CURS 10 – AG

Transformări ortogonale. Endomorfisme simetrice. Forma Jordan

Def $(E, \langle \cdot, \cdot \rangle)$ s.v.e.r., $f \in End(E)$

$f \in O(E) \Leftrightarrow \langle f(x), f(y) \rangle = \langle x, y \rangle, \forall x, y \in E$

$O(E)$ = mult. transf. ortogonale

- $f \in O(E) \Leftrightarrow A = [f]_{R,R} \in O(n), \forall R$ reper ortonormat

a) $\det A = 1$ f s.n. de speță 1.

b) $\det A = -1$ f s.n. de speță 2.

Clasificare

1) $n = 1$ $\quad O(E) = \{ id_E, -id_E \}$

2) $n = 2$. $\quad \exists$ un reper $R = \{e_1, e_2\}$ (reper ortonormat)

a.î. a) $\det A = 1$ $\quad A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = A_\varphi, \varphi \in [-\pi, \pi)$

$f = R_\varphi$ rotație de $\measuredangle$ orientat φ

$Tr A = 2\cos\varphi$ = invariant.

b) $\det A = -1$ $\quad A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$f = s$ simetrie față de $\langle \{e_2\} \rangle = \langle \{e_1\}^\perp \rangle$. $\quad \langle \{e_2\} \rangle = \langle \{e_1\} \rangle^\perp$

3) $n = 3$ $\quad A = [f]_{R,R} \in O(3)$

$P(\lambda) = \det(A - \lambda I_3) = 0$ (pol. de grad 3, cu coef reali) polinomul are cel puțin 1 răd. reală $\lambda \Rightarrow \lambda = \pm 1$. Fie e_1 = versorul propriu pt λ

$f(e_1) = \lambda e_1 \Rightarrow \langle \{e_1\} \rangle$ subsp. invar. al lui f

$\Rightarrow \langle \{e_1\} \rangle^\perp$ = subsp. invariant al lui f și $f|_{\langle \{e_1\} \rangle^\perp} : \langle \{e_1\} \rangle^\perp \to \langle \{e_1\} \rangle^\perp$ transf. ortog.

<u>OBS</u> $f \in O(E)$

1) $U \trianglelefteq E$ subsp. invariant (i.e $f(U) \subseteq U$) $\Rightarrow$ $f(U) = U$ $\Rightarrow$ $U^{\perp} \subseteq E$ subsp invariant

$f/_{U^{\perp}} : U^{\perp} \longrightarrow U^{\perp}$ transf. ortog

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I. $\det A = 1$

a) $\lambda = 1 \Rightarrow f(e_1) = e_1$ $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \tilde{A} & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

$\tilde{A} = [f|_{\langle\{e_1\}\rangle^{\perp}}]_{R', R'}$ $\quad R' = \{e_2, e_3\}$

$\det \tilde{A} = 1 \Rightarrow \tilde{A} = A_\varphi$

b) $\lambda = -1 \Rightarrow f(e_1) = -e_1$, $\quad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & & \\ 0 & \tilde{A} & \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\det \tilde{A} = -1$

$\{e_3, e_1, e_2\} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\pi & -\sin\pi \\ 0 & \sin\pi & \cos\pi \end{pmatrix}$

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$f(x_1, x_2, x_3) = (x_1,\ x_2\cos\varphi - x_3\sin\varphi,\ x_2\sin\varphi + x_3\cos\varphi)$

f = rotație de un orientat φ în planul $\langle\{e_1\}\rangle^{\perp}$ și axă $\langle\{e_1\}\rangle$

$\operatorname{Tr} A = 1 + 2\cos\varphi$ invariant

Axa: $f(x) = x$

Ex II $\det A = -1$ (f de speță 2)

a) $\lambda = 1 \Rightarrow f(e_1) = e_1$ $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tilde{A} & \\ 0 & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\det \tilde{A} = -1$

În raport cu $\{e_2, e_1, e_3\}$ matricea este $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos 0 & -\sin 0 \\ 0 & \sin 0 & \cos 0 \end{pmatrix}$

b) $\lambda = -1 \Rightarrow f(e_1) = -e_1$ $\quad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \tilde{A} & \\ 0 & & \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$.

$\det \tilde{A} = 1$

$f: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x_1, x_2, x_3) = (-x_1, x_2\cos\varphi - x_3\sin\varphi, x_2\sin\varphi + x_3\cos\varphi)$.

$f = s \circ R_\varphi$, R_φ = rotație de $\measuredangle \varphi$ orientat în planul $\langle\{e_1\}\rangle^\perp$ și axă $\langle\{e_1\}\rangle$

s = simetrie față de $\langle\{e_1\}\rangle^\perp$

$\mathrm{Tr} A = -1 + 2\cos\varphi$ invariant la schimbarea reperelor

Axa: $f(x) = -x$.

4) $\dim E = n \geq 4$

$\exists$ un reper ortonormat ai $A = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & -1 & & \\ & & & & & & A_{\varphi_1} & \\ & & & & & & & \ddots \; A_{\varphi_k} \end{pmatrix}$ (s ori 1, r ori -1, 0 în rest)

$A_{\varphi_j} = \begin{pmatrix} \cos\varphi_j & -\sin\varphi_j \\ \sin\varphi_j & \cos\varphi_j \end{pmatrix}, \; j = \overline{1,k}$

$s + r + 2k = n$

<u>Teorema Cartan</u> $f \in O(E)$, $f \neq id_E$, $n \geq 2$

f se scrie ca o compunere de cel mult n simetrii ortogonale față de hiperplane

$f(0)=?$, f de 2 ori derivab. și $\boxed{f(0)=5}$

$f'(x) = -2x f(x) \;\; |'$

— " —

Ex $(\mathbb{R}^3, g_0)$

Fie $f:\mathbb{R}^3 \to \mathbb{R}^3$ $\quad g_0 : \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R},\ g_0(x,y) = x_1y_1 + x_2y_2 + x_3y_3$.

$$f(x_1,x_2,x_3) = \frac{1}{3}\left(2x_1 + x_2 - 2x_3,\ -2x_1 + 2x_2 - x_3,\ x_1 + 2x_2 + 2x_3\right)$$

a) $f \in O(E)$

b) Să se determine $\exists\ \mathcal{R} = \{e_1, e_2, e_3\}$ reper ortonormat în $\mathbb{R}^3$ ai $[f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$.

Sol

$f \in \mathrm{End}(\mathbb{R}^3)$

$f \in O(E) \iff A = [f]_{\mathcal{R}_0, \mathcal{R}_0} \in O(3)$

1) $A \cdot A^T = I_3$; 2) $\det A = \pm 1$

$$A = \frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}$$

$$A\cdot A^T = \frac{1}{9}\begin{pmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ -2 & -1 & 2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = I_3$$

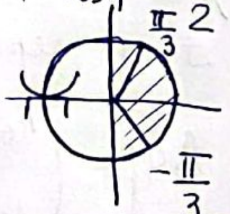
$\Rightarrow A \in O(3)$

$\det(\alpha A) = \alpha^n \det A,\ A \in \mathcal{M}_n(\mathbb{R})$

$$\det A = \frac{1}{27}\begin{vmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{vmatrix} = 1 \Rightarrow f \in O(E)$$ de speță 1.

b) $\mathrm{Tr}\,A = \frac{6}{3} = 2 = 1 + 2\cos\varphi \Rightarrow 2\cos\varphi = 1 \Rightarrow \cos\varphi = \frac{1}{2}$

$\varphi \in \left\{\frac{\pi}{3}, -\frac{\pi}{3}\right\}$.



Det. axa: $f(x) = X$

$$\begin{cases} \frac{1}{3}(2x_1 + x_2 - 2x_3) = x_1 \\ \frac{1}{3}(-2x_1 + 2x_2 - x_3) = x_2 \\ \frac{1}{3}(x_1 + 2x_2 + 2x_3) = x_3 \end{cases} \Rightarrow \begin{cases} -x_1 + x_2 - 2x_3 = 0 \\ -2x_1 - x_2 - x_3 = 0 \\ x_1 + 2x_2 - x_3 = 0 \end{cases}$$

$$\det\begin{pmatrix} -1 & 1 & -2 \\ -2 & -1 & -1 \\ 1 & 2 & -1 \end{pmatrix} = 0$$

$$\begin{cases} -x_1 + x_2 = 2x_3 \\ -2x_1 - x_2 = x_3 \end{cases} \oplus$$
$$-3x_1 \quad / \quad = 3x_3$$

$x_1 = -x_3$

$x_2 = 2x_3 - x_3 = x_3$

$$\{ (x_1, x_2, x_3) = \underset{\underset{x_3(-1,1,1)}{\parallel}}{(-x_3, x_3, x_3)}, \ x_3 \in \mathbb{R} \}$$

$\|x\| = \sqrt{g(x,x)}$

$g = g_0$, $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$e_1 = \frac{1}{\sqrt{3}}(-1,1,1)$ versorul axei

$$\langle\{e_1\}\rangle^{\perp} = \{ x \in \mathbb{R}^3 \mid \underset{\underset{-x_1+x_2+x_3}{\parallel}}{g_0((x_1,x_2,x_3),(-1,1,1))} = 0 \}$$

$$= \{ (x_2+x_3, x_2, x_3) = x_2 \underbrace{(1,1,0)}_{f_2} + x_3 \underbrace{(1,0,1)}_{f_3}, \ x_2, x_3 \in \mathbb{R} \}$$

$\mathbb{R}^3 = \langle\{e_1\}\rangle \oplus \langle\{e_1\}\rangle^{\perp}$

$\{f_2, f_3\}$ reper arbitrar în $\langle\{e_1\}\rangle^{\perp}$

Aplicăm procedeul Gram-Schmidt

$$\underset{\text{reper } \forall}{\{f_2, f_3\}} \longrightarrow \underset{\text{reper ortogonal}}{\{e_2', e_3'\}} \longrightarrow \underset{\text{reper ortonormat}}{\{e_2, e_3\}}$$

$e_2' = f_2 = (1,1,0)$

$$e_3' = f_3 - \frac{g_0(f_3, e_2')}{g_0(e_2', e_2')} e_2' = (1,0,1) - \frac{1}{2}(1,1,0)$$

$$= \left(\frac{1}{2}, -\frac{1}{2}, 1\right) = \frac{1}{2}(1,-1,2)$$

OBS $u = \alpha u'$, $\alpha > 0$

$$\frac{u}{\|u\|} = \frac{\alpha u'}{\|\alpha u'\|} = \frac{\cancel{\alpha} u'}{\cancel{\alpha}\|u'\|}$$

$$\begin{cases} e_2 = \frac{e_2'}{\|e_2'\|} = \frac{1}{\sqrt{2}}(1,1,0) \\ e_3 = \frac{1}{\sqrt{6}}(1,-1,2) \end{cases}$$

$\mathcal{R} = \left\{ e_1 = \frac{1}{\sqrt{3}}(-1,1,1),\ e_2 = \frac{1}{\sqrt{2}}(1,1,0),\ e_3 = \frac{1}{\sqrt{6}}(1,-1,2) \right\}$ = 5

reper ortonormat în $\mathbb{R}^3$ aî

$$A' = [f]_{R,R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & A_\varphi & \end{pmatrix} \quad \varphi \in \left\{ -\frac{\pi}{3}, \frac{\pi}{3} \right\}$$

Teorema (Cauchy-Buniakowski-Schwarz)

$(E, \langle \cdot, \cdot \rangle)$ s.v.e.r, $x, y \in E$

$\Rightarrow |\langle x, y \rangle| \leq \|x\| \cdot \|y\|$

"=" $\Leftrightarrow \{x, y\}$ este SLD.

<u>Dem"</u>

Dacă $x = 0_E$ sau $y = 0_E \Rightarrow 0 = 0$

Dacă $x \neq 0_E$ și $y \neq 0_E$.

$\langle x+\lambda y, x+\lambda y \rangle = \|x+\lambda y\|^2 \geq 0, \forall \lambda \in \mathbb{R}$

$\|x\|^2 + \lambda^2 \|y\|^2 + 2\lambda \langle x, y \rangle$

$\lambda^2 \|y\|^2 + 2\lambda \langle x, y \rangle + \|x\|^2 \geq 0, \forall \lambda \in \mathbb{R} \Rightarrow \Delta_\lambda \leq 0$

$\Delta_\lambda = 4\langle x, y \rangle^2 - 4\|x\|^2 \cdot \|y\|^2$

$\langle x, y \rangle^2 \leq \|x\|^2 \cdot \|y\|^2 \Rightarrow$

$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$

Dem că "=" $\Leftrightarrow \{x, y\}$ este SLD.

"$\Rightarrow$" $\exists \lambda_0 \in \mathbb{R}$ aî $\langle x+\lambda_0 y, x+\lambda_0 y \rangle = 0 \Rightarrow x + \lambda_0 y = 0$

$\Rightarrow \{x, y\}$ SLD

"$\Leftarrow$" $\exists a \in \mathbb{R}$ aî $y = ax$

$\langle x, y \rangle = a\langle x, x \rangle = a\|x\|^2 \Rightarrow |\langle x, y \rangle| = |a| \cdot \|x\|^2$

$\|x\| \cdot \|y\| = \|x\| \cdot |a| \cdot \|x\| = |a| \cdot \|x\|^2 \quad \Rightarrow$ "="

Def $(E, \langle \cdot, \cdot \rangle)$ s.v.e.r.

$f \in End(E)$ s.n. simetric $\Leftrightarrow \langle x, f(y) \rangle = \langle f(x), y \rangle \quad \forall x, y \in E$

Not $Sim(E)$ = mult. endomorfismelor simetrice.

Prop $f \in End(E)$

$f \in Sim(E) \Leftrightarrow A = [f]_{R,R}$ este simetrica $\forall R$ = reper ortonormat în E.

Dem $R = \{e_1, .., e_n\}$ reper ortonormat $\langle e_i, e_j \rangle = \delta_{ij} \quad \forall i,j = \overline{1,n}$

$A = [f]_{R,R}$

$f \in Sim E \Leftrightarrow \langle e_i, f(e_j) \rangle = \langle f(e_i), e_j \rangle \quad \forall i,j = \overline{1,n}$

$$\langle e_i, \sum_{k=1}^{n} a_{kj} e_k \rangle = \langle \sum_{r=1}^{n} a_{ri} e_r, e_j \rangle$$

$$\sum_{k=1}^{n} a_{kj} \underbrace{\langle e_i, e_k \rangle}_{\delta_{ik}} = \sum_{r=1}^{n} a_{ri} \underbrace{\langle e_r, e_j \rangle}_{\delta_{rj}} \Leftrightarrow a_{ij} = a_{ji}$$

$A = A^T$

OBS $R = \{e_1, .., e_n\} \xrightarrow{C} R' = \{e_1', .., e_n'\}$ repere ortonormat

$C \in O(n)$ $\quad A' = [f]_{R',R'}$

$A' = C^{-1} A C = C^T A C$

$A'^T = (C^T A C)^T = C^T A^T (C^T)^T = C^T A C = A'$

Prop $f \in Sim(E) \Rightarrow$ vectorii proprii coresp. la valori proprii distincte sunt ortogonali.

Dem Fie $\lambda \neq \mu$ valori proprii distincte

Fie $\underset{\neq 0_E}{x}$, $\underset{\neq 0_E}{y}$ vectori proprii coresp, $f(x) = \lambda x$, $f(y) = \mu y$

5

$<x, f(y)> = <f(x), y>$

$<x, \mu y> = <\lambda x, y> \Rightarrow <x,y> (\lambda \overset{\neq}{-} \mu) = 0 \Rightarrow <x,y>=0$

Prop

Fie $f \in Sim(E)$, $U \subseteq E$ subsp. invariant al lui $f \Rightarrow U^{\perp}$ — " —

și $f/_{U^{\perp}} : U^{\perp} \to U^{\perp}$ este endom. simetric.

Dem

Dacă $f(U) \subseteq U$. Dem că $f(U^{\perp}) \subseteq U^{\perp}$

Fie $x \in U$, $y \in U^{\perp}$

$\underset{U}{\overset{\cap}{<x}}, \underset{f(U^{\perp})}{\overset{\cap}{f(y)}}> = <\underset{U}{\overset{\cap}{f(x)}}, \underset{U^{\perp}}{\overset{\cap}{y}}> = 0 \Rightarrow f(y) \in U^{\perp}$

$\Rightarrow f(U^{\perp}) \subseteq U^{\perp}$

Teoremă $f \in Sim(E) \Rightarrow$ toate rădăcinile polinomului caracteristic sunt reale.

Teoremă

Dacă $f \in Sim(E)$, atunci $\exists$ un reper R format din versori proprii ai $[f]_{R,R}$ = diagonală.

Dem

Fie R_0 reper ~~arbitrar~~ ortonormat, $A = [f]_{R_0,R_0}$

$P(\lambda) = \det(A - \lambda I_n)$ polinomul caracteristic

$\Rightarrow$ toate răd. sunt reale.

Fie λ_1 o rădăcină și e_1 = versorul propriu

$f(e_1) = \lambda_1 e_1 \Rightarrow <\{e_1\}>$ este subspațiu invariant

$\Rightarrow <\{e_1\}>^{\perp}$ — " —

$f/_{<\{e_1\}>^{\perp}}$ este endom. simetric. Fie λ_2 valoare propr. și e_2 versor propriu ; $<e_1, e_2> = 0$

$f(e_1) = \lambda_1 e_1$
$f(e_2) = \lambda_2 e_2$ $\Rightarrow \langle \{e_1, e_2\} \rangle$ un ssp. invariant

$\Rightarrow \langle \{e_1, e_2\} \rangle^{\perp}$ — || —

$f|_{\langle\{e_1,e_2\}\rangle^{\perp}}$ este endomorfism simetric

Se repetă raționamentul și după n fași construim $R = \{e_1, .., e_n\}$
& $e_1, .., e_n$ sunt mutual ortogonali $\Big\} \Rightarrow R$ este SLI

$|R| = n$

$\Rightarrow R$ este reper ortonormat

$f(e_i) = \lambda_i e_i, \forall i = \overline{1,n}$ $\quad [f]_{R,R} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ diagonală

<u>OBS</u> $(E, \langle \cdot, \cdot \rangle)$

a) $f \in Sim(E) \Rightarrow \dim V_{\lambda_i} = m_i, \ i = \overline{1,k}$

$\lambda_1, .., \lambda_k$ valori proprii dist

$m_1, .., m_k$ = multiplicitatile,

$E = V_{\lambda_1} \oplus ... \oplus V_{\lambda_k}$

$R = R_1 \cup .. \cup R_k$, unde R_i reper ortonormat în V_{λ_i} $\quad i = \overline{1,k}$

b) $A = A^T \longrightarrow f \in Sim(E)$, $[f]_{R,R} = A$

$\searrow Q : E \longrightarrow \mathbb{R}$, $Q(x) = X^T A X$

$\langle f(x), x \rangle = Q(x)$, $\forall x \in E$.

c) $R_0 = \{e_1^0, .., e_n^0\} \xrightarrow{C} R = \{e_1, .., e_n\}$ repere ortonormate.

$h : E \to E$ liniară

$h(e_i^0) = e_i$, $\forall i = \overline{1,n}$. $\quad C \in O(n)$

$[h]_{R_0, R_0} = C$; $e_i = \sum_{k=1}^{n} c_{ki} e_k^0$, $h \in O(E)$.

Ex $(\mathbb{R}^3, g_0)$, $f \in End(\mathbb{R}^3)$

$$[f]_{R_0,R_0} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = A$$

a) $f \in Sim(\mathbb{R}^3)$

b) $Q : \mathbb{R}^3 \to \mathbb{R}$ forma pătratică asociată ($Q(x) = \langle f(x), x \rangle$)

Să se aducă Q la o formă canonică printr-o transformare ortogonală.

Sol

a) $A = A^T \Rightarrow f \in Sim(\mathbb{R}^3)$

$f(x) = (x_1 + x_2 - x_3,\ x_1 + x_2 - x_3,\ -x_1 - x_2 + x_3)$

b) $Q : \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 2x_2x_3$

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^3 - \sigma_1 \lambda^2 + \sigma_2 \lambda - \sigma_3 = 0 \Rightarrow \lambda^3 - 3\lambda^2 = 0 \Rightarrow \lambda^2(\lambda - 3) = 0$

$\sigma_1 = Tr A = 3$, $\sigma_2 = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

$\sigma_3 = \det A = 0$

1) $\lambda_1 = 0$, $m_1 = 2$

2) $\lambda_2 = 3$, $m_2 = 1$.

$V_{\lambda_1} = \{X \in \mathbb{R}^3 / AX = 0\} = \{(x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R}\}$.

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 + x_2 - x_3 = 0$

$x_3 = x_1 + x_2$

$V_{\lambda_1} = \langle \{ \underbrace{(1,0,1)}_{f_1}, \underbrace{(0,1,1)}_{f_2} \} \rangle$ $\quad \dim V_{\lambda_1} = 3 - rg A = 3 - 1 = 2$

Aplicăm procedeul Gram-Schmidt

$e_1 = f_1$; $e_2 = f_2 - \dfrac{g_0(f_2, e_1)}{g_0(e_1, e_1)} e_1 =$

$$e_2=(0,1,1)-\frac{1}{2}(1,0,1)=\left(-\frac{1}{2},1,\frac{1}{2}\right)=$$
$$=\frac{1}{2}\underline{(-1,2,1)}$$

$$e_1'=\frac{1}{\sqrt2}(1,0,1),\ e_2'=\frac{1}{\sqrt6}(-1,2,1)$$

$\mathcal{R}_1=\{e_1',e_2'\}$ reper ortonormat în V_{λ_1}

$$V_{\lambda_2}=\{x\in\mathbb{R}^3 \mid AX=3X\}$$
$$(A-3I_3)X=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}-2&1&-1\\1&-2&-1\\-1&-1&-2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$\dim V_{\lambda_2}=1$

$$\begin{cases}-2x_1+x_2=x_3 & \mid\cdot 2\\ x_1-2x_2=x_3\end{cases}\ \oplus$$
$$-3x_1\quad /\quad =3x_3$$

$x_1=-x_3$

$x_2=x_3-2x_3=-x_3$

$$V_{\lambda_2}=\{(-x_3,-x_3,x_3)=x_3(-1,-1,1)\mid x_3\in\mathbb{R}\}.$$

$\mathcal{R}_2=\{e_3'=\frac{1}{\sqrt3}(-1,-1,1)\}$ reper ortonormat în V_{λ_2}.

$$\mathbb{R}^3=V_{\lambda_1}\oplus V_{\lambda_2}$$

$\mathcal{R}=\mathcal{R}_1\cup\mathcal{R}_2$ reper ortonormat a.î. $[f]_{\mathcal{R},\mathcal{R}}=\begin{pmatrix}0&&\\&0&\\&&3\end{pmatrix}$

$Q(x)=3(x_3')^2$ $\quad$ $(1,0)$ signatura

$$\mathcal{R}_0=\{e_1^0,e_2^0,e_3^0\}\xrightarrow{C}\mathcal{R}=\{\underbrace{\tfrac{1}{\sqrt2}(1,0,1)}_{e_1'},\underbrace{\tfrac{1}{\sqrt6}(-1,2,1)}_{e_2'},\underbrace{\tfrac{1}{\sqrt3}(-1,-1,1)}_{e_3'}\}$$

$h\in O(\mathbb{R}^3)$

$h(e_i^0)=e_i$, $i=\overline{1,3}$

$$C=\begin{pmatrix}\frac{1}{\sqrt2}&-\frac{1}{\sqrt6}&-\frac{1}{\sqrt3}\\0&\frac{2}{\sqrt6}&-\frac{1}{\sqrt3}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\end{pmatrix}$$

$h:\mathbb{R}^3\longrightarrow\mathbb{R}^3$

$$h(x)=\left(\frac{1}{\sqrt2}x_1-\frac{1}{\sqrt6}x_2-\frac{1}{\sqrt3}x_3,\ \frac{2}{\sqrt6}x_2-\frac{1}{\sqrt3}x_3,\ \frac{1}{\sqrt2}x_1+\frac{1}{\sqrt6}x_2+\frac{1}{\sqrt3}x_3\right)$$

Def Fie $\lambda \in \mathbb{C}$

S.n <u>bloc Jordan de ordin p asociat lui λ</u> matricea $J_p(\lambda) = \begin{pmatrix} \lambda & & 0 \\ 1 & \ddots & \\ 0 & 1 & \lambda \end{pmatrix} \in \mathcal{M}_p(\mathbb{C})$, $p \geq 2$

$J_1(\lambda) = (\lambda)$

$p = 3$ $\begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix} = J_3(\lambda)$; $p = 2$ $\begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} = J_2(\lambda)$

S.n. matrice Jordan de ordin n matricea

$$J = \begin{pmatrix} J_{p_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{p_t}(\lambda_t) \end{pmatrix} \in \mathcal{M}_n(\mathbb{C})$$

$p_1 + \dots + p_t = n$.

<u>Teoremă</u>

$(V, +, \cdot)/\mathbb{C}$ sp. vect.

$\forall f \in End(V)$, $\exists$ un reper R a.î. $[f]_{R,R} =$

$$= J = \begin{pmatrix} J_{p_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{p_t}(\lambda_t) \end{pmatrix}, p_1 + \dots + p_t = n$$

sau $\forall A \in \mathcal{M}_n(\mathbb{C})$, $\exists C \in GL(n, \mathbb{C})$ a.î. $A' = C^{-1} A C$ este o matrice Jordan J.

Scrierea blocurilor Jordan pe diagonală este unică, modulo permutarea blocurilor Jordan pe diagonală!