\subsection{Mass Conservation}

Mass conservation (MC) stands as a cornerstone principle for modeling glacier and ice sheet dynamics. It operates on the fundamental physical concept of invariant total flux as ice traverses from accumulation zones through ablation areas to the glacier terminus. MC specifically addresses the continuity of ice volume throughout the glacial system. This approach ensures that the quantitative relationship between accumulation, flow, and ablation remains physically consistent across the entire glacial domain. The MC technique is particularly valuable for constraining models and filling data gaps in glaciological studies, and has proven effective for reconstructing bed topography in regions where direct measurements are sparse~\cite{Morlighem_2017, Morlighem_2020}.

For proper implementation, MC requires some contemporary measurements of ice thickness at the inflow boundary to adequately constrain the system~\cite{Morlighem_Goldberg_2023}. This constraint provides the necessary initial conditions for the model. Once established, the depth integration of the continuity equation, along with boundary conditions at the ice base and surface, lead to a mass transport equation~\cite{Morlighem_Goldberg_2023}. The equation can then be solved to provide insights into ice flux throughout the glacier system.

A key example of MC usage is Thickness Estimation by a Lagrangian Validated Interpolation Scheme (TELVIS), a sophisticated, streamline interpolation method)~\cite{TELVIS_2011}. TELVIS leverages our understanding of ice dynamics and MC to produce a more robust and accurate interpolation of sparse ice thickness data in East Antarctica's Aurora Subglacial Basin (ASB). Building up on up on the work of Warner and Budd (2000) who developed an ice-thickness estimation model based on MC, Glen's flow law, the shallow-ice approximation (SIA) and the relationships between ice thickness, ice flux and surface elevation gradient~\cite{Warner_Budd_2000}.

TELVIS uses the magnitude of the ice flux to estimate the ice thickness. The flow is assumed to be in the direction of steepest descent of the surface topography~\cite{TELVIS_2011}. The method employs a Lagrangian balance flux code that assumes the ice sheet is in local equilibrium so that the accumulation rate and the downstream advection are in balance~\cite{TELVIS_2011}. To allow for local variations of ice flow conditions, including possible departures from the shallow-ice approximation, TELVIS introduces a locally tunable parameter $\kappa$ which assimilates local ice thickness observations. The mass-conserved ice fluxes allow the method to infer ice thickness in data-sparse regions by considering how much ice is likely flowing through a particular location based on accumulation and surface slope. The refined morphology revealed by TELVIS shows a more extensive marine-based ice sheet with deeper connections to the interior than previously known~\cite{TELVIS_2011}, highlighting the potential susceptibility of the ASB to ocean warming. In general, it underscores the importance of developing ice sheet modeling based on detailed marginal and basal boundary conditions for this region.

Another important implementation of MC is exemplified by BedMachine Antarctica a high-resolution bed topography map that incorporates ice thickness data, gravity-derived bathymetry, satellite-derived ice velocity, and surface mass balance~\cite{Morlighem_2020}. This product reveals previously unknown basal features, including stabilising ridges and retrograde slopes, which have major implications for assessing glacier vulnerability~\cite{Morlighem_2020}. BedMachine offers a detailed and physically consistent depiction of the continent's bed, redefining high- and lower-risk sectors for rapid ice loss~\cite{Morlighem_2020}.

A high spatial-resolution mesh is used to infer ice thickness by combining sparse ice thickness data from radar sounding profilers with dense, high spatial-resolution ice velocity data~\cite{Morlighem_2011}. Morlighem et al. (2011) describe the ``balance thickness'' calculation for the glacier model in BedMachine as a steady hyperbolic partial differential equation of first order. This equation states that the ice flux divergence is balanced by the apparent mass balance, which is the difference between surface mass balance and basal melting rate. At inflow boundaries, ice thickness must be constrained to match observed values, which provides a necessary boundary condition for each flow line. The equation is solved using a stream-line upwinding finite element method, this is appropriate for hyperbolic equations that can be difficult to solve numerically~\cite{Morlighem_2011}. The approach also used an objective function that minimizes the misfit between observed and modeled ice thickness along flight tracks, reducing errors in the apparent mass balance and depth averaged velocity. A projected gradient descent algorithm minimizes their objective function while keeping the depth-averaged velocity within certain tolerances and the apparent mass balance within error margins. They stop optimization when the cost function reaches a value comparable to the error in ice thickness measurements~\cite{Morlighem_2011}.

MC techniques provide physically consistent approaches to mapping topographical features. The implementations discussed here: TELVIS and BedMachine Antarctica demonstrate that MC principles enable researchers to transform sparse observational data into improved high-resolution bed topography maps.

**\item linear perturbation theory (e.g., Ockenden et al. (2022)) % Ana**

Ice bed features often show up as subtle patterns on the ice surface topography above them. Bed conditions such as geology, hydrology and sediment distribution play an important role in controlling ice flow and behavior; these conditions are often combined into a parameter known as `slipperiness'~\cite{Ockenden_2022}. Inversion methods are used to fill in the gaps in data by attempting to reconstruct bed properties such as topography, basal slipperiness, and roughness from what can be observed on the surface. The inversion method developed by Ockenden et al (2022) builds on the work by Gudmundsson in~\cite{Gudmundsson_2008} and its based on a steady-state linear perturbation analysis of the shallow-ice-stream equations (SSA). The method derives transfer functions that capture the relationship between time-invariant Fourier transforms of bed topography, slipperiness, with surface topography and horizontal velocity components. By introducing perturbations on this linear model Ockenden et al. are able to reveal how small changes in the model parameters affect ice flow. The assumptions in this work are a linear viscous medium, non-linear sliding law, steady-state conditions, and spatially constant zero-order solutions. The equations are solved analytically and minimised using a weighted least-squares approach. The derived transfer functions describe how perturbations in basal topography and slipperiness affect surface topography and velocity. These relationships are expressed as a series of Fourier transform ($\hat{}$) equations ($\hat{s} = T_{sb}\hat{b} +

T_{sc}\hat{c}$, $\hat{u} = T_{ub}\hat{b} + T_{uc}\hat{c}$, $\hat{v} = T_{vb}\hat{b} + T_{vc}\hat{c}$) which form an over-determined system. A weighted least-squares solution minimizes the differences between observed and predicted values, expressed as $\Sigma_s(s_{\mathrm{obs}} - s_{\mathrm{pred}})^2 + \Sigma_u(u_{\mathrm{obs}} - u_{\mathrm{pred}})^2 + \Sigma_v(v_{\mathrm{obs}} - v_{\mathrm{pred}})^2$, resulting in a matrix solution that becomes problematic when features are aligned with ice flow direction or at small wavelengths. The method is able to generate a bed topography and slipperiness and works by identifying disturbances to surface flow caused by obstacles or sticky patches in the ice bed, this method is applicable wherever the shallow-ice-stream equations hold and surface data are available, even where ice thickness is not well known a priori. Ockenden et al use a combination of high resolution satellite datasets calibrated by sparser field datasets such as REMA surface digital elevation model (DEM) at $\approx$8 m resolution and NASA ITS_LIVE velocity at $\approx$120 m resolution. Ockenden et al. assess the performance of the model by comparing the topography output from the inversion to radar grids and flight lines over Thwaites Glacier in West Antarctica, with the method performing particularly well in areas with moderate topographic gradients in the central trunk of glaciers and medium-wavelength bedrock features~\cite{Ockenden_2022}, though it faces limitations in cases of steep topography where the SSA breaks down, and it lacks validation data for slipperiness predictions.

**\item EnKF % Ana**

The Ensemble Kalman Filter (EnKF) is a data assimilation technique used to estimate the state of a system when you have both: 1. A model of how the system evolves over time and 2. Observations of the system that might be incomplete or contain errors. The EnKF method combines these two factors into an ensemble of multiple paths each representing plausible system states that get updated whenever new observations become available.  At each time step, the analysis uses the observations and their uncertainty to produce a mean system state that is closer to the observations and with a lower uncertainty, this process is repeated until the end of the data assimilation window is finished\cite{Gillet_Chaulet_2020, Choi_2025}. Traditional methods require calculating large covariance matrices or using linearised models, which can be computationally expensive; in comparison, the EnKF avoids this by representing multiple possible uncertainty states in the model with each member in the ensemble. The ensemble members are each run forward in time using the model. When new observations are obtained, each ensemble member is updated using a version of the Kalman filter equations. This creates a discontinuous trajectory that's informed by past and present observations. Nevertheless, it is possible to extend it to "smoother" versions that consider future observations too. This method is characterised by its efficiency, as it can be parallelised, and works with any forward model without modification, which is why several open-source implementations exist~\cite{Gillet_Chaulet_2020}.
Important limitations of EnKF include undersampling of the systems analysed, but this is often solved by methods such as localisation or inflation~\cite{Gillet_Chaulet_2020}. EnFK is a method optimal only for Gaussian distributions and linear models but have been effective for assimilating diverse observations into complex, large-scale and non-linear models~\cite{Gillet_Chaulet_2020, Carrassi_2018}.

An effective application of EnKF is demonstrated in Gillet-Chaulet's 2020 work, which analyzes its performance in tracking the transient evolution of surface and velocity in a one-dimensional marine ice sheet model with grounding line (GL) migration. This simplified study revealed that surface observations alone were sufficient to track GL movement during the study period, eliminating the need for direct GL position measurements. The research further shows that extending the data assimilation window until the glacier clearly enters unstable retreat significantly narrows the range of simulation predictions. Before this approach can be widely implemented in ice sheet modeling, however,  Gillet-Chaulet notes the need for improved methods to assess uncertainties in both the models and the observations used for data assimilation.

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