

Aria. Volume. Distanță

[CURS 12] - AG

- $\{u, v\}$ SLI , $w = u \times v$

$$\|w\| = \|u\| \cdot \|v\| \cdot \sin \alpha = \text{Aria paralelogramului}$$

1) Aria unui triunghi

$$A_{\Delta ABC} = \frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\| = \sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} e_1 & e_2 & e_3 \\ b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \end{vmatrix}^2 = (\Delta_1, \Delta_2, \Delta_3)$$

Ex A(1,0,1), B(0,-1,0), C(0,1,1)

$$A_{\Delta ABC} = ?$$

$$\overrightarrow{AB} = (-1, -1, -1)$$

$$\overrightarrow{AC} = (-1, 1, 0)$$

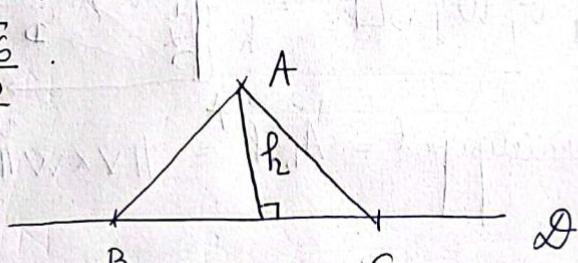
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -1 & -1 & -1 \\ -1 & 1 & 0 \end{vmatrix} = (1, 1, -2)$$

$$A_{\Delta} = \frac{1}{2} \sqrt{1+1+4} = \frac{\sqrt{6}}{2}$$

2) dist (A, \mathcal{D})

(M₁)

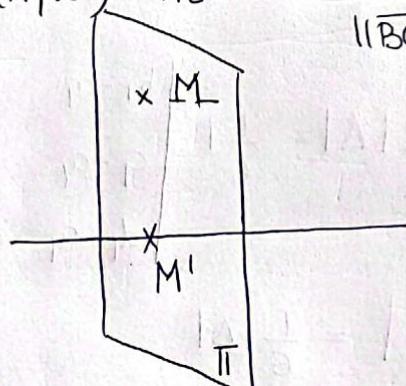
[CURS 12]



$$A_{\Delta ABC} = \frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2} h \|\overrightarrow{BC}\|$$

$$\text{dist}(A, \mathcal{D}) = h = \frac{\|\overrightarrow{AB} \times \overrightarrow{AC}\|}{\|\overrightarrow{BC}\|} = \frac{\sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}}{\sqrt{(a-b_1)^2 + (c-b_2)^2 + (c-b_3)^2}}$$

(M₂)



$\pi \not\perp \mathcal{D}$

$\mathcal{D} \perp \pi$

$$\mathcal{D} \cap \pi = \{M'\}$$

$$\text{dist}(M, \mathcal{D}) = \text{dist}(M, M')$$

Exemplu $\mathcal{D}: \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3 - 1}{2} = t$, A(1,1,0)

dist(A, \mathcal{D})

SOL (M₁) $\mathcal{D}: \begin{cases} x_1 = t \\ x_2 = -t \\ x_3 = 1 + 2t \end{cases}$

$$t=0 \Rightarrow B(0, 0, 1)$$

$$t=1 \Rightarrow C(1, -1, 3)$$

$$\overrightarrow{AB} = (-1, -1, 1)$$

$$\overrightarrow{AC} = (0, 2, 3)$$

$$\overrightarrow{BC} = (1, -1, 2)$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -1 & -1 & 0 \\ 0 & 2 & -2 \end{vmatrix} = (-1, 3, 2)$$

$$\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \sqrt{1+9+4} = \sqrt{14}$$

$$\|\overrightarrow{BC}\| = \sqrt{6}$$

$$\text{dist} = \frac{\sqrt{14}}{\sqrt{6}} = \sqrt{\frac{7}{3}} = \sqrt{\frac{21}{3}}$$

$$M_2 \quad A(1,1,0) \in \pi, \quad N_{\pi} = M_{\mathcal{D}} = (1, -1, 2)$$

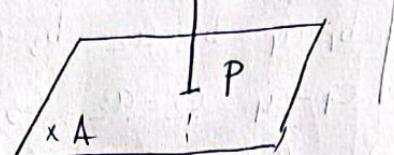
$$\pi: x_1 - x_2 + 2x_3 + d = 0$$

$$A(1,1,0) \in \pi \Rightarrow 1 - 1 + 0 + d = 0 \Rightarrow d = 0 \Rightarrow \pi: x_1 - x_2 + 2x_3 = 0$$

$\mathcal{D} \cap \pi$

$$\begin{cases} x_1 = t \\ x_2 = -t \\ x_3 = 1+2t \end{cases} \Rightarrow t + t + 2(1+2t) = 0 \Rightarrow 6t = -2 \Rightarrow t = -\frac{1}{3}$$

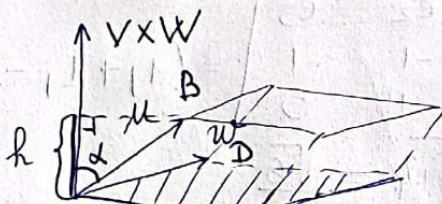
$$P\left(-\frac{1}{3}, \frac{1}{3}, 1-\frac{2}{3}\right)$$



$$\text{dist}(A, \mathcal{D}) = \text{dist}(A, P) = \sqrt{\left(\frac{1}{3} - 1\right)^2 + \left(\frac{1}{3} - 1\right)^2 + \left(\frac{1}{3} - 0\right)^2}$$

Volume

$\{u, v, w\}$ SLI



$$\checkmark \text{parallelipiped} = A_b \cdot h = \|v \times w\| \cdot h = |\langle u, v \times w \rangle|$$

$$= |u \wedge v \wedge w| = \|u\| \cdot |\cos \alpha|$$

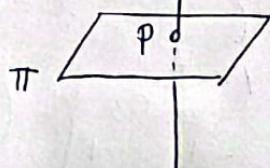
$$= 1 \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = |\Delta|$$

$$\checkmark A_{ABCD} = \frac{1}{6} \checkmark \text{parallelip} = \frac{1}{6} |\Delta| = \frac{1}{6} \begin{vmatrix} b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \\ d_1 - a_1 & d_2 - a_2 & d_3 - a_3 \end{vmatrix}$$

$$= \frac{1}{6} \begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix} = \frac{1}{6} |\Delta|$$

OBS $A, B, C, D = \text{nonplanar} \Leftrightarrow |\Delta| = 0$.

- $\text{dist}_{\pi}(A, \pi)$
- $\text{dist}_{\pi}(A, P)$



$\mathcal{D} \perp \pi$

$$M_{\mathcal{D}} = N_{\pi}$$

$$\mathcal{D} \cap \pi = \{P\}$$

$$\text{Ex} \quad \pi : x_1 - 2x_2 + 3x_3 + 1 = 0, \quad A(1, 2, 3)$$

$$\text{dist}(A, \pi) = ?$$

SOL

$$\mathcal{D} \perp \pi, \quad A \in \mathcal{D} \Rightarrow \mathcal{D} : \frac{x_1 - 1}{1} = \frac{x_2 - 2}{-2} = \frac{x_3 - 3}{3} = t$$

$$N_{\pi} = (1, -2, 3) = u_{\mathcal{D}}$$

$$\mathcal{D} : \begin{cases} x_1 = 1+t \\ x_2 = 2-2t \\ x_3 = 3+3t \end{cases} \quad \mathcal{D} \cap \pi : 1 + \underline{t} - 2(2 - 2\underline{t}) + 3(3 + 3\underline{t}) + 1 = 0$$

$$t + 4t + 9t + 1 - 4 + 9 + 1 = 0 \Rightarrow 14t = -7 \Rightarrow t = -\frac{1}{2}$$

$$P\left(\frac{1}{2}, 3, \overbrace{3 - \frac{3}{2}}^{3/2}\right)$$

$$\text{dist}(A, \pi) = \text{dist}(A, P) = \sqrt{\left(\frac{1}{2} - 1\right)^2 + (3 - 2)^2 + \left(\frac{3}{2} - 3\right)^2}$$

$$= \sqrt{\frac{1}{4} + 1 + \frac{9}{4}} = \frac{\sqrt{14}}{2}$$

Hipercuadrica în spațiu afin \mathbb{R}^n

Def $(\mathbb{R}^n, \mathbb{R}^n/\mathbb{R}, \varphi)$ (sau $(\mathbb{R}^n, (\mathbb{R}^n/\mathbb{R}, g_0), \varphi)$)

$\mathbb{R} = \{0; e_1, \dots, e_n\}$ reprez cartezian.

T.n. hipercuadrica în \mathbb{R}^n L.G al funcțiilor $P(x_1, \dots, x_n)$ ai'

$$\Gamma: f(x_1, \dots, x_n) = a_{11}x_1^2 + \dots + a_{nn}x_n^2 + 2a_{12}x_1x_2 + \dots + 2a_{n-1,n}x_{n-1}x_n + 2b_1x_1 + \dots + 2b_nx_n + c = 0.$$

$$\Gamma: X^T A X + 2B X + C = 0$$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \in M_n(\mathbb{R})$$

$$B = (b_1 \dots b_n)$$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\tilde{A} = \left(\begin{array}{c|c} A & B^T \\ \hline B & C \end{array} \right)$$

$$r = \text{rg } A, \quad r' = \text{rg } \tilde{A}$$

$$r \leq r' \leq r+2.$$

$$\Delta = \det(A)$$

$$\Delta = \det(\tilde{A})$$

1) $\Delta \neq 0 \Rightarrow \Gamma$ hipercuadrica nedegenerată

2) $\Delta = 0 \Rightarrow \Gamma$ - degenerată

OBS a) $(\mathbb{R}^n, (\mathbb{R}/\mathbb{R}, \varphi))$

$\Gamma_1 \sim \Gamma_2$ afin echivalente $\Leftrightarrow \exists: \mathbb{R}^n \rightarrow \mathbb{R}^n$ transf. afina
 $\exists: x' = CX + D, C \in GL(n, \mathbb{R})$

a) $\Gamma_2 = \exists(\Gamma_1)$; invariante afini: $\frac{\Delta}{\delta}, \varepsilon, \varepsilon'$

b) $(\mathbb{R}^n, (\mathbb{R}/\mathbb{R}, g_0), \varphi)$

$\Gamma_1 \equiv \Gamma_2$ congruente metric $\Leftrightarrow \exists: \mathbb{R}^n \rightarrow \mathbb{R}^n$ izometrie

$\exists: x' = CX + D, C \in O(n)$

al ~~$\Gamma_1 \not\equiv \Gamma_2$~~ $\Gamma_2 = \exists(\Gamma_1)$; invariante metrici
 $\frac{\Delta}{\delta}, \varepsilon, \varepsilon', \Delta, \delta$.

OBS $m=2$ Γ conică

$m=3$ Γ cuadratice.

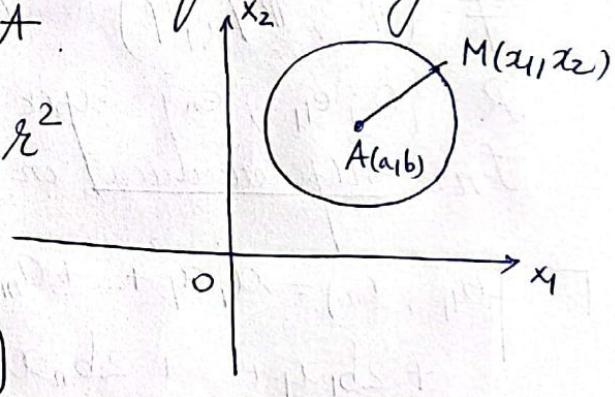
Conice ca locuri geometrice

1) Cercul $C(A(a, b), r) = LG$ al punctelor egale depărtate de un punct fix A .

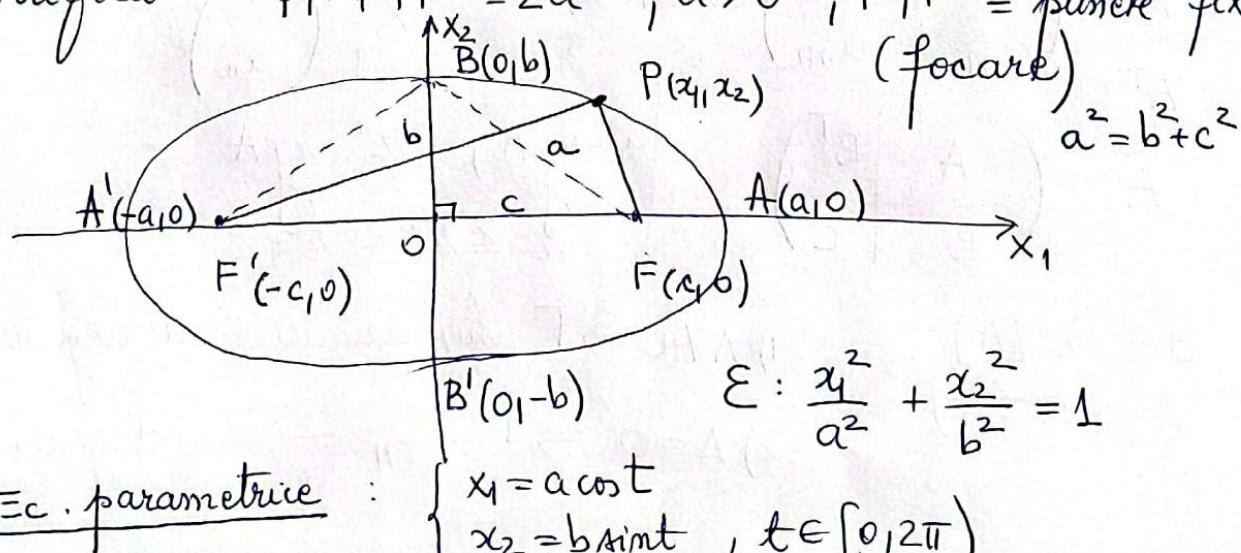
$$AM=r \Leftrightarrow (x_1-a)^2 + (x_2-b)^2 = r^2$$

Ec. parametrice.

$$\begin{cases} x_1 - a = r \cos t \\ x_2 - b = r \sin t, t \in [0, 2\pi] \end{cases}$$



2) Elipsa este LG al punctelor P din plan care verifică $PF + PF' = 2a$, $a > 0$, F, F' = puncte fixe (focare)



Ec. parametrice

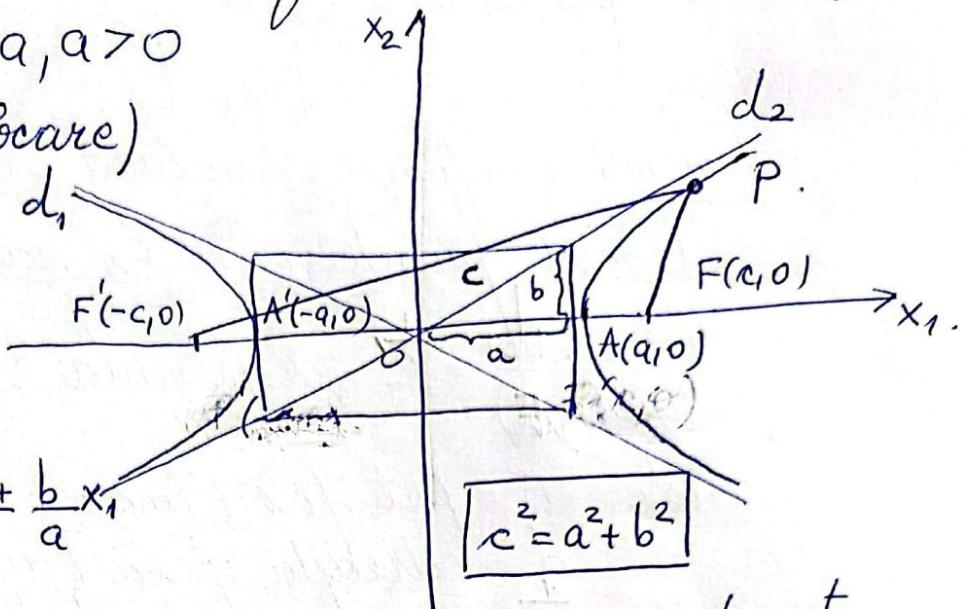
$$\begin{cases} x_1 = a \cos t \\ x_2 = b \sin t, t \in [0, 2\pi] \end{cases}$$

③ Hiperbola este LG al punctelor $P \in E_2$ care verifică

$$|PF - PF'| = 2a, a > 0$$

F, F' puncte fixe (focare)

$$\mathcal{H}: \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1.$$



$$d_1 \cup d_2 : x_2 = \pm \frac{b}{a} x_1$$

(asimptotele)

OBS Ec. parametrice: $\begin{cases} x_1 = a \operatorname{cht} t \\ x_2 = b \operatorname{sht} t \end{cases}$

$$\operatorname{cht} t = \frac{e^t + e^{-t}}{2}$$

$$\operatorname{sht} t = \frac{e^t - e^{-t}}{2}$$

$$\operatorname{ch}^2 t - \operatorname{sh}^2 t = 1$$

$$t \in \mathbb{R}.$$

④ Parabola = LG al punctelor $P \in E_2$ care verifică

$$\frac{\operatorname{dist}(P, F)}{\operatorname{dist}(P, d)} = 1,$$

F = punct fix (focar)

d = dreapta fixa (direcție)

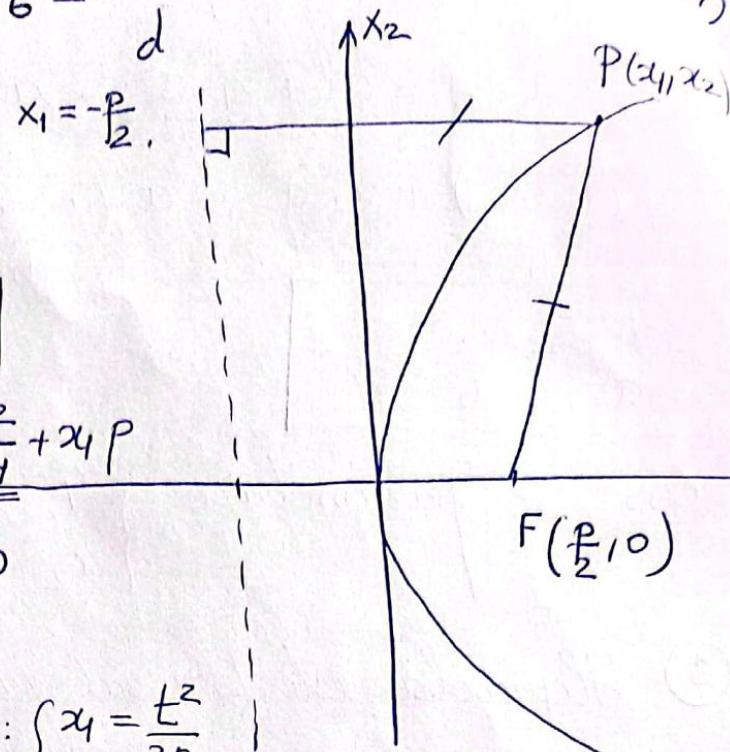
$F \notin d$.

$$\frac{\text{dist}(P_1, F)}{\text{dist}(P_1, d)} = 1$$

$$\sqrt{(x_1 - \frac{p}{2})^2 + x_2^2} = |x_1 + \frac{p}{2}|$$

$$\underline{x_1^2 + \frac{p^2}{4}} - x_1 p + x_2^2 = \underline{x_1^2 + \frac{p^2}{4}} + x_1 p$$

$$\boxed{Q: x_2^2 = 2px_1} \quad | p > 0$$



OBS. Ec parametrică: $\begin{cases} x_1 = \frac{t^2}{2p}, \\ x_2 = t \end{cases}$

Teorema (definirea unitară a conicelor nedegenerate)
LG al punctelor $P \in E_2$ care verifică

$\frac{\text{dist}(P, F)}{\text{dist}(P, d)} = e$, reprezintă o conică nedegenerată,

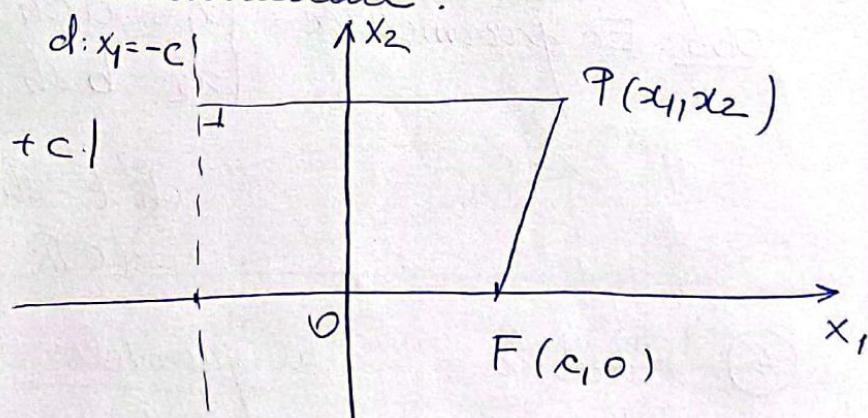
unde F = pct fix (focar)

d = dreptă fixă (direcție), $F \notin d$

e = s.m. excentricitate.

Dem.,

$$\text{d: } x_1 = -c \quad \sqrt{(x_1 - c)^2 + x_2^2} = e|x_1 + c|$$



$$e^2(x_1^2 + c^2 - 2x_1 c + x_2^2) = e^2(x_1^2 + 2x_1 c + c^2)$$

$$e^2(x_1^2 + x_2^2 - 2x_1 c) = e^2(x_1^2 + 2x_1 c) + e^2(c^2 - c^2) = 0.$$

OBS

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$$\text{Conica } \Gamma: f(x_1, x_2) = \frac{a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2 +}{+ 2b_1x_1 + 2b_2x_2 + c} = 0$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = A^T, \tilde{A} = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{12} & a_{22} & b_2 \\ b_1 & b_2 & c \end{pmatrix}$$

$$\delta = \det A, \Delta = \det \tilde{A}$$

$$\Gamma: \underline{x_1^2(1+e^2)} + \underline{x_2^2(-2x_1)} \underline{c(1+e^2)} + \underline{c^2(1+e^2)} = 0$$

$$A = \begin{pmatrix} 1+e^2 & 0 \\ 0 & 1 \end{pmatrix}, \tilde{A} = \begin{pmatrix} 1+e^2 & 0 & -c(1+e^2) \\ 0 & 1 & 0 \\ -c(1+e^2) & 0 & c^2(1+e^2) \end{pmatrix}$$

Γ nedegenerata $\Leftrightarrow \Delta = \det \tilde{A} \neq 0$

$$\Delta = e^2 \begin{vmatrix} 1+e^2 & -c(1+e^2) \\ -c(1+e^2) & c^2(1+e^2) \end{vmatrix} = [c^2(1-e^2)^2 - c^2(1+e^2)^2]e^2 = c^2(-4e^4) = -4c^2e^2 \neq 0$$

$F \notin d \Rightarrow e \neq 0.$ $\Rightarrow \Gamma$ nedegenerata
 $c \neq 0.$

(elipsă, hiperbolă, parabolă)

OBS

$$\textcircled{1} \quad \mathcal{E}: \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

$$e = \frac{c}{a} < 1$$

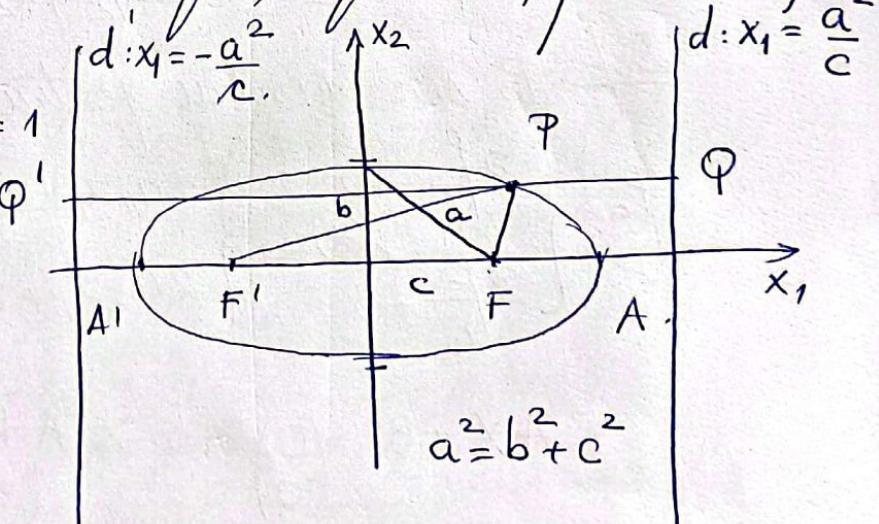
(excentricitatea)

$$d \cup d': x_1 = \pm \frac{a^2}{c}$$

directoare.

$$\frac{a^2}{c} > a \Leftrightarrow a > c$$

$$\Rightarrow PF + PF' = \frac{c}{a} \cdot 2 \frac{a^2}{c} = 2a;$$



$$\textcircled{A} \quad \frac{PQ}{PF} = \frac{a}{c} \Rightarrow PF = \frac{c}{a} PQ$$

$$\frac{PQ'}{PF'} = \frac{c}{a} \Rightarrow PF' = \frac{c}{a} PQ'$$

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$$\textcircled{2} \quad \text{J6: } \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1.$$

$$c^2 = a^2 + b^2$$

$$e = \frac{c}{a} > 1.$$

$$\frac{a^2}{c} < a \Leftrightarrow a < c$$

$$\frac{PQ}{PF} = \frac{a}{c} \Rightarrow PF = \frac{c}{a} PQ$$

$$\frac{PQ'}{PF'} = \frac{a}{c} \Rightarrow PF' = \frac{c}{a} PQ'$$

$$|PF - PF'| = \frac{c}{a}, \quad |PQ - PQ'| = \frac{c}{a} \cdot \frac{2a^2}{c} = 2a.$$

$$\textcircled{3} \quad \text{J7: } x_1^2 = 2px_1,$$

$$e = 1.$$

Aducerea la forma canonica a conicelor
cu centru unic

$$\Gamma: X^T A X + 2BX + C = 0,$$

$$\Gamma: a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2 + 2b_1x_1 + 2b_2x_2 + C = 0$$

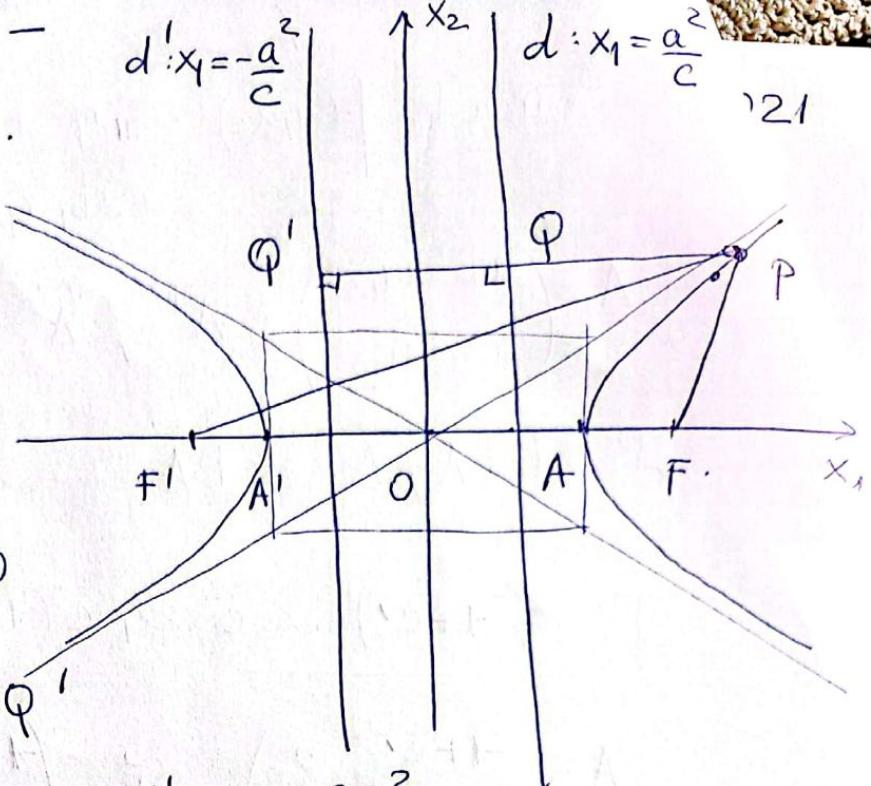
Def P_0 s.n. centru pentru $\Gamma \Leftrightarrow \forall P \in \Gamma \Leftrightarrow f_{P_0}(P) \in \Gamma$

$$P_0: \begin{cases} \frac{\partial f}{\partial x_1} = 0 \\ \frac{\partial f}{\partial x_2} = 0 \end{cases} \stackrel{*}{\Rightarrow} \begin{cases} 2a_{11}x_1 + 2a_{12}x_2 + 2b_1 = 0 \\ 2a_{12}x_1 + 2a_{22}x_2 + 2b_2 = 0 \end{cases}$$

$$AX + B^T = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow X^T A + B = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$\delta = \det A \neq 0 \Rightarrow \textcircled{*}$ are sol unică.



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Prop Dacă $\delta \neq 0$, atunci $f(x_1^0, x_2^0) = \frac{\Delta}{\delta}$, unde $P_0(x_1^0, x_2^0)$ este centrul paralelei Γ .

$$f(x_1^0, x_2^0) = X_0^T A X_0 + 2 B X_0 + C = \\ = \underbrace{(X_0^T A + B)}_{= (X_0^T A + B) X_0} + B X_0 + C = B X_0 + C = \frac{\Delta}{\delta} \text{ (dem).}$$

$$b_1 x_1^0 + b_2 x_2^0 + C = \frac{\Delta}{\delta} \text{ (dem)} \\ x_1^0 = \frac{|-b_1 \quad a_{12}|}{\delta}, \quad x_2^0 = \frac{|a_{11} \quad -b_1|}{|a_{12} \quad -b_2|} \quad (1)$$

$$\frac{\Delta}{\delta} = \frac{1}{\delta} \begin{vmatrix} a_{11} & a_{12} & \begin{pmatrix} b_1 \\ b_2 \\ C \end{pmatrix} \\ a_{12} & a_{22} & \begin{pmatrix} b_1 \\ b_2 \\ C \end{pmatrix} \\ b_1 & b_2 & \begin{pmatrix} b_1 \\ b_2 \\ C \end{pmatrix} \end{vmatrix} = \\ = \frac{1}{\delta} \cdot b_1 \begin{vmatrix} a_{12} & a_{22} \\ b_1 & b_2 \end{vmatrix} - \frac{b_2}{\delta} \begin{vmatrix} a_{11} & a_{12} \\ b_1 & b_2 \end{vmatrix} + \frac{C}{\delta} \delta \quad (2)$$

$$b_1 x_1^0 + b_2 x_2^0 + C = \frac{b_1}{\delta} \begin{vmatrix} -b_1 & a_{12} \\ -b_2 & a_{22} \end{vmatrix} + \frac{b_2}{\delta} \begin{vmatrix} a_{11} & -b_1 \\ a_{12} & -b_2 \end{vmatrix} + C \quad (3)$$

(I) $\delta \neq 0 \Rightarrow f(x_1^0, x_2^0) = \frac{\Delta}{\delta}$

• Dacă $(R^2, R^2/R, \varphi)$ sp. afim.

$$R = \{0; e_1, e_2\} \xrightarrow[\text{translatie}]{} R' = \{P_0; e_1, e_2\} \xrightarrow[\text{transf. afim.}]{} R'' = \{P_0; e'_1, e'_2\}.$$

$| \theta: X = X' + X_0.$

$$\theta(\Gamma): (X' + X_0)^T A (X' + X_0) + 2B(X' + X_0) + C = 0.$$

$$\underbrace{X'^T A X'}_{\text{mmm}} + \underbrace{X_0^T A X'}_{\text{mmm}} + \underbrace{X'^T A X_0}_{\text{mmm}} + \underbrace{X_0^T A X_0}_{\text{mmm}} + \underbrace{2B X'}_{\text{mmm}} + \underbrace{2B X_0}_{\text{mmm}} + \underbrace{C}_{\text{mmm}} = 0$$

$| \theta(\Gamma): X'^T A X' + \frac{\Delta}{\delta} = 0$

$Q: \mathbb{R}^2 \rightarrow \mathbb{R}$, $Q(x) = x^T A x'$ forma patratica

Aducem Q la forma canonica (met Gauss)

$$Q(x) = \lambda_1 x_1''^2 + \lambda_2 x_2''^2$$

$$\text{G: } x' = Cx'', \quad C \in GL(2, \mathbb{R})$$

$$\text{G} \circ \theta(\Gamma): \lambda_1 x_1''^2 + \lambda_2 x_2''^2 + \frac{\Delta}{S} = 0.$$

Γ'

Γ, F' conice afini echivalente
 $x \rightarrow x' + x_0 \rightarrow Cx'' + x_0$ (transf. afina)

- $(E_2, (E_2, \langle \cdot, \cdot \rangle), \varphi)$ sf. afini euclidian.

$$Q: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad Q(x) = x'^T A x'$$

\exists un refer ortonormal format din vectori proprii ai $A = \text{diagonala}$.

a) $P(\lambda) = \det(A - \lambda I_2) = 0$

$$\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0.$$

$$\lambda_1 \neq \lambda_2, \quad m_1 = m_2 = 1$$

$$\forall \lambda_i = \langle \{e'_i\} \rangle \quad i=1,2 \quad \langle e'_i, e'_j \rangle = \delta_{ij}$$

$$e'_1 = (l_1, m_1), \quad e'_2 = (l_2, m_2) \quad i, j = 1, 2$$

$$R = \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \quad (\det R = 1, \text{ refer pozitiv orientat})$$

$$\text{G: } x' = Rx'' \quad \text{isometrie (de spatiu 1).}$$

$$\text{G} \circ \theta(\Gamma): \lambda_1 x_1''^2 + \lambda_2 x_2''^2 + \frac{\Delta}{S} = 0$$

Γ, Γ' conice congruente metric.

$$x = x' + x_0, \quad x' = Rx'' \Rightarrow \text{G} \circ \theta: x = Rx'' + x_0.$$

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b) $\lambda_1 = \lambda_{2,1}$ $m_1 = 2$

$$\sqrt{\lambda_1} = \langle \{f_1, f_2\} \rangle \quad \text{Apliām Gram-Schmidt}$$
$$\Rightarrow \{e_1', e_2'\} \text{ ir vien ortonormat.}$$

Aplicări

Ex1 ($\delta \neq 0$)

În sp. euclidian E_2 se consideră conica:

$$\Gamma: f(x) = \frac{7x_1^2 - 8x_1x_2 + x_2^2 - 6x_1 - 12x_2 - 9}{\Delta} = 0$$

Să se aducă la o formă canonica, utilizând
împărțirea. Reprez. grafică

SOL

$$A = \begin{pmatrix} 7 & -4 \\ -4 & 1 \end{pmatrix}, \tilde{A} = \begin{pmatrix} 7 & -4 & -3 \\ -4 & 1 & -6 \\ -3 & -6 & -9 \end{pmatrix}$$

$$\delta = \det A = 7 - 16 = -9 \neq 0 \quad (\Gamma \text{ are centru unic})$$

$$\Delta = \det \tilde{A} = -9 \cdot 36 \neq 0 \quad (\Gamma \text{ nedegenerată})$$

Det. centrul conicei

$$P_0: \begin{cases} \frac{\partial f}{\partial x_1} = 0 \\ \frac{\partial f}{\partial x_2} = 0 \end{cases} \Rightarrow \begin{cases} 14x_1 - 8x_2 - 6 = 0 \\ -8x_1 + 2x_2 - 12 = 0 \end{cases}$$

$$\begin{cases} 7x_1 - 4x_2 = 3 \\ -4x_1 + x_2 = 6 \end{cases} \quad | \cdot 4$$

$$\begin{array}{r} 7x_1 - 4x_2 = 3 \\ -16x_1 + 4x_2 = 24 \\ \hline -9x_1 = 27 \end{array}$$

$$x_1 = -3$$

$$x_2 = 6 + 12 = -6$$

$$P_0(-3, -6)$$

$$R = \{O; e_1, e_2\} \xrightarrow[\text{translație}]{\theta} R' = \{P_0; e_1, e_2\} \xrightarrow[\text{rotare}]{\tau} R'' = \{P_0; e'_1, e'_2\}$$

$$\theta: x = x' + x_0, \quad x_0 = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$$

$$\theta(\Gamma): x'^T A x' + \frac{\Delta}{\delta_{11} 36} = 0$$

$$\theta(\Gamma) : \underbrace{7x_1'^2 - 8x_1'x_2' + x_2'^2}_{\text{ }} + 36 = 0.$$

$$Q : \mathbb{R}^2 \rightarrow \mathbb{R}, Q(x) = 7x_1'^2 - 8x_1'x_2' + x_2'$$

Aplicăm met. val. proprii

$$\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0 \Rightarrow \lambda^2 - 8\lambda - 9 = 0$$

$$(\lambda+1)(\lambda-9) = 0$$

$$1. \lambda_1 = -1, 2. \lambda_2 = 9.$$

$$V_{\lambda_1} = \left\{ x \in \mathbb{R}^2 \mid Ax = -x \right\} \\ (A + I_2)x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-4x_1 + 2x_2 = 0 \Rightarrow x_2 = 2x_1$$

$$V_{\lambda_1} = \left\{ (x_1, 2x_1) = x_1(1, 2), x_1 \in \mathbb{R} \right\}$$

$$e_1' = \frac{1}{\sqrt{5}}(1, 2)$$

$$V_{\lambda_2} = \left\{ x \in \mathbb{R}^2 \mid Ax = 9x \right\} \\ (A - 9I_2)x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & -4 \\ -4 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - 4x_2 = 0 \Rightarrow x_1 = -2x_2$$

$$V_{\lambda_2} = \left\{ (-2x_2, x_2) = x_2(-2, 1), x_2 \in \mathbb{R} \right\}$$

$$e_2' = \frac{1}{\sqrt{5}}(-2, 1).$$

$$\text{Z: } x' = Rx'' \quad , \quad R = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \in SO(2).$$

$$z_0 \theta : x = Rx'' + x_0 \quad (\text{igometrică}).$$

$$z_0 \theta(\Gamma) : -x_1''^2 + 9x_2''^2 + 36 = 0$$

$$\frac{x_1''^2}{36} - \frac{x_2''^2}{4} = 1 \quad (\text{hiperbolă}).$$

$$e_1' = \frac{1}{\sqrt{5}}(1|2)$$

$$\mathbf{e}_2^{-1} = \frac{1}{\sqrt{5}} (-2, 1)$$

$$\frac{x_1^{11^2}}{36} - \frac{x_2^{11^2}}{4} = 1.$$

$$a=6, b=2$$

