CURS 4 - AG

Repere. Coordonate în raport cu un reper. Operații cu subspații vectoriale

Def $(V,+,\cdot)_{/\mathbb{K}}$ sp. vectorial finit generat, $R=\{e_1,\dots,e_n\}$ bază

R s.n. <u>reper</u> $\iff$ R este o bază ordonată

Prop $(V,+,\cdot)_{/\mathbb{K}}$ sp. vect. f. generat, $\dim_{\mathbb{K}} V = n$, $R=\{e_1,\dots,e_n\}$ reper.

$\Rightarrow \forall x\in V, \exists!\ (x_1,\dots,x_n)\in\mathbb{K}^n$ (coordonatele sau componentele lui x în raport cu reperul R)

a.î. $x = x_1e_1+\dots+x_ne_n$

Dem $V = \langle R\rangle$ (R este SG)

$\forall x\in V$, $\exists\ x_1,\dots,x_n\in\mathbb{K}$ a.î. $x = x_1e_1+\dots+x_ne_n$.

<u>Unicitate</u> Pp. prin abs $\exists\ x_1',\dots,x_n'\in\mathbb{K}$ a.î. $x=x_1'e_1+\dots+x_n'e_n$.

$x_1e_1+\dots+x_ne_n = x_1'e_1+\dots+x_n'e_n \Rightarrow$

$(x_1-x_1')e_1+\dots+(x_n-x_n')e_n = 0 \xRightarrow{R \text{ este SLI}} \begin{cases} x_1-x_1'=0\\ \vdots \\ x_n-x_n'=0\end{cases} \Rightarrow$

$x_k = x_k',\ \forall k=\overline{1,n}$

Modificarea coordonatelor unui vector la schimbarea reperului

Fie $R=\{e_1,\dots,e_n\} \xrightarrow{A} R'=\{e_1',\dots,e_n'\}$ repere în $(V,+,\cdot)_{/\mathbb{K}}$

A = matrice de trecere.

$e_i' = \sum_{j=1}^{n} a_{ji}e_j,\ \forall i=\overline{1,n}$

$x = \sum_{j=1}^{n} x_je_j$

$x = \sum_{i=1}^{n} x_i'e_i' = \sum_{i=1}^{n} x_i'\left(\sum_{j=1}^{n} a_{ji}e_j\right) = \sum_{j=1}^{n}\underline{\left(\sum_{i=1}^{n} a_{ji}x_i'\right)}e_j$

$x_j = \sum_{i=1}^{n} a_{ji}x_i',\ \forall j=\overline{1,n} \iff \begin{pmatrix} x_1\\ \vdots\\ x_n\end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n}\\ \vdots & & \\ a_{n1} & \dots & a_{nn}\end{pmatrix}\begin{pmatrix} x_1'\\ \vdots\\ x_n'\end{pmatrix} \iff \boxed{X = AX'}$

Prop $A \in GL(n, \mathbb{K})$

Dem

- $$\mathcal{R} \xrightarrow{A} \mathcal{R}' \xrightarrow{B} \mathcal{R}'' \Rightarrow C = AB$$

$$\mathcal{R} = \{e_1, \dots, e_n\}, \quad \mathcal{R}' = \{e'_1, \dots, e'_n\}, \quad \mathcal{R}'' = \{e''_1, \dots, e''_n\}, \quad \mathcal{R} \xrightarrow{C} \mathcal{R}''$$

$$e''_k = \sum_{i=1}^{n} c_{ik} e_i, \quad \forall k = \overline{1,n}$$

$$e''_k = \sum_{j=1}^{n} b_{jk} e'_j = \sum_{j=1}^{n} b_{jk} \left(\sum_{i=1}^{n} a_{ij} e_i \right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} b_{jk} \right) e_i$$

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}, \quad \forall i,k = \overline{1,n} \Leftrightarrow C = AB$$

- $$\begin{rcases} \mathcal{R} \xrightarrow{A} \mathcal{R}' \xrightarrow{B} \mathcal{R}, \quad \mathcal{R} \xrightarrow{I_n} \mathcal{R} & AB = I_n \\ \mathcal{R}' \xrightarrow{B} \mathcal{R} \xrightarrow{A} \mathcal{R}', \quad \mathcal{R}' \xrightarrow{I_n} \mathcal{R}' & B \cdot A = I_n \end{rcases} \Rightarrow A \text{ inversabilă}, \quad B = A^{-1}$$

Def $\mathcal{R} \xrightarrow{A} \mathcal{R}'$ repere <u>orientate la fel</u> $\Leftrightarrow \det A > 0$.

<u>orientate opus</u> $\Leftrightarrow \det A < 0$.

<u>OBS</u> Relatia „a fi orientate" este o relatie de echivalentă:

1) reflexivă $\mathcal{R}$ la fel orientat cu $\mathcal{R}$ ($\mathcal{R} \xrightarrow{I_n} \mathcal{R}$, $\det I_n = 1 > 0$)

2) simetrică: $\mathcal{R}$ la fel orientat cu $\mathcal{R}' \Rightarrow$ $\mathcal{R}'$ la fel orientat cu $\mathcal{R}$.

$$\mathcal{R} \xrightarrow{A} \mathcal{R}', \quad \det A > 0$$

$$\mathcal{R}' \xrightarrow{A^{-1}} \mathcal{R} \quad \det(A^{-1}) = \frac{1}{\det A} > 0$$

3) tranzitivă $\mathcal{R} \xrightarrow{A} \mathcal{R}'$, $\det A > 0$; $\mathcal{R}' \xrightarrow{B} \mathcal{R}''$, $\det B > 0$ $\Rightarrow \mathcal{R} \xrightarrow{C = AB} \mathcal{R}''$

$$\det(AB) = \det A \cdot \det B > 0$$

Pe multimea reperelor se consideră 2 clase de echivalentă. A alege o orientare = a preciza un reper poz. orientat.

Criterii de LI

$(V,+,\cdot)/\mathbb{K}$ sp vect n-dim.

Fie $S=\{v_1,\dots,v_m\}\subset V$ sistem de vectori, $m\le n$

S este SLI $\Leftrightarrow$ rangul matricei componentelor vectorilor din S, în raport cu $\forall$ reper, este maxim i.e. m.

Dem

Fie $R=\{e_1,\dots,e_n\}$ reper în V

$$v_i=\sum_{j=1}^{n} v_{ji}e_j,\ \forall i=\overline{1,m}$$

S este SLI $\Leftrightarrow [\forall a_1,\dots,a_m\in\mathbb{K}: a_1v_1+\dots+a_mv_m=0_V \Rightarrow a_1=\dots=a_m=0_{\mathbb{K}}]$

$$\sum_{i=1}^{m} a_iv_i=0_V \Leftrightarrow \sum_{i=1}^{m} a_i\Big(\sum_{j=1}^{n} v_{ji}e_j\Big)=0_V \Leftrightarrow$$

$$\sum_{j=1}^{n}\underline{\Big(\sum_{i=1}^{m} v_{ji}a_i\Big)}e_j=0_V \xrightarrow[SLI]{R}$$

$$\Rightarrow \sum_{i=1}^{m} v_{ji}a_i=0,\ \forall j=\overline{1,n}$$

SLO de n ecuații cu m $(m\le n)$ necunoscute: $a_1,\dots,a_m$

SLO are sol unică nulă: $(a_1,\dots,a_m)=(0,\dots,0)\Leftrightarrow$

$$rg(v_{ji})_{\substack{j=\overline{1,n}\\ i=\overline{1,m}}}=m=\text{maxim},\quad C=(v_{ji})_{\substack{j=\overline{1,n}\\ i=\overline{1,m}}}$$

Dem că rangul nu depinde de reperul ales (este un invariant).

Fie $\underset{\{e_1,\dots,e_n\}}{R}\xrightarrow{A}\underset{\{e'_1,\dots,e'_n\}}{R'}$ $\quad e'_k=\sum_{j=1}^{n} a_{jk}e_j,\ \forall k=\overline{1,n}$

$$v_i=\sum_{k=1}^{n} v'_{ki}e'_k=\sum_{k=1}^{n} v'_{ki}\Big(\sum_{j=1}^{n} a_{jk}e_j\Big)=$$

$$v_i=\sum_{j=1}^{n}\underline{\Big(\sum_{k=1}^{n} a_{jk}v'_{ki}\Big)}e_j \qquad C=AC',\ C'=(v'_{ki})$$

$$v_i=\sum_{j=1}^{n}\underline{v_{ji}}e_j \qquad rg\,C\underset{A\in GL(n,\mathbb{K})}{=}rg(AC')=rg\,C'$$

$\boxed{\text{Aplicatie}}$ $(\mathbb{R}^2, +, \cdot)_{/\mathbb{R}}$, $\mathcal{R}_0 = \{\overset{e_1}{(1,0)}, \overset{e_2}{(0,1)}\}$ reper canonic

$\mathcal{R}' = \{ e_1' = (2,1),\ e_2' = (3,0)\}$.

a) $\mathcal{R}'$ este reper.

b) $\mathcal{R}_0 \xrightarrow{A} \mathcal{R}'$, $\mathcal{R}' \xrightarrow{B} \mathcal{R}_0$, $A, B = ?$

Sunt $\mathcal{R}_0, \mathcal{R}'$ la fel orientate?

c) Fie $x = (1,2)$. Să se afle coordonatele lui x în raport cu $\mathcal{R}_0$ și $\mathcal{R}$.

SOL

PROP $\dim V = n$, $S = \{v_1, \dots, v_n\}$, card $S = n$

UAE 1) S bază.

2) S este SLI

3) S este SG.

$\dim_{\mathbb{R}} \mathbb{R}^2 = 2$, card $\mathcal{R}' = 2$. Dem că $\mathcal{R}'$ este SLI

$e_1' = (2,1) = 2(1,0) + 1(0,1) = \underline{2} e_1 + \underline{1} \cdot e_2$ $\quad A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$

$e_2' = (3,0) = \underline{3} e_1 + \underline{0} e_2$

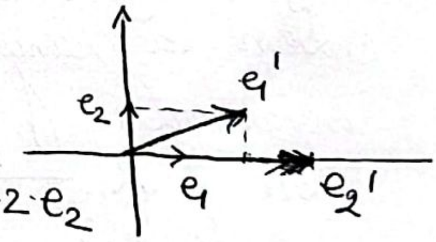
$\mathcal{R}_0 \xrightarrow{A} \mathcal{R}'$ $\quad rg A = 2 = \max \overset{crit}{\underset{LI}{\Longrightarrow}} \mathcal{R}'$ este SLI $\overset{PROP}{\Longrightarrow}$

$\mathcal{R}'$ este reper. $\quad \det A = -3 < 0$

$\mathcal{R}' \xrightarrow{B} \mathcal{R}_0$, $B = A^{-1}$ $\quad A^T = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$

$B = \frac{1}{-3} \begin{pmatrix} 0 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$

$\mathcal{R}_0, \mathcal{R}'$ sunt opus orientate.



c) $x = (1,2) = 1(1,0) + 2(0,1) = 1 \cdot e_1 + 2 \cdot e_2$

$(1,2)$ coordonatele sau componentele lui x în raport cu reperul canonic $\mathcal{R}_0$

$\underset{(1,2)}{\overset{\parallel}{x}} = x_1' e_1' + x_2' e_2' = x_1'(2,1) + x_2'(3,0) = (2x_1' + 3x_2', x_1')$

$\begin{cases} 2x_1' + 3x_2' = 1 \\ x_1' = 2 \end{cases} \rightarrow x_2' = \frac{1}{3}(1-4) = -1$

$(x_1', x_2') = (2, -1)$ coordonatele lui x în rap. cu reperul $\mathcal{R}'$

OBS $X = AX' \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \overset{-5-}{=} \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = B \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2x_1' + 3x_2' \\ x_1' \end{pmatrix}$

Aplicație $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$, $R_0 = \{(1,0,0), (0,1,0), (0,0,1)\}$ reper canonic.

$S = \{(1,2,3), (-1,1,0)\}$ SLI

$S' = \{(1,2,3), (-1,1,0), (1,5,6)\}$ SLD.

SOL

$rg \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 0 \end{pmatrix} = 2 = max \Leftrightarrow S$ este SLI

$\det \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 5 \\ 3 & 0 & 6 \end{pmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} = 0 \Rightarrow rg \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 5 \\ 3 & 0 & 6 \end{pmatrix} \neq 3$

$\Rightarrow S'$ este SLD.

$S \subset S'$ este un SLI maximal.

Operații cu subspații vectoriale

$(V, +, \cdot)/\mathbb{K}$, $V' \subseteq V$ subm. nevidă

$V' \subset V$ subspațiu vect $\Leftrightarrow ax + by \in V'$, $\forall a, b \in \mathbb{K}$, $\forall x, y \in V'$

EX $(\mathbb{R}^2, +, \cdot)/\mathbb{R}$, $V' = \{(x,y) \in \mathbb{R}^2 \mid x + y = 2\}$ este sp vect? NU.

$0_{\mathbb{R}^2} \notin V'$

Prop $(V, +, \cdot)/\mathbb{K}$, $V_1, V_2 \subseteq V$ ssp vect $\Rightarrow V_1 \cap V_2 \subseteq V$ ssp vect

Dem

$\forall a, b \in \mathbb{K}$, $\forall x, y \in V_1 \cap V_2 \Rightarrow x, y \in V_1$ și $x, y \in V_2$ $\Rightarrow ax + by \in V_1$ și $ax + by \in V_2$ $\Rightarrow$

$\Rightarrow ax + by \in V_1 \cap V_2$.

OBS În general, $V_1 \cup V_2 \subseteq V$ nu este subspațiu vectorial.

Def $\underset{\overset{\parallel}{\text{not}}}{\langle V_1 \cup V_2 \rangle} = \left\{ \sum_{i=1}^{n} a_i x_i \,,\ x_i \in V_1 \cup V_2,\ a_i \in \mathbb{K},\ \forall i = \overline{1,n},\ n \in \mathbb{N} \right.$

$V_1 + V_2$ (spațiul vect. generat de $V_1 \cup V_2$, $V_1, V_2 \subseteq V$ subspații vect.)

Prop $V_1 + V_2 = \{ v_1 + v_2 \,,\ v_1 \in V_1,\ v_2 \in V_2 \}$

Dem

„$\subseteq$" $V_1 + V_2 = \langle V_1 \cup V_2 \rangle \subseteq \{ v_1 + v_2,\ v_1 \in V_1,\ v_2 \in V_2 \}$

Fie $\underset{V_1+V_2}{\underset{\cap}{x}} = \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{m} a_i x_i + \sum_{j=m+1}^{n} a_j x_j = \underset{V_1}{\underset{\cap}{x_1}} + \underset{V_2}{\underset{\cap}{x_2}}$

Considerăm că $x_1, \dots, x_m \in V_1$ (altfel renumerotăm indicii)

$x_{m+1}, \dots, x_n \in V_2$

„$\supseteq$" $\underset{\text{exemplu de comb. liniară.}}{\overset{\parallel}{\{ v_1 + v_2,\ v_1 \in V_1,\ v_2 \in V_2 \}}} \subseteq \langle V_1 \cup V_2 \rangle$

Def $(V, +, \cdot)/\mathbb{K}$, $V_1, V_2 \subseteq V$ ssp vect

Spunem că $V_1 + V_2$ este <u>sumă directă</u> și notăm $V_1 \oplus V_2$

$\Leftrightarrow V_1 \cap V_2 = \{0_V\}$

Prop $V_1 + V_2$ este sumă directă i.e. $V_1 \oplus V_2 \Leftrightarrow$

$\forall x \in V_1 + V_2$, $\exists!\ x_1 \in V_1$, $x_2 \in V_2$ a.î. $x = x_1 + x_2$

Dem „$\Rightarrow$" $V_1 \oplus V_2 \Leftrightarrow V_1 \cap V_2 = \{0_V\}$

Pp. abs. $\exists$ $x_1', x_1 \in V_1$ (dist), $x_2', x_2 \in V_2$ (dist) a.î. $x = \underline{x_1} + \underline{\underline{x_2}} = \underline{x_1'} + \underline{\underline{x_2'}}$

$\underset{V_1}{\underset{\cap}{x_1 - x_1'}} = \underset{V_2}{\underset{\cap}{x_2' - x_2}} \in V_1 \cap V_2 = \{0_V\} \Rightarrow \begin{array}{l} x_1 = x_1' \\ x_2 = x_2' \end{array}$ Pp este falsă

$\Rightarrow$ scrierea este unică.

„$\Leftarrow$" $x \in V_1 + V_2$ se scrie în mod unic $x = \underset{V_1}{\underset{\cap}{x_1}} + \underset{V_2}{\underset{\cap}{x_2}}$

Pp. prin abs $\exists$ $\underset{0_V}{\underset{\#}{u}} \in V_1 \cap V_2$

$x = \underbrace{x_1 - u}_{\cap\, V_1} + \underset{V_2}{\underset{\cap}{x_2 + u}}$ scrierea nu e unică ↯.

$\boxed{\text{Exemplu}}$ $(V = \mathcal{M}_n(\mathbb{R}), +, \cdot)/\mathbb{R}$.

$V_1 = \{A \in V \mid Tr(A) = 0\}$

$V_2 = \{A \in V \mid A = \alpha I_n,\ \alpha \in \mathbb{R}\}$

a) V_1, V_2 subsp. vect

b) $V = V_1 \oplus V_2$

Sol

a) $\forall A, B \in V_1$, $\forall a, b \in \mathbb{R}$; $Tr(aA + bB) = a\underset{0}{\underset{\shortparallel}{Tr(A)}} + b\underset{0}{\underset{\shortparallel}{Tr(B)}} = 0$

$\Rightarrow aA + bB \in V_1 \Rightarrow V_1 \subset V$ ssp v.

$\forall A, B \in V_2$, $\forall a, b \in \mathbb{R}$ $\quad A = \alpha I_n$, $B = \beta I_n$ $\Rightarrow aA + bB = (\alpha a + \beta b) I_n \in V_2$

b). Fie $A \in V_1 \cap V_2 \Rightarrow \left.\begin{array}{l} Tr(A) = 0 \\ A = \alpha I_n \Rightarrow Tr(A) = n\alpha \end{array}\right\} \Rightarrow \alpha = 0 \Rightarrow A = O_n$

$V_1 \oplus V_2 \subset V$ (din construcție)

Dem $V \subset V_1 \oplus V_2$

$\forall A \in V$, $A = \underbrace{\left(A - \frac{1}{n} Tr(A) I_n\right)}_{A_1 \in V_1} + \underbrace{\frac{1}{n} Tr(A) I_n}_{A_2 \in V_2}$

$\alpha = \frac{1}{n} Tr(A)$

$Tr(A_1) = Tr(A) - \frac{1}{n} Tr(A) \underbrace{Tr(I_n)}_{n} = 0 \Rightarrow A_1 \in V_1$

$\mathcal{M}_n(\mathbb{R}) = V_1 \oplus V_2$. , $\dim_{\mathbb{R}} \mathcal{M}_n(\mathbb{R}) = n^2$

$\dim_{\mathbb{R}} V_1 = n^2 - 1$, $\dim_{\mathbb{R}} V_2 = 1$.

Teoremă $(V, +, \cdot)/\mathbb{K}$ sp. vect, $n = \dim_{\mathbb{K}}$, $\mathcal{R} = \{e_1, .., e_n\}$ reper în V.

$x \in V$, $x = x_1 e_1 + \dots + x_n e_n$, $A \in \mathcal{M}_{m,n}(\mathbb{K})$

$S(A) = \{(x_1, .., x_n) \in \mathbb{K}^n \mid \underset{(m,n)}{A}\underset{(n,1)}{X} = \underset{(m,1)}{O}\}$

1) $S(A) \subseteq \mathbb{K}^n$ subsp vect

2) $\dim_{\mathbb{K}} S(A) = n - rg A$

Dem
1) $S(A) \subset \mathbb{K}^n$ subsp. vect.

Fie $(x_1,\dots,x_n), (x_1',\dots,x_n') \in S(A)$ $\quad AX=0$, $AX'=0$

$\forall a,b \in \mathbb{K}$.

$A(aX+bX') = a\,AX + b\,AX' = 0 \Rightarrow a(x_1,\dots,x_n)+b(x_1',\dots,x_n') \in S(A)$

2) Dem că $\dim S(A) = n-k$, $k = rg(A)$

$A \in \mathcal{M}_{m,n}(\mathbb{K})$, $k \le \min\{m,n\}$, $p = n-k$.

Fie $\Delta = \begin{vmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \\ a_{k1} & \dots & a_{kk} \end{vmatrix} \neq 0$ (minor principal)

(eventual renumerotăm indicii)

$x_1,\dots,x_k$ = var principale

$x_{k+1} = \lambda_1, \dots, x_n = \lambda_p$ var secundare.

$$\begin{cases} a_{11}x_1 + \dots + a_{1k}x_k = -a_{1k+1}\lambda_1 - \dots - a_{1n}\lambda_p \\ \vdots \\ a_{k1}x_1 + \dots + a_{kk}x_k = -a_{kk+1}\lambda_1 - \dots - a_{kn}\lambda_p. \end{cases}$$

$$x_1 = \frac{\Delta_{x_1}}{\Delta}, \quad \Delta_{x_1} = \begin{vmatrix} -a_{1k+1}\lambda_1 - \dots - a_{1n}\lambda_p & a_{12} & \dots & a_{1k} \\ \vdots & \vdots & & \\ -a_{kk+1}\lambda_1 - \dots - a_{kn}\lambda_p & a_{k2} & \dots & a_{kk} \end{vmatrix}$$

$$\begin{cases} x_1 = \frac{\Delta_{11}}{\Delta}\lambda_1 + \dots + \frac{\Delta_{1p}}{\Delta}\lambda_p. \\ \vdots \\ x_k = \frac{\Delta_{k1}}{\Delta}\lambda_1 + \dots + \frac{\Delta_{kp}}{\Delta}\lambda_p \end{cases}$$

$(x_1,\dots,x_k, x_{k+1},\dots,x_n) =$

$= \left(\frac{\Delta_{11}}{\Delta}\lambda_1 + \dots + \frac{\Delta_{1p}}{\Delta}\lambda_p, \dots, \frac{\Delta_{k1}}{\Delta}\lambda_1 + \dots + \frac{\Delta_{kp}}{\Delta}\lambda_p, \lambda_1, \dots, \lambda_p\right)$ $(*)$

$= \lambda_1 \underbrace{\left(\frac{\Delta_{11}}{\Delta}, \dots, \frac{\Delta_{k1}}{\Delta}, 1, 0, \dots 0\right)}_{y_1} + \dots + \lambda_p \underbrace{\left(\frac{\Delta_{1p}}{\Delta}, \dots, \frac{\Delta_{kp}}{\Delta}, 0, \dots, 1\right)}_{y_p}$

$(x_1,\dots,x_k, x_{k+1},\dots,x_n) = \lambda_1 y_1 + \dots + \lambda_p y_p$, $\mathcal{B} = \{y_1,\dots,y_p\}$ e SG.

• $\mathcal{B}$ este SI

$\forall \lambda_1, \dots, \lambda_p \in \mathbb{R}$ ai $\lambda_1 y_1 + \dots + \lambda_p y_p = 0_{\mathbb{K}^n} \xrightarrow{(*)} \lambda_1 = \dots = \lambda_p = 0$

Deci $\mathcal{B}$ este bază pt $S(A)$

$\dim_{\mathbb{K}} S(A) = \text{card } \mathcal{B} = p = n - k = n - rg A$

$\boxed{\text{Aplicație}}$ $(\mathbb{R}^3, +, \cdot)_{/\mathbb{R}}$

$V' = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \begin{cases} x - y + z = 0 \\ 2x + y - z = 0 \end{cases} \right\}$

a) $\dim V' = ?$; b) Precizați o bază în V'

$V' = S(A)$ $\quad A = \left(\boxed{\begin{matrix} 1 & -1 \\ 2 & 1 \end{matrix}} \begin{matrix} 1 \\ 1 \end{matrix} \right) \Rightarrow rg A = 2$

$\dim V' = 3 - 2 = 1.$

x, y = var principale, $z = \alpha$ var secundară

$\begin{cases} x - y = -\alpha \\ 2x + y = \alpha \end{cases}$

$3x \quad = 0 \quad (+)$

$x = 0$

$y = \alpha$

$(x, y, z) \in \{ (0, \alpha, \alpha) \mid \alpha \in \mathbb{R} \}$

$= \{ \alpha (0, 1, 1), \alpha \in \mathbb{R} \} \Rightarrow \mathcal{B}$ este SG

$\mathcal{B} = \{ (0, 1, 1) \}$ bază în V'.

<u>T. Grassmann</u> $(V, +, \cdot)_{/\mathbb{K}}$, $V_1, V_2 \subseteq V$ ssp vect

$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$

Conv: $\dim \{ 0_V \} = 0$

$\dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2.$

<u>Def</u> $(V, +, \cdot)_{/\mathbb{K}}$, $V_1, V_2 \subseteq V$ ssp v.

Dacă $V = V_1 \oplus V_2$, atunci V_1 s.n. ssp. complementar lui V_2

V_2 —"— V_1

<u>OBS</u> subsp. complementar nu este unic.

OBS 1) $V_1 \oplus V_2$, $\mathcal{B}_i$ bază în V_i, $i=\overline{1,2} \Rightarrow \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ bază în $V_1 \oplus V_2$!

2) V sp. vect f. generat și $\mathcal{B}$ bază în V.

Partiționăm $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, $\left.\begin{array}{l} V_1 = \langle \mathcal{B}_1 \rangle \\ V_2 = \langle \mathcal{B}_2 \rangle \end{array}\right\} \Rightarrow V = V_1 \oplus V_2$

$\boxed{EX}$. $V = (\mathbb{R}^3, +, \cdot)_{/\mathbb{R}}$

$V_1 = \{(x,y,z) \in \mathbb{R}^3 \mid z = 0\}$

$V_2 = \left\{(x,y,z) \in \mathbb{R}^3 \mid \begin{cases} x = 0 \\ y = 0 \end{cases}\right\} \Rightarrow \mathbb{R}^3 = V_1 \oplus V_2 = V_1 \oplus V_2'$

$V_2' = \{(x,x,x) \in \mathbb{R}^3 \mid x \in \mathbb{R}\}$.

SOL $V_1 = \{(x,y,0) \mid x,y \in \mathbb{R}\} = \{x(1,0,0) + y(0,1,0) \mid x,y \in \mathbb{R}\}$

$\left.\begin{array}{l} \mathcal{B}_1 = \{(1,0,0), (0,1,0)\} \text{ SG pt } V_1 \\ rg\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = 2 = \max \Rightarrow \text{SLI} \end{array}\right\} \Rightarrow \mathcal{B}_1$ bază în V_1.

$V_2 = \{(0,0,z) \mid z \in \mathbb{R}\} = \{z(0,0,1), z \in \mathbb{R}\}$

$\mathcal{B}_2 = \{(0,0,1)\}$ bază în V_2.

$V_2' = \{x(1,1,1), x \in \mathbb{R}\}$, $\mathcal{B}_2' = \{(1,1,1)\}$.

OBS Dacă completăm $\mathcal{B}_1$ la o bază în $\mathbb{R}^3$

$\mathcal{B}_1 \cup \{v\}$ $\qquad \langle\{v\}\rangle = W$ subsp. complementar lui V_1.

$rg\left(\boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}} \; \boxed{\begin{matrix} 1 \\ 0 \\ 1 \end{matrix}}\right) = 3$ $\qquad v = (1,0,1)$, de exemplu.

$\boxed{EX}$ $(\mathbb{R}^4, +, \cdot)$ $\qquad V' = \{(a,b,c,0) \mid a,b,c \in \mathbb{R}\}$

$V'' = \{(0,0,d,e) \mid d,e \in \mathbb{R}\}$.

Este V'' subsp. complementar al lui V'?

$(0,0,1,0) \in V' \cap V''$ $\qquad$ nu e $V' \oplus V''$

$\dim V' = 3$, $\dim V'' = 2$, $\dim \mathbb{R}^4 = 4$.

$V' + V'' \subseteq \mathbb{R}^4$; $\dim(V' + V'') = 3 + 2 - 1 = 4$

$\Rightarrow V' + V'' = \mathbb{R}^4$.

$\boxed{T_2 \text{ curs}}$

1. a) $\mathcal{M}_n(\mathbb{R}) = \mathcal{M}_n^s(\mathbb{R}) \oplus \mathcal{M}_n^a(\mathbb{R})$

   b) Precizați $\dim_{\mathbb{R}} \mathcal{M}_n^s(\mathbb{R})$, $\dim_{\mathbb{R}} \mathcal{M}_n^a(\mathbb{R})$

2. a) $V' = \{ f \in \mathcal{F}(\mathbb{R}) \mid f \text{ bij} \} \subset \left(\mathcal{F}(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ funcție} \}, +, \cdot\right)_{/\mathbb{R}}$

   b) $V'' = \{ P \in \mathbb{R}_3[X] \mid \text{grad } P = 2 \} \subset (\mathbb{R}_3[X], +, \cdot)_{/\mathbb{R}}$

   Precizați dacă V', V'' sunt subspații vect.

3. $(\mathcal{M}_2^{\Delta}(\mathbb{R}), +, \cdot)$ $\quad \mathcal{R}_0 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ reper canonic

   $\mathcal{R}' = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$

   a) $\mathcal{R}'$ reper în $\mathcal{M}_2^{\Delta}(\mathbb{R})$. Sunt $\mathcal{R}'$ și $\mathcal{R}_0$ la fel orientate?

   b) $\mathcal{R}_0 \xrightarrow{A} \mathcal{R}'$, $\mathcal{R}' \xrightarrow{B} \mathcal{R}_0$, $A, B = ?$