

## phase Shifts and cross-section

$\psi(r) \rightarrow$  partial waves  
(decomposed)

if  $V(r) = 0$ :

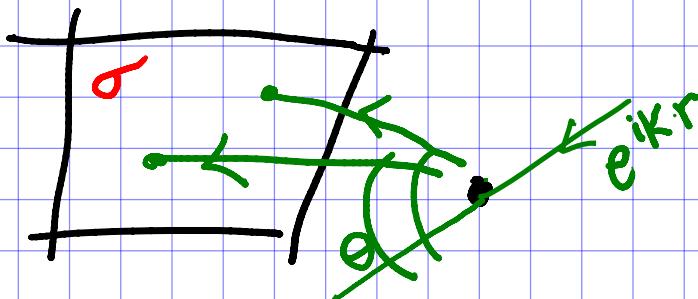
$$\sum_i^{\infty} \text{partial wave}_i = e^{ikz}$$

elif  $V(r) \neq 0$ :

$$\text{partial wave}_i = \text{partial wave}_i + \delta_e$$

$\therefore$  interference

which causes the probability  
to depend upon the angle  
cross-section  $\sigma$



$$\therefore R^{ik\hat{r}} = e^{ikz} = e^{ikr \cos \theta}$$

Spherical Harmonics

$$= \sum_{l,m} g_{l,m}(r) Y_{l,m}(\theta, \phi)$$

$$Y_{l,0} = P_l(\cos \theta)$$

(Legendre polynomial)

when  $m=0$

Solutions to free Schrödinger eqn for angular momentum  $\ell$  are:

$j_\ell(kr)$  - Bessel function\*  
 (\*regular at origin)

$n_\ell(kr)$  - Neumann function

—Partial wave expansion—

$$e^{ikr \cos \theta} = \sum_l i^l (2l+1) j_l$$

we use this expansion in asymptotic normalisation for scattering states

by expanding  $f(\theta)$ :

$$f(\theta) = \sum_l (2l+1) a_l(k) P_l(\cos \theta)$$

partial Scattering amp.

⋮  
⋮

$$\text{we get } \sigma = 4\pi \sum_l (2l+1) |a_l(k)|^2$$

when  $r \rightarrow \infty$

$$j_\ell(kr) \rightarrow (kr)^{-\frac{1}{2}} \sin\left(kr - \frac{\ell\pi}{2}\right)$$

we have a combination of incoming and outgoing spherical waves

\*single valued  
odd finite derivative

$$\psi_{sc} = e^{ikz} + f(\theta) e^{\frac{ikr}{r}}$$

Asymptotic Behaviour

$$u_\ell(k, r) = \sin(kr - \frac{l\pi}{2} + \delta_\ell(k))$$

with amplitude

$$\psi_{sc} \xrightarrow[r \rightarrow \infty]{} \sum_l c_l \sin(kr - \frac{l\pi}{2} + \delta_\ell(k))$$

$$\therefore \psi_{sc} = e^{-i(kr - \frac{l\pi}{2} + \delta_\ell)} P_l(\cos\theta) \quad (2)$$

matching coefficients:

$$\frac{c_l}{2i} = \frac{2l+1}{2ik} e^{i(\delta_\ell + \frac{l\pi}{2})}$$

$$\text{we arrive at } S_\ell = 1 + 2ik a_\ell(k)$$

S-matrix

Solving for partial amplitudes ...

$$\therefore S_\ell = e^{2if_\ell(k)}$$

$$\therefore A_\ell = \frac{e^{2i\delta_\ell} - 1}{2ik}$$

$$\therefore A_\ell = \frac{e^{i\delta_\ell} \sin \delta_\ell}{k}$$

we can see that  $f(\theta)$

is interference between incoming plane

phase shifted outgoing (scattered) spherical wave

$\therefore$  if phase shifts are zero  $\Rightarrow$  no scattering amplitude

$$\text{then } \sigma = \sum_l 4\pi(2l+1) \frac{\sin^2 \delta_\ell(k)}{k^2}$$

Notice: for any  $l$ :  $\boxed{S_\ell = \frac{\pi}{2} + n\pi}$

is max contribution to phase

We can write

$$a_l = \frac{e^{i\delta_l} \sin \delta_l(k)}{k} = \frac{\sin \delta_l(k)}{k e^{i\delta_l}}$$

$$= \frac{\sin \delta_l(k)}{k(\cos \delta_l - i \sin \delta_l)} = \frac{1}{k \frac{\cos \delta_l}{\sin \delta_l} - ik}$$

$$\therefore a_l = \frac{1}{k \cot(\delta_l) - ik}$$

we can write the scattering cross-section as:

$$\boxed{\sigma = \sum_l \frac{4\pi(2l+1)}{k^2 + k^2 \cot^2 \delta_l(k)}}$$

Scattering lengths