

# Faster than Hermitian time-evolution

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# Outline

Introduction

Time evolution

Brachistochrone problem

Conclusion

Appendix

# Contents

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Brachistochrone problem

Conclusion

Appendix

# Hilbert spaces

A Hilbert space is a vector space that can be infinite dimensional.

Vector spaces are equipped with an inner product.

Inner product:

Define a distance function.

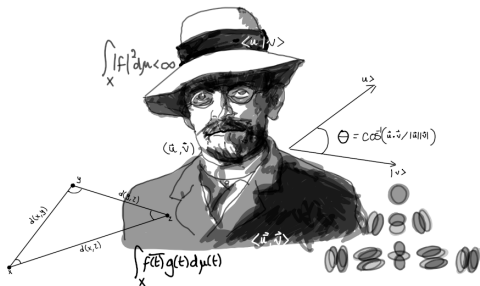


Fig.3: Hilbert space stuff

# Hamiltonians as Observables

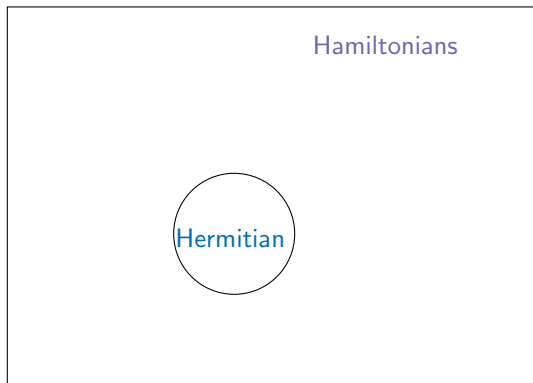


Fig.1: The set of all possible Hamiltonians.

1. Observables are **self-adjoint** operators  $\{\hat{O}, \hat{H}, \dots\}$
2. Real energy spectrum with defined lowest energy
3. A state  $\psi$  of the quantum system is a unit vector of  $\hat{H}$
4. Expectation values of observables are given by the inner-product.
5. Unitarity

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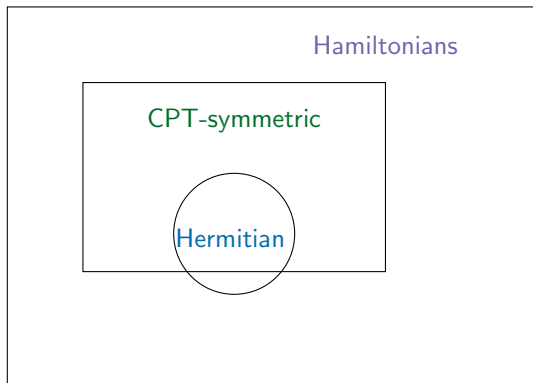


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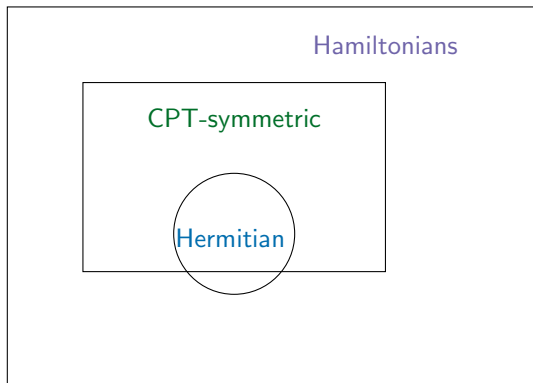


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# CPT - symmetry in short

Hilbert space  $\rightarrow \hat{H} = \hat{H}^{CPT}$

$\mathcal{C} \rightarrow$  charge conjugation,

$\mathcal{P} \rightarrow$  spatial inversion,

$\mathcal{T} \rightarrow$  complex conjugation and time reversal.

Inner product:  $(\vec{u}, \vec{v}) = CPT(\vec{u}) \cdot \vec{v}$ ,

If the eigenfunctions of a  $\mathcal{PT}$ -symmetric Hamiltonian are **not** also eigenfunctions of the  $\mathcal{PT}$  operator we say the Hamiltonian possesses **broken  $\mathcal{PT}$ -symmetry**.



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Interesting physical phenomena occurs in the broken symmetry region

# Contents

Introduction

Time evolution

Brachistochrone problem

Conclusion

Appendix

# Time evolution

$$\vec{\psi}_i \rightarrow \vec{\psi}_f$$

## Time evolution

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# Contents

Introduction

Time evolution

Brachistochrone problem

Conclusion

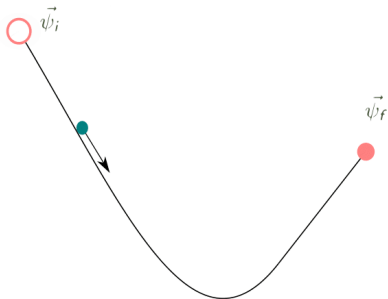
Appendix



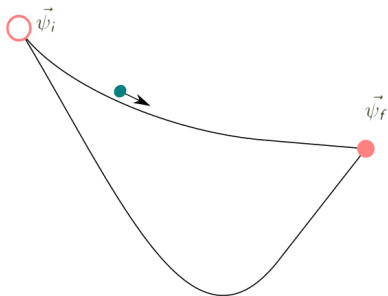
# The Quantum Brachistochrone problem



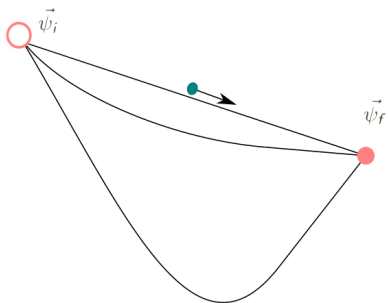
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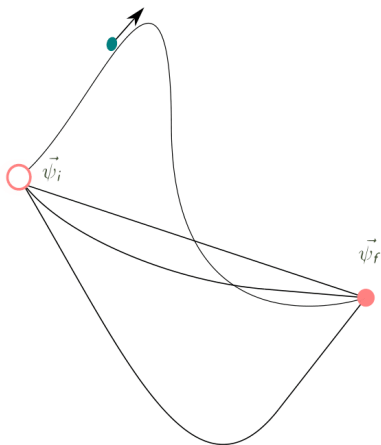
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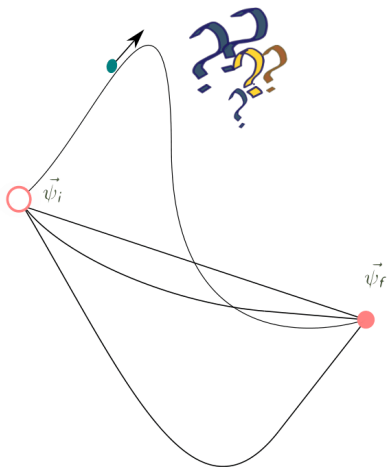
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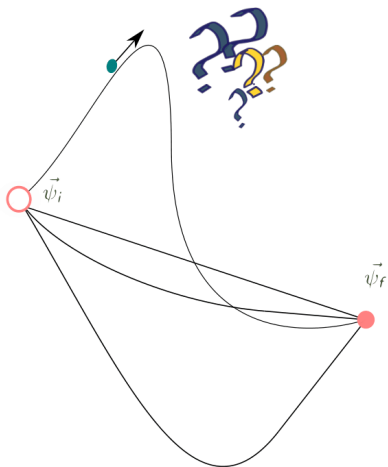
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brákhistos khrónos:  
“shortest time”

Fig.2: A particle travels from left to right in time  $t$ ,  
Can we make this trip nearly instantaneous?

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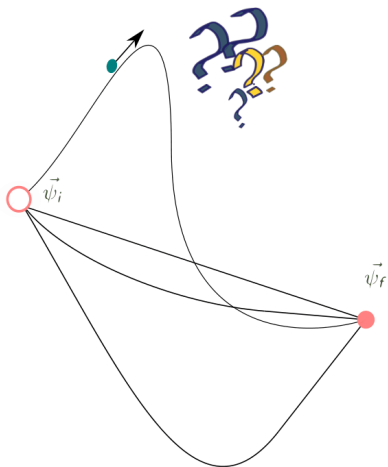


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How fast can we evolve?

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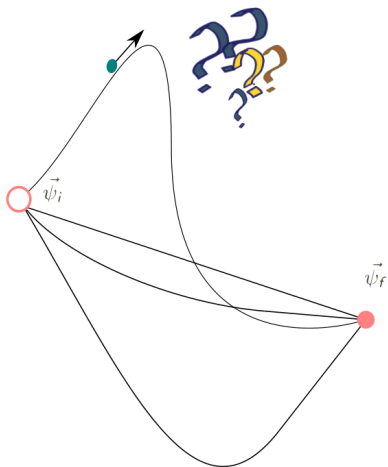


1. We want the shortest time step possible

Fig.2: A particle travels from left to right in time  $t$ ,  
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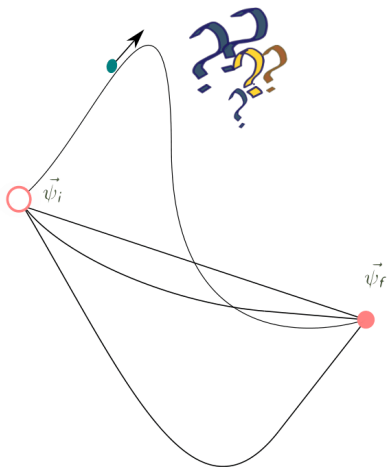
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1. We want the shortest time step possible
2. Complex non-Hermitian Hamiltonians:  
time-optimal evolution

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# The Quantum Brachistochrone problem



1. We want the shortest time step possible
2. Complex non-Hermitian Hamiltonians:  
time-optimal evolution
3. Uncertainty principle violation?

Fig.2: A particle travels from left to right in time  $t$ ,  
Can we make this trip nearly instantaneous?

# The Maths

$$\vec{\psi}_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{\psi}_f = \begin{pmatrix} a \\ b \end{pmatrix}, \quad |a|^2 + |b|^2 = 1,$$

Hermitian case (not CPT-symmetric)

$$\hat{H} = \begin{pmatrix} s & re^{-i\theta} \\ re^{i\theta} & u \end{pmatrix}, \quad \{r, s, \theta, u\} \in \mathbb{R},$$

Energy constraint

$$\omega^2 = (s - u)^2 + 4r^2,$$

Time evolution

$$\vec{\psi}_f = \begin{pmatrix} a \\ b \end{pmatrix} = e^{\frac{-i(s+u)t}{2\hbar}} \begin{pmatrix} \cos(\frac{\omega t}{2\hbar}) - i\frac{s-u}{\omega} \sin(\frac{\omega t}{2\hbar}) \\ -i\frac{2r}{\omega} e^{i\theta} \sin(\frac{\omega t}{2\hbar}) \end{pmatrix}.$$

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$$\vec{\psi}_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{\psi}_f = \begin{pmatrix} a \\ b \end{pmatrix}, \quad |a|^2 + |b|^2 = 1,$$

CPT-symmetric case

$$\tilde{H} = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}, \quad \{r, s, \theta\} \in \mathbb{R},$$

Energy constraint

$$\omega^2 = 4s^2 - 4r^2 \sin^2 \theta > 0,$$

Time evolution

$$\vec{\psi}_f = \begin{pmatrix} a \\ b \end{pmatrix} = \frac{e^{\frac{-itr \cos \theta}{\hbar}}}{\cos \alpha} \begin{pmatrix} \cos(\frac{\omega t}{2\hbar} - \alpha) \\ -i \frac{2r}{\omega} e^{i\theta} \sin(\frac{\omega t}{2\hbar}) \end{pmatrix}.$$

# Rabi frequency

The frequency of fluctuations in the populations of two atomic energy levels.

It is proportional to the strength of the **coupling** between the light's and the atomic transition's frequencies.

# The geometry of the Hilbert space

# Contents

Introduction

Time evolution

Brachistochrone problem

Conclusion

Appendix

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# References



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# Contents

Introduction

Time evolution

Brachistochrone problem

Conclusion

Appendix

# Broken and unbroken CPT-symmetry

# Quantum systems with gain and loss

# Constructing the C operator

# The maths in-depth

# More CPT quantum theory

## 1. Balanced open systems

Imaginary potential contributions  $\rightarrow$  source and drain terms.

## 2. Possible experimental ideas

Embedding the non-Hermitian CPT symmetric system into a larger structure described by a Hermitian Hamiltonian.