## Seminar 2

① $A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{pmatrix}$

Det A = ? (Th Laplace pt $p=2$, $c_{1,2}$ fixate respectiv $l_{2,3}$ fixate

$$\det A = (-1)^{1+2+1+2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} + (-1)^{1+2+1+3}\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix}\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} +$$

$$+ (-1)^{1+2+1+4}\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix}\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + (-1)^{1+2+2+3}\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix}\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} +$$

$$+ (-1)^{1+2+2+4}\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix}\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + (-1)^{1+2+3+4}\begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix}\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -5$$

$$\det A = (-1)^{2+3+1+2}\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix}\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} + (-1)^{2+3+1+3}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}\begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} +$$

$$+ (-1)^{2+3+1+4}\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix}\begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix} + (-1)^{2+3+2+3}\begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix}\begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix} +$$

$$+ (-1)^{2+3+2+4}\begin{vmatrix} 1 & 4 \\ 5 & -1 \end{vmatrix}\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + (-1)^{2+3+3+4}\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix}\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} = -5$$

③ Să se arate că:

$$A = \begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (a-1)^6$$

$$\det A \overset{L_1-L_4}{\underset{\substack{L_2-L_4 \\ L_3-L_4}}{=}} \begin{vmatrix} a^3-1 & (3a^2-3) & (3a-3) & 0 \\ (a^2-1) & (a^2+2a-3) & (2a-2) & 0 \\ a-1 & (2a-2) & (a-1) & 0 \\ 1 & 3 & 3 & 1 \end{vmatrix} =$$

$$(-1)^{4+4}\cdot 1 \cdot \begin{vmatrix} (a-1)(a^2+a+1) & 3(a-1)(a+1) & 3(a-1) \\ (a-1)(a+1) & (a-1)(a+3) & 2(a-1) \\ a-1 & 2(a-1) & a-1 \end{vmatrix} =$$

$$= (a-1)^3 \begin{vmatrix} a^2+a+1 & 3(a+1) & 3 \\ a+1 & a+3 & 2 \\ 1 & 2 & 1 \end{vmatrix} \overset{C_1-C_3}{\underset{C_2-2C_3}{=}}$$

$$= (a-1)^3 \cdot \begin{vmatrix} a^2+a-2 & 3(a-1) & 0 \\ a-1 & a-1 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= (a-1)^3 \cdot (-1)^{3+3} \cdot 1 \begin{vmatrix} (a-1)(a+2) & 3(a-1) \\ a-1 & a-1 \end{vmatrix}$$

$$= (a-1)^5 \begin{vmatrix} a+2 & 3 \\ 1 & 1 \end{vmatrix} = (a-1)^5 (a+2-3)$$

$$= (a-1)^5 \cdot (a-1) = (a-1)^6 \quad (A).$$

**(4)** Fie $A = (a_{ij})_{i,j=\overline{1,n}}$, $a_{ij} = \min\{i,j\}$, $1 \le i, j \le n$. Să se arate că $\Delta_n = \det(A) = 1$

pt $n=2 \Rightarrow A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \Rightarrow \Delta_2 = 2-1 = 1$

pt $n=3 \Rightarrow A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \Rightarrow \Delta_3 = 1 = \Delta_2$

$$\Delta_n = \det(A) = \Delta_{n-1} = \dots = \Delta_2 = 1$$

pt $n \Rightarrow A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ \vdots \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$; $\det A = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & & & & \\ 1 & 1 & 2 & \dots & n-1 \end{vmatrix}$

$C_k - C_1$, $k = \overline{2,n}$ $\qquad = \Delta_{n-1} = \dots = \Delta_2$

**(5)** $A = \begin{pmatrix} a & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1-a \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$ $\quad rg A = ?$

$$\det A = \begin{vmatrix} a & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1-a \end{vmatrix} \underset{C_3 - C_2}{\overset{C_1 - C_2}{=}} \begin{vmatrix} a-1 & 1 & 1 \\ 0 & 1 & 0 \\ -2 & 1 & -a \end{vmatrix}$$

$$= (-1)^{2+2} \cdot 1 \begin{vmatrix} a-1 & 1 \\ -2 & -a \end{vmatrix} = -(a-2)(a+1)$$

cI : $\det A \neq 0 \Rightarrow r_A = 3,\ a \in \mathbb{R} - \{-1, 2\}$

cII : $\det A = 0$

a) $a = -1 \Rightarrow A = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$

$d_2 = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \neq 0 \Rightarrow r_A = 2$

b) $a = 2 \Rightarrow A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$

$d_2 = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \neq 0 \Rightarrow r_A = 2$

(2) $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & a & 1 \\ 0 & 1 & 3 & b \end{pmatrix} \in \mathcal{M}_{3,4}(\mathbb{R})$

$a, b = ?$ a.î $r_A = 2$.

$d_1 = 1 \neq 0$

$d_2 = \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -4 \neq 0$

$d_{3_1} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & a \\ 0 & 1 & 3 \end{vmatrix} = 0 \Rightarrow$

$d_{3_2} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & b \end{vmatrix} = 0$

$d_{3_1} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & a \\ 0 & 1 & 3 \end{vmatrix} = 0 + 6 + 0 - 0 - a - 12 = -a - 6 = 0$

$\begin{matrix} 1 & 2 & 3 \\ 2 & 0 & a \end{matrix}$

$\Rightarrow a = -6$

$d_{3_2} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & b \end{vmatrix} = 0 + 2 + 0 - 0 - 1 - 4b = 0$

$\begin{matrix} 1 & 2 & 1 \\ 2 & 0 & 1 \end{matrix}$

$4b = 1 \Rightarrow b = \frac{1}{4}$

$\Rightarrow a = -6 ;\ b = \frac{1}{4}$

3) $A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 4 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Să se det. forma eşalon (redusă). Precizaţi rangul

$$A \underset{\substack{L_1 \to L_3 \\ L_3 \to L_1}}{\sim} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & 1 \\ 3 & 1 & 2 \end{pmatrix} \underset{L_3 - 3L_1}{\sim} \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & 1 & \frac{1}{4} \\ 0 & -2 & 2 \end{pmatrix} \underset{L_3 + 2L_2}{\sim} \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & \frac{1}{4} \\ 0 & 0 & \boxed{\frac{5}{2}} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$L_2 - \frac{1}{4} \cdot L_3$

$L_1 - L_2$

8) b) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$A^{-1} = \frac{1}{\det A} \cdot A^*$

$A^{-1}$ ( algoritmul Gauss Jordan)

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0 + 1 + 1 + \cancel{1} - \cancel{1} + 0 = 2$$
$$\begin{matrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{matrix}$$

$A^t = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$$A^* = \begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 0 & -2 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$A$ $(A \mid I_3) \sim (B \mid C)$ — formă eșalon redusă

$A$ inversabilă $\quad B = I_3$

$C = A^{-1} \Rightarrow (A \mid I_3) \sim (I_3 \mid A^{-1})$

Gauss Jordan

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} L_2 - L_1 \\ L_3 - L_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right) \underset{L_2 : (-2)}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

$$\underset{L_1 - L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

e) $A^{-1} = ?$ (Hamilton Cayley)

$$A^3 - \overset{0}{\overset{\shortparallel}{\sigma_1}} A^2 + \overset{-4}{\overset{\shortparallel}{\sigma_2}} A - \overset{2}{\overset{\shortparallel}{\sigma_3}} I_3 = O_3 \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$\sigma_1 = \operatorname{Tr} A = 0$

$$\sigma_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -2 + (-1) + (-1) = -4$$

$\sigma_3 = \det A = 2$

$A^3 - 4A - 2I_3 = O_3 \mid A^{-1}$

$A^2 - 4I_3 - 2A^{-1} = O_3$

$\Rightarrow A^{-1} = \frac{1}{2}(A^2 - 4I_3)$

$$\sigma_2 = \sum_{i<j} \begin{vmatrix} a_{ij} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$$

1 2
1 3
2 3