Seminar 11

Transformări ortogonale. Endomorfisme simetrice

(1) $(\mathbb{R}^3, g_0)$ s.v.e.r.

$$A = [f]_{R_0,R_0} = \frac{1}{9}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix}$$

R_0 – rep canonic

a) $f \in O(\mathbb{R}^3)$ e speța 2 i.e. $f = s_0 R_\varphi$

b) $\varphi = ?$ axa $= ?$

c) $R = \{e_1, e_2, e_3\}$ ortonormat a.î.

$$[f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

a) $$A \cdot A^t = \left(\frac{1}{9}\right)^2 \begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix}$$

$$= \frac{1}{81}\begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix} = I_3 \Rightarrow A \in O(3)$$

$\Rightarrow f \in O(\mathbb{R}^3)$

$$\left(\frac{1}{9}\right)^3 \begin{vmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{vmatrix} \xrightarrow[C_3+4C_2]{C_1-8C_2} \begin{vmatrix} 0 & 1 & 0 \\ -63 & 8 & 36 \\ -36 & 4 & 9 \end{vmatrix} = -1$$

$\Rightarrow f$ de speța 2

b) $\text{Tr}\, A' = -1 + 2\cos\varphi = 1 \Rightarrow 2\cos\varphi = 2 \Rightarrow$

$\cos\varphi = 1 \Rightarrow \varphi \in [-\pi, \pi]$

$\Rightarrow \varphi = 0$

$f : \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x) = \frac{1}{9}(8x_1 + x_2 - 4x_3,\ x_1 + 8x_2 + 4x_3,\ -4x_1 + 4x_2 - 7x_3)$$

$$= (x_1, x_2, x_3) \quad / 9$$

$\underbrace{f(x) = -x}_{\text{speța 2.}} \Rightarrow \begin{cases} 8x_1 + x_2 - 4x_3 = -9x_1 \\ x_1 + 8x_2 + 4x_3 = -9x_2 \\ -4x_1 + 4x_2 - 7x_3 = -9x_3 \end{cases}$

$$\begin{cases} 17x_1 + x_2 - 4x_3 = 0 \\ x_1 + 17x_2 + 4x_3 = 0 \\ -4x_1 + 4x_2 + 2x_3 = 0 \end{cases} \quad A = \left(\begin{array}{ccc|c} 17 & 1 & -4 & 0 \\ 1 & 17 & 4 & 0 \\ -4 & 4 & 2 & 0 \end{array}\right)$$

$$\det A = 2\begin{vmatrix} 17 & 1 & -2 \\ 1 & 17 & 2 \\ -4 & 4 & 1 \end{vmatrix} \underset{C_1+2C_2}{\overset{C_1+2C_3}{=}} \begin{vmatrix} 9 & 9 & -4 \\ 9 & 9 & 4 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 9 & 9 \\ 9 & 9 \end{vmatrix} = 0$$

$$\begin{cases} x_1 + 17x_2 = 4x_3 \quad / 4 \\ -4x_1 + 4x_2 = -2x_3 \end{cases}$$

$$\Rightarrow \begin{aligned} x_2 &= -\frac{1}{4}x_3 \\ x_1 &= \frac{1}{4}x_3 \end{aligned}$$

Axa: $\left\{\left(\frac{1}{4}x_3, -\frac{1}{4}x_3, x_3\right)\right\}, x_3 \in \mathbb{R}$

$= \frac{x_3}{4}\{(1, -1, 4)\} \quad x_3 \in \mathbb{R}$

$g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}, \; g_0(x,y) = x_1y_1 + x_2y_2 + x_3y_3$

Norma: $\|x\| = \sqrt{g_0(x,y)} = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$e_1 = \frac{1}{3\sqrt{2}}(1, -1, 4)$ versorul axei

c) $\langle\{e_1\}\rangle = \{x \in \mathbb{R}^3 \mid g_0(x, e_1) = 0\}$

$\Rightarrow x_1 - x_2 + 4x_3 = 0$ planul $\perp$ pe axa

$(x_1, x_1 + 4x_3, x_3), \; x_1, x_3 \in \mathbb{R}$

$\{x_1(1, 1, 0) + x_3(0, 4, 1)\}$

$\langle\{(1,1,0), (0,4,1)\}\rangle \quad f_2, f_3$

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f_2, f_3 - reper arbitrar în $\langle \{e_1\} \rangle^{\perp}$

$e_2' = f_2$

$e_3' = f_3 - \frac{\langle f_3, e_2' \rangle}{\langle e_2', e_2' \rangle} \cdot e_2' = (0,4,1) - \frac{\cancel{4}^{2}}{\cancel{2}}(1,1,0)$

$= (0,4,1) - (2,2,0) = (-2,2,1)$

$e_2 = \frac{1}{\sqrt{2}}(1,1,0)$

$e_3 = \frac{1}{3}(-2,2,1)$ se raporte vectorul la norma lui.

În raport cu $R = \{ e_1 = \frac{1}{3\sqrt{2}}(1,-1,4), e_2 = \frac{1}{\sqrt{2}}(1,1,0),$
$e_3 = \frac{1}{3}(-2,2,1) \}$

$$[f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. $(\mathbb{R}^3, g_0)$, $f \in End(\mathbb{R}^3)$

$$A = [f]_{R_0, R_0} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

a) Dem că $f \in Sim(\mathbb{R}^3)$. Det f

b) Să se afle $Q : \mathbb{R}^3 \to \mathbb{R}$ forma pătratică asoci

c) Să se aducă Q la forma canonică, efectuând o transformare ortogonală h

(i.e. o schimbare de repere ortonormate)

a) Observăm $A = A^t \Rightarrow$ matr sim $\Rightarrow A \in \mathcal{M}_3^S$

$\Leftrightarrow f \in Sim(\mathbb{R}^3)$

$f(x) = (x_1 + x_3, x_2, x_1 + x_3)$

b) $Q(x) = X^t A X = x_1^2 + x_2^2 + x_3^2 + 2x_1x_3$

c) $P(\lambda) = \det(A - \lambda I_3) = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (-1)^{2+2} \cdot (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$= (1-\lambda)\left[(1-\lambda)^2 - 1\right] = (1-\lambda)\lambda(\lambda-2) = 0$

$\lambda_1 = 1 \in \mathbb{R}, \; m_1 = 1$

$\lambda_2 = 0 \in \mathbb{R}, \; m_2 = 1$

$\lambda_3 = 2 \in \mathbb{R}, \; m_3 = 1$

$V_{\lambda_1} = \{x \in \mathbb{R}^3, \; f(x) = \lambda_1 x = x\}$

$AX = X \Leftrightarrow (A - I_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 = 0$
$x_2 \in \mathbb{R}$
$x_3 = 0$

$\{(0, x_2, 0) = x_2(0,1,1)\; x_2 \in \mathbb{R}\}$

$<(0,1,1)> = e_1$

$V_{\lambda_2} = \{x \in \mathbb{R}^3, \; f(x) = \lambda_2 x = 0\}$

$AX = 0$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 + x_3 = 0$
$x_2 = 0$
$\Rightarrow x_1 = -x_3$

$V_{\lambda_2} = \{(-x_3, 0, x_3) = x_3(-1, 0, 1), \; x_3 \in \mathbb{R}\}$

$<(-1,0,1)> \Rightarrow e_2 = \frac{1}{\sqrt{2}}(-1,0,1)$

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$V_{\lambda_3} = \{ x \in \mathbb{R}^3 \mid f(x) = 2x \}$

$(A - 2I_3)x = 0$

$$\begin{cases} -x_1 + x_3 = 0 \\ x_2 = 0 \\ x_1 - x_3 = 0 \end{cases} \Rightarrow x_1 = x_3$$

$= \{ (x_1, 0, x_1) = x_1 (1, 0, 1) \}$

$< \{ (1, 0, 1) \} >$

$\|$

$e_3 = \frac{1}{\sqrt{2}} (1, 0, 1)$

$R = \{ e_1, e_2, e_3 \}$ reper ort a.î. $[f]_{R,R} = A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

$Q(x) = x_1'^2 + 2x_3'^2$ sign $(2, 0)$

$$C = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$h(x) = \left(-\frac{1}{\sqrt{2}} x_2 + \frac{1}{\sqrt{2}} x_3, x_1, \frac{1}{\sqrt{2}} x_2 + \frac{1}{\sqrt{2}} x_3 \right)$