Solutions for delay differential equations in business cycle macroeconomic models

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Abstract

This paper aims to generalize the modeling with delay differential equations first presented in Kalecki's seminal business cycle model. For context, we first resume some of Kalecki's contributions to macroeconomics, focusing on the principle of effective demand and the innovative proposition of a temporal lag between stages of the investment process. These aspects constitute the framework for his dynamic modeling of the economic cycle, which is then summarized in a delay differential equation for investment. By defining and proving existence and uniqueness theorems for the solutions to this sort of equation, we offer a more rigorous resolution path for the model than that of Kalecki's original work. In sum, the main contribution of this paper to the existing literature is the simple and general resolution of the delay differential equation. A series of business cycle models can then replicate this modeling strategy to better capture macroeconomic fluctuations.

1 Introduction

Michal Kalecki made several original contributions to macroeconomics. Three key ones are his formulation of the principle of effective demand, his studies on the relationship between savings and investment, and his formal models of capitalist dynamics. For Kalecki, the principle of effective demand creates cyclical fluctuations in the economic system, especially through variations in investment (KALECKI, 1935; 1938; 2003 [1945]). His models also include distributional aspects and are based on mark-up pricing mechanisms. Moreover, Kalecki is particularly recognized for pioneering the formalization of economic cycles and growth using advanced mathematical models, hence having contributed to the studies of modeling itself (Baxley and Moorhouse, 2002; Franke, 2017).

The relevance of the mathematical modeling used by Kalecki relates to the context of his initial works and how it remains being used until now. In the 1930s, when the first works of Kalecki were published, the adjustment dynamics of the economy was being investigated. For that purpose, dynamical systems of both difference and differential equations were increasingly used. Despite this rising interest and the fact that this mathematical framework had been formalized by Henri Poincaré in the late nineteenth century, a greater use of dynamic systems was only seen in economics in the 1960s and 1970s, in the context of the accelerated development of computational methods (SHONE, 2002). Since then, this type of dynamical modeling has persisted, especially in the heterodox tradition that follows the works of Kalecki and Keynes (e.g., Skott, 2008; Setterfield, 2018).

However, the specific modeling used in Kalecki (1935) - the delay differential equation - has not yet been extensively explored even in the heterodox literature. This type of equation is useful since it simultaneously incorporates aspects of differential and difference equations. As Gandolfo (1971) and Franke (2017) argue, this feature of combining both types of dynamic equations allows for a more accurate representation of time, since the equations encompass phenomena that are approximately continuous, but have discrete lags, [1] - like investment decisions, as Kalecki (1935) proposes.

Some recent studies have tried to encompass this type of modeling in different ways, while still starting from Kalecki (1935) original paper (e. g., Manfredi and Fanti, 2004; Ballestra, Guerrini and Pacelli, 2013; Krawiec and Szydlowski, 2017; Franke, 2017 and Murakami, 2018). In general, these works extend the model, applying it to either business cycle or growth frameworks, and give it a refined mathematical treatment, such as exploring Hopf bifurcations. However, the theory regarding delay differential equations is not widespread even in mathematical textbooks and courses, much less in economics. Therefore, neither the original model nor these extensions (half of which are written by mathematicians) are easily understood, restraining their appropriation by others interested in the subject of economic dynamics. Moreover, the solution for the delay differential equation that Kalecki offered in the original 1935 paper is not ideal. Kalecki relies on a transformation of known forms of similar equations to solve it, not exploring all the possible cases that characterize the system and are important to interpret it fully.

Therefore, it is relevant to have a formal, general and simple presentation and resolution of Kalecki's (1935) delay differential equation for investment, which is the contribution of this paper. This is done in three steps. First, Kalecki's key contributions to macroeconomics are revisited. Among these are the principle of effective demand and the proposition of a temporal lag between stages of investment, which are the theoretical basis of this modeling of the economic cycle. Second, this paper presents the theorems for the existence and uniqueness of the model's solutions. This step is necessary because the latter are different from those of ordinary differential equations, as they require more information to be solved. The resulting solutions of the model, explored in the third step, are more rigorous than those of Kalecki's original paper. The implications of the presented solutions reinforces that the economic system is subject to endogenous cyclical fluctuations.

[1]Despite these advantages, it is important to note that the modeling of delay differential equations to which this work refers is that of linear models. See Blatt (1983) for a more broad discussion on the adequacy of linear and nonlinear models to economics.

2 Kalecki's Model: cycle and investment

2.1 Kalecki's Principle of Effective Demand

In his analysis of the business cycle[2], Kalecki sees investment as the main determinant of the cycle, according to the principle of effective demand. This principle is used to deny Say's Law, whose perspective of accounting identities that equals the product to the expense is the same as in a household, considering some given and constant revenue. Kalecki, in turn, proposes that the correct perspective on the relationship between these identities is expressed by the principle of effective demand. According to Possas (1999), the elaboration of this principle by Kalecki is such that in a monetary economy the decision to spend is the only autonomous one in the economic transactions and, thus, this expense will determine a revenue of the same magnitude - in a closed economy without government. It can be said, then, that aggregate spending determines the aggregate supply in a certain accounting period.

In this regard, the principle of effective demand does not impose the necessity of the existence of a static equilibrium. That happens because it is only necessary to understand that the expenditure will determine certain income [3] The concept of equilibrium is no longer necessary to determine, at least theoretically, the macroeconomic variables. Possas (1999) argues that these variables are then determined by a unilateral causal relationship, from expenditure to income.

Hence, Kalecki's macroeconomics is characterized by accounting identities modified by the implicit incorporation of the unilateral determination of income by expenditure. In Kalecki (1990 [1933]), the economy is analyzed in terms of social classes: capitalists and workers. Thus, the national income Y is, on the one hand, the sum of real gross profits of the capitalist P, and wages W of the workers:

$$Y = P + W. \tag{1}$$

And, on the other hand, in a closed economy without government, national income is the sum of the reproduction and expansion of fixed capital and the increase of stocks A - taking as null the savings and the capitalist incomes of the workers -, of capitalists' consumption C_c and of workers' consumption C_w:

$$Y = A + C_c + C_w, \tag{2}$$

in which the unidirectional causality from the right side of the equation to the left should

[2]This paper will cover the original formulation of Kalecki's business cycle model described in the paper "Esboço de uma Teoria do Ciclo Econômico" (KALECKI, 1990 [1933]) originally published in Polish and "A Macrodynamic Theory of Business Cycles" (KALECKI, 1935), published in English - although this general theory is presented with changes in later works, such as Kalecki (1971a [1954], 1971c [1954]). This first version is relevant as it is the starting point for Kalecki's subsequent work in business cycles, already presenting his pioneering formal approach.

[3]According to Possas (1999), Keynes (1985 [1936]) explores this notion by defining how the expenditure incurred, in determining income, also determines the level of activity and employment.

be emphasized, so that by the accounting identity the income is equal to the expenses that compose the product, but it is also determined by these.

Since, by abstraction, workers' propensity to consume is equal to one, $C_w = W$ and from (1) and (2) the same unidirectional determination of capitalist income by their expenditures is perceived, as indicated by the equation below:

$$P = C_c + A. \tag{3}$$

In this context, capitalist spending, both with consumption and investment, is fundamental to the cycle, since it is not necessarily equivalent to previous profits. And for that reason, it differs from the workers' spending, which is only a reflection of the variations in the national income. That is, as the principle of effective demand holds, product fluctuations are the result of the variation in consumption and investment expenditures from capitalists. Therefore, it is the class of capitalists who, through their decisions about their spending, determines the national income, given a certain income distribution. And the distributive factors, in turn, determine the income of the workers. [4]

According to Libânio (2002), one of the implications of the validity of the principle of effective demand, as opposed to Say's Law, is the existence of involuntary unemployment and idle capacity in an economic system due to insufficient demand. In this regard, it is worth emphasizing that the principle of effective demand and the acceptance of the existence of involuntary unemployment are also present in Keynes' work. Despite the numerous differences between Keynes and Kalecki, such as the economic tradition in which they were formed [5], both construct this principle with the same general idea, denying Say's Law. The difference lies in the theoretical course of its formulation. Possas (1999) describes that Keynes relates it to the ex-ante determination of employment and production, while Kalecki considers employment as a result of a certain level of demand, with a focus on the ex-post income result.

However, it is investment, as part of the capitalist spending, that becomes the main determinant of the cycle for the theory developed by Kalecki. This is justified by its autonomy in relation to a previous income - which makes it a key variable for the determination of income level and its variations (POSSAS, 1999) - in addition to its greater variability in comparison with the other components of aggregate expenditure - from the author's assumption that consumption is more stable.

[4]Libânio (2002) synthesizes the vision of Kalecki (1954) of this distributive thinking as "[i]n particular, the share of wages in national income is the inverse of the degree of monopoly. Therefore, aggregate consumption and employment are shaped by autonomous expenses of profit earners jointly with workers' consumption, which depends on the determinants of distribution" (p. 33). See Kalecki (1938) for the author's original formulation of his income distribution theory and Kalecki (1971a [1954], 1971b [1954]) for a revised discussion of these ideas.

[5]See Sawyer (1985) and Libânio (2002) for the description of the different backgrounds of the authors and its importance to understand and locate their work in context. See also Asimakopolus (1983) for a comparison between the two authors' approaches on investment theory.

2.2 Investment and Economic Dynamics

Kalecki is one of the authors who recognize the relevance of the role of investment in the determination of economic cycles by trying to analyze the similarities of the various cycles and to represent them through a dynamic model. For this purpose, he contributes to the field of macrodynamics by defining variables in different periods in the same model. The importance of this model is also related to the context of the study of macroeconomics since this approach of time contrasts with that used in mainstream economics, dominated by comparative static models - that is, without the temporal component - to analyze equilibrium solutions.

Libânio (2002) argues that Kalecki analyzes the cycle guided by two questions: what determines investment and what is the influence of investment on profits and national income. The model developed is characterized by cycles generated endogenously "(...) by the mutual interaction of the positive effect of higher income on investment and the negative influence of a greater capital stock on the decision to invest" (p. 32). As already mentioned, Kalecki's theory does not demand concepts of equilibrium, attempting to capture the nature of the instability of the capitalist economy as a result of this dual role that belongs to investment. He hypothesizes that the system moves with a cyclical tendency when considering a time lag between the decision to invest and the finished investment, which justifies the choice of using a specific modeling strategy to incorporate this lag.

In the model's formulation, Kalecki makes some assumptions. He considers a closed economy, without population growth, technological change or economic policy. Besides, he also assumes there is no trend. These assumptions are made to note the inherently cyclical component of the economic system, which is independent of the aforementioned factors.

The profit P is given by the sum of the capitalist consumption C and the gross accumulation A, as shown in equation (3). It is also assumed that the consumption of the capitalists is relatively inelastic, being then divided between a fixed component B_0 and a component dependent on the gross profits:

$$C = B_0 + \lambda P \tag{4}$$

where B_0 is a positive constant and $0 < \lambda < 1$.

And from equations (3) and (4) it is obtained:

$$\begin{aligned} P &= B_0 + \lambda P + A \\ P &= \frac{B_0 + A}{1 - \lambda}. \end{aligned} \tag{5}$$

It is noticeable that the relation indicating that the income of the capitalist depends on the autonomous component of consumption and the production of investment goods is maintained. In other words, the equations above illustrate an important part of Kalecki's model: the consumption function and the multiplier. [6]

[6] In this context, Gandolfo (1971) emphasizes that this model introduces before Keynes (1985 [1936]) the future instruments used by him in his theory, namely the consumption function and the multiplier.

That being said, concerning investment, Kalecki defines the existence of an average gestation period equivalent to the time interval demanded to the construction of this particular investment - determined by the structural properties of the economy. This interval represents the time lag between the decision to invest and the delivery of the finished equipment and is equal to the positive constant θ. According to Possas and Baltar (1981), this discrepancy stems from the fact that investment is not only seen as a function of the behavior of capitalists but is also approached in terms of a "decision to invest", while the investment itself will only be finished after the time indicated. They argue that this characteristic of Kalecki's model gives it a significant superiority over the usual capital adjustment models, not only in recognizing in this temporal lag a logical requirement of dynamics but also in giving it a clear and objective economic content (p. 140).

Thus, three phases of the investment process can be distinguished: the decision and order of the investment, both for reproduction and expansion of the capital stock, with volume by time denoted by I; the gross capital accumulation, A; and finished equipment deliveries, D. From this distinction, the intuitive relationship between I and D is:

$$D(t) = I(t - \theta). \tag{6}$$

$Z(t)$, in turn, equals the sum of all orders I made between $t - \theta$ and t. As each order requires a θ time period to be met, all orders placed in this period were not met at its end - when, however, all equipment ordered before was installed. This means that:

$$Z(t) = \int_{t-\theta}^{t} I(\tau)d\tau. \tag{7}$$

Also, the production of capital goods A in time t is equal to an average, in continuous terms, of the orders in the interval $(t - \theta, t)$:

$$A(t) = \frac{1}{\theta}\int_{t-\theta}^{t} I(\tau)d\tau. \tag{8}$$

Given these equations, taking $Z(t)$ as the total quantity of orders in time t, since each order takes θ to be met, then $1/\theta$ of its volume must be complete in each unit of time - assuming that the production of orders has a steady rhythm. [7] Thus,

$$A = \frac{Z}{\theta}. \tag{9}$$

Regarding the change of the capital stock, called K, its first derivative with respect to time is equivalent to its net change, such that

$$K'(t) = D(t) - U, \tag{10}$$

where U indicates the physical depreciation of capital, considered constant during the economic cycle. Kalecki divides the economic cycle into two parts, of which the first is characterized by K greater than the average K and the second by the inverse relation. In

[7]Kalecki sees this fact as a consequence of the existence of the period θ. Gandolfo (1971) argues that this must be taken as a separate assumption.

other words, in the first phase, there is more capital surplus than the average. However, the capital increase is characterized by new assets, with "life" much more extensive than the extension of the economic cycle. Therefore, fluctuations of U are considered negligible.

Finally, Kalecki proposes an investment function. The decision to invest is determined by the expected profitability. This profitability will be estimated, on the one hand, by the gross profitability P/K, in turn, estimated from the profitability of the existing equipment; and, on the other hand, by the interest rate i. However, for Kalecki these variables do not directly influence I, but instead the I/K ratio, its level relative to the capital stock. If P and K grow at the same rate, P/K remains unchanged, whereas I is likely to increase in the same proportion of P and K (KALECKI, 1990 [1933], p. 34). Then, the following relation holds:

$$\frac{I}{K} = f\left(\frac{P}{K}, i\right), \tag{11}$$

in which f is an increasing and decreasing function of P/K and i, respectively.

Kalecki makes another simplification by assuming that the interest rate is an increasing function of P/K. Thus, i is pro-cyclical, rising during the recovery and falling during the recession, and the general business conditions are represented by the gross profit P/K - also procyclical- then,

$$\frac{I}{K} = F\left(\frac{P}{K}\right). \tag{12}$$

The increase in the interest rate is also assumed to be sufficiently slow relative to gross profitability to F be an increasing function. Since, according to equation (5), P is proportional to $(B_0 + A)$, then

$$\frac{I}{K} = \phi\left(\frac{B_0 + A}{K}\right), \tag{13}$$

and assuming that ϕ is a linear function, this equation can be written as

$$\frac{I}{K} = m\left(\frac{B_0 + A}{K}\right) - n, \tag{14}$$

where m and n are positive constants. The constant m must be positive, given ϕ being an increasing function. As for n, Kalecki assumes that the investment comes close to zero if the rate of profit falls to a given minimum, implying $n > 0$ - a fundamental condition for the economic cycle to occur. Rearranging:

$$I = m(B_0 + A) - nK, \tag{15}$$

according to which the volume of orders I is an increasing function of the gross accumulation A and decreasing of the capital stock K. And this equation allows Kalecki to establish a delay differential equation for $I(t)$.

To sum up, there are relations between I, A, D and K and the time t,

$$\begin{aligned} D(t) &= I(t - \theta) && (16) \\ A(t) &= \frac{1}{\theta}\int_{t-\theta}^{t} I(\tau)d\tau && (17) \\ K'(t) &= D(t) - U, && (18) \end{aligned}$$

resulting from capitalist techniques of production, besides

$$I = m(B_0 + A) - nK, \tag{19}$$

resulting from the interdependence between investments and the earnings of existing equipment. From these relations, Kalecki determines the relationship between I and t. Differentiating (19) and (17) in relation to time yields:

$$I'(t) = mA'(t) - nK'(t) \tag{20}$$

$$A'(t) = \frac{I(t) - I(t-\theta)}{\theta}, \tag{21}$$

and from (16) and (18)

$$K'(t) = I(t-\theta) - U. \tag{22}$$

substituting (21) and (22) in (20) yields:

$$I'(t) = \frac{m}{\theta}[I(t) - I(t-\theta)] - n[I(t-\theta) - U]. \tag{23}$$

Finally, denoting by $J(t)$ the deviation of $I(t)$ from U, and taking into account that $J'(t) = I'(t)$, equation (23) can be transformed as follows:

$$J'(t) = \frac{m}{\theta}[J(t) - J(t-\theta)] - n[J(t-\theta)] \tag{24}$$

or even as

$$(m + \theta n)J(t-\theta) = mJ(t) - \theta J'(t), \tag{25}$$

which is a delay differential equation, linear and with constant coefficients. With that in mind, Kalecki (1935) concludes that "[t]he solution of that equation will enable us to express $J(t)$ as a function of t and to find out which, if any, are the endogenous cyclical fluctuations in our economic system" (p. 332).

Hence, Kalecki explains the dynamics of the business cycle incorporating a time lag into the investment function. As seen, his modeling of the cycle consists of a linear model, which would suggest that cycles would be exogenous to the system. However, by taking the time lag and modeling the system with the differential equation that includes this delay, the system will have continuous endogenous oscillations. A solution that is a function of the investment decision (I), expressed in terms of t and the other parameters of the model, is then sought to analyze if the model presents an oscillatory, cyclic behavior.

3 Delay Differential Equations

3.1 Definition and Theorem of Existence and Uniqueness of Solutions

The Delay Differential Equations are of the following type:

$$x'(t) = f[t, x(t), x(t-h)], \tag{26}$$

where $x : \mathbb{R} \to \mathbb{R}^\eta$ and h, which represents the temporal lag, is a positive constant. In the one-dimensional case ($\eta = 1$), which will be addressed here, this equation corresponds to Kalecki's modeling of the business cycle.

Generally, the solutions of ordinary differential equations without delay are uniquely defined if the initial conditions, which define the solution at some initial point t_0, are given. In particular, under some assumptions about the function f, a differential equation in one dimension admits a single solution for each initial value given $x_0 = x(t_0)$.

However, unlike these, determining the solutions of delay differential equations requires more information: it depends not only on the knowledge of x at time t_0 but also on the knowledge of x in the previous interval between $t_0 - h$ and t_0. Therefore, to determine the solution of a delay differential equation, it is necessary to specify an initial function that establishes the behavior of the system before t_0, more specifically in a period corresponding to the delay.

Consider, for example, the ordinary differential equation:

$$x'(t) = x(t)$$

whose family of solutions is given by $x(t) = ce^t$, where c is an arbitrary constant. The constant c is completely and uniquely determined by fixing the value $x_0 \in \mathbb{R}$ of the solution, for example, at an instant $t_0 = 0$, that is, $x(0) = x_0$. In this case, unique solution to the initial value problem is given by $x(t) = x_0 e^t$.

On the other hand, taking the time delay $h = 1$ and $t_0 = 0$, the following delay differential equation is obtained:

$$x'(t) = x(t-1)\,.$$

For a solution to $t \geq 0$ to be determined, time intervals with length equivalent to the delay are considered to construct the solution in each interval. Thus, if $t \in [0, 1]$, the differential equation above can be integrated, yielding:

$$x(t) = x(0) + \int_0^t x(s-1)ds,$$

with s being an arbitrary instant in which x is continuous.

Thus, the conclusion is that the function needs to be known in the interval before $t = 0$ and not only in an initial instant so that:

$$x(t) = \varphi_0(t) \text{ for } t \in [-1, 0]$$

where $\varphi_0 \in \mathcal{C}([-1, 0], \mathbb{R})$ represents the initial condition. If φ_0 is known, it is possible to obtain $x(t)$ with $t \in [0, 1]$. Now, defining $\varphi_1(t) = \varphi_0(0) + \int_0^t \varphi_0(s-1)ds$, $t \in [0, 1]$. In the case of $t \in [1, 2]$:

$$x(t) = x(1) + \int_1^t x(s-1)ds = \varphi_1(1) + \int_1^t \varphi_1(s-1)ds,$$

that is, the solution is known in the range $[1, 2]$. Taking $t \in [2, 3]$ and repeating this reasoning successively, the solution of this problem is obtained in the range $[-1, \infty)$. In sum, if $x'(t) = f(t, x(t), x(t-1))$, x in $[-1, 0]$ is needed as initial data, not being sufficient to know its value in the instant $t_0 = 0$, as in the case of the ordinary differential equation.

With the delay fixed at $0 \leq h < \infty$, the definitions of the delay differential equation and its solutions can be formally established. Those definitions are needed to state the existence and uniqueness theorems that allow solutions to be obtained for the equation in question, according to Onuchic (1977) and Estevam (2012). [8]

For this, fix the delay $0 \leq h < \infty$. Given $0 \leq H \leq \infty$, consider $\mathcal{C}_H = \{\varphi \in \mathcal{C}$, such that $||\varphi|| < H\}$, where $\mathcal{C} = \mathcal{C}([-h, 0], \mathbb{R})$ is the Banach space of continuous functions from $[-h, 0]$ to $\mathbb{R}$ with the norm $||\varphi|| = \sup_{-h \leq \theta \leq 0} |\varphi(\theta)|$, where $|.|$ denotes the norm of $\mathbb{R}$.

Given $0 < A \leq \infty$, and $x(t)$ continuous on $[t_0 - h, t_0 + A]$ with values in $\mathbb{R}$. For each t, $t_0 \leq t < t_0 + A$, define x_t as the function from $\mathcal{C}$ given by $x_t(\theta) = x(t + \theta)$ for $-h \leq \theta \leq 0$. If $x \in \mathcal{C}([t_0 - h, t_0 + A], \mathbb{R}^n)$, the application $F : [t_0, t_0 + A] \to \mathcal{C}([-h; 0]; \mathbb{R})$ with $F(t) = x_t$ is a continuous function.

From this, two definitions can be established to enunciate the two theorems that allow obtaining solutions for the equation in question:

Definition 1. *Let* $f : [0, \infty] \times \mathcal{C}_H \to \mathbb{R}^n$ *be an application, the equation*

$$x'(t) = f(t, x_t) \tag{27}$$

is called a Delay Differential Equation.

Definition 2. *The function* $x(t)$*, continuous on* $[t_0 - h, t_0 + A), A > 0$*, differentiable on* $[t_0, t_0 + A]$*, is called a solution of (27) with initial function* ψ *at* t_0*, if:*

i) $x_t \in \mathcal{C}_H$*, for* $t_0 \leq t < t_0 + A$*;*

ii) $x_{t_0} = \psi$*, i.e.,* $x_{t_0}(\theta) = \psi(\theta)$*, for* $\theta \in [-h, 0]$*;*

iii) $x'(t) = f(t, x_t)$*, for* $t_0 \leq t < t_0 + A$*.*

From these definitions, the theorem for the existence and uniqueness of solutions can be enunciated.

Theorem 1. *Let* $f(t, \varphi)$ *be continuous and locally Lipschitz with respect to* φ *on* $[0, \infty) \times \mathcal{C}_H$*. Then, for any* $t_0 \geq 0, \psi \in \mathcal{C}_H$*, there exist* $A > 0$ *and a function* $x(t)$ *defined on* $[t_0 - h, t_0 + A)$*, which is a solution of (27) with initial function* ψ *at* t_0*. Furthermore, this solution is unique.*

In sum,

$$\begin{cases} x'(t) = f(t, x_t), & \text{for } t \in [t_0, t_0 + A]; \\ x_{t_0} = \psi(\theta), & \text{for } \theta \in [-h, 0]. \end{cases}$$

Finally, as this paper focuses only on the linear case, the uniqueness of the solutions can be better specified as follows.

[8] Where the precise definitions and demonstrations can be found.

Theorem 2. *Let $h > 0$ and a_1 and a_2 be $\mathcal{C}^1$ functions on $[0, d]$. Let ψ be a $\mathcal{C}^1$ function on $[-h, 0]$. Then, there exists a unique function $x(t)$ satisfying the system*

$$\begin{cases} x'(t) = a_1(t)x(t) + a_2(t)x(t-h) \text{ for } t \in [0, h] \\ x(t) = \psi(t) \text{ for } t \in [-h, 0] \end{cases}$$

From this formalization, this paper can proceed to obtain solutions to the linear delay differential equation established by Kalecki (1935) and how the question of the initial conditions can be treated in this case. It is worth emphasizing that the following study of the solutions of this model follows a more rigorous and less restrictive theoretical course than the one used by Kalecki (1935), who relies on a transformation to the form of the equations found in Tinbergen (1931) for its resolution. The solutions that will be presented here are found using less formal and theoretical constraints. Besides, the approach presented here has the advantage of allowing this procedure to be replicated by works with similar modeling strategies that use this type of equation to capture time more accurately.

3.2 Solutions of Linear Delay Differential Equations

As previously discussed, Kalecki's analysis of the economic system leads to the linear delay differential equation with constant coefficients given by

$$\begin{cases} J'(t) = \dfrac{m}{\theta}J(t) - \dfrac{m+\theta n}{\theta}J(t-\theta), & \text{for} \quad \theta > 0 \quad \text{in} \quad [0, \infty) \\ J_0(t) = \psi(t), & \text{in} \quad [-\theta, 0] \end{cases} \tag{28}$$

where J is the unknown function, m and n are positive constants, θ is a given positive lag and the initial condition in given by the function ψ.

Without loss of generality, the time scale of (28) can be changed employing a change of variables. In order to do that,

$$\tau = \frac{t}{\theta}$$

can be taken and, for simplicity, also

$$a = m + \theta n.$$

The problem can be then rewritten as

$$\begin{cases} x'(\tau) = mx(\tau) - ax(\tau - 1); \\ x_0(\tau) = \psi(\tau), & \text{in} \quad [-1, 0] \end{cases} \tag{29}$$

according to which the solution of the original problem is given by $J(t) = x(t/\theta)$.

It is important to notice that since the differential equation in (29) is linear, by the superposition principle, if $x_1(\tau)$ and $x_2(\tau)$ are solutions of the equation, so is the linear combination $A_1x_1(\tau) + A_2x_2(\tau)$. Therefore, the general solution has the form $x(\tau) = \sum_k A_k x_k$, for whichever values of the constants A_k and where x_k are solutions of the equation (BOYCE; DIPRIMA, 2006).

Also by the linearity of the equation, the solutions wanted are exponential of the form

$$x(\tau) = e^{\rho\tau},$$

for ρ real or complex. Substituting this function in equation (29) yields:

$$\rho e^{\rho\tau} = me^{\rho\tau} - ae^{(\rho\tau)-(\rho)}$$

so that ρ must be the solution of the characteristic equation

$$\rho = m - ae^{-\rho}. \tag{30}$$

It can be reinforced that if ρ is a root of the characteristic equation (30), then the function of the form $x(\tau) = e^{\rho\tau}$ satisfies (29). The equation (30) is a transcendental equation, which has infinite complex roots. That is, for every solution of the characteristic equation (30) ρ_k, a solution of the delay differential equation in (29) can be obtained, $x_k(\tau) = e^{\rho_k t}$. Therefore,

$$x(\tau) = \sum_{k=1}^{\infty} D_k e^{\rho_k \tau} \tag{31}$$

is also a solution of (29), where D_k are constant coefficients that must be adjusted to fit the solution to the initial condition ψ given in (29).

At first, the solutions of the characteristic equation in $\mathbb{C}$ must be found. Then, ρ is taken as:

$$\rho = \beta + i\alpha, \quad i = \sqrt{-1},$$

where α e β are real. With that the equation (30) takes the form

$$\beta + i\alpha = m - ae^{-\beta}e^{-i\alpha}$$

where it is noticeable that since the transcendental equation (30) is real, if ρ is a solution, so is $\overline{\rho}$. Considering Euler's formula, which states $e^{-i\alpha} = \cos\alpha - i\sin\alpha$, the equation above can be written as:

$$\beta + i\alpha = m - ae^{-\beta}(\cos\alpha - i\sin\alpha),$$

such that separating the real and imaginary parts of the equation, respectively, yields the following real equations:

$$\begin{aligned} \beta &= m - ae^{-\beta}\cos\alpha \\ \alpha &= ae^{-\beta}\sin\alpha. \end{aligned} \tag{32}$$

Hence, solutions to the above system are sought with real α and β.

3.2.1 Real roots ($\alpha = 0$)

At first, the case with real solutions of the characteristic equation is considered, which returns purely exponential solutions of the differential equation. This happens when $\alpha = 0$ and thus $\beta = m - ae^{-\beta}$. That is, in this case

$$e^{-\beta} = \frac{m - \beta}{a},$$

with $m, a > 0$ and $a > m$. Denoting:

$$f(\beta) = e^{-\beta} + \frac{\beta - m}{a},$$

the roots of f are wanted. To do so, the following limits are taken:

$$\lim_{\beta \to +\infty} f(\beta) = +\infty$$
$$\lim_{\beta \to -\infty} f(\beta) = +\infty,$$

whose calculation indicates that the function f has a minimum at the point given by

$$f'(\beta) = -e^{-\beta} + \frac{1}{a} = 0.$$

The minimum occurs in $e^{-\beta} = 1/a$, that is, $\beta = \ln a$. And as

$$f(\ln a) = \frac{1}{a} + \frac{\ln a - m}{a},$$

the minimum value of f can be positive, zero or negative. Therefore, f may have no real roots, have a single root, or have two distinct roots, depending on the value of m and a. From this observation, it can be verified for which values of m and a there is no solution, there is only one solution or there are two solutions to the characteristic equation. For simplicity:

$$m - \ln a = C, \tag{33}$$

in order to notice the distinct results that occur for $C > 1$, $C = 1$ and $C < 1$.

If $C = 1$, the characteristic equation has only one real root β. In this case, $x_1(\tau) = e^{\beta\tau}$ will be a solution. Nonetheless, $x_2(\tau) = \tau e^{\beta\tau}$ is also a solution. This statement can be verified by replacing $x_2(\tau)$ in the differential equation in (29), keeping in mind that $x'(\tau) = e^{\beta\tau} + \beta\tau e^{\beta\tau}$ and $x(\tau - 1) = \tau e^{\beta\tau} e^{-\beta} - e^{\beta\tau} e^{-\beta}$, so that

$$\tau e^{\beta\tau}(\beta - m + ae^{\beta}) + e^{\beta\tau}(1 - ae^{-\beta}) = 0.$$

And considering the argument developed until here, it is known that $\beta - m + ae^{\beta}$ and $1 - ae^{-\beta}$ are zero, because when C=1, from equation $e^{-\beta} = (m - \beta)/a$ and $\beta = \ln a$ are verified. Thus, $e^{\beta\tau}$ e $\tau e^{\beta\tau}$ are solutions.

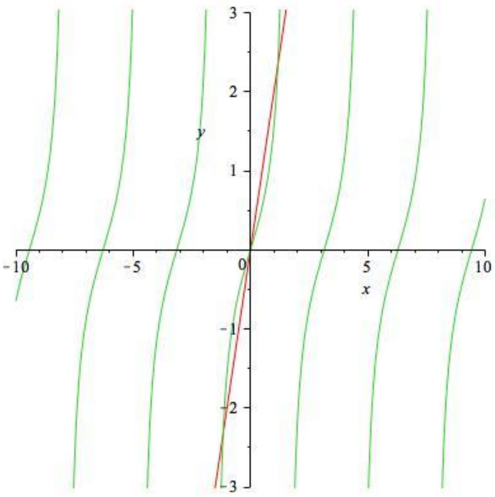
If $C > 1$, there is $f(\beta) = 0$ in two distinct points β and β', that is, there are two distinct solutions, $x_1(\tau) = e^{\beta\tau}$ and $x_2(\tau) = e^{\beta'\tau}$. Finally, if $C < 1$, f is always positive and thus, it does not have roots. So there is no purely exponential solution in this case.

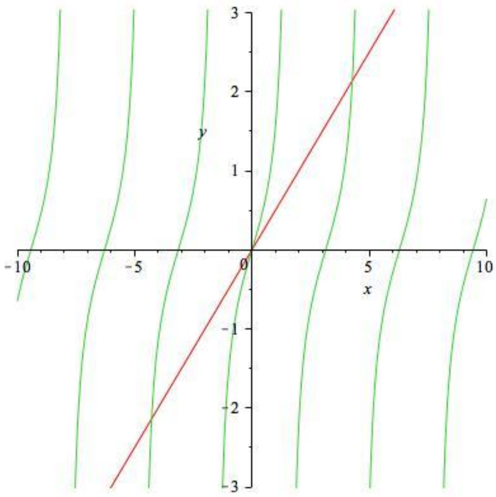
In sum, for $\alpha = 0$, according to the values of m and a, if $C \geq 1$ there are solutions $x(\tau) = D_1 e^{\beta\tau} + D_2 e^{\beta'\tau}$ or $x(\tau) = D_1 e^{\beta\tau} + D_2 \tau e^{\beta\tau}$.

3.2.2 Purely imaginary roots ($\beta = 0$)

Secondly, the question analyzed is the one when the characteristic exponents are purely imaginary and hence, the associated solutions are periodic. In this case, solutions of the characteristic equation have $\beta = 0$. Thus, the equations in (32) become:

Figure 1: $\tan\alpha = \alpha/m$: $m < 1$ (left), $m > 1$ (right)





Source: author's elaboration.

$$\begin{cases} m = a\cos\alpha \\ \alpha = a\sin\alpha, \end{cases}$$

that is, $\tan\alpha = \alpha/m$. It is worth remembering that if α is a solution, so is $-\alpha$, then it is enough to identify the positive solutions.

As can be seen in Figure (1), there are always infinite solutions, which are the intersections of the graph of $\tan\alpha$ with the straight line α/m, for every interval of α in $(k\pi + \pi/2, (k+1)\pi + \pi/2)$. However, if the slope of the line $(1/m) < 1$, there is a solution in every branch, including the first one ($k = 0$), while if $(1/m) > 1$ there is not a solution in this first branch.

Thus, in this case there are infinite solutions of the differential equation of the form

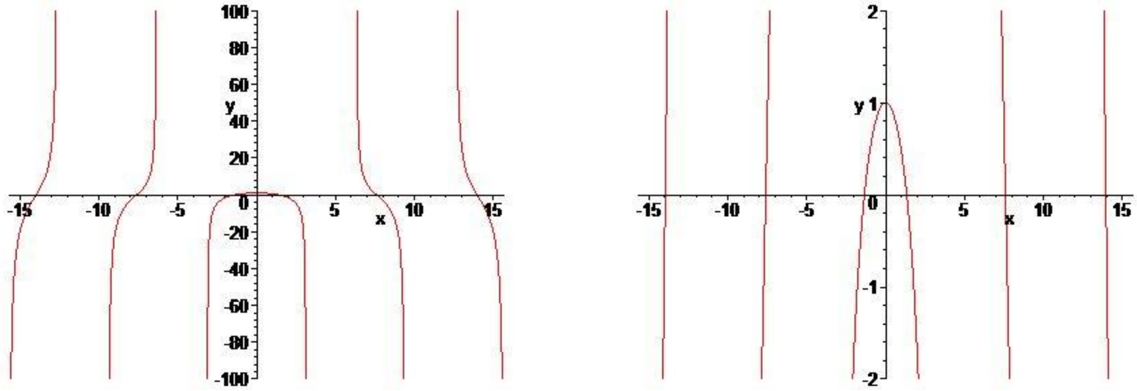
$$x_k(\tau) = A_k\cos\alpha_k\tau + B_k\sin\alpha_k\tau.$$

3.2.3 Other roots ($\alpha, \beta \neq 0$)

Finally, the solutions for when β and α are both different from zero are analyzed. From the second equation in (32):

$$e^{\beta} = \frac{a\sin\alpha}{\alpha} \tag{34}$$

Figure 2: Graph of $g(\alpha)$ in two different vertical scales



Source: author's elaboration.

what substituting in the first equation of (32) yields

$$\ln\left(\frac{a\sin\alpha}{\alpha}\right) = m - \frac{\alpha}{\sin\alpha}\cos\alpha$$

such that

$$\ln\left(\frac{\sin\alpha}{\alpha}\right) + \frac{\alpha}{\tan\alpha} = m - \ln a.$$

That is, the equation to be solved is

$$g(\alpha) = \ln\left(\frac{\sin\alpha}{\alpha}\right) + \frac{\alpha}{\tan\alpha} = C \tag{35}$$

where the constant $C = m - \ln a$, as defined in (33). So, the values of α are given by the intersections of the graph of $g(\alpha)$ with the horizontal line C. Using L'Hôpital's rule:

$$\lim_{\alpha\to 0} g(\alpha) = \lim_{\alpha\to 0} \ln\left(\frac{\sin\alpha}{\alpha}\right) + \lim_{\alpha\to 0}\frac{\alpha\cos\alpha}{\sin\alpha} = 1$$

as it is observed in Figure 2.

Besides that, it is clear that the function $g(\alpha)$ is only defined where $\sin\alpha > 0$, that is, in intervals of the form $(2j\pi, (2j+1)\pi)$, for j integer. And in which one of these intervals, except the first one, $g(\alpha)$ decreases from ∞ to $-\infty$.

For each α, the real part of the characteristic exponent, β, if given from (34) by

$$\beta = \ln a + \ln\left(\frac{\sin\alpha}{\alpha}\right) = m - \frac{\alpha}{\tan\alpha}.$$

Thus, given the parameters m and a, a solution of the delay differential equation (29) is given by the sum of the solutions corresponding to all solutions $\beta_k \pm i\alpha_k$ of the characteristic equation (32):

$$x(\tau) = x_0(\tau) + e^{\beta_1\tau}(F_1\sin\alpha_1\tau + G_1\cos\alpha_1\tau) + e^{\beta_2\tau}(F_2\sin\alpha_2\tau + G_2\cos\alpha_2\tau) + \dots$$

where

$$x_0(\tau) = \begin{cases} 0 & \text{if } C < 1 \\ e^{\beta_0 \tau}(F_0 + \tau G_0) & \text{if } C = 1 \\ F_0 e^{\beta_0 \tau} + G_0 e^{\beta_0' \tau} & \text{if } C > 1 \end{cases} \tag{36}$$

besides, the constants F_k and G_k must be determined in order to adjust the initial condition $x(\tau) = \psi(\tau)$ in $[-1, 0]$, and the β_k are organized in decreasing order.

For $C < 1$ and considering a sufficiently large and distant from the initial interval time, the predominant term in the solution will be the one corresponding to β_1, the largest of them all. Therefore, the behaviour of the following expression can be studied:

$$x(\tau) = e^{\beta_1 \tau}(F_1 \sin \alpha_1 \tau + G_1 \cos \alpha_1 \tau). \tag{37}$$

This solution represents vibrations of $x(\tau)$ with an amplitude that depends on β_1, increasing if $\beta_1 > 0$, decreasing if $\beta_1 < 0$ or constant if $\beta_1 = 0$. The period of the oscillations depends on α_1, which can be in any interval (see Figure 2).

Nonetheless, when $C = m - \ln(m + \theta n) < 1$, the period and the rate of growth/decrease no longer depend on the form of the function x (or J) in the initial interval. Therefore, the parameters m and n of the investment function (15), related to the gross accumulation (A) and the capital stock (K), respectively, as well as θ, the temporal delay, will asymptotically determine the period of oscillation and the rate of progression/regression of the amplitude of the oscillations.

The oscillations can be seen as independent from the short-term behavior of the variables in the model, either of growth or decrease, being linked only to the investment decisions - since they only depend on the parameters of the investment function. The economic system is then subject to the occurrence of cyclical variations. And the cycle, in turn, expresses the endogenous movement linked to the effective demand.

On the other hand, when the parameters are such that $C = m - \ln(m + \theta n) \geq 1$, it is necessary to take into account the presence of the purely exponential term corresponding to x_0 in (36) and discussed in the section (36). For these parameter values, this term is dominant and therefore, in this case, there are no cyclical variations in the system.

4 Consequences for the Business Cycle

4.1 Adjusting the parameters of the model

As shown in the last section, starting from the modelling of the Kalecki cycle (1935), the resolution of its synthesis equation (28) leads to the conclusion that when $m - \ln(m + \theta n) < 1$, cyclic oscillations occur in the economic system. In this case, the solution has the form as in (37):

$$x(\tau) = e^{\beta_1 \tau}(F_1 \sin \alpha_1 \tau + G_1 \cos \alpha_1 \tau). \tag{38}$$

The relevance of this result can be analyzed from the gradual recovery of the variables of interest for the interpretation of the model. First, it is worth remembering that $x(\tau) =$

$x(t/\theta) = J(t)$ such that solution (38) becomes

$$J(t) = e^{\beta_1 \frac{t}{\theta}} \left(F_1 \sin \alpha_1 \frac{t}{\theta} + G_1 \cos \alpha_1 \frac{t}{\theta} \right),$$

which can be rewritten as:

$$J(t) = e^{\beta_1 \frac{t}{\theta}} F_1 \sin \left(\alpha_1 \frac{t}{\theta} + \varphi \right). \tag{39}$$

In this solution, in turn, a translation in time is made, taking $\varphi = 0$, resulting in the function $J(t)$ below

$$J(t) = e^{\beta_1 \frac{t}{\theta}} F_1 \sin \alpha_1 \frac{t}{\theta}, \tag{40}$$

and taking the meaning of $J(t)$, which refers to the deviation of the volume of investment orders (I) in relation to the physical depreciation of capital (U) yields

$$I(t) - U = F_1 e^{\beta_1 \frac{t}{\theta}} \sin \alpha_1 \frac{t}{\theta}. \tag{41}$$

From this moment on, relations that allow the calibration of the parameters by applying empirical values are sought, returning to the results of Kalecki (1935). In general terms, it is important to adjust the parameters $m, n,$, and θ such that $C = m - \ln(m + \theta n) < 1$, to determine the behaviour of the cycle numerically.

To do so, in his original work, the author assumes that cycles have a constant amplitude. This assumption is based on his observation of the regular growth/decrease of the cyclical oscillations' amplitude that occur historically, which would suggest this as the closest condition to the reality [9]. Therefore, it is assumed that $\beta = 0$ and consequently $I(t) - U$ becomes:

$$I(t) - U = r \sin \alpha_1 \frac{t}{\theta}, \tag{42}$$

where r is the constant amplitude.

From the assumption that $\beta = 0$, it is possible to rewrite the relations that determine α_1 and that relate it to m and n, as seen in section (3.2.2), as follows:

$$cos\alpha_1 = \frac{m}{m + \theta n} \tag{43}$$

$$\frac{\alpha_1}{\tan \alpha_1} = m. \tag{44}$$

Nevertheless, the parameters m and n are also related by (19):

$$I = m(B_0 + A) - nK,$$

which relates the volume of orders I to the gross accumulation A and the capital stock K through the parameters of interest m and n.

[9] Frisch and Holme (1935) criticize this assumption of Kalecki (1935). For the authors, even in this modeling, cyclical fluctuations would be maintained by exogenous shocks, due to the instability of endogenous fluctuations. However, it is noted here that Kalecki aims to show the instability intrinsic to the economic system, derived from the endogenous forces present in this system. Thus, this assumption does not invalidate his contribution.

Besides that, the other relationships concerning the investment theory - which, in addition to the volume of orders I and the gross accumulation A, also involve the volume of orders delivered D - can be resumed from (16), (17) and (18):

$$\begin{aligned} D(t) &= I(t-\theta) \\ A(t) &= \frac{1}{\theta}\int_{t-\theta}^{t} I(\tau)d\tau \\ K'(t) &= D(t) - U. \end{aligned}$$

From these relations, Kalecki (1935) states that the average of I, A and D during the cycle, for the case of constant amplitude, is U, around which I, A e D would oscillate. The author also establishes $K'(t) = K - K_0$. Thus, equation (19) presented above can be rewritten as:

$$U = m(B_0 + U) - nK_0, \tag{45}$$

$$n = (m-1)\frac{U}{K_0} + m\frac{B_0}{K_0}. \tag{46}$$

Therefore, the values of U/K_0 and B_0/K_0 are sought to find m, n, and θ, to determine the length of the cycle. According to Kalecki (1935), U is the depreciation, the demand for equipment restoration; K_0 is the average volume of equipment - thus, U/K_0 is the depreciation rate - and B_0 is the constant part of capitalist consumption.

The author calculates these values based on German statistics for the 1930s, obtaining a gestation period $\theta = 0.6$ years, amortization rate $U/K_0 = 0.05$ and $B_0/K_0 = 0,13$. Substituting these values into equations (43), (44) and (46), he obtains $m = 0.95$, $n = 0.121$, and $\alpha_1 = 0.378$. In this case, the cycle lasts

$$T = \frac{2\pi}{\alpha_1}\theta = \frac{2\pi}{0,378}0,6 = 10$$

, 10 years. [10] Replacing the values found, the investment function then becomes:

$$I - U = r\sin 0,63t \tag{47}$$

and the other equations of the model can also be adjusted: [11]

$$\begin{aligned} A - U &= 0,98r\sin 0,63(t-0,3) \\ D - U &= r\sin 0,63(t-0,6) \\ K - K_0 &= -1,59r\cos 0,63(t-0,6). \end{aligned}$$

Thus, the behavior of the economic cycle mechanism is modeled, and the same can be interpreted according to these results.[12]

[10]A period that, according to Possas (1999), other models of the macroeconomic cycle sought.

[11]From the adjusted equations, the author simulates the mechanism of the economic cycle, sketching curves that illustrate it, which can be found in the author's original work of 1935.

[12]Possas (1999) emphasizes the plausibility of this model because of precisely this trajectory generated, considered "realistic" from the data collected by Kalecki (1935).

4.2 The business cycle dynamics

As was seen in the course of this paper, the intuition associated with the cycle's operating mechanism is the interaction between investment, income, and capital accumulation. When investment I increases, the production of these goods also increases, by the principle of effective demand, stimulating new I. Nevertheless, while the accumulation A is greater than I there is an increase in the capital stock K, with a negative effect on the profitability of I and the own volume I. Initially, this effect compresses the rate of I, and this fall will lead to the declining phase of the cycle.

In sum, the cycle is caused by the dual role of investment. While investment generates demand and new investment, it also increases the capital stock, affecting its profitability and reducing the volume of new investments. Kalecki (1990 [1933]) expands the interpretation of this analysis by decomposing the cycle into four phases: recovery, expansion, recession, and depression. [13] Kalecki (1990 [1933]) defines these phases starting with the recovery.

Kalecki (1990 [1933], p. 38) means by recovery the period in which investment orders exceed the level of replacement needs. However, the capital equipment has not yet begun to expand, because the deliveries of new equipment are even smaller than the replacement needs. The production of capital goods A grows, although the volume of equipment K is still decreasing, and it follows that investment orders I rise sharply.

Thus, an increase in equipment orders causes an increase in the production of investment goods, A. The increase in A, in turn, causes an increase in investment activity, as seen in equation (19). However, after a time interval of θ, the volume of equipment begins to grow. This increase marks the expansion, the next phase, where deliveries of new equipment exceed replacement needs. The increase in K initially restricts the growth of new orders to soon make them decline, followed by a decrease in the production of new goods, A.

In the following phases, the process described is reversed. In the recession, the investment orders, I, are below the level necessary to cover the restoration of the equipment. However, the volume of existing equipment is still growing, because deliveries are larger than demand. As production of capital goods continues to decline, with K increasing, investment orders fall sharply. Following is the depression, in which investment orders are below the replacement level, reducing by K. Initially, this decrease smooths the decay in orders, while in the second part of the depression it causes the return of production of investment goods, taking the cycle back to the recovery.

Hence, the investment activity does not stabilize either at a level that exceeds capital replacement needs or at a level below that would provide adequate replacement, never adjusting to the desired level of capacity utilization. According to Possas and Baltar (1981), because of the capitalist competition, it is as if the economy as a whole was pursuing the equilibrium target which, however, is never reached. Therefore, the system does not

[13] Regarding this aspect, it is worth emphasizing that these phases should be understood as phases in which the cycle decomposes, not as its components. The cycle is understood in its integrity, and these phases are outlined only from its global understanding. As Possas and Baltar (1981, p. 149) argue, for the same reason there is no reference of any kind to moments of cyclical reversal, since all moments in this process could also be considered as points of reversal, without loss of generality.

establish itself at an equilibrium, since even if investment orders remain at a constant level, the production of capital goods will change as capital equipment expands, when investment is greater than replacement needs. Under such conditions, however, investment orders will start to decrease and the stability of investment will be disrupted (KALECKI, 1990 [1933], p. 36).

From this understanding of the cycle mechanism, some conclusions of this model can be highlighted. First, Kalecki's business cycle model is a natural unfolding of his conception of effective demand. According to this view, the investigation of the cycle depends only on investment, the main component of its dynamics, and its contradictory behavior. As a result, as Possas (1999) points out, the model explains the dynamic mechanism associated with effective demand, which is responsible for producing cyclical fluctuations in the system under certain conditions. Therefore, the cycle can be explained only by this behavior of effective demand, without resorting to technical progress or structural change, depending solely on the usual movement of economic activity. Besides, the system would also not tend to an equilibrium in a strict sense. On the contrary, the capitalist cycle would have the property of being dynamically unstable, not moving between equilibria at any moment, only continuously fluctuating.

However, according to Possas and Baltar (1981), in addressing the question of growth Kalecki asserts that it is not guaranteed that the cyclical movement of the capitalist system will be expressed in a long-term expansion - when there are no other exogenous movements that alter the interaction of the variables through effective demand. In this regard, Possas (1999) argues that Kalecki divides the dynamics of the capitalist economy into two theoretically distinct components. The first is related to the effective demand, which produces fluctuations in the economic system. The second would be structural change, which changes the parameters of the model, such as technical progress, innovations and the opening of new markets. Structural change is the component of tendency - exogenous to this model - that would generate unstable trajectories from a structural point of view and would be a necessary condition for the expansion of the system. The combination of these two components would produce an integrated trajectory, by the "sum" of these effects, since the model is linear.

Nonetheless, the theory of business cycles reviewed in this paper reveals that in the absence of structural change, the capitalist economy would not show a positive trend, but a trajectory of fluctuations around a steady state, forming a pure cycle (POSSAS, 1999, p. 40). And this movement, as analyzed here, is explained by the lagged, and therefore doubled, role of investment.

5 Concluding Remarks

This paper revisited Kalecki's (1935) business cycle model, focusing on his modeling with a delay differential equation, which we solved in a general way. To do that, we resumed some of Kalecki's most important contributions. For instance, his analysis of the economic cycle is guided by the determinants of investment and the factors under its influence.

His 1935 model is simple, consisting only of a consumption function, a multiplier, and an investment function. However, the dynamics is inserted by the proposition that there is an average gestation period for investment, θ, the time interval between the decision to invest and the delivery of the completed equipment. With that, Kalecki obtains that the investment decision is a growing function of the gross accumulation $A(m)$ and of the capital stock $K(n)$, a relation that allows the establishment of a delay differential equation for investment I as a function of time t.

By presenting the theory that specifically concerns the delay differential equation, we were able to find solutions for the model in a more general way than in the original work of the author. Delay differential equations, unlike ordinary differential equations, require broader initial conditions so that their solutions can be obtained - which depend not only on the knowledge of the function at the initial instant t_0 but also on its knowledge in a certain interval before $t_0, t - \theta$. This resolution path is the main contribution of this paper to the literature. On the one hand, the generality of this resolution makes it applicable to other studies that use similar models. This paper also implicitly argued, therefore, that delay differential equations can be better appropriated by macroeconomics, since their aspects of both difference and differential equations allow them to capture time in a more precise way. On the other hand, the implications of the solutions reinforce that the economic system is subject to endogenous cyclical fluctuations. These fluctuations are linked to the interaction between investment, income, and capital accumulation through effective demand, without depending on the "initial conditions" of the functions and/or variables representing, for example, consumption and investment. In sum, this paper presented a set of theoretical concepts and modeling strategies that can be appropriated to obtain new understandings about the conformations of the capitalist economy.

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