Semana 2 - Discussion of Bolzano - Weierstrass Property

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Premise: Talk about Bolzano - Weierstrass Property and Weierstrass TheoremR^n, and how it applies to compactness.

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1 Bolzano - Wierstrass property

Bolzano-Wierstrass Theorem: Every bounded sequence of real numbers has a convergent subsequence.

Talking more about sequences of real numbers.
The first big result regarding the limits of sequences was shown by Karl Weierstrass.
Not all sequences converge. The sequence -1, 1, -1... for example, does not converge. But notice, there's a subsequence that do.
In fact, we will see that there is a condition that guarentees that it does exists: The **Bolzano-Weierstrass Theorem**.

> "Every bounded subset of real numbers has a convergent subsequence."

A set with this property is a compact set, or a sequentialy compact set.

1.1 Discussion of Bolzano-Wierstrass Property

- The Bolzano-Wierstrass Theorem is also true in R^n :

Every bounded sequence in R^n has a convergent subsequence.

1.2 Proof of Bolzano-Wierstrass property

Proof. Let x_n be a bounded sequence and without loss of generality assume that every term of the sequence lies in the interval [0, 1]. Divide [0, 1] into two intervals, [0, 1/2] and [1/2 , 1]. (Note: this is not a partition of [0, 1].) At least

one of the halves contains infinitely many terms of x_n;
denote that interval by I_1, which has length 1 2 , and let x_n be the subsequence of x_n consisting of every term that lies in I_1. Now divide I_1 into two halves, each of length $1/4 = (1/2)^2$, at least one of which contains infinitely many terms of the (sub)sequence x'_n, and denote that half by I_2. Let x''_n be the subsequence of x'_n consisting of all of the terms that lie in I_2. Continuing in this way, we construct a sequence of nested intervals $I_1 I_2$ . . . , where the length of I_n is $(1/2)n$, and each interval contains an infinite number of terms of the original sequence x_n. Finally, we construct a subsequence z_n of x_n made up of one term from each interval In. This subsequence is clearly Cauchy: $\forall N : m, n > N \Rightarrow |zmzn| < (1/2)N$. Therefore the subsequence z_n converges.

2 Wierstrass Theorem

Given the Bolzano - Weierstrass property, we can see that Wierstrass Theorem as a corollary:

> **Corollary: (Weierstrass Theorem)** Each function f continuous in a closed and bounded interval $[a, b]$ attains there at least once its maximum.

2.1 Proof of Weierstrass Theorem

Proof. Notice that $[a, b]$ has Bolzano-Weierstrass property, because it is a bounded sequence of R, and therefore, it is compact.
$f([a, b])$ is also compact, because it is a closed and bounded set of R as well. Since $f([a, b])$ is a bounded subset of R, it has both a least upper bound M and a greatest lower bound m; and since $f([a, b])$ is closed, it contains m and M. Therefore m = $\min f([a, b])$ and M = $\max f([a, b])$.