Phase shifts and cross-section

$\psi(r) \longrightarrow$ partial waves (decomposed)

if $V(r) = 0$:

$$\sum_{i}^{\infty} \text{partial wave}_i = e^{ikz}$$

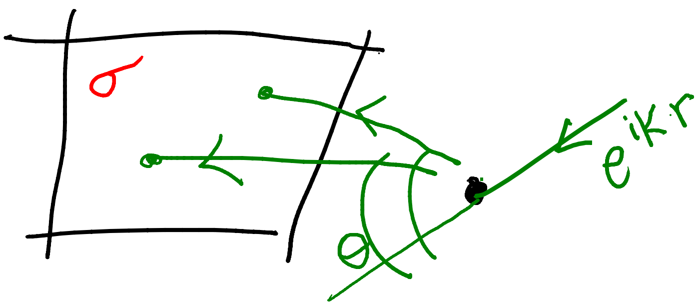
elif $V(r) \neq 0$:

$$\text{partial wave}_i = \text{partial wave}_i + \delta_\ell$$

$\therefore$ interference

which causes the probability to depend upon the angle

cross-section σ



$$\therefore e^{i\vec{k}\cdot\vec{r}} = e^{ikz} = e^{ikr\cos\theta}$$

Spherical Harmonics

$$= \sum_{l,m} g_{\ell,m}(r)\, Y_{\ell,m}(\theta,\phi)$$

when $m=0$ $\quad Y_{\ell,0} = P_\ell(\cos\theta)$ (legendre polynomial)

Solutions to free schrödinger eqn for angular momentum ℓ are:

$j_\ell(kr)$ – Bessel function*
(*regular at origin)

$n_\ell(kr)$ – Neumann function

—Partial wave expansion—

$$e^{ikr\cos\theta} = \sum_\ell i^\ell (2\ell+1) j_\ell$$

we use this expansion in asymptotic normalisation for scattering states by expanding $f(\theta)$:

$$f(\theta) = \sum_\ell (2\ell+1) a_\ell(k) P_\ell(\cos\theta)$$

partial scattering amp.

⋮

we get $\sigma = 4\pi \sum_\ell (2\ell+1) |a_\ell(k)|^2$

when $r \to \infty$

$$j_\ell(kr) \to (kr)^{-1} \sin\left(kr - \frac{\ell\pi}{2}\right)$$

we have a combination of incoming and outgoing spherical waves

*single valued and finite derivative.

$$\psi_{sc} = e^{ikz} + f(\theta)\frac{e^{ikr}}{r}$$

Asymptotic Behaviour

$$u_\ell(k,r) = \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell(k)\right)$$

with amplitude

$$\psi_{sc} \xrightarrow[r\to\infty]{} \frac{1}{r}\sum_\ell c_\ell \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell(k)\right)$$

$$\therefore \psi_{sc} = e^{-i\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right)} P_\ell(\cos\theta) \quad . \quad (2)$$

matching coefficients:

$$\frac{c_\ell}{2i} = \frac{2\ell+1}{2ik} e^{i\left(\delta_\ell + \frac{\ell\pi}{2}\right)}$$

we arrive at $S_\ell = 1 + 2ik\, a_\ell(k)$

S-matrix $\therefore S_\ell = e^{2i\delta_\ell(k)}$

Solving for partial amplitudes ... $\therefore a_\ell = \dfrac{e^{2i\delta_\ell} - 1}{2ik}$

$$\therefore a_\ell = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k}$$

we can see that $f(\theta)$ is interference between incoming plane phase shifted outgoing (scattered) spherical wave

$\therefore$ if phase shifts are zero $\Rightarrow$ no scattering amplitude

then $$\sigma = \sum_\ell 4\pi(2\ell+1)\frac{\sin^2\delta_\ell(k)}{k^2}$$

Notice: for any ℓ: $\boxed{\delta_\ell = \frac{\pi}{2} + n\pi}$

is max contribution to phase

we can write

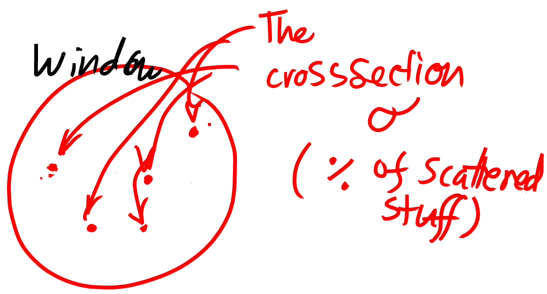
$$a_\ell = \frac{e^{i\delta_\ell}\sin\delta_\ell(k)}{k} = \frac{\sin\delta_\ell(k)}{k\,e^{-i\delta_\ell}}$$

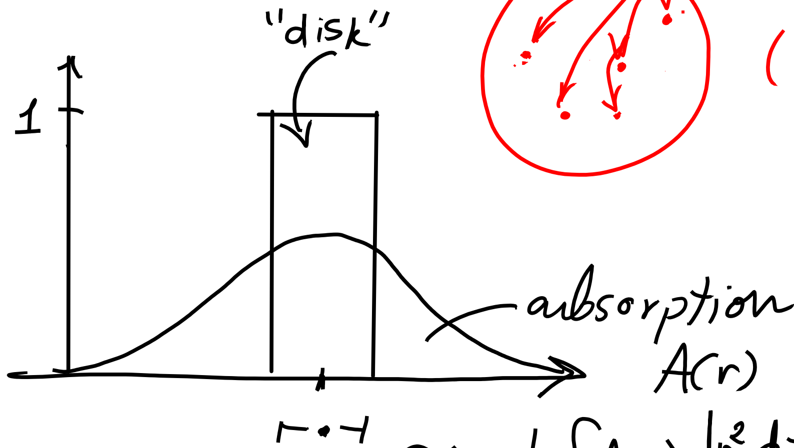
$$= \frac{\sin\delta_\ell(k)}{k(\cos\delta_\ell - i\sin\delta_\ell)} = \frac{1}{k\frac{\cos\delta_\ell}{\sin\delta_\ell} - ik}$$

$$\therefore\ a_\ell = \frac{1}{k\cot(\delta_\ell) - ik}$$

we can write the scattering cross-section as:

$$\boxed{\sigma = \sum_\ell \frac{4\pi(2\ell+1)}{k^2 + k^2\cot^2\delta_\ell(k)}}$$





$$\sigma = \frac{1}{\Delta z}\int A(r)\,dr^2\,dz$$