SEMINAR 9 Forme pătratice. Metoda Gauss, Metoda Jacobi

1. $Q:\mathbb{R}^3 \to \mathbb{R}$, $Q(x) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$.

a) G = matricea asociată lui Q în raport cu $R_0 = \{e_1, e_2, e_3\}$

b) $g:\mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}$ forma polară as.

c) Aduceți Q la o formă canonică. (Gauss/Jacobi).
Este Q o poziție definită? Generalizare

$g:\mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}$ formă biliniară simetrică $g(x,x) = Q(x)$ $\forall x \in \mathbb{R}^3$

$$g(x,y) = \sum_{i,j=1}^{n} x_i y_j \quad \sum_{i=1}^{n} g_{ii} x_i y_i + 2\sum_{i<j} g_{ij} x_i y_j \Rightarrow$$

$$\Rightarrow Q(x) = \sum_{i=1}^{n} g_{ii} x_i^2 + 2\sum_{i<j} g_{ij} x_i x_j$$

$$G = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

$$g(x,y) = 1\cdot x_1y_1 + 1\cdot x_2y_2 + 1\cdot x_3y_3 + \frac{1}{2}x_1y_2 + \frac{1}{2}x_1y_3 + \frac{1}{2}x_2y_1 + \frac{1}{2}x_2y_3 + \frac{1}{2}x_3y_1 + \frac{1}{2}x_3y_2$$

Obs: $g(x,y) = \frac{1}{2}(Q(x+y) - Q(x) - Q(y))$

Metoda Gauss:

$$\det(G) = \begin{vmatrix} 2 & 2 & 2 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2} \neq 0$$

$$Q(x) = \left(x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3\right)^2 \underbrace{- \frac{1}{4}x_2^2 - \frac{1}{4}x_3^2 - \frac{1}{2}x_2x_3 + x_2^2 + x_3^2 + x_2x_3}$$

$$+ \frac{3}{4}x_2^2 + \frac{3}{4}x_3^2 + \frac{1}{2}x_2x_3 = \frac{3}{4}\left(x_2^2 + \frac{2}{3}x_2 x_3\right) + \frac{3}{4}x_3^2 =$$

$$= \left(x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3\right)^2 + \frac{3}{4}\left(x_2 + \frac{1}{3}x\right)^3 +$$

$$= \frac{3}{4}\left(x_2 + \frac{1}{3}x_3\right)^2 + \frac{8}{12}x_3^2$$

Fie schimbarea de reper $x_1' = (x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3)$

$x_2' = (x_2 + \frac{1}{3}x_3)$

$x_3' = x_3^2$

$Q(x) = x_1'^2 + \frac{3}{4}x_2'^2 + \frac{2}{3}x_3'^2$ $(3,0)$ signatura

$$\left.\begin{aligned} x_1'' &= x_1' \\ x_2'' &= \sqrt{\tfrac{5}{2}}\, x_2' \\ x_3'' &= \sqrt{\tfrac{2}{3}}\, x' \end{aligned}\right| \Rightarrow Q(x) = x_1''^2 + x_2''^2 + x_3''^2$$

(forma normala)

Este $\underbrace{\text{poz. def}}_{\text{pozitiv definită}}$ (signatura $= (n,0)$)

Metoda Jacobi

$\Delta_1 = |1| = 1$

$\Delta_2 = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{3}{4} \neq 0$

$\Delta_3 = \det(G) = \frac{1}{2} \neq 0$

$(\exists)$ un reper a.î. $Q(x) = \frac{1}{\Delta_1}x_1'^2 + \frac{\Delta_1}{\Delta_2}x_2'^2 + \frac{\Delta_2}{\Delta_3}x_3'^2 =$

$= x_1'^2 + \frac{4}{3}x_2'^2 + \frac{3}{2}x_3'^2$

$\Rightarrow$ signatura este $(3,0)$

Generalizare:

$Q: \mathbb{R}^n \to \mathbb{R}$, $Q(x) = \sum_{i=1}^{n} x_i^2 + \sum_{i<j} x_i x_j$ $(n,0)$ signatura

2. $Q: \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = 2x_1x_2 - 6x_1x_3 - 6x_2x_3$

Să se aducă la o formă canonică (met. Ga[uss]

Jacobi). Precizați signatura

$$G = \begin{pmatrix} 0 & 1 & -3 \\ 1 & 0 & -3 \\ -3 & -3 & 0 \end{pmatrix}, \quad \det(G) = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & -3 \\ -3 & -3 & -9 \end{vmatrix} =$$

$$= -1 \begin{vmatrix} 1 & -3 \\ -3 & -9 \end{vmatrix} = 18 \Rightarrow \text{rang } G = 3$$

Fie schimbarea de reper: $x_1' = x_1 + x_2 \Rightarrow x_1 = \frac{1}{2}(x_1' + x_2')$

$x_2' = x_1 - x_2$ $\quad x_2 = \frac{1}{2}(x_1' - x_2')$

$x_3' = x_3 \Rightarrow x_3 = x_3'$

$$Q(x) = 2x_1x_2 - 6x_3(x_1+x_2) = \frac{1}{2}(x_1'^2 - x_2'^2) - 6x_1'x_3' =$$

$$= \frac{1}{2}(x_1' - 12x_1'x_3') - \frac{1}{2}x_2'^2 =$$

$$= \frac{1}{2}(x_1' - 6x_3')^2 - 18x_3'^2 - \frac{1}{2}x_2'^2$$

$$\left.\begin{aligned} x_1'' &= x_1' - 6x_3' \\ x_2'' &= x_2' \\ x_3'' &= x_3' \end{aligned}\right| \Rightarrow Q(x) = \frac{1}{2}x_1''^2 - \frac{1}{2}x_2''^2 - 18x_3''^2$$

$\Rightarrow (1,2)$ signatura

Metoda Jacobi

$\Delta_1 = 0 \Rightarrow$ nu se poate aplica această metodă.

8. $g : M_2(\mathbb{R}) \times M_2(\mathbb{R}) \to \mathbb{R}$, $g(X,Y) = 2Tr(XY) - Tr(X)\cdot Tr(Y)$

a) $g \in L^s(M_2(\mathbb{R}), M_2(\mathbb{R}); \mathbb{R})$

b) G matricea asociată lui g în raport cu R_0.

c) $Q : M_2(\mathbb{R}) \to \mathbb{R}$, forma pătratică asociată

d) Să se aducă Q la o f. canonică.

$Tr(XY) = Tr(YX) \Rightarrow g(XY) = g(YX) \Rightarrow g$ simetrică

$g(aX + bZ, Y) \overset{?}{=} a g(X,Y) + b g(Z,Y)$, $(\forall)\, a,b \in \mathbb{R}$

$(\forall)\, X, Y, Z \in M_2(\mathbb{R})$

$Tr(aX + bZ) = a Tr(X) + b Tr(Z)$

$g(aX + bZ, X) = 2Tr((aX + bZ)Y) - Tr(aX + bZ)Tr(Y)$

$= 2a\,Tr(XY) + 2b\,Tr(ZX) - a\,Tr X\, Tr X - b\,Tr(Z)\,Tr(X)$

$= a\,g(X,Y) + b\,g(Z,X)$

g simetrică

g liniară în primul arg. $\Big| \Rightarrow$ biliniară

$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$

$Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$

$XY = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} =$

$= \begin{pmatrix} x_1y_1 + x_2y_3 & x_1y_2 + x_2y_4 \\ x_3x_1 + x_4y_3 & x_3y_2 + x_4y_4 \end{pmatrix}$

$g(X,Y) = 2(x_1y_1 + x_2y_3 + x_3y_2 + x_4y_4) - (x_1y_1 + x_1y_4 + x_4\ldots$

$= x_1y_1 + x_4y_4 + 2x_2y_3 + 2x_3y_2 - x_1y_4 - x_4y_1$

$G = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$

c) $Q(x) = x_1^2 + x_4^2 + 4x_2x_3 - 2x_1x_4$

Gauss $= x_1^2 - 2x_1x_4 + x_4^2 + 4x_2x_3 = (x_1 - x_4)^2 - x_4^2 + x_4^2 + \ldots$

$= (x_1 - x_4)^2$

$x_1' = x_1$

$x_2' = x_2 + x_3 \Rightarrow x_2 = \frac{1}{2}(x_2' + x_3')$

$x_3' = x_2 - x_3 \Rightarrow x_3 = \frac{1}{2}(x_2' - x_3')$

$x_4' = x_4$

$Q(x) = (x_1' - x_4')^2 + 4 \cdot \frac{1}{4}(x_2'^2 - x_3'^2) = (x_1' - x_4')^2 + x_2'\ldots$

$Q(x) = x_1''^2 + x_2''^2 - x_3''^2$

$x_1'' = x_1' - x_4'$

$x_2'' = x_2'$

$x_3'' = x_3'$

$x_4'' = x_4'$

signatura $= (2,1)$

$g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$g(x, y) = (x_1y_1 - x_2y_2 - x_1y_3 - x_3y_1 + 2x_2y_3 + 2x_3y_2)$

a) $G = ?$ (G = matricea în raport cu R_0)

b) $\operatorname{Ker} g = ?$ Este g nedegenerată?

c) $G' = ?$ a.î. G' = matricea asociată lui g în raport cu $R' = \{ e_1' = (1,1,1),\ e_2' = (1,2,1),\ e_2' = (0,0,1) \}$

a) $G = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix}$, $\det G = 0 + 0 + 0 + 1 - 4 = -3 \neq 0$

b) $\operatorname{Ker} g = \{ x \in \mathbb{R}^3 \mid g(x,y) = 0,\ (\forall)\, y \in \mathbb{R}^3 \}$

$$\Rightarrow (*) \begin{cases} g(x, e_1) = 0 \Rightarrow x_1 - x_3 = 0 \\ g(x, e_2) = 0 \Rightarrow -x_2 + 2x_3 = 0 \\ g(x, e_3) = 0 \Rightarrow -x_1 + 2x_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$(*)$ are sol. unică nulă $\Rightarrow \operatorname{Ker} g = \{ 0_{\mathbb{R}} \} \Rightarrow g$ nedegenerată

c) $R_0 \xrightarrow{C} R'$

$e_1' = e_1 + e_2 + e_3$

$e_2' = e_1 + 2e_2 + e_3$

$e_3' = e_3$

$C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$G' = C^t G C =$

$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Scanned with CamScanner