Faster than Hermitian time-evolution

by Carl M Bender

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Jesper Levinsen
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Outline

Contents

Hilbert spaces

A Hilbert space is a vector space that can be infinite dimensional.

Vector spaces are equipped with an inner product.

Inner product:
Define a distance function.

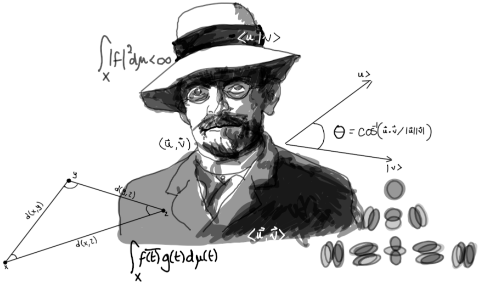


Fig.3: Hilbert space stuff

Hamiltonians as Observables

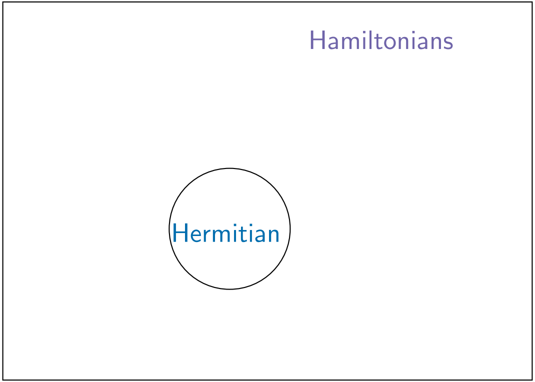
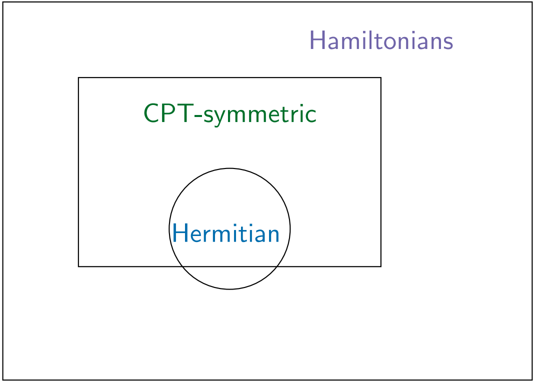


Fig.1: The set of all possible Hamiltonians.

1. Observables are **self-adjoint** operators $\{\hat{O}, \hat{H}, ...\}$
2. Real energy spectrum with defined lowest energy
3. A state ψ of the quantum system is a unit vector of $\hat{H}$
4. Expectation values of observables are given by the inner-product.
5. Unitarity

Hamiltonians as Observables



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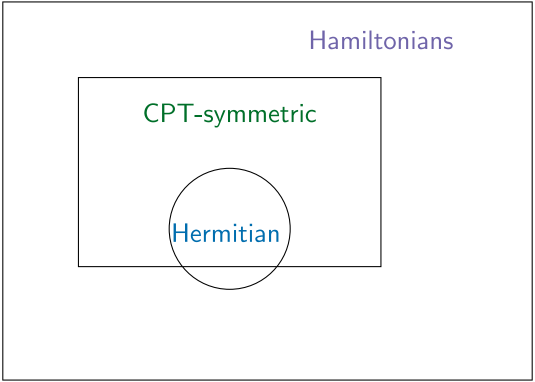

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CPT - symmetry in short

Hilbert space $\rightarrow \hat{H} = \hat{H}^{\mathcal{CPT}}$

$$\begin{aligned}
\mathcal{C} &\rightarrow \text{charge conjugation,} \\
\mathcal{P} &\rightarrow \text{spatial inversion,} \\
\mathcal{T} &\rightarrow \text{complex conjugation and time reversal.}
\end{aligned}$$

Inner product: $(\vec{u}, \vec{v}) = \mathcal{CPT}(\vec{u}) \cdot \vec{v}$,

If the eigenfunctions of a $\mathcal{PT}$-symmetric Hamiltonian are **not** also eigenfunctions of the $\mathcal{PT}$ operator we say the Hamiltonian possesess **broken $\mathcal{PT}$-symmetry**.

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Interesting physical phenomena occurs in the broken symmetry region

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Time evolution

$$\vec{\psi}_i \to \vec{\psi}_f$$

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1. Hermitian quantum mechanics:

$$\hat{U} = e^{-i\hat{H}t/\hbar},$$

where $t > 0$.

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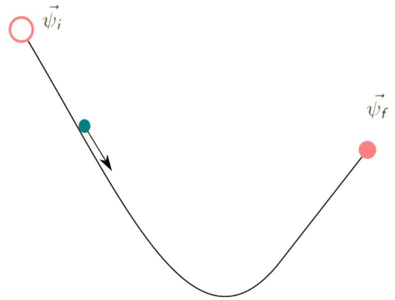

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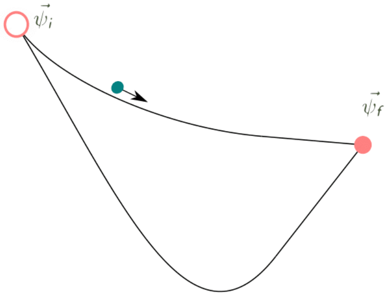

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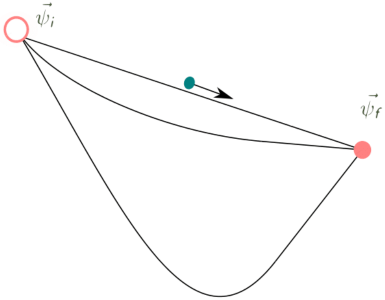

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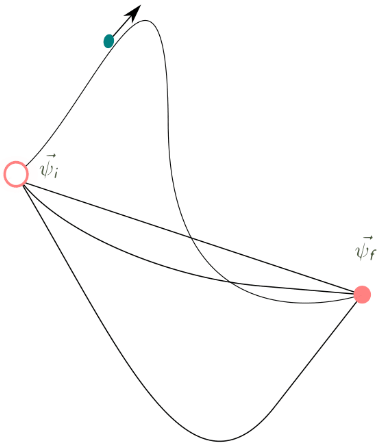

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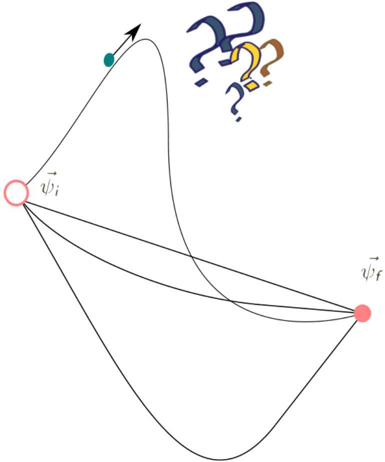

Fig.2: A particle travels from left to right in time t,
Can we make this trip nearly instantaneous?

βράχιστος χρόνος
brákhistos khrónos:
“shortest time”

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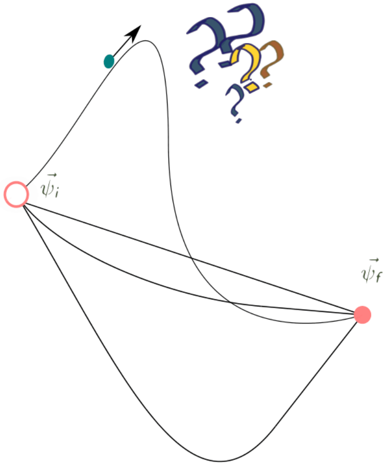

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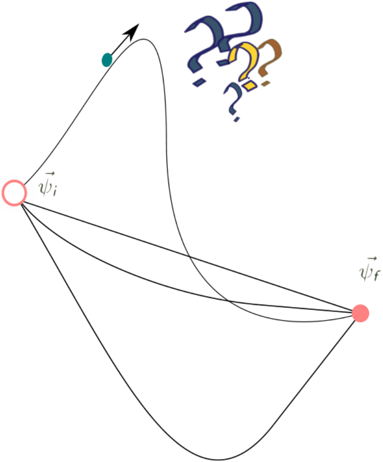
How fast can we evolve?

The Quantum Brachistochrone problem



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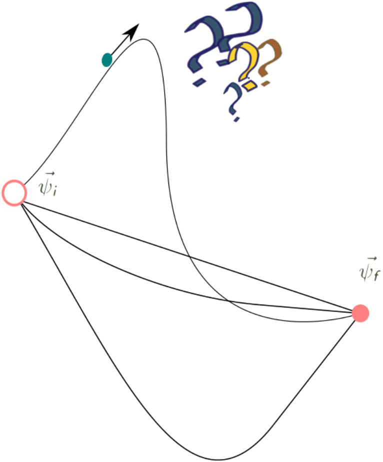
1. We want the shortest time step possible

The Quantum Brachistochrone problem



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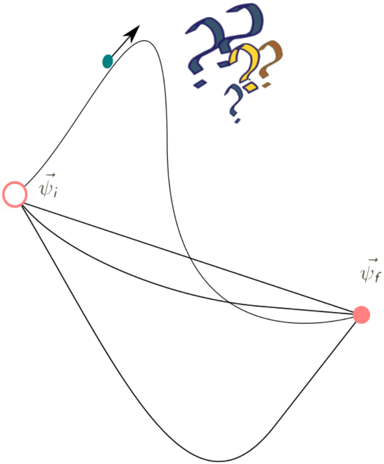

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Can we make this trip nearly instantaneous?

1. We want the shortest time step possible
2. Complex non-Hermitian Hamiltonians:
   time-optimal evolution
3. Uncertainty principle violation?

The Maths

$$\vec{\psi_i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{\psi_f} = \begin{pmatrix} a \\ b \end{pmatrix}, \quad |a|^2 + |b|^2 = 1,$$

Hermitian case (not CPT-symmetric)

$$\hat{H} = \begin{pmatrix} s & re^{-i\theta} \\ re^{i\theta} & u \end{pmatrix}, \quad \{r, s, \theta, u\} \in \mathbb{R},$$

Energy constraint

$$\omega^2 = (s-u)^2 + 4r^2,$$

Time evolution

$$\vec{\psi_f} = \begin{pmatrix} a \\ b \end{pmatrix} = e^{\frac{-i(s+u)t}{2\hbar}} \begin{pmatrix} \cos(\frac{\omega t}{2\hbar}) - i\frac{s-u}{\omega}\sin(\frac{\omega t}{2\hbar}) \\ -i\frac{2r}{\omega}e^{i\theta}\sin(\frac{\omega t}{2\hbar}) \end{pmatrix}.$$

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CPT-symmetric case

$$\tilde{H} = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}, \quad \{r, s, \theta\} \in \mathbb{R},$$

Energy constraint

$$\omega^2 = 4s^2 - 4r^2 \sin^2\theta > 0,$$

Time evolution

$$\vec{\psi}_f = \begin{pmatrix} a \\ b \end{pmatrix} = \frac{e^{\frac{-itr\cos\theta}{\hbar}}}{\cos\alpha} \begin{pmatrix} \cos(\frac{\omega t}{2\hbar} - \alpha) \\ -i\frac{2r}{\omega} e^{i\theta} \sin(\frac{\omega t}{2\hbar}) \end{pmatrix}.$$

Rabi frequency

The geometry of the Hilbert space

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References

C. M. Bender.
Faster than hermitian time evolution.
Symmetry, Integrability and Geometry: Methods and Applications (2007).
DOI: 10.3842/sigma.2007.126.

J. J. Sakurai.
Modern Quantum Mechanics (Revised Edition).
Addison Wesley, 1st ed. edition (1993).

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The maths in-depth