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Real-time state estimator without noise covariance matrices knowledge – fast minimum norm filtering algorithm

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Abstract: The digital filtering technology has been widely applied in a majority of signal processing applications. For the linear systems with state-space model, Kalman filter provides optimal state estimates in the sense of minimum-mean-squared errors and maximum-likelihood estimation. However, only with accurate system parameters and noise statistical properties, the estimation obtained by standard Kalman filter is the optimal state estimate. Most of time, the exact noise statistical properties could not be obtained as a priori information or even wrong statistical properties may be captured by the offline method. This may lead to a poor performance (even divergence) of Kalman filtering algorithm. In this study, a novel real-time filter, named as fast minimum norm filtering algorithm, has been proposed to deal with the case when the covariance matrices of the process and measurement noises were unknown in the linear time-invariant systems with state-space model. Tests have been performed on numerical examples to illustrate that the fast minimum norm filtering algorithm could be used to obtain acceptable precision state estimation in comparison with the standard Kalman filter for the discrete-time linear time-invariant systems.

1 Introduction

State estimation is a key part in most applications of model-based control techniques since feedback control is usually designed based on accurate state information in most existing control approaches, such as model predictive control and linear quadratic regulator. For example, in [1] a high-gain observer is adopted to cope with the problem of estimating partial state information for the purpose of synchronised tracking control, and it is well known that linear state observers can be designed to generate asymptotically accurate state estimates for a deterministic linear system. For stochastic linear systems with state-space model, Kalman filtering algorithm is known as a common optimal state estimating method and has been widely used in many applications, such as navigation [2], communication [3] and fault diagnosis [4]. In Kalman's pioneering work [5, 6], state estimate and prediction problems were described in a different versions; he also proposed a method for the optimal solution of this general problem when system model is linear, precisely known and the statistical properties of process and measurement noises are obtained precisely in advance.

However, the requirements of Kalman filtering algorithm can be seldom completely satisfied in the practical engineering systems for a variety of reasons. In order to use Kalman filtering algorithm to deal with different state estimating problems, a series of modified Kalman filter algorithms were proposed to resolve certain practical issues. Stanley F. Schmidt proposed extend Kalman filter (EKF) [7, 8] algorithm to solve filtering and prediction problems of non-linear systems encountered in Apollo program in the 1960s. On the basis of the idea of Taylor expansion, EKF is designed to approximate non-linear system by linear model and then use standard Kalman filter to estimate system states for the linearised system model in real time. Such a modified filtering algorithm was proved to be very useful in many applications, and hence EKF algorithm was ever regarded as a kind of standard filtering method for non-linear systems for a long time and has attracted huge amount of research interests [9, 10]. Then Carlson [11] presented federal Kalman filter [12] algorithm to deal with multi-system optimal data fusion. On the basis of the covariance matrix of the estimation error, different estimations from various measurement systems were used to obtain more precise real-time state estimate for multiple systems with state-space model.

Since the estimation obtained by EKF algorithm has unacceptable precision when system model is strongly non-linear, unscented Kalman filter (UKF) [13] was introduced by Julier and Uhlmann in 1997, and later applied in various areas, such as induction motor drives [14] and UAV attitude estimation [15]. In recent years, some variants of UKF were proposed in the literature [16, 17]. When the system model has uncertainty in the parameters, the modified Kalman filter algorithms above were not suitable to be used to obtain state estimation in real time. Robust Kalman filter [18–21] was proposed to solve the problems when the transfer and/or measurement matrices have uncertainty error with a bounded norm in a linear system with state-space model. Adaptive Kalman filter (AKF) [22, 23] was used to deal with the cases where process and measurement noise statistical properties are uncertain.

Standard Kalman filter is optimal in the sense of minimum-mean-squared errors and maximum-likelihood estimation, provided that the system model is linear and precisely known a priori and the process and measurement noises are independent and Gaussian with known covariance matrices and zero mathematical expectation. However, in practice these requirements can seldom be completely satisfied because of the following reasons:

(a) The mathematical system models of engineering systems are usually non-linear and/or time varying although many of them may be approximated by linear system models.
(b) Even if the practical system in consideration is linear and/or time-invariant, the system model may not be exactly known with accurate system parameters. In practice, model parameters may be approximately identified by applying some system identification methods offline through the data obtained via extensive experiments. However, this approach is usually expensive and does not

guarantee accurate system identification, which may make standard Kalman filter not working very well for the identified model. Furthermore, if the practical system is in fact time varying, the approach of system identification usually fails.
(c) Standard Kalman filter requires that the process and measurement noises are zero-mean random noises. However, in some applications the noise may be biased and its mean or mathematical expectation may be unknown. In such cases, further noise modelling is often needed and it is possible to use Kalman filter by augmenting the mean of the noise as an extra state or other modified methods.
(d) In most cases, the covariance matrices of process and measurement noises cannot be exactly obtained in advance. Therefore to apply the standard Kalman filter we must first take effort to obtain the statistical properties of the process and measurement noises, which are usually calculated from extensive engineering testing experiments. To deal with this problem, an alternative approach is to simply use larger covariance matrices to represent the a priori knowledge on the process and measurement noises.
(e) In practice, the probability distribution of process noise or measurement noise may not be normal distribution, that is, 'non-Gaussian' systems, which often results that the performance of standard Kalman filter may degrade much or even become divergent.

Owing to the above mentioned issues, to resolve these practical problems to some extent some extensions or variants of standard Kalman filter have been proposed in the literature. For example, for issue (a), EKF is one approach to apply technique of Kalman filter to non-linear systems based on the idea of Taylor expansion, whereas UKF [13] and particle filters [24, 25] are two typical approaches to handle filtering problem for non-linear/non-Gaussian systems addressed by issue (a) or issue (e). Comparing with other issues, issue (c) may be relatively easy to resolve, and one typical approach has been mentioned above if we have some physical or a priori knowledge on the sources of possible errors in the sensors. As to issue (d), there exists one approach called AKF [22, 26, 27], whose idea is to adaptively estimate the uncertain statistical properties of the noises and combine the Kalman filter with the modified covariance matrices. Kalman filter based on support vector machine [28] may be regarded as another example to deal with the issue (d). As to challenging issue (b), a general framework of finite-model Kalman filtering (FMKF) was introduced based on the idea that the large model uncertainty may be restricted by a finite set of known different models, and under this framework, a simple yet effective finite-model Kalman filter, termed as MVDP-FMKF, was discussed extensively according to the idea of adaptive switching via the minimum vector distance principle [29].

In this paper, we will present a new algorithm to solve the state estimation problem for the discrete-time linear time-invariant systems when the covariance matrices of both the process and measurement noises were completely unknown. Since the process and measurement noise covariance matrices could not be obtained in advance, the standard Kalman filter and AKF became not suitable to be used to solve the state estimation problem in such cases any more. Since most modified filtering algorithms are in fact based on Kalman filter, the results from these modified filters are thus always suboptimal. Referring to the framework of standard Kalman filter, the so-called fast minimum norm filtering algorithm proposed in this paper tries to design a new framework to solve the filtering problem when the covariance matrices of process and measurement noises were absent in the state-space model. Finally, numerical simulation examples were presented to illustrate that the fast minimum norm filtering algorithm can effectively deal with the problem of state estimation when the process and measurement noise covariance matrices were completely unknown for the discrete-time linear time-invariant (LTI) systems.

The rest of this paper is organised as follows: in Section 2, problem formulation is performed for the fast minimum norm filtering algorithm and some assumptions are made for convenience of further analysis. In Section 3, the fast minimum norm filtering algorithm is presented to obtain state estimation of discrete-time LTI systems in the absence of process and measurement covariance matrices. For the convenience of discussion, the problem has been classified into four different cases; in Section 4, numerical simulation studies on a number of examples will be carried out to illustrate the effectiveness of the fast minimum norm filtering algorithm; finally, in Section 5, concluding remarks are presented to summarise the fast minimum norm filtering algorithm and our prospective further work to improve the proposed algorithm.

2 Problem formulation

A linear state-space model is often used in practice to approximate true plants. Consider the following typical state-space model

$$\begin{cases} x_k = A_k x_{k-1} + B_k u_{k-1} + w_{k-1} \\ y_k = C_k x_k + D_k u_{k-1} + v_k \end{cases} \tag{1}$$

where x_k is the system state vector, u_k represents the system input, A_k stands for the state transfer matrix, C_k is the measurement matrix, y_k is the system measurement, w_k is the process noise and v_k is the measurement noise.

For linear plant (1), the control signal u_k is often designed as a linear controller, for example, $u_k = K_k x_k$, thus we may obtain the resulting closed-loop system without explicit control inputs. Hence, without loss of generality, we assume that assume that $u_k \equiv 0$. The system model should be simplified as follows

$$\begin{cases} x_k = A_k x_{k-1} + w_{k-1} \\ y_k = C_k x_k + v_k \end{cases} \tag{2}$$

The linear filtering problem is trying to obtain system state estimates in real time according to the measurement sequence based on the system model. Kalman filter could obtain the optimal state estimation based on the measurement sequence when the state-space model of the linear system and the statistical properties of both the process and measurement noises have been precisely obtained. However, the a priori requirements of Kalman filter can be hardly completely satisfied in practice. This may degrade the performance of the Kalman filter or even cause the estimation error divergent with the incorrect system parameters. In general, state transfer matrix A_k could be often obtained by the relationship of the system states or kinetic relationship among each variable of the state. The measurement matrix C_k is on behalf of the mathematical relationship between the system state and measurement signal. The system process noise sequence $\{w_k\}$ represents dynamic uncertainty. Since the filter deals with the measurement signal in real time, the statistical properties of the random variable w_k can be seldom determined in advance. The measurement noise sequence $\{v_k\}$ represents the credibility of the measurement signal. When the measurement signal is directly obtained from the sensors, the statistical properties of measurement noise could be determined by the accuracy of measurement physical sensors. In other cases, the statistical properties of the random variable v_k can hardly be determined in practice. However, these statistical properties play an important role in the linear filtering algorithm, such as Kalman filter. Kalman filter is not suitable for obtaining the state estimates without the statistical properties of w_k and v_k.

In this paper, we are trying to deal with the linear system state estimation problem when the statistical properties of w_k and w_k were completely unknown for the state-space system model.

Mathematically speaking, in this paper we are going to investigate this fundamental problem under the following technical assumptions:

Assumption 1: For simplicity, supposing that the discrete-time linear system is time invariant. Equivalently, the system could be described as follows

$$\begin{cases} x_k = Ax_{k-1} + w_{k-1} \\ y_k = Cx_k + v_k \end{cases} \tag{3}$$

where x_k is the system state, A is the state transfer matrix, C is the measurement matrix, y_k is the system measurement, w_k is the process noise and v_k is the measurement noise.

Assumption 2: For the discrete-time LTI system (3), supposing that the transfer matrix A and measurement matrix C are constant and known, and the controllability (regarding the process noise as control input) and observability matrices

$$\mathcal{Q}_c = [e, Ae, \ldots, A^{p-1}e] \tag{4}$$

$$\mathcal{Q}_o = [C; CA; \ldots; CA^{p-1}] \tag{5}$$

are of full rank, where $e = [1, 1, \ldots, 1]^{\mathrm{T}}$ and p is the least number such that the $\mathcal{Q}_c$ and $\mathcal{Q}_o$ are of full rank.

Assumption 3: The mathematical expectation of the variables w_k and v_k is equal to 0. The state transfer matrix A and measurement matrix C could be exactly obtained.

Problem 1: In the absence of the statistical covariance matrices of processing and measurement noises, how to design a real-time filtering algorithm for the discrete-time LTI systems with state-space model (3)?

For the discrete-time LTI systems, the state estimation Kalman filter is a kind of optimal estimation in the sense of least mean squares and maximum likelihood when the system parameters and noise statistical properties could be determined a priori. However, the statistical properties of the process and measurement noises were hardly determined or even cannot be obtained in advance in most engineering systems. Then various modified filtering algorithms based on Kalman filter had been presented in order to use Kalman filter to deal with real-time practical state estimation problem. In engineering practice, one frequently used method is to use the upper bound of noise covariance matrix replacing the correct one in the Kalman filter. The side effect of this method is that the state estimation from the filter is not optimal any more and this method may even cause the filter divergent if the upper value is not suitable for the system model. For the reasons above, the idea of adaptive control was introduced into state estimation problem to deal with the case where the noise statistical properties could not be completely determined a priori. According to the measurement sequence, adaptive regulating principle was designed a priori to adjust the covariance matrix of state estimation error based on the idea of adaptive control. The above two methods were not suitable to be used to obtain the state estimation if the process and measurement noise covariance matrices were completely unknown. Besides, all of these modified algorithms were based on the framework of standard Kalman filter and the results from these filters were always suboptimal and the stability is difficult to be established in mathematics.

In this paper, we will focus on the filtering problem when the process and measurement covariance matrices were completely unknown and propose a new algorithm, named as fast minimum norm filtering algorithm, which is based on a new framework compared with the standard Kalman filter, and two theorems will be presented to illustrate that the covariance matrix of estimation error of the fast minimum norm filtering algorithm has a certain upper bound.

3 Fast minimum norm filtering algorithm

In this section, a new algorithm will be presented to cope with the problem described in the previous section. When the mean of the noise random variable is equal to 0 and the covariance matrix is completely unknown in advance, we try to formulate a new framework which is different from Kalman filer and present a new algorithm to deal with the state estimation problem without the knowledge noise on covariance matrices for the discrete-time LTI systems with state-space model.

For simplicity yet without loss of generality, the following LTI system is considered

$$\begin{cases} x_k = Ax_{k-1} + w_{k-1} \\ y_k = Cx_k + v_k \end{cases} \tag{6}$$

where $w_k \in R^{n\times 1}$, $x_k \in R^{n\times 1}$, $v_k \in R^{m\times 1}$, $y_k \in R^{m\times 1}$, $A \in R_{n\times n}$ and $C \in R^{m\times n}$ are process noise, state, measurement noise, output, transfer matrix and output matrix, respectively. Supposing that $E(w_k) = 0$, $E(w_k) = 0$ and $\mathrm{Cov}(w_k) = Q$, $\mathrm{Cov}(v_k) = R$ but Q and R are completely unknown a priori, where $E(\cdot)$ is the mathematical expectation and $\mathrm{Cov}(\cdot)$ is the covariance matrix. We hope to estimate the unknown states $\{x_k\}$ with the only available output data.

To resolve Problem 1, a new algorithm named as fast minimum norm filtering algorithm will be presented in this section and the algorithm can be divided into two steps: (i) computing $\hat{x}_{k,k-p}$ based on previous measurements where $\hat{x}_{k,k-p}$ is the prediction of the system state x_k at the kth time step; and (ii) using the new measurement to modify the prediction $\hat{x}_{k,k-p}$ and obtain the real-time estimation $\hat{x}_k$ at the kth time step based on the measurement up to and including the kth measurement data.

3.1 Time update

From Assumption 2, it can be obtained that the system observation matrix is

$$Q_o = [C; CA; \ldots; CA^{p-1}] \tag{7}$$

and $\mathrm{rank}(Q_o) = n$, where n is the dimension of the system state.

Introducing the notation

$$Q_{ob} \triangleq [C; CA; \ldots; CA^{p}] \tag{8}$$

It can be obtained that $\mathrm{rank}(Q_{ob}) = n$.

Then the least-square estimation of the variable x_{k-p} could be obtained as follows

$$\hat{x}^{ls}_{k-p} = [Q^{\mathrm{T}}_{ob} Q_{ob}]^{-1} Q^{\mathrm{T}}_{ob} Y_k \tag{9}$$

where $Y_k = [y_{k-p}, y_{k-p+1}, \ldots, y_k]^{\mathrm{T}}$.

The estimation of x_k at $(k-p)$th step can be obtained as

$$\hat{x}_{k,k-p} = A^p \hat{x}^{ls}_{k-p} \tag{10}$$

From Assumption 3, we have $E[w_k] = 0$ and $E[v_k] = 0$. Then it can be obtained that $\hat{x}^{ls}_{k-p}$ is a kind of unbiased estimation of the variable x_{k-p}, and $\hat{x}_{k,k-p}$ is an unbiased estimation of x_k, according to the discrete-time LTI systems with state-space model as (6).

3.2 Measurement update

It is well known that Kalman filtering algorithm is a general method of using the measurement to modify the time prediction of state estimate as the following equation

$$\begin{aligned} \hat{x}_k &= \hat{x}_{k,k-p} + \Delta\hat{x}_k \\ &= \hat{x}_{k,k-p} + K_k(y_k - C\hat{x}_{k,k-p}) \end{aligned} \tag{11}$$

where K_k is the Kalman filter gain, that is, the optimal gain under the mean of least square with exact system parameter. According to the formulation of Kalman filter, K_k is coupled with the noise covariance matrices Q and R. However, the optimal K_k is hard to obtain when the noise covariance matrices are completely unknown. For such cases, we are trying to present some effective choice of modified gain K_k to make the incremental correction $\Delta\hat{x}_k$ minimum norm and easy to be formulated in this paper.

To this end, the measurement updating part is divided into four different cases and we design different measurement updating methods separately for each case for the discrete-time LTI system (6).

3.2.1 Case I: $m \geq n$ and $\text{rank}(C) = n$:

For this case, it can be obtained that the measurement matrix is of full column rank. Referring to the framework of Kalman filter algorithm, time updating and measurement updating parts are re-designed for Problem 1.

In this case, since the measurement matrix C is of full column rank, the filtering gain can be obtained as follows

$$K_k = [C^{\mathrm{T}}C]^{-1}C^{\mathrm{T}} \tag{12}$$

So the state estimation at the kth time step could be obtained

$$\begin{aligned}\hat{x}_k &= \hat{x}_{k,k-1} + \Delta\hat{x}_k \\ &= \hat{x}_{k,k-p} + K_k(y_k - C\hat{x}_{k,k-p})\end{aligned} \tag{13}$$

Remark 1: To intuitively understand the gain in (12), a simple example would be presented in this remark when the measurement matrix $C = [1 \quad 1 \quad 1]^{\mathrm{T}}$. From (12), we have that $K_k = [\frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}]^{\mathrm{T}}$, which means that

$$\Delta\hat{x}_k = \begin{bmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{bmatrix}^{\mathrm{T}}(y_k - C\hat{x}_{k,k-p}) \tag{14}$$

since the measurement noise covariance matrix is completely unknown. In such case, the K_k choice is consistent with the commonly used method of taking mean of sensor measurements.

3.2.2 Case II: $m \geq n$ and $\text{rank}(C) = r < n$:

It can be seen that this case is different from Case I and the solution for Case I is not suitable any more.

Similar to Case I, it is hard to construct a filtering gain. Hence, we try to modify the measurement equation in order to find suitable filtering gain K_k. Since $\text{rank}(C) = r < n$, the singular value decomposition of measurement matrix C can be given by

$$C = U\begin{bmatrix}\varepsilon_{r\times r} & 0 \\ 0 & 0_{(m-r)\times(n-r)}\end{bmatrix}V^{\mathrm{T}} \tag{15}$$

where U and V is $m \times m$ unitary matrix, $\varepsilon_{r\times r} = \text{diag}(\delta_1, \delta_2, \ldots, \delta_r)$ and δ_i is the singular value of the matrix C .

Introducing the notation

$$\Delta z_k \triangleq (U^{\mathrm{T}}(y_k - \hat{y}_k))_r \tag{16}$$

where $(U^{\mathrm{T}}(y_k - \hat{y}_k))_r$ denotes the first r rows of the matrix $(U^{\mathrm{T}}(y_k - \hat{y}_k))$

$$C_{SVD} \triangleq [\varepsilon_{r\times r} \quad 0]V^{\mathrm{T}} \tag{17}$$

$$v_k' \triangleq (U^{\mathrm{T}}v_k)_r \tag{18}$$

Then, it can be obtained that

$$\Delta z_k = C_{SVD}\Delta x_k + v_k' \tag{19}$$

where $\Delta x_k = x_k - \hat{x}_{k,k-p}$

Since $\text{rank}(C_{SVD}) = r$ and $C_{SVD} \in R^{r\times n}$, $C_{SVD}C_{SVD}^{\mathrm{T}}$ is invertible, therefore the minimum norm solution of the variable Δx_k could be obtained as follows

$$\Delta\hat{x}_k = C_{SVD}^{\mathrm{T}}(C_{SVD}C_{SVD}^{\mathrm{T}})^{-1}\Delta z_k \tag{20}$$

Then the state estimation at the kth time step could be obtained

$$\hat{x}_k = \hat{x}_{k,k-p} + C_{SVD}^{\mathrm{T}}(C_{SVD}C_{SVD}^{\mathrm{T}})^{-1}\Delta z_k \tag{21}$$

3.2.3 Case III: $m < n$ and $\text{rank}(C) = m$:

From (6), it can be obtained that

$$y_k - \hat{y}_k = C[x_k - \hat{x}_{k,k-p}] + v_k \tag{22}$$

where $\hat{y}_k = C\hat{x}_{k,k-p}$.

Introducing the notation, $\Delta x_k \triangleq x_k - \hat{x}_{k,k-p}$.

Since $\text{rank}(C) = m$, the minimum norm solution of the variable Δx_k could be obtained by

$$K_k = C^{\mathrm{T}}(CC^{\mathrm{T}})^{-1} \tag{23}$$

$$\Delta\hat{x}_k = K_k(y_k - \hat{y}_k) \tag{24}$$

Then, the state estimation could be obtained as follows

$$\hat{x}_k = \hat{x}_{k,k-p} + K_k(y_k - \hat{y}_k) \tag{25}$$

Remark 2: For clarity of understanding, a simple example is presented in such case when the measurement matrix $C = [1 \quad 0 \quad 0]$. From (23), we have that $K_k = [1 \quad 0 \quad 0]^{\mathrm{T}}$. Obviously there is no correct information for the second and third elements of state time prediction in the $\Delta\hat{x}_k$ since it can be obtained that there is no information of the second and third elements of system state in the measurement and the measurement cannot be used to update the state prediction. In such case, the K_k choice is also consistent with the intuitive physical method.

3.2.4 Case IV: $m < n$ and $\text{rank}(C) = r < m$:

From (6), it can be obtained that

$$y_k - \hat{y}_k = C[x_k - \hat{x}_{k,k-p}] + v_k \tag{26}$$

where $y_k = C\hat{x}_{k,k-p}$.

Since $\text{rank}(C) = r < n$, the singular value decomposition of measurement matrix C is given by

$$C = U\begin{bmatrix}\varepsilon_{r\times r} & 0 \\ 0 & 0_{(m-r)\times(n-r)}\end{bmatrix}V^{\mathrm{T}} \tag{27}$$

where U and V is $m \times m$ unitary matrix, $\varepsilon_{r\times r} = \text{diag}(\delta_1, \delta_2, \ldots, \delta_r)$ and δ_i is the singular value of the matrix C.

Introducing the notation

$$\Delta z_k \triangleq (U^{\mathrm{T}}(y_k - \hat{y}_k))_r \tag{28}$$

where $(U^{\mathrm{T}}(y_k - \hat{y}_k))_r$ represents the first r rows of the matrix $(U^{\mathrm{T}}(y_k - \hat{y}_k))$. Let

$$C_{SVD} \triangleq [\varepsilon_{r\times r} \quad 0]V^{\mathrm{T}} \tag{29}$$

$$v_k' \triangleq (U^{\mathrm{T}}v_k)_r \tag{30}$$

Then, it can be obtained that

$$\Delta z_k = C_{SVD}\Delta x_k + v_k' \tag{31}$$

where $\Delta x_k = x_k - \hat{x}_{k,k-p}$.

Algorithm 1

Initialisation: $x_0 = 0$
Input: the measurement sequence $\{y_k\}$
Output: state estimated sequence $\{\hat{x}_k\}$

1: Calculating the matrix Q_{ob} as in Eq.(8)
2: Calculating the matrix $Q_t = [Q_{ob}^T Q_{ob}]^{-1} Q_{ob}^T$
3: Switching cases by judging the measurement matrix C
  Case I: $K_k = [C^T C]^{-1} C^T$
  Case II: $K_k = C_{SVD}^T (C_{SVD} C_{SVD}^T)^{-1}$
  Case III: $K_k = C^T (CC^T)^{-1}$
  Case VI: $K_k = C_{SVD}^T (C_{SVD} C_{SVD}^T)^{-1}$
4: **for** $k = 1$ to N **do**
5:   Calculating the state prediction $\hat{x}_{k,k-p}$ as Eq.(9)
6:   Switching cases by judging the measurement matrix C
   Case I: calculating the state estimate $\hat{x}_k$ as in Eq.(13)
   Case II: calculating the Δz as in Eq.(16), then obtained the state estimate $\hat{x}_k$ as in Eq.(21)
   Case III: calculating the state estimate $\hat{x}_k$ as in Eq.(25)
   Case VI: calculating the Δz as in Eq.(28), then obtained the state estimate $\hat{x}_k$ as in Eq.(32)
7: **end for**
8: Return $\{\hat{x}_k\}$

Fig. 1 *FMNF($\{y_k\}$, A, B, C)*

Noting that $\mathrm{rank}(C_{SVD}) = r$ and $C_{SVD} \in R^{r\times n}$, the minimum norm solution of the variable Δx_k could be obtained as follows

$$\Delta \hat{x}_k = C_{SVD}^{\mathrm{T}} (C_{SVD} C_{SVD}^{\mathrm{T}})^{-1} \Delta z_k \tag{32}$$

The state estimation $\hat{x}_k$ at the kth step could be obtained

$$\hat{x}_k = \hat{x}_{k,k-p} + C_{SVD}^{\mathrm{T}} (C_{SVD} C_{SVD}^{\mathrm{T}})^{-1} \Delta z_k \tag{33}$$

For Problem 1, it can be divided into four different cases according to the measurement matrix and the estimating results could be obtained by different means presented in this section.

Two mathematical theorems were presented in order to illustrate the closed-loop properties of the fast minimum norm filtering algorithm.

Theorem 1: Under Assumptions 1–3, for Problem 1, the estimation $\hat{x}_k$ obtained by (8)–(33) for different four cases is an unbiased estimate of the system state x_k at the kth step in the discrete-time LTI system (6).

Proof: It can be seen that the estimated state $\hat{x}_k$ could be obtained as follows

$$\hat{x}_k = \hat{x}_{k,k-p} + K_k[y_k - C\hat{x}_{k,k-p}] \tag{34}$$

where $\hat{x}_{k,k-p}$ is the prediction of x_k based on the measurement sequence and K_k is the filtering gain.

Introducing the notations

$$\varepsilon_{k,\mathrm{pre}} \triangleq x_k - \hat{x}_{k,k-p} \tag{35}$$

$$\varepsilon_k \triangleq x_k - \hat{x}_k \tag{36}$$

From (34), it can be obtained that

$$\begin{aligned} E[\varepsilon_k] &= E[x_k - \hat{x}_k] \\ &= E[x_k - \hat{x}_{k,k-p} - K_k[y_k - C\hat{x}_{k,k-p}]] \\ &= E[[I - K_kC][x_k - \hat{x}_{k,k-p}] - K_k v_k] \\ &= [I - K_kC]E[\varepsilon_{k,\mathrm{pre}}] + K_k E[v_k] \end{aligned} \tag{37}$$

From Assumption 3, we have $E[v_k] = 0$. Then $E[\varepsilon_k] = 0$ if $E[\varepsilon_{k,k-p}] = 0$. In other word, the estimation $\hat{x}_k$ obtained by (34) is unbiased if $E[\varepsilon_{k,\mathrm{pre}}] = 0$.

From the time updating process of the fast minimum norm filtering algorithm, it can be obtained that $\hat{x}_{k,k-p}$ is a kind of unbiased estimation for the state x_k and then the estimation obtained by the fast minimum norm filtering algorithm will be unbiased. □

Remark 3: Obviously, it can be seen that the initial state estimation $\hat{x}_0$ is not needed for the new algorithm from the process of time updating. We just need p measurements as a 'built-in' parameter for the fast minimum norm filtering algorithm. The estimation error of the initial state has no effect on the unbiased property of the new algorithm.

Theorem 2: Under Assumptions 1–3, for Problem 1, the estimate $\hat{x}_k$ obtained by (8)–(33) for different four cases, has a certain covariance matrix, which is determined by the measurement matrix and covariance matrix of the measurement noise in the discrete-time LTI system (6).

Proof: From the framework of the fast minimum norm filtering algorithm, it can be obtained that

$$\begin{aligned} E[\varepsilon_k \varepsilon_k^{\mathrm{T}}] &= E[[x_k - \hat{x}_k][x_k - \hat{x}_k]^{\mathrm{T}}] \\ &= [I - K_kC]E[\varepsilon_{k,k-p}\varepsilon_{k,k-p}^{\mathrm{T}}][I - K_kC]^{\mathrm{T}} + K_kRK_k^{\mathrm{T}} \end{aligned} \tag{38}$$

For Case I, it can be obtained that

$$K_k = (C^{\mathrm{T}}C)^{-1}C^{\mathrm{T}} \tag{39}$$

Substituting (39) into (38), we obtain

$$E[\varepsilon_k\varepsilon_k^{\mathrm{T}}] = (C^{\mathrm{T}}C)^{-1}C^{\mathrm{T}}R((C^{\mathrm{T}}C)^{-1}C^{\mathrm{T}})^{\mathrm{T}} \tag{40}$$

Therefore, for Case I, the covariance matrix of estimation error is determined by the measurement matrix C and measurement noise covariance matrix R.

For Case III, we have

$$K_k = C^{\mathrm{T}}(CC^{\mathrm{T}})^{-1} \tag{41}$$

Substituting (41) into (38), then multiplying matrix C and multiplying C^{T} to (38), it can be obtained that

$$CE[\varepsilon_k\varepsilon_k^{\mathrm{T}}]C^{\mathrm{T}} = R \tag{42}$$

Hence for Case III the covariance matrix of estimation error is constrained by (42), which is governed by the measurement matrix C and measurement noise covariance matrix R.

For Cases II and IV, they can be treated as Case III with different measurement equations obtained by the singular value decomposition method. Therefore the estimation error covariance matrix is determined by the equivalent measurement equation.

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Finally, we can conclude that the covariance matrix of the estimation error will always be determined by C and R. □

Remark 4: For the LTI systems, compared with the standard Kalman filter algorithm, the fast minimum norm filtering algorithm works well when the covariance matrices of the process and measurement noises were completely unknown. Besides, the fast minimum norm filtering algorithm can deal with the case where the initial state estimation cannot be obtained in advance. However, the estimation precision of the fast minimum norm filtering algorithm was relevant with measurement matrix and measurement noise covariance matrix though the covariance matrix of the measurement noise is not necessary in the process of the fast minimum norm filtering algorithm.

4 Numerical examples

In this section, three numerical examples were presented to illustrate the filtering performance of the fast minimum norm filtering algorithm. In order to test the estimating results, the standard Kalman filter was chosen to compare with the fast minimum norm filtering algorithm.

4.1 Example I: measurement feedback simulation

This simulation example is for the case where matrix C is a 'fat' matrix (columns greater than rows). The discrete-time LTI system model is given as follows

$$\begin{cases} x_k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_{k-1} + w_{k-1} \\ y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k \end{cases} \tag{43}$$

where $x_0 = [0 \quad 0]^{\mathrm{T}}$, $E(w_k) = 0$, $E(v_k) = 0$, $\mathrm{Cov}(w_k) = \mathrm{diag}(0.02, 0.02)$ and $\mathrm{Cov}(v_k) = 0.01$. Here $E(\cdot)$ is the mathematical expectation and $\mathrm{Cov}(\cdot)$ stands for the covariance matrix.

Different from the standard Kalman filter algorithm, it is supposed that the process and measurement noise covariance matrices

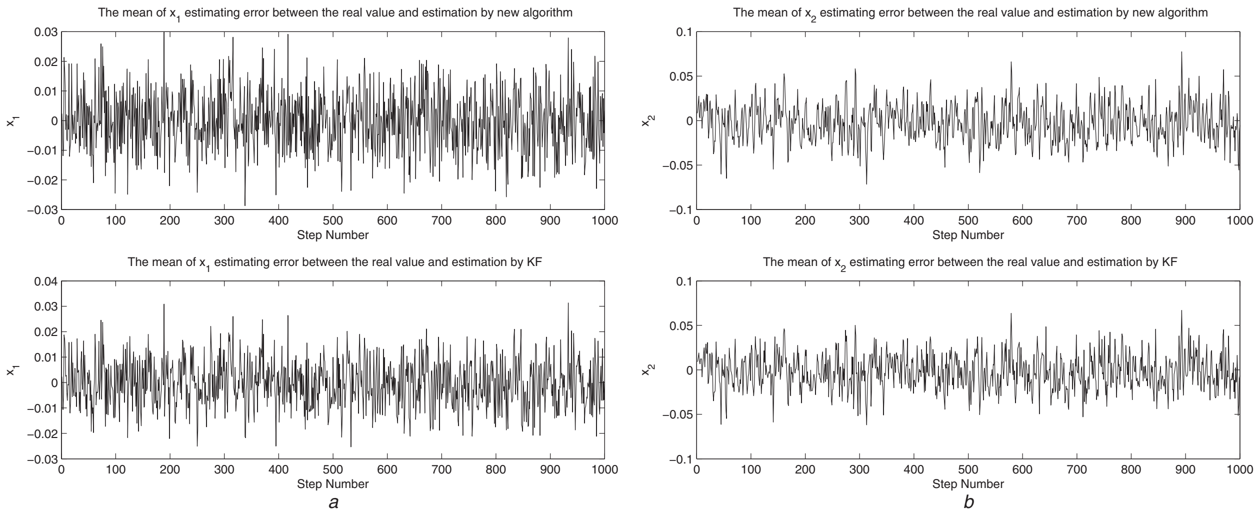

Fig. 2 *Mean of state estimation error at different times (Gaussian noise)*

a Results for state x_1
b Results for state x_2
c Results for state x_3

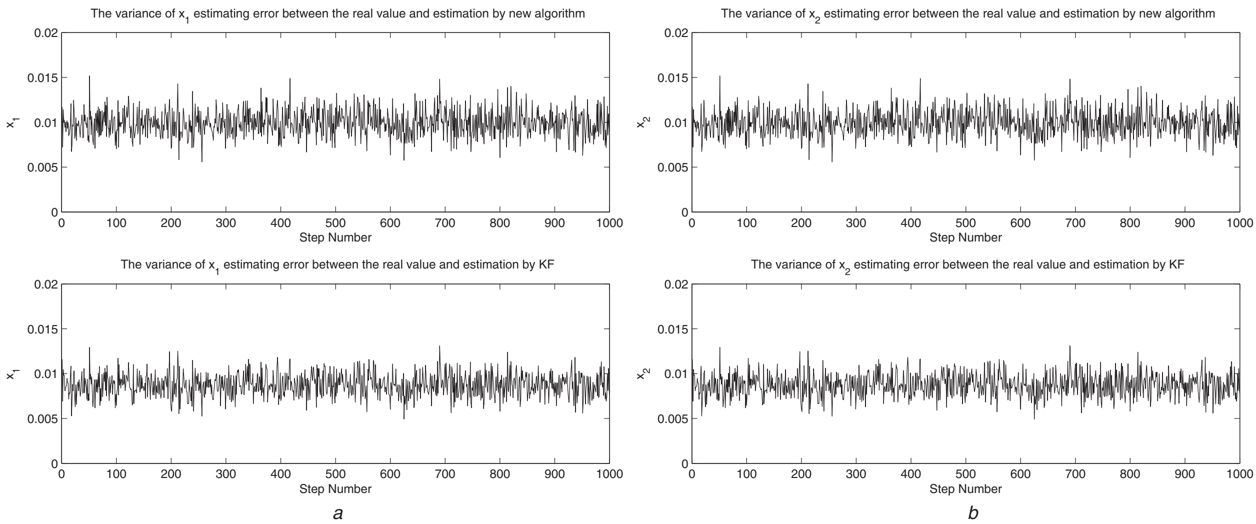

Fig. 3 *Variance of state estimation error at different times (Gaussian noise)*

a Results for state x_1
b Results for state x_2
c Results for state x_3

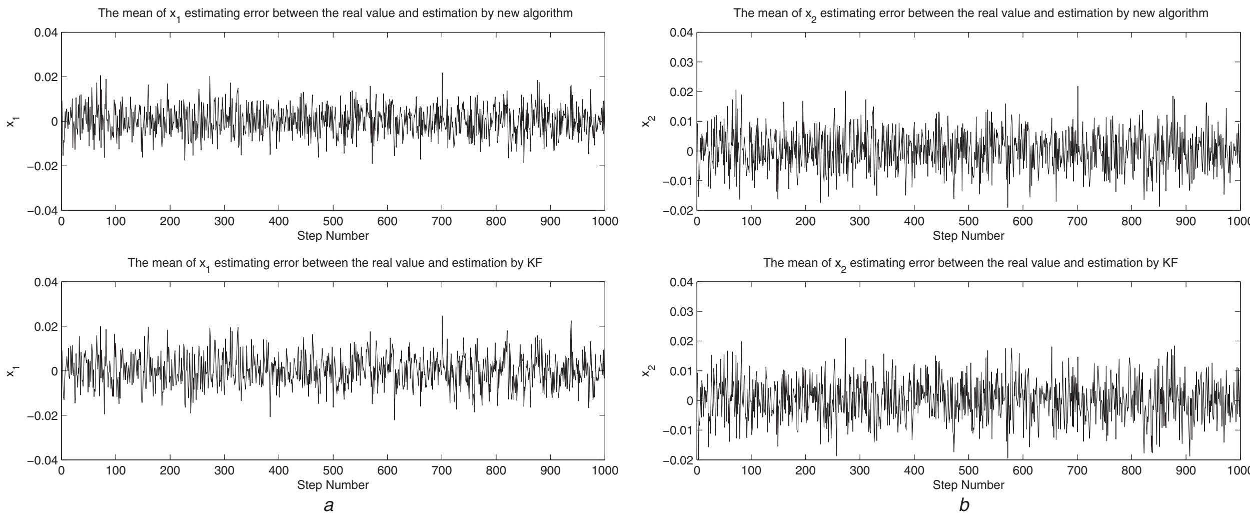

Fig. 4 *Mean of state estimation error at different times (Gaussian noise)*

a Results for state x_1
b Results for state x_2
c Results for state x_3

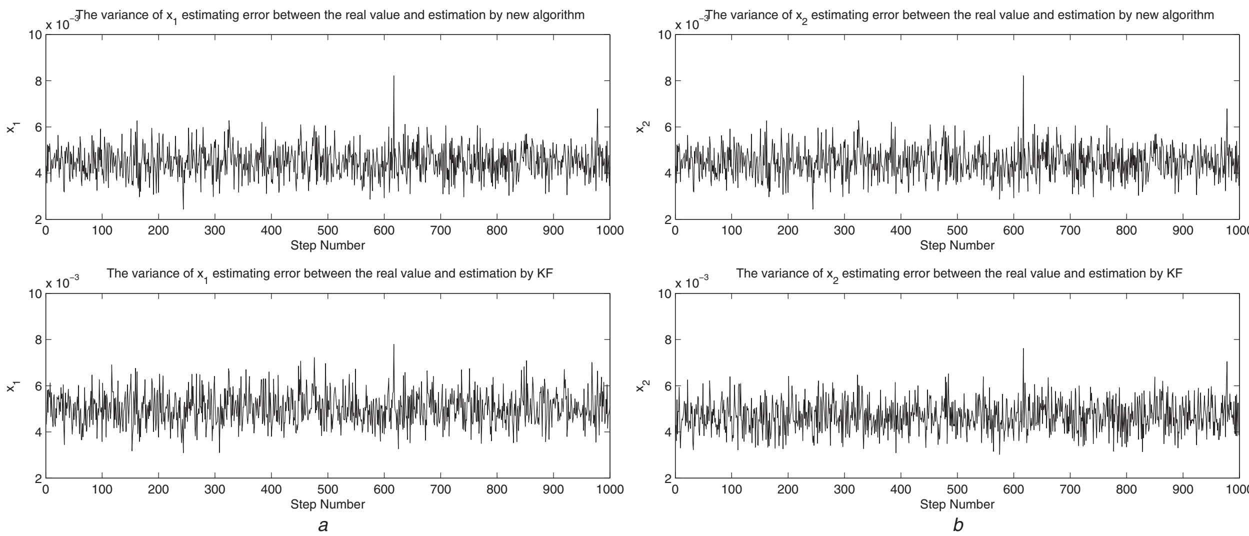

Fig. 5 *Variance of state estimation error at different times (Gaussian noise)*

a Results for state x_1
b Results for state x_2
c Results for state x_3

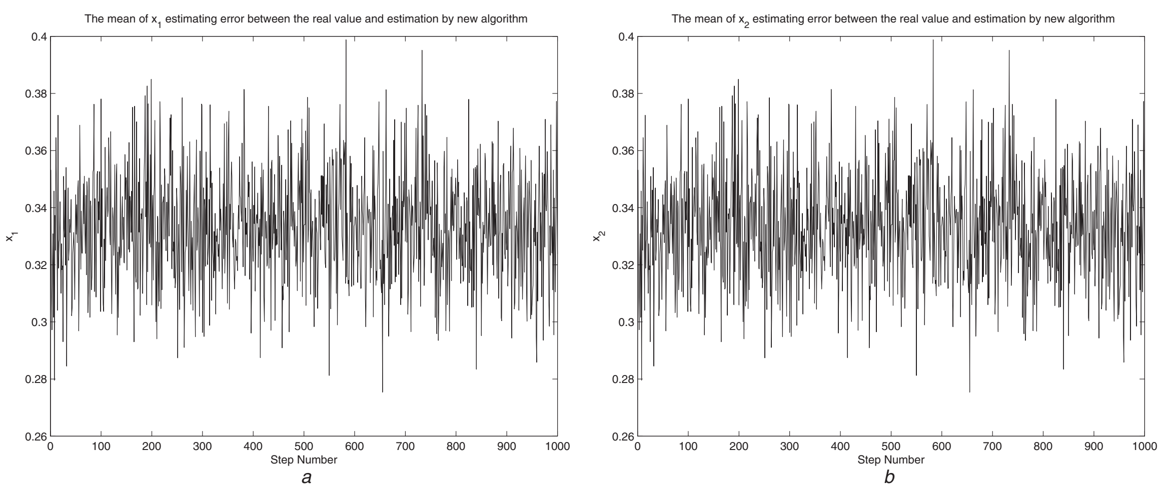

Fig. 6 *Mean of state estimation error at different times (non-Gaussian noise)*

a Results for state x_1
b Results for state x_2
c Results for state x_3

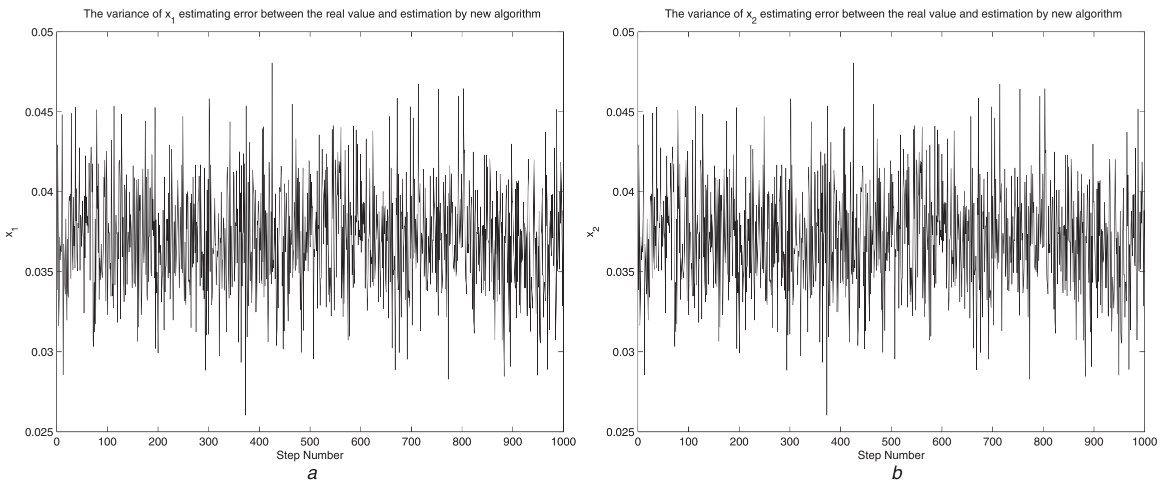

Fig. 7 *Variance of state estimation error at different times (non-Gaussian noise)*

a Results for state x_1
b Results for state x_2
c Results for state x_3

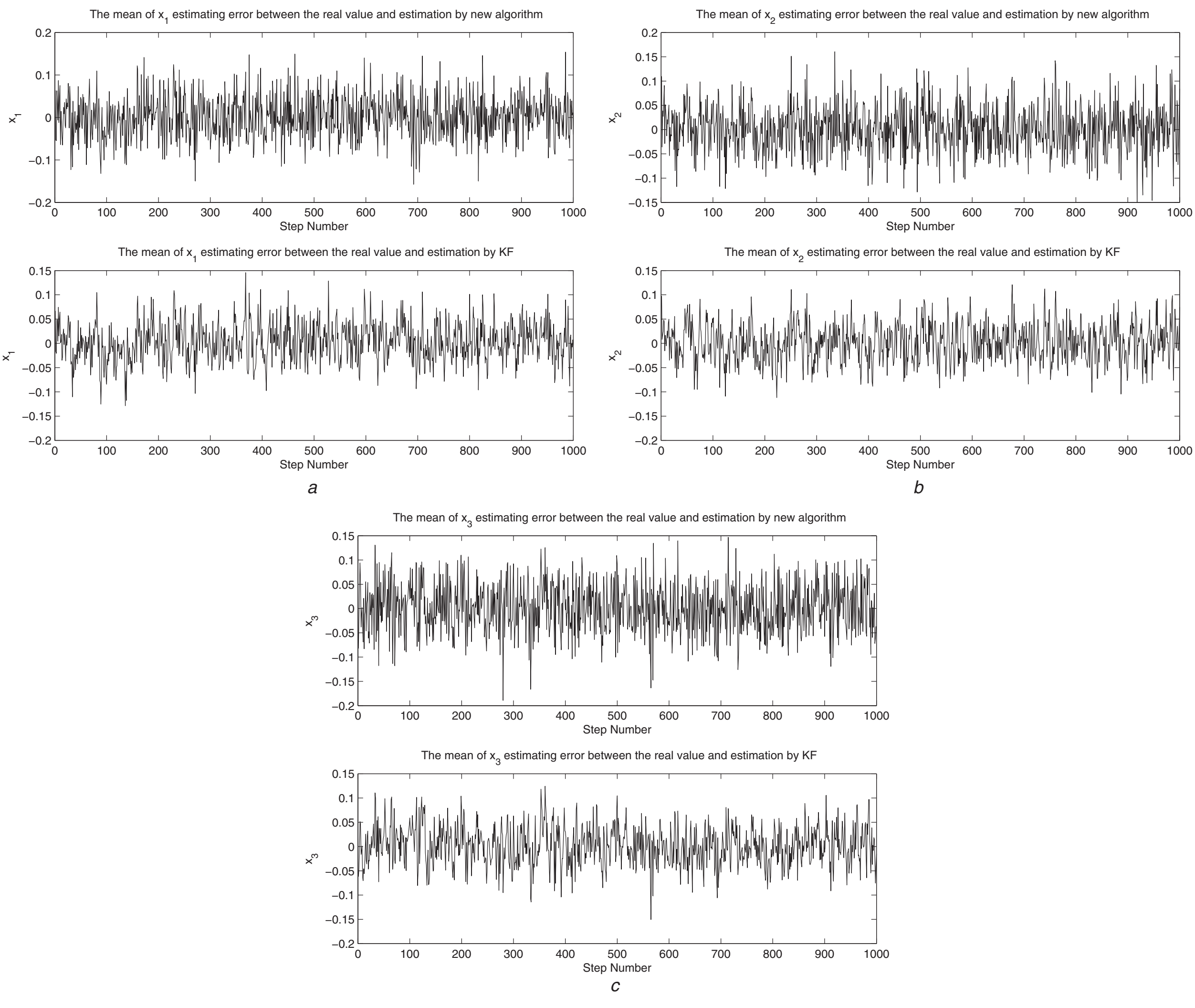

Fig. 8 *Mean of state estimation error at different times (Gaussian noise)*

a Results for state x_1
b Results for state x_2
c Results for state x_3

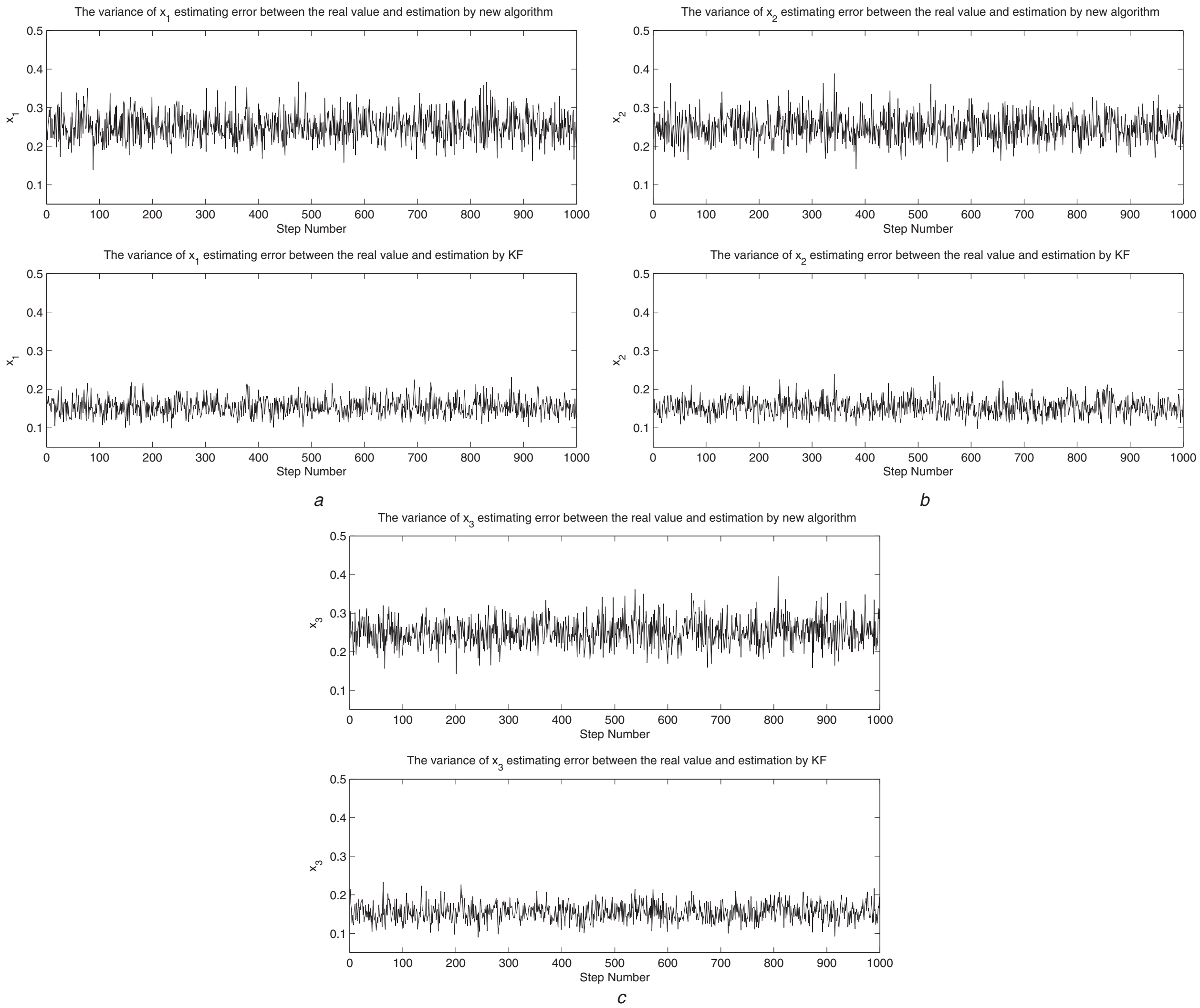

Fig. 9 *Variance of state estimation error at different times (Gaussian noise)*

a Results for state x_1
b Results for state x_2
c Results for state x_3

were completely unknown in the fast minimum norm filtering algorithm. From Algorithm 1 (see Fig. 1), it can be obtained that

$$K_k = [C^{\mathrm{T}}C]^{-1}C^{\mathrm{T}} = [1 \quad 0]^{\mathrm{T}} \tag{44}$$

$$Q_t = [Q_{\mathrm{ob}}^{\mathrm{T}}Q_{\mathrm{ob}}]^{-1}Q_{\mathrm{ob}}^{\mathrm{T}} = \begin{bmatrix} 0.8333 & 0.3333 & -0.1667 \\ -0.5000 & 0 & 0.5000 \end{bmatrix} \tag{45}$$

Then the real-time state estimate sequence could be obtained by Fig. 1. The simulation experiment is carried out based on Monte Carlo method with 100 random simulations. The filtering results are shown as in Figs. 2 and 3.

From Figs. 2 and 3, it can be seen that the filtering results illustrate that the fast minimum norm filtering algorithm could be used to track the state signal trend in such case and the estimation has an acceptable performance compared with the standard Kalman filter with exact system parameters. This simulation illustrates the acceptable performance of the fast minimum norm filtering algorithm in the case where matrix C is a 'fat' matrix (columns greater than rows).

4.2 Example II: data fusion simulation

This simulation example is for the case where matrix C is a 'tall' matrix (columns less than rows). In this example, measurements from multiple sensors are available, and the goal is to recover the unknown states by fusing the data of sensors. The discrete-time LTI system model is given as follows

$$\begin{cases} x_k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_{k-1} + w_{k-1} \\ y_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} x_k + v_k \end{cases} \tag{46}$$

where $x_0 = [0 \quad 0]^{\mathrm{T}}$, $E(w_k) = 0$, $E(v_k) = 0$, $\mathrm{Cov}(w_k) = \mathrm{diag}(0.02, 0.02)$ and $\mathrm{Cov}(v_k) = \mathrm{diag}(0.01, 0.01, 0.01)$. $E(\cdot)$ is the mathematical expectation and $\mathrm{Cov}(\cdot)$ stands for the covariance matrix.

From Fig. 1, it can be obtained that

$$K_k = C^{\mathrm{T}}(CC^{\mathrm{T}})^{-1} = \begin{bmatrix} 0.6667 & -0.3333 & 0.3333 \\ -0.3333 & 0.6667 & 0.3333 \end{bmatrix} \tag{47}$$

$$Q_t = [Q_{\mathrm{ob}}^{\mathrm{T}}Q_{\mathrm{ob}}]^{-1}Q_{\mathrm{ob}}^{\mathrm{T}} = \begin{bmatrix} 0.5 & -0.25 \\ -0.25 & 0.25 \\ 0.25 & 0 \\ 0.25 & 0 \\ -0.25 & 0.25 \\ 0 & 0.25 \end{bmatrix}^{\mathrm{T}} \tag{48}$$

Then, the real-time state estimate sequence could be obtained by Fig. 1. The simulation experiment is carried out based on Monte Carlo method with 100 random simulations. The filtering results are shown as in Figs. 4 and 5, which verified the performance of the fast minimum norm filtering algorithm without knowing the covariance matrices of the process and measurement noises a priori. Roughly speaking, the performance of the fast minimum norm filtering algorithm is comparable with that of the standard Kalman filter (optimal filter) with exact system parameters. This simulation illustrates the acceptable performance of the fast minimum norm filtering algorithm in the case where matrix C is a 'tall' matrix (columns less than rows).

In order to test the robustness of the new algorithm, another simulation example is presented when the measurement noise does not satisfy the demand of Gaussian distribution. Supposing that the measurement noise is the sample sequence from a uniform distribution as following

$$v_k^i \sim U[0, 1] \quad i \in \{1, 2, 3\} \quad \text{independent with each other} \tag{49}$$

The simulation experiment is carried out based on Monte Carlo method with 100 random simulations. The filtering results are shown as in Figs. 6 and 7, which verified the performance of the fast minimum norm filtering algorithm when the measurement noise cannot satisfy the demand of Gaussian distribution.

4.3 Example III: integrated navigation simulation

The data fusion algorithm plays an important role in the integrated navigation systems. For instance, some positioning systems are always used to be integrated with inertial navigation systems to suppress the navigation error of the inertial navigation systems divergence. The speed and acceleration information of the vehicle could be obtained by the inertial navigation systems, and the position information could be obtained by other aided positioning systems. Then the position information can be used to suppress the inertial navigation error divergence.

For brevity, we adopt a 'lineland' (one-dimensional space) system model to demonstrate the application of the new algorithm in integrated navigation systems

$$\begin{cases} \begin{bmatrix} x_k \\ \dot{x}_k \\ \ddot{x}_k \end{bmatrix} = \begin{bmatrix} 1 & T & 0.5T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{k-1} \\ \dot{x}_{k-1} \\ \ddot{x}_{k-1} \end{bmatrix} + w_{k-1} \\ \begin{bmatrix} y_k \\ \dot{y}_k \\ \ddot{y}_k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ \dot{x}_k \\ \ddot{x}_k \end{bmatrix} + v_k \end{cases} \tag{50}$$

where T is the sample period, x_k is the vehicle position, $\dot{x}_k$ is the vehicle speed, $\ddot{x}_k$ is the acceleration and $[y_k \quad \dot{y}_k \quad \ddot{y}_k]^{\mathrm{T}}$ is the measurement signal.

Supposing the $E(w_k) = E(v_k) = 0$, $T = 0.05\,\mathrm{s}$ and

$$\mathrm{Cov}(w_k) = Q = \begin{bmatrix} 0.25 & 0.04 & 0.04 \\ 0.04 & 0.25 & 0.04 \\ 0.04 & 0.04 & 0.25 \end{bmatrix} \tag{51}$$

$$\mathrm{Cov}(v_k) = R = \begin{bmatrix} 0.25 & 0.04 & 0.04 \\ 0.04 & 0.25 & 0.04 \\ 0.04 & 0.04 & 0.25 \end{bmatrix} \tag{52}$$

However, Q and R are completely unknown in the data fusion of the integrated navigation systems.

The simulation experiment is carried out based on Monte Carlo method with 100 random simulations. The simulation results are presented in Figs. 8 and 9. This simulation illustrates the acceptable performance of the fast minimum norm filtering algorithm in the case where matrix C is a square matrix.

5 Conclusion

To solve the state estimation problem when the covariance matrices of process and measurement noises were completely unknown for the discrete-time LTI systems with state-space model, a new algorithm, named as fast minimum norm filtering algorithm, has been presented in this paper. For different four cases, various measurement updating methods have been proposed for the discrete-time LTI systems with state-space model.

Compared with the standard Kalman filter with accurate model, state estimation by the fast minimum norm filtering algorithm has an acceptable performance with less computing burden. Two mathematical theorems have been presented to illustrate that the covariance matrix of estimation error by the fast minimum norm filtering algorithm is determined by the covariance matrices of measurement noise and process noise.

The method proposed in this paper does not require any a priori information on the knowledge of the covariance matrices of measurement noise and process noise, hence this new filtering algorithm would be very useful in case where such knowledge are not available. And the extensive simulations have shown that this proposed filtering algorithm indeed has a comparable good performance in comparison with a standard optimal Kalman filter using the accurate covariance matrices of measurement noise and process noise. Therefore, the fast minimum norm filtering algorithm may potentially result in promising wide applications in practice like Kalman filter.

From several typical simulation examples, it can be seen that the new algorithm could obtain acceptable state estimates when the noise covariance matrices are either diagonal or almost diagonal with equal diagonal values. In practice, the constant noise covariance matrices are always used to replace the real covariance matrices of process and measurement noise. If the chosen covariance matrices are not suitable, the Kalman filter will degrade much or even not work. However, the new algorithm proposed in this paper could be used to obtain the more acceptable state estimates than Kalman filter with unsuitable covariance matrices. Besides, the new algorithm has a Kalman-like style and is easy to be used in engineering applications and hence, it could obtain the real-time state estimates in certain conditions.

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7 References

1 Cui, R., Ren, B., Ge, S.S.: 'Synchronised tracking control of multi-agent system with high-order dynamics', *IET Control Theory Appl.*, 2012, **6**, (5), pp. 603–614
2 Wu, W., Chen, S., Qin, A.Q.: 'Online estimation of ship dynamic flexure model parameters for transfer alignment', *IEEE Trans. Control Syst. Technol.*, 2013, **21**, (5), pp. 1666–1678
3 Mostofi, Y., Murray, R.: 'To drop or not to drop: Design principles for Kalman filtering over wireless fading channels', *IEEE Trans. Autom. Control*, 2009, **54**, (2), pp. 376–381
4 Xu, H., Mannor, S.: 'A Kalman filter design based on the performance/robustness tradeoff', *IEEE Trans. Autom. Control*, 2009, **54**, (5), pp. 1171–1175
5 Kalman, R.E.: 'A new approach to linear filtering and prediction problems', *Trans. ASME, D, J. Basic Eng.*, 1960, **82**, pp. 35–45
6 Kalman, R.E., Bucy, R.S.: 'New results in linear filtering and prediction theory', *Trans. ASME, D, J. Basic Eng.*, 1961, **83**, pp. 95–107
7 Schmidt, S.F.: 'The Kalman filter: its recognition and development for aerospace applications', *AIAA J. Guid. Control*, 1981, **4**, (1), pp. 4–7
8 Bogosyan, S., Barut, M., Gokasan, M.: 'Braided extended Kalman filters for sensorless estimation in induction motors at high-low/zero speed', *IET Control Theory Appl.*, 2007, **1**, (4), pp. 987–998
9 Khanesar, M.A., Kayacan, E., Teshnehlab, M., Kaynak, O.: 'Extended Kalman filter based learning algorithm for type-2 fuzzy logic systems and its experimental evaluation', *IEEE Trans. Ind. Electron.*, 2012, **59**, (11), pp. 4443–4455
10 Kluge, S., Reif, K., Brokate, M.: 'Stochastic stability of the extended Kalman filter with intermittent observations', *IEEE Trans. Autom. Control*, 2010, **55**, (2), pp. 514–518

11 Carlson, N.: 'Fast triangular formulation of the square root filter', *AIAA J.*, 1973, **11**, (9), pp. 1259–1265

12 Carlson, N.A.: 'Federated square root filter for decentralized parallel processors', *IEEE Trans. Aerosp. Electron. Syst.*, 1990, **26**, (3), pp. 517–525

13 Julier, S.J., Uhlmann, J.K.: 'New extension of the Kalman filter to nonlinear systems'. Proc. of the Int. Society for Optical Engineering, vol. 3068, (SPIE-Int. Soc. Opt. Eng, Orlando, FL, USA, April 1997), pp. 182–193

14 Jafarzadeh, S., Lascu, C., Fadali, M.S.: 'State estimation of induction motor drives using the unscented Kalman filter', *IEEE Trans. Ind. Electron.*, 2012, **59**, (11), pp. 4207–4216

15 Marina, H., Pereda, F.J., Giron-Sierra, J.M., Espinosa, F.: 'UAV attitude estimation using unscented Kalman filter and TRIAD', *IEEE Trans. Ind. Electron.*, 2012, **59**, (11), pp. 4465–4474

16 Bhaumik, S., Sadhu, S., Ghoshal, T.K.: 'Risk-sensitive formulation of unscented Kalman filter', *IET Control Theory Appl.*, 2009, **3**, (4), pp. 375–382

17 Chang, L., Hu, B., Chang, G., Li, A.: 'Marginalised iterated unscented Kalman filter', *IET Control Theory Appl.*, 2012, **6**, (6), pp. 847–854

18 Petersen, I.R., Savkin, A.V.: 'Robust Kalman filtering for signals and systems with large uncertainties' (Birkhause Boston, 1999)

19 Hassibi, B., Sayed, A.H., Kailath, T.: 'H_∞ optimality of the LMS algorithm', *IEEE Trans. Signal Process.*, 1996, **44**, (2), pp. 267–280

20 Ra, W.S., Whang, I.H., Park, J.B.: 'Non-conservative robust Kalman filtering using a noise corrupted measurement matrix', *IET Control Theory Appl.*, 2009, **3**, (9), pp. 1226–1236

21 Chen, B., Yu, L., Zhang, W.A.: 'Robust Kalman filtering for uncertain state delay systems with random observation delays and missing measurements', *IET Control Theory Appl.*, 2011, **5**, (17), pp. 1945–1954

22 Mehra, R.K.: 'On the identification of variances and adaptive Kalman filtering', *IEEE Trans. Autom. Control*, 1970, **15**, (2), pp. 175–184

23 Xiong, K., Zhang, H., Liu, L.: 'Adaptive robust extended Kalman filter for non-linear stochastic systems', *IET Control Theory Appl.*, 2008, **2**, (3), pp. 239–250

24 Doucet, A., De Freitas, N., Gordon, N.: 'Sequential Monte Carlo Methods in Practice' (Springer, 2001)

25 Zhong, J., Fung, Y.F.: 'Case study and proofs of ant colony optimisation improved particle filter algorithm', *IET Control Theory Appl.*, 2012, **6**, (5), pp. 689–697

26 Mohamed, A.H., Schwarz, K.P.: 'Adaptive Kalman filtering for INS/GPS', *J. Geodesy*, **73**, (4), pp. 193–203, 1999

27 Azimi-Sadjadi, M.R., Xiao, R.R., Yu, X.: 'Neural network decision directed edge-adaptive Kalman filter for image estimation', *IEEE Trans. Image Process.*, 1999, **8**, (4), pp. 589–592

28 Xiao, X., Feng, B., Wang, B.: 'On-line realization of SVM Kalman filter for MEMS gyro'. Proc. of the Third Int. Conf. on Measuring Technology and Mechatronics Automation, vol. 2., (IEEE Computer Society, Shanghai, China, 2011), pp. 768–770

29 Feng, B., Ma, H.B., Fu, M.Y., Wang, S.T.: 'A framework of finite-model Kalman filter with case study: MVDP-FMKF algorithm', *Acta Autom. Sin.*, 2013, **39**, (8), pp. 1246–1256