Interference Alignment as a Rank Constrained Rank Minimization

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***Abstract*—We show that the maximization of the sum degrees-of-freedom for the static flat-fading multiple-input multiple-output (MIMO) interference channel (IC) is equivalent to a rank constrained rank minimization problem (RCRM), when the signal subspaces span all available dimensions. The rank minimization corresponds to maximizing interference alignment (IA) so that interference spans the lowest dimensional subspace possible. The rank constraints account for the useful signal subspaces spanning all available spatial dimensions. That way, we reformulate the IA requirements to requirements involving ranks. Then, we present a convex relaxation of the RCRM problem inspired by recent results in compressed sensing and low-rank matrix completion theory that rely on approximating rank with the nuclear norm. We show that the convex envelope of the sum of ranks of the interference matrices is the normalized sum of their corresponding nuclear norms and replace the rank constraints with asymptotically equivalent and tractable ones. We then tune our heuristic relaxation for the multicell interference channel. We experimentally show that in many cases the proposed algorithm attains perfect interference alignment and in some cases outperforms previous approaches for finding precoding and receive matrices for interference alignment.**

***Index Terms*—Convex relaxations, interference alignment, interference channel, rank minimization.**

I. Introduction

A RECENT information-theoretic breakthrough established that at the high signal-to-noise ratio (SNR) regime every user in a K-user wireless interference network can enjoy half the capacity of the interference free case [2]. Therefore, interference is not a fundamental limitation for such networks since it accounts for only constant scaling of the interference free case capacity, provided that it is sufficiently mitigated. Such a surprising result is possible when interference alignment is employed. IA is a sophisticated technique first presented in [4] and subsequently utilized in [2] as a means of showing the achievability of $\frac{K}{2}$ degrees-of-freedom (DoF) for the K-user interference channel. The DoF of an interference channel can be interpreted as the multiplexing gain, i.e., the number of interference free signaling dimensions, including time, frequency, or space.

Intuitively, IA serves as a means for obtaining as many interference free dimensions for communication as possible and in practice stands for designing the transmit and receive strategies for each transmitter-receiver pair of a wireless network [2]–[8]. For the case of static flat-fading MIMO channels, such as the ones studied in [5] and [6], where all the transmitters and the receivers have perfect channel knowledge, the flexibility is confined to designing the transmit precoding and receive matrices that maximize the number of interference free *spatial* dimensions. Unfortunately, such matrices are NP-hard to obtain [13], closed form solutions have been found only for a few special cases, such as [2] and [12], and the problem is open for fixed dimensions [10]. Even characterizing the feasibility of IA is a highly nontrivial task, as discussed in recent work [10], [13], [23], [24]. The hardness in either finding IA solutions, or even deciding for feasibility, is the cost of the problem's over constrained and nonconvex nature. A review of the current status of IA techniques is presented in [3].

As an alternative to finding closed-form solutions, several algorithmic approaches have been proposed in the literature, such as [9], [14], [15], and [17]. Many of those methods aim to minimize the *interference leakage* at each receiver so that —at best case— interference alignment is perfectly attained. The suggested insight for their effectiveness is that when interference alignment is feasible, then interference leakage will be zero and such algorithms may obtain the optimal solutions. Although a meaningful metric to optimize, we show in this paper, that interference leakage is not the tightest approximation to the notion of multiplexing gain, or DoF, which is typically the desired objective. When perfect alignment of interferences is not attained, the objective remains to maximize the available spatial DoF, that is the prelog factor of the capacity at the high-SNR regime.

In this work, we present a variational characterization of IA in terms of signal and interference subspace ranks. Specifically, we pose full-rank constraints on the useful signal subspaces and minimize the rank of the interference subspaces. The full-rank constraints ensure that useful signal subspaces span all available spatial dimensions. The rank minimization guarantees that the interference subspaces collapse to the minimum dimensions possible. This variational characterization, even though it is harder to solve exactly than solving a set of bilinear equations for IA, as the ones in [9] and [10], suggests a natural relaxation that leads to a good, albeit suboptimal, algorithm.

Motivated by the rank minimization framework, we establish a new heuristic for near optimal interference alignment. Using recent results from the rank minimization and compressed

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sensing literature [19], [20], along with cues from settings where a sparsity cost function is replaced with an ℓ_1-norm [21], [22], we suggest that the aggregate of the nuclear norms of the interference matrices is the best cost function in terms of convex functions. This cost function is equivalent to the ℓ_1-norm of the singular values of the interference matrices. As intuition suggests, the ℓ_1-norm minimization of the interference singular values will provide sparse solutions, translating to more interference free signaling dimensions. We show that the leakage minimization techniques presented in [9] and [14], and [15] minimize the ℓ_2-norm of the interference singular values which accounts for "low energy" solutions rather than sparse ones. To deal with the (nonconvex) full-rank constraints, we suggest that positivity constraints on minimum eigenvalues of positive definite matrices serve as a well motivated approximation.

An interesting aspect of our approach is its inherent robustness to a challenging caveat that the leakage minimization approaches face: for some cases of interest, such as symbol extended (i.e., multiple time/frequency slot) interference channels, the iterative leakage minimization approaches may converge to zero dimensional, or rank deficient, signal subspaces, yielding zero or very low multiplexing gain [16]. This defficiency is due to the fact that the beamforming and zero-forcing matrices are constructed only as a function of the interference links and not the direct links. Our heuristic avoids these singularities due to its positivity constraints on eigenvalues, which explicitly enforce the signal subspaces to be full-rank; these constraints involve all direct links of the network. Hence, our approach can be used to construct nondegenerate IA solutions for any channel structure.

Then, we extend our heuristic algorithm to the K-cell interference channel [11], where each cell consists of several users and show that additional affine constraints on the precoding matrices need to be posed. The affine constraints are tractable and we add them in a straightforward manner on top of our K-user IC heuristic. This approximation is again motivated by the fact that IA can still be posed as a rank constrained rank minimization on an interference channel with structural constraints on the beamforming matrices.

The last section contains an experimental evaluation of the proposed algorithm. Our experiments suggest that the proposed scheme is optimal in terms of multiplexing gain for many setups where IA is feasible. In some cases, when IA is either not feasible or hard to attain, it provides extra interference free dimensions compared to the leakage minimization and max-SINR approaches [9], [15]. Furthermore, our results indicate that our algorithm is robust to diagonal channel structures which can cause singularities to the leakage minimization algorithms. We conclude with a discussion on further research directions.

II. System Model

We consider static flat-fading K-user MIMO interference wireless systems consisting of K transmitters and K receivers. We assume that each transmitting user is equipped with M_{t} transmit antennas, each receiver with M_{r} receive antennas, and all K transmitting users are synchronizing their transmissions. Each user, say $k \in \mathcal{K} = \{1, \ldots, K\}$, wishes to communicate a symbol vector $\mathbf{x}_k \in \mathbb{C}^{d \times 1}$ to its associated receiver, where d represents the multiplexing gain desired by each transmitter-receiver pair. To aid intuition, we note that d can be perceived as the number of signal space dimensions that are *free* of interference. Prior to transmitting, user $k \in \mathcal{K}$ linearly precodes its symbol vector to obtain $\mathbf{s}_k \triangleq \mathbf{V}_k \mathbf{x}_k$, where $\mathbf{V}_k \in \mathbb{C}^{M_{\mathrm{t}} \times d}$ denotes the precoding matrix whose d columns are linearly independent. We consider signal vectors with an expected power constraint $E\left\{\|\mathbf{s}_k\|^2\right\} = P > 0$, for all $k \in \mathcal{K}$. The downconverted and pulse matched received signal at receiver k is given by

$$\begin{aligned}\mathbf{y}_k &\triangleq \mathbf{H}_{k,k}\mathbf{s}_k + \sum_{l=1, l\neq k}^{K} \mathbf{H}_{k,l}\mathbf{s}_l + \mathbf{w}_k \\ &= \mathbf{H}_{k,k}\mathbf{V}_k\mathbf{x}_k + \sum_{l=1, l\neq k}^{K} \mathbf{H}_{k,l}\mathbf{V}_l\mathbf{x}_l + \mathbf{w}_k \qquad (1)\end{aligned}$$

where $\mathbf{H}_{i,j} \in \mathbb{C}^{M_{\mathrm{r}} \times M_{\mathrm{t}}}$ represents the channel "processing" between the jth transmitter and the ith receiver and $\mathbf{w}_k \in \mathbb{C}^{M_{\mathrm{r}} \times 1}$ denotes the zero-mean complex additive white Gaussian noise vector with covariance matrix $\sigma_k^2 \mathbf{I}_{M_{\mathrm{r}}}$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix and $i, j, k \in \mathcal{K}$. Each receiver $k \in \mathcal{K}$, linearly processes the received signal to obtain

$$\mathbf{U}_k^H \mathbf{y}_k = \mathbf{U}_k^H \mathbf{H}_{k,k}\mathbf{V}_k\mathbf{x}_k + \mathbf{U}_k^H \sum_{l=1, l\neq k}^{K} \mathbf{H}_{k,l}\mathbf{V}_l\mathbf{x}_l + \mathbf{U}_k^H \mathbf{w}_k$$

where $\mathbf{U}_k \in \mathbb{C}^{M_{\mathrm{r}} \times d}$ is the corresponding linear receive filter with d linearly independent columns. In the following, $\{\mathbf{X}_l\}_{l=1, l\neq k}^{K}$ denotes the horizontal concatenation of matrices $\mathbf{X}_1, \ldots, \mathbf{X}_{k-1}, \mathbf{X}_{k+1}, \ldots \mathbf{X}_K$. Here, $\operatorname{span}\left(\mathbf{U}_k^H \mathbf{H}_{k,k}\mathbf{V}_k\right)$ constitutes the useful signal subspace in which receiver $k \in \mathcal{K}$ expects to observe the symbols transmitted by transmitter k, while $\operatorname{span}\left(\left\{\mathbf{U}_k^H \mathbf{H}_{k,l}\mathbf{V}_l\right\}_{l=1, l\neq k}^{K}\right)$ is the subspace where all interference is observed. We denote this K-user MIMO interference channel as an $(M_{\mathrm{r}} \times M_{\mathrm{t}}, d)^K$ system, in the same manner as in [10], where all signal subspaces span all available dimensions. For all the cases considered we assume $d \leq \min(M_{\mathrm{t}}, M_{\mathrm{r}})$.

For practical reasons one might consider $\mathbf{V}_k^H \mathbf{V}_k = \frac{P}{d}\mathbf{I}_d$, for all $k \in \mathcal{K}$. This might be a setting where each column of $\mathbf{V}_k$ represents a beamforming (or signature) vector assigned to a user in a group (or a cell) of d users and enforces orthogonality among user interference subspaces. Accordingly, we may as well assume that the columns of each receive filter $\mathbf{U}_k$, for all receivers $k \in \mathcal{K}$, form a d-dimensional orthonormal basis, if practical interest requires such a construction.

Remark 1: We note that for simplicity we made the assumption that the number of transmit antenna elements are the same at all transmitters, or receivers, and all users wish to transmit $d > 0$ symbols. However, the results that follow can be easily carried to the case where transmitters and receivers might not have the same number of antennas or identical multiplexing-gain requirements.

III. Interference Alignment as a Rank Constrained Rank Minimization

In this section, we show that for each user $k \in \mathcal{K}$, the maximum achievable multiplexing gain (or DoF) can be put in the form of an RCRM problem. We do this by restating the interference alignment conditions using a rank minimization framework. The rank minimization aims for the maximum interference suppression possible, while the rank constraints enforce each user's signal subspace to span exactly d spatial dimensions worth of communication. We show that minimizing the rank of the interference is equivalent to maximizing the multiplexing gain (or sum-DoF) in an $(M_{\mathrm{r}} \times M_{\mathrm{t}}, d)^K$ system, where each useful signal space spans d dimensions. Then, we use this framework to develop a new algorithm and compare its tightness to existing interference leakage minimization approaches.

We begin by stating the perfect IA requirements of [9]. In an $(M_{\mathrm{r}} \times M_{\mathrm{t}}, d)^K$ system, for all $k \in \mathcal{K}$, IA requires that

$$\mathbf{U}_k^H \mathbf{H}_{k,l} \mathbf{V}_l = \mathbf{0}_{d\times d}, \quad \forall l \in \mathcal{K}\backslash k \tag{2}$$

$$\text{rank}\left(\mathbf{U}_k^H \mathbf{H}_{k,k} \mathbf{V}_k\right) = d \tag{3}$$

where (2) enforces all interference subspaces to have zero dimensions and (3) enforces the useful signal to span all d dimensions.

Remark 2: Observe that (2) is a set of bilinear equations in the unknown linear precoding and linear receive filters. Recently, a feasibility question has been raised as to whether a system can satisfy these IA requirements [10]. Yetis *et al.* introduce in [10] the notion of proper systems. An $(M_{\mathrm{r}} \times M_{\mathrm{t}}, d)^K$ system is called *proper*, when the number of variables is less than or equal to the number of equations, which is equivalent to the condition $M_{\mathrm{r}} + M_{\mathrm{t}} - d(K+1) \geq 0$; if this condition is not satisfied, then the system is called improper. When the channel coefficients are "generic", i.e., selected at random and i.i.d. from continuous distributions, then improper systems do not admit perfect IA (i.e., d is not achievable), almost surely [10], [23], [24]. Many proper systems, e.g., cases like $d = 1$, or $M = N$, admit IA almost surely [10], [23], [24]. However, "properness" does not always imply that IA is feasible [10], nevertheless, in many interesting cases it serves as a useful proxy for IA feasibility. Finding precoding and receive matrices satisfying the IA conditions, for arbitrary channel matrices, is computationally intractable. In particular, Razaviyayn *et al.* [13] recently established that for an arbitrary K-user MIMO interference channel, checking the achievability of a certain multiplexing gain tuple $\{d_1, \dots, d_K\}$ is NP-hard, when each user and transmitter has more than 2 antennas. Therefore, solving the perfect IA set of bilinear equations in a single-shot manner cannot be performed efficiently in the general case. For these cases the maximization of the number of interference free dimensions is still a goal one should be aiming for. Hence, we believe that a more general framework for the IA problem is needed, that not only captures the notion of what is the maximum multiplexing gain, but can also quantify what one should be optimizing to achieve this gain. In the following, we introduce a computationally challenging optimization which might be even harder than solving a set of bilinear IA equations, however, it provides helpful cues for good heuristic schemes and possibly a better understanding of the problem.

We continue with rewriting (2)

$$\begin{aligned} & \mathbf{U}_k^H \mathbf{H}_{k,l} \mathbf{V}_l = \mathbf{0}_{d\times d}, \quad \forall l \in \mathcal{K}\backslash k \\ \iff & \left[\left\{\mathbf{U}_k^H \mathbf{H}_{k,l} \mathbf{V}_l\right\}_{l=1, l\neq k}^K\right] = \left[\mathbf{0}_{d\times d} \dots \mathbf{0}_{d\times d}\right] \\ \iff & \mathbf{U}_k^H \left[\left\{\mathbf{H}_{k,l} \mathbf{V}_l\right\}_{l=1, l\neq k}^K\right] = \mathbf{0}_{d\times (K-1)d} \end{aligned}$$

and defining the *signal* and *interference* matrices for all $k \in \mathcal{K}$

$$\mathbf{S}_k \triangleq \mathbf{U}_k^H \mathbf{H}_{k,k} \mathbf{V}_k \in \mathbb{C}^{d\times d} \tag{4}$$

$$\mathbf{J}_k \triangleq \mathbf{U}_k^H \left[\left\{\mathbf{H}_{k,l} \mathbf{V}_l\right\}_{l=1, l\neq k}^K\right] \in \mathbb{C}^{d\times (K-1)d}. \tag{5}$$

The space spanned by the columns of $\mathbf{S}_k$ is the subspace in which the kth receiver expects to observe the transmitted signal $\mathbf{x}_k$. Accordingly, the space spanned by the columns of $\mathbf{J}_k$ accounts for the interference subspace at receiver $k \in \mathcal{K}$. We restate (2) and (3) in terms of ranks

$$\text{rank}\left(\mathbf{J}_k\right) = 0 \tag{6}$$

$$\text{rank}\left(\mathbf{S}_k\right) = d \tag{7}$$

for all $k \in \mathcal{K}$. The following definition motivates the rank framework, since the spatial degrees of freedom for a given user k can be stated in terms of ranks.

Definition 1: Let $\{\mathbf{V}_l\}_{l=1}^K$ be a given set of precoding filters and $\mathbf{U}_k$ a given reveive filter employed by receiver $k \in \mathcal{K}$. Then, the multiplexing gain of user k for these sets is

$$d_k \triangleq \text{ rank}\left(\mathbf{S}_k\right) - \text{ rank}\left(\mathbf{J}_k\right) \tag{8}$$

where we assume that $\text{rank}\left(\mathbf{S}_k\right) \geq \text{rank}\left(\mathbf{J}_k\right)$, or else $d_k = 0$.

To maximize the per user multiplexing gain, we have to design transmit and receive strategies in a sophisticated way such that this difference of ranks in d_k is maximized. More precisely, for an $(M_{\mathrm{r}} \times M_{\mathrm{t}}, d)^K$ system where each user aims for d interference free dimensions, the following set of K "parallel" rank constrained rank minimization problems has to be solved:

$$\min_{\{\mathbf{V}_l\}_{l=1, l\neq k}^K, \mathbf{U}_k} \quad \text{rank}\left(\mathbf{J}_k\right) \tag{9}$$

$$\text{s.t.}: \ \text{rank}\left(\mathbf{S}_k\right) = d. \tag{10}$$

When perfect IA is feasible (9) and (10) can find it, or else the best possible solution, in terms of interference free dimensions, will be obtained for user $k \in \mathcal{K}$, assuming we can solve the above optimization.

Remark 3: As noted before, for the constant K-user MIMO interference channel it is known that the maximum multiplexing gain using linear transmit and receive schemes cannot exceed $d^* = \frac{M_{\mathrm{t}} + M_{\mathrm{r}}}{K+1}$, almost surely [10], [23], [24]. In that regard, when aiming for $d \leq d^*$, and d^* is achievable, we should expect that the rank of each interference can go down to zero, as perfect IA requires. Hence, why use the rank formulation? As we see in the following, the fact that the rank formulation completely captures the notion of per user multiplexing gain, assists us in tightly approximating this objective. The nature of our relaxation is expected to favor solutions where the interference has low rank, instead of low energy (i.e., small interference leakage). This is of particular interest when perfect IA might be feasible but hard to obtain, when it is not feasible but the best possible solution is required, or for interference channels where the notion of properness is not clearly defined, such as cellular networks.

Since we cannot solve in parallel the K optimization problems defined in (9) and (10), we will aim to maximize the sum of interference free dimensions through the following RCRM:

$$\boxed{\begin{array}{ll}\mathcal{P}: & \\ & \min_{\substack{\{\mathbf{V}_l\}_{l=1}^K \\ \{\mathbf{U}_l\}_{l=1}^K}} \sum_{k=1}^K \operatorname{rank}(\mathbf{J}_k) \\ \text{s.t.:} & \operatorname{rank}(\mathbf{S}_k) = d, \quad \forall k \in \mathcal{K}.\end{array}}$$

The orthogonality constraints on the precoding and receive matrices are omitted in the above RCRM. This is due to the fact that we can always linearly transform them so that the columns of each of these matrices are mutually orthogonal. Specifically, we can rewrite the precoding matrices as $\mathbf{V}_k = \mathbf{Q}_k^{(v)}\mathbf{R}_k^{(v)}$, where $\mathbf{Q}_k^{(v)} \in \mathbb{C}^{M_\mathrm{t}\times d}$ is an orthonormal basis for the column space of $\mathbf{V}_k$ and $\mathbf{R}_k^{(v)} \in \mathbb{C}^{d\times d}$ is the matrix of coefficients participating in the linear combinations yielding the columns of $\mathbf{V}_k$. Then we may use $\sqrt{\frac{P}{d}}\mathbf{Q}_k^{(v)}$ as the precoding matrix. Accordingly, we use orthonormal matrices $\mathbf{Q}_k^{(u)}$ constructed by decomposing $\mathbf{U}_k$ to $\mathbf{Q}_k^{(u)}\mathbf{R}_k^{(u)}$, where $\mathbf{Q}_k^{(u)} \in \mathbb{C}^{M_\mathrm{r}\times d}$ and $\mathbf{R}_k^{(u)} \in \mathbb{C}^{d\times d}$. Observe that $\operatorname{span}\left(\mathbf{Q}_k^{(v)}\right) = \operatorname{span}(\mathbf{V}_k)$ and $\operatorname{span}\left(\mathbf{Q}_k^{(u)}\right) = \operatorname{span}(\mathbf{U}_k)$, for all $k \in \mathcal{K}$. Moreover, the ranks of the interference and signal matrices are oblivious to full-rank linear transformations, that is

$$\begin{aligned}
&\operatorname{rank}(\mathbf{J}_k) \\
&\quad= \operatorname{rank}\left(\left(\mathbf{R}_k^{(u)}\right)^H \left(\mathbf{Q}_k^{(u)}\right)^H \left[\{\mathbf{H}_{k,l}\mathbf{V}_l\}_{l=1,l\neq k}^K\right]\right) \\
&\quad= \operatorname{rank}\left(\left(\mathbf{Q}_k^{(u)}\right)^H \left[\left\{\mathbf{H}_{k,l}\mathbf{Q}_l^{(v)}\right\}_{l=1,l\neq k}^K\right]\right. \\
&\qquad\qquad \left.\times \operatorname{blkdiag}\left(\left\{\mathbf{R}_l^{(v)}\right\}_{l=1,l\neq k}^K\right)\right) \\
&\quad= \operatorname{rank}\left(\left(\mathbf{Q}_k^{(u)}\right)^H \left[\left\{\mathbf{H}_{k,l}\mathbf{Q}_l^{(v)}\right\}_{l=1,l\neq k}^K\right]\right)
\end{aligned} \tag{11}$$

where $\operatorname{blkdiag}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$ denotes the block diagonal matrix that has as ith diagonal block the matrix $\mathbf{A}_i$. The above equalities hold since $\operatorname{rank}\left(\mathbf{R}_l^{(u)}\right) = \operatorname{rank}\left(\mathbf{R}_l^{(v)}\right) = d$. Furthermore, we have

$$\begin{aligned}
\operatorname{rank}(\mathbf{S}_k) &= \operatorname{rank}\left(\left(\mathbf{R}_k^{(u)}\right)^H \left(\mathbf{Q}_k^{(u)}\right)^H \mathbf{H}_{k,k}\mathbf{Q}_k^{(v)}\mathbf{R}_k^{(v)}\right) \\
&= \operatorname{rank}\left(\left(\mathbf{Q}_k^{(u)}\right)^H \mathbf{H}_{k,k}\mathbf{Q}_k^{(v)}\right) = d.
\end{aligned}$$

Hence, orthogonalization is always possible.

Remark 3: Observe that we can generalize $\mathcal{P}$ to the case where transmitter k is equipped with $M_{\mathrm{t},k}$ antennas, receiver k with $M_{\mathrm{r},k}$ antennas, and the k user's signal space spans d_k dimensions, $k \in \mathcal{K}$. This generalized version of $\mathcal{P}$ can be used to decide the achievability of any multiplexing gain tuple $\{d_1, d_2, \ldots, d_K\}$: if this tuple is achievable the cost function of $\mathcal{P}$ will be zero. However, this is an NP-hard problem for the case of $M_{\mathrm{t},k}, M_{\mathrm{r},k} > 2$ [13]. Therefore, the generalized version of $\mathcal{P}$ has to be at least as hard as determining the achievability of a multiplexing gain tuple.

To conclude, we have established that the minimization of the sum of the interference dimensions under full-rank signal space constraints is equivalent to maximizing the sum multiplexing gain of a static flat-fading MIMO interference channel. There exist various regimes of difficulty for this RCRM problem. There are tractable regimes, where one randomly selects the precoding or receive matrices and constructs the receive or precoding matrices with columns that exactly fit in the null space of the interference or reciprocal interference matrices. This is possible when either $M_\mathrm{r} \geq Kd$ or $M_\mathrm{t} \geq Kd$ hold, i.e., when there are enough antennas (i.e., dimensions) so that each stream populates its own subspace and standard zero-forcing is optimal. Various closed form solutions for special cases of the problem have been also introduced in the literature [12]. Moreover, for i.i.d. diagonal channel matrices, the symbol extension method presented in [2] corresponds to instances of the RCRM problem that are efficiently approximated to within a vanishing gap of multiplexing gain with respect to the number of extensions. However, generally such solutions are NP-hard to obtain and the RCRM problem cannot be solved efficiently. In the next section, we provide a heuristic that approximates $\mathcal{P}$.

IV. A Nuclear Norm Heuristic

In the previous section, we have shown that the multiplexing gain maximization of a K-user MIMO IC can be recast as an RCRM, where the precoding and receive matrices are the optimization variables. To approach this nonconvex and intractable problem, we propose convex surrogates for the cost function and the feasible solution set. Then, we use a coordinate descent approach, where we alternatively and iteratively optimize over the transmit and then over the receive matrices.

We begin by obtaining the tightest convex approximation to the cost function of $\mathcal{P}$. We have

$$\begin{aligned}
&\overline{\operatorname{conv}}\left(\sum_{k=1}^K \operatorname{rank}(\mathbf{J}_k)\right) \\
&\quad= \overline{\operatorname{conv}}\left(\operatorname{rank}(\operatorname{blkdiag}(\mathbf{J}_1, \ldots, \mathbf{J}_K))\right) \\
&\quad= \frac{1}{\mu}\|\operatorname{blkdiag}(\mathbf{J}_1, \ldots, \mathbf{J}_K)\|_* \\
&\quad= \frac{1}{\mu}\sum_{k=1}^K \|\mathbf{J}_k\|_* = \frac{1}{\mu}\sum_{k=1}^K\sum_{i=1}^d \sigma_i(\mathbf{J}_k)
\end{aligned} \tag{12}$$

where $\overline{\operatorname{conv}}(f)$ denotes the convex envelope of a function f, $\|\mathbf{A}\|_* = \sum_{i=1}^{\operatorname{rank}(\mathbf{A})}\sigma_i(\mathbf{A})$ is the nuclear norm of a matrix $\mathbf{A}$, which accounts for the sum (i.e., the ℓ_1-norm) of the singular values of $\mathbf{A}$, and $\sigma_i(\mathbf{A})$ is the ith largest singular value of $\mathbf{A}$. Then, the normalized sum of nuclear norms in (12) is the convex envelope of the sum of interference ranks, when the maximum singular value of the interference matrices is upper bounded by $\mu > 0$ [19].

Before we proceed with our heuristic, we wish to provide some insights on the algorithms presented in [9] and [14] and try to motivate our use of the nuclear norm as a cost function. In [9] and [14], the authors provide alternating minimization algorithms that aim to minimize the total interference leakage at each receiver. The sum of interference leakage is defined as

$$\sum_{k=1}^{K} \operatorname{tr}\left\{\mathbf{U}_k^H \mathbf{Q}_k \mathbf{U}_k\right\} \tag{13}$$

where

$$\mathbf{Q}_k = \sum_{l=1, l\neq k}^{K} \frac{P}{d} \mathbf{H}_{k.l} \mathbf{V}_l \mathbf{V}_l^H \mathbf{H}_{k,l}^H. \tag{14}$$

At each step of the optimization, either the precoding or the receive matrices are fixed, and minimization of (13) is performed over the free variables. These solution matrices are subject to orthogonality constraints, i.e., $\mathbf{V}_k^H \mathbf{V}_k = \frac{P}{d}\mathbf{I}_d$ and $\mathbf{U}_k^H \mathbf{U}_k = \mathbf{I}_d$, for all $k \in \mathcal{K}$. Observe, that if we plug (14) in (13) we get

$$\begin{aligned}
&\sum_{k=1}^{K} \operatorname{tr}\left\{\mathbf{U}_k^H \left(\sum_{l=1, l\neq k}^{K} \frac{P}{d} \mathbf{H}_{k,l} \mathbf{V}_l \mathbf{V}_l^H \mathbf{H}_{k,l}^H\right) \mathbf{U}_k\right\} \\
&= \frac{P}{d} \sum_{k=1}^{K} \operatorname{tr}\left\{\mathbf{U}_k^H \left[\{\mathbf{H}_{k,l}\mathbf{V}_l\}_{l=1,l\neq k}^{K}\right]\right. \\
&\qquad\qquad \left.\times \left[\{\mathbf{H}_{k,l}\mathbf{V}_l\}_{l=1,l\neq k}^{K}\right]^H \mathbf{U}_k\right\} \\
&= \frac{P}{d} \sum_{k=1}^{K} \operatorname{tr}\left\{\mathbf{J}_k \mathbf{J}_k^H\right\} = \frac{P}{d} \sum_{k=1}^{K} \|\mathbf{J}_k\|_F^2 \\
&= \frac{P}{d} \sum_{k=1}^{K} \sum_{i=1}^{d} \sigma_i^2\left(\mathbf{J}_k\right)
\end{aligned} \tag{15}$$

where $\|\mathbf{A}\|_F$ is the Frobenius norm of $\mathbf{A}$ and the constant $\frac{P}{d}$ can be dropped in the minimization. Therefore, the interference leakage metric is the ℓ_2-norm of the singular values of all interference matrices, i.e., the sum of "interference energy" at all receivers.

Experimentally, it has been observed that the alternating leakage minimization yields perfect IA solutions for various instances of proper systems. When there is no guarantee for IA (i.e., for specific proper systems), or when the leakage minimization does not converge fast, then the ℓ_2-norm alternating minimization approaches are expected to generate solutions that yield "low energy", albeit not low rank, interference subspaces. Potentially, these interference subspaces will span more dimensions than the lowest possible, i.e., the full multiplexing gain of the system will not be utilized.

In our heuristic, instead of minimizing the ℓ_2-norm of the interference singular values, we aim at minimizing their ℓ_1-norm. We believe that minimizing the ℓ_1-norm of the interference singular values may favor low-rank solutions, as it is provably the case for affine constrained, rank minimization problems [19], [20]. Low rank solutions in our context translate to higher multiplexing gain, therefore to higher data-rates for the high-SNR regime, which motivates the use of the nuclear norm as a tractable cost function. However, we should note that our RCRM, although a rank minimization itself, does not fit exactly in the affine constrained, rank minimization framework of [20] and exact solution guarantees cannot be used for our algorithm in a straightforward manner.

We continue by observing that, even for the nuclear norm cost function, the problem remains challenging for two reasons: i) although the nuclear norm is convex in either of the two sets of input matrices, if we try to minimize over both sets at the same time, the cost function is no longer convex; and ii) we have to satisfy the rank constraints that define a nonconvex set. The nonconvexity of the cost function is due to the fact that the elements of the transmit and receive matrices appear in bilinear terms. To avoid the bilinear terms, we relax the optimization and follow a coordinate descent approach by alternating between optimizing over the transmit and then over the receive matrices.

Even under the coordinate descent relaxation, the rank constraints do not imply a convex feasible set. However, we observe that we can enforce the rank constraints of $\mathcal{P}$, without loss of generality with respect to the solutions of $\mathcal{P}$, by constraining the signal space matrices to be positive definite. The positivity constraints define a convex (but open) set of solutions. Specifically, we can replace the constraint $\operatorname{rank}(\mathbf{S}_k) = d$ with the following:

$$\mathbf{S}_k \succ \mathbf{0}_{d\times d} \tag{16}$$

where $\mathbf{S}_k \succ \mathbf{0}_{d\times d}$ denotes that matrix $\mathbf{S}_k$ is Hermitian positive definite, i.e., $\mathbf{S}_k = \mathbf{S}_k^H$ and $\lambda_{\min}(\mathbf{S}_k) > 0$, for all $k \in \mathcal{K}$, where $\lambda_{\min}(\mathbf{S}_k)$ is the minimum eigenvalue of $\mathbf{S}_k$. In the following lemma, we show that any solution of $\mathcal{P}$ can be transformed to one that yields positive definite signal space matrices, while preserving the same sum-rank cost function.

Lemma 1: Let $\{\mathbf{V}_l\}_{l=1}^K$ and $\{\mathbf{U}_l\}_{l=1}^K$ be feasible solutions of $\mathcal{P}$ that give a cost function of ρ. Then, there exists a set of matrices $\hat{\mathbf{U}}_k$ such that the cost function of $\mathcal{P}$ is ρ and the signal space matrix is positive definite, for all $k \in \mathcal{K}$.[1]

Proof: Let a new set of receive matrices

$$\{\hat{\mathbf{U}}_k\}_{k=1}^K = \{\mathbf{U}_k\mathbf{S}_k\}_{k=1}^K \tag{17}$$

Then

$$\begin{aligned}
\operatorname{rank}\left(\hat{\mathbf{S}}_k\right) &= \operatorname{rank}\left(\left(\mathbf{U}_k\mathbf{S}_k\right)^H\right)\mathbf{H}_{k,k}\mathbf{V}_k) \\
&= \operatorname{rank}\left(\mathbf{S}_k^H\mathbf{U}_k^H\mathbf{H}_{k,k}\mathbf{V}_k\right) \\
&= \operatorname{rank}\left(\mathbf{S}_k^H\mathbf{S}_k\right) = d
\end{aligned}$$

and the new signal matrices are positive definite, that is

$$\lambda_{\min}\left(\hat{\mathbf{S}}_k\right) = \lambda_{\min}\left(\mathbf{S}_k^H\mathbf{S}_k\right) > 0$$

where $\hat{\mathbf{S}}_k$ denotes the signal space under the new $\hat{\mathbf{U}}_k$ receive matrices, for all $k \in \mathcal{K}$. Moreover

$$\begin{aligned}
\operatorname{rank}(\mathbf{J}_k) &= \operatorname{rank}\left(\mathbf{S}_k^H\mathbf{J}_k\right) \\
&= \operatorname{rank}\left(\mathbf{S}_k^H\mathbf{U}_k^H\left[\{\mathbf{H}_{k,l}\mathbf{V}_l\}_{l=1,l\neq k}^K\right]\right) \\
&= \operatorname{rank}\left(\hat{\mathbf{U}}_k^H\left[\{\mathbf{H}_{k,l}\mathbf{V}_l\}_{l=1,l\neq k}^K\right]\right) \\
&= \operatorname{rank}\left(\hat{\mathbf{J}}_k\right)
\end{aligned}$$

[1]We can obtain an analogous result for the precoding matrices.

due to the fact that multiplying $\mathbf{J}_k$ with a square full-rank matrix does not change the rank of $\mathbf{J}_k$. Hence, under the new set of receive matrices the cost function reevaluates to ρ and the constraints of $\mathcal{P}$ are satisfied. □

In our heuristic, we approximate the open set $\mathbf{S}_k \succ \mathbf{0}_{d\times d}$ with the closed one

$$\mathbf{S}_k \succeq \mathbf{0}_{d\times d} \text{ and } \lambda_{\min}(\mathbf{S}_k) \geq \epsilon \tag{18}$$

where $\epsilon > 0$. Although, the above closed set is always a subset of the open one, as ϵ gets closer to 0, the two sets asymptotically overlap.

To summarize the above, i) we relaxed the rank cost function to a nuclear norm one, ii) we exchanged the rank constraints with tractable ones, and iii) we have decided to solve our optimization using a coordinate descent approach. The steps of our approximation algorithm follow. First, we arbitrarily select the receive matrices. Then, we solve the following convex optimization problem:

$$\boxed{\begin{aligned} &\mathcal{A}_{\mathbf{V}}\left(\{\mathbf{U}_l\}_{l=1}^K\right): \\ &\qquad\qquad \min_{\{\mathbf{V}_l\}_{l=1}^K} \sum_{k=1}^K \|\mathbf{J}_k\|_* \\ &\qquad\qquad \text{s.t.: } \mathbf{S}_k \succeq \mathbf{0}_{d\times d}, \\ &\qquad\qquad \lambda_{\min}(\mathbf{S}_k) \geq \epsilon, \quad \forall k \in \mathcal{K}. \end{aligned}}$$

We use its solution as an input to $\mathcal{A}_{\mathbf{U}}\left(\{\mathbf{V}_l\}_{l=1}^K\right)$, and then feed the solutions of this optimization back to $\mathcal{A}_{\mathbf{V}}\left(\{\mathbf{U}_l\}_{l=1}^K\right)$. We continue by iterating this process. This procedure is stated below as algorithm $\mathcal{A}(n)$, where n is the number of iterations.

$\mathcal{A}(n)$:

1: `initialize` $\{\mathbf{U}_l\}_{l=1}^K$

2: `for` n `iterations`

3: $\{\mathbf{V}_l\}_{l=1}^K \leftarrow \mathcal{A}_{\mathbf{V}}\left(\{\mathbf{U}_l\}_{l=1}^K\right)$

4: $\{\mathbf{U}_l\}_{l=1}^K \leftarrow \mathcal{A}_{\mathbf{U}}\left(\{\mathbf{V}_l\}_{l=1}^K\right)$

5: `orthogonalize` $\{\mathbf{V}_l\}_{l=1}^K, \{\mathbf{U}_l\}_{l=1}^K$

Observe that at every step the cost function will not increase. Therefore, the heuristic will converge in terms of the cost function, but that does not imply convergence in terms of the receive and transmit matrices: at each step the local minimizer may not be unique. In the simulations section, we provide quantitative performance results for our proposed approximation and observe that it indeed favors low-rank interference solutions.

Remark 5: We would like to note a key difference between our iterative approach and the approaches of [9] and [14]. For channels that do not have the block diagonal structure, (3) almost surely holds when using the alternating leakage minimization approaches, due to the precoding and receive orthogonality constraints and the random nature of the medium. However, this probabilistic argument does no longer hold when considering channels with block diagonal structure. That is, zero interference leakage may be obtained using alternating leakage minimization, but at the same time the signal subspaces may be confined to less than d dimensions not obeying (3) [16]. This means that the signal matrix $\mathbf{S}_k$ will be rank deficient, i.e., the multiplexing gain potential of the system may not be fully exploited. In our approach, we explicitly enforce $\mathbf{S}_k$ to be full-rank, so that (3) is always satisfied for *any* set of channel matrices that can support it.

To conclude this section, we have introduced an alternating nuclear norm minimization scheme to approximate the sum multiplexing gain maximization in the K-user MIMO interference channel. Our approximation is motivated by our RCRM framework. In our approach, we relax the rank cost function to its convex envelope, the nuclear norm of interference singular values. Then, we approximate the full-rank constraints with positivity constraints on the minimum eigenvalue of the signal space matrices that is (without loss of generality) restricted to be positive definite, and show that asymptotically this is a tight relaxation with respect to the initial RCRM objective.

V. Interference Alignment for Cellular Networks

In this section, we consider the case of the K-cell interference channel as presented in [11], where each cell supports d users. We tailor our algorithm for this problem by adding extra affine constraints on the entries of the precoding matrices. Such constraints correspond to the fact that each user $u \in \{1,\ldots,d\}$, in a cell $k \in \mathcal{K}$, wishes to transmit only one symbol using a beamforming vector $\mathbf{v}_{k,u} \in \mathbb{C}^{\frac{M_t}{d}\times 1}$, where we assume that $\frac{M_t}{d}$ is a positive integer. For this system, the received signal at receiver $k \in \mathcal{K}$ is given by

$$\begin{aligned} \mathbf{y}_k &= \sum_{u=1}^d \mathbf{H}_{k,k}^{(u)}\mathbf{v}_{k,u}x_{k,u} + \sum_{l=1,l\neq k}^K \sum_{u=1}^d \mathbf{H}_{k,l}^{(u)}\mathbf{v}_{l,u}x_{l,u} + \mathbf{w}_k \\ &= \mathbf{H}_{k,k}\mathbf{V}_k\mathbf{x}_k + \sum_{l=1,l\neq k}^K \mathbf{H}_{k,l}\mathbf{V}_l\mathbf{x}_l + \mathbf{w}_k \end{aligned} \tag{19}$$

where $\mathbf{H}_{k,l}^{(u)} \in \mathbb{C}^{M_r\times\frac{M_t}{d}}$ represents the channel between user u of cell l and receiver k, $\mathbf{H}_{k,l} = \left[\mathbf{H}_{k,l}^{(1)}\ldots\mathbf{H}_{k,l}^{(d)}\right] \in \mathbb{C}^{M_r\times M_t}d$ and

$$\mathbf{V}_k = \begin{bmatrix} \mathbf{v}_{k,1} & \cdots & \mathbf{0}_{\frac{M_t}{d}\times 1} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{\frac{M_t}{d}\times 1} & \cdots & \mathbf{v}_{k,d} \end{bmatrix}$$

where $\mathbf{v}_{k,u}$ represents the beamforming vector used by user u of cell k, for $k,l \in \mathcal{K}$. This multicell interference model is equivalent to a general K-user MIMO interference channel, where symbol $x_{k,u}$ of the symbol vector $\mathbf{x}_k = [x_{k,1}\ldots x_{k,d}]^T$ is transmitted only from some subset of $\frac{M_t}{d}$ transmit antennas.

Ideally, we would like to solve $\mathcal{P}$ with the extra affine constraints corresponding to zeros at the appropriate entries of $\mathbf{V}_k$. We propose the following approximation: run $\mathcal{A}(n)$, where $\mathcal{A}_{\mathbf{V}}\left(\{\mathbf{U}_l\}_{l=1}^K\right)$ has added affine constraints

$$[\mathbf{V}_k]_{\mathcal{I}_u,u} = \mathbf{0}_{K(d-1)\times 1} \quad \forall u \in \{1,2,\dots d\} \tag{20}$$

where

$$\mathcal{I}_u = \left\{1,\dots,(u-1)\frac{M_t}{d}\right\} \cup \left\{u\frac{M_t}{d}+1,\dots,M\right\} \tag{21}$$

$\forall u \in \{1,\dots,d\}$, are the sets of indices (rows) where the precoding matrices have zero entries. Observe that no further approximation is made to $\mathcal{P}$ due to the tractability of the affine constraints. In the simulations section, we observe that the cellular case heuristic yields solutions with low rank interference matrices, resulting in high sum-rate. We would like to note that the above framework can be easily extended to the case where each user of a cell transmits more than 1 symbol.

VI. Simulations

A. Interference Channel

In this experimental evaluation we run simulations for a $(4\times 8, d=1,3)^3$, a $(6\times 6, d=1,3)^3$, a 2 time slot symbol extended single antenna, 3-user interference channel, with $d=1$, and a $(4\times 18, d=1,2)^{10}$ system. All MIMO systems considered are proper, i.e., $d \leq \frac{M_t+M_r}{K+1}$; we show that although the first system is proper, it is not expected to admit perfect IA solutions for $d=3$ in a random and i.i.d. channel setting. In our simulations, we allocate power $\frac{10^{\frac{P}{10}}}{d}$ to each column of the precoding matrices, where $P \in \{0,10,20,\dots,80\}$ dB, and set the real noise power level to $\sigma^2 = 1$. We present results averaged over 200 channel realizations, where each channel element is drawn i.i.d. from a real Gaussian distribution with mean zero and variance 1. We plot the sum-rate of each system and the average number of interference free dimensions per user, i.e., the average multiplexing gain per user. The sum rates that we plot are computed using the formula $R = \sum_{i=1}^{K}\frac{1}{2}\log\det\left(\mathbf{I}_d + \left(\mathbf{I}_d + \mathbf{J}_k\mathbf{J}_k^H\right)^{-1}\mathbf{S}_k\mathbf{S}_k^H\right)$. The number of interference free dimensions (for normalized input $\mathbf{V}_k$ and output $\mathbf{U}_k$ matrices) is calculated as the number of singular values of $\mathbf{S}_k$ with value greater than 10^{-6}, minus the number of singular values of $\mathbf{J}_k$ that are greater than 10^{-6}. We should note that this metric does not account for the rate slope at the low SNR regime.

1) 3-User Interference Channel: First, we consider a $(4\times 8, d=1,3)^3$ and a $(6\times 6, d=1,3)^3$ system. For each simulation, we run 5 iterations of our algorithm, 10^4 iterations of the minimum interference leakage algorithm, and 10^4 iterations of the max-SINR algorithm. We also calculate an orthogonalized version of the max-SINR outputs, which is denoted as max-SINR with QR in the figures. To run $\mathcal{A}(5)$ we set $\epsilon = 0.1$ and use the CVX toolbox [25] in MATLAB.

Remark 6: In terms of calculations, at each iteration our proposed algorithm solves 2 semidefinite programs, the leakage minimization $2\cdot K$ eigenvalue decompositions, and the max-SINR performs $2\cdot d\cdot K$ matrix inversions. For our proposed scheme and the max-SINR with QR algorithm, $2K$ QR factorizations are performed to orthogonalize the output precoding and receive matrices. In the following experiments, we select the specific number of iterations for each approach so that all three take comparable times to run using MATLAB, on a 2.53 Ghz Intel Core 2 Duo processor with 4 GB of RAM. We would also like to note that each iteration of our proposed scheme solves 2 semidefinite programs, which in their own respect are iterative processes, therefore, a straightforward comparison of iteration numbers might not be fair at this point. Moreover, the algorithms that we compare against are based on built-in functions of MATLAB, whereas our scheme relies on a toolbox that is not optimized for our framework. For future work, it may be interesting to develop a custom script that solves our optimization, which would serve for a more realistic comparison in terms of running times.

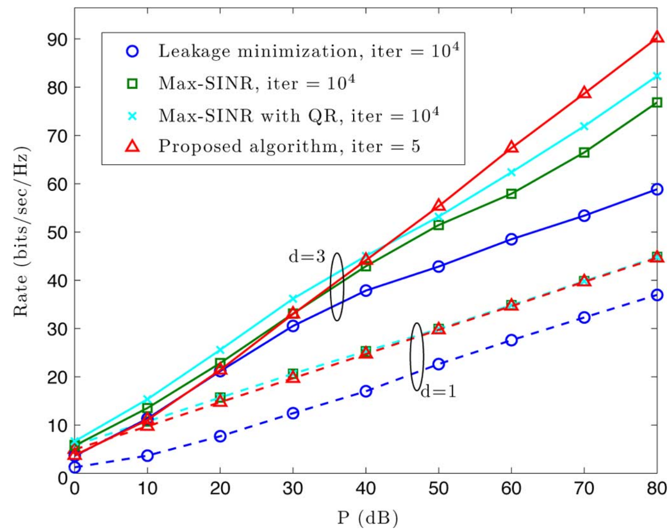

Fig. 1. Sum-rate versus P, for a 3-user system, $M_r = 4$, $M_t = 8$, and $d = 1,3$.

In Figs. 1 and 2, we consider a $(4\times 8, d=1,3)^3$ system and plot versus P the sum rate and the average per user multiplexing gain, or available interference free signaling dimensions achieved by the interference leakage minimization algorithm, the max-SINR approach of [9] and [14], and our scheme, respectively. Although the system here is proper, perfect IA is not feasible, and the total number of interference free dimensions in the considered system cannot exceed 8. This is due to the fact that we can consider as a valid outerbound for d, the case where the 2 users fully cooperate, which can be seen as an equivalent 2-user MIMO IC, for which we have an outerbound on multiplexing gain [5]. This bound yields $3d \leq \min\left(M_t+2M_t, \max\left(M_t,2M_r\right), \max\left(2M_t,M_r\right)\right) = 8$.

In Fig. 1, we observe for the $d=1$ case that at low to moderate SNRs, the max-SINR solution outperforms both our proposed algorithm and the leakage minimization with respect to the achievable data rate. When we shift to higher SNRs above 40 dB, our proposed algorithm matches the performance of max-SINR (with or without QR) and both schemes offer a slight rate benefit compared to the leakage minimization approach. For $d=3$, we observe the same trend at the low-SNR regime, where in this case both max-SINR and leakage minimization

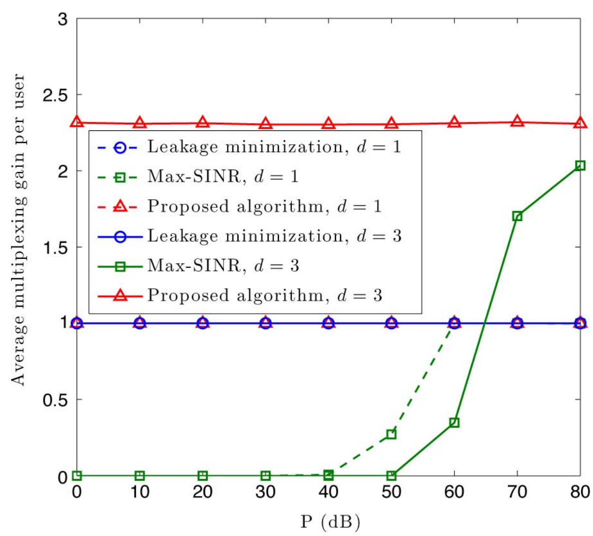

Fig. 2. Average number of interference free dimensions per user versus P, for a 3-user system, $M_{\rm r}=4$, $M_{\rm t}=8$, and $d=1,3$.

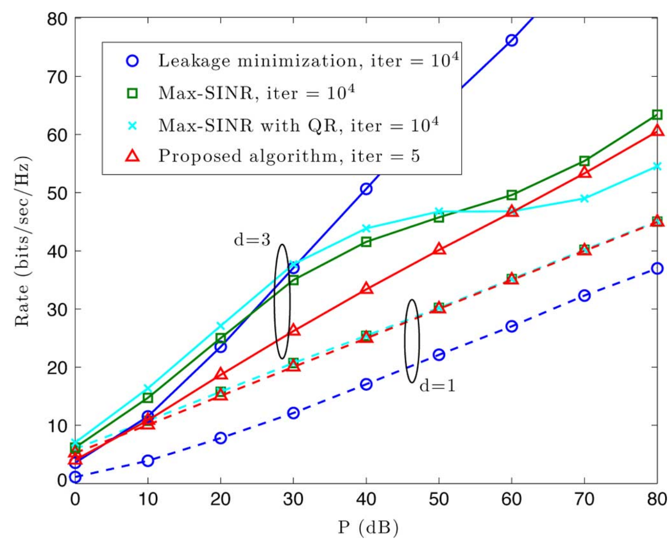

Fig. 3. Sum-rate versus P, for a 3-user system, $M_{\rm r}=6$, $M_{\rm t}=6$, and $d=1,3$.

schemes achieve higher rates compared to our approach. However, the benefits of our algorithm become apparent for SNRs above 40 dB where the extra multiplexing gain achieved yields higher data rates compared to both the max-SINR and leakage minimization approaches. In Fig. 2, observe that for $d=1$ both our scheme and the leakage minimization achieve exactly $d=1$ average per user multiplexing gain for all SNRs, while the max-SINR algorithm exhibits a more adaptive behavior which yields 1 interference free dimension at the high SNR regime, where the multiplexing gain becomes an important factor of the SINR metric. For $d=3$, both the max-SINR and leakage minimization algorithms do not achieve more than 1 per user multiplexing gain, for 10^4 iterations. In this case, our nuclear norm approach seems to favor sparse solutions that yield an average of 2.333 multiplexing gain per user that results to higher data rates.

In Figs. 3 and 4, we consider a $(6\times 6, d=1,3)^3$ system. In Fig. 3 for $d=1$, the rates when using the max-SINR and our proposed algorithm approximately match for SNRs greater than 40 dB and both schemes offer a constant gap rate advantage compared to the leakage minimization approach. For, $d=3$ the max-SINR algorithm outperforms both our proposed algorithm and the leakage minimization approach, for SNRs up to approximately 25 dB. For the SNR regime above 30 dB, our algorithm does not provide extra multiplexing gain for 5 iterations, while the interference leakage provides more and yields substantially higher data rates compared to both max-SINR and our approach. In Fig. 4, we observe that leakage minimization achieves more interference free dimensions compared to our algorithm and the max-SINR approach which results to higher achievable per user multiplexing gain. In many instances of this experiment the leakage minimization achieved a multiplexing gain of $d=3$ per user, however it seems that the number of iterations we used did not allow the algorithm to always converge to an optimal solution. This is why the multiplexing gain curve of Fig. 4 has an oscillating behavior.

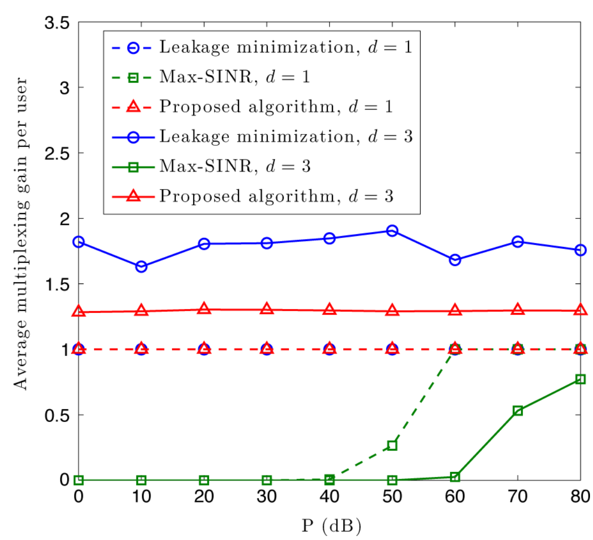

Fig. 4. Average number of interference free dimensions per user versus P, for a 3-user system, $M_{\rm r}=6$, $M_{\rm t}=6$, and $d=1,3$.

Remark 7: In general, the experiments that we run suggest 3 regimes of "behavior" for our algorithm: 1) when $d<\frac{M_{\rm t}+M_{\rm r}}{K+1}$, i.e., when there are strictly more unknowns than equations in the IA conditions, all algorithms seem to get the maximum multiplexing gain (at least in the high-SNR regime); the experimental feasibility of these proper systems may be attributed to the fact that indeed they admit an IA solution, 2) when a system is "borderline" proper, i.e., $d=\frac{M_{\rm t}+M_{\rm r}}{K+1}$ and IA is feasible, as is the $N=M=6$ case [23], the leakage minimization and max-SINR algorithms require a substantial number of iterations, however, they converge to a multiplexing-gain optimal solution; for this case our algorithm performs worse and does not seem to achieve perfect IA, and 3) for systems that are proper, but not feasible, as is the case of the one in Figs. 1 and 2, our algorithm outputs nontrivial multiplexing gain in comparison to the leakage minimization and max-SINR approaches.

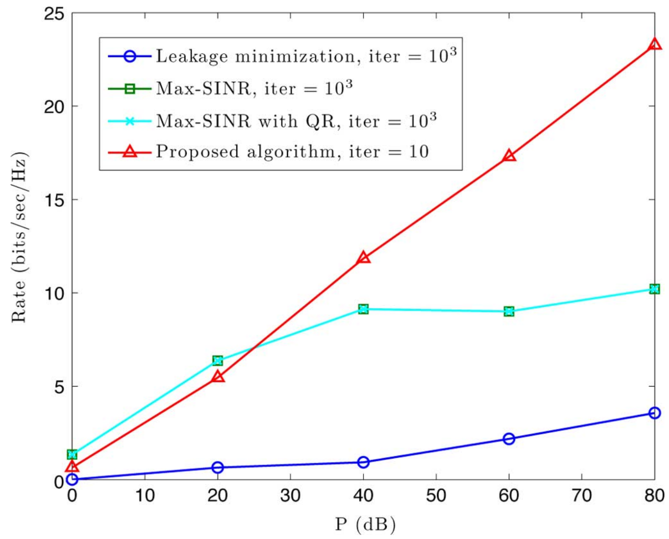

Fig. 5. Sum-rate versus P, for a 3-user system, single antenna system using a length 2 symbol extension, where $d = 1$.

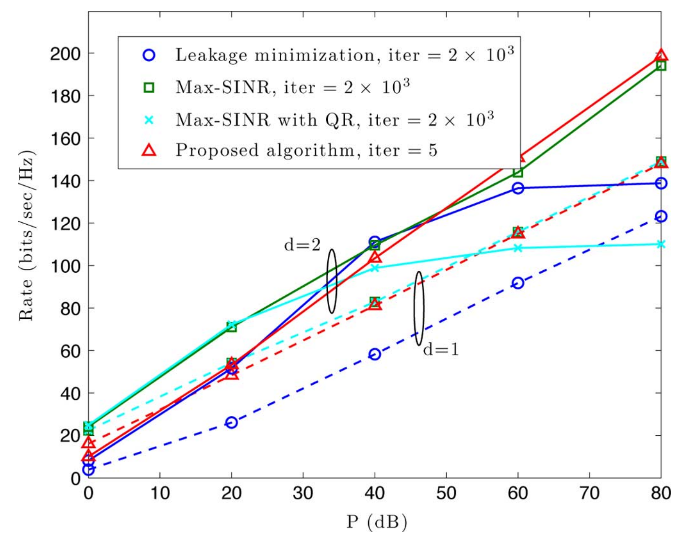

Fig. 6. Sum-rate versus P, for a 10-user system, $M_{\mathrm{r}} = 4$, $M_{\mathrm{t}} = 18$, and $d = 1, 2$.

2) 3-User, Symbol Extended, Interference Channel: In Fig. 5, we consider a single antenna interference channel, which we extend across 2 time slots and plot the sum rate versus P of the interference leakage minimization algorithm, the max-SINR approach, and our scheme, respectively, and set $d = 1$. We run the interference leakage and max-SINR approaches for 10^4 iterations and our algorithm for 5. For SNRs up to 25 dB, the max-SINR solution gives slightly higher sum-rate compared to our proposed algorithm and the leakage minimization. However, due to the diagonal structure the interference leakage scheme generates beamforming and receive matrices that result in a rank deficient signal subspace, as was also exhibited in [16]. These signal space deficiencies are avoided in our approach due to the explicit positivity constraint on the minimum eigenvalue of each signal space. We observe that in the high-SNR regime, our approach offers a substantial rate increase compared to max-SINR and leakage minimization approaches.

3) 10-User Interference Channel: In Fig. 6, we consider a $(4 \times 18, d = 1, 2)^{10}$ system and plot the sum rate versus P of the interference leakage minimization algorithm, the max-SINR approach and our scheme, respectively. Due to the size of the problem parameters, in this part, we run the interference leakage and max-SINR approaches for $2 \cdot 10^3$ iterations. For the $d = 1$ case, we observe a similar trend to the 3-user system: at the low to moderate SNR regime the max-SINR solution outperforms both our proposed algorithm and the leakage minimization, with respect to the achievable data rate. Then, the rate performance of the max-SINR and our approach seem to match and are slightly higher compared to the rate achieved by the leakage minimization. For $d = 2$, we observe that up to the SNR of 40 dB the max-SINR achieves higher data rate compared to our approach and the leakage minimization. Past the 40 dB mark, the leakage minimization and max-SINR with QR exhibit a zero slope sum-rate curve. Eventually, it seems that the extra multiplexing gain obtained by our algorithm here offers slightly better performance compared to the max-SINR at the high-SNR regime. However, we would like to note that it might be the case that none of the iterative algorithms has converged, due to the limited number of iterations.

B. Cellular Interference Channel

In Fig. 7, we consider a 4-cell interference channel, where each cell has 2 users. Each user in the cell is equipped with 4 antennas, and each receiver has 8 receive antennas. For this system, we compare our algorithm with intercell orthogonalization of the 2 users, i.e., each transmitting user has a beamformer orthogonal to the other user in the cell, and the receiver side performs interference zero-forcing (ZF). We also compare against a time-division multiple-access (TDMA) variant scheme that employs intercell orthogonalization and received interference zero-forcing; for this scheme, 2 out of the 4 cells are transmitting at each time. The ZF matrices used by the comparison schemes have columns selected as the d eigenvectors associated with the d smallest eigenvalues of the interference correlation matrix, like the one defined in (14). We run 200 experiments for different channel realizations and plot the data-rate versus $P \in \{0, 20, 40, 60, 80\}$ dB. Our proposed algorithm runs for 1, 2, and 10 iterations.

For 1 iteration, the performance of our scheme is comparable to orthogonalization and beamforming (BF), which is reasonable since 1 iteration accounts for tuning the beamforming and zero-forcing matrices once. We observe that there is an increase in data rate for 2 iterations (approximately 1.5 times more sum-rate at 80 dB) and 10 iterations (approximately 2.5 times more sum-rate at 80 dB), where our algorithm outperforms the user orthogonalization and ZF scheme, but is comparable to TDMA. The time-division scheme, due to its full multiplexing gain, achieves higher data rates in many SNR regimes, compared to our proposed scheme for 1 and 2 iterations and to user orthogonalization and interference ZF.

In Fig. 8, we simulate an 8-cell interference channel, where each cell has 2 users, each of them has 6 antennas and each receiver has 14 antennas. Again, we run our algorithm for 1, 2, and 10 iterations and compare it with user orthogonalization and

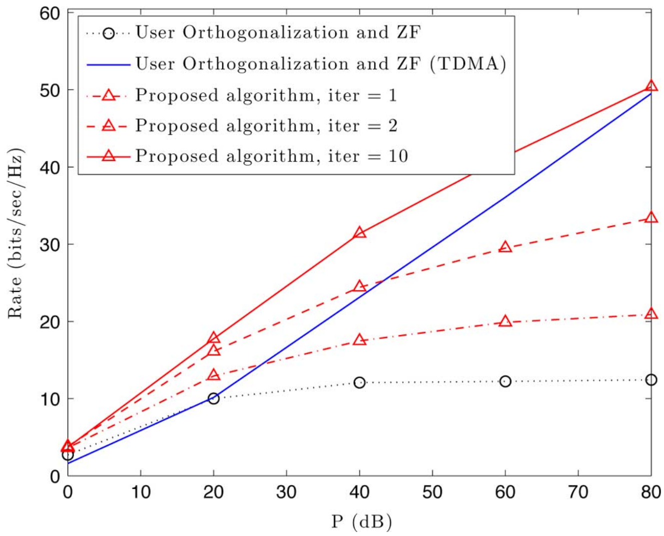

Fig. 7. Sum-rate versus P, for a 4-cell interference channel, with 2 users per cell, 2 transmit antennas per user, and 6 antennas per receiver.

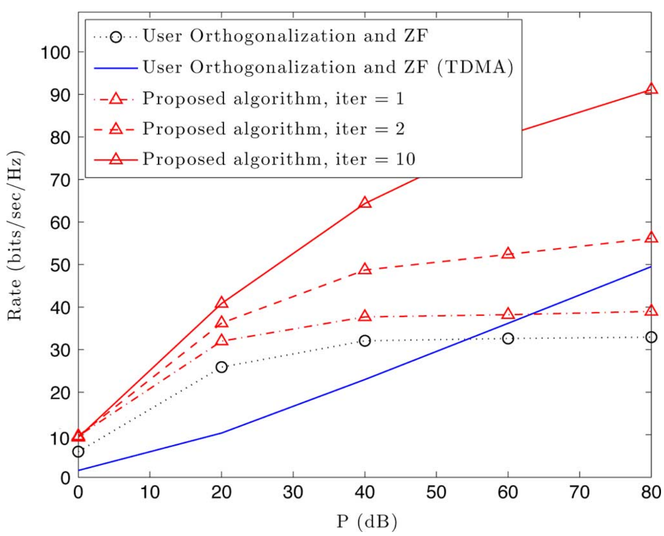

Fig. 8. Sum-rate versus P, for a 8-cell interference channel, with 2 users per cell, 6 transmit antennas per user, and 14 antennas per receiver.

interference ZF at the receivers, and TDMA with user orthogonalization and interference ZF at the receivers. We note that here the TDMA scheme requires 4 time-slots to operate without interference. For SNRs below 50 dB, we observe that even the conventional user orthogonalization and receive interference ZF strategy outperforms the time division scheme. At the SNR of 80 dB, our scheme, for 10 iterations, offers approximately 2.5 times more rate than the simple ZF strategy and almost 2 times more rate compared to the TDMA scheme.

VII. Conclusion

To conclude, in this work we reformulated the interference alignment problem as a rank constrained, rank minimization. This framework allowed us to introduce individually tight convex relaxations for the cost function and constraints of the RCRM problem. Our heuristic was inspired by the nuclear norm relaxation of rank introduced in [19], however, in this paper we did not establish theoretical guarantees under which this relaxation is tight. Such a theoretical investigation would be a very interesting open problem for future research.

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