1 Part 2 - Discussion of Bolzano Wierstrass Theorem

1.1 Bolzano - Wierstrass property

Bolzano-Wierstrass Theorem: Every bounded sequence of real numbers has a convergent subsequence.

A set in a metric space is said to have the Bolzano-Wierstrass property if every sequence in it has a convergent subsequence.

That means that compact sets in R^n are characterized by Bolzano-Wierstrass Property.

1.1.1 Discussion of Bolzano-Wierstrass Property

- Instead of verifing if a sequence converges, whit Bolzano-Wierstrass we can see if a sequence has a convergent subsequence.

For example, the sequence -1, 1, -1, 1, .. doesn't converge, but has a subsequence that converges 1, 1, 1, ...

- The Bolzano-Wierstrass Theorem is also true in R:

Every bounded sequence in R^n has a convergent subsequence.

This property fails in spaces different from R^n, such as the sequences in infinite dimensional spaces, and others.
Although, it works for metric spaces.

Definition: A metric space is compact if Bolzano-Wierstrass property holds.

Obs.: Given a closed and bounded set in a metric space, it is not necessarily compact. Example: (X, d) s.t. x$\neq$y $\Rightarrow$ d(x,y) = 1 and X = N This space is closed and bounded but not compact.

1.1.2 Proof of Bolzano-Wierstrass property

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1.2 Wierstrass Theorem

> Each function continuous in a limited [equivalent to modern-day "closed and bounded"] interval attains there at least once its maximum.

1.3 Discussion of Wierstrass Theorem

1.3.1 Proof of Wierstrass Theorem

Proof.

1.4 Bolzano Lemma

If a property M does not apply to all values of a vari- able quantity x, but to all those that are smaller than a certain u, there is always a quantity U which is the greatest of those of which it can be asserted that all smaller x possess the property M.

1.4.1 Discussion of Bolzano Lemma

1.4.2 Proof of Bolzano Lemma

Proof.