Faster than Hermitian Quantum Mechanics

by Carl M Bender et al

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Jesper Levinsen
Meera Parish

Outline

Contents

Hilbert spaces

A Hilbert space is a vector space that can be infinite dimensional.

Vector spaces are equipped with an inner product.

Inner product:
Define a distance function.

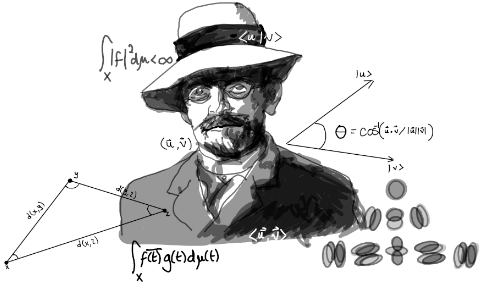

Fig.1: Hilbert space stuff

Hamiltonians as Observables

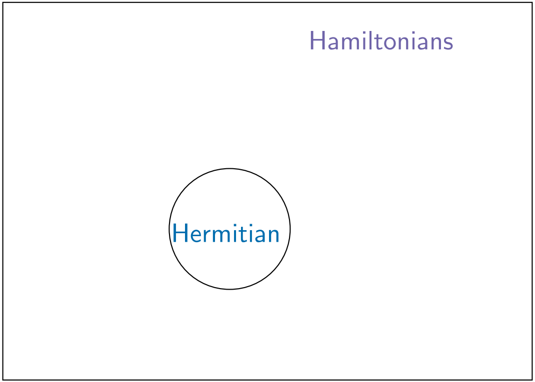

Fig.1: The set of all possible Hamiltonians.

1. Observables are **self-adjoint** operators.
2. A state ψ of the quantum system is a unit vector of $\hat{H}$.
3. $\hat{H}$ has a real energy spectrum with defined lowest energy.
4. Unitarity.

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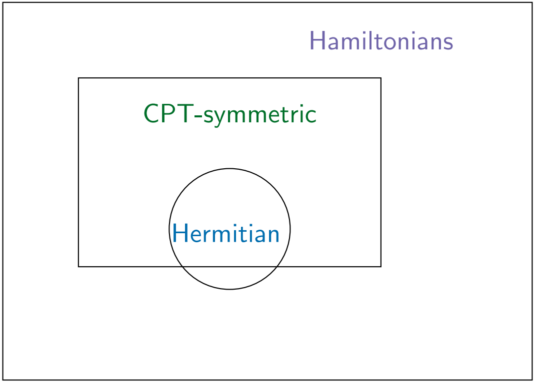

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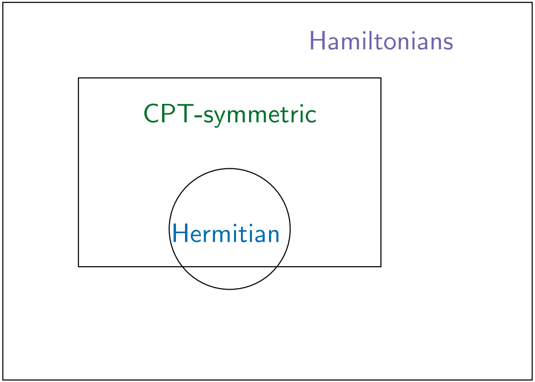

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CPT - symmetry in short

$$\mathcal{C} \rightarrow \text{charge conjugation},$$
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Interesting physical phenomena occurs in the broken symmetry region

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$$\vec{\psi}_i \rightarrow \vec{\psi}_f$$

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2. CPT quantum mechanics: ?

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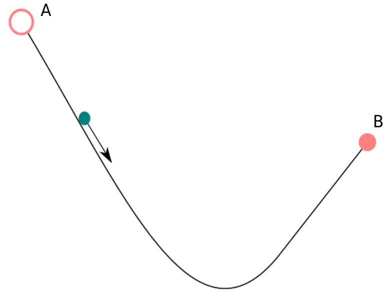

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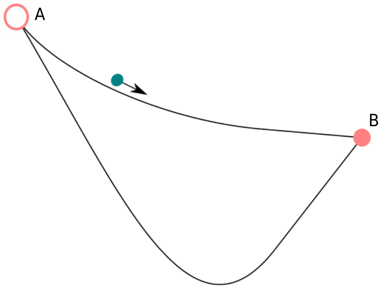

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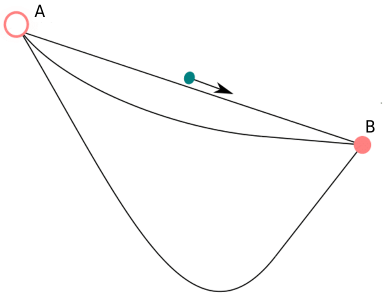

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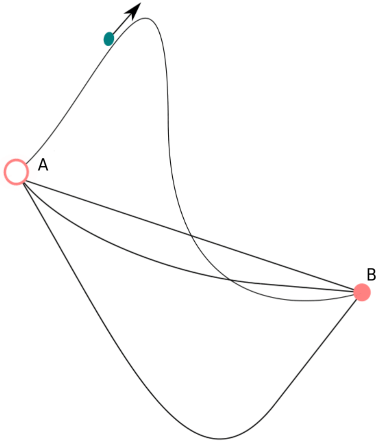

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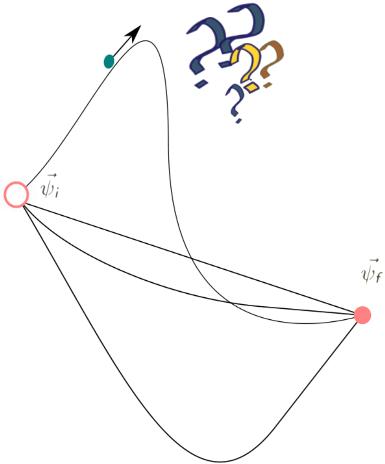

Fig.2: A particle travels from left to right in time t,
Can we make this trip nearly instantaneous?

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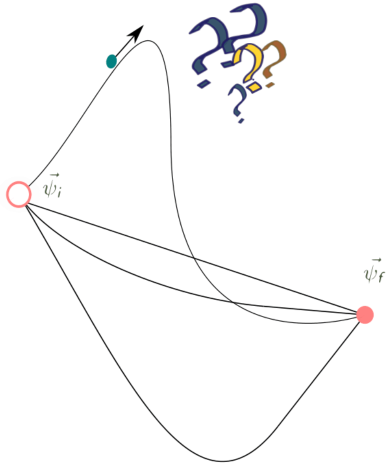

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How fast can we evolve a state?

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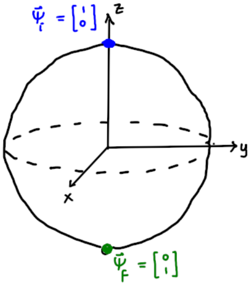

Fig.3: Bloch sphere with initial and final states

This space is spanned by $\vec{\psi}_i$ and $\vec{\psi}_f$.

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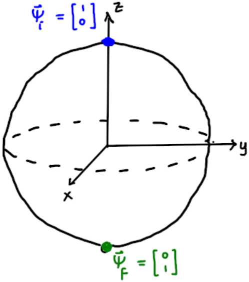

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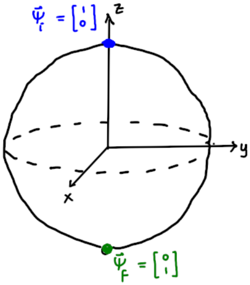

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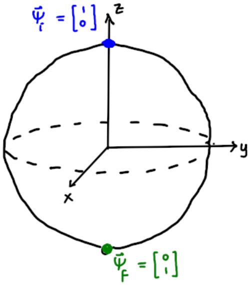

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Will a complex non-Hermitian Hamiltonian give time-optimal evolution?

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where $\sin(\alpha) = \frac{r}{s}\sin(\theta)$.

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Then $t \rightarrow 0$ when $\alpha \rightarrow -\frac{\pi}{2}$.
but we also require $s, r >> 1$.

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1. orthogonal $\rightarrow$ Hermitian inner product

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We can choose α to create a “wormhole” effect in Hilbert space.

Contents

Conclusion

1. This paper demonstrates the theoretical differences in both $\mathcal{CPT}$-symmetric and conventional Hermitian quantum mechanics.
2. The "wormhole" effect could be of importance in design and implementation of fast quantum computation and comunication algorithms.
3. There could be quantum protection mechanisms limiting the applicability of Hilbert-space "wormholes".

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