

\mathcal{PT} -symmetric quantum mechanics

HONOURS LITERATURE REVIEW

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11th November 2022

Abstract

IN LESS THAN 100 WORDS

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Introduction

In “The principles of quantum mechanics”, Paul Dirac advises us that “it is important to remember that science is concerned only with *observable* things and that we can observe an object only by letting it interact with some outside influence” [1]. This means that in order to study valid physical systems, these systems must satisfy the fundamental postulates of quantum mechanical theory. Nearly all these postulates are based in physical properties. For example, one postulate establishes that time evolution of a quantum system must be *unitary* (i.e. probability conserving). Another postulate requires that the energy spectrum of the system is bounded below so a lowest energy state can be measured. In quantum mechanics, the Hamiltonian operator (\hat{H}) encapsulates the total energy of a system. A system’s energy spectrum is required to be bounded and real in order to be measurable. According to the fundamental postulates of the standard theory, if an operator \hat{O} satisfies the mathematical property known as Hermiticity –these operators are also known as self-adjoint in mathematics– then \hat{O} has the property that its effect on the vectors of the Hilbert space in which \hat{O} is defined is independent of the order in which \hat{O} acts on said vectors[2]

$$\hat{O} |\psi\rangle = \langle\psi| \hat{O}^\dagger. \quad (1.1)$$

Despite of the correctness of Hermiticity, some believe that perhaps we have ended up with an overly restrictive quantum theory. Therefore the aim of this review is to summarize the present developments of a quantum theory that uses an alternative postulate to Hermiticity. This alternative postulate is referred to as space–time reflection symmetry (\mathcal{PT} symmetry)[3].

1.1 \mathcal{PT} -symmetry

In the late nineties, Carl Bender and colleagues presented a \mathcal{PT} -symmetric theory of quantum mechanics. Their aim was to explain a conjecture on the reality and positiveness of the spectrum of a non-Hermitian Hamiltonian proposed by Bessis[4], this Hamiltonian spectrum was surprising since it was widely thought that non-Hermitian Hamiltonians would not have physically valid energy spectra. In very general terms, \mathcal{PT} -symmetric theory can be viewed as an analytical continuation of the conventional theory from real into the complex phase space[5].

An important question that must be answered, is whether a \mathcal{PT} -symmetric Hamiltonian defines a valid physical theory of quantum mechanics. By a physical theory we mean that the following conditions must be satisfied; The energy spectra of a system described by \hat{H} must be real and bounded below. Another condition is related to the probabilistic interpretation of the norm of a state, a norm must be always positive to be

a valid probability. Finally, time evolution under the theory must be unitary. This means that as a state vector evolves in time the state's probability does not leak away[3][6].

1.1.1 The \mathcal{PT} operator

The \mathcal{PT} operator is the anti-linear operator composed of the linear parity operator (\mathcal{P}), which performs spatial reflection, and the anti-linear time-reversal operator (\mathcal{T}). These operators act on position and momentum operators in the following form

$$\begin{aligned}\mathcal{P}: \quad \hat{x} &\rightarrow -\hat{x}, & \hat{p} &\rightarrow -\hat{p}, \\ \mathcal{T}: \quad \hat{x} &\rightarrow \hat{x}, & \hat{p} &\rightarrow -\hat{p}, \quad i \rightarrow -i.\end{aligned}\tag{1.2}$$

Some Hamiltonians may not be symmetric under \mathcal{P} or \mathcal{T} separately, but Hamiltonians that remain invariant under the influence of the \mathcal{PT} operator are labelled as \mathcal{PT} -symmetric. A Hamiltonian \hat{H} will possess unbroken \mathcal{PT} symmetry if it's eigenstates are simultaneously eigenstates of the \mathcal{PT} operator, or in other words, if \hat{H} and the \mathcal{PT} operator commute. If the \mathcal{PT} operator and \hat{H} do not commute we say that the Hamiltonian's \mathcal{PT} -symmetry is broken[6][7][3]. If the symmetry is unbroken, then the eigenspectrum of \hat{H} is fully real and bounded below—This is also known as exact \mathcal{PT} -symmetry—. The effectiveness of \mathcal{PT} symmetry as a tool to investigate the spectra of some non-Hermitian Hamiltonians has been proved rigorously in various works, such as Dorey et al[8], Bender and Boettcher[4], Brody[9], Bender and Mannheim [10], Bender et al[5], Mostafazadeh[11][12] amongst others.

1.2 A new inner product

To ensure that the eigenstates $\{|\psi_n\rangle\}$ of a \mathcal{PT} -symmetric Hamiltonian \tilde{H} are orthogonal, it is necessary to specify an inner product. A possible choice for the inner product associated with \tilde{H} might be[5]

$$\langle\psi_n|\psi_m\rangle \equiv \int_c dx [\psi_n(x)]^{\mathcal{PT}} \psi_m(x) = \int_c dx [\psi_n(-x)]^* \psi_m(x)\tag{1.3}$$

where c is a complex contour that starts and ends within specific boundary conditions (see section ??). Unfortunately this obvious choice for the \mathcal{PT} inner product is not correct because it does not guarantee a positive definite norm.

1.2.1 The \mathcal{CPT} inner product

To be able to describe precisely the nature of \mathcal{PT} -symmetric quantum mechanics, we must delve briefly into the inner-product under which our theory satisfies the postulates of conventional quantum mechanics. It is important to note that \mathcal{PT} -symmetric quantum mechanics is a kind of ‘bootstrap’ theory[6], since infinitely many inner-products exist for a given vector space, we can construct an inner product whose associated norm is positive definite by design. This inner-product is in general dependent on the characteristics of the Hamiltonian in question and it guarantees that the underlying dynamics of any \mathcal{PT} -symmetric Hamiltonian satisfies unitarity[3]. Firstly, it is necessary to solve for the eigenstates of the Hamiltonian before knowing the Hilbert space and consequentially the associated inner product. To guarantee a positive norm for our theory, we will construct

a new linear operator \mathcal{C} that commutes with both \hat{H} and \mathcal{PT} . We use the symbol \mathcal{C} to represent this symmetry because its properties are similar to those of the charge conjugation operator in particle physics[6].

1.2.2 The \mathcal{C} operator

When the \mathcal{PT} -symmetry of \hat{H} is unbroken, then \hat{H} and \mathcal{PT} commute. This statement is equivalent to saying that the eigenfunctions $\phi_n(x)$ of \hat{H} are simultaneously eigenstates of the \mathcal{PT} operator[13]

$$\mathcal{PT}\phi_n(x) = \lambda_n\phi_n(x), \quad (1.4)$$

Where λ_n is a pure phase. Without loss of generality, for each n the phase can be absorbed into $\phi_n(x)$ and this makes the eigenvalue of the \mathcal{PT} operator unity[13]:

$$\mathcal{PT}\phi_n(x) = \phi_n^*(-x) = \phi_n(x). \quad (1.5)$$

There is strong numerical evidence of the completeness of the eigenfunctions $\phi_n(x)$ [7][13][14]. In the coordinate basis, the completeness statement reads:

$$\sum_n (-1)^n \phi_n(x) \phi_n(y) = \delta(x - y), \quad x, y \in \mathbb{R} \quad (1.6)$$

the unconventional $(-1)^n$ factor in 1.6 can be explained if we define the \mathcal{PT} inner product as

$$\begin{aligned} (f, g) &= \int dx [\mathcal{PT}f(x)] g(x) \\ &= \int dx f^*(-x) g(x). \end{aligned} \quad (1.7)$$

where the integral above follows a path in the complex plane. Under this definition the eigenstate norms alternate in sign depending on the value n . This means that the metric associated with the \mathcal{PT} inner product is indefinite[13][15]. In quantum theory, the norm of states is interpreted as a probability and this means that the indefinite metric described above presents a serious problem for the validity of \mathcal{PT} -symmetric quantum theory. The solution to this problem lies in finding an interpretation for the negative valued norms[5].

\mathcal{PT} -symmetry breaking

Most model \mathcal{PT} -symmetric Hamiltonians \tilde{H} in the literature admit tuneable parameters such that there will be regions in the parameter space where the eigenstates of \tilde{H} are not \mathcal{PT} -symmetric[14]. This means that a system described by \tilde{H} admits two phases, one for broken and another for unbroken \mathcal{PT} -symmetry. In the unbroken phase eigenstates of \tilde{H} become degenerate and this implies that they lose completeness[14]. The literature claims that for any theory with unbroken \mathcal{PT} -symmetry there exists a symmetry of the Hamiltonian that describes the negative and positive norm states. To describe this symmetry of \tilde{H} it is necessary to construct a linear operator denoted by \mathcal{C} [3][7][13]. When represented in position space \mathcal{C} is a sum over the energy eigenstates of \tilde{H} :

$$\mathcal{C} = \sum_n \phi_n(x) \phi_n(y) \quad (1.8)$$

From this definition, and from the relation $(\phi_m(y), \phi_n(y)) = \int dy \phi_m(y) \phi_n(y)$ we can verify that the eigenvalues of \mathcal{C} are ± 1

$$\begin{aligned} \mathcal{C}\phi_n(x) &= \int dy \mathcal{C} \phi_n(y) \\ &= \sum_m \phi_m(x) \int dy \phi_m(y) \phi_n(y) \\ &= (-1)^n \phi_n(x) \end{aligned} \quad (1.9)$$

The \mathcal{PT} norm signatures can therefore be interpreted as the “charge” of the states, while \mathcal{C} is the operator used to measure this charge[13].

The \mathcal{C} operator commutes with the \mathcal{PT} operator, but it does not commute with the parity operator \mathcal{P} . Notice that \mathcal{C} and \mathcal{P} operators are square roots of $\delta(x - y)$ the unity operator[7]

$$\mathcal{P}^2 = \mathcal{C}^2 = \mathbb{1}, \quad (1.10)$$

where $\mathcal{P} \neq \mathcal{C}$, since \mathcal{P} is real and \mathcal{C} is complex valued[3][13].

Using the newly constructed \mathcal{C} operator, we can redesign the \mathcal{PT} inner product to suit the conventional probabilistic interpretation of the vector norms in quantum mechanics

$$(f, g) = \int dx [\mathcal{CPT}f(x)] g(x). \quad (1.11)$$

1.2.3 2×2 \mathcal{PT} -symmetric Hamiltonian

An example of the results of unbroken \mathcal{PT} -symmetric theory that is often found in the literature describes a simple finite non-Hermitian but \mathcal{PT} -symmetric Hamiltonian

$$\tilde{H} = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix} \quad (1.12)$$

where the parameters r, s and θ are real. In this setup the \mathcal{T} operator performs complex conjugation and the parity operator is written as

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (1.13)$$

The Hamiltonian in 1.12 allows two parametric regions. When $s^2 < r^2 \sin^2 \theta$ the eigenvalues of \tilde{H} form a complex conjugate pair. This is the broken \mathcal{PT} -symmetry region. Otherwise, when $s^2 \geq r^2 \sin^2 \theta$, then the eigenvalues are real. This is the region of unbroken \mathcal{PT} -symmetry region. In the unbroken symmetry region the \mathcal{PT} normalised eigenstates of both \tilde{H} and \mathcal{PT} are

$$|E_+\rangle = \frac{1}{\sqrt{2 \cos \alpha}} \begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix} \quad \text{and} \quad |E_-\rangle = \frac{i}{\sqrt{2 \cos \alpha}} \begin{pmatrix} e^{-i\alpha/2} \\ -e^{i\alpha/2} \end{pmatrix}, \quad (1.14)$$

we use the relation $\sin \alpha = \frac{r}{s} \sin \theta$. The corresponding \mathcal{C} operator for this Hamiltonian is

$$\mathcal{C} = \frac{1}{\cos \alpha} \begin{pmatrix} i \sin \alpha & 1 \\ 1 & -i \sin \alpha \end{pmatrix}, \quad (1.15)$$

This \mathcal{C} operator commutes with \tilde{H} and satisfies $\mathcal{C} = \hat{\mathbb{1}}$. The eigenvalues of \mathcal{C} are precisely the signs of the \mathcal{PT} norms of the corresponding eigenstates[13]. Using this \mathcal{C} operator we can guarantee that the \mathcal{CPT} inner product in the two-dimensional Hilbert space spanned by $|E_\pm\rangle$ is positive definite.

Time evolution

The aim of this section is to illustrate the dynamics of non-Hermitian \mathcal{PT} -symmetric Hamiltonians. In Hermitian quantum mechanics, time-evolution of states is done by using the unitary operator $\hat{U} = e^{-i\hat{H}t/\hbar}$. Because the \hat{U} operator is unitary this means that as the state $\vec{\psi}$ evolves in time, its norm remains constant. This is useful in quantum mechanics because the norm of a state is interpreted as a probability, and therefore unitarity guarantees that a state's probability does not grow or decay with time. As explained in section 1.2.1, when the \mathcal{PT} -symmetry of \hat{H} is unbroken we have positive definite \mathcal{CPT} norm states. This means that time evolution under the unbroken \mathcal{PT} -symmetric framework is also unitary[2][7][12]. The unitary time evolution of a state $\vec{\psi}$ from time $t = 0$ to an arbitrary time t is written as

$$\vec{\psi}(t) = \hat{U}\vec{\psi}(0). \tag{2.1}$$

Project statement

My project is concerned with the dynamical processes of non-Hermitian systems.

3.1 Parametric family of PTsNHH

3.2 Investigating \mathcal{PT} -symmetric time evolution

Here I include some of the progress I made in my work on the dynamics of Hamiltonians under the \mathcal{PT} -symmetric Framework.

Conclusion

Appendix

Bi-orthogonality and completeness

In the case of a \mathcal{PT} -symmetric Hamiltonian \tilde{H} , it is possible for two (or more) complex eigenvalues to cross. At the crossing point the eigenvalues are degenerate and this is accompanied by the coalescence of the eigenstates of \tilde{H} . If eigenstates of \tilde{H} become degenerate this implies that they lose completeness[14]. This special situation is associated with a branch point in the complex energy plane and is commonly termed an “exceptional point”[16]. Moreover, sufficiently close to the branch point the eigenfunctions of the coalescing eigenvalues are nearly self-orthogonal.

We need completeness and closure relations in order to develop, for example, perturbation theory and scattering theories for non-Hermitian Hamiltonians and in order to be able to solve the Schrödinger equation by numerical methods[16].

Suppose that a diagonalizable non-Hermitian Hamiltonian \tilde{H} has a discrete spectrum and it commutes with the \mathcal{PT} operator. This means that \tilde{H} has unbroken \mathcal{PT} -symmetry. A Hilbert space \mathcal{H} spanned by the eigenvectors of \tilde{H} exists, but because \tilde{H} is non-Hermitian –with respect to the standard inner product of \mathcal{H} – the time evolution generated by \tilde{H} will not be unitary[11]. Nevertheless, as explained in section 1.2.2. For the assumed \tilde{H} above it is possible to construct a different complete positive-definite inner product by constructing and using the antisymmetric \mathcal{C} operator that corresponds to \tilde{H} .

The assumed diagonalizability condition of \tilde{H} may be viewed as a physical requirement without which an energy eigenbasis would not exist. To our knowledge all known non-Hermitian Hamiltonians that are used in physical applications are diagonalizable and therefore admit a complete *bi-orthonormal* set of eigenvectors. This set of bi-orthonormal eigenvectors is $\{(|\psi_n, a\rangle, |\phi_n, a\rangle)\}$ and will satisfy the following defining relations[17]

$$\tilde{H} |\psi_n, a\rangle = E_n |\psi_n, a\rangle, \quad \tilde{H}^\dagger |\phi_n, a\rangle = E_n^* |\phi_n, a\rangle, \quad (5.1)$$

$$\langle \phi_m, b | \psi_n, a \rangle = \delta_{mn} \delta_{ab}, \quad (5.2)$$

$$\sum_n \sum_{a=1}^{d_n} |\psi_n, a\rangle \langle \phi_n, a| = \hat{\mathbb{I}} \quad (5.3)$$

$$\tilde{H} = \sum_n \sum_{a=1}^{d_n} E_n |\psi_n, a\rangle \langle \phi_n, a|, \quad \tilde{H}^\dagger = \sum_n \sum_{a=1}^{d_n} E_n^* |\phi_n, a\rangle \langle \psi_n, a| \quad (5.4)$$

where n and a are the spectral degeneracy levels, d_n is the degree of degeneracy of E_n and $\hat{\mathbb{I}}$ is the identity operator[17].

Equivalent Hamiltonians with distinct symmetries

In parallel to the work of Bender is the research of Mostafazadeh who introduced the notion of pseudo-Hermiticity in [11]. A Hamiltonian is said to be pseudo-Hermitian with respect to a positive-definite, Hermitian operator η if it satisfies

$$\tilde{H}^\dagger = \eta^{-1} \tilde{H} \eta. \quad (5.5)$$

In the case of Hamiltonians with \mathcal{PT} -symmetry, the role of η is played by the combined action of \mathcal{P} and \mathcal{C} . A convenient way to write the \mathcal{C} operator was proposed in [18]

$$\mathcal{C} = e^Q \mathcal{P} = \eta^{-1} \mathcal{P}, \quad (5.6)$$

here Q is an antisymmetric Hermitian operator. Mostafazadeh [12] has shown that the square root of the positive-definite Hermitian operator $\eta = e^{-Q}$ can be used to transform any non-Hermitian Hamiltonian with unbroken \mathcal{PT} -symmetry into a spectrally equivalent Hermitian Hamiltonian by means of a unitary “similarity transformation” [19][20]. The transformation is as follows, the invertible operator $\rho = \sqrt{\eta}$ acts on the non-Hermitian \mathcal{PT} -symmetric Hamiltonian \tilde{H} and returns an equivalent Hermitian Hamiltonian \hat{H}

$$\hat{H} = \rho^{-1} \tilde{H} \rho = e^{-Q/2} \tilde{H} e^{Q/2}. \quad (5.7)$$

To verify that the similar Hamiltonian \hat{H} is Hermitian we take the Hermitian conjugate of \hat{H}

$$\begin{aligned} \hat{H}^\dagger &= (e^{-Q/2} \tilde{H} e^{Q/2})^\dagger, \\ &= e^{Q/2} \tilde{H}^\dagger e^{-Q/2}, \end{aligned} \quad (5.8)$$

If we “swap” Hermiticity: \tilde{H}^\dagger in 5.8 for \mathcal{PT} -symmetry, and we use equation ??

$$\begin{aligned} \hat{H}^\dagger &= e^{Q/2} \mathcal{P} \tilde{H} \mathcal{P}^{-1} e^{-Q/2}, \\ &= e^{-Q/2} \mathcal{C} \tilde{H} \mathcal{C}^{-1} e^{Q/2}, \end{aligned} \quad (5.9)$$

Finally we recall that \mathcal{C} and \tilde{H} commute

$$\hat{H}^\dagger = e^{-Q/2} \tilde{H} e^{Q/2} = \hat{H}. \quad (5.10)$$

Mostafazadeh [11], conjectures that because \mathcal{CPT} -symmetry satisfies the postulates of quantum mechanics –whilst only “swapping” the Hermiticity of a Hamiltonian by \mathcal{CPT} -symmetry– then this must mean that non Hermitian \mathcal{CPT} -symmetric theories are equivalent to certain non local Hermitian field theories. This kind of transformation is explored in references [11],[21],[17],[20],[22]. In general this transformation is extremely complicated and so it is quite impractical to perform. Hence, while the mapping from \mathcal{PT} -symmetric to Hermitian Hamiltonians is of great theoretical interest, it does not have much practical value[6]. Another important fact is that this kind of transformation is a similarity and not a unitary transformation. Thus, while the eigenvalues of both Hamiltonians are identical, the relationships between vectors are changed; pairs of vectors that are orthogonal are mapped into pairs of vectors that are not orthogonal[6][23].

One-to-one equivalence:

PT-symmetric and Hermitian quantum properties

In the case of unbroken \mathcal{PT} -symmetry of a Hamiltonian \tilde{H} with eigenfuctions ψ_n then the \mathcal{PT} operator eigenfunctions are: $\chi_n(x) = e^{-i\alpha/2}\psi_n(x)$

Properties	Hermitian	\mathcal{PT} -symmetry (unbroken)
Eigenvalues and Eigenstates	$\hat{H} \psi_n\rangle = E_n \psi_n\rangle$	$\tilde{H} \psi_n\rangle = E_n \psi_n\rangle,$ $\langle\phi_n \tilde{H}^\dagger = \langle\phi_n E_n^*$
Inner product (ignoring degenerate spectra)	$\langle\psi \phi\rangle = (\psi\rangle)^\dagger \cdot \phi\rangle$	$(\psi \phi) = (\mathcal{CPT} \psi\rangle)^T \cdot \psi\rangle$
Orthonormality of states	$\langle\psi_n \psi_m\rangle = \delta_{mn}$	$(\psi_n \psi_m) = \delta_{mn}$
Completeness of states	$\sum_{n=0}^{\infty} [\psi(x)]^* \psi(y) = \delta(x - y)$	$\sum_{n=0}^{\infty} (-1)^n \chi_n(x) \chi_n(y) = \delta(x - y)$
Hamiltonian, Greens and zeta functions	$H(x, y) = \sum_{n=0}^{\infty} E_n [\psi_n(x)]^* \psi_n(y),$ $G(x, y) = \sum_{n=0}^{\infty} \frac{1}{E_n} [\psi_n(x)]^* \psi_n(y),$ $\int dx G(x, x) = \sum_{n=0}^{\infty} \frac{1}{E_n}$	$H(x, y) =$ $\sum_{n=0}^{\infty} (-1)^n E_n \chi_n(x) \chi_n(y),$ $G(x, y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{E_n} \chi_n(x) \chi_n(y),$
Time evolution	$\hat{U} = e^{-i\hat{H}t/\hbar}$	$\hat{U} = e^{-i\tilde{H}t/\hbar}$

Stokes wedges and boundary conditions

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