CURS 9 - AG. 8.12.2022

Spatii vectoriale euclidiene. Procedeul Gram-Schmidt Transformări ortogonale

Def $(V, +, \cdot)_{\mathbb{R}}$ sp. v. real

$g: V\times V \to \mathbb{R}$ s.n. produs scalar $\Leftrightarrow$ 1) $g \in L^s(V, V; \mathbb{R})$
2) g este poz. def.

OBS A da un produs scalar $\Leftrightarrow$ a declara un reper ortonormat $R=\{e_1,.., e_n\}$

"$\Rightarrow$" g produs scalar $R=\{e_1,.., e_n\}$ reper ortonormat

$$g(e_i, e_j) = \delta_{ij}, \forall i,j = \overline{1,n}$$

"$\Leftarrow$" R reper ortonormat $g: V\times V \to \mathbb{R}$ produs scalar

$g(e_i, e_j) = \delta_{ij}$

Prelungim prin liniaritate în ambele argumente.

$$g(x, y) = g\left(\sum_{i=1}^n x_i e_i, \sum_{j=1}^n y_j e_j\right) = \sum_{i,j=1}^n \overbrace{g(e_i, e_j)}^{\delta_{ij}} x_i y_j$$

$$= \sum_{i=1}^n x_i y_i$$

Exemplu $g_0: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $g_0(x, y) = \sum_{i=1}^n x_i y_i$; g_0, $\langle \cdot, \cdot \rangle$ produs scalar canonic.

$R_0 = \{e_1,.., e_n\}$ reperul canonic este ortonormat

$g_0(e_i, e_j) = \delta_{ij}$.

Def $(\mathbb{R}^3, g_0)$ sp. vect euclidian real (cu str. canonică).

Fie $S = \{x, y\} \subset \mathbb{R}^3$

Definim $z = x \times y$ (produsul vect.) astfel

1) S este SLD $\Rightarrow z = 0$

2) S este SLI atunci a) $\|z\|^2 = \begin{vmatrix} \langle x, x\rangle & \langle x, y\rangle \\ \langle y, x\rangle & \langle y, y\rangle \end{vmatrix}$

b) $\langle z, x\rangle = 0$, $\langle z, y\rangle = 0$

c) reperul $\mathcal{R} = \{x, y, z\}$ este pozitiv orientat (la fel orientat ca și reperul canonic)

$\left(\mathcal{R}_0 \xrightarrow{A} \mathcal{R}, \quad \det A > 0\right)$

<u>OBS</u> $z = x \times y$ este un „determinant formal"

$$z = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = e_1 \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - e_2 \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + e_3 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

$$= (x_2 y_3 - x_3 y_2,\ x_3 y_1 - x_1 y_3,\ x_1 y_2 - x_2 y_1)$$

$$z = \begin{vmatrix} x_1 & y_1 & e_1 \\ x_2 & y_2 & e_2 \\ x_3 & y_3 & e_3 \end{vmatrix}$$

<u>Prop</u> a) $x \times y = -y \times x$

b) $(x \times y) \times z = \langle x, z\rangle y - \langle y, z\rangle x$

c) $\sum\limits_{x,y,z}^{c} (x \times y) \times z = (x \times y) \times z + (y \times z) \times x + (z \times x) \times y = 0$

(identitatea Jacobi)

<u>Def</u> (produsul mixt)

$(\mathbb{R}^3, g_0)$, $\{x, y, z\} \subset \mathbb{R}^3$

$$z \wedge x \wedge y = \langle z, x \times y\rangle = \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & \cancel{z_2}\, x_2 & \cancel{z_3}\, x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \langle x, y \times z\rangle = x \wedge y \wedge z$$

<u>Ex</u> $(\mathbb{R}^3, g_0)$, $u = (1, -1, 2)$, $v = (0, 1, 3)$, $w = (1, 1, 0)$

a) $u \times v$; b) $w \wedge u \wedge v$

<u>SOL</u> a) $u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{vmatrix} =$

$\operatorname{rg}\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 3 \end{pmatrix} = 2$

$\{u, v\}$ SLI

$$= e_1\begin{vmatrix} -1 & 2 \\ 1 & 3\end{vmatrix} - e_2\begin{vmatrix} 1 & 2 \\ 0 & 3\end{vmatrix} + e_3\begin{vmatrix} 1 & -1 \\ 0 & 1\end{vmatrix} = (-5, -3, 1)$$

b) $w \wedge u \wedge v = \langle w, u \times v) = -5-3+0 = -8$

sau $w \wedge u \wedge v = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{vmatrix}$

OBS $R = \{u, v, u\times v\}$ reper poz. orientat în $\mathbb{R}^3$

$R_0 \xrightarrow{A} R = \{u = (1,-1,2), v = (0,1,3), \underset{}{u\times v} = (-5,-3,1)\}$

$A = \begin{pmatrix} 1 & 0 & -5 \\ -1 & 1 & -3 \\ 2 & 3 & 1 \end{pmatrix}$ $\det A > 0$.

Problemă

$R \longrightarrow R' \longrightarrow R''$

reper $\forall$ reper ortogonal reper ortonormat

Teoremă (procedeul Gram-Schmidt)

Fie $(E, \langle\cdot,\cdot\rangle)$ s.v.e.r

Fie $R = \{f_1, .., f_n\}$ reper arbitrar în E.

$\Rightarrow \exists R' = \{e_1, .., e_n\}$ reper ortogonal, a.î. $Sp\{f_1, .., f_i\} = Sp\{e_1, .., e_i\}$ $\forall i = \overline{1,n}$

Dem Dem. este inductivă.

$f_1 \neq 0$; $e_1 = f_1$

Construim $e_2 = f_2 + \alpha_{21} e_1$

$\langle e_2, e_1\rangle = 0 \Rightarrow 0 = \langle f_2, e_1\rangle + \alpha_{21} \overbrace{\langle e_1, e_1\rangle}^{\|f_1\|^2}$

$\alpha_{21} = -\dfrac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle}$.

$e_2 = f_2 - \dfrac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} \cdot e_1$

$\Rightarrow \begin{cases} f_1 = e_1 \\ f_2 = \dfrac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1 + e_2 \end{cases} \Rightarrow Sp\{f_1, f_2\} = Sp\{e_1, e_2\}$.

Pp. adev P_{k-1}. Dem P_k.

$\{e_1, \dots, e_{k-1}\}$ vectori mutual ortogonali și

$Sp\{e_1, \dots, e_i\} = Sp\{f_1, \dots, f_i\}$, $i = \overline{1, k-1}$

Construim $e_k = f_k + \sum_{i=1}^{k-1} \alpha_{ki} e_i$

$0 = \langle e_k, e_j \rangle = \langle f_k + \sum_{i=1}^{k-1} \alpha_{ki} e_i, e_j \rangle =$

$\forall j = \overline{1, k-1}$

$= \langle f_k, e_j \rangle + \sum_{i=1}^{k-1} \alpha_{ki} \langle e_i, e_j \rangle$

$0 = \langle f_k, e_j \rangle + \alpha_{kj} \langle e_j, e_j \rangle \Rightarrow \alpha_{kj} = - \frac{\langle f_k, e_j \rangle}{\langle e_j, e_j \rangle}$

$\forall j = \overline{1, k-1}$

$$e_k = f_k - \sum_{i=1}^{k-1} \frac{\langle f_k, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$

$$\begin{cases} f_1 = e_1 \\ f_2 = \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + e_2 \\ \vdots \\ f_k = \frac{\langle f_k, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \dots + \frac{\langle f_k, e_{k-1} \rangle}{\langle e_{k-1}, e_{k-1} \rangle} e_{k-1} + e_k \end{cases} \Rightarrow$$

$Sp\{f_1, \dots, f_i\} = Sp\{e_1, \dots, e_i\}$, $\forall i = \overline{1, k}$

R' vect. mutual $\perp$ $\xRightarrow{\text{PROP}}$ R' este SLI $\Rightarrow$ reper.

Am construit

$$R = \{f_1, \dots, f_n\} \underset{A}{\overset{A^{-1}}{\rightleftarrows}} R' = \{e_1, \dots, e_n\}$$

$$\begin{cases} f_1 = e_1 \\ f_2 = \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + e_2 \\ \vdots \\ f_n = \frac{\langle f_n, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \dots + \frac{\langle f_n, e_{n-1} \rangle}{\langle e_{n-1}, e_{n-1} \rangle} e_{n-1} + e_n \end{cases}$$

$$A = \begin{pmatrix} 1 & \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} & \dots & \frac{\langle f_n, e_1 \rangle}{\langle e_1, e_1 \rangle} \\ 0 & 1 & & \vdots \\ \vdots & \vdots & & \frac{\langle f_n, e_{n-1} \rangle}{\langle e_{n-1}, e_{n-1} \rangle} \\ 0 & 0 & & 1 \end{pmatrix}$$

$\det A = 1 > 0$.

OBS reper $\forall$ — reper ortogonal — reper ortonormat

$$\mathcal{R}=\{f_1,\dots,f_n\} \xrightarrow{A^{-1}} \mathcal{R}'=\{e_1,\dots,e_n\} \xrightarrow{B} \mathcal{R}''=\left\{\frac{e_1}{\|e_1\|},\dots,\frac{e_n}{\|e_n\|}\right\}$$

$$\det(A^{-1}) = \frac{1}{\det A} = 1 \qquad B=\begin{pmatrix}\frac{1}{\|e_1\|} & & 0\\ & \ddots & \\ 0 & & \frac{1}{\|e_n\|}\end{pmatrix}$$

$\mathcal{R}, \mathcal{R}', \mathcal{R}''$ sunt la fel orientate.

$$\det B = \frac{1}{\|e_1\|\cdots\|e_n\|} > 0$$

OBS $\left\|\frac{v}{\|v\|}\right\| = \frac{1}{\|v\|}\cdot\|v\| = 1$

Def $(E, \langle\cdot,\cdot\rangle)$ s.v.e.r.

a) $x\in E$, $\{x\}^\perp = \{y\in E \mid \langle x,y\rangle = 0\} \subset E$ subsp.v.

b) $\underset{\text{ssp.v}}{U\subseteq E} \Rightarrow U^\perp = \{y\in E \mid \langle x,y\rangle=0,\ \forall x\in U\} \subset E$ ssp.v.

Ex. $(\mathbb{R}^3, g_0)$, $u=(1,2,-1)$

a) $u^\perp = ?$ b) Det. un reper ortonormat în $u^\perp$

Sol

a) $u^\perp = \{x\in\mathbb{R}^3 \mid \underset{x_1+2x_2-x_3}{g_0(x,u)} = 0\} = \{(x_1,x_2,x_1+2x_2) \mid x_1,x_2\in\mathbb{R}\}$

$x_3 = x_1+2x_2$ $\qquad (x_1,x_2,x_1+2x_2) = x_1(1,0,1)+x_2(0,1,2)$

$\dim u^\perp = 2$ $\qquad \mathcal{R}=\{f_1=(1,0,1),\ f_2=(0,1,2)\}$ SG

$\Rightarrow \mathcal{R}$ reper $\forall$ în $u^\perp$. Aplicăm Gram-Schmidt

$e_1 = f_1 = (1,0,1)$

$e_2 = f_2 - \frac{\langle f_2,e_1\rangle}{\langle e_1,e_1\rangle}\cdot e_1 = (0,1,2) - \frac{\cancel{2}}{\cancel{2}}(1,0,1) = (-1,1,1)$

$$\mathcal{R}=\{f_1,f_2\} \longrightarrow \mathcal{R}'=\{e_1,e_2\} \longrightarrow \mathcal{R}''=\left\{\frac{1}{\sqrt{2}}e_1,\ \frac{1}{\sqrt{3}}e_2\right\}$$

$\|x\| = \sqrt{g_0(x,x)} = \sqrt{x_1^2+x_2^2+x_3^2}$

$\left\{\frac{1}{\sqrt{2}}(1,0,1),\ \frac{1}{\sqrt{3}}(-1,1,1)\right\}$ reper ortonormat în $u^\perp$

Prop $(E, \langle \cdot, \cdot \rangle)$ s.v.e.r, $U \underset{ssp.v.}{\subset} E$

$\Rightarrow E = U \oplus U^{\perp}$ (scrierea este unică)

$U^{\perp}$ complement ortogonal (este unic)

Dem

$U + U^{\perp} \subseteq E$ (din constr.)

Fie $x \in U \cap U^{\perp} \Rightarrow \underbrace{\langle x, x \rangle}_{\|x\|^2} = 0 \Rightarrow x = 0$ $\Big| \Rightarrow U \oplus U^{\perp} \subseteq E$ (1)

Fie $v \in E$; $\dim U = k$.

Fie $R_0 = \{e_1, \dots, e_k\}$ reper ortonormat în U

- $v' = v - \sum_{j=1}^{k} \langle v, e_j \rangle e_j \Rightarrow$

$v = v' + \underbrace{\sum_{j=1}^{k} \langle v, e_j \rangle e_j}_{v''}$. Dem că $v' \in U^{\perp}$

Arătăm că $v'' \in U$

$\langle v', e_i \rangle = 0, \forall i = \overline{1,k}$

$\langle v', e_i \rangle = \langle v, e_i \rangle - \sum_{j=1}^{k} \langle v, e_j \rangle \underbrace{\langle e_j, e_i \rangle}_{\delta_{ij}}$

$= \langle v, e_i \rangle - \langle v, e_i \rangle = 0, \quad \forall i = \overline{1,k}$

Fie $x \in U$, $x = \sum_{i=1}^{k} x_i e_i$

$\langle v', x \rangle = \sum_{i=1}^{k} x_i \langle v', e_i \rangle = 0 \Rightarrow v' \in U^{\perp}$

$v = \underset{U^{\perp}}{v'} + \underset{U}{v''} \Rightarrow E \subseteq U \oplus U^{\perp}$ (2)

(1), (2) $\Rightarrow$ avem $E = U \oplus U^{\perp}$

Ex Fie $(\mathbb{R}^4, g_0)$, $U = \left\{ x \in \mathbb{R}^4 \;\middle|\; \begin{cases} x_1 - x_2 + x_3 = 0 \\ x_1 + x_2 - x_4 = 0 \end{cases} \right\}$

a) $U^{\perp}$; b) $R = R_1 \cup R_2$ reper ortonormat în $\mathbb{R}^4$

R_1, resp R_2 reper ortonormat în U, resp $U^{\perp}$

SOL.

$$A = \left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \end{array}\right) \quad rg\, A = 2$$

$$\dim U = 4-2 = 2$$

$$\begin{cases} x_3 = -x_1 + x_2 \\ x_4 = x_1 + x_2 \end{cases}$$

$$U = \{ (x_1, x_2, -x_1+x_2, x_1+x_2) \mid x_1, x_2 \in \mathbb{R} \}$$

$$= x_1 \underbrace{(1,0,-1,1)}_{f_1} + x_2 \underbrace{(0,1,1,1)}_{f_2}$$

$\{f_1, f_2\}$ SG, $\dim U = 2$ $\Rightarrow$ $\{f_1, f_2\}$ reper în U

$\langle f_1, f_2 \rangle = 0+0-1+1 \Rightarrow \{f_1, f_2\}$ reper ortogonal.

$\mathcal{R}_1 = \{ \frac{1}{\sqrt{3}}(1,0,-1,1), \frac{1}{\sqrt{3}}(0,1,1,1) \}$ reper ortonormat în U

$$U^\perp = \{ x \in \mathbb{R}^4 \mid g_0(x,y) = 0, \forall y \in U \}$$

$$= \left\{ x \in \mathbb{R}^4 \;\middle|\; \begin{cases} g_0(x, f_1) = 0 \\ g_0(x, f_2) = 0 \end{cases} \right\}$$

$$= \left\{ x \in \mathbb{R}^4 \;\middle|\; \begin{cases} x_1 - x_3 + x_4 = 0 \\ x_2 + x_3 + x_4 = 0 \end{cases} \right\} \quad A' = \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$x_1 = +x_3 - x_4$$
$$x_2 = -x_3 - x_4$$

$$U^\perp = \{ (x_3 - x_4, -x_3 - x_4, x_3, x_4) \mid x_3, x_4 \in \mathbb{R} \}$$

$$= x_3 \underbrace{(1,-1,1,0)}_{f_3} + x_4 \underbrace{(-1,-1,0,1)}_{f_4}$$

$\langle f_3, f_4 \rangle = 0 \Rightarrow \{f_3, f_4\}$ SLI, dar $\{f_3, f_3\}$ SL

$\{f_3, f_4\}$ este reper ortogonal în $U^\perp$

$\mathcal{R}_2 = \{ \frac{1}{\sqrt{3}} f_3, \frac{1}{\sqrt{3}} f_4 \}$ reper ortonormat în $U^\perp$

$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ reper orton. în $\mathbb{R}^4$

Transformări ortogonale.

Def $(E_i, \langle \cdot,\cdot \rangle_i)$ $i=\overline{1,2}$ sp. v. e. r. Aplicația liniară $f: E_1 \to E_2$ s.n. <u>aplicație ortogonală</u> $\Leftrightarrow$

$$(*) \quad \langle f(x), f(y) \rangle_2 = \langle x, y \rangle_1, \ \forall x, y \in E_1$$

Prop Dacă $f: E_1 \to E_2$ apl. ortogonală $\Rightarrow$

1) $\| f(x) \|_2 = \|x\|_1, \ \forall x \in E_1$

2) f injectivă

Dem

1) $(*)$ $y = x \Rightarrow \langle f(x), f(x) \rangle_2 = \langle x, x \rangle_1 \Rightarrow \| f(x) \|_2^2 = \|x\|$

$\Rightarrow \| f(x) \|_2 = \|x\|$.

2) f inj $\Leftrightarrow \mathrm{Ker} f = \{0_{E_1}\}$.

Fie $x \in \mathrm{Ker} f \Rightarrow f(x) = 0_{E_2} \Rightarrow \underbrace{\| f(x) \|_2}_{\|x\|_1} = \underbrace{\| 0_{E_2} \|}_{0}$

$\Rightarrow x = 0_{E_1} \Rightarrow f$ inj

Def $(E, \langle \cdot,\cdot \rangle)$ sp v e r, $f \in \mathrm{End}(E)$

f s.n. <u>transformare ortogonală</u> $\Leftrightarrow \langle f(x), f(y) \rangle = \langle x, y \rangle$ $\forall x, y \in E$.

Not $O(E) = \{ f: E \to E \mid f \text{ transf. ortog.} \}$.

Prop Fie $f \in \mathrm{End}(E)$

$f \in O(E) \Leftrightarrow \| f(x) \| = \|x\|, \ \forall x \in E$

Dem

"$\Rightarrow$" cf. prop anterioară

"$\Leftarrow$" $f \in \mathrm{End}(E)$ a.î. $\| f(x) \| = \|x\|, \ \forall x \in E \Rightarrow f \in O(E)$

$\| f(x+y) \|^2 = \| x + y \|^2$

$\langle f(x) + f(y), f(x) + f(y) \rangle = \langle x + y, x + y \rangle$

$\cancel{\| f(x) \|^2} + \cancel{\| f(y) \|^2} + 2 \langle f(x), f(y) \rangle = \cancel{\|x\|^2} + \cancel{\|y\|^2} + 2 \langle x, y \rangle$

$\Rightarrow \langle f(x), f(y) \rangle = \langle x, y \rangle$

OBS $f \in O(E)$, $\mathcal{R} = \{e_1, .., e_n\}$ reper ortonormat în E

$A = [f]_{\mathcal{R},\mathcal{R}}$ $\quad f(e_i) = \sum_{k=1}^{n} a_{ki} e_k$, $\forall i = \overline{1,n}$

$< f(e_i), f(e_j) > = < e_i, e_j > = \delta_{ij}$

$\shortparallel$

$< \sum_{k=1}^{n} a_{ki} e_k, \sum_{r=1}^{n} a_{rj} e_r > = \sum_{k,r=1}^{n} a_{ki} a_{rj} \overbrace{< e_k, e_r >}^{\delta_{kr}}$

$\sum_{k=1}^{n} a_{ki} a_{kj} = \delta_{ij} \Rightarrow A^T A = I_n \Rightarrow A \in O(n)$

$\mathcal{R} = \{e_1, .., e_n\} \xrightarrow{C} \mathcal{R}' = \{e_1', .., e_n'\}$ repere ortonormate
$C \in O(n)$

$A' = [f]_{\mathcal{R}',\mathcal{R}'}$

$A' = C^{-1} A C = C^T A C$

$A'^T \cdot A' = (C^T A C)^T (C^T A C) = C^T A^T \underbrace{C C^T}_{I_n} A C = I_n$

Prop $f \in End(E)$

$f \in O(E) \Leftrightarrow$ matricea asociat, în rap cu $\forall$ reper, este ortog.

• OBS $f \in O(E) \Leftrightarrow$ o schimbare de repere ortonormate

$\mathcal{R} = \{e_1, .., e_n\}$ ortonormat $\xrightarrow{A} \mathcal{R}' = \{e_1', .., e_n'\}$ reper ortonormat

$A = [f]_{\mathcal{R},\mathcal{R}} \in O(n)$

$f(e_i) = e_i' = \sum_{k=1}^{n} a_{ki} e_k$, $\forall i = \overline{1,n}$

$f(x) = f(\sum_{i=1}^{n} x_i e_i) = \sum_{i=1}^{n} x_i f(e_i) = \sum_{i=1}^{n} x_i e_i' = x'$

Prop $f \in O(E) \Rightarrow$ valorile proprii $\in \{-1, 1\}$.

Dem Fie λ = val. proprie $\Rightarrow \exists \underset{\neq 0_E}{x} \in E$ aî $f(x) = \lambda x$

$\langle \underset{\lambda x}{f(x)}, \underset{\lambda x}{f(x)} \rangle = \langle x, x \rangle \Rightarrow \lambda^2 \|x\|^2 = \|x\|^2 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = .$
$\lambda \in \mathbb{R}$

<u>Prop</u> $f \in O(E)$, $U \subseteq E$ subsp. invariant al lui f

i.e $f(U) \subseteq U$.

1) $f(U) = U$

2) $U^{\perp} \subseteq E$ subsp. invar al lui f

3) $f/_{U^{\perp}} : U^{\perp} \longrightarrow U^{\perp}$ transf. ortog.

<u>Dem</u>

1) $f : U \longrightarrow f(U)$ bij + lin $\Rightarrow$ izom. de sp vect.

$$\left.\begin{array}{l} \dim U = \dim f(U) \\ f(U) \subseteq U \end{array}\right\} \Rightarrow f(U) = U.$$

2) Dem că $f(U^{\perp}) \overset{?}{\subseteq} U^{\perp}$ $\quad \underset{\Downarrow}{U^{\perp}} \ \underset{\Downarrow}{U}$

Fie $x \in U^{\perp} \Rightarrow f(x) \in U^{\perp}$

$\langle f(x), \underset{U}{\overset{\in}{z}} \rangle = \langle f(x), f(y) \rangle = \langle x, y \rangle = 0 \Rightarrow$

$z \in U = f(U) \Rightarrow \exists y \in U$ aî $z = f(y)$

$f(x) \in U^{\perp}$

Deci $U^{\perp} \subseteq E$ ssp invariant $\overset{1)}{\Rightarrow} f(U^{\perp}) = U^{\perp}$

3) $f/_{U^{\perp}} : U^{\perp} \longrightarrow U^{\perp}$ transf. ortog.

<u>Clasificare transf. ortogonale</u>

① $n = 1$ $\quad O(E) = \{ id_E, -id_E \}$.

② $n = 2$. $\quad f \in O(E)$, $\underset{O(2)}{A} = [f]_{R,R}$ $\quad R = \{e_1, e_2\}$ reper ortonormat

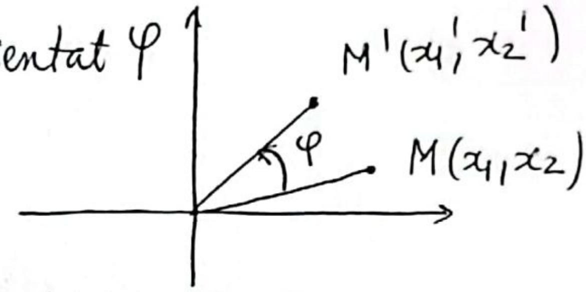
a) $\det A = 1$ $\quad A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$, $\varphi \in (-\pi, \pi]$

f este de speța 1

$$A=\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$f:\mathbb{R}^2\to\mathbb{R}^2,\ f(x_1,x_2)=(\overbrace{x_1\cos\varphi-x_2\sin\varphi}^{x_1'},\ \overbrace{x_1\sin\varphi+x_2\cos\varphi}^{x_2'})$$

f rotație în plan de $\measuredangle$ orientat φ



$\operatorname{Tr} A = 2\cos\varphi$ invariant la schimbarea reperului

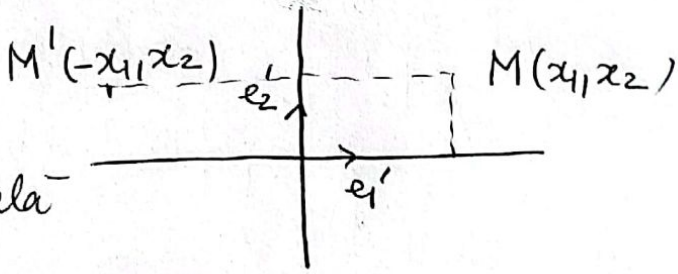
b) $\det A=-1$ i.e. $f\in O(E)$ de speță 2

$\exists$ un reper $R'=\{e_1', e_2'\}$ a.î. $[f]_{R',R'}=\left(\begin{array}{c|c}-1 & 0\\ 0 & 1\end{array}\right)$

<u>OBS</u> $\det A=-1$ $\quad A=\begin{pmatrix}\cos\varphi & \sin\varphi\\ \sin\varphi & -\cos\varphi\end{pmatrix}$

$$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$f:\mathbb{R}^2\to\mathbb{R}^2,\ f(x_1,x_2)=(-x_1,x_2)$$



f este simetria ortogonală față de $\langle e_1'\rangle^{\perp}$

$f(e_1')=-e_1'$, $f(e_2')=e_2'$ $\quad\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$

<u>Teoremă</u> $n=2$ $\quad f\in O(E)$, $f\neq id_E$

$\Rightarrow f$ se poate scrie ca o compunere de 2 simetrii ortogonale față de drepte

<u>Dem</u>

1) f de speță 1 $\Rightarrow \det(A_f)=1$

Fie s simetrie ortog $\Rightarrow \det(A_s)=-1$.

$s'=s\circ f$ de speța 2 $\Rightarrow s\circ s'=f$

$s\circ s=id$

2) f de speță 2

$f=s$

sim. ortog.