C_7 - AG

Proiecții și simetrii

Def $p \in End(V)$ s.n. proiecție pe V_1, de-a lungul lui V_2
dacă $p : V = V_1 \oplus V_2 \to V = V_1 \oplus V_2$

$$p(\underset{V_1}{v_1} + \underset{V_2}{v_2}) = v_1$$

Prop $p \in End(V)$
p proiecție $\Leftrightarrow p \circ p = p$.

Def $s \in End(V)$ s.n. simetrie (sau involuție) $\Leftrightarrow s \circ s = id_V$.

Prop $(V, +, \cdot)/_{\mathbb{K}}$, $ch\mathbb{K} \neq 2$ (i.e $1+1 \neq 0$)
p proiecție $\Leftrightarrow s = 2p - id_V$ este simetrie.

Dem
"$\Rightarrow$" Ip: p proiecție $\Rightarrow p \circ p = p$.
Dem că $s \circ s = id_V$

$$s \circ s = (2p - id_V) \circ (2p - id_V) = 4 p \circ p - 2p - 2p + id_V$$
$$= 4p - 4p + id_V = id_V.$$

"$\Leftarrow$" Ip s = simetrie $\Rightarrow s \circ s = id_V$.
Dem că $p \circ p = p$.

$$\underset{id_V}{\underset{\parallel}{s \circ s}} = 4 p \circ p - 4p + id_V \Rightarrow p \circ p = p.$$

OBS $p : V_1 \oplus V_2 \to V_1$ proiecția pe V_1

$$s(v) = 2p(v) - v = 2v_1 - (v_1 + v_2) = v_1 - v_2$$

$v = v_1 + v_2$
s este simetria față de V_1

OBS $V = V_1 \oplus V_2$, $V_1 = Im\, p$, $V_2 = Ker\, p$, $p(\underset{V_1}{v_1} + \underset{V_2}{v_2}) = v_1$

$R_1 = \{e_1, .., e_k\}$ reper în V_1 $\quad p(e_i) = e_i, \forall i = \overline{1,k}$

$R_2 = \{e_{k+1}, .., e_n\}$ reper în V_2 $\quad p(e_j) = 0$

$$[p]_{R,R} = A_p = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right) = \begin{pmatrix} \boxed{\begin{matrix} 1 & \cdots & 0 \\ 0 & & 0 \\ 0 & & 1 \end{matrix}} & \begin{matrix} 0 & & 0 \\ 0 & & 0 \\ \vdots & & \vdots \end{matrix} \\ \begin{matrix} 0 & \cdots & 0 \end{matrix} & \begin{matrix} 0 & & 0 \end{matrix} \end{pmatrix} \notin O(n)$$

$R = R_1 \cup R_2$ reper în $V = V_1 \oplus V_2$

$$A_p \cdot A_p^T \neq I_n.$$

$s(e_i) = 2p(e_i) - e_i = e_i$, $\forall i = \overline{1,k}$

$s(e_j) = 2p(e_j) - e_j = -e_j$ $\forall j = \overline{k+1,n}$

$$[s]_{R,R} = A_s = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & -I_{n-k} \end{array}\right) \in O(n)$$

$$A_s \cdot A_s^T = I_n.$$

Aplicație

$(\mathbb{R}^3, +, \cdot)/\mathbb{R}$, $V' = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - 2x_3 = 0\}$

$s : V' \oplus V'' \longrightarrow V' \oplus V''$ simetria față de V'

$s(x' + x'') = x' - x''$

Calculați $s(1,2,5)$

SOL

$x_1 + x_2 - 2x_3 = 0 \Rightarrow x_1 = -x_2 + 2x_3$

$V' = \{(-x_2 + 2x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\}$

$\phantom{V' = \{} = x_2 \underbrace{(-1,1,0)}_{e_1'} + x_3 \underbrace{(2,0,1)}_{e_2'}$

$R' = \{e_1', e_2'\}$ reper în V'. Extindem la $R = R' \cup R''$ reper în $\mathbb{R}^3$

$$rg\begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 3 \qquad V'' = \langle\{(1,0,0)\}\rangle$$

$(1,2,5) = \overbrace{a(-1,1,0) + b(2,0,1)}^{x'} + \overbrace{c\underbrace{(1,0,0)}_{e_3'}}^{x''}$, (a,b,c) coordonatele lui $(1,2,5)$ în raport cu reperul R.

$(1,2,5) = (-a + 2b + c, a, b)$

$a = 2$

$b = 5$

$-2 + 10 + c = 1 \Rightarrow c = 3 - 10 = -7$

$\Rightarrow x' = 2(-1,1,0) + 5(2,0,1) = (8,2,5)$

$x'' = -7(1,0,0) = (-7,0,0)$

$$s(1,2,5) = s(x' + x'') = x' - x'' = (8,2,5) + (7,0,0) = (15,2,5)$$

Vectori proprii. Valori proprii. Diagonalizare

Problemă $(V,+,\cdot)/_{\mathbb{K}}$, $f \in End(V)$

$\exists\, R=\{e_1,..,e_n\}$ reper în V a.î. $[f]_{R,R} = A = diag = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$?

$f(e_1) = \lambda_1 e_1$

$\vdots$

$f(e_n) = \lambda_n e_n$.

Def $f \in End(V)$

$x \neq 0_V$ s.n. vector propriu $\Leftrightarrow \exists\, \lambda \in \mathbb{K}$ aî $f(x) = \lambda x$ (valoare proprie)

OBS $f(0_V) = 0_V = \lambda \cdot 0_V$

Not $V_\lambda = \{x \in V \mid f(x) = \lambda x\}$ subspațiul propriu coresp. valorii proprii λ.

Prop

a) $V_\lambda \subseteq V$ subsp vect.

b) V_λ = subspațiu invariant i.e. $f(V_\lambda) \subseteq V_\lambda$

Dem

a) $\forall x, y \in V_\lambda$, $\forall a, b \in \mathbb{K}$ $\Rightarrow ax + by \in V_\lambda$

$f(ax+by) = a\underbrace{f(x)}_{\lambda x} + b\underbrace{f(y)}_{\lambda y} = \lambda(ax+by) \Rightarrow V_\lambda \subseteq V$ ssp v.

b) Fie $x \in V_\lambda \Rightarrow f(x) = \lambda x \in V_\lambda \Rightarrow f(V_\lambda) \subseteq V_\lambda$

Polinomul caracteristic

$f \in End(V)$, $R = \{e_1,..,e_n\}$ reper $\forall$ în V, $A = [f]_{R,R}$

Fie x vector propriu coresp. val. proprii λ

$f(x) = \lambda x$

$f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i f(e_i) = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} a_{ji} e_j\right)$

$= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ji} x_i\right) e_j$ $\Bigg|$ $\Rightarrow \sum_{i=1}^{n} a_{ji} x_i = \lambda x_j = \sum_{j=1}^{n} \lambda \delta_{ji} x_i$

$\lambda x = \lambda \sum_{j=1}^{n} x_j e_j$

$$\sum_{i=1}^{n} (a_{ji} - \lambda \delta_{ji}) x_i = 0, \ \forall j = \overline{1,n}$$

SLO cu sol nenule $\Rightarrow \det((a_{ij} - \lambda \delta_{ij})) = 0$

$p(\lambda) = \det(A - \lambda I_n) = 0$

$p(\lambda) = (-1)^n [\lambda^n - \sigma_1 \lambda^{n-1} + \dots + (-1)^n \sigma_n] = 0$

σ_k = suma minorilor diag. de ord k, $k = \overline{1,n}$

$\sigma_1 = Tr(A), \dots, \sigma_n = \det(A)$

<u>Prop</u> polinomul caract. este invariant la sch. reperului

$R \xrightarrow{C} R'$ $\quad [f]_{R,R} = A, \quad [f]_{R',R'} = A'$

$A' = C^{-1} A C$

$\det(A' - \lambda I_n) = \det(\underbrace{C^{-1} A C - \lambda C^{-1} I_n C}_{C^{-1}(A - \lambda I_n) C})$

$= \det(C^{-1}) \cdot \det(A - \lambda I_n) \det C = \det(A - \lambda I_n)$

$\square$ valorile proprii = rădăcinile din $\mathbb{K}$ ale polinomului caract

<u>Exemplu</u> $(\mathbb{R}^2, +, \cdot)/\mathbb{R}$, $J: \mathbb{R}^2 \to \mathbb{R}^2$, $J(x_1, x_2) = (-x_2, x_1)$.

$[J]_{R_0, R_0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$

$P(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

J nu are valori proprii

<u>OBS</u>

$P(\lambda) = 0 \Rightarrow (\lambda - \lambda_1)^{m_1} \cdot \ldots \cdot (\lambda - \lambda_r)^{m_r} = 0$

$\lambda_1, \dots, \lambda_r$ sunt răd. distincte,

$m_1, \dots, m_r$ = multiplicitati. $\quad m_1 + \dots + m_r = n$

$\sigma(f) = \{\lambda_1, \dots, \lambda_r\}$?

$Spec(f) = \{\underbrace{\lambda_1 = \dots = \lambda_1}_{m_1 \text{ ori}} < \dots < \underbrace{\lambda_r = \dots = \lambda_r}_{m_r \text{ ori}}\}$

Prop Vectorii proprii coresp. la valori proprii distincte formează un SLI

Dem
Dem prin ind după nr de vectori proprii

$x \neq 0$ vect proprii $\Rightarrow \{x\}$ SLI

P_{n-1}: Pp. prop adev pt $n-1$ vect. proprii coresp. la val pr. dist

Dem P_n adev.

Fie $v_1, \dots, v_n$ vect. proprii coresp. la val pr dist $\lambda_1, \dots, \lambda_n$

Dem că $\{v_1, \dots, v_n\}$ este SLI

$(*)$ $a_1 v_1 + \dots + a_n v_n = 0_V \; | f$

$$f(a_1 v_1 + \dots + a_n v_n) = a_1 f(v_1) + \dots + a_n f(v_n) = \boxed{a_1 \lambda_1 v_1 + \dots + a_n \lambda_n v_n = 0} \quad (1)$$

Considerăm $\lambda_n \neq 0_{\mathbb{K}}$ (eventual renumerotăm indicii)

$$(*) \; | \cdot \lambda_n \Rightarrow \boxed{a_1 \lambda_n v_1 + \dots + a_{n-1} \lambda_n v_{n-1} + a_n \lambda_n v_n = 0} \quad (2)$$

$$(1) - (2) \Rightarrow a_1 \underbrace{(\lambda_1 - \lambda_n)}_{\neq 0_{\mathbb{K}}} v_1 + \dots + a_{n-1} \underbrace{(\lambda_{n-1} - \lambda_n)}_{\neq 0_{\mathbb{K}}} v_{n-1} = 0 \xRightarrow[\{v_1, \dots, v_{n-1}\} \text{SLI}]{P_{n-1}}$$

$$\Rightarrow a_1 = \dots = a_{n-1} = 0 \xRightarrow{(1)} a_n \underbrace{\lambda_n v_n}_{\neq 0_{\mathbb{K}}} = 0 \Rightarrow a_n = 0$$

$$\Rightarrow a_i = 0, \; \forall i = \overline{1,n} \Rightarrow \{v_1, \dots, v_n\} \text{ SLI}$$

Prop $f \in End(V)$, λ = valoare proprie $\Rightarrow \dim V_\lambda \leq m_\lambda$

Dem $\dim V_\lambda = n_\lambda$. Fie $\{e_1, \dots, e_{n_\lambda}\}$ reper în V_λ

Extindem la un reper $R = \{e_1, \dots, e_{n_\lambda}, e_{n_\lambda + 1}, \dots, e_n\}$ reper în V

$x \in V_\lambda \Rightarrow f(x) = \lambda x$

$f(e_1) = \lambda e_1$

$\vdots$

$f(e_{n_\lambda}) = \lambda e_{n_\lambda}$

$$f(e_j) = \sum_{k=1}^{n} a_{kj} e_k, \; j = \overline{n_\lambda + 1, n}$$

$$[f]_{R,R} = \left(\begin{array}{cccc|c} \lambda & 0 & \cdots & 0 & \\ 0 & \lambda & & \vdots & \\ \vdots & \vdots & & \lambda & \\ \hline & & & 0 & \\ 0 & 0 & & 0 & \end{array}\right)$$

$$P(X) = \det(A - XI_n) = \begin{vmatrix} \begin{matrix} \lambda - X & & 0 \\ & \ddots & \\ 0 & & \lambda - X \end{matrix} & * \\ 0 & \end{vmatrix}$$

$P(X) = (\lambda - X)^{n_\lambda} Q(X) \Rightarrow m_\lambda \geq n_\lambda = \dim V_\lambda$.

! <u>Teoremă</u> $f \in End(V)$

$\exists$ un reper $R = \{e_1, .., e_n\}$ în V aî $[f]_{R,R}$ diagonală $\Longleftrightarrow$

(1) toate răd polinomului caracteristic $\in \mathbb{K}$

$(\lambda_1, .., \lambda_r \in \mathbb{K}$, răd dist ale pol caract$)$

(2) dimensiunile subsp. proprii = multiplicitatile valorilor proprii coresp

$(\dim V_{\lambda_i} = m_i, \forall i = \overline{1,r}, m_1 + ... + m_r = n)$

Dem

"$\Rightarrow$" Ip. $\exists R = \{e_1, .., e_n\}$ reper aî

$[f]_{R,R} = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix} \in \mathcal{M}_n(\mathbb{K})$

Eventual renumerotăm $A = \begin{pmatrix} \overbrace{\lambda_1 \ldots \lambda_1}^{m_1} & & 0 \\ & \ddots & \\ 0 & & \underbrace{\lambda_r \ldots \lambda_r}_{m_r} \end{pmatrix} \in \mathcal{M}_n$

$\lambda_1 < .. < \lambda_r$ distincte $\Rightarrow \lambda_1, .., \lambda_r \in \mathbb{K}$.

$$\det(A - \lambda I_n) = \begin{vmatrix} \lambda_1 - \lambda & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_1 - \lambda & & & \\ & & & \ddots & & \\ & & & & \lambda_r - \lambda & \\ 0 & & & & & \lambda_r - \lambda \end{vmatrix} =$$

$= (\lambda_1 - \lambda)^{m_1} \cdot \ldots \cdot (\lambda_r - \lambda)^{m_r}$

$\lambda_1, .., \lambda_r$ răd ale pol. caract.

cf prop anterioare: $\dim V_{\lambda_i} \leq m_i, \forall i = \overline{1,r}$ (1)

$R_1 = \{e_1, .., e_{m_1}\} \subset V_{\lambda_1}$ $\quad f(e_1) = \lambda_1 e_1$

$\quad \vdots$

$\quad f(e_{m_1}) = \lambda_1 e_{m_1}$

$\dim V_{\lambda_1} \geq m_1$ (2)

Din (1), (2) $\Rightarrow \dim V_{\lambda_1} = m_1$. Analog $\dim V_{\lambda_i} = m_i, \forall i = \overline{2,r}$

"$\Leftarrow$" Ip $\lambda_1, \ldots, \lambda_r \in \mathbb{K}$ (răd. dist. ale pol. caract)
$\dim V_{\lambda_i} = m_i$, $\forall i = \overline{1,r}$, $m_1 + \ldots + m_r = n$.

Construim $R = \{e_1, \ldots, e_n\}$ ai $[f]_{R,R} = \text{diag}$.

Fie R_i reper în V_{λ_i}, $i = \overline{1,r}$ și $R = R_1 \cup \ldots \cup R_r$.

$|R| = m_1 + \ldots + m_r = n = \dim V$

Dem că R este SLI.

$$\underbrace{\sum_{i=1}^{m_1} a_i e_i}_{f_1 \in V_{\lambda_1}} + \ldots + \underbrace{\sum_{j=m_1+\ldots+m_{r-1}+1}^{n} a_j e_j}_{f_r \in V_{\lambda_r}} = 0$$

$R_1 = \{e_1, \ldots, e_{m_1}\}$ reper în V_{λ_1}
$\vdots$
$R_r = \{e_{m_1+\ldots+m_{r-1}+1}, \ldots, e_n\}$ în V_{λ_r}.

Ip. prin abs $f_{i_1}, \ldots, f_{i_p}$ nenuli (vect proprii coresp. la valori proprii dist) $\Rightarrow$ SLI
$f_{i_1} + \ldots + f_{i_p} = 0$
} Contrad.

Ip. este falsă $\Rightarrow$ $f_1 = 0 \Rightarrow \sum_{i=1}^{m_1} a_i e_i = 0 \xRightarrow{R_1 \text{ reper}} a_1 = \ldots = a_{m_1} = 0$
$\vdots$
$f_r = 0 \Rightarrow \sum_{j=m_1+\ldots+m_{r-1}+1}^{n} a_j e_j = 0 \xRightarrow{R_r \text{ reper}} a_i = 0$, $i = \overline{m_1+\ldots+m_{r-1}+1, n}$

Deci $a_i = 0$, $\forall i = \overline{1,n}$ $R = R_1 \cup \ldots \cup R_r$ SLI
$|R| = n$
} $\Rightarrow$ R reper în V

$$A = [f]_{R,R} = \begin{pmatrix} \overbrace{\lambda_1 \ldots \lambda_1}^{m_1 \text{ ori}} & & \\ & \ddots & \\ & & \overbrace{\lambda_r \ldots \lambda_r}^{m_r} \end{pmatrix}$$

<u>OBS</u> $V = V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_r}$

Ex $f:\mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1, x_2+x_3, 2x_3)$.

Determinați un reper $\mathcal{R}$ în $\mathbb{R}^3$ a.î. $[f]_{\mathcal{R},\mathcal{R}}$ diagonală.

SOL $\mathcal{R}_0 = \{e_1, e_2, e_3\}$ reperul canonic

$$[f]_{\mathcal{R}_0,\mathcal{R}_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2+x_3 \\ 2x_3 \end{pmatrix}$$

$f(e_1) = f(1,0,0) = (1,0,0) = e_1 + 0e_2 +$

$f(e_2) = f(0,1,0) = (0,1,0) = e_2$

$f(e_3) = f(0,0,1) = (0,1,2)$
$= e_2 + 2e_3$

Polinomul caracteristic

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda) = 0$$

$\lambda_1 = 1$, $m_1 = 2$

$\lambda_2 = 2$, $m_2 = 1$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_1 x\}$

$AX = X \Rightarrow (A - I_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$x_3 = 0$

$V_{\lambda_1} = \{\underset{x_1(1,0,0) + x_2(0,1,0)}{\underbrace{(x_1, x_2, 0)}} \mid x_1, x_2 \in \mathbb{R}\} = \langle\{e_1, e_2\}\rangle \Rightarrow \dim V_{\lambda_1} = 2 = m_1$

$V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_2 x\}$

$AX = 2X \Rightarrow (A - 2I_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{cases} -x_1 = 0 \\ -x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = x_3 \end{cases}$

$V_{\lambda_2} = \{(0, x_2, x_2) = x_2(0,1,1)\} = \langle\{(0,1,1)\}\rangle \quad \dim V_{\lambda_2} = 1 = m_2$

$\mathcal{R} = \{(1,0,0), (0,1,0), (0,1,1)\}$.

$$[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 2 \end{pmatrix}$$

OBS

$$\begin{tikzcd} R_0 \arrow[r, "A"] \arrow[d, "C"'] & R_0 \arrow[d, "C"] \\ R \arrow[r, "A'"] & R \end{tikzcd}$$

$$\underline{A' = C^{-1} A C} \Rightarrow A = C A' C^{-1}$$

A' = matrice diag.

$$A^n = C A'^n C^{-1}$$

Ex $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x) = (3x_1 + 2x_2, -x_1)$

$A = [f]_{R_0, R_0}$. Calculați A^n.

SOL $A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$ $\quad P(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = 0$

$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda_1 = 1, \; m_1 = 1; \quad \lambda_2 = 2, \; m_2 = 1$

$V_{\lambda_1} = \{ x \in \mathbb{R}^2 \mid f(x) = x \}$

$AX = X \Leftrightarrow (A - I_2) X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-x_1 - x_2 = 0 \Rightarrow x_2 = -x_1$

$V_{\lambda_1} = \{ (x_1, -x_1) = x_1 (1, -1) \} = \langle \{(1,-1)\}$

$V_{\lambda_2} = \{ x \in \mathbb{R}^2 \mid f(x) = 2x \}$

$AX = 2X \Rightarrow (A - 2I_2) X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$

$V_{\lambda_2} = \{ (-2x_2, x_2) \mid x_2 \in \mathbb{R} \} = \langle \{(-2,1)\} \rangle$

$R = \{ (1,-1), (-2,1) \} = R_1 \cup R_2$; $A' = [f]_{R,R} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$R_0 = \{ e_1 = (1,0), e_2 = (0,1) \} \xrightarrow{C} R = \{ e_1' = (1,-1), e_2' = (-2,1) \}$

$e_1' = (1,-1) = e_1 - e_2$

$e_2' = (-2,1) = -2e_1 + e_2$

$C = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$, $\det C = 1 - 2 = -1$

$A^n = C A'^n C^{-1}$

$C^T = \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$, $C^* = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

$C^{-1} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$

$$A^n = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}}_{\begin{pmatrix} -1 & -2 \\ -2^n & -2^n \end{pmatrix}} = \begin{pmatrix} -1 + 2^{n+1} & -2 + 2^{n+1} \\ 1 - 2^n & 2 - 2^n \end{pmatrix}$$

$$A^n = (CA'C^{-1})(CA'C^{-1}) \cdots (CA'C^{-1}) = CA'^n C^{-1}$$

Forme biliniare. Forme pătratice.

Def $g : V \times V \to \mathbb{K}$ s.n. formă biliniară $\Leftrightarrow$ g liniară în fiecare argument.

a) $g(ax+by, z) = a g(x,z) + b g(y,z)$

b) $g(x, ay+bz) = a g(x,y) + b g(x,z)$, $\forall x,y,z \in V$, $\forall a,b \in \mathbb{K}$.

OBS $(L(V,V;\mathbb{K}) = \{g : V\times V \to \mathbb{K} \mid g \text{ formă biliniară}\}, +, \cdot)/\mathbb{K}$ sp. vectorial.

$g : V\times V \to \mathbb{K}$ s.n. formă simetrică $\Leftrightarrow g(x,y) = g(y,x)$

s.n. formă antisimetrică $\Leftrightarrow g(x,y) = -g(y,x)$ $\forall x,y \in V$

OBS g formă simetrică (resp antisimetrică) + liniară într-un argument $\Rightarrow$ biliniară

$L^s(V,V;\mathbb{K}) = \{g \in L(V,V;\mathbb{K}) \mid g \text{ simetrică}\}$

$L^a(V,V;\mathbb{K}) = \{g \in L(V,V;\mathbb{K}) \mid g \text{ antisimetrică}\}$ $\subset L(V,V;\mathbb{K})$ ssp vect

Matricea asociată unei forme biliniare.

$\mathcal{R} = \{e_1, \dots, e_n\}$ reper în V, $g_{ij} = g(e_i, e_j)$, $G = (g_{ij})_{i,j=\overline{1,n}}$ matricea asoc. lui g în raport cu $\mathcal{R}$.

$$g(x,y) = g\left(\sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{n} y_j e_j\right) = \sum_{i,j=1}^{n} x_i y_j g_{ij} = X^T G Y$$

$$= (x_1 \dots x_n)\begin{pmatrix} g_{11} & \dots & g_{1n} \\ \vdots & & \\ g_{n1} & \dots & g_{nn}\end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_n\end{pmatrix}$$

$\mathcal{R} \xrightarrow{C} \mathcal{R}'$ $\quad G' = C^T G C$ matricea asoc. lui g în rap cu $\mathcal{R}'$

Prop Rangul matricei este un invariant, la sch. reperului.

$rg G' = rg(C^T G C) = rg G$

Ex $g : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \; g(x,y) = x_1 y_1 + x_2 y_2 = X^T I_2 Y.$

$\mathcal{R}_0 = \{e_1, e_2\}$ $\quad g(\underset{(1,0)}{e_1}, \underset{(1,0)}{e_1}) = 1, \; g(e_1, e_2) = 0, \; g(e_2, e_2) = 1$

$G = I_2$

$g(x,y) = g(y,x) \qquad g \in L^s(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$