA

Temă 2 - Curs

$\mathbb{R}) = M_n^S(\mathbb{R}) \oplus M_n^a(\mathbb{R})$

b) $Ker(f) = \{ x \in \mathbb{R}^3 / AX = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \} = S(A)$

$\dim Ker = 3 - rg\, A = 3 - 2 = 1$

(f nu e injectivă)

$\dim \mathbb{R}^3 = \dim Ker(f) + \dim Im f$

$3 = 1 + \dim Im f$

$\dim Im f = 2$

(f nu e surjectivă)

c) $B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix} \Big| \begin{matrix} 0 \\ 0 \end{matrix}$

$rg\, B = 2 \Rightarrow \dim V' = 3 - rg\, B = 3 - 2 = 1$

$f\big|_{V'} : V' \to \mathbb{R}^3$

$\underbrace{\dim V'}_{=1} = \dim Ker(f)\big|_{V'} + \dim f(V')$

$\begin{cases} x_1 - x_2 = -x_3 \quad \cdot(-1) \\ x_1 + 2x_2 = x_3 \end{cases}$

$3x_2 = 2x_3$

$x_2 = \frac{2}{3} x_3$

$x_1 = -\frac{1}{3} x_3$

$V' = \{ (-\frac{1}{3}x_3, \frac{2}{3}x_3, x_3) \}$

$= \{ \frac{x_3}{3} (-1, 2, 3) \; x_3 \in \mathbb{R} \}$

$V' = \langle \{ (-1, 2, 3) \} \rangle$

$f(-1,2,3) = (-1+4+3, 1-4-3, -1+2+3) = (6, -6, 4) \neq$

$\neq (0,0,0) = 0_{\mathbb{R}^3}$

$f(V') = \langle \{ (6, -6, 4) \} \rangle$

$\dim V' = 1$

$f : \mathbb{R}_2[x] \to \mathbb{R}_1[x]$, $[x]$ liniară

1) $f(x+2) = x+1 = f(x) + f(2) = f(x) + 2f(1) = x+1$

$f(-x^2+3) = 2x+3 = -f(x^2) + 3f(1) = 2x+3$

$f(2x+5) = -x+1 = 2f(x) + 5f(1) = -x+1$

Det f.

$R = \{1, x, x^2\}$ reper în $R_2[X]$

$R' = \{1, x\}$ reper în $\mathbb{R}_1[x]$

$$\begin{cases} 2f(1) + f(x) = x+1 \\ 3f(1) - f(x^2) = 2x+3 \\ 5f(1) + 2f(x) = -x+1 \end{cases}$$

$f(1) = -3x - 1$

$f(x) = 7x + 3$

$f(x^2) = (11x+6) = -11x - 6$

$f(a_0 + a_1 X + a_2 X^2) = a_0 f(1) + a_1 f(x) + a_2 f(x^2)$

$a_0(-3x-1) + a_1(7x+3) + a_2(-11x-6)$

$= -a_0 + 3a_1 - 6a_2 + x(-3a_0 + 7a_1 - 11a_2)$

$[f]_{R,R'} = A = \begin{pmatrix} -1 & 3 & -6 \\ -3 & 7 & -11 \end{pmatrix} \in \mathcal{M}_{2,3}(\mathbb{R})$

__OBS__ : Când se cere f în cazul de mai sus, ne trb. elem unui reper pt a scrie polinomul.