Interference Alignment as a Rank Constrained Rank Minimization

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Abstract—We show that the maximization of the sum degrees-of-freedom (DoF) for the static flat-fading multiple-input multiple-output (MIMO) interference channel is equivalent to a rank constrained rank minimization (RCRM) problem, when the signal spaces span all available dimensions. The rank minimization corresponds to maximizing interference alignment (IA) such that interference spans the lowest dimensional subspace possible. The rank constraints account for the useful signal spaces spanning all available spatial dimensions. That way, we reformulate all IA requirements to requirements involving ranks. Then, we present a convex relaxation of the RCRM problem inspired by recent results in compressed sensing and low-rank matrix completion theory that rely on approximating rank with the nuclear norm. We show that the convex envelope of the sum of ranks of the interference matrices is the sum of their corresponding nuclear norms and introduce convex constraints that are asymptotically equivalent to the rank constraints for the initial problem. We also show that our heuristic relaxation can be also tuned to the multi-cell interference channel. Furthermore, we experimentally show that the proposed algorithm outperforms previous approaches for finding precoding and zero-forcing matrices for interference alignment.

I. INTRODUCTION

A recent information-theoretic breakthrough established that at the high signal-to-noise (SNR) regime every user in a K-user wireless interference network can enjoy half the capacity of the interference free case [1]. Therefore, interference is not a fundamental limitation for such networks since it accounts for only constant scaling of the interference free case capacity, provided that it is sufficiently mitigated. Such a surprising result is possible when interference alignment is employed. IA is a sophisticated technique first presented in [3] and subsequently utilized in [1] as a means of showing the achievability of $\frac{K}{2}$ degrees-of-freedom (DoF) for the K-user interference channel. The DoF for any channel can be perceived as the interference free dimensions, including time, frequency, or space.

Intuitively, IA serves as a means for obtaining as many interference free dimensions for communication as possible and in practice stands for designing the transmit and receive strategies for each user-receiver pair of a wireless network [1]-[7]. For the case of static flat-fading MIMO channels, such as the ones studied in [4] and [5], where both transmitter and receiver have perfect channel knowledge, the flexibility

is confined in designing the transmit precoding and receive zero-forcing matrices to maximize the achievable *spatial* DoF. Unfortunately, such matrices are a challenge to obtain, closed form solutions have been found only for a few special cases such as [1] and [11], and the problem is open for limited dimensions [9]. Even characterizing the feasibility of perfect IA is a highly non-trivial task as discussed in recent work [9]. The hardness to either find perfect IA solutions or even decide for feasibility is the cost of the problem's over constrained nature. A review of the current status of IA techniques is presented in [2].

As an alternative to closed form designs, several algorithmic approaches have been proposed in the literature such as [8], [12], and [13], that aim to minimize the *interference leakage* at each receiver so that -at best case- interference alignment is perfectly attained. The suggested insight for their effectiveness is that when *perfect* interference alignment is possible then interference leakage will be zero and such algorithms may obtain the optimal solutions. Although a fair metric to optimize, as we show in this paper, interference leakage is not the tightest approximation to the notion of DoF which is typically the desired objective. Even when perfect alignment of interferences is impossible, the objective remains to maximize the available spatial DoF, that is the prelog factor of the capacity at the high-SNR regime.

In this work, we present a formulation for the IA conditions in terms of signal and interference space ranks. Specifically, we pose full rank constraints on the useful signal spaces and minimize the rank of the interference spaces. The full rank constraints ensure useful signal spaces spanning all available spatial dimensions. The rank minimizations guarantee interference spaces collapsing to the smallest dimensional subspaces possible. We show that under the full rank constraints, minimizing the sum of ranks of the interference matrices is equivalent to maximizing the sum of spatial DoF for static flat-fading MIMO systems.

We continue by establishing a new heuristic for near optimal interference alignment and maximization of the sum of spatial DoF. Using results from [14] and [15] we provide the tightest convex approximation for the sum of interference space ranks. We suggest that the sum of the nuclear norms of all interference matrices is the best objective function in

terms of convex functions and is equivalent to the ℓ_1-norm of the singular values of the interference matrices. As intuition suggests, the ℓ_1-norm minimization of the singular values of interference matrices will provide sparse solutions, translating to more interference free signaling dimensions. Interestingly, we show that the leakage minimization techniques presented in [8], [12], and [13] minimize the ℓ_2-norm of the singular values of the interference matrices which accounts for "lowest" energy solutions rather than sparse ones. To deal with the (non-convex) full rank constraints we suggest that positivity constraints on minimum eigenvalues of positive definite matrices serve as a well motivated approximation. We extend our approximation algorithm to the K-cell interference channel [10] where each cell consists of several users and show that additional affine constraints on the precoding matrices are posed. The last section contains a preliminary experimental evaluation of the proposed algorithm. Our experiments suggest that the proposed scheme is optimal for many setups where perfect IA is possible. Furthermore, it provides extra DoF compared to the leakage minimization approach, when perfect IA is not attained. We conclude with a discussion on further research directions.

II. System Model

We consider static flat-fading MIMO interference wireless systems consisting of K users, in the set $\mathcal{K} \triangleq \{1, 2, \ldots, K\}$. We assume that each user is equipped with M_k transmit antennas, each receiver with N_k receive antennas, and all K users are synchronizing their transmissions. Each user, say $k \in \mathcal{K}$, wishes to communicate a symbol vector $\mathbf{x}_k \in \mathbb{C}^{d_k \times 1}$ to its associated receiver, where d_k represents the "pursued" DoF by user k, that is the number of symbols it wishes to transmit. To aid intuition, we note that achievable DoF can be perceived as the number of signal space dimensions *free* of interference. Prior to transmitting, user $k \in \mathcal{K}$ linearly precodes its symbol vector $\mathbf{s}_k \triangleq \mathbf{V}_k \mathbf{x}_k$, where $\mathbf{V}_k \in \mathbb{C}^{M_k \times d_k}$ denotes the precoding matrix whose d_k columns are linearly independent. We consider power constrained signals such that $E\left\{\|\mathbf{s}_k\|^2\right\} \leq P_k$, for all $k \in \mathcal{K}$. The downconverted and pulse matched received signal at a given channel use by receiver k is given by

$$\mathbf{y}_k \triangleq \mathbf{H}_{k,k}\mathbf{s}_k + \sum_{l=1,l\neq k}^{K} \mathbf{H}_{l,k}\mathbf{s}_l + \mathbf{w}_k, \qquad (1)$$

where $\mathbf{H}_{i,j} \in \mathbb{C}^{N_j \times M_i}$ represents the channel "processing" between the ith user and the jth receiver and $\mathbf{w}_k \in \mathbb{C}^{N_k \times 1}$ denotes the zero-mean complex additive white Gaussian noise vector with covariance matrix $\sigma_k^2 \mathbf{I}_{N_k}$, where $i, j, k \in \mathcal{K}$. Each receiver $k \in \mathcal{K}$, linearly processes the received signal to obtain $\mathbf{U}_k^H \mathbf{y}_k$ where $\mathbf{U}_k \in \mathbb{C}^{N_k \times d_k}$ is the corresponding linear zero-forcing filter with linearly independent d_k columns. Hence, for each receiver $k \in \mathcal{K}$, span $\left(\mathbf{U}_k^H \mathbf{H}_{k,k} \mathbf{V}_k\right)$ constitutes the useful signal space in which it expects to observe all symbols transmitted by user k, while span $\left(\left\{\mathbf{U}_k^H \mathbf{H}_{l,k} \mathbf{V}_l\right\}_{l=1,l\neq k}^{K}\right)$

is the space where all interference is observed. We denote by $\{\mathbf{X}_l\}_{l=1,l\neq k}^{K}$ the horizontal concatenation of matrices $\mathbf{X}_1, \ldots, \mathbf{X}_{k-1}, \mathbf{X}_{k+1}, \ldots \mathbf{X}_K$.

For simplicity we assume that the number of transmit antenna elements are the same at all transmitters $M_j = M_i = M$; the same goes for the number of receive antenna elements at the K receivers $N_j = N_i = N$, for all $i, j \in \mathcal{K}$. Moreover, we set $d_i = d_j = d > 0$, $P_i = P_j = P > 0$, and $\sigma_i^2 = \sigma_j^2 = \sigma^2 > 0$, for all $i, j \in \mathcal{K}$. We denote this K-user MIMO interference channel as an $(N \times M, d)^K$ system, in the same manner as in [9]. For all the cases considered we assume $d \leq \min(M, N)$.

For practical reasons one might consider $\mathbf{V}_k^H \mathbf{V}_k = \frac{P}{d} \mathbf{I}_d$, for all $k \in \mathcal{K}$. This might be a setting where each column of $\mathbf{V}_k$ represents a beamforming (or signature) vector assigned to d users of a group (cell) and enforces orthogonality among user signal subspaces. Accordingly, we may as well assume that the columns of each zero-forcing filter $\mathbf{U}_k$, for all receivers $k \in \mathcal{K}$, form an d-dimensional orthonormal basis, if practical interest requires such construction.

III. Interference Alignment as a Rank Constrained Rank Minimization

In this section we show that for each user $k \in \mathcal{K}$, the maximum achievable DoF can be put in the form of a RCRM problem. We pose new conditions that account for *maximum* IA instead of *perfect* IA as presented in [3]. Maximum interference alignment stands for the maximum interference suppression possible, when the signal space of each user is required to span exactly d spatial dimensions worth of communication. The perfect IA conditions being stricter require that interference signals are nulled out using linear processing of the received signal at each receiver, when the useful signal space spans d dimensions. If perfect IA requirements are met, then the signal transmitted by user $k \in \mathcal{K}$ is distinguishable by receiver k in an interference free space worth of d spatial communication dimensions. Generalizing the framework of perfect IA, we present maximum IA in the form of ranks of matrices. We show that the RCRM formulation is more general than that of perfect IA and is equivalent to the DoF maximization in an $(N \times M, d)^K$ system. Then, we use this framework to develop a new approximation algorithm and compare its tightness to existing interference leakage minimization approaches.

We begin by stating the perfect IA requirements. For all $k \in \mathcal{K}$ we require

$$\mathbf{U}_k^H \mathbf{H}_{l,k} \mathbf{V}_l = \mathbf{0}_{d \times d}, \quad \forall l \in \mathcal{K}\backslash k \qquad (2)$$

$$\text{rank}\left(\mathbf{U}_k^H \mathbf{H}_{k,k} \mathbf{V}_k\right) = d, \qquad (3)$$

where (2) enforces interference space to have zero dimensions posterior to zero-forcing and (3) enforces the useful signal to span d dimensions. Observe that (2) is a set of bilinear equations in the unknown precoding and zero forcing filters. Recently, a feasibility question has been raised as to whether a system admits perfect interference alignment or not. An intuitive claim suggests that as long as the number of unknowns

is more than or equal to the number of equations, then perfect IA should be feasible. An $(N \times M, d)^K$ system is *proper*, i.e. perfect IA should be feasible, when $N + M - d(K+1) \geq 0$ [9]. However, for cases of multiple symbol extensions for the MIMO case, or multi-cell systems as the ones presented in [10] the concept of a proper system is not established and still the maximum number of spatial DoF is an achievement one should be aiming for. We continue with rewriting (2)

$$\mathbf{U}_k^H \mathbf{H}_{l,k} \mathbf{V}_l = \mathbf{0}_{d \times d}, \quad \forall l \in \mathcal{K} \backslash k$$

$$\Leftrightarrow \left[\left\{ \mathbf{U}_k^H \mathbf{H}_{l,k} \mathbf{V}_l \right\}_{l=1, l \neq k}^K \right] = [\mathbf{0}_{d \times d} \ldots \mathbf{0}_{d \times d}]$$

$$\Leftrightarrow \mathbf{U}_k^H \left[\left\{ \mathbf{H}_{l,k} \mathbf{V}_l \right\}_{l=1, l \neq k}^K \right] = \mathbf{0}_{d \times (K-1)d},$$

defining the *signal* and *interference* matrices for all $k \in \mathcal{K}$

$$\mathbf{J}_k \left(\left\{ \mathbf{V}_l \right\}_{l=1, l \neq k}^K, \mathbf{U}_k \right) \triangleq \mathbf{U}_k^H \left[\left\{ \mathbf{H}_{l,k} \mathbf{V}_l \right\}_{l=1, l \neq k}^K \right], \quad (4)$$

$$\mathbf{S}_k \left(\mathbf{V}_k, \mathbf{U}_k \right) \triangleq \mathbf{U}_k^H \mathbf{H}_{k,k} \mathbf{V}_k, \quad (5)$$

and restating (2) and (3) for all $k \in \mathcal{K}$ as

$$\mathrm{rank} \left(\mathbf{J}_k \left(\left\{ \mathbf{V}_l \right\}_{l=1, l \neq k}^K, \mathbf{U}_k \right) \right) = 0, \quad (6)$$

$$\mathrm{rank} \left(\mathbf{S}_k \left(\mathbf{V}_k, \mathbf{U}_k \right) \right) = d. \quad (7)$$

For ease of notation we hereafter refer to the signal and interference matrices as $\mathbf{S}_k$ and $\mathbf{J}_k$, for all $k \in \mathcal{K}$. The space spanned by the columns of $\mathbf{S}_k$ is the space in which the kth receiver expects to observe the transmitted signal $\mathbf{x}_k$. Accordingly, the space spanned by the columns of $\mathbf{J}_k$ consist the interference space at receiver $k \in \mathcal{K}$. The following lemma characterizes the spatial degrees of freedom for a given user k of a $(N \times M, d)^K$ static flat-fading MIMO interference system as a function of space ranks, when given precoding and zero-forcing filters are employed.

Lemma 1: Let $\{\mathbf{V}_l\}_{l=1}^K$ be a given set of precoding filters and $\mathbf{U}_k$ be a given zero-forcing filter employed by user $k \in \mathcal{K}$. Then, the achievable spatial DoF by user k for these sets is

$$\hat{d}_k \triangleq \mathrm{rank} \left(\mathbf{S}_k \right) - \mathrm{rank} \left(\mathbf{J}_k \right). \quad (8)$$

Apparently, the aim of the entire network is to correspondingly maximize the per user DoF by choosing transmit and receive strategies in a sophisticated way. We can generalize the multiple requirements of (6) and (7) to a set of K "parallel" solutions to the following rank constrained rank minimization problems

$$\min_{\substack{\{\mathbf{V}_l\}_{l=1, l \neq k}^K \\ \mathbf{U}_k}} \mathrm{rank} \left(\mathbf{J}_k \right) \quad (9)$$

$$\text{s.t.: } \mathrm{rank} \left(\mathbf{S}_k \right) = d. \quad (10)$$

The following lemma clarifies that (9) and (10) are more general than (6) and (7).

Lemma 2: Let $\mathrm{rank} \left(\mathbf{J}_k \right) = 0$ be feasible for all $k \in \mathcal{K}$ and some set of beamforming and zero-forcing matrices. Then, $\min \mathrm{rank} \left(\mathbf{J}_k \right) = 0$. Moreover, let $\min \mathrm{rank} \left(\mathbf{J}_k \right) = \rho_k$ such that $\mathrm{rank} \left(\mathbf{J}_k \right) = 0$ not be feasible for for any set of

beamforming and zero-forcing matrices. Then, $d - \rho_k$ degrees of freedom are achievable by user k if (7) holds.

In all cases, optimal solution sets to the aforementioned "parallel" optimization problems yield precoding and zero-forcing matrices that guarantee the maximum number of interference free dimensions per user, that is the maximum DoF achievable. However, it is not trivial to solve in parallel a set of such optimization problems. Alternatively, we can maximize the sum DoF of all K users through the following RCRM.

$$\boxed{\begin{array}{c} \mathcal{P}: \\ \\ \min_{\substack{\{\mathbf{V}_l\}_{l=1}^K \\ \{\mathbf{U}_l\}_{l=1}^K}} \sum_{k=1}^K \mathrm{rank} \left(\mathbf{J}_k \right) \\ \\ \text{s.t.: } \mathrm{rank} \left(\mathbf{S}_k \right) = d, \quad \forall k \in \mathcal{K}. \end{array}}$$

Theorem 1: A solution to $\mathcal{P}$ maximizes the sum of spatial degrees of freedom for an $(N \times M, d)^K$ static flat-fading MIMO interference channel.

Proof: For all selections of $\mathbf{V}_1, \ldots, \mathbf{V}_K \in \mathbb{C}^{M \times d}$ and $\mathbf{U}_1, \ldots, \mathbf{U}_K \in \mathbb{C}^{N \times d}$ satisfying the constraints of $\mathcal{P}$ we have that

$$\sum_{k=1}^K \hat{d}_k = \sum_{k=1}^K \left(\mathrm{rank} \left(\mathbf{S}_k \right) - \mathrm{rank} \left(\mathbf{J}_k \right) \right) = Kd - \sum_{k=1}^K \mathrm{rank} \left(\mathbf{J}_k \right)$$

$$\Leftrightarrow \arg\max_{\substack{\{\mathbf{V}_l\}_{l=1}^K \\ \{\mathbf{U}_l\}_{l=1}^K}} \sum_{k=1}^K \hat{d}_k = \arg\max_{\substack{\{\mathbf{V}_l\}_{l=1}^K \\ \{\mathbf{U}_l\}_{l=1}^K}} \left\{ Kd - \sum_{k=1}^K \mathrm{rank} \left(\mathbf{J}_k \right) \right\}$$

$$\Leftrightarrow \arg\max_{\substack{\{\mathbf{V}_l\}_{l=1}^K \\ \{\mathbf{U}_l\}_{l=1}^K}} \sum_{k=1}^K \hat{d}_k = \arg\min_{\substack{\{\mathbf{V}_l\}_{l=1}^K \\ \{\mathbf{U}_l\}_{l=1}^K}} \sum_{k=1}^K \mathrm{rank} \left(\mathbf{J}_k \right).$$

The proof is complete. $\square$

The orthogonality constraints for the precoding and zero-forcing matrices where not stated as optimization constraints since one can always linearly transform the columns into their associated orthonormal column basis consisting of d column vectors. Since $d \leq \min(M, N)$ we can rewrite $\mathbf{V}_k = \mathbf{Q}_k^{(v)} \mathbf{R}_k^{(v)}$, where $\mathbf{Q}_k^{(v)} \in \mathbb{C}^{M \times d}$ is an orthonormal basis for the column space of $\mathbf{V}_k$ and $\mathbf{R}_k^{(v)} \in \mathbb{C}^{d \times d}$ is the matrix of coefficients participating in the linear combinations yielding the columns of $\mathbf{V}_k$. Then we may use $\sqrt{\frac{P}{d}} \mathbf{Q}_k^{(v)}$ as the precoding matrix. Accordingly, we use orthonormal matrices $\mathbf{Q}_k^{(u)}$ constructed by decomposing $\mathbf{U}_k$ to $\mathbf{Q}_k^{(u)} \mathbf{R}_k^{(u)}$, where $\mathbf{Q}_k^{(u)} \in \mathbb{C}^{N \times d}$ and $\mathbf{Q}_k^{(u)} \in \mathbb{C}^{d \times d}$. Observe that $\mathrm{span} \left(\mathbf{Q}_k^{(v)} \right) = \mathrm{span} \left(\mathbf{V}_k \right)$ and $\mathrm{span} \left(\mathbf{Q}_k^{(u)} \right) = \mathrm{span} \left(\mathbf{U}_k \right)$ for all $k \in \mathcal{K}$. Moreover, the ranks of the interference and signal matrices remain the same under column transformations

$$\mathrm{rank} \left(\mathbf{J}_k \right)$$

$$= \mathrm{rank} \left(\left(\mathbf{R}_k^{(u)} \right)^H \left(\mathbf{Q}_k^{(u)} \right)^H \left[\left\{ \mathbf{H}_{l,k} \mathbf{V}_l \right\}_{l=1, l \neq k}^K \right] \right)$$

$$= \mathrm{rank} \left(\left(\mathbf{Q}_k^{(u)} \right)^H \left[\left\{ \mathbf{H}_{l,k} \mathbf{Q}_l^{(v)} \right\}_{l=1, l \neq k}^K \right] \mathrm{blkdiag} \left(\left\{ \mathbf{R}_l^{(v)} \right\}_{l=1, l \neq k}^K \right) \right)$$

$$= \mathrm{rank} \left(\left(\mathbf{Q}_k^{(u)} \right)^H \left[\left\{ \mathbf{H}_{l,k} \mathbf{Q}_l^{(v)} \right\}_{l=1, l \neq k}^K \right] \right), \quad (11)$$

where $\text{blkdiag}\left(\mathbf{A}_1, \ldots, \mathbf{A}_n\right)$ denotes the block diagonal matrix that has as ith diagonal block the matrix $\mathbf{A}_i$, the third and last equality hold due to the fact that $\text{rank}\left(\mathbf{R}_l^{(u)}\right) = \text{rank}\left(\mathbf{R}_l^{(v)}\right) = d$. Furthermore, we have

$$
\begin{aligned}
\text{rank}\left(\mathbf{S}_k\right) &= \text{rank}\left(\left(\mathbf{R}_k^{(u)}\right)^H \left(\mathbf{Q}_k^{(u)}\right)^H \mathbf{H}_{k,k} \mathbf{Q}_k^{(v)} \mathbf{R}_k^{(v)}\right) \\
&= \text{rank}\left(\left(\mathbf{Q}_k^{(u)}\right)^H \mathbf{H}_{k,k} \mathbf{Q}_k^{(v)}\right) = d.
\end{aligned}
$$

Therefore, orthogonalization is always possible when $d \leq \min(M, N)$.

To conclude, we have established that the minimization of sum of ranks of the interference matrices under full rank signal space constraints is equivalent to maximizing the sum DoF of a static flat-fading MIMO channel. There exist various instances and regimes of difficulty for this problem. There exist tractable regimes where one randomly selects the precoding or zero-forcing matrices matrices and constructs the zero-forcing or precoding matrices with columns that are in the nullspace of the interference or reciprocal interference matrices. This is possible when either $d \leq N - (K-1)d$, or $d \leq M - (K-1)d$ holds. Moreover, when the channel matrices are diagonal, the symbol extension method presented in [1] creates instances of the RCRM problem that can be efficiently solved and determine the maximum achievable sum DoF. However, for many instances and regimes such solutions are not established and the RCRM problem cannot be efficiently solved. In the next section we provide a heuristic that approximates $\mathcal{P}$.

IV. A NUCLEAR NORM HEURISTIC

In the previous section we establish that maximizing the sum DoF of a static MIMO interference network is equivalent to solving a RCRM, where the precoding and zero-forcing matrices are the optimization variables. To approach such a highly nonconvex problem we use convex approximations for the cost function and constraints of $\mathcal{P}$. We begin by obtaining the tightest convex approximation to the cost function of $\mathcal{P}$. We have

$$
\begin{aligned}
\overline{\text{conv}}\left(\sum_{k=1}^{K} \text{rank}\left(\mathbf{J}_k\right)\right) &= \overline{\text{conv}}\left(\text{rank}\left(\text{blkdiag}\left(\mathbf{J}_1, \ldots, \mathbf{J}_K\right)\right)\right) \\
&= \frac{1}{\mu} \left\|\text{blkdiag}\left(\mathbf{J}_1, \ldots, \mathbf{J}_K\right)\right\|_* \qquad (12) \\
&= \frac{1}{\mu} \sum_{k=1}^{K} \left\|\mathbf{J}_k\right\|_* = \frac{1}{\mu} \sum_{k=1}^{K} \sum_{i=1}^{d} \sigma_i\left(\mathbf{J}_k\right),
\end{aligned}
$$

where $\overline{\text{conv}}(f)$ denotes the convex envelope of a function f, $\|\mathbf{A}\|_* = \sum_{i=1}^{\text{rank}(\mathbf{A})} \sigma_i(\mathbf{A})$ is the nuclear norm of a matrix $\mathbf{A}$ which accounts for the sum of the singular values of $\mathbf{A}$, and $\sigma_i(\mathbf{A})$ is the ith largest singular value of $\mathbf{A}$. This nuclear norm is the convex envelope of the rank function when we can bound the maximum singular value of the interference matrices by $\mu > 0$ [14]. Therefore, if we assume that this is possible we are operating on the following sets of interference matrices $\left\{\mathbf{J}_k \in \mathbb{C}^{d \times (K-1)d}, \forall k \in \mathcal{K}; \max_{k \in \mathcal{K}} \sigma_1\left(\mathbf{J}_k\right) \leq \mu\right\}$.

Before we proceed with approximating nonconvex rank constraints by a convex feasible set, we provide some insights

on the algorithms presented in [8] and [12]. These algorithms aim to minimize the total interference leakage at each receiver, a metric defined as

$$
\sum_{k=1}^{K} \text{tr}\left\{\mathbf{U}_k^H \mathbf{Q}_k \mathbf{U}_k\right\}, \qquad (13)
$$

where

$$
\mathbf{Q}_k = \sum_{l=1, l \neq k}^{K} \frac{P}{d} \mathbf{H}_{l,k} \mathbf{V}_l \mathbf{V}_l^H \mathbf{H}_{l,k}^H \qquad (14)
$$

If we plug (14) in (13) we get

$$
\begin{aligned}
&\sum_{k=1}^{K} \text{tr}\left\{\mathbf{U}_k^H \left(\sum_{l=1, l \neq k}^{K} \frac{P}{d} \mathbf{H}_{l,k} \mathbf{V}_l \mathbf{V}_l^H \mathbf{H}_{l,k}^H\right) \mathbf{U}_k\right\} \\
&= \frac{P}{d} \sum_{k=1}^{K} \text{tr}\left\{\mathbf{U}_k^H \left[\{\mathbf{H}_{l,k} \mathbf{V}_l\}_{l=1, l \neq k}^{K}\right] \left[\{\mathbf{H}_{l,k} \mathbf{V}_l\}_{l=1, l \neq k}^{K}\right]^H \mathbf{U}_k\right\} \\
&= \frac{P}{d} \sum_{k=1}^{K} \text{tr}\left\{\mathbf{J}_k \mathbf{J}_k^H\right\} = \frac{P}{d} \sum_{k=1}^{K} \left\|\mathbf{J}_k\right\|_F^2 = \frac{P}{d} \sum_{k=1}^{K} \sum_{i=1}^{d} \sigma_i^2\left(\mathbf{J}_k\right),
\end{aligned}
$$

where $\|\mathbf{A}\|_F$ is the Frobenius norm of $\mathbf{A}$ and the constant $\frac{P}{d}$ can be dropped in the minimization. The constraints for the minimization of such metric are in the form of orthogonality of the columns of the precoding and zero-forcing matrices. We note that this nonconvex problem, yields matrices for perfect IA, when such is possible; when not, it results in the "smallest energy" interference spaces possible. For such solutions interference may be weak, nonetheless, it is not confined to span limited dimensions. Potentially, it spans more dimensions than the sparsest possible solution.

We continue our approximation by obtaining constraints with convex feasible set. We approximate the constraint, $\text{rank}\left(\mathbf{S}_k\right) = d, \ \forall k \in \mathcal{K}$ with

$$
\lambda_{\min}\left(\mathbf{S}_k\right) > \epsilon, \qquad (15)
$$

$$
\mathbf{S}_k \succeq \mathbf{0}_{d \times d}, \qquad (16)
$$

where $\lambda_{\min}\left(\mathbf{S}_k\right)$ is the minimum eigenvalue of $\mathbf{S}_k$, $\epsilon \geq 0$, and $\mathbf{S}_k \succeq \mathbf{0}_{d \times d}$ denotes that matrix $\mathbf{S}_k$ is hermitian positive semidefinite; that is for all $k \in \mathcal{K}$ we have $\left\{\mathbf{S}_k \in \mathbb{C}^{d \times d} \mid \mathbf{S}_k = \mathbf{S}_k^H \text{ and } \mathbf{z}^H \mathbf{S}_k \mathbf{z} \geq 0, \forall \mathbf{z} \in \mathbb{C}^d\right\}$. We note that the minimum eigenvalue constraint serves as a tractable constraint yielding a convex feasible solution set. This approximation might seem stricter than the rank constraints, however, for the initial cost function of the sum of ranks we show that it is not for $\epsilon = 0$.

Lemma 3: Let $\{\mathbf{V}_l\}_{l=1}^{K}$ and $\{\mathbf{U}_l\}_{l=1}^{K}$ be any feasible pair of sets of precoding and zero-forcing matrices such that $\text{rank}(\mathbf{J}_k) = \rho_k$ and $\text{rank}(\mathbf{S}_k) = d \Leftrightarrow \lambda_{\min}(\mathbf{S}_k) \neq 0$ for any $k \in \mathcal{K}$. Then, for any such feasible point of $\mathcal{P}$ there exists a feasible pair of sets for (15) and (16), that is

$$
\{\hat{\mathbf{V}}_l\}_{l=1}^{K} = \{\mathbf{V}_l \mathbf{V}_l^H \mathbf{H}_{l,l}^H \mathbf{U}_l\}_{l=1}^{K} \qquad (17)
$$

$$
\text{and } \{\hat{\mathbf{U}}_l\}_{l=1}^{K} = \{\mathbf{V}_l^H \mathbf{H}_{l,l}^H \mathbf{U}_l^H \mathbf{U}_l\}_{l=1}^{K}, \qquad (18)
$$

not affecting the cost function when $\epsilon = 0$.

Proof: Observe that such selections yield positive definite signal matrices for all $k \in \mathcal{K}$

$$\mathbf{U}_{k,k}^{H}\mathbf{H}_{k,k}\hat{\mathbf{V}}_{k} = \mathbf{U}_{k}^{H}\mathbf{H}_{k,k}\mathbf{V}_{k}\mathbf{V}_{k}^{H}\mathbf{H}_{k,k}^{H}\mathbf{U}_{k} \succ 0,$$
$$\hat{\mathbf{U}}_{k,k}^{H}\mathbf{H}_{k,k}\mathbf{V}_{k} = \mathbf{V}_{k}^{H}\mathbf{H}_{k,k}^{H}\mathbf{U}_{k}^{H}\mathbf{U}_{k}\mathbf{H}_{k,k}\mathbf{V}_{k} \succ 0.$$

Moreover, we have

$$\begin{aligned}
\text{rank}\left(\mathbf{U}_{k}^{H}\mathbf{H}_{k,k}\mathbf{V}_{k}\right) &= \text{rank}\left(\mathbf{U}_{k}^{H}\mathbf{H}_{k,k}\mathbf{V}_{k}\mathbf{V}_{k}^{H}\mathbf{H}_{k,k}^{H}\mathbf{U}_{k}^{H}\right) \\
&= \text{rank}\left(\mathbf{V}_{k}\mathbf{H}_{k,k}^{H}\mathbf{U}_{k}^{H}\mathbf{U}_{k}^{H}\mathbf{H}_{k,k}\mathbf{V}_{k}\right) = d, \\
\text{span}\left(\mathbf{V}_{k}\right) &= \text{span}\left(\mathbf{V}_{k}\mathbf{V}_{k}^{H}\mathbf{H}_{k,k}^{H}\mathbf{U}_{k}^{H}\right), \\
\text{and span}\left(\mathbf{U}_{k}\right) &= \text{span}\left(\mathbf{U}_{k}\mathbf{U}_{k}^{H}\mathbf{H}_{k,k}^{H}\mathbf{V}_{k}\right).
\end{aligned}$$

This is due to $\text{rank}\left(\mathbf{V}_{k}^{H}\mathbf{H}_{k,k}^{H}\mathbf{U}_{k}\right) = \text{rank}\left(\mathbf{V}_{k}\right) = \text{rank}\left(\mathbf{U}_{k}^{H}\mathbf{H}_{k,k}^{H}\mathbf{V}_{k}^{H}\right) = \text{rank}\left(\mathbf{U}_{k}\right) = d$ for all $k \in \mathcal{K}$. Hence, any linear full rank column transformation of $\mathbf{V}_{k}$ and $\mathbf{U}_{k}$ is span and rank preserving, i.e. it does not affect the cost function of $\mathcal{P}$. □

Having provided approximations for both the cost function and constraints of $\mathcal{P}$ we are stating the steps of our algorithm. We first decide to arbitrarily select either the precoding or the zero-forcing matrices depending on whether $M \geq N$ or not. The motivation for this step is that we wish to have as many free optimization variables as possible. Next, we solve the following convex optimization assuming that $M \geq N$

$$\boxed{\begin{aligned}
&\mathcal{A}: \\
&\min_{\{\mathbf{V}_{l}\}_{l=1}^{K}} \sum_{k=1}^{K} \|\mathbf{J}_{k}\|_{*} \\
&\text{s.t.: } \lambda_{\min}\left(\mathbf{S}_{k}\right) > \epsilon, \\
&\qquad \mathbf{S}_{k} \succeq \mathbf{0}_{d\times d}, \; \forall k \in \mathcal{K}.
\end{aligned}}$$

The arguments maximizing $\mathcal{A}$ are then orthogonalized using a QR factorization as proposed in Section III. Alternatively, If $M < N$ holds, then we arbitrarily select the precoding matrices, set as optimization variables the zero-forcing matrices, and proceed accordingly. In the simulations section we provide quantitative results for the performance of our proposed approximation.

V. MAXIMUM INTERFERENCE ALIGNMENT FOR CELLULAR NETWORKS

In this section we briefly discuss the case of K cell interference channel as presented in [10], where each cell supports d users. We tailor our algorithm for this problem by adding extra affine constraints on the entries of the precoding matrices. Such constraints correspond to the fact that each user u in a cell k wishes to transmit only one symbol using a beamforming vector $\mathbf{v}_{k,u} \in \mathbb{C}^{\frac{M}{d}\times 1}$, where we assume $\frac{M}{d} \in \mathbb{N}^{*}$. The analogous to a general K-user MIMO interference channel would be transmitting symbol $x_{k,u}$ of the symbol vector $\mathbf{x}_{k} = [x_{k,1}\ldots x_{k,d}]^{T}$ only from say the uth $\frac{M}{d}$ transmit

antennas, assuming spatial ordering. Therefore, all precoding (or beamforming) matrices have the following structure

$$\mathbf{V}_{k} = \begin{bmatrix} \mathbf{v}_{k,1} & \cdots & \mathbf{0}_{\frac{M}{d}\times 1} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{\frac{M}{d}\times 1} & \cdots & \mathbf{v}_{k,d} \end{bmatrix}, \qquad (19)$$

where $\mathbf{v}_{k,u}$ represents the beamforming vector used by user u of cell k, for all $k \in \mathcal{K}$, such that the received signal at receiver k is given by

$$\mathbf{y}_{k} = \sum_{u=1}^{d} \mathbf{H}_{k,k}^{(u)}\mathbf{v}_{k,u}x_{k,u} + \sum_{l=1,l\neq k}^{K} \sum_{u=1}^{d} \mathbf{H}_{l,k}^{(u)}\mathbf{v}_{l,u}x_{l,u} + \mathbf{w}_{k},$$

where $\mathbf{H}_{l,k} = \left[\mathbf{H}_{l,k}^{(1)}\ldots\mathbf{H}_{l,k}^{(d)}\right]$, for $l,k \in \mathcal{K}$. Ideally, we have to solve $\mathcal{P}$ with the extra affine constraints corresponding to zeros at entries of $\mathbf{V}_{k}$ accounting to antenna elements not transmitting symbols of other than their associated users. Such special structure may not be trivially tackled by the algorithms minimizing the interference leakage since they only guarantee orthogonality of the precoding and zero-forcing sets of matrices. We propose the following approximation posterior to arbitrarily selecting the zero-forcing matrices: assuming $M \geq N$, we solve $\mathcal{A}$ with the additional constraints $[\mathbf{V}_{k}]_{\mathcal{I}_{u},u} = \mathbf{0}_{K(d-1)\times 1} \; \forall u \in \{1,2,\ldots d\}$ where $\mathcal{I}_{u} = \left\{1,\ldots,(u-1)\frac{M}{d}\right\} \cup \left\{u\frac{M}{d}+1,\ldots,M\right\}$ represents the sets of indices (rows) that the precoding matrices have zero entries, for all $u \in \{1,2,\ldots,d\}$. Observe that no further approximations (than the ones in $\mathcal{A}$) are made to $\mathcal{P}$ due to affine constraints being tractable.

VI. SIMULATIONS AND CONCLUSIONS

In this preliminary experimental evaluation we test our algorithm for a $(5 \times 11, d)^{3}$ MIMO interference system where $d \leq 4$, such that all systems are proper. We allocate $\frac{P}{d}$ to each column of the precoding matrices, where $10\log_{10}P = [0:10:50]$ dB, and set the noise power to $\sigma^{2} = 1$. We run simulations for 1000 channel realizations where each channel element is drawn i.i.d. from a real Gaussian distribution with mean zero and variance 1. The sum rate we plot is computed as $R = \sum_{i=1}^{K} \frac{1}{2}\log\det\left(\mathbf{I}_{d} + \left(\mathbf{I}_{d} + \mathbf{J}_{k}\mathbf{J}_{k}^{H}\right)^{-1}\mathbf{S}_{k}\mathbf{S}_{k}^{H}\right)$. For each simulation, we run 100 iterations of the minimum interference leakage algorithm. To run $\mathcal{A}$ we set $\epsilon = 10^{-2}$ and use the CVX toolbox [16].

In Fig. 1 and Fig. 2, we plot the sum rate and interference leakage of our scheme versus the interference leakage minimization algorithm of [12], respectively, for $d = 2,3,4$. In Fig. 2 we observe that, as expected, the proposed algorithm performs worse in terms of interference leakage. However in terms of sum rate and average DoF per user (shown in Table I), the proposed algorithm outperforms leakage minimization. Observe that when both algorithms achieve perfect IA (for $d = 2,3$) the proposed scheme results in slightly higher sum rate compared to leakage minimization. The benefits become more substantial however when perfect IA is feasible, but not achieved by any of the two schemes. In this case the
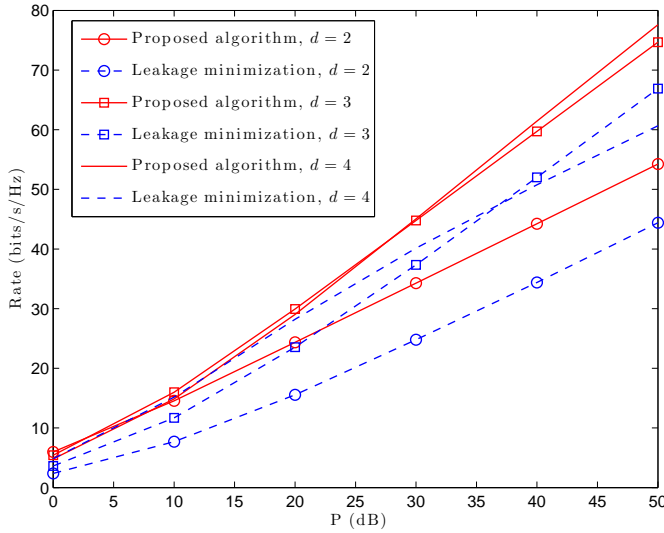
Fig. 1. Rate vs. P, for a $(5 \times 11, d = 2, 3, 4)^3$ system.

d	proposed algorithm	leakage minimization
1	1	1
2	2	2
3	3	3
4	3.213	2

TABLE I
AVERAGE PER USER DoF

Fig. 2. Interference leakage vs. P for a $(5 \times 11, d = 4)^3$ system.

sparsity of the proposed relaxation creates more degrees of freedom as shown in Table I. The fractional numbers are due to averaged results over all channels and SNR values, i.e. 3.213 corresponds to cases where some users achieve 4 DoF and some less, such that the sum DoF is between $3 \cdot 3$ and $3 \cdot 4$. Observe that although perfect IA is feasible for the case of $d = 4$, the leakage minimization algorithm for 100 iterations cannot achieve it. Apparently, when this happens, 3 beams should be used instead of 4.

To conclude, we would like to note that while the proposed algorithm is inspired by the nuclear norm relaxation of rank introduced in [14], in this paper we did not establish theoretical guarantees under which this relaxation is tight. Such a theoretical investigation would be a very interesting open problem for future research. Moreover, we need to extend the experiments to more channel setups, such as the time extended version of the MIMO case and the multi-cell interference channel. Furthermore, several more approaches need to be studied for comparison such as the maximum signal-to-interference-and-noise ratio (SINR) algorithms of [8] and [13]. Finally, we note that it would be interesting to study the performance of an iterative version of our proposed scheme.

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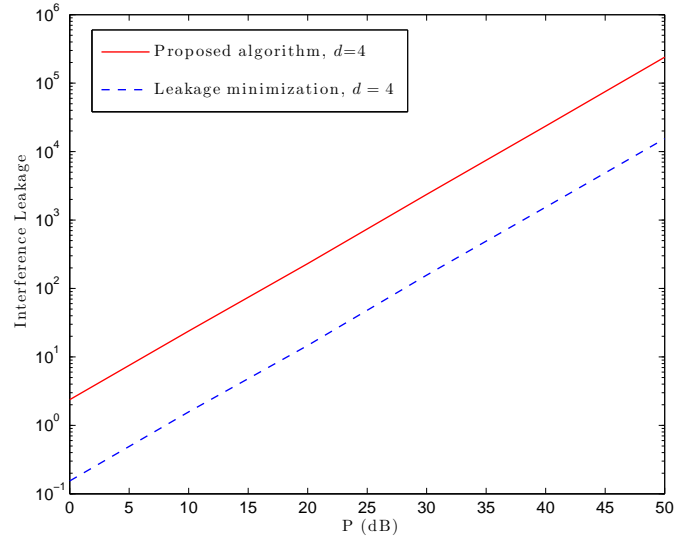
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