Faster than Hermitian time-evolution

by Carl M Bender

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Jesper Levinsen
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Outline

Contents

Hilbert spaces

A Hilbert space is a vector space that can be infinite dimensional.

Vector spaces are equipped with an inner product.

Inner product:
Define a distance function.

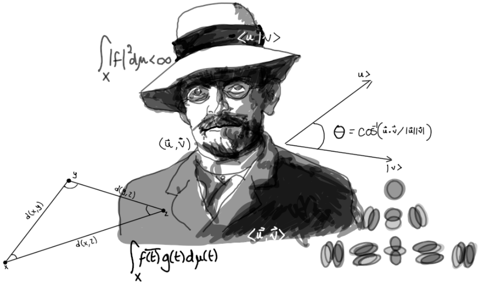

Fig.3: Hilbert space stuff

Hamiltonians as Observables

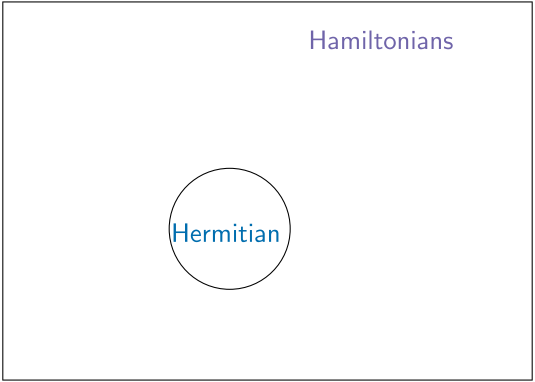

Fig.1: The set of all possible Hamiltonians.

1. Observables are **self-adjoint** operators $\{\hat{O}, \hat{H}, ...\}$
2. Real energy spectrum with defined lowest energy
3. A state ψ of the quantum system is a unit vector of $\hat{H}$
4. Expectation values of observables are given by the inner-product.
5. Unitarity

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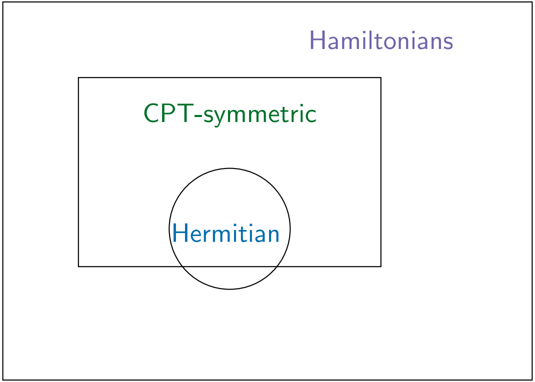

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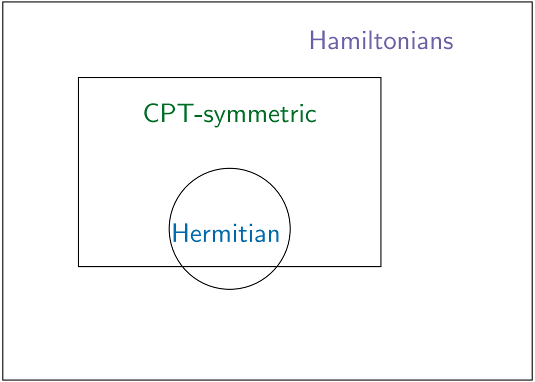

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CPT - symmetry in short

Hilbert space $\rightarrow \hat{H} = \hat{H}^{\mathcal{CPT}}$

$$\mathcal{C} \rightarrow \text{charge conjugation},$$
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Interesting physical phenomena occurs in the broken symmetry region

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2. CPT quantum mechanics: ?

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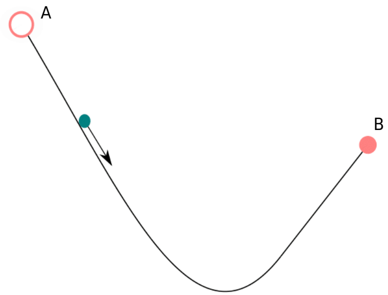

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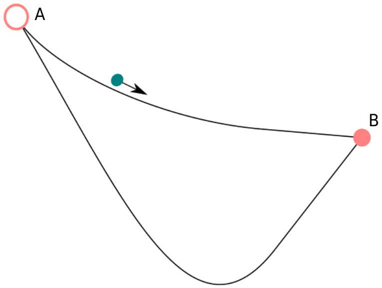

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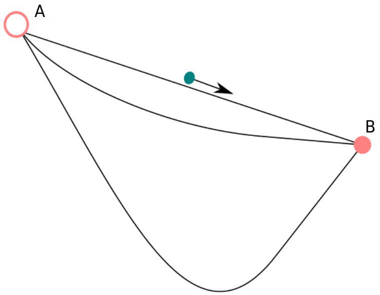

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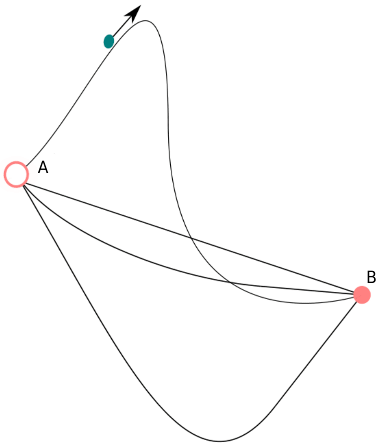

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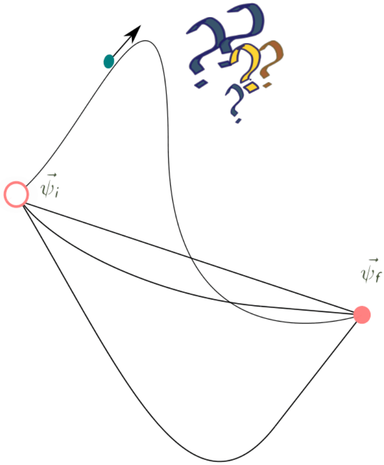

Fig.2: A particle travels from left to right in time t,
Can we make this trip nearly instantaneous?

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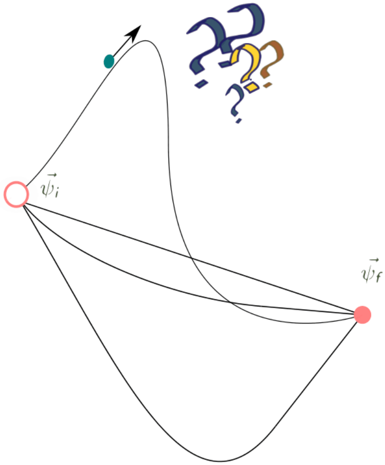

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How fast can we evolve a state?

A simple quantum brachistochrone problem

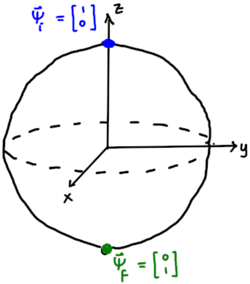

Fig.3: Bloch sphere with initial and final states

This space is spanned by $\vec{\psi}_i$ and $\vec{\psi}_f$.

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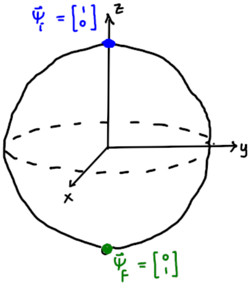

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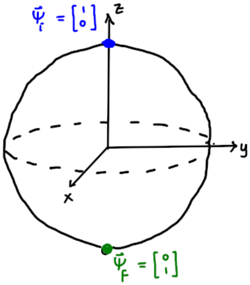

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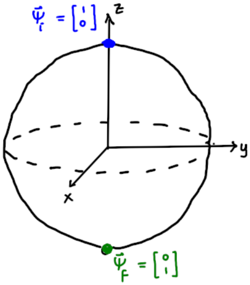

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Will a complex non-Hermitian Hamiltonian give time-optimal evolution?

The Maths

$$\vec{\psi}_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{\psi}_f = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

.

Hermitian case

$$\hat{H} = \begin{pmatrix} s & re^{-i\theta} \\ re^{i\theta} & u \end{pmatrix}, \{r, s, \theta, u\} \in \mathbb{R},$$

Energy constraint

$$\omega^2 = (s-u)^2 + 4r^2,$$

Time evolution

$$\begin{pmatrix} a \\ b \end{pmatrix} = e^{\frac{-i(s+u)t}{2\hbar}} \begin{pmatrix} \cos(\frac{\omega t}{2\hbar}) - i\frac{s-u}{\omega}\sin(\frac{\omega t}{2\hbar}) \\ -i\frac{2r}{\omega}e^{i\theta}\sin(\frac{\omega t}{2\hbar}) \end{pmatrix}.$$

CPT-symmetric case

$$\tilde{H} = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}, \quad \{r, s, \theta\} \in \mathbb{R},$$

Energy constraint

$$\omega^2 = 4s^2 - 4r^2\sin^2\theta > 0,$$

Time evolution

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{e^{\frac{-itr\cos\theta}{\hbar}}}{\cos\alpha} \begin{pmatrix} \cos(\frac{\omega t}{2\hbar} - \alpha) \\ -i\sin(\frac{\omega t}{2\hbar}) \end{pmatrix}.$$

where $\sin(\alpha) = \frac{r}{s}\sin(\theta)$.

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For all $r > 0$ subject to $\omega^2 = (s-u)^2 + 4r^2$ for a fixed ω.

$$\therefore \tau = \frac{2\hbar}{\omega}\arcsin(|b|) = \frac{\pi\hbar}{\omega}$$

is the minimum *passage time*.

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Then $t \rightarrow 0$ when $\alpha \rightarrow -\frac{\pi}{2}$.

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We can choose α to create a "wormhole" effect in Hilbert space.

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Conclusion

1. This paper demonstrates the theoretical differences in both $\mathcal{PT}$-symmetric and conventional Hermitian quantum mechanics.
2. The "wormhole" effect could be of importance in design and implementation of fast quantum computation and comunication algorithms.
3. There could be quantum protection mechanisms limiting the applicability of Hilbert-space "wormholes".

References

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