Forecasting Emerging Pandemic Risks Through Random-Walk Metropolis–Hastings Monte Carlo Simulations

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Abstract

This theoretical framework presents a Bayesian approach for pandemic risk forecasting using Random-Walk Metropolis–Hastings Monte Carlo simulations. The methodology combines logistic regression with adaptive Markov Chain Monte Carlo sampling to provide robust uncertainty quantification in emerging pathogen risk assessment.

1 Theoretical Development

1.1 Bayesian Logistic Regression Framework

The probabilistic foundation for pandemic risk classification begins with the likelihood function for binary outcomes. For each observation i with features $\mathbf{x}_i \in R^p$ and pandemic status $y_i \in \{0, 1\}$, we have:

$$p(y_i|\mathbf{x}_i, \boldsymbol{\beta}) = \text{Bernoulli}(y_i|\sigma(\mathbf{x}_i^T\boldsymbol{\beta})) \tag{1}$$

where the logistic sigmoid function $\sigma(z) = (1 + \exp(-z))^{-1}$ ensures probabilistic outputs between 0 and 1. The complete data likelihood for N independent observations is:

$$p(\mathbf{y}|\mathbf{X}, \boldsymbol{\beta}) = \prod_{i=1}^{N} \left[\sigma(\mathbf{x}_i^T\boldsymbol{\beta})\right]^{y_i} \left[1 - \sigma(\mathbf{x}_i^T\boldsymbol{\beta})\right]^{1-y_i} \tag{2}$$

1.2 Prior Specification and Posterior Derivation

We specify independent Gaussian priors for each parameter to regularize the solution:

$$p(\beta_j) = \mathcal{N}(\beta_j|\mu_0, \tau_0^{-1}), \quad j = 1, \ldots, p \tag{3}$$

with $\mu_0 = 0$ and $\tau_0 = 1$ defining weakly informative priors. The joint prior distribution is:

$$p(\boldsymbol{\beta}) = \prod_{j=1}^{p} p(\beta_j) = \left(\frac{\tau_0}{2\pi}\right)^{p/2} \exp\left(-\frac{\tau_0}{2}\sum_{j=1}^{p}(\beta_j - \mu_0)^2\right) \tag{4}$$

Applying Bayes' theorem, the posterior distribution combines likelihood and prior:

$$p(\boldsymbol{\beta}|\mathbf{X}, \mathbf{y}) \propto p(\mathbf{y}|\mathbf{X}, \boldsymbol{\beta}) \cdot p(\boldsymbol{\beta}) \tag{5}$$

The log-posterior, more suitable for computational implementation, becomes:

$$\log p(\boldsymbol{\beta}|\mathbf{X}, \mathbf{y}) = \sum_{i=1}^{N} \left[y_i \mathbf{x}_i^T \boldsymbol{\beta} - \log(1 + \exp(\mathbf{x}_i^T \boldsymbol{\beta})) \right] \tag{6}$$

$$- \frac{\tau_0}{2} \sum_{j=1}^{p} (\beta_j - \mu_0)^2 + \text{constant} \tag{7}$$

1.3 Random-Walk Metropolis–Hastings Algorithm

The intractability of the posterior normalization constant necessitates Markov Chain Monte Carlo methods. The Random-Walk Metropolis–Hastings (RWMH) algorithm generates samples from the posterior through an iterative accept-reject mechanism.

Algorithm 1 Random-Walk Metropolis–Hastings for Bayesian Logistic Regression

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Input: Data $\mathbf{X}$, $\mathbf{y}$; prior parameters $\mu_0$, $\tau_0$; iterations $T$
Output: Posterior samples $\{\boldsymbol{\beta}^{(1)}, \ldots, \boldsymbol{\beta}^{(T)}\}$
Initialize $\boldsymbol{\beta}^{(0)} \leftarrow \mathbf{0}$
Initialize proposal variances $\sigma_j^2$ for $j = 1, \ldots, p$
for $t = 1$ to $T$ do
    for $j = 1$ to $p$ do
        Propose new value: $\beta_j^* \sim \mathcal{N}(\beta_j^{(t-1)}, \sigma_j^2)$
        Create proposal vector: $\boldsymbol{\beta}^* = (\beta_1^{(t)}, \ldots, \beta_j^*, \ldots, \beta_p^{(t)})$
        Compute log-posterior ratio:
        $\Delta = \log p(\boldsymbol{\beta}^*|\mathbf{X}, \mathbf{y}) - \log p(\boldsymbol{\beta}^{(t-1)}|\mathbf{X}, \mathbf{y})$
        Compute acceptance probability: $\alpha = \min(1, \exp(\Delta))$
        Sample $u \sim \text{Uniform}(0, 1)$
        if $u < \alpha$ then
            $\beta_j^{(t)} \leftarrow \beta_j^*$                                ▷ Accept proposal
        else
            $\beta_j^{(t)} \leftarrow \beta_j^{(t-1)}$                          ▷ Reject proposal
        end if
    end for
end for
```

1.4 Detailed Algorithmic Components

1.4.1 Log-Posterior Calculation

The key computational step involves evaluating the log-posterior ratio Δ. For numerical stability, we implement:

$$\log p(\boldsymbol{\beta}|\mathbf{X}, \mathbf{y}) = \sum_{i=1}^{N} [y_i z_i - \log(1 + \exp(z_i))] \tag{8}$$

$$- \frac{\tau_0}{2} \sum_{j=1}^{p} (\beta_j - \mu_0)^2 \tag{9}$$

where $z_i = \mathbf{x}_i^T \boldsymbol{\beta}$, with clipping $z_i \in [-250, 250]$ to prevent numerical overflow.

1.4.2 Adaptive Proposal Mechanism

The proposal variance σ_j^2 adapts dynamically based on empirical acceptance rates:

$$\sigma_j^{(t+1)} = \begin{cases} 0.9 \cdot \sigma_j^{(t)} & \text{if } \hat{\alpha}_j < 0.15 \\ 1.1 \cdot \sigma_j^{(t)} & \text{if } \hat{\alpha}_j > 0.35 \\ \sigma_j^{(t)} & \text{otherwise} \end{cases} \tag{10}$$

where $\hat{\alpha}_j$ is the acceptance rate for parameter j over a moving window of iterations.

1.4.3 Multiple Chain Implementation

We employ K independent chains with dispersed initializations:

$$\boldsymbol{\beta}_k^{(0)} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_k), \quad k = 1, \ldots, K \tag{11}$$

with $\boldsymbol{\Sigma}_k$ chosen to ensure exploration of different posterior modes.

1.5 Convergence Diagnostics

The Gelman–Rubin statistic $\hat{R}$ assesses convergence across multiple chains:

$$W = \frac{1}{K} \sum_{k=1}^{K} s_k^2 \quad \text{(within-chain variance)} \tag{12}$$

$$B = \frac{N}{K-1} \sum_{k=1}^{K} (\bar{\beta}_k - \bar{\beta})^2 \quad \text{(between-chain variance)} \tag{13}$$

$$\hat{V} = \frac{N-1}{N} W + \frac{1}{N} B \quad \text{(marginal posterior variance)} \tag{14}$$

$$\hat{R} = \sqrt{\frac{\hat{V}}{W}} \tag{15}$$

where $\hat{R} \approx 1$ indicates convergence, and values > 1.1 suggest inadequate mixing.

1.6 Posterior Inference

After discarding burn-in samples B, we obtain the posterior sample:

$$\mathcal{S} = \{\boldsymbol{\beta}^{(B+1)}, \ldots, \boldsymbol{\beta}^{(T)}\} \tag{16}$$

Point estimates derive from posterior summaries:

$$\hat{\boldsymbol{\beta}}_{\text{mean}} = \frac{1}{|\mathcal{S}|} \sum_{\boldsymbol{\beta} \in \mathcal{S}} \boldsymbol{\beta} \tag{17}$$

$$\hat{\boldsymbol{\beta}}_{\text{median}} = \text{median}(\mathcal{S}) \tag{18}$$

$$\text{HPD}_{1-\alpha}(\beta_j) = [L, U] \quad \text{where} \int_L^U p(\beta_j | \mathbf{X}, \mathbf{y}) d\beta_j = 1 - \alpha \tag{19}$$

The Highest Posterior Density (HPD) interval provides the shortest interval containing $(1 - \alpha) \times 100\%$ of posterior probability.

1.7 Prediction and Uncertainty Propagation

For new data $\mathbf{x}^*$, the posterior predictive distribution integrates over parameter uncertainty:

$$p(y^* = 1|\mathbf{x}^*, \mathbf{X}, \mathbf{y}) = \int \sigma(\mathbf{x}^{*T}\boldsymbol{\beta})p(\boldsymbol{\beta}|\mathbf{X}, \mathbf{y})d\boldsymbol{\beta} \tag{20}$$

Approximated via Monte Carlo integration:

$$\hat{p}(y^* = 1|\mathbf{x}^*, \mathbf{X}, \mathbf{y}) \approx \frac{1}{|\mathcal{S}|}\sum_{\boldsymbol{\beta}\in\mathcal{S}} \sigma(\mathbf{x}^{*T}\boldsymbol{\beta}) \tag{21}$$

This Bayesian approach naturally propagates parameter uncertainty into predictive probabilities, providing calibrated uncertainty estimates for pandemic risk assessments.