Faster than Hermitian time-evolution

by Carl M Bender

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Jesper Levinsen
Meera Parish

Outline

Contents

Hilbert spaces

A Hilbert space is a vector space that can be infinite dimensional.

Vector spaces are equipped with an inner product.

Inner product:
Define a distance function.

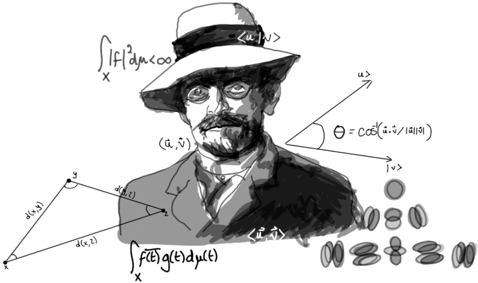


Fig.3: Hilbert space stuff

Hamiltonians as Observables

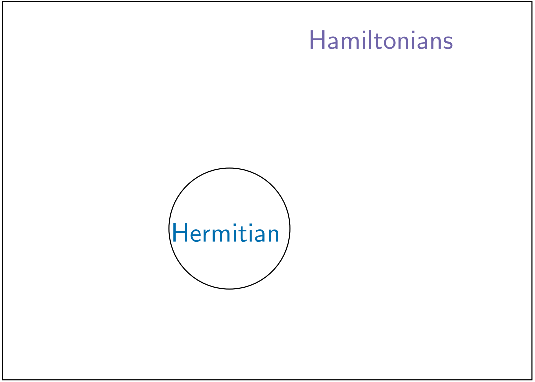
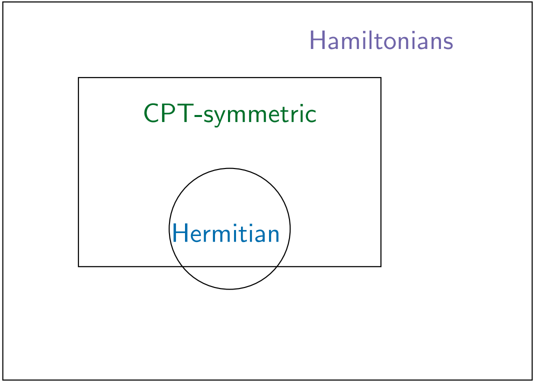


Fig.1: The set of all possible Hamiltonians.

1. Observables are **self-adjoint** operators $\{\hat{O}, \hat{H}, ...\}$
2. Real energy spectrum with defined lowest energy
3. A state ψ of the quantum system is a unit vector of $\hat{H}$
4. Expectation values of observables are given by the inner-product.
5. Unitarity

Hamiltonians as Observables



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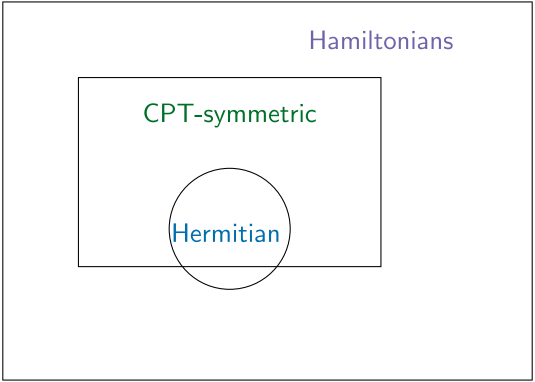

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CPT - symmetry in short

Hilbert space $\rightarrow \hat{H} = \hat{H}^{\mathcal{CPT}}$

$$\mathcal{C} \rightarrow \text{charge conjugation},$$
$$\mathcal{P} \rightarrow \text{spatial inversion},$$
$$\mathcal{T} \rightarrow \text{complex conjugation and time reversal}.$$

Inner product is determined dynamically in terms of the Hamiltonian.

If the eigenfunctions of a $\mathcal{PT}$-symmetric Hamiltonian are **not** also eigenfunctions of the $\mathcal{PT}$ operator we say the Hamiltonian possesess **broken $\mathcal{PT}$-symmetry**.

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Interesting physical phenomena occurs in the broken symmetry region

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Time evolution

$$\vec{\psi}_i \rightarrow \vec{\psi}_f$$

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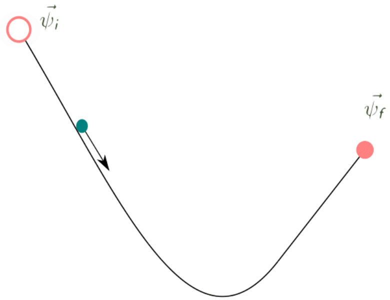

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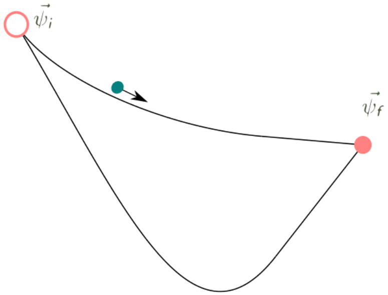

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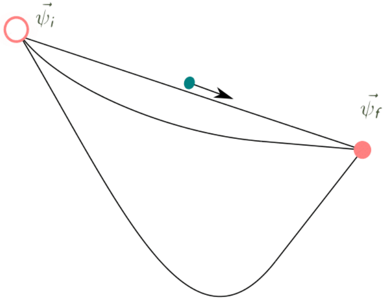

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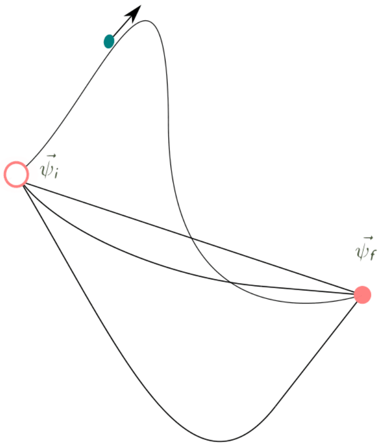

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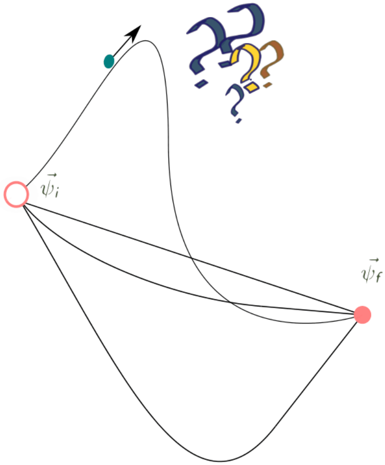

Fig.2: A particle travels from left to right in time t,
Can we make this trip nearly instantaneous?

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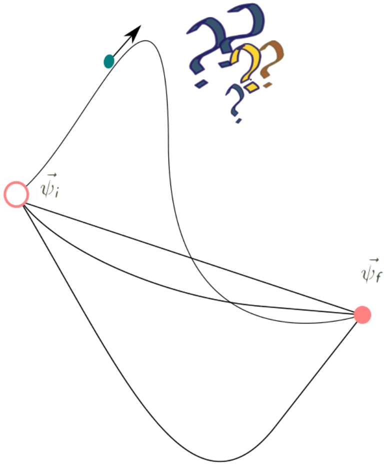

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How fast can we evolve?

Two-dimensional Quantum Brachistochrone problem

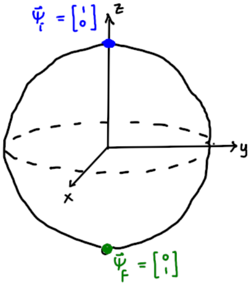

Fig.3: Bloch sphere with initial and final states

This space is spanned by $\vec{\psi}_i$ and $\vec{\psi}_f$.

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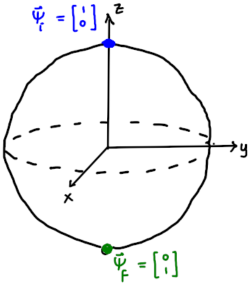

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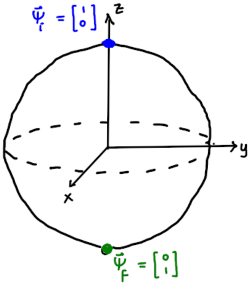

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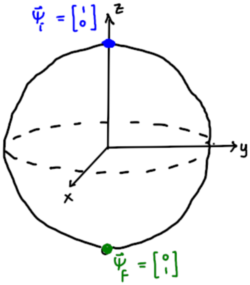

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Will a complex non-Hermitian Hamiltonian give time-optimal evolution?

The Maths

$$\vec{\psi}_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{\psi}_f = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

.

Hermitian case

$$\hat{H} = \begin{pmatrix} s & re^{-i\theta} \\ re^{i\theta} & u \end{pmatrix}, \{r, s, \theta, u\} \in \mathbb{R},$$

Energy constraint

$$\omega^2 = (s-u)^2 + 4r^2,$$

Time evolution

$$\begin{pmatrix} a \\ b \end{pmatrix} = e^{\frac{-i(s+u)t}{2\hbar}} \begin{pmatrix} \cos(\frac{\omega t}{2\hbar}) - i\frac{s-u}{\omega}\sin(\frac{\omega t}{2\hbar}) \\ -i\frac{2r}{\omega}e^{i\theta}\sin(\frac{\omega t}{2\hbar}) \end{pmatrix}.$$

CPT-symmetric case

$$\tilde{H} = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}, \quad \{r, s, \theta\} \in \mathbb{R},$$

Energy constraint

$$\omega^2 = 4s^2 - 4r^2\sin^2\theta > 0,$$

Time evolution

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{e^{\frac{-itr\cos\theta}{\hbar}}}{\cos\alpha} \begin{pmatrix} \cos(\frac{\omega t}{2\hbar} - \alpha) \\ -i\sin(\frac{\omega t}{2\hbar}) \end{pmatrix}.$$

where $\sin(\alpha) = \frac{r}{s}\sin(\theta)$.

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For all $r > 0$ subject to $\omega^2 = (s-u)^2 + 4r^2$ for a fixed ω.

$$\therefore \tau = \frac{2\hbar}{\omega}\arcsin(|b|) = \frac{\pi\hbar}{\omega}$$

is the minimum *passage time*.

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Then $t \to 0$ when $\alpha \to -\frac{\pi}{2}$.

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1. orthogonal $\rightarrow$ Hermitian inner product
2. not orthogonal $\rightarrow$ CPT inner product

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References

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Contents

Rabi frequency

The frequency of fluctuations in the populations of two atomic energy levels.

It is proportional to the strength of the **coupling** between the light's and the atomic transition's frequencies.

Broken and unbroken CPT-symmetry

Quantum systems with gain and loss

Constructing the C operator

The maths in-depth

More CPT quantum theory

1. Balanced open systems

   Imaginary potential contributions $\rightarrow$ source and drain terms.

2. Possible experimental ideas

   Embedding the non-Hermitian CPT symmetric system into a larger structure described by a Hermitian Hamiltonian.