Faster than Hermitian Quantum Mechanics

by Carl M Bender et al

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Outline

Contents

Hilbert spaces

A Hilbert space is a vector space that can be infinite dimensional.

Vector spaces are equipped with an inner product.

Inner product:
Define a distance function.

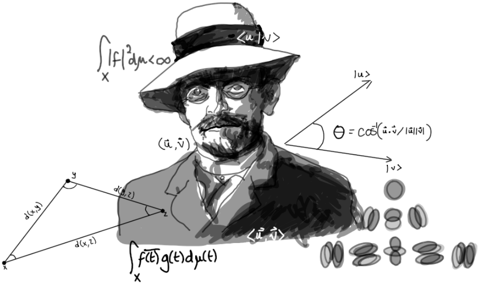


Fig.1: Hilbert space stuff

Hamiltonians as Observables

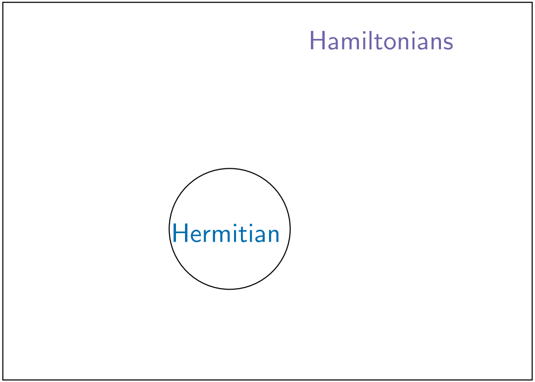
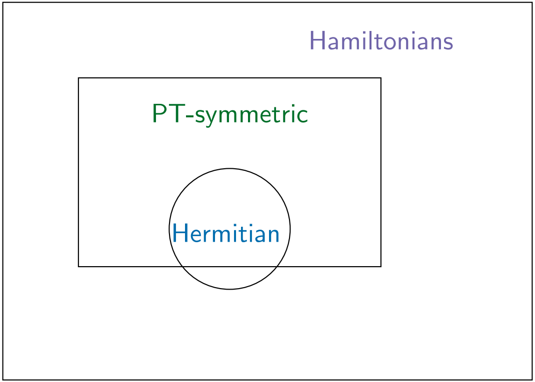


Fig.2: The set of all possible Hamiltonians.

1. Observables are **self-adjoint** operators.
2. $\hat{H}$ has a real energy spectrum with defined lowest energy.
3. Unitarity.

Hamiltonians as Observables



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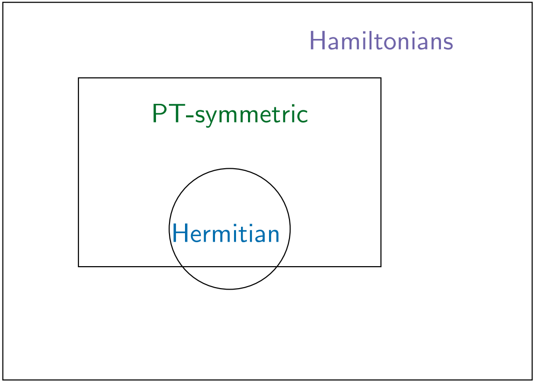

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PT - symmetry in short

$$\mathcal{P} \to \text{spatial inversion},$$
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Hilbert space $\to \hat{H} = \hat{H}^{\mathcal{PT}}$

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If the eigenfunctions of a $\mathcal{PT}$-symmetric Hamiltonian are **not** also eigenfunctions of the $\mathcal{PT}$ operator we say the Hamiltonian possesess **broken $\mathcal{PT}$-symmetry**.

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Interesting physical phenomena occurs in the broken symmetry region.

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   $$\hat{U} = e^{-i\hat{H}t/\hbar},$$

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2. PT quantum mechanics: ?

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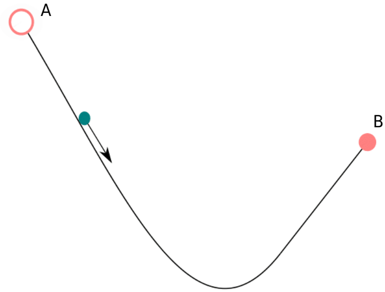

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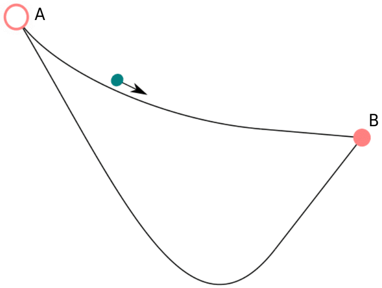

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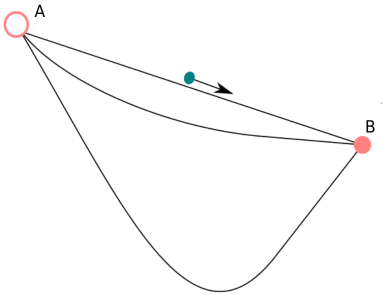

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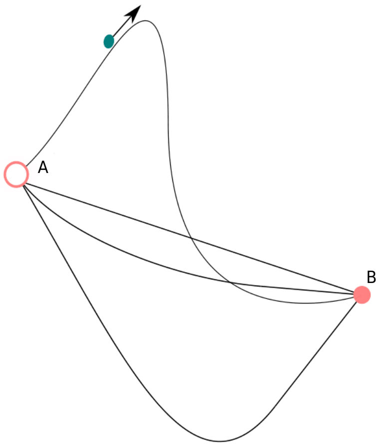

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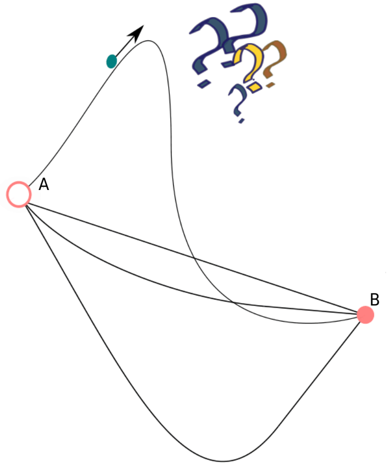

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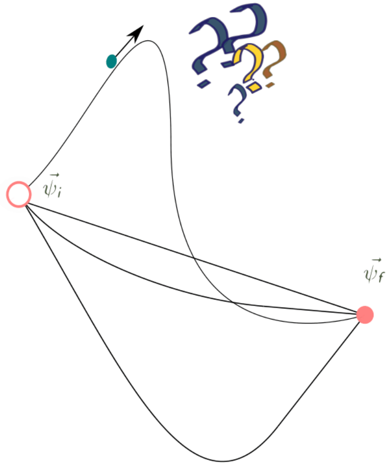

Fig.3: A particle travels from left to right in time t,
Can we make this trip nearly instantaneous?

βράχιστος χρόνος
brákhistos khrónos:
“shortest time”

How fast can we evolve a state?

A simple quantum brachistochrone problem

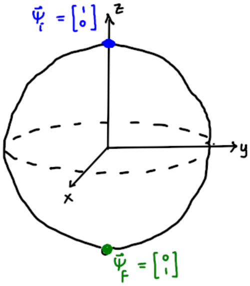

Fig.4: Bloch sphere with initial and final states

This space is spanned by $\vec{\psi}_i$ and $\vec{\psi}_f$.

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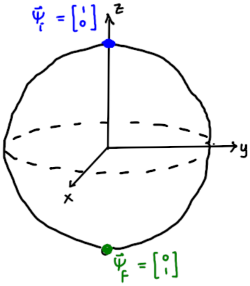

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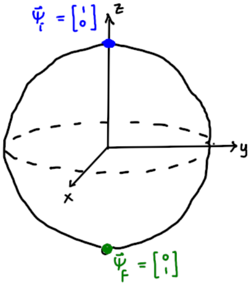

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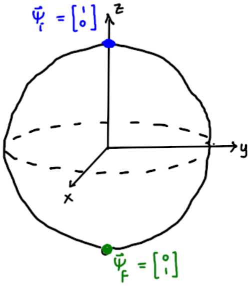

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Will a complex non-Hermitian Hamiltonian give time-optimal evolution?

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where $\sin(\alpha) = \frac{r}{s}\sin(\theta)$.

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Since ω is fixed, $\omega^2 = 4s^2 - 4r^2\sin^2(\theta)$
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Then $t \rightarrow 0$ when $\alpha \rightarrow -\frac{\pi}{2}$.
but we also require $s, r >> 1$.

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The geometry of the space

$\mathcal{PT}$ inner products are defined depending on the Hamiltonian used.

$$\vec{\psi}_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{\psi}_f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

1. orthogonal $\rightarrow$ Hermitian inner product
   vector separation: $\delta = 2 \arccos |\langle \psi_f | \psi_i \rangle| = \pi$
2. not orthogonal $\rightarrow$ PT inner product
   vector separation: $\delta = \pi - 2|\alpha|$

We can choose α to create a "wormhole" effect in Hilbert space.

Contents

Conclusion

1. This paper demonstrates the theoretical differences in both $\mathcal{PT}$-symmetric and conventional Hermitian quantum mechanics.
2. The "wormhole" effect could be of importance in design and implementation of fast quantum computation and comunication algorithms.
3. There could be quantum protection mechanisms limiting the applicability of Hilbert-space "wormholes".

References

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Broken PT-symmetry phenomena

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Light Stops at Exceptional Points

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Almost twenty years ago, light was slowed down to less than 10^{-7} of its vacuum speed in a cloud of ultracold atoms of sodium. Upon a sudden turn-off of the coupling laser, a slow light pulse can be imprinted on cold atoms such that it can be read out and converted into a photon again. In this process, the light is stopped by absorbing it and storing its shape within the atomic ensemble. Alternatively, the light can be stopped at the band edge in photonic-crystal waveguides, where the group speed vanishes. Here, we extend the phenomenon of stopped light to the new field of parity-time (PT) symmetric systems. We show that zero group speed in PT symmetric optical waveguides can be achieved if the system is prepared at an exceptional point, where two optical modes coalesce. This effect can be tuned for optical pulses in a wide range of frequencies and bandwidths, as we demonstrate in a system of coupled waveguides with gain and loss.

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Broken PT-symmetry phenomena

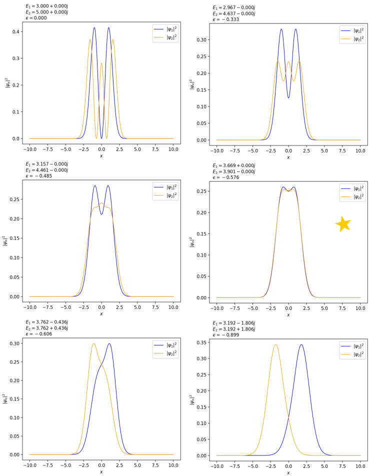

Fig 5. The probability density of the wave functions corresponding to the first and second excited states of the harmonic oscillator. States' probability densities are symmetric about zero. As the exceptional point parametric value is approached from the left, the probability densities begin to overlap in space. We can see the densities of the wave functions behave as mirror images of each other after we pass the exceptional point in the negative direction.

Constructing the PT inner product

The PT inner product is **not positive definite**.
To patch up this problem, we use the $\mathcal{C}$ operator:

$$\mathcal{C} \rightarrow \text{charge conjugation}$$

$$\mathcal{C}^2 = 1, \quad [\mathcal{C}, \hat{H}] = 0, \quad [\mathcal{C}, \mathcal{PT}] = 0.$$

We construct $\mathcal{C}$ from the $\hat{H}$ eigenstates ψ_n.

$$\mathcal{C} = \sum_n \psi_n(x)\psi_n(y)$$

Inner product:

$$(\mathcal{CPT}\psi_n(x)) \cdot \psi_m(y) = \mathcal{C}\psi_n^*(-x)) \cdot \psi_m(y)$$