Honours Literature review

$\mathcal{PT}$-symmetric quantum mechanics

Ana Fabela Hinojosa [1]

Supervisors:
Dr. Jesper Levinsen
Prof. Meera Parish
School of Physics & Astronomy

August 2022

[1]acfab1@student.monash.edu.au

Contents

Abstract

Introduction

This work reflects on a possible extension to the canonical formalism of quantum mechanics. The main goal of this extension is to allow us to access a much larger class of interesting Hamiltonians which are non-Hermitian but nevertheless, physical and therefore possibly of great importance to the advancement of new and physically significant theories.

In "The principles of quantum mechanics", Paul Dirac advises us that"it is important to remember that science is concerned only with *observable* things and that we can observe an object only by letting it interact with some outside influence"[1]. This means that in order to study valid physical systems, these systems must satisfy the fundamental postulates of quantum mechanical theory. Nearly all these postulates are based in physical properties. For example, one postulate establishes that time evolution of a quantum system must be *unitary* (i.e. probability conserving), another requires that the energy spectrum of the system is bounded below so a lowest energy state can be measured. In quantum mechanics, the Hamiltonian operator ($\hat{H}$) encapsulates the total energy of a system. As explained already, a system's energy spectrum is required to be bounded and real in order to be measurable. According to the fundamental axioms, if the operator $\hat{H}$ satisfies the mathematical property known as Hermiticity (these operators are also known as self-adjoint in mathematics), then $\hat{H}$ will be an adequate physical observable. Interestingly, the Hermiticity postulate stands out from others in the conventional theory of quantum mechanics because it is mathematical rather than physical in its character[2]. An operator $\hat{O}$ that is Hermitian has the property that its effect on the vectors of the Hilbert space in which $\hat{O}$ is defined is independent of the order in which $\hat{O}$ acts on said vectors[3]

$$\hat{O}\,|\psi\rangle = \langle\psi|\,\hat{O}^{\dagger} \tag{1.1}$$

Despite of the correctness of Hermiticity, since this property depends on the definition of the inner-product used, some believe that perhaps we have ended up with an overly restrictive quantum mechanical theory. The aim of this review is to summarize the present developments of a more physical alternative to Hermiticity. This postulate is referred to as space–time reflection symmetry ($\mathcal{PT}$ symmetry)[4].

1.1 $\mathcal{PT}$-symmetry

In the late nineties, Carl Bender et al presented a $\mathcal{PT}$-symmetric theory of quantum mechanics, their aim was to explain a conjecture on the reality and positiveness of the spectrum of a non-Hermitian Hamiltonian proposed by Bessis[5]. This new $\mathcal{PT}$-symmetric theory can be viewed as an analytical continuation of the conventional theory from real into the complex phase space[6].

The critical question that must be answered, is whether a $\mathcal{PT}$-symmetric Hamiltonian defines a valid physical theory of quantum mechanics. By a physical theory we mean that the energy spectra of a system described by $\hat{H}$ must be real and bounded below, and since the norm of a state is interpreted as a probability, this norm must be positive. Finally, we must show that the time evolution of the theory is unitary. This means that as a state vector evolves in time the state's probability does not leak away[4][2].

1.1.1 The $\mathcal{PT}$ operator

The $\mathcal{PT}$ operator is the anti-linear operator composed of the linear parity operator ($\mathcal{P}$), which performs spatial reflection, and the anti-linear time-reversal operator ($\mathcal{T}$). These operators act on position and momentum operators in the following form

$$\begin{aligned} &\mathcal{P}: \quad \hat{x} \to -\hat{x}, \quad \hat{p} \to -\hat{p}, \\ &\mathcal{T}: \quad \hat{x} \to \hat{x}, \qquad \hat{p} \to -\hat{p}, \quad \mathrm{i} \to -\mathrm{i}. \end{aligned} \tag{1.2}$$

Some Hamiltonians may not be symmetric under $\mathcal{P}$ or $\mathcal{T}$ separately, but Hamiltonians that remain invariant under the influence of the antilinear $\mathcal{PT}$ operator are labelled as $\mathcal{PT}$-symmetric. A Hamiltonian will possess unbroken $\mathcal{PT}$ symmetry if it's eigenfunctions $\psi_n(x)$ are simultaneously eigenstates of the $\mathcal{PT}$ operator, otherwise we say that the symmetry is broken[2][7][4]. If the symmetry is unbroken, then the eigenspectrum of $\hat{H}$ is fully real and bounded below. The effectiveness of $\mathcal{PT}$ symmetry as a tool to investigate the spectra of some non-Hermitian Hamiltonians has been proved rigorously in various works, such as Dorey et al[8], Bender and Boettcher[5], Brody[9], Bender and Mannheim [10], Bender et al[6], Mostafazadeh[11][12].

1.1.2 The $\mathcal{CPT}$ inner product

To be able to describe precisely the nature of $\mathcal{PT}$-symmetric quantum mechanics, we must delve briefly into the inner-product under which our theory satisfies the axioms of conventional quantum mechanics. It is important to note that $\mathcal{PT}$-symmetric quantum mechanics is a kind of 'bootstrap' theory[2], since infinitely many inner-products exist for a given vector space, we can construct an inner product whose associated norm is positive definite, this inner-product is in general dependent on the characteristics of the Hamiltonian in question and it guarantees that the underlying dynamics of any $\mathcal{PT}$-symmetric Hamiltonian satisfies unitarity[4]. Firstly, it is necessary to solve for the eigenstates of the Hamiltonian before knowing the Hilbert space and the associated inner product. To guarantee a positive norm for our theory, we will construct a new linear operator $\mathcal{C}$that commutes with both the Hamiltonian ($\hat{H}$) and $\mathcal{PT}$, we use the symbol $\mathcal{C}$ to represent this symmetry because it's properties are similar to those of the charge conjugation operator in particle physics[2].

$$\langle\psi|\chi\rangle^{\mathcal{CPT}} = \int dx\, \psi^{\mathcal{CPT}}(x)\chi(x) \tag{1.3}$$

Time evolution

Consequences and applications

Conclusion

Bibliography

[1] P. A. M. Dirac. *THE PRINCIPLES OF QUANTUM MECHANICS*. International series of monographs on physics (Oxford, England). Clarendon Pr., Oxf., 4th ed. edition (1958). 1

[2] C. M. Bender. *Making sense of non-Hermitian Hamiltonians*. Reports on Progress in Physics **70**, 947 (2007). DOI: 10.1088/0034-4885/70/6/r03. 1, 2

[3] K. Jones-Smith. *Non-Hermitian Quantum Mechanics*. PhD thesis, Case Western Reserve University (2010). 1

[4] C. M. Bender, D. C. Brody, and H. F. Jones. *Must a Hamiltonian be Hermitian?* American Journal of Physics **71**, 1095 (2003). DOI: 10.1119/1.1574043. 1, 2

[5] C. M. Bender and S. Boettcher. *Real Spectra in Non-Hermitian Hamiltonians Having PT-Symmetry*. Phys. Rev. Lett. **80**, 5243 (1998). DOI: 10.1103/PhysRevLett.80.5243. 1, 2

[6] C. M. Bender, S. Boettcher, and P. N. Meisinger. *Pt-symmetric quantum mechanics*. Journal of Mathematical Physics **40**, 2201 (1999). DOI: 10.1063/1.532860. 1, 2

[7] C. M. Bender, D. C. Brody, and H. F. Jones. *Complex Extension of Quantum Mechanics*. Phys. Rev. Lett. **89**, 270401 (2002). DOI: 10.1103/PhysRevLett.89.270401. 2

[8] P. Dorey, C. Dunning, and R. Tateo. *Spectral equivalences, Bethe ansatz equations, and reality properties in PT-symmetric quantum mechanics*. Journal of Physics A: Mathematical and General **34**, 5679 (2001). DOI: 10.1088/0305-4470/34/28/305. 2

[9] D. C. Brody. *Consistency of PT-symmetric quantum mechanics*. Journal of Physics A: Mathematical and Theoretical **49**, 10LT03 (2016). DOI: 10.1088/1751-8113/49/10/10lt03. 2

[10] C. M. Bender and P. D. Mannheim. *PT-symmetry and necessary and sufficient conditions for the reality of energy eigenvalues*. Physics Letters A **374**, 1616 (2010). DOI: 10.1016/j.physleta.2010.02.032. 2

[11] A. Mostafazadeh. *Pseudo-Hermiticity versus PT symmetry: The necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian*. Journal of Mathematical Physics **43**, 205 (2002). DOI: 10.1063/1.1418246. 2

[12] A. Mostafazadeh. *Exact PT-symmetry is equivalent to Hermiticity*. Journal of Physics A: Mathematical and General **36**, 7081 (2003). DOI: 10.1088/0305-4470/36/25/312. 2