

Preference Restrictions in Computational Social Choice: A Survey

Edith Elkind
Northwestern University

Martin Lackner
WU Wien

Dominik Peters
CNRS

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Social choice becomes easier on restricted preference domains such as single-peaked, single-crossing, and Euclidean preferences. Many impossibility theorems disappear, the structure makes it easier to reason about preferences, and computational problems can be solved more efficiently. In this survey, we give a thorough overview of many classic and modern restricted preference domains and explore their properties and applications. We do this from the viewpoint of computational social choice, letting computational problems drive our interest, but we include a comprehensive discussion of the economics and social choice literatures as well. Particular focus areas of our survey include algorithms for recognizing whether preferences belong to a particular preference domain, and algorithms for winner determination of voting rules that are hard to compute if preferences are unrestricted.

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1 Introduction

Social choice studies aggregation of preferences with the aim of making group decisions. An important part of social choice is voting theory, which designs voting rules that identify high-quality societal decisions, and formulates desirable properties of such rules. Computational social choice builds on this theory by studying computational tasks that arise in voting contexts. Among these tasks, the most important one is winner determination, i.e., computing the output of a voting rule given voters' preferences. While for many well-known voting rules winner determination is a straightforward matter of counting ballots and adding numbers, for other rules identifying the winner(s) is computationally challenging, especially when the decision space is combinatorial in nature. Other computational tasks concern elicitation, processing and analysis of preference data.

Famously, voting theory is riddled with impossibility theorems. Arrow's (1950) impossibility theorem concerning rank aggregation is the best-known, but for the purposes of voting, a theorem of Gibbard (1973) and Satterthwaite (1975) is perhaps more problematic. This theorem shows that when there are three or more alternatives, every non-trivial voting rule can be manipulated by strategic voters who misrepresent their preferences. The underlying problem was identified much earlier, by Condorcet (1785). He observed that even when individual voters are fully rational (in the sense of having transitive preferences), their collective judgment may be irrational and contain cyclic (paradoxical) preferences.

Example 1: *Condorcet cycles: Pairwise majority judgments can be cyclic.*

v_1	v_2	v_3	
a	b	c	Three voters rank the alternatives a , b , and c from the most-preferred to the least-preferred. A majority of voters prefers a over b , a majority prefers b over c , and a majority prefers c over a . Thus, the majority judgment contains a cycle.
b	c	a	
c	a	b	

Condorcet cycles are the primary source of paradoxes in voting: in their absence, there is a clear winning alternative and the majority judgment is well-behaved. Some empirical results suggest that Condorcet cycles are rare in real elections (Niemi, 1970; Gehrlein, 1983; Feld and Grofman, 1992; Radcliff, 1994; Van Deemen and Vergunst, 1998). Thus, it is natural to look for contexts in which we can be assured that paradoxical majorities are avoided and in which the impossibility theorems do not apply.

The first, and most famous, such context was identified by Black (1948) (and independently discovered by Arrow, 1951). Black considers the case where the alternatives to be ranked lie on a one-dimensional axis; this occurs, for example, when we vote over the value of some numerical quantity. In those cases, it is natural to assume that votes will be *single-peaked*, i.e., voters prefer values that are close to their favorite value. When preferences follow this pattern, it can be shown that Condorcet cycles are impossible. In addition, there are voting rules that cannot be manipulated.

Following Black (1948), social choice theorists have identified other domains of preferences that lead to similar positive results. Among others, such domains include generalizations of single-peakedness to allow for tree-shaped alternative spaces, the single-crossing property which assumes that voters (rather than alternatives) form a one-dimensional spectrum, and value restricted domains, obtained by simply forbidding the Condorcet cycle from Example 1 to occur as a subprofile.

Starting with the seminal work of [Bartholdi III et al. \(1989\)](#), computer scientists began to realize that important voting-related computational problems may be algorithmically challenging, in the sense of being NP-complete or even harder. For example, it is difficult to determine the winner in elections that use the voting rules proposed by [Dodgson \(1876\)](#) or [Kemeny \(1959\)](#). In computational complexity theory, when faced with a formally intractable problem, a common approach is to look for “islands of tractability”. These are restricted classes of inputs for which polynomial-time algorithms exist. In the context of voting, a natural place to look for such islands is to start with classes that had already been suggested by social choice theorists in their quest to circumvent impossibility theorems. This research program was initiated by [Walsh \(2007\)](#) and turned out to be fruitful: in many cases restrictions like single-peakedness do make computation easier.

Interestingly, while purely social choice-theoretic issues (such as manipulability or majority cycles) vanish as soon as we assume that voters’ preferences belong to a suitable restricted domain, many of the algorithms for voting-related problems require the knowledge of the respective structural relationship among voters and alternatives (such as the order of candidates witnessing that the profile is single-peaked). This means that, to make use of these algorithms, one also needs an efficient procedure to discover whether a given preference profile has the required structural property and to find a respective witness. Consequently, the problem of designing such procedures has received a considerable amount of attention, too, resulting in polynomial-time algorithms for recognizing preferences that belong to several prominent restricted domains. These recognition algorithms are also useful for gaining a deeper understanding of a preference data set, by identifying unexpected structure underlying the preferences. This can make the data set more interpretable.

Contents of This Survey

In this survey, we provide an overview of research on preference restrictions. While we focus on contributions of computational social choice, we briefly discuss the most important results from ‘pure’ social choice theory as well. For many results, we have included proofs or detailed proof sketches, to provide a convenient reference. Throughout, we state open questions that provide important avenues for future research.

- [Section 3](#) begins the survey by defining major preference domains that have been studied. We point out useful equivalent definitions and logical relationships. We also discuss implications on the structure of the majority relation, and the existence of strategyproof voting rules.
- [Section 4](#) studies recognition algorithms that decide, for a given preference profile, whether it is a member of one of the preference domains defined in [Section 3](#). This topic is a major focus of this survey, and in many cases we give full details of the best-known algorithms for these tasks. In several cases, the analysis of these algorithms further illuminates properties of the different domains; in particular we can sometimes characterize all possible witnesses for membership of a given profile in a given domain.
- [Section 5](#) gives an overview of several papers that have found characterizations of preference domains in terms of forbidden subprofiles.
- [Section 6](#) investigates the winner determination problem under the assumption that voters’ preferences are drawn from a restricted domain. This section starts by considering several famous single-winner voting rules that are known to be NP-hard to evaluate for unrestricted

preferences, and then moves on to computationally challenging multi-winner rules. Multi-winner rules elect a committee of candidates, and this class of rules has been intensely studied in recent years.

- [Section 7](#) focuses on the computational complexity of various problems concerning manipulation and control. This literature has found that some types of manipulative attacks on elections are computationally difficult to pull off (in the worst case), which might provide some protection against these attacks. However, subsequent work showed that many of those hardness results do not hold when preferences are structured.
- In [Section 8](#) we briefly discuss other contexts in which restricted preference domains have been considered, including weak orders (and specifically dichotomous preferences) and preference elicitation. We also mention results on counting structured profiles and computing the probability of random profiles being structured.
- [Section 9](#) concludes the survey by listing several directions for future research.

Technical Contributions

In writing this survey, we have included some new results that are not published elsewhere. We feel that this survey is an appropriate place to describe these results.

- In [Section 4.1](#), we present a linear-time algorithm for recognizing whether a preference profile is single-peaked. While this algorithm is directly based on the algorithms proposed by [Doignon and Falmagne \(1994\)](#) and [Escoffier et al. \(2008\)](#), we believe that our version is the simplest to implement, and our analysis of this algorithm is the clearest one available.
- In [Section 4.2](#), we describe an algorithm that recognizes whether a preference profile is single-crossing. This algorithm runs in time $O(mn \log m)$, and is faster than all previously known algorithms.
- In [Section 3.9](#), we discuss multidimensional extensions of the single-peaked domain. We identify a new variant of an existing definition, and prove [Theorem 3.21](#), which shows that all profiles with at most $2^{2^{d-1}}$ alternatives are d -dimensional single-peaked.

Feedback Welcome!

This is a draft version. In assembling this survey, we will doubtlessly have omitted relevant work, and introduced errors. We would appreciate any feedback and suggestions you might have for us. Please email to mail@dominik-peters.de. We hope you enjoy reading the survey!

2 Preliminaries and Notation

We write $[n]$ to denote the set $\{1, \dots, n\}$.

Relations and Profiles Let A be a finite set of m *alternatives* (or *candidates*). Our survey will deal with ways to analyze *preferences* over this set. Formally, we will consider binary relations over A , i.e., subsets of $A \times A$. Given a binary relation R and alternatives $x \in A$ and $y \in A$, we interpret $(x, y) \in R$ to mean that x is favored over y by R . We will often use symbols such as \succsim and \succ to denote binary relations, and write $a \succsim b$ instead of $(a, b) \in \succsim$ in this case.

A *weak order* is a binary relation \succsim over A that is reflexive, complete and transitive. A weak order can be used to describe a preference relation: we have $x \succsim y$ if x is weakly preferred to y . It could be that both $x \succsim y$ and $y \succsim x$, in which case \succsim is indifferent between x and y . A weak order with no indifferences is a *linear order*. In other words, a linear order is a weak order that, in addition, is antisymmetric, so that $x \succsim y$ and $y \succsim x$ imply $x = y$. A linear order is a *ranking* of the alternatives, and thus there are $|A|!$ many different linear orders on A . In what follows, we will often use the terms ‘preferences’, ‘(preference) ordering’, ‘(preference) order’ and ‘(preference) ranking’ interchangeably.

If \succsim is a weak order, then we use \succ to refer to its *strict part*, that is, $x \succ y$ if and only if $x \succsim y$ but $y \not\succsim x$. When \succsim is a linear order, then \succ is its irreflexive part. Whenever we use a strict relation symbol such as \succ , $>$, or $<$, we are referring to the strict part of an order, so that these relations are always irreflexive.

Given a weak order \succsim over A and two disjoint sets $A_1, A_2 \subseteq A$, we write $A_1 \succsim A_2$ if for all alternatives $a \in A_1$ and $b \in A_2$ it holds that $a \succsim b$, i.e., every element of A_1 is weakly preferred to every element of A_2 . If \succsim is a binary relation, then the relation \succsim' is the *reverse* of \succsim if $a \succsim b \Leftrightarrow b \succsim' a$.

Typically, we will be interested in *collections* of orders, and notions of what it means for such a collection to be structured. Such collections are referred to as *preference profiles*, or simply *profiles*.

Definition 1. A *profile* $P = (v_1, \dots, v_n)$ over A is a list of linear orders over A . The elements of $N = \{1, \dots, n\}$ are called *voters*, and we associate voter $i \in N$ with the order v_i , which we call the *vote* of voter i . For convenience, we write $a \succ_i b$ whenever $(a, b) \in v_i$, i.e., when voter i strictly prefers alternative a to alternative b .

We will always write m for the number of alternatives and n for the number of voters. We denote the *rank* of alternative a in the vote of voter i by $\text{rank}_i(a)$; we have $\text{rank}_i(a) = |\{b : b \succ_i a\}| + 1$. Further, we write $\text{top}(v_i)$ for the alternative that is ranked first in the linear order v_i , i.e., $\text{rank}_i(\text{top}(v_i)) = 1$.

Majority Relation and Condorcet Winners When there are only two alternatives x and y , then following May’s seminal analysis (May, 1952), we should compare them by checking whether there is a majority of voters who prefer x to y , in which case x beats y in a majority comparison. More generally, we can analyze a given profile in a pairwise fashion, considering pairs $\{x, y\} \subseteq A$ of alternatives separately. This approach gives rise to the idea of the majority relation. Formally, given a profile P over A , we define its *majority relation* \succ_{maj} to be a relation over A such that

$$a \succ_{\text{maj}} b \iff |i \in N : a \succ_i b| \geq |i \in N : b \succ_i a|.$$

We write $a \succ_{\text{maj}} b$ if $a \succcurlyeq_{\text{maj}} b$ and not $b \succcurlyeq_{\text{maj}} a$. Thus, $a \succ_{\text{maj}} b$ if and only if a strict majority of voters has $a \succ b$ in the profile P . Alternative a is a *weak Condorcet winner* if $a \succcurlyeq_{\text{maj}} b$ for all $b \in A \setminus \{a\}$ and a *strong Condorcet winner* if $a \succ_{\text{maj}} b$ for all $b \in A \setminus \{a\}$. As is well known, the relation \succ_{maj} need not be transitive, a feature that leads to a lot of trouble. If the majority relation is not transitive, we say that $\succcurlyeq_{\text{maj}}$ contains a *Condorcet cycle*, as in the following example.

Example 2: *Condorcet winners and Condorcet cycles.*

v_1	v_2	v_3	v_4	v_5	v_6	
a	a	e	b	d	b	In profile $P = (v_1, \dots, v_6)$ there are two weak Condorcet winners, namely, alternatives a and b , as it holds that $\{a, b\} \succ_{\text{maj}} \{c, d, e\}$, $a \succcurlyeq_{\text{maj}} b$, and $b \succcurlyeq_{\text{maj}} a$. The majority relation is, however, not transitive: it holds that $c \succ_{\text{maj}} d \succ_{\text{maj}} e \succ_{\text{maj}} c$, a Condorcet cycle. Also, as there are two weak Condorcet winners, it follows that there is no strong Condorcet winner.
c	b	c	a	e	e	
d	d	d	c	c	c	
e	e	a	d	b	d	
b	c	b	e	a	a	

Alternative and Voter Deletion Suppose that $A' \subseteq A$ and let v be a linear order. Then v can be restricted to A' in a natural way: define $v|_{A'} := v \cap (A' \times A')$, i.e., intuitively, we remove all candidates in $A \setminus A'$. Similarly, we can restrict a profile P by restricting each vote in it: $P|_{A'} := (v_1|_{A'}, \dots, v_n|_{A'})$. We say that $P|_{A'}$ is obtained from P by *alternative deletion*. Similarly, if $N' \subseteq N$, the profile $P' = (v_i : i \in N')$ is said to be obtained from P by *voter deletion*.

Algorithms and Computational Complexity We assume that the reader is familiar with basic concepts of algorithm analysis (e.g., big- \mathcal{O} notation and runtime analysis) and computational complexity (e.g., NP-hardness and reductions).

3 Domain Restrictions

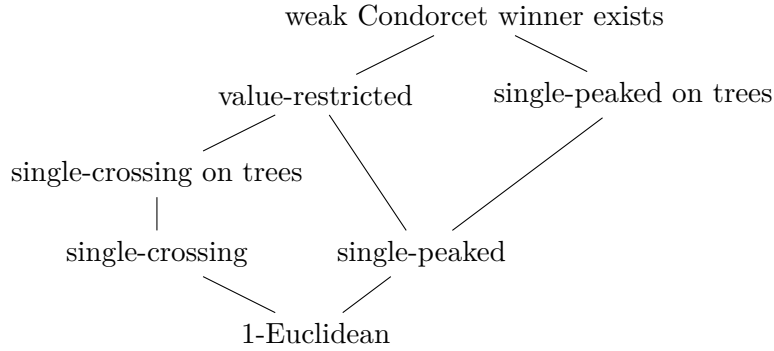


Figure 1: Some domain restrictions that guarantee a weak Condorcet winner, and inclusion relationships.

In this section, we will present the definitions of several prominent domain restrictions, and discuss some of their basic properties. Within the economics and social choice literature, domain restrictions were found to be interesting primarily because of their implications for the structure of the majority relation. For example, the majority relation of a single-peaked profile cannot have a Condorcet cycle. Also, some domain restrictions admit well-behaved voting rules: e.g., for the single-peaked domain we have the median voter rule, which is strategyproof.

As an overview, in [Figure 1](#) we display the relationship among domain restrictions that guarantee a weak Condorcet winner. The 1-Euclidean domain is the most restrictive domain that we consider here, whereas the value-restricted and single-peaked on trees domains are very general. The figure does not mention the multidimensional single-peaked domain, the d -Euclidean domain for $d \geq 2$, and the domain of profiles that are single-peaked on a circle, because these domains do not rule out the presence of Condorcet cycles.

Before we start our discussion of specific domain restrictions, let us take a meta-level view, and define what we formally mean by a *domain restriction* (which we also call a *restricted domain* or a *preference domain*). Most abstractly, a domain restriction is a property of a profile. Typically, this property imposes some structure on this profile. For most of this survey, we will consider notions of structure only in the context of profiles of *linear* orders. Thus, except for some remarks in [Section 8](#), we will not consider preferences that have indifferences. Hence, our formal definition of a domain restriction is as follows.

Definition 2. A *domain restriction* is a set of profiles of linear orders.

For example, the domain of single-peaked profiles is the set of profiles P for which there exists an axis \triangleleft such that P is single-peaked on \triangleleft (we will discuss the meaning of single-peakedness just below in [Section 3.2](#)).

Our definition of a domain restriction is a departure from much of the discussion of domain restrictions in the social choice literature. There, a domain typically refers to a set of allowed *votes*, rather than a set of profiles. Under this view, a profile is structured if each voter reports a linear order that belongs to the respective domain. Our notion of a domain is more general; in particular, we allow domain restrictions that are not Cartesian products of restrictions on individual voters' preferences. To appreciate this distinction, observe that the domain of profiles that are single-peaked on some *fixed* axis \triangleleft is a Cartesian product, whereas the domain of all

single-peaked profiles is not.

An important property that is satisfied by many—but not all—domain restrictions is closure under deleting voters and alternatives. A domain with this property is called *hereditary*; for such domains, taking a subprofile of a structured profile yields another structured profile.

Definition 3. A domain restriction X is *hereditary* if for every profile P in X and every profile P' that can be obtained from P by voter and alternative deletion it holds that P' belongs to X .

As we will see, the domain of single-peaked profiles is hereditary, and the same is true for Euclidean or single-crossing profiles. A counter-example is the domain of profiles that admit a weak Condorcet winner.

Example 3: *The property of having a weak Condorcet winner is not hereditary.*

v_1	v_2	v_3	v_4	
a	a	b	c	Alternative a is a weak Condorcet winner. However, if we remove v_1 , we obtain the Condorcet cycle $a \succ_{\text{maj}} b \succ_{\text{maj}} c \succ_{\text{maj}} a$, and consequently this reduced profile does not have a weak Condorcet winner.
b	b	c	a	
c	c	a	b	

Hereditary domain restrictions are of particular importance for characterizations via forbidden subprofiles, as we will see in [Section 5](#).

3.1 Condorcet Winners

The concept of a Condorcet winner dates back to the 18th century; it was proposed by the Marquis de Condorcet, who argued that a voting rule should elect a Condorcet winner whenever it exists. However, it is well-known that Condorcet winners may fail to exist, since there are profiles whose majority relation contains cycles. Thus, this concept defines an interesting domain restriction: let $\mathcal{D}_{\text{Condorcet}}$ denote the set of profiles that have a (strong) Condorcet winner. For an odd number of voters, all the other domain restrictions that we consider in this survey (such as, e.g., single-peaked profiles) are *subdomains* of $\mathcal{D}_{\text{Condorcet}}$, because each of these restrictions guarantees the existence of a strong Condorcet winner as long as the number of voters is odd. However, while $\mathcal{D}_{\text{Condorcet}}$ is a large domain compared to the other domains considered here, it is somewhat less interesting to us, because it does not imply much *structure* in individuals' preferences, and because it does not imply tractability results for the winner determination problems of many important voting rules that we consider in [Section 6](#). Nevertheless, in this section, we will briefly discuss some properties of $\mathcal{D}_{\text{Condorcet}}$ since they have implications for the subdomains we will study in more detail later on.

There exists an obvious voting rule for the profiles in $\mathcal{D}_{\text{Condorcet}}$: the *Condorcet rule*, which elects the Condorcet winner of the profile. Formally, a *voting rule* is a function $f : \mathcal{D} \rightarrow A$ that maps each profile from some domain \mathcal{D} to a unique winning alternative. We write $f_{\text{Condorcet}} : \mathcal{D}_{\text{Condorcet}} \rightarrow A$ for the Condorcet rule, where $f_{\text{Condorcet}}(P)$ is the Condorcet winner of $P \in \mathcal{D}_{\text{Condorcet}}$. An important, but difficult to achieve, property of a voting rule is *strategyproofness*, which requires that voters cannot manipulate the voting rule by misrepresenting their preferences. Formally, we say that a voting rule f is *manipulable* if there exist profiles $P = (\succ_1, \dots, \succ_i, \dots, \succ_n)$ and $P' = (\succ_1, \dots, \succ'_i, \dots, \succ_n)$, both members of the domain, that only differ in the preferences of the i th voter, and such that $f(P') \succ_i f(P)$. Thus, if \succ_i is the truthful preference order of voter

i , this voter can obtain a strictly preferred outcome by submitting the non-truthful preference order \succ'_i instead. If a voting rule is not manipulable, then it is *strategyproof*.

While Gibbard (1973) and Satterthwaite (1975) showed that any (surjective) voting rule defined on the domain of *all* preference profiles must be dictatorial, there are restricted domains where strategyproofness can be achieved. The domain of profiles admitting a Condorcet winner is an example; in fact, one can check that the Condorcet rule is strategyproof.

Proposition 3.1. *The Condorcet rule $f_{\text{Condorcet}}$ is strategyproof.*

Proof. For a contradiction, suppose there are profiles $P = (\succ_1, \dots, \succ_i, \dots, \succ_n)$ and $P' = (\succ_1, \dots, \succ'_i, \dots, \succ_n)$ such that

$$f_{\text{Condorcet}}(P) = a, \quad f_{\text{Condorcet}}(P') = b, \quad \text{and } b \succ_i a.$$

Since a is the Condorcet winner at P , there is a strict majority $N' \subseteq N$ of voters who prefer a to b in P . As $b \succ_i a$, we have $i \notin N'$, so all voters in N' also prefer a to b in P' , forming a strict majority. This is a contradiction with b being the Condorcet winner at P' . \square

A similar argument also establishes that $f_{\text{Condorcet}}$ is resistant to manipulation by *groups of voters* (Moulin, 1991, Lemma 10.3).

It turns out that $f_{\text{Condorcet}}$ is essentially the only voting rule defined on $\mathcal{D}_{\text{Condorcet}}$ that is strategyproof, at least if the number n of voters is odd.

Theorem 3.2 (Campbell and Kelly, 2002, 2016). *Suppose that the number of voters is odd. Let f be a non-dictatorial and surjective voting rule defined on $\mathcal{D}_{\text{Condorcet}}$. Then f is strategyproof if and only if $f = f_{\text{Condorcet}}$.*

Campbell and Kelly (2015) show an analog of their theorem for an even number of voters if one strengthens non-dictatorship to anonymity, and surjectivity to neutrality. Peters (2019, Theorem 3.6) proves a different version for an even number of voters using anonymity and Pareto optimality.

Notice that the Campbell–Kelly theorem may not hold for domains that are smaller than $\mathcal{D}_{\text{Condorcet}}$. For example, as we will discuss below, there are many more strategyproof voting rules defined for single-peaked profiles only.

3.2 Single-Peaked Preferences

Let us now turn to what is arguably the most famous domain restriction: the single-peaked domain. Consider a situation where voters need to decide among different possible quantities of a numerical measure: it might be a parliament deciding on a tax rate, a firm’s board deciding on the price for a new product, or housemates deciding on the optimal setting of the thermostat. As Black (1948) noted, in such situations it is reasonable to expect that each decision maker first identifies her optimum value of the measure under consideration, and that she is less and less happy the further the chosen quantity deviates from her optimum. For example, it would be surprising if a politician whose most preferred income tax rate is 20% had 70% as his second-favorite tax rate, and 40% as his third-favorite tax rate. Rather, we expect his preference curve to have a single peak like the solid lines in Figure 2. Another situation where preferences can be expected to have this shape is in a political election where the candidates can be placed on a left-to-right spectrum, with left-wing voters preferring left-wing candidates, and right-wing voters preferring right-wing candidates.

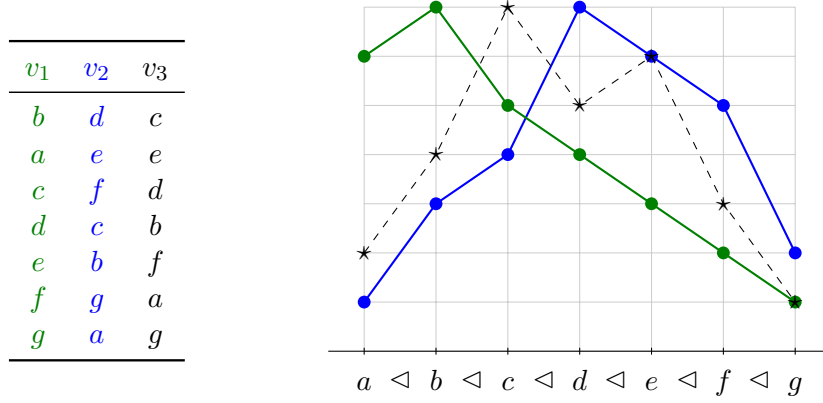


Figure 2: Votes v_1 and v_2 , shown as solid lines, are single-peaked with respect to \triangleleft . The vote v_3 , depicted as a dashed line, is not single-peaked with respect to \triangleleft . While a profile consisting of v_2 and v_3 is single-peaked (d and e have to be flipped on \triangleleft), there is no ordering of the candidates for which the profile (v_1, v_3) is single-peaked.

Definition As we will see, single-peaked profiles have many desirable properties. Let us start by giving a formal definition. See Figure 2 for an example.

Definition 4. Let P be a profile over A and let \triangleleft be a linear order over A . A linear order v_i over A is *single-peaked on \triangleleft* if for every pair of alternatives $a, b \in A$ with $\text{top}(v_i) \triangleleft b \triangleleft a$ or $a \triangleleft b \triangleleft \text{top}(v_i)$ we have $b \succ_i a$. A profile P over A is *single-peaked with respect to \triangleleft* if every vote in P is single-peaked on \triangleleft , and it is *single-peaked* if there exists a linear order \triangleleft over A such that P is single-peaked with respect to \triangleleft .

In what follows we will often refer to a linear order \triangleleft over A that witnesses the single-peaked property as an *axis*.

This definition requires that, as we move away from a voter's most-preferred alternative either to the left or to the right, the voter becomes less and less enthusiastic. Thus, plotting a “preference curve”, which depicts the voter's preference intensity, as in Figure 2, yields a single-peaked shape, giving rise to this domain restriction's name (Black, 1948).

To reason about this domain restriction, it is useful to have several alternative ways of thinking about it. For example, a feature of the preference curves shown in Figure 2 is that they do not contain any valleys, as shown in Figure 3 on the right. Formally, given an axis \triangleleft , a vote \succ_i has a *valley* if there is a triple $a \triangleleft b \triangleleft c$ of alternatives such that $a \succ_i b$ and $c \succ_i b$. Proposition 3.3 below shows that a linear order has no valleys with respect to \triangleleft if and only if it is single-peaked on \triangleleft . It also shows that it suffices to check that there are no ‘local’ valleys where the alternatives a, b, c are next to each other on the axis \triangleleft . This condition is used by some of the dynamic programming algorithms presented later in this survey (see, e.g., Theorem 4.39).

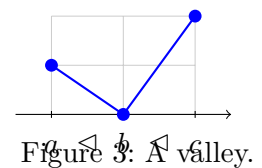


Figure 3: A valley.

One can also view single-peakedness as a *convexity* condition. Given an axis \triangleleft , a subset $A' \subseteq A$ of alternatives is *convex* if and only if it is an interval of the axis \triangleleft , that is, for any triple of alternatives a, b, c with $a \triangleleft b \triangleleft c$ and $a, c \in A'$ we have $b \in A'$. By looking at Figure 2, we can see that if we take any *prefix* of the votes v_1 and v_2 , then this prefix is an interval of \triangleleft . For example, the four most-preferred alternatives in v_2 are $\{d, e, f, c\}$, and this set forms an interval

of \triangleleft . Again, [Proposition 3.3](#) establishes that all prefixes of a vote are intervals of \triangleleft if and only if this vote is single-peaked on \triangleleft .

Proposition 3.3. *Let \succ be a vote over A , and let \triangleleft be a linear order over A . The following conditions are equivalent:*

- (1) *The vote \succ is single-peaked on \triangleleft .*
- (2) *For all $a, b, c \in A$ such that $a \triangleleft b \triangleleft c$ and these alternatives form an interval of \triangleleft , we do not have both $a \succ b$ and $c \succ b$, that is, there is no ‘local’ valley in \succ .*
- (3) *For all $a, b, c \in A$ such that $a \triangleleft b \triangleleft c$, we do not have both $a \succ b$ and $c \succ b$, that is, there is no valley in \succ .*
- (4) *For each $c \in A$, the set $\{a \in A : a \succcurlyeq c\}$ is an interval on \triangleleft .*

Proof. (1) \rightarrow (2): Assume \succ is single-peaked on \triangleleft . We will actually prove the stronger condition (3). Assume towards a contradiction that there exists a triple $a, b, c \in A$ with $a \triangleleft b \triangleleft c$, $a \succ b$ and $c \succ b$. Let x be the top-ranked candidate in \succ ; note that $x \neq b$. If $x \triangleleft b$ then $x \triangleleft b \triangleleft c$ and $c \succ b$; if $b \triangleleft x$ then $a \triangleleft b \triangleleft x$ and $a \succ b$. In either case we obtain a contradiction with the assumption that \succ is single-peaked on \triangleleft .

(2) \rightarrow (3): We show that if \succ has a valley, then it also has a local valley. Define the *width* of a valley $a \triangleleft b \triangleleft c$ as the number of alternatives that appear between a and c on \triangleleft ; a valley is local if and only if its width is 1. Now, suppose a, b, c form a valley of width $w > 1$ in \succ . Assume without loss of generality that a and b are non-adjacent on \triangleleft , i.e., $a \triangleleft a' \triangleleft b \triangleleft c$ for some $b' \in A$. But then either $a' \succ b$, in which case a', b, c form a valley of width at most $w - 1$, or $b \succ a'$, in which case a, a', b form a valley of width at most $w - 1$; proceeding in this fashion, we arrive at a local valley.

(3) \rightarrow (4): Assume towards a contradiction that there exists an alternative c such that $S = \{a \in A : a \succcurlyeq c\}$ is not an interval of \triangleleft , i.e., there exist alternatives $d, e, f \in A$ such that $d \in S$, $e \notin S$, $f \in S$, and $d \triangleleft e \triangleleft f$. Consequently, $d \succcurlyeq c \succ e$ and $f \succcurlyeq c \succ e$, which contradicts Condition (3).

(4) \rightarrow (1): Let $a, b \in A$ with $x \triangleleft b \triangleleft a$ or $a \triangleleft b \triangleleft x$, where x is the top-ranked candidate in \succ . By assumption, $S = \{c \in A : c \succcurlyeq a\}$ is an interval of \triangleleft . By construction of S we have $a \in S$, $x \in S$ and hence $b \in S$. Thus, $b \succ a$, and therefore \succ is single-peaked on \triangleleft . \square

[Definition 4](#) follows the spirit of the original definition due to [Black \(1948\)](#). Condition (3) is essentially how [Arrow \(1951\)](#) formulated single-peakedness. Condition (4), which is stated in terms of the connectedness of prefixes (also known as *upper contour sets*) is often an elegant way of reasoning about single-peaked preferences, and it generalizes well to other settings (see [Section 3.5](#) and [Section 8](#)).

Consider a profile P over A that is single-peaked with respect to an axis \triangleleft , and a triple of distinct alternatives $A' = \{a, b, c\} \subseteq A$. The no-valley property implies that there exist an alternative $x \in A'$ such that no voter ranks x below all other alternatives in A' : indeed, if $a \triangleleft b \triangleleft c$ then a vote \succ with $a \succ b$, $c \succ b$ would have a valley at b . In particular, there can be at most two different alternatives in A that are ranked last by some voter. This is also easy to conclude from the convexity property: only the endpoints of the axis can be ranked last. This observation can often be used to show that a given profile is not single-peaked.

Majority Relation The most famous property of single-peaked profiles is that they admit a Condorcet winner, and that the Condorcet winner is the top alternative of the *median voter*, i.e., the voter in the middle when one orders the voters according to the position of their top-ranked alternatives on \triangleleft . This property is often used by political scientists and in public choice to reason about decision over quantities such as tax rates.

Proposition 3.4 (Median Voter Theorem). *Every single-peaked profile has a weak Condorcet winner.*

Proof. In what follows, given an axis \triangleleft and two votes v, v' , we write $\text{top}(v) \trianglelefteq \text{top}(v')$ to denote that either $\text{top}(v) \triangleleft \text{top}(v')$ or $\text{top}(v) = \text{top}(v')$. First, we reorder the votes in $P = (v_1, \dots, v_n)$ so that $\text{top}(v_1) \trianglelefteq \text{top}(v_2) \trianglelefteq \dots \trianglelefteq \text{top}(v_n)$, set $\ell := \lceil \frac{n+1}{2} \rceil$, and let $c := \text{top}(v_\ell)$ be the top-ranked alternative of (one of) the median voter(s). We claim that c is a weak Condorcet winner. To see this, consider any other alternative $a \in A$.

- If $a \triangleleft c$, then the voters $\ell, \ell + 1, \dots, n$ all prefer c to a , and these voters form a weak majority.
- If $c \triangleleft a$, then the voters $1, \dots, \ell - 1, \ell$ all prefer c to a , and these voters form a weak majority.

Hence, c is a weak Condorcet winner, as required. \square

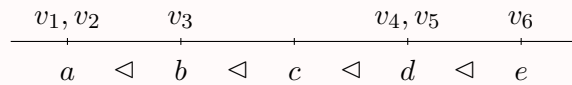
If the number of voters in P is odd, then there is a unique Condorcet winner. If the number of voters in P is even, then the set of weak Condorcet winners is the interval of \triangleleft with endpoints given by the top choices of voters $\lfloor \frac{n+1}{2} \rfloor$ and $\lceil \frac{n+1}{2} \rceil$.

Example 4: Median voters

Consider a single-peaked profile with the axis $a \triangleleft b \triangleleft c \triangleleft d \triangleleft e$. We have six voters with $\text{top}(v_1) = \text{top}(v_2) = a$, $\text{top}(v_3) = b$, $\text{top}(v_4) = \text{top}(v_5) = d$, and $\text{top}(v_6) = e$. Consequently, it holds that

$$\text{top}(v_1) \trianglelefteq \text{top}(v_2) \trianglelefteq \text{top}(v_3) \trianglelefteq \text{top}(v_4) \trianglelefteq \text{top}(v_5) \trianglelefteq \text{top}(v_6).$$

This profile has three weak Condorcet winners: alternatives b, c , and d . Indeed, since $\lfloor \frac{n+1}{2} \rfloor = 3$ and $\lceil \frac{n+1}{2} \rceil = 4$, all alternatives in the interval $[\text{top}(v_3), \text{top}(v_4)] = \{b, c, d\}$ are weak Condorcet winners.



As we will now see, the single-peaked property constrains the majority relation \succ_{maj} even further: it implies that \succ_{maj} is transitive as long as the number of voters is odd. We give two different proofs, one of which works by repeated application of the Median Voter Theorem.

Corollary 3.5. *If P is a single-peaked profile with an odd number of voters then its majority relation is transitive.*

Proof by induction. By induction on m , the number of alternatives. The result is obvious for $m = 1$. If $m > 1$, let c be the (unique) Condorcet winner of P . Let $P' = P_{A \setminus \{c\}}$. Since P is single-peaked, P' is single-peaked as well. By the inductive hypothesis, the majority relation

of P' is transitive; denote it by \succ' . Define \succ by setting $a \succ b$ if and only if (i) $a \succ' b$ and $a, b \in A \setminus \{c\}$ or (ii) $a = c$ and $b \in A \setminus \{c\}$. Then \succ is the majority relation of P , and it is transitive. \square

Proof by contradiction. Suppose for a contradiction that the majority relation is not transitive, so that $a \succ_{\text{maj}} b \succ_{\text{maj}} c \succ_{\text{maj}} a$ for some alternatives a, b, c . Assume without loss of generality that $a \triangleleft b \triangleleft c$. Note that any two strict majorities intersect in at least one voter. Thus, since $a \succ_{\text{maj}} b$ and $c \succ_{\text{maj}} a$, there exists a voter i with $c \succ_i a \succ_i b$, a contradiction with the no-valley property. \square

Corollary 3.5 depends on the assumption that the number of voters is odd. In the following example with four voters, the relation \succ_{maj} is not transitive. However, our second proof can be adjusted to show that its strict part \succ_{maj} is always transitive. A relation whose strict part is transitive is called *quasi-transitive*, and hence we can say that the majority relation of any single-peaked profile is quasi-transitive.

Example 5: *The weak majority relation of a single-peaked profile may not be transitive for even n*

v_1	v_2	v_3	v_4
b	b	c	c
a	a	b	b
c	c	a	a

This profile is single-peaked with respect to $a \triangleleft b \triangleleft c$. The majority relation \succ_{maj} is not transitive as it satisfies $a \succ_{\text{maj}} c$ and $c \succ_{\text{maj}} b$, yet all voters prefer b to a . Still, alternatives b and c are weak Condorcet winners, and the strict majority relation \succ_{maj} is transitive.

A domain restriction is said to have the *representative voter property* (Rothstein, 1991) if for every structured profile P it holds that its majority relation \succ_{maj} is transitive, and there is some *representative* voter v_i in P whose preferences coincide with \succ_{maj} . However, as the following example shows, the single-peaked domain does not have this property even if the number of voters is odd.

Example 6: *Single-peaked profiles may not have the representative voter property*

v_1	v_2	v_3
a	b	c
b	c	b
c	d	a
d	a	d

This profile is single-peaked with respect to $a \triangleleft b \triangleleft c \triangleleft d$. The majority relation of this profile satisfies $b \succ_{\text{maj}} c \succ_{\text{maj}} a \succ_{\text{maj}} d$, but no voter has this preference ranking.

The reason is that when we repeatedly applied the Median Voter Theorem in the first proof of **Corollary 3.5**, different voters may have appeared in the median position at different stages of the process. We will soon see that *single-crossing* preferences do satisfy the representative voter property. However, for single-peaked preferences, a weaker property holds. Consider the majority relation in **Example 6**: while no voter submitted this ordering as their vote, we can see that the majority ranking is single-peaked with respect to the axis \triangleleft . We will now argue that this is not a coincidence.

Proposition 3.6. *The single-peaked domain is closed. That is, if P is a profile with an odd number of voters that is single-peaked with respect to \triangleleft , then its majority relation \succ_{maj} is*

single-peaked on \triangleleft .

Proof. One can prove this using the techniques of Puppe and Slinko (2019), i.e., by arguing that the single-peaked domain is a *maximal* Condorcet domain, and then appealing to their Lemma 2.1. Here is a direct proof, which is due to Moulin (1991, Lemma 11.4). By Proposition 3.4 the majority relation has a unique peak $\text{top}(\succ_{\text{maj}})$, namely, the Condorcet winner. Now consider two alternatives $a, b \in A$ with $\text{top}(\succ_{\text{maj}}) \triangleleft b \triangleleft a$ (the case $a \triangleleft b \triangleleft \text{top}(\succ_{\text{maj}})$ is similar). To establish that \succ_{maj} is single-peaked on \triangleleft , we need to show that $b \succ_{\text{maj}} a$. Now, since the Condorcet winner is the peak of the median voter, a strict majority $N' \subseteq N$ of voters have their peak located at or to the left of $\text{top}(\succ_{\text{maj}})$. Since all the voters in N' have single-peaked preferences, we then must have $b \succ_i a$ for all $i \in N'$. Since N' forms a strict majority, we have $b \succ_{\text{maj}} a$, as required. \square

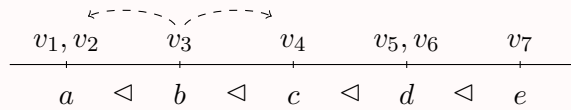
Proposition 3.6 implies that some rank aggregation rules like Kemeny’s rule always output a single-peaked ranking. Bredereck et al. (2022) study the class of rank aggregation rules that preserve single-peakedness, and note their applications in an opinion diffusion model.

In addition to the majority relation, one can also consider a weighted version, namely the collection of *majority margins*; the majority margin of a over b is defined to be $m_{ab} = |i \in N : a \succ_i b| - |i \in N : b \succ_i a|$. Thus, $m \succ_{\text{maj}} b$ if and only if $m_{ab} > 0$. Smeulders et al. (2014) and Spanjaard and Weng (2016) have characterized collections of majority margins that can be induced by a single-peaked preference profile and have given efficient algorithms for recognizing such collections.

Strategyproof Social Choice Consider the domain of single-peaked profiles with an odd number of voters. For odd n , every single-peaked profile admits a Condorcet winner. Hence, by Proposition 3.1, a voting rule that is defined on this domain and returns the Condorcet winner of the input profile is strategyproof. Note that in this case the Condorcet winner is the top choice of the median voter.

Example 7: *The median rule is strategyproof.*

It is instructive to see why the median voter rule is strategyproof. Consider a single-peaked profile with an odd number of voters. As an example, consider the following distribution of voter peaks.



The median voter is voter 4, and so the median voter rule returns c . Of course, voter 4 is not interested in manipulating, since her favorite alternative is elected. Consider any other voter, say voter 3, whose truthful peak is b . If voter 3 were to report a peak further to the left (such as a), then this would not affect the position of the median voter, so this is not a successful manipulation. If voter 3 were to report a peak further to the right (such as c or d), then either the median would not change, or it would move further to the right. Since the preferences of voter 3 are single-peaked, the latter change would lead to a worse outcome for her. Hence voter 3 cannot manipulate, and a similar argument shows that this is the case for all other voters as well.

However, on this smaller domain, there may be additional strategyproof voting rules. In this context, instead of considering the domain of all profiles that are single-peaked, it is arguably more natural to fix an axis \triangleleft over A , and take the domain of profiles single-peaked with respect to \triangleleft . We can interpret this setting as having a commonly-known structure of the alternative space; potential manipulators must submit preferences that conform with this structure. (If we considered voting rules defined for single-peaked profiles without a fixed axis, manipulations could change the axis.) In this model, there are other examples of strategyproof rules: for instance, the rule returning the leftmost reported peak is strategyproof (the argument is the same as for the median voter rule in the example above), as is any other order statistic. [Moulin \(1980\)](#) gave a characterization of strategyproof voting rules defined for profiles single-peaked with respect to a fixed axis \triangleleft . Note that his result does not require the number of voters n to be odd.

Theorem 3.7 ([Moulin, 1980](#)). *Fix a set of voters N and a set of alternatives A , and let \mathcal{D} be the set of profiles single-peaked with respect to the axis \triangleleft . Then $f : \mathcal{D} \rightarrow A$ is anonymous, Pareto-optimal, and strategyproof if and only if there exist alternatives $\alpha_1, \dots, \alpha_{n-1} \in A$ such that for all profiles $P \in \mathcal{D}$, we have*

$$f(P) = \text{median}_{\triangleleft}(\text{top}(v_1), \dots, \text{top}(v_n), \alpha_1, \dots, \alpha_{n-1}).$$

The social choice functions identified in this characterization are called *generalized median rules*. These rules take a median of the voters' reported top choices together with $n - 1$ fixed values. The fixed values $\alpha_1, \dots, \alpha_{n-1}$ are often interpreted as the reported top choices of $n - 1$ “phantom voters”. As [Thomson \(2018, p. 78\)](#) suggests, one can also interpret this result as follows: for each “extremist” profile (in which $n - i$ voters report the leftmost and i voters report the rightmost alternative of \triangleleft) we can choose an arbitrary output alternative α_i so that $\alpha_1 \triangleleft \dots \triangleleft \alpha_{n-1}$; then there is a unique extension of these choices to the full domain \mathcal{D} that is strategyproof.

[Moulin \(1980\)](#) proves his result using the additional assumption of f being “tops-only” (so that f only depends on voters' top-ranked alternatives). Later work has shown that any strategyproof and onto rule on \mathcal{D} must be tops-only ([Barberà and Jackson, 1994](#); [Weymark, 2011](#)), so this assumption can be dropped.

Theorem 3.7 continues to hold when the set of alternatives is \mathbb{R} , or any subset of \mathbb{R} ([Weymark, 2011](#)). It also holds when replacing strategyproofness by group strategyproofness (i.e., resistance to manipulation by groups of voters), since these properties are equivalent on the single-peaked domain ([Moulin, 1980](#)). An analog of **Theorem 3.7** holds when weakening Pareto optimality to the tops-only condition, in which case the class of anonymous and strategyproof rules consists of the generalized median rules with $n + 1$ rather than $n - 1$ phantom voters ([Moulin, 1980](#)). The class of strategyproof rules without the anonymity requirement has also been characterized, though the description of this class involves 2^n parameters and is more complicated ([Moulin, 1980](#); [Weymark, 2011](#)).

Within the class of generalized median rules, the most commonly considered ones are the order statistics (e.g., left-most peak, median peak) which can be obtained by having all the phantoms at extreme positions (only the left-most or right-most alternative). But other mechanisms in this class are also of interest. For example if the set of alternatives is the interval $[0, 1]$, and we place phantom voters at $\alpha_1 = \frac{1}{n}, \alpha_2 = \frac{2}{n}, \dots, \alpha_{n-1} = \frac{n-1}{n}$, we obtain the *uniform phantom mechanism* (also known as the *linear median*), which approximates the rule that selects the average of the peaks ([Caragiannis et al., 2016](#); [Freeman et al., 2021](#); [Caragiannis et al., 2024](#); [Jennings et al., 2023](#)).

Counting It is easy to see that a single-peaked profile could be single-peaked with respect to several different axes. Indeed, *every* single-peaked profile will be single-peaked with respect to at least two different axes, since reversing the axis preserves single-peakedness. If we consider a profile consisting of just a single preference order \succ , we can see that this profile is single-peaked on 2^{m-1} different axes. Indeed, we can start building a partial axis for \succ by placing the peak of \succ on the line, and then process the rest of the vote from top to bottom. For each alternative, we can choose whether to put it to the left or to the right of the alternatives placed so far. For example, the profile $P = (a \succ b \succ c)$ is single-peaked with respect to the axes $a \triangleleft b \triangleleft c$, $b \triangleleft a \triangleleft c$, $c \triangleleft a \triangleleft b$, and $c \triangleleft b \triangleleft a$. On the other hand, a profile containing two reverse preference orders is only single-peaked on precisely two different axes (which coincide with these two orders). We will study the collection of all axes that make a given profile single-peaked in more detail in [Section 4.1](#), where we will see that the size of this collection is always a power of 2, and that any two axes can be obtained from each other by (repeatedly) reversing certain intervals of the axis.

A converse counting problem fixes an axis \triangleleft and asks how many preference orders are single-peaked on this axis. For this case, too, the answer is 2^{m-1} : this is easy to see by induction on m , observing that for a preference order that is single-peaked on a fixed axis \triangleleft , there are two choices available for the m -th position, namely, the two outermost alternatives of \triangleleft .

Sampling Given a fixed axis \triangleleft , how can we randomly sample one of the 2^{m-1} single-peaked preferences on \triangleleft uniformly at random? [Walsh \(2015\)](#) gives an algorithm that solves this problem. It works from the outside in: note that the last-ranked alternative in any single-peaked preference is either the left-most or the right-most alternative. Thus, the algorithm flips a coin to decide the last-ranked alternative. Say it is the left-most. Note that the second-to-last alternative in the preference ranking must now be either the second-left-most alternative or the right-most alternative. Again, the algorithm flips a coin to decide which of these two it is. The algorithm continues in the same fashion.

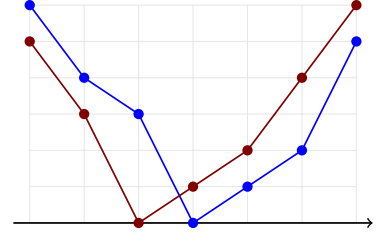
An earlier sampling algorithm of [Conitzer \(2009\)](#) can also be used to sample single-peaked preferences, but it doesn't sample them uniformly at random. Conitzer's algorithm works by first choosing one of the alternatives uniformly at random to be the most-preferred alternative (i.e., the peak). It then repeatedly decides by a coin flip whether the next-most-preferred alternative will be immediately to the left or immediately to the right of the set of alternatives the algorithm has already ranked. This algorithm oversamples rankings whose peaks are at the ends of the axis ([Walsh, 2015](#)). For example, only 1 out of 2^{m-1} rankings has the left-most alternative as its peak, but Conitzer's algorithm gives this ranking a probability of $1/m$.

Both sampling algorithms are frequently used in numerical experiments within computational social choice ([Boehmer et al., 2024](#)), since they tend to give qualitatively different results. Interestingly, Conitzer's algorithm produces elections that are similar to those obtained when sampling one-dimensional Euclidean preference profiles, with alternatives and voters placed on positions in the interval $[0, 1]$ uniformly at random.

Further Properties [Puppe \(2018\)](#) has characterized the domain of all preferences that are single-peaked on a fixed axis \triangleleft among all (Cartesian) domains that guarantee a transitive majority relation: namely, this domain is the only one that is minimally rich (every alternative appears in a top position in some order), connected (with respect to the natural betweenness relation), and contains two completely reversed preference orders.

Finally, observe that the domain of single-peaked profiles is hereditary: closure under voter deletion is immediate from the definition, and closure under alternative deletion is clear from the no-valley condition.

Single-Caved Preferences A related domain restriction is the domain of *single-caved* preferences, also known as *single-dipped* preferences, which is obtained by reversing single-peakedness. Thus, a profile is single-caved if there exists an axis \triangleleft such that every voter has a least-preferred alternative, and their preferences increase as we move away from this minimum. As an example, such preferences could arise in situations where we



need to decide on the location of a public bad, such as a polluting factory, and people would like this facility to be placed as far away from their own location as possible.

Recall that if R is a binary relation, then the relation R' is the *reverse* of R if $(a, b) \in R \Leftrightarrow (b, a) \in R'$. The reverse of a profile P is obtained by taking the reverse of every vote in P .

Definition 5. A profile P of linear orders is *single-caved* if the reverse of P is single-peaked.

The class of strategyproof voting rules on this domain is much smaller than in the case of single-peaked preferences: all such rules have a range of size at most two, and select only between the left-most and the right-most alternative (Manjunath, 2014; Barberà et al., 2012).

3.3 Single-Crossing Preferences

In the previous section, we have considered a notion of structure that was imposed on the alternative set: we assumed that the alternative space is one-dimensional. Another natural idea is to require the set of *voters* to be one-dimensional. This approach gives rise to the notion of *single-crossing preferences*, which we will study in this section.

We defined a preference profile $P = (\succ_1, \dots, \succ_n)$ as an ordered list of votes. This already suggests a one-dimensional structure on the set $[n]$ of voters. What should it mean for the voters' preferences to respect this structure? Suppose that the leftmost voter and the rightmost voter disagree on the order of some alternatives, so that $a \succ_1 b$, but $b \succ_n a$ for some $a, b \in A$. Then we expect that voters who “tend left” agree with voter 1, and that voters who “tend right” agree with voter n . Thus, we require that there is a voter $i \in [n]$ such that $a \succ_1 b, \dots, a \succ_i b$, and $b \succ_{i+1} a, \dots, b \succ_n a$. Thus, the preferences over the pair $\{a, b\}$ cross only once as we scan the profile from left to right. We will allow the cross-over voter i to be different for different alternative pairs. Formally, a profile is single-crossing if it satisfies the following condition.

Definition 6. A profile $P = (v_1, \dots, v_n)$ over A is *single-crossing with respect to the given ordering* if for every pair of alternatives $a, b \in A$ both sets $\{i \in [n] : a \succ_i b\}$ and $\{i \in [n] : b \succ_i a\}$ are (possibly empty) intervals of $[n]$. A profile $P = (v_1, \dots, v_n)$ over A is *single-crossing* if the votes in P can be permuted so that the permuted profile is single-crossing with respect to the given ordering.

Example 8: *A single-crossing profile*

v_1	v_2	v_3	v_4	v_5
a	b	b	d	d
b	a	d	b	c
c	d	a	c	b
d	c	c	a	a

When a profile is single-crossing with respect to the given ordering, this admits an attractive visualization. For each alternative $a \in A$, we draw a “trajectory” through the positions in the profile in which a appears. If the profile is single-crossing with respect to the given ordering, any two trajectories will cross at most once.

In which situations can we expect to observe single-crossing preferences? Within economics, such preferences arise in models of income taxation under common assumptions (Mirrlees, 1971; Roberts, 1977; Rothstein, 1990). Specifically, if voters are ordered by increasing income, and some voter prefers a higher tax rate to a lower tax rate, then it stands to reason that all lower-income voters would agree that the higher rate is preferable to the lower rate (e.g., because those voters would obtain higher benefits under a redistributive regime). In many other contexts, there is a natural one-dimensional ordering of voters induced by a parameter θ (e.g., in terms of income, productivity, a discount factor, years of education, etc.); if the utility of the alternatives exhibits increasing differences in θ , the resulting profile will be single-crossing. The foregoing discussion is based on the exposition of Saporiti (2009), who provides references to several models in which single-crossing preferences appear.

Proposition 3.8. *The domain of single-crossing preferences is hereditary, that is, closed under deleting voters and alternatives.*

Majority Relation As we have seen, single-peaked profiles with an odd number of voters always have a transitive majority relation, and the Median Voter Theorem holds. This has been a major reason why social choice theorists have studied this domain restriction. Single-crossing profiles enjoy the same guarantee: for an odd number of voters, their majority relation is transitive. Moreover, in contrast to single-peaked profiles, single-crossing profiles with an odd number of voters always have a representative voter, i.e., a voter whose preference relation is identical to the majority relation.

Proposition 3.9 (Representative Voter Theorem, Rothstein, 1991). *Suppose P is single-crossing. If the number of voters is odd, $n = 2k - 1$, then the preference order of the median voter k coincides with the majority relation. Thus, the majority relation is transitive.*

Proof. We will argue that the majority relation agrees with the preferences of the k -th voter on every pair of alternatives. Let $a, b \in A$, and suppose $a \succ_k b$. By the single-crossing property, it cannot be the case that both $b \succ_1 a$ and $b \succ_n a$; assume without loss of generality that $a \succ_1 b$. Then, by applying the single-crossing property again, we conclude that voters $2, \dots, k-1$ also prefer a to b , i.e., there are at least $k > n/2$ voters who rank a above b . \square

In particular, Proposition 3.9 implies that in a single-crossing profile with an odd number of voters the top alternative of the median voter is a strong Condorcet winner.

When the number of voters is even, $n = 2k$, there are two median voters, i.e., k and $k+1$, and the majority relation is the intersection of the relations v_k and v_{k+1} . Thus, $a \succ_{\text{maj}} b$ if and only if both $a \succ_k b$ and $a \succ_{k+1} b$. This implies that the strict part of the majority relation is transitive, and hence the majority relation is quasi-transitive, though it may fail to be transitive (e.g., the profile in Example 5, which was used to show that the weak majority relation of a

single-peaked profile may fail to be transitive, is both single-peaked and single-crossing). Also, an argument similar to the proof of [Proposition 3.9](#) shows that the top choices of voters k and $k + 1$ are weak Condorcet winners (and if these voters rank the same candidate first, then this candidate is a strong Condorcet winner).

Strategyproof Social Choice The definition of a ‘strategyproof voting rule on the single-crossing domain’ is somewhat subtle. Which rankings do we allow as admissible manipulative votes? Indeed, if we assume that the ordering of voters is given externally (i.e., the voters are ordered by an observable parameter, such as their age, and cannot change this parameter), then it is natural to require the manipulator’s vote to be consistent with her position in the voter ordering. Under this assumption, if the number of voters is odd, then the median voter rule is trivially strategyproof: the median voter has no incentive to manipulate, and no other voter can change who the median voter is (and hence voters cannot change the election outcome).

A more permissive approach is to allow the manipulator to submit a vote that is not necessarily consistent with her original position; however, we still require the resulting profile to remain single-crossing (i.e., the profile should become single-crossing with respect to the given ordering once the manipulator moves to an appropriate position in the voter ordering). The median voter rule (for n odd) remains strategyproof in this case: while a voter can now change who the median voter is, no such change can be beneficial to her; the argument is similar to the one for single-peaked preferences. To extend the analysis beyond the median voter rule, [Saporiti \(2009\)](#) considers the following model. Let P be a maximal single-crossing profile, i.e., if we add to P a linear order that is not present in P , then P stops being single-crossing. Let S be the set of linear orders present in P . Then we consider voting rules defined on the domain $\mathcal{D} = S^n$ of n -voter profiles where each voter’s preference is taken from S . Further, let $T \subseteq A$ be the set of alternatives that are ranked in the top position by at least one order in S . [Saporiti \(2009\)](#) proves that a voting rule f is anonymous, unanimous, and strategyproof if and only if f is a generalized median rule (as defined in the statement of [Theorem 3.7](#)), where each phantom voter α_i votes for an alternative in T .

Counting In the case of single-peaked preferences, we have seen that a profile can be single-peaked with respect to exponentially many different axes. In contrast, a profile can only be single-crossing with respect to at most two essentially different orderings of the voters.

Proposition 3.10 (see, e.g., [Elkind and Faliszewski, 2014](#)). *Suppose $P = (v_1, \dots, v_n)$ is single-crossing with respect to the given ordering, and all linear orders appearing in P are pairwise distinct. Then the ordering of voters making P single-crossing is unique up to reversal.*

Proof. Note first that if a profile $P = (v_1, \dots, v_n)$ is single-crossing with respect to the given ordering, then so is the reversed profile (v_n, \dots, v_1) .

We show that every other permutation of P results in some pairs of alternatives crossing at least twice. We proceed by induction on n . The case $n \leq 2$ is trivial. Now, suppose $n \geq 3$. Since $v_1 \neq v_2$, there is a pair a, b of alternatives such that $a \succ_1 b$, but $b \succ_2 a$. As P is single-crossing with respect to (v_1, \dots, v_n) , it follows that $b \succ_j a$ for all $j \neq 1$. Thus, voter 1 must be at one end of any single-crossing ordering of P .

Now, the profile $P' = (v_2, \dots, v_n)$ is also single-crossing with respect to the given ordering. By the inductive hypothesis, the only other ordering making P' single-crossing is (v_n, \dots, v_2) . Thus, we need to show that profiles (v_2, \dots, v_n, v_1) and (v_1, v_n, \dots, v_2) are not single-crossing with respect to the given ordering. Since $v_2 \neq v_n$, there is a pair c, d of alternatives such

that $c \succ_2 d$, but $d \succ_n c$. Since P is single-crossing, we must have $c \succ_1 d$. But then both in (v_2, \dots, v_n, v_1) and in (v_1, v_n, \dots, v_2) alternatives c and d cross at least twice, as desired. \square

If the preference orders in a profile are not pairwise distinct, then there are more than two admissible orderings, because we can swap positions of identical votes. However, [Proposition 3.10](#) implies that this is the only freedom we have.

Another difference between single-peaked and single-crossing preferences is how ‘permissive’ these preference restrictions are. Recall that a single-peaked profile can contain up to 2^{m-1} different preference orders. In contrast, a single-crossing profile contains at most quadratically many different votes.

Proposition 3.11. *A single-crossing profile contains at most $\binom{m}{2} + 1$ distinct votes.*

Proof. Consider a single-crossing profile $P = (v_1, \dots, v_n)$ in which all votes are pairwise distinct, and a vote v_i in P . As v_i is different from its successor, there must be a pair of alternatives a, b with $a \succ_i b$ but $b \succ_{i+1} a$. By the single-crossing property, v_i is the *last* vote in P to rank a above b . Thus, we can label each of the first $n - 1$ votes in P with a distinct pair of alternatives, so $n - 1 \leq \binom{m}{2}$. \square

Example 9: *A single-crossing profile with $\binom{m}{2} + 1$ distinct votes.*

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
a	b	b	b	b	c	c	c	d	d	e
b	a	c	c	c	b	d	d	c	e	d
c	c	a	d	d	d	b	e	e	c	c
d	d	d	a	e	e	e	b	b	b	b
e	e	e	e	a	a	a	a	a	a	a

For each $m \geq 1$, there is a single-crossing profile with $\binom{m}{2} + 1$ different votes. An example with $m = 5$ is shown. To construct such profiles, label v_1 so that $x_1 \succ_1 \dots \succ_1 x_m$. In the following $m - 1$ votes, alternative x_1 is moved down one rank at a time until it is at rank m . In the next $m - 2$ votes, alternative x_2 is moved down until it is at rank $m - 1$, and so on. See also [Bredereck et al. \(2013\)](#).

Comparison to Single-Peakedness Intuitively speaking, it seems that when a profile is single-crossing, this has implications on the structure of the alternative set as well: Alternatives that are highly-ranked by voters on the left seem like “left-wing” alternatives, and similarly for alternatives highly-ranked by voters towards the right. As the following example shows, there are single-crossing profiles that are not single-peaked. However, this implication *almost* holds: if a single-crossing profile is also *minimally rich* (meaning that every alternative is top-ranked by some voter; another term used for this constraint is ‘narcissistic’), then it is also single-peaked, as we will see in [Section 3.8](#).

Example 10: *A profile that is single-crossing, but not single-peaked.*

v_1	v_2	v_3
a	c	c
b	a	b
c	b	a

This profile is single-crossing with respect to the given ordering (v_1, v_2, v_3) . It is not single-peaked, however, since three different alternatives occur in bottom-most positions. (Further, as we will see in [Section 3.8](#), this profile cannot be single-peaked because b 's trajectory has a valley.)

The reverse implication does not hold either: there are single-peaked profiles that are not single-crossing.

Example 11: *A profile that is single-peaked, but not single-crossing.*

v_1	v_2	v_3	v_4
a	a	b	b
b	b	a	a
c	d	c	d
d	c	d	c

This profile is single-peaked with respect to $c \triangleleft b \triangleleft a \triangleleft d$. It is not single-crossing, however. For an ordering of voters to be single-crossing, we need

- v_1 and v_2 to be adjacent (because of $\{a, b\}$),
- v_3 and v_4 to be adjacent (because of $\{a, b\}$),
- v_1 and v_3 to be adjacent (because of $\{c, d\}$),
- v_2 and v_4 to be adjacent (because of $\{c, d\}$).

This is impossible.

The profile in [Example 11](#) is an important example of a profile that is not single-crossing. It will make another appearance in [Section 5](#) as a forbidden subprofile of the single-crossing domain.

Together, these examples show that the single-peaked and single-crossing conditions are logically independent. However, there are interesting domain restrictions that are strengthenings of both conditions, such as 1-Euclidean preferences ([Section 3.4](#)), the conjunction of the two conditions ([Section 3.8](#)), and top-monotonicity ([Barberà and Moreno, 2011](#)).

3.4 Euclidean Preferences

We now consider preference profiles that can be ‘embedded’ into d -dimensional Euclidean space. Precisely, a preference profile is d -Euclidean if we can assign every voter and every alternative a point in \mathbb{R}^d so that voters prefer those alternatives that are closer to them (according to the usual Euclidean metric ρ_d) to those that are further away. This characterization of preferences has intuitive appeal: considering \mathbb{R}^d as a continuous ‘policy space’, within which alternatives can vary along multiple dimensions, each voter is identified with an *ideal point* ([Bennett and Hays, 1960](#)). The best alternative for the voter is the one that minimizes the distance to the ideal policy. We could also think of a facility location problem, where a single facility needs to be placed somewhere on a plane, with each decision maker preferring the facility to be placed as close to them as possible ([Hotelling, 1929](#)).

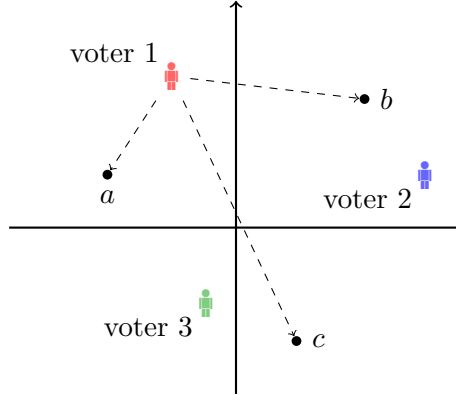


Figure 4: A 2-Euclidean embedding of the Condorcet profile.

Example 12: *2-Euclidean preferences*

Consider the 2-Euclidean profile illustrated in Figure 4. Both voters and alternatives are located in a two-dimensional plane. The (ordinal) preferences of voters are determined by their closeness to alternatives, based on the Euclidean distance ρ_2 . For example, for voter 1 we have $\rho_2(1, a) < \rho_2(1, b) < \rho_2(1, c)$. Hence, $a \succ_1 b \succ_1 c$.

v_1	v_2	v_3
a	b	c
b	c	a
c	a	b

Let $\|\cdot\|$ refer to the usual Euclidean ℓ_2 -norm on \mathbb{R}^d , that is

$$\|(x_1, \dots, x_d)\| = \|(x_1, \dots, x_d)\|_2 = \sqrt{x_1^2 + \dots + x_d^2}.$$

Definition 7. A profile P is *d-Euclidean* (where $d \geq 1$) if there is a map $x : N \cup A \rightarrow \mathbb{R}^d$ such that

$$a \succ_i b \iff \|x(i) - x(a)\| < \|x(i) - x(b)\| \quad \text{for all } i \in N \text{ and } a, b \in A.$$

Thus, voter i prefers those alternatives which are closer to i according to the embedding x .

This preference representation has been introduced by Coombs (1950, 1964) as *unidimensional unfolding* in the psychometrics literature (for the 1-dimensional case), and was later also studied in the multidimensional case (Bennett and Hays, 1960; Hays and Bennett, 1961). The extensive multidimensional unfolding literature tries to find an approximate d -Euclidean representation of a given preference profile, usually by minimizing a loss function using local search.

Majority Relation The Condorcet paradox profile is 2-Euclidean, as seen in Example 12. Therefore, being d -Euclidean does not guarantee a transitive majority relation or the existence of a Condorcet winner for every $d \geq 2$. (It does guarantee these properties for $d = 1$, as we discuss below.) In fact, Escoffier et al. (2022) prove that McGarvey’s theorem (McGarvey, 1953) holds for 2-Euclidean preferences, so that every possible majority tournament can be induced as the majority relation of a profile that is 2-Euclidean. Thus, this preference restriction does not impose any structure on the majority relation.

Choice of Metric Outside of facility location type problems, in many cases the space \mathbb{R}^d does not have a natural geometric interpretation, and hence the ℓ_2 distance is not necessarily a good modeling assumption: using other metrics on \mathbb{R}^d , such as ℓ_1 and ℓ_∞ , may also be sensible. For example, in politics, the d dimensions may encode the positions of a candidate on a variety of independent issues, and the best measure of a voter’s distance to a candidate may be the sum of distances along each individual dimension, captured by the ℓ_1 distance. For further discussion of the merits of using the ℓ_1 norm, see Eguia (2011) and the references therein. If voters are pessimistic, and focus on the issue where they have the most disagreement with a candidate, then the ℓ_∞ distance would be appropriate. However, preferences that are driven by these alternative metrics are not well-studied, and so for the rest of this survey we will focus on the usual ℓ_2 distance. Note that for the case of $d = 1$, all three of these metrics give the same definition, because ℓ_1 , ℓ_2 , and ℓ_∞ coincide on the line.

Number of Distinct Preferences Given an embedding of the set of alternatives in d -dimensional space, how many different preference rankings are Euclidean? The answer is $O(m^{2d})$ (Jamieson and Nowak, 2011), which one can determine by counting the number of cells in a hyperplane arrangement (Halperin and Sharir, 2017, Section 28.1.1). For two dimensions, this means that there are up to $\Theta(m^4)$ different preference orders compatible with a given embedding of the alternatives (Bennett and Hays, 1960). For the two-dimensional case, the same bound also applies under the ℓ_1 and ℓ_∞ metrics (Escoffier et al., 2024a, Theorem 6.1).

Sufficient Dimension Every preference profile is d -Euclidean for a sufficiently large dimension d . In particular, it is easy to see that every profile over m alternatives is $(m - 1)$ -Euclidean. For example, we can place the alternatives in the vertices of the standard $(m - 1)$ -dimensional simplex; then, by placing the voters at appropriate points within the simplex, we can induce all orders over the m alternatives.

For given values of n and m , Bogomolnaia and Laslier (2007) ask which dimension $d = d(n, m)$ is sufficient to guarantee that all profiles with m alternatives and n voters are d -Euclidean. By the argument above, we know that $d(n, m) \leq m - 1$. Another simple argument establishes that $d(n, m) \leq n$.

Proposition 3.12 (Bogomolnaia and Laslier, 2007). *Every profile with n voters is n -Euclidean.*

Proof. We give a Euclidean embedding $x : N \cup A \rightarrow \mathbb{R}^n$. Each voter’s preferences are encoded on a separate axis. For $i \in N$ and $a \in A$, we set $x(a)_i = -|\{b \in A : b \succ_i a\}|$. Thus, on axis i , voter i ’s top-ranked alternative has coordinate -1 , the second-ranked alternative has coordinate -2 , and so on.

Next, for some fixed $M > 0$, and for all $i \in N$, we define $x(i)_i = M$ and $x(i)_j = 0$ for $j \neq i$. Then it is easy to check that, for sufficiently large M , x gives an n -Euclidean embedding of the profile. \square

To obtain a lower bound, we need examples of profiles that are not embeddable for a given dimension d . Bogomolnaia and Laslier (2007) show that a generalization of the Condorcet profile gives such an example.

Proof. P is single-crossing since all voters i with $x(i) < (x(a) + x(b))/2$ have $a \succ_i b$, and all voters i with $x(i) > (x(a) + x(b))/2$ have $b \succ_i a$.

Define an axis \triangleleft by setting $a \triangleleft b$ if and only if $x(a) < x(b)$. We claim that P is single-peaked with respect to \triangleleft . To see this, note that for each $i \in N$ and each $c \in A$, the set $\{a \in C : a \succ_i c\}$ equals $\{a \in C : x(a) \in I\}$ for some interval $I \subseteq \mathbb{R}$ of real numbers. Thus $\{a \in C : a \succ_i c\}$ is an interval of \triangleleft as well, and hence P is single-peaked with respect to \triangleleft by [Proposition 3.3](#) (4). \square

One might think that the conjunction of the single-peaked condition and the single-crossing condition is sufficient to guarantee the existence of a 1-Euclidean embedding, i.e., that the converse of [Proposition 3.13](#) is true. However, this is not the case: there are profiles that are single-peaked and single-crossing but not 1-Euclidean, as the following example shows. Thus, the 1-Euclidean domain imposes additional geometric constraints beyond the combinatorial information in the single-peaked axis and the single-crossing voter ordering.

Example 14: *A profile that is single-peaked and single-crossing, but not 1-Euclidean*

v_1	v_2	v_3
b	d	d
c	e	e
d	c	f
e	b	c
a	a	b
f	f	a

This profile is single-peaked with respect to the axis $a \triangleleft b \triangleleft c \triangleleft d \triangleleft e \triangleleft f$ (and its reverse), and single-crossing with respect to the given ordering of the voters. Moreover, it can be verified that it is not single-peaked with respect to any other axes (see, e.g., [Section 4.1](#)), so if there exists an embedding x witnessing that this profile is 1-Euclidean, it has to be the case that $x(a) < x(b) < \dots < x(f)$ or $x(f) < x(e) < \dots < x(a)$; without loss of generality, we assume the former.

Since voter 1 prefers e to a and b to c , we have

$$(x(a) + x(e))/2 < x(1) < (x(b) + x(c))/2.$$

Considering the preferences of voters 2 and 3, we obtain

$$(x(c) + x(d))/2 < x(2) < (x(a) + x(f))/2$$

$$(x(b) + x(f))/2 < x(3) < (x(d) + x(e))/2.$$

Adding these inequalities yields $x(a) + x(b) + x(c) + x(d) + x(e) + x(f) < x(a) + x(b) + x(c) + x(d) + x(e) + x(f)$, a contradiction.

The profile shown in [Example 14](#) is minimal in the sense that there are no examples with two voters and six alternatives, and no examples with three voters and five alternatives. Indeed, for five or fewer alternatives, [Chen and Grottko \(2021\)](#) prove that a profile is 1-Euclidean if and only if it is single-peaked and single-crossing. In addition, they show that every single-peaked profile with two voters is 1-Euclidean.

3.5 Preferences Single-Peaked on a Tree

[Demange \(1982\)](#) observed that some of the good properties of the single-peaked domain can be preserved even when we allow the underlying axis to assume a more complicated shape. Specifically, Demange defined a notion of being single-peaked on a *tree*.¹

¹One can also define single-peakedness on other graphs, such as on cycles (see [Section 3.6](#)) and on general graphs (see, e.g., [Nehring and Puppe, 2007](#)).

Definition 8 (Demange, 1982). Let $T = (A, E)$ be a tree with vertex set A . A linear order v_i over A is *single-peaked on T* if for every pair of distinct alternatives $a, b \in A$ such that a lies on the unique $\text{top}(v_i)$ – b path in T we have $a \succ_i b$. A profile P over A is *single-peaked on T* if every vote in P is single-peaked on T . A profile P is *single-peaked on a tree* if there exists a tree T such that P is single-peaked on T .

Thus, a profile is single-peaked (in the sense of Section 3.2) if and only if it is single-peaked on a tree that is a path. In particular, the property of being single-peaked on a tree is less demanding than the property of being single-peaked.

One of the motivating examples for the single-peaked domain is the problem of locating a facility along a line (such as a road) when voters prefer the facility to be close to them. When the road network has a tree structure, we can expect the voters' preferences to be single-peaked on a tree.

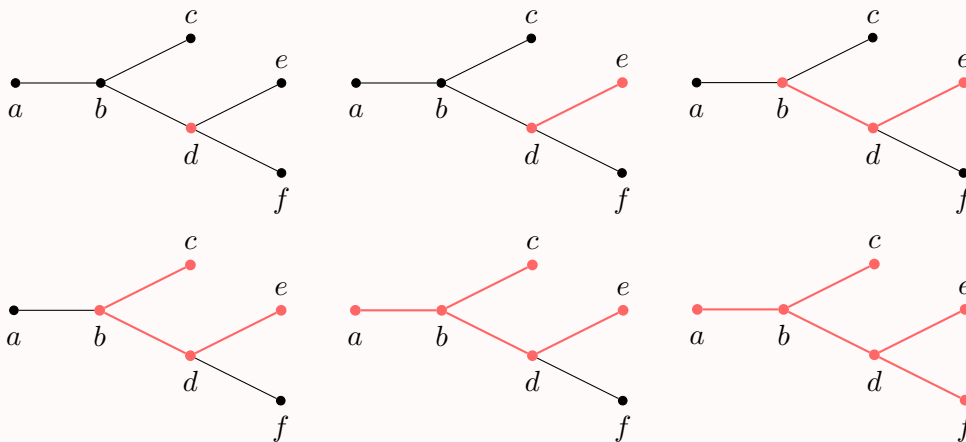
Just like we did for single-peaked preferences, we can phrase the definition of being single-peaked on a tree in several different ways. In the following proposition, definition (2) is the one used by Demange (1982), and definition (4) is the one used by Trick (1989).

Proposition 3.14. Let $T = (A, E)$ be a tree. The following are equivalent:

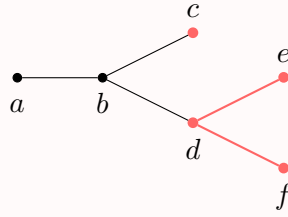
- (1) P is single-peaked on T .
- (2) For every subtree $T' \subseteq T$ which is a path, $P|_{T'}$ is single-peaked.
- (3) For each $i \in N$, and all $a, b, c \in A$ such that a, b, c lie on a common path in T , we do not have both $a \succ_i b$ and $c \succ_i b$.
- (4) For each $i \in N$ and each $c \in A$, the set $\{a \in C : a \succ_i c\}$ is connected in T .

Example 15: A vote single-peaked on a tree

The vote *debcaf* is single-peaked on the below tree, as we can verify by using condition (4) of Proposition 3.14, i.e., by checking that the following sets are all connected in the tree: $\{d\}$, $\{d, e\}$, $\{d, e, b\}$, $\{d, e, b, c\}$, $\{d, e, b, c, a\}$, and $\{d, e, b, c, a, f\}$.

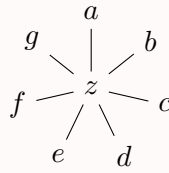


On the other hand, the vote *defcba* is not single-peaked on this tree, because the set $\{d, e, f, c\}$ of the top 4 alternatives is not connected in the tree.



Example 16: *Profiles single-peaked on a star*

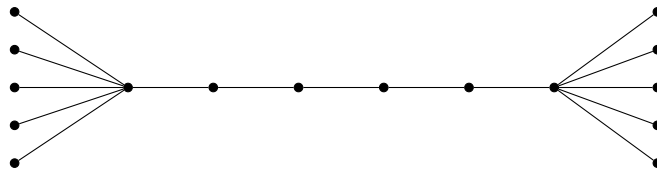
Consider the star T with center alternative z and leaf alternatives a, \dots, g . Which orders are single-peaked on T ?



Consider a voter whose preferences are single-peaked on this tree. If her ranking begins with z , she can rank the other alternatives in an arbitrary order: any such ranking is single-peaked on T . But suppose her top choice is a leaf alternative such as a . Then z must be the second alternative in her ranking, because the set consisting of her top two choices must be connected in T ([Proposition 3.14](#) (4)). After ranking a and z , the voter can order the remaining alternatives arbitrarily.

Thus, precisely the orders in which the center vertex is ranked first or second are single-peaked on a star. Hence, there are $2(m-1)! = \Theta(m!)$ orders single-peaked on a star—many more than the $\Theta(2^m)$ orders that are single-peaked on a line.

Allowing trees as the underlying structure of the alternatives also enables us to handle certain configurations that are ‘almost’ single-peaked on a line. For example, in politics, while the traditional left-to-right spectrum of political parties has substantial explanatory power, this classification can break down, especially at the extremes. For instance, it can be difficult to say whether a party focused on the environment is more left-wing than a party focused on women’s rights, or vice versa. The following kind of tree might capture the situation better:



With this tree, voters are free to order the extreme alternatives as they like, but there is still a noticeable left-to-right ordering over the more moderate choices.

Majority Relation We have seen that the single-peaked domain is attractive in that, for an odd number of voters, it guarantees a Condorcet winner, and, moreover, a transitive majority relation. The generalization to trees preserves the first of these guarantees.

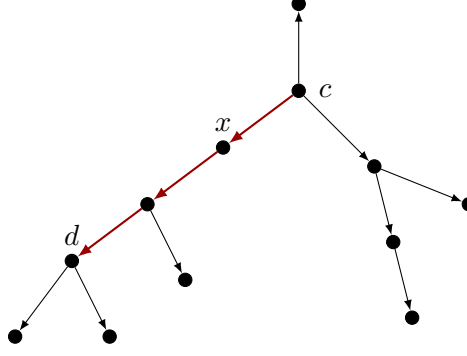


Figure 6: Illustration of the proof of [Proposition 3.15](#). The edges of the tree T are oriented according to the majority relation. The source vertex of the resulting directed acyclic graph can be shown to be a Condorcet winner.

Proposition 3.15 ([Demange, 1982](#)). *Every profile P with an odd number of voters that is single-peaked on a tree has a Condorcet winner.*

Proof. Fix a profile P that is single-peaked on a tree T , and turn T into a directed graph by orienting its edges according to the majority relation, so that there is an arc $a \rightarrow b$ if and only if $\{a, b\}$ is an edge of T and $a \succ_{\text{maj}} b$ in P . See [Figure 6](#) for an illustration. Since T is a tree, the resulting directed graph D is acyclic. Hence it must have a source vertex $c \in A$, i.e., a vertex that dominates all of its neighbors. We claim that c is a Condorcet winner. Indeed, consider an alternative $d \in A \setminus \{c\}$. If d is a neighbor of c , then $c \succ_{\text{maj}} d$ since c is a source. Otherwise, consider the (unique) path $c - x - \dots - d$ between c and d in T . Since c is a source, we have $c \succ_{\text{maj}} x$, and so there is a strict majority $N' \subseteq N$ of voters who all prefer c to x . Consider a voter $i \in N'$. Since her preferences remain single-peaked when restricted to the path $c - x - \dots - d$, by the no-valley property she prefers x to d , and therefore, by transitivity, $c \succ_i d$. Hence $c \succ_{\text{maj}} d$. \square

Interestingly, to determine which alternative is the Condorcet winner, it suffices to know the plurality scores of all the alternatives (i.e., the number of voter ranking it top), since this information is enough to decide how to orient the edges in the tree according to the majority relation ([Xu, 2024](#)). Indeed, for an edge $\{a, b\}$ of the tree T , if we delete the edge, we obtain a graph with two connected components. We can compute the total plurality score of the alternatives in each component. Then $a \succ_{\text{maj}} b$ if and only if the total plurality score in the component including a is strictly higher than the total plurality score of the other component.

A slight modification of this proof shows that profiles that are single-peaked on a tree and have an even number of voters still have a weak Condorcet winner ([Demange, 1982](#)). Further, this preference domain is maximal in guaranteeing the existence of a Condorcet winner, in the following sense. Take any tree $T = (A, E)$ and let L be the set of all linear orders that are single-peaked on T . [Demange \(1982\)](#) shows that if we add any linear order to L , then there exists a profile that consists of preferences drawn from this enlarged set and does not have a Condorcet winner.

However, if a profile is single-peaked on a tree, its majority relation may fail to be transitive (unlike for profiles that are single-peaked on a path). Examples are easy to construct, for instance using a star: recall that single-peakedness on a star is not very restrictive ([Example 16](#)). In fact, one can show that paths are the only trees that guarantee a transitive majority relation.

Proposition 3.16 (Demange, 1982). *No tree other than a path guarantees a transitive majority relation.*

Proof. We have seen in Corollary 3.5 that paths do guarantee a transitive majority relation. Suppose $T = (A, E)$ is a tree that is not a path, so that there is some vertex $x \in A$ with at least three neighbors. Suppose $a, b, c \in A$ are three distinct neighbors of x in T . Then consider the following profile:

v_1	v_2	v_3
x	x	x
a	b	c
b	c	a
c	a	b
\vdots	\vdots	\vdots

Assume that the votes have been completed so as to be single-peaked on T . The resulting profile has x as the Condorcet winner, but contains a Condorcet cycle $a \succ_{\text{maj}} b \succ_{\text{maj}} c \succ_{\text{maj}} a$ below x . \square

Similarly to Proposition 3.6, one can show that if P is single-peaked on T and its majority relation \succ_{maj} forms a linear order, then \succ_{maj} is also single-peaked on T . Indeed, a vote is single-peaked on T if and only if it is single-peaked on every path in T , and we can apply Proposition 3.6 to each such path.

Strategyproof Voting Rules Danilov (1994) characterizes the class of strategy-proof voting rules for preferences single-peaked on a specific tree T . Like in Moulin’s (1980) characterization for the single-peaked case, any such rule is a median of voters’ peaks and ‘phantom’ peaks, for an appropriate definition of ‘median’. Peters et al. (2021) generalize this result to randomized rules and to more general graphs.

Counting As we have seen, there are $\Theta(m!)$ different rankings that are single-peaked on a star, but only $\Theta(2^m)$ rankings that are single-peaked on a path. Conversely, since a profile can be single-peaked on exponentially many different axes, it can also be single-peaked on exponentially many different trees. The collection of all such trees admits a concise representation, as we will discuss in Section 4.4.

In our discussion of single-peaked preferences, we saw that every single-peaked profile is single-peaked with respect to at least two axes, since reversing an axis preserves single-peakedness. However, from a graph-based perspective, ‘reversing’ the vertices of a path does not change the graph $T = (A, E)$. Thus, it is possible for a profile to be single-peaked on a unique tree. In particular, Trick (1989) has shown that every narcissistic profile that is single-peaked on a tree is single-peaked on a unique tree.

Sampling Sliwinski and Elkind (2019) give an efficient algorithm that given a fixed tree T , samples a preference single-peaked on T uniformly at random. The algorithm can be seen as a generalization of the idea behind the algorithm of Conitzer (2009) for single-peaked preferences, except that it samples the vote’s peak with probabilities carefully chosen to ensure that preferences are sampled uniformly at random.

Hereditariness It is clear from the definition that the domain of profiles single-peaked on a tree is closed under voter deletion. However, as the following example shows, it is not closed under deleting alternatives. Thus, this domain is not hereditary.

Example 17: *The domain of profiles single-peaked on trees is not closed under alternative deletion*

v_1	v_2	v_3	This profile is single-peaked on a star with center x . However, deleting alternative x results in a profile that does not have a Condorcet winner, so it cannot be single-peaked on a tree.
x	x	x	
a	b	c	
b	c	a	
c	a	b	

While the domain of preferences single-peaked on trees is not closed under alternative deletion, it has a weaker property: if a profile is single-peaked on some tree $T = (A, E)$, and some alternative $a \in A$ has degree 2 in T , then deleting a yields a profile that is again single-peaked on a tree (namely, we delete a from T and join the two neighbors of a by an edge). Similarly, we can delete leaves (vertices of degree 1) from T .

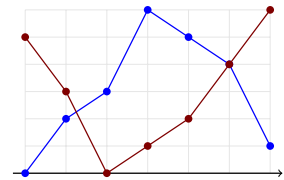
An interesting direction for future work is to consider the domain of profiles that are single-peaked on a tree, and, moreover, remain single-peaked on some tree, no matter which (and how many) alternatives we delete.

3.6 Preferences Single-Peaked on a Circle

In the previous section, we explored the consequences of extending the notion of single-peaked preferences from paths to trees. Another class of graphs that are, in some sense, similar to paths are cycles. Motivated by this intuition, [Peters and Lackner \(2020\)](#) introduced the domain of preferences single-peaked on a circle. In such profiles, the alternatives can be arranged in a circle, and each voter “cuts” the circle at some point, obtaining a line, so that the voter’s preferences are then single-peaked on this line. Note that different voters may cut the circle at different points. In our discussion, we follow the exposition of [Peters and Lackner \(2020\)](#).

Motivating Examples In some cases, the alternative set comes with an obvious cyclic structure, for example, when deciding on a recurring meeting time (with weekdays forming a cycle), or a time for a daily event (on a 24-hour cycle). In facility location, when locating a facility along the boundary of an area (such as a city), one can often view this boundary as a circle.

Another important subclass of preferences single-peaked on a circle is formed by mixtures of single-peaked and single-dipped preferences on a common axis. Such profiles are single-peaked on the circle obtained by gluing together the ends of the axis. Mixtures of this form occur naturally in facility location on a line, where voters may disagree whether the facility (e.g., a school or a hospital) is a good (and want it to be as close as possible to their own position) or a bad (and want it to be far away).

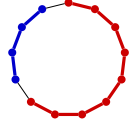


Definition We say that two axes \triangleleft and \triangleleft' are *cyclically equivalent* if there is a value $\ell \in [m]$ such that we can write $a_1 \triangleleft a_2 \triangleleft a_3 \triangleleft \dots \triangleleft a_m$ and $a_\ell \triangleleft' a_{\ell+1} \triangleleft' \dots \triangleleft' a_m \triangleleft' a_1 \triangleleft' \dots \triangleleft' a_{\ell-1}$.

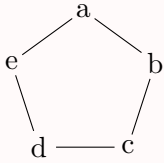
For an axis \triangleleft , we then define the *circle* $C(\triangleleft)$ of \triangleleft to be the set of axes cyclically equivalent to \triangleleft . Any set C of axes that can be written as $C = C(\triangleleft)$ for some \triangleleft is called a *circle*. For example, $C = \{a \triangleleft b \triangleleft c, b \triangleleft' c \triangleleft' a, c \triangleleft'' a \triangleleft'' b\}$ is a circle. Note that “cutting” a circle C at a point yields an axis $\triangleleft \in C$. We say that \triangleleft *starts at* $a \in A$ if $a \triangleleft b$ for all $b \in A \setminus \{a\}$.

Definition 9. Let C be a circle. A vote v_i is *single-peaked on C* if there is an axis $\triangleleft \in C$ such that v_i is single-peaked with respect to \triangleleft . A preference profile P is *single-peaked on a circle (SPOC)* if there exists a circle C such that every vote $v_i \in P$ is single-peaked on C .

Intuitively, a vote v_i is single-peaked on C if C can be cut so that v_i is single-peaked on the resulting line. Again, we can give an equivalent definition in terms of connected sets. An *interval* $I \subseteq A$ of a circle C is a set that is an interval of one of the axes $\triangleleft \in C$ of the circle. Then a vote v_i is single-peaked on a circle C if and only if each prefix $\{a \in A : a \succ_i c\}$ of v_i is an interval of C . Note that the complement $A \setminus I$ of an interval I of C is again an interval. Thus, a weak order \succ is single-peaked on C if and only if its reverse is also single-peaked on C .



Example 18: *Single-peaked preference orders on a circle with 5 alternatives*



Consider the circle with candidates a, b, c, d, e in that order. The eight preference orders that are single-peaked on that circle and that rank a on top are: $abcde, abced, abecd, abedc, aebcd, aebdc, aedbc, aedcb$. Hence there are $5 \times 8 = 40$ preference orderings single-peaked on this circle.

Majority Relation One can easily check that the Condorcet paradox profile $(x \succ_1 y \succ_1 z, y \succ_2 z \succ_2 x, z \succ_3 x \succ_3 y)$ on 3 alternatives is single-peaked on the circle $x \triangleleft y \triangleleft z$. Therefore, being single-peaked on a circle does not guarantee a transitive majority relation or the existence of a Condorcet winner. In fact, [Peters and Lackner \(2020\)](#) prove that McGarvey’s theorem ([McGarvey, 1953](#)) holds for preferences single-peaked on a circle: every possible majority tournament can be induced as the majority relation of a profile that is single-peaked on a circle. Thus, this preference restriction does not impose any structure on the majority relation. Notably, even if the majority relation is transitive, it need not be single-peaked on a circle itself, in contrast to the case of preferences single-peaked on lines and trees.

Strategyproof Social Choice We have seen that, for any given tree (e.g., a path), the domain of preferences single-peaked on that tree admits a non-trivial strategyproof voting rule. In contrast, the results of [Kim and Roush \(1980\)](#) and [Sato \(2010\)](#) show that there is no (resolute) voting rule defined on profiles single-peaked on a circle that is non-dictatorial, onto, and satisfies strategyproofness. In fact, they prove that this result holds for an even more restricted domain consisting only of the $2m$ orders which traverse the circle clockwise and counterclockwise starting from each possible alternative. Note that the orders used in this proof are heavily ‘directional’: if we associate each agent with a position on the circle, then for a ‘clockwise’ agent her top choice is the first alternative located clockwise from her and her bottom choice is the first alternative located counterclockwise from her (and similarly for ‘counterclockwise’ agents). Still, a similar dictatorship result can be proved for orders that are ‘Euclidean’ on a circle, where preferences decrease uniformly in both directions from the peak ([Schummer and Vohra, 2002](#)). It can also be shown that, with these Euclidean orders, the *random dictatorship* rule is *group-strategyproof*.

(Alon et al., 2010b), and there is an intriguing randomized mechanism that is strategyproof and provides a $3/2$ -approximation to the egalitarian social welfare (Alon et al., 2010a).

Counting and Sampling Given a fixed circle C , the number of preference orders single-peaked on C is $m \cdot 2^{m-2}$ (OEIS Foundation Inc., 2007). One can sample among these orders uniformly at random by using the analog of the Conitzer (2009) algorithm for single-peaked preferences: sample the peak uniformly at random, then repeatedly flip a coin to decide whether the next-most-preferred alternative should be adjacent to the set of already-ranked alternatives in clockwise or counterclockwise direction. This leads to a selection that is uniformly at random for the circular context (Boehmer et al., 2024).

Single-Crossing on a Circle Constantinescu and Wattenhofer (2022) give a definition of preferences that are single-crossing on a circle, which is equivalent to preferences that cross at most twice. Like for single-peaked on a circle, McGarvey’s theorem also holds for single-crossing on a circle. Constantinescu and Wattenhofer (2022) give a polynomial-time recognition algorithm for this domain, and propose polynomial-time algorithms for evaluating Young’s rule and the Chamberlin–Courant rule on this domain.

3.7 Preferences Single-Crossing on a Tree

The single-crossing domain also allows a generalization from the line to trees, as proposed by Clearwater et al. (2015) and Kung (2015). (In fact, Clearwater et al. (2015), and, subsequently, Puppe and Slinko (2019), also consider preferences that are single-crossing on *median graphs*, but we will not discuss this larger domain.) This restricted domain consists of profiles where voters correspond to the vertices of a tree, and for each pair of alternatives, the voters’ preferences cross along a single edge.

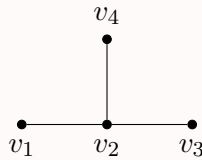
Definition 10. A profile P of linear orders is *single-crossing on the tree* $T = (N, E)$ if for each pair $a, b \in A$ of alternatives, the sets $\{i \in N : a \succ_i b\}$ and $\{i \in N : b \succ_i a\}$ are connected in T . A profile P is *single crossing on a tree* if there exists a tree T such that P is single-crossing on T .

Clearly, a profile is single-crossing in the sense of Section 3.3 if and only if it is single-crossing on a tree that is a path.

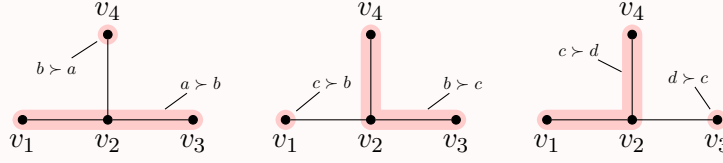
Example 19: A profile that is single-crossing on a tree, but not on a path

v_1	v_2	v_3	v_4
a	a	a	b
c	b	b	a
b	c	d	c
d	d	c	d

It can be checked that this profile is not single-crossing. However, it is single-crossing on the following tree:



Indeed, for each pair of alternatives, the relevant sets of voters are all connected:



The following observation shows that a profile is single-crossing on a tree if and only if it is single-crossing on every path in that tree; it can be seen as a partial analog of [Proposition 3.14](#).

Proposition 3.17. *A profile P is single-crossing on a tree T if and only if for every path Q in T the restriction of P to Q is single-crossing.*

Proof. Suppose that for some pair of alternatives $a, b \in A$ the set of voters who prefer a to b in T is not connected. Let i and j be two such voters, and let Q be the (unique) path in T that connects them. Then some voter on Q prefers j to i and hence the restriction of P to Q is not single-crossing.

Conversely, consider a path Q in T such that the restriction of P to that path is not single-crossing. Then there are two voters i and j on Q and a pair of alternatives $a, b \in A$ such that $a \succ_i b$, $a \succ_j b$, but there is a voter k that lies between i and j on Q such that $b \succ_k a$. Since T is a tree, one can only travel from i to j in T along Q , so the set of voters who prefer a to b is not connected in T . \square

Majority Relation For single-peakedness, we saw that, in generalizing from paths to trees, we lost transitivity of the majority relation, retaining only the existence of Condorcet winners. For single-crossingness, on the other hand, the move to trees preserves the transitivity guarantee. In fact, the Representative Voter Theorem still holds; in [Example 19](#), voter v_2 is the representative voter.

Theorem 3.18 ([Clearwater et al., 2014](#)). *Let P be a profile with an odd number of voters that is single-crossing on a tree $T = (N, E)$. Then the majority relation of P is transitive, and P contains a vote that coincides with the majority relation.*

Proof. For a pair $a, b \in A$ of alternatives, let $N_{ab} = \{i \in N : a \succ_i b\}$. Since P is single-crossing on T , each set N_{ab} is connected in T .

Write $\mathcal{M} = \{N_{ab} : a, b \in A \text{ such that } a \succ_{\text{maj}} b\}$. Note that any two sets $N_{ab}, N_{cd} \in \mathcal{M}$ must have non-empty intersection, since any two strict majorities intersect. Hence, by the Helly property of connected subsets of a tree (see, e.g., [Golumbic, 2004](#)), we must have $\bigcap_{a \succ_{\text{maj}} b} N_{ab} \neq \emptyset$. The voters in this intersection are representative voters. \square

Hereditarity It is easy to see that a profile single-crossing on a tree remains single-crossing on the same tree if we delete alternatives. However, the domain of preferences single-crossing on trees is not closed under deleting voters. In fact, [Clearwater et al. \(2014\)](#) show that if P is a profile single-crossing on a tree, then all subprofiles of P are single-crossing on a tree if and only if P is in fact single-crossing on a path.

Example 20: *The domain of preferences single-crossing on trees is not closed under voter deletion*

v_1	v_2	v_3	v_4
a	a	a	b
c	b	b	a
b	c	d	c
d	d	c	d

(on any tree) and hence the domain of preferences single-crossing on trees is not closed under voter deletion.

Revisiting the profile in [Example 19](#), we see that this profile is no longer single-crossing on a tree if we delete the central voter v_2 . To see this, notice that a three-voter profile is single-crossing on a tree if and only if it is single-crossing (because a tree with three vertices must be a path). Can we order v_1, v_3 and v_4 so that the resulting profile is single-crossing in the given order? In particular, which voter can be placed second? Not v_1 (because of b, c), not v_3 (because of c, d), and not v_4 (because of a, b), so the answer is ‘no’. We conclude that (v_1, v_3, v_4) is not single-crossing

3.8 Single-Peaked and Single-Crossing (SPSC) Preferences

Every 1-Euclidean profile is single-peaked and single-crossing, but the converse is not true when there are 6 or more alternatives ([Example 14](#), [Chen and Grottko, 2021](#)). Even though the 1-Euclidean domain does not coincide with the domain of preference profiles that are both single-peaked and single-crossing (SPSC for short), we can still view SPSC as a ‘combinatorial’ version of 1-Euclidean preferences. [Elkind et al. \(2020a\)](#) undertook the task of understanding more closely how the two concepts interact. A key role in their approach is played by so-called narcissistic profiles: A preference profile P over A is *narcissistic* if for every $c \in A$, there is a voter $i \in N$ with $\text{top}(i) = c$. The term ‘narcissistic’ is chosen because such profiles arise in situations where voters and alternatives coincide, and each voter ranks herself first; some authors use the term ‘minimally rich’ instead. We start with an interesting observation on the impact of narcissism on single-crossing profiles, which is discussed by [Puppe \(2018\)](#) and implicit in [Barberà and Moreno \(2011, Theorem 1\)](#).

Proposition 3.19. *A narcissistic single-crossing profile $P = (v_1, \dots, v_n)$ is single-peaked with respect to the axis $\triangleleft = \succ_1$ given by the preference order of the first voter.*

Proof. Suppose for the sake of contradiction that there are alternatives $a \succ_1 b \succ_1 c$ with respect to which voter $j > 1$ has a valley, i.e., $a \succ_j b$ and $c \succ_j b$. Because P is narcissistic, there is some voter i with $\text{top}(i) = b$, and hence $b \succ_i c$. Since P is single-crossing we must have $i < j$. But also $b \succ_i a$, and so, since P is single-crossing, we must have $i > j$. This is a contradiction. \square

However, the converse is not true: clearly, there are SPSC profiles that are not narcissistic. Indeed, we can start with a narcissistic SPSC profile and delete all voters whose top choice belongs to some non-empty subset of A . The resulting profile is not narcissistic, but it remains SPSC, because both the single-peaked domain and the single-crossing domain are closed under voter deletion. It turns out that all SPSC profiles can be generated in this way.

Theorem 3.20 ([Elkind et al., 2020a, Theorem 9](#)). *A preference profile is both single-peaked and single-crossing if and only if it can be obtained from a narcissistic single-crossing profile by deleting voters.*

Proof sketch. We already know one direction from [Proposition 3.19](#): every profile obtained by deleting voters from a narcissistic single-crossing profile is SPSC. The other direction can be

shown by a constructive argument: given an SPSC profile, we can extend it to a narcissistic one, while preserving the single-crossing property. \square

Elkind et al. (2020a, Theorem 10) also give an $O(m^2n)$ time algorithm that checks whether a given profile P is single-peaked and single-crossing, and, if so, returns a narcissistic single-crossing profile $P' \supseteq P$.

As an alternative characterization, Elkind et al. (2014, Corollary 12) show that a profile P is SPSC if and only if for every alternative $c \in A$, there is a vote v_i with $\text{top}(v_i) = c$ such that $P + v_i$ is single-crossing (for an appropriate reordering of the voters). Finally, Elkind et al. (2020a, Proposition 11) show that in an SPSC profile written in a single-crossing order, the ‘trajectories’ of all alternatives are single-peaked, in the sense that if $a = \text{top}(v_k)$ for some $k \in [n]$ then for all $i, j \in [n]$ such that $1 \leq i < j \leq k$ or $k \leq j < i \leq n$ we have $\text{rank}_i(a) \geq \text{rank}_j(a)$.

3.9 Multidimensional Single-Peaked Preferences

Single-peaked preferences are described in terms of ordering the alternatives along a single dimension—the axis. It is natural to ask whether this concept can be extended to higher dimensions in a way that preserves some of its desirable features. Barberà et al. (1993) were the first to introduce a multidimensional analog of the single-peaked domain. Their definition of multidimensional single-peaked preferences is based on the assumption that alternatives are points on a grid and every grid point is an alternative; thus, the alternative set has a very specific structure. Sui et al. (2013) define d -dimensional single-peaked preferences for a general model with any alternative set. Here, we consider their definition and a slight variant. The domain defined by Sui et al. (2013), which corresponds to condition (MD1) in the following definition, turns out not to be closed under alternative deletion. Our variant (MD2) is hereditary.

Definition 11. Let P be a profile over A and let $\vec{\triangleleft} = (\triangleleft_1, \dots, \triangleleft_d)$ be a d -tuple of linear orders over A . For three alternatives a, b, c , we write $a \vec{\triangleleft} b \vec{\triangleleft} c$ if b is between a and c on every axis, i.e., for every $k \in [d]$ it holds that either $a \triangleleft_k b \triangleleft_k c$ or $c \triangleleft_k b \triangleleft_k a$.

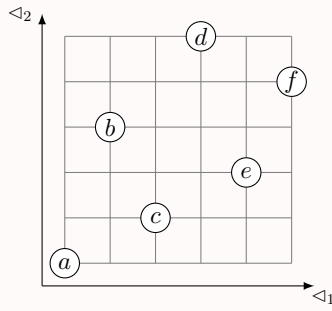
(MD1) Vote v_i over A is *d -dimensional single-peaked with respect to $\vec{\triangleleft}$* if for every pair of alternatives $a, b \in A$ with $a \vec{\triangleleft} b \vec{\triangleleft} \text{top}(v_i)$ it holds that $b \succ_i a$.

(MD2) Vote v_i over A is *d -dimensional hereditary single-peaked with respect to $\vec{\triangleleft}$* if for every triple of alternatives $a, b, c \in A$ with $a \vec{\triangleleft} b \vec{\triangleleft} c$ we do not have both $a \succ_i b$ and $c \succ_i b$.

A profile P over A is *d -dimensional (hereditary) single-peaked with respect to $\vec{\triangleleft}$* if every vote in P is d -dimensional (hereditary) single-peaked with respect to $\vec{\triangleleft}$, and it is *d -dimensional (hereditary) single-peaked* if there exists a d -tuple $\vec{\triangleleft}$ of linear orders over A such that P is d -dimensional (hereditary) single-peaked with respect to $\vec{\triangleleft}$.

It is easy to see that if P satisfies (MD2) then it also satisfies (MD1): the constraint in (MD1) is obtained by reformulating the constraint in (MD2) for triples of the form $(a, b, \text{top}(v_i))$ only.

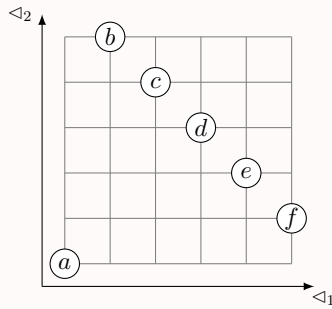
By Proposition 3.3, in the one-dimensional case the conditions (MD1) and (MD2) are equivalent, and indeed single-peakedness (according to Definition 4) is the same as one-dimensional (hereditary) single-peakedness.

Example 21: *Preferences that are two-dimensional single-peaked*

both $a = \text{top}(\succ)$ and e .

Take axes $a \triangleleft_1 b \triangleleft_1 c \triangleleft_1 d \triangleleft_1 e \triangleleft_1 f$ and $a \triangleleft_2 c \triangleleft_2 e \triangleleft_2 b \triangleleft_2 f \triangleleft_2 d$. Then the alternatives can be displayed on a grid as shown. The order $a \succ c \succ e \succ b \succ d \succ f$ satisfies both (MD1) and (MD2). The order $d \succ f \succ c \succ e \succ b \succ a$ satisfies (MD1), but it fails (MD2) because $c \vec{\triangleleft} e \vec{\triangleleft} f$ yet e is ranked below both c and f . (To convince yourself that e is between c and f in the sense that $c \vec{\triangleleft} e \vec{\triangleleft} f$, note that e is in the box spanned by corner points c and f). Finally, the order $a \succ b \succ e \succ d \succ c \succ f$ fails (MD1) (and hence also (MD2)), because $a \vec{\triangleleft} c \vec{\triangleleft} e$ yet c is ranked below

It is immediate that (MD2) is hereditary: removing an alternative x does not change the set of triples of alternatives $a, b, c \in A \setminus \{x\}$ such that $a \vec{\triangleleft} b \vec{\triangleleft} c$, and for each such triple the voters' preferences do not change either. However, for (MD1) this argument does not go through: if we remove an alternative x such that $x = \text{top}(v_i)$ for some voter i , in the new profile the constraint in (MD1) should now apply to triples of alternatives involving i 's top choice in $A \setminus \{x\}$. Indeed, the following example shows that (MD1) is not hereditary for $d \geq 2$, and thus (MD1) and (MD2) are not equivalent for $d \geq 2$.

Example 22: *The two-dimensional single-peaked domain defined by (MD1) is not hereditary*

Consider the profile P over $A = \{a, b, c, d, e, f\}$ that contains all 5! votes that rank a first. Consider axes $a \triangleleft_1 b \triangleleft_1 c \triangleleft_1 d \triangleleft_1 e \triangleleft_1 f$ and $a \triangleleft_2 f \triangleleft_2 e \triangleleft_2 d \triangleleft_2 c \triangleleft_2 b$. Then P satisfies (MD1), since for all $x, y \in A \setminus \{a\}$ it is not the case that $x \vec{\triangleleft} y \vec{\triangleleft} a$. Hence, this profile is two-dimensional single-peaked. However, if we remove candidate a from the profile, the resulting profile $P|_{A \setminus \{a\}}$ is no longer two-dimensional single-peaked. Indeed, this profile contains all possible orderings of alternatives in $A \setminus \{a\}$. Hence, for it to be two-dimensional single-peaked, there has to exist a 2-tuple of linear orders $\vec{\triangleleft} = (\triangleleft_1, \triangleleft_2)$ such that there is no triple of alternatives $x, y, z \in A \setminus \{a\}$ with $x \vec{\triangleleft} y \vec{\triangleleft} z$. However, it can be shown that for five alternatives no such 2-tuple exists. (That the restricted profile violates (MD1) also follows from the tightness claim of [Theorem 3.21](#).)

We observe that if a profile is d -dimensional (hereditary) single-peaked, then it is also d' -dimensional (hereditary) single-peaked for $d' > d$. Indeed, if a d' -tuple $\vec{\triangleleft}' = (\triangleleft_1, \dots, \triangleleft_d, \triangleleft_{d+1}, \dots, \triangleleft_{d'})$ is obtained from a d -tuple $\vec{\triangleleft} = (\triangleleft_1, \dots, \triangleleft_d)$ by adding additional axes, then $a \vec{\triangleleft}' b \vec{\triangleleft}' c$ implies $a \vec{\triangleleft} b \vec{\triangleleft} c$, i.e., adding axes can only reduce the number of triples of alternatives to which the constraints in (MD1) and (MD2) apply, making these constraints easier to satisfy. This suggests that, as d grows, the domain of d -dimensional single-peaked profiles becomes quite large. The following result confirms this intuition: it shows that every profile is d -dimensional (hereditary) single-peaked for sufficiently large d .²

Theorem 3.21. *Let $d \geq 1$. All profiles with at most $2^{2^{d-1}}$ alternatives are d -dimensional (hereditary) single-peaked. This bound is tight: The profile containing all linear orders over*

²We thank Jan Kyncl for pointing us to the results used in the proof.

$2^{2^{d-1}} + 1$ alternatives is not d -dimensional (hereditary) single-peaked.

Proof. Let $d \geq 1$ be fixed. We rely on a result by de Bruijn (published by Kruskal (1953), with further proof details provided by Alon et al. (1985)), stated here in a restricted form. We say that a sequence (x_1, \dots, x_m) with $x_j \in \mathbb{R}^d$, $j \in [m]$, is *monotonic* if it is component-wise monotonic, i.e., for all $k \in [d]$ it holds that $(x_{1,k}, \dots, x_{m,k})$ is either weakly increasing or weakly decreasing. The result states that every sequence (x_1, \dots, x_m) with $x_j \in \mathbb{R}^d$ and $m \geq 2^{2^d} + 1$ contains a monotonic subsequence of length 3. On the other hand, there exists a sequence of length 2^{2^d} that does not contain monotonic subsequences of length 3; Alon et al. (1985) explicitly construct a sequence with this property that additionally satisfies $x_{i,k} \neq x_{j,k}$ for all $k \in [d]$ and all $i, j \in [m]$ with $i \neq j$.

Let $m \leq 2^{2^{d-1}}$ and $A = [m]$. To prove that all profiles with m alternatives are d -dimensional (hereditary) single-peaked, we will construct a d -tuple of linear orders $\vec{\triangleleft} = (\triangleleft_1, \dots, \triangleleft_d)$ such that there is no triple of alternatives $a, b, c \in A$ with $a \vec{\triangleleft} b \vec{\triangleleft} c$; then the constraints in (MD1) and (MD2) are trivially satisfied.

To this end, let (x_1, \dots, x_m) be a sequence with $x_j \in \mathbb{R}^{d-1}$ that does not contain monotonic subsequences of length 3 and such that for each $k \in [d-1]$ all elements of $X_k = \{x_{1,k}, \dots, x_{m,k}\}$ are pairwise distinct. We transform this sequence into a d -tuple $\vec{\triangleleft}$ as follows. First, we take \triangleleft_1 to be $1 \triangleleft_1 2 \triangleleft_1 \dots \triangleleft_1 m$. Then, for each $k \in [d-1]$, we set $i \triangleleft_{k+1} j$ if $x_{i,k} < x_{j,k}$. Note that $i \triangleleft_{k+1} j \triangleleft_{k+1} \ell$ if the sequence $(x_{i,k}, x_{j,k}, x_{\ell,k})$ is increasing, and $\ell \triangleleft_{k+1} j \triangleleft_{k+1} i$ if the sequence $(x_{i,k}, x_{j,k}, x_{\ell,k})$ is decreasing. Let $\vec{\triangleleft} = (\triangleleft_1, \dots, \triangleleft_d)$. Then a triple of alternatives $i, j, \ell \in [m]$ satisfies $i \vec{\triangleleft} j \vec{\triangleleft} \ell$ if and only if (x_i, x_j, x_ℓ) is a monotone subsequence of length three in the input sequence, and such a subsequence does not exist.

Conversely, suppose that $m > 2^{2^{d-1}}$, and let P be the profile containing all possible votes over $[m]$. Furthermore, let $\vec{\triangleleft} = (\triangleleft_1, \dots, \triangleleft_d)$ be an arbitrary d -tuple of linear orders over $[m]$; by renaming the alternatives, we can assume that $1 \triangleleft_1 \dots \triangleleft_1 m$. We are going to identify a triple $i, j, \ell \in [m]$ with $i \vec{\triangleleft} j \vec{\triangleleft} \ell$; this shows that P is not d -dimensional (hereditary) single-peaked with respect to $\vec{\triangleleft}$, because P contains a vote of the form $i \succ \dots \succ j$. For this purpose we construct a sequence (x_1, \dots, x_m) of vectors in $[m]^{d-1}$ as follows. For each $k \in [d]$, we set $x_{i,k}$ to be the rank of i in \triangleleft_{k+1} . By de Bruijn's theorem, the sequence (x_1, \dots, x_m) contains a monotone subsequence (x_i, x_j, x_ℓ) . Further, we have $i \triangleleft_1 j \triangleleft_1 \ell$, and for each $k \in [d-1]$ we have $x_{i,k} < x_{j,k} < x_{\ell,k}$ if and only if $i \triangleleft_{k+1} j \triangleleft_{k+1} \ell$ and $x_{\ell,k} < x_{j,k} < x_{i,k}$ if and only if $\ell \triangleleft_{k+1} j \triangleleft_{k+1} i$. Thus, we have $i \vec{\triangleleft} j \vec{\triangleleft} \ell$. This concludes the proof. \square

Consequently, all profiles with 4 alternatives are 2-dimensional single-peaked, all profiles with up to 16 alternatives are 3-dimensional single-peaked, etc. This immediately implies that many nice properties of the 1-dimensional single-peaked domain do not extend to two or more dimensions. For example, since all profiles on 3 alternatives are 2-dimensional single-peaked, so is a profile that induces a cyclic majority relation, and hence transitivity of the majority relation cannot be guaranteed. The result might also partially explain the empirical findings of Sui et al. (2013), who show that in real-world election datasets, one can find 2-dimensional axes on which 47–65% voters are single-peaked (in the (MD1) sense); 2-dimensional single-peakedness is perhaps a less demanding condition than one might expect.

d -Euclidean Preferences and d -Dimensional Single-Peaked Preferences We know that every 1-Euclidean profile is single-peaked. It is thus natural to ask if this relationship holds in higher dimensions.

Suppose first that we have a d -Euclidean profile P where the set of voters coincides with the

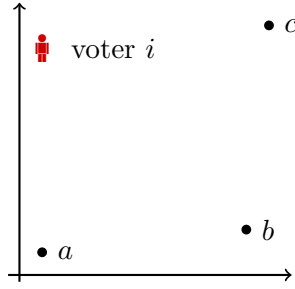


Figure 7: A difficulty in translating 2-Euclidean profiles to 2-dimensional single-peaked profiles.

set of candidates. Then the answer to our question is ‘yes’. Indeed, we can define the d axes $\triangleleft_1, \dots, \triangleleft_d$ in a natural way: given two candidates a and b , located at (x_1, \dots, x_d) and (y_1, \dots, y_d) , respectively, for each $k \in [d]$ we set $a \triangleleft_k b$ if and only if $x_k \leq y_k$. Let $\vec{\triangleleft} = (\triangleleft_1, \dots, \triangleleft_d)$. Denote the vote of candidate c by \succ_c . We have $c \succ_c a$ for all $a \in A \setminus \{c\}$. Further, $c \vec{\triangleleft} a \vec{\triangleleft} b$ implies that for each $k \in [d]$ the k -th coordinate of a lies between the k -th coordinate of c and the k -th coordinate of b , which means that c prefers a to b . Thus, P is d -dimensional single-peaked.

However, if the set of voters may differ from the set of candidates, this construction no longer works. Indeed, suppose that $d = 2$, candidate a is located at $(0, 0)$, candidate b is located at $(9, 1)$, and candidate c is located at $(10, 10)$ (compare Figure 7). If we define axes \triangleleft_1 and \triangleleft_2 as we did in the previous paragraph, we obtain $a \vec{\triangleleft} b \vec{\triangleleft} c$. However, a voter i with 2-Euclidean preferences who is located at $(0, 9)$ has a as her top choice and prefers c to b , so her preferences are not two-dimensional single-peaked with respect to $\vec{\triangleleft}$. Moreover, if this voter were to join the election as a candidate, her preferences would not be two-dimensional hereditary single-peaked with respect to $\vec{\triangleleft}$.

Of course, the example in the previous paragraph only shows that voter i ’s preferences fail to be two-dimensional single-peaked for a specific choice of axes; clearly, a single vote over three alternatives is always single-peaked with respect to a suitable axis. However, it is not clear whether every d -dimensional Euclidean profile admits a suitable collection of d axes.

3.10 Value Restriction

Since Black (1948) and Arrow (1951) introduced the single-peaked condition, social choice theorists looked for less restrictive conditions that still entailed transitivity of the majority relation. Sen (1966) proposed value restriction, and Sen and Pattanaik (1969) showed that value restriction is, in a sense we will discuss below, the largest preference domain that guarantees transitivity.

Definition 12. A profile P over A is *value-restricted* if for every triple $a, b, c \in A$ of alternatives, there is an element in $\{a, b, c\}$ that among the alternatives $\{a, b, c\}$ is either never ranked first, or never ranked second, or never ranked third in any vote $v_i \in P$.

This definition can be equivalently phrased in terms of a *forbidden subprofile*: a profile is value-restricted if and only if it does not contain the Condorcet profile as a subprofile. In Section 5 we will see similar characterizations of other domain restrictions.

Proposition 3.22. A profile P over A is value-restricted if and only if there do not exist alternatives $a, b, c \in A$ and voters $i, j, k \in N$ such that the restriction of P to these voters and alternatives forms a Condorcet profile, i.e.,

- $a \succ_i b \succ_i c$,
- $b \succ_j c \succ_j a$,
- $c \succ_k a \succ_k b$.

v_i	v_j	v_k
a	b	c
b	c	a
c	a	b

Proof. If there is a triple of alternatives a, b, c and a triple of voters i, j, k such that the restriction of P to these voters and alternatives forms a Condorcet profile, then clearly P is not value-restricted.

Conversely, suppose P is not value-restricted, as witnessed by some triple $a, b, c \in A$, and let P' be the restriction of P to $\{a, b, c\}$. If P' contains all six different linear orders over $\{a, b, c\}$ then in particular it contains a Condorcet profile. Thus, suppose some order, say, $a \succ b \succ c$, does not appear in P' . As P' is not value-restricted, it has to contain a linear order where a is ranked first, and it can only be $a \succ c \succ b$. Also, it has to contain a linear order where b is ranked second, and it can only be $c \succ b \succ a$. Finally, it has to contain a linear order where c is ranked last, and it can only be $b \succ a \succ c$. But then $a \succ c \succ b$, $c \succ b \succ a$ and $b \succ a \succ c$ form a Condorcet profile. \square

Majority Relation The defining feature of value-restricted profiles is that, because they exclude Condorcet subprofiles, they avoid cycles in the majority relation, as we will now formally show.

Proposition 3.23. *The majority relation of a value-restricted profile with an odd number of voters is transitive.*

Proof. Consider a profile P with an odd number of voters whose majority relation is not transitive, so that there are $a, b, c \in A$ with $a \succ_{\text{maj}} b \succ_{\text{maj}} c \succ_{\text{maj}} a$. We will show that P is not value-restricted. Indeed, for a majority of voters we have $a \succ b$, and for a majority of voters we have $b \succ c$; as any two strict majorities intersect, there must be a voter i with $a \succ_i b \succ_i c$. Similarly, there are majorities with $b \succ c$ and with $c \succ a$, so there is a voter j with $b \succ_j c \succ_j a$. And again, similarly, there is a voter k with $c \succ_k a \succ_k b$. Together, voters i, j , and k witness that P is not value-restricted. \square

Fix a set A of alternatives and a number n of voters. A domain \mathcal{D} of n -voter preference profiles over A is said to have *product structure* if there exists a subset L of the set of linear orders over A such that $\mathcal{D} = L^n$. Thus, a profile is part of \mathcal{D} if and only if each voter chooses their preference order from the set L . For example, one such domain is the domain of all n -voter profiles where each vote is single-peaked on a specific axis \triangleleft . Within the universe of domains with product structure, the value-restricted domain turns out to be the unique maximal domain that guarantees a transitive majority relation (for n odd).

Proposition 3.24. *Let \mathcal{D} be a domain of n -voter profiles that has product structure, where $n \geq 3$ is odd. If every profile in \mathcal{D} has a transitive majority relation, then every profile in \mathcal{D} is value-restricted.*

Proof. Since \mathcal{D} has product structure, we can write $\mathcal{D} = L^n$ for some set L of linear orders. Suppose for the sake of contradiction that there is some profile in \mathcal{D} that is not value-restricted. Then, by **Proposition 3.22**, L must contain three orders $\succ_i, \succ_j, \succ_k \in L$ such that $a \succ_i b \succ_i c$, $b \succ_j c \succ_j a$, and $c \succ_k a \succ_k b$. Since n is odd and $n \geq 3$, we can write $n = 2s + 3$ for some $s \geq 0$. Now, consider the profile P containing $s + 1$ copies of \succ_i , $s + 1$ copies of \succ_j , and one copy of \succ_k . Then $P \in \mathcal{D}$, but the majority relation of P satisfies $a \succ_{\text{maj}} b \succ_{\text{maj}} c \succ_{\text{maj}} a$, a contradiction. \square

The result of [Proposition 3.24](#) has led some authors to discount conditions such as single-peakedness as less interesting or useful, because value restriction allows strictly more profiles while retaining the guarantee of a transitive majority relation. This is a fair point if one is mainly interested in avoiding the impossibilities of Arrow and Gibbard–Satterthwaite, but value-restriction does not imply enough structure on the profile to be useful for many other applications, such as efficiently evaluating multi-winner rules.

It is easy to misinterpret [Propositions 3.23](#) and [3.24](#) as saying that a profile P is value-restricted if and only if it has a transitive majority relation. The following example shows that this is not true, and in particular the converse of [Proposition 3.23](#) does not hold. However, if a profile P has the property that its majority relation is transitive, and every way of changing the multiplicities (weights) of the preference orders occurring in P leads to a profile whose majority relation is still transitive, then P must be value-restricted by [Proposition 3.24](#).

Example 23: *A profile whose majority relation is transitive, but which is not value-restricted*

v_1	v_2	v_3	v_4	v_5	v_6
a	b	c	c	c	c
b	c	a	a	a	a
c	a	b	b	b	b

There are profiles that are not value-restricted, but still have a transitive majority relation: Here, the voters with $c \succ a \succ b$ ‘overwhelm’ the Condorcet cycle.

Three notions that are closely related to value restriction are sometimes discussed in the literature: best-restricted, medium-restricted, and worst-restricted profiles. Each of these conditions is stronger than being value-restricted, so trivially each best/medium/worst-restricted profile again admits a transitive majority relation.

Definition 13. A profile P over A is *best-/medium-/worst-restricted* if for every triple $a, b, c \in A$ of alternatives, there is an element in $\{a, b, c\}$ that is never ranked first/second/last, respectively, in the restriction of P to $\{a, b, c\}$.

Observe that, by the no-valley property, every single-peaked profile is worst-restricted. However, the converse is not true: the two-voter profile with votes $a \succ b \succ c \succ d$ and $d \succ b \succ c \succ a$ is worst-restricted (indeed, any two-voter profile is worst-restricted), but it is easy to see that it is not single-peaked. We will encounter this profile again in [Section 5](#), where we discuss forbidden subprofiles. Similarly, every single-caved profile is best-restricted, but the converse is not true.

Counting Value-restriction makes most sense in the context of domains $\mathcal{D} = L^n$ with product structure. In this case, sets L that induce value-restricted profiles are known as *Condorcet domains*, and they have been intensely studied; see [Puppe and Slinko \(2024\)](#) for a recent survey. It is particularly interesting to ask how big the set L can be: for a given m , what is the maximum number $f(m)$ of different orders that L can contain without including the forbidden subprofile? This question is combinatorially challenging and has inspired much work (see

m	$f(m)$	$f_{\text{SP}}(m)$	$f_{\text{SC}}(m)$
3	4	4	4
4	9	8	7
5	20	16	11
6	45	32	16
7	100	64	22
8	224	128	29

[Monjardet, 2009](#), for a survey). It is easy to see that $f(3) = 4$: the 6 possible linear orders over three profiles split into two Condorcet cycles, and we need to eliminate one order from each. It is also not hard to check that $f(4) = 9$. [Fishburn \(1996\)](#) showed that $f(5) = 20$ and later that

$f(6) = 45$ (Fishburn, 2002). Computing the values of $f(m)$ for $m \geq 7$ remains open. To prove the lower bounds, Fishburn (1996, 2002) introduced two schemes that produce large L -sets: the *alternating scheme* and the *replacement scheme*. The alternating scheme produces the unique maximum value-restricted sets for $m = 5$ and $m = 6$, but for $m \geq 16$, the replacement scheme is better. Fishburn conjectured that the alternating scheme is also optimal for $m = 7$, which would imply that $f(7) = 100$ which was confirmed by Galambos and Reiner (2008). However, for $m = 8$, the alternating scheme has size 222, while the largest domain contains 224 orders (Leedham-Green et al., 2024). In the table, we show the known values of $f(m)$ and contrast with the maximum number $f_{\text{SP}}(m) = 2^{m-1}$ of different orders in a single-peaked profile and with the maximum number $f_{\text{SC}}(m) = \binom{m}{2} + 1$ of different orders in a single-crossing profile.

3.11 Group-Separable Preferences

Group-separable preferences were introduced by Inada (1964, 1969). Intuitively, the voters' preferences are group-separable if every subset of at least two alternatives can be split into two groups, so that each voter ranks one group above the other (but voters may differ in which of the two groups they prefer). This idea is formalized as follows.

Definition 14. A profile P is *group separable* if for every set of alternatives $A' \subseteq A$ there is a proper subset $B \subset A'$ such that for every voter $i \in N$ we have either $B \succ_i (A' \setminus B)$ or $(A' \setminus B) \succ_i B$.

Example 24: *Group-separable preferences.*

v_1	v_2	v_3	v_4
Austin	Nice	Calgary	Bergen
Calgary	Madrid	Austin	Lisbon
Bergen	Lisbon	Nice	Madrid
Nice	Bergen	Madrid	Nice
Madrid	Austin	Lisbon	Calgary
Lisbon	Calgary	Bergen	Austin

In this instance, when comparing cities, voters first consider the continent they are located on, with some voters preferring Europe to North America, and some voters preferring North America to Europe. Within Europe, the voters distinguish between Southern Europe (Nice, Lisbon, and Madrid) and Northern Europe (Bergen). Within Northern America the voters distinguish between the USA and Canada, and within South-

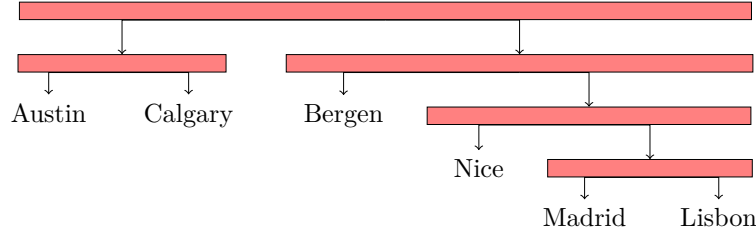
ern Europe they distinguish between France and the Iberic Peninsula; within the latter, they distinguish between Spain and Portugal.

Note that Austin, Bergen, Calgary and Nice are weak Condorcet winners, but Madrid and Lisbon lose to Nice in a pairwise comparison.

Hereditariness and Relationship to Other Domains It is immediate from the definition that the domain of group-separable preferences is closed under voter and alternative deletion. Moreover, group separability is related to another domain restriction we have discussed, namely the domain of medium-restricted preferences.

Proposition 3.25. *Every group-separable profile is medium-restricted.*

Proof. Suppose a profile P fails to be medium-restricted, i.e., there is a triple of alternatives a, b, c such that in the restriction of P to $A' = \{a, b, c\}$ each alternative is ranked second in at least one vote. Now suppose for the sake of contradiction that P is group-separable, and let

Figure 8: The clone tree decomposition of [Example 24](#).

$(B, A' \setminus B)$ be the associated partition of A' ; we can assume without loss of generality that $B = \{a, b\}$. But then no voter can rank c second among a, b , and c , a contradiction. \square

We note that the converse of [Proposition 3.25](#) is not true: a two-voter profile $a \succ_1 b \succ_1 c \succ_1 d$, $c \succ_2 a \succ_2 d \succ_2 b$ is medium-restricted, but not group-separable. This profile will appear again in [Section 5](#), where we characterize the group-separable domain in terms of forbidden minors.

Majority Relation Let us apply [Definition 14](#) to the entire set of alternatives A : let B be a proper subset of A such that some voters rank B above $A \setminus B$, while other voters rank $A \setminus B$ over B . We can assume that a weak majority of voters prefer B to $A \setminus B$. Thus, if B is a singleton, then the unique candidate in B is a weak Condorcet winner, and otherwise we can recursively partition B so as to eventually arrive at a weak Condorcet winner. As the domain of group-separable preferences is hereditary, we can remove the weak Condorcet winner and apply the same argument to the remaining alternatives; this establishes that the weak majority relation of a group-separable profile is transitive.

Characterization via Clone Tree Decomposition Group-separable profiles are conveniently described in terms of *clone sets*. We say that a set of alternatives $C \subseteq A$ is a *clone set with respect to a profile P* if $1 < |C| < |A|$ and for every voter i , every alternative $b \in A \setminus C$ and every pair of alternatives $c, c' \in C$ it is not the case that $c \succ_i b \succ_i c'$; that is, alternatives in C form a contiguous block in each voter's ranking. Observe that a clone set may be a subset of another clone set: e.g., in [Example 24](#) both $\{\text{Lisbon, Madrid}\}$ and $\{\text{Lisbon, Madrid, Nice}\}$ are clone sets.

The clone sets of a given profile can be succinctly described by its *clone decomposition tree* ([Elkind et al., 2012](#)). To define this concept formally, we need the machinery of PQ-trees ([Booth and Lueker, 1976](#)). A *PQ-tree* over a set of alternatives A is an ordered rooted tree whose leaves are labeled with elements of A so that each label is used exactly once (i.e., there is a bijection between A and the leaves of T) and whose internal nodes belong to one of the two types (P-nodes and Q-nodes). There are two types of permissible operations on a PQ-tree: we can either reorder the children of a P-node in an arbitrary way, or flip the order of the children of a Q-node. A PQ-tree T *generates* a ranking r of A if r can be obtained by performing zero or more permissible operations and then listing the leaves of the resulting ordered tree from left to right; let $R(T)$ denote the set of all rankings generated by T . Given a node x of a tree T over A , we let $L(x)$ denote the set of labels of all leaves of T that are descendants of x . Note that for each node x the alternatives in $L(x)$ form a contiguous block in every ranking in $R(T)$; in fact, this is also the case for any set $L(x_i) \cup \dots \cup L(x_j)$, where x_i, \dots, x_j are consecutive children of some Q-node of T .

We say that a PQ-tree T is a *clone tree decomposition* for a profile P over A if for each $B \subseteq A$ it holds that B is a clone set with respect to P if and only if $B = L(x)$ for some node x of T or

$B = L(x_i) \cup \dots \cup L(x_j)$, where x_i, \dots, x_j are consecutive children of some Q-node. Each profile admits a unique clone decomposition tree, which can be computed in polynomial time (Elkind et al., 2012). The clone tree decomposition of the profile in Example 24 is shown in Figure 8.

The definition of group-separable preferences can be rephrased in terms of clone sets: it says that for each $A' \subseteq A$ there is a proper subset $B \subset A'$ such that both B and $A' \setminus B$ are clone sets with respect to $P|_{A'}$. Building on this idea, Karpov (2019a) provides a characterization of group-separable profiles in terms of the properties of their clone decompositions trees: specifically, he establishes that a profile is group-separable if and only if all internal nodes of its clone decomposition tree are Q-nodes.

Counting Recall that a single-peaked profile can contain at most 2^{m-1} distinct rankings, whereas a single-crossing profile may contain at most $\binom{m}{2} + 1$ distinct rankings. Interestingly, the maximum number of distinct rankings in a group-separable profile over a set A is the same as in a single-peaked profile over A .

Proposition 3.26. *A group-separable profile over m alternatives may contain at most 2^{m-1} distinct rankings, and this bound is tight.*

Proof. For the upper bound, we proceed by induction on m . For $m = 2$, the profile that consists of two distinct rankings is group-separable. Now, suppose our claim has been established for $m' < m$. Consider a group-separable profile P over a set of alternatives A , $|A| = m$, such that all rankings in P are pairwise distinct. Since P is group-separable, there exists a set B , $1 \leq |B| \leq m - 1$ such that in each ranking in P either B appears above $A \setminus B$ or $A \setminus B$ appears above B . By construction, the profiles $P|_B$ and $P|_{A \setminus B}$ are group-separable. Thus, by the inductive hypothesis, $P|_B$ contains at most $2^{|B|-1}$ distinct rankings and $P|_{A \setminus B}$ contains at most $2^{m-|B|-1}$ distinct rankings. Now, note that each ranking in P is obtained by ‘stacking’ a ranking from $P|_B$ and a ranking from $P|_{A \setminus B}$ in one of the two possible ways. Hence, P contains at most $2 \cdot 2^{|B|-1} \cdot 2^{m-|B|-1} = 2^{m-1}$ distinct rankings.

To show that this bound is tight, we explain how to build a group-separable profile over m alternatives that contains 2^{m-1} distinct rankings. Again, we proceed by induction. The case $m = 2$ is immediate. Now, suppose we have built a group-separable profile $P_{\text{gs-max}}^{m-1}$ over $\{a_1, \dots, a_{m-1}\}$ that contains 2^{m-2} distinct rankings. For each vote v in $P_{\text{gs-max}}^{m-1}$, we construct two rankings over $\{a_1, \dots, a_m\}$: both rank the alternatives a_1, \dots, a_{m-1} in the same order as v does, but one ranks a_m first, while the other ranks a_m last. By construction, the resulting profile $P_{\text{gs-max}}^m$ contains 2^{m-1} distinct rankings.

To see that $P_{\text{gs-max}}^m$ is group-separable, consider a subset of alternatives $A' \subseteq \{a_1, \dots, a_m\}$ with $|A'| \geq 2$. We need to argue that there is a subset B , $1 \leq |B| \leq |A'| - 1$, such that each voter’s preferences are either of the form $B \succ A' \setminus B$ or of the form $A' \setminus B \succ B$. Now, if $a_m \in A'$, we can simply take $B = \{a_m\}$. On the other hand, if $a_m \notin A'$ then $A' \subseteq \{a_1, \dots, a_{m-1}\}$ and the existence of B follows from the assumption that $P_{\text{gs-max}}^{m-1}$ is group-separable. \square

The clone decomposition tree of the profile $P_{\text{gs-max}}^m$ is a caterpillar, i.e., a binary tree such that each internal node has one child that is a leaf. Each vote in this profile can be encoded by a binary string of length $m - 1$. Specifically, to generate a vote from a binary string (b_1, \dots, b_{m-1}) , we think of this vote as an m -by-1 array, with i -th entry containing the alternative ranked in the i -th position, and then place a_m, \dots, a_1 in that array, so that a_i is placed in the top-most available position if $b_i = 1$ and in the bottom-most available position if $b_i = 0$ (with a_1 being placed in the last available position). Yet another way to think about $P_{\text{gs-max}}^m$ is that there is a

one-to-one correspondence between votes in $P_{\text{gs-max}}^m$ and subsets of $A \setminus \{a_1\}$: a subset $B \subseteq A \setminus \{a_1\}$ corresponds to a vote in which all elements of B are ranked in top $|B|$ positions in decreasing order of indices, all elements of $(A \setminus \{a_1\}) \setminus B$ are ranked in the bottom $m - 1 - |B|$ positions in increasing order of indices, and a_1 is ranked in between in position $|B| + 1$. These observations show that $P_{\text{gs-max}}^m$ is neither single-peaked nor single-crossing for $m = 3$; also, they will be useful in [Section 6](#), where we use them to show that many computationally difficult problems remain hard if voters' preferences are group separable.

We note that [Karpov \(2019a\)](#) provides an exact formula for the number of group-separable profiles with n voters and m alternatives, as well as an expression for the number of such profiles that are narcissistic.

4 Recognition

domain	recognition	
single-peaked	$O(mn)$	Theorem 4.2
single-crossing	$O(nm \log m)$	Theorem 4.8
1-Euclidean	in P	via solving an LP, see Section 4.3
d -Euclidean, $d \geq 2$	$\exists \mathbb{R}$ -complete	Section 4.3
single-peaked on a tree	$O(mn)$	Theorem 4.19
single-peaked on a circle	$O(mn)$	Thm. 2 in Peters and Lackner (2020)
single-crossing on a tree	$O(m^2 n^3)$	Thm. 6 in Clearwater et al. (2014)
single-peaked and single-crossing	$O(nm \log m)$	Theorems 4.2 and 4.8
single-caved	$O(mn)$	Theorem 4.2
value/best/medium/worst-restricted	$O(m^3 n)$	Prop. 6 in Elkind and Lackner (2014)
group-separable	$O(mn)$	Section 4.7
d -dim. (hereditary) single-peaked	open	Open Problem 1

Table 1: The fastest known algorithms for recognizing domain restrictions for profiles with n voters and m alternatives.

Each of the many domain restrictions introduced in [Section 3](#) is associated with a natural algorithmic question: given a profile P , does P belong to this restricted domain? Such *recognition* problems have been studied extensively and, in many cases, complexity classifications and algorithms exist. Nonetheless, important questions remain open. In this section, we consider restricted domains listed in [Section 3](#), survey the known results for their corresponding recognition problems, and point out open problems along the way. For an overview, we refer the reader to [Table 1](#).

For computational purposes, the appeal of the restricted domains is that many popular voting rules that are computationally hard in general admit efficient winner determination algorithms when their inputs are drawn from such domains (see [Section 6](#)). Many of these algorithms require a certificate that the input profile is structured; such a certificate can be provided, e.g., by an axis \triangleleft that makes the input single-peaked, or an ordering of voters that makes it single-crossing. This is another reason why we are interested in recognition algorithms: without them, it would be impossible to make use of specialized winner determination procedures.

From this perspective, it is important to have recognition algorithms that are as fast as possible. Indeed, the cost of a slow recognition algorithm may outweigh the benefits of a dedicated fast winner determination procedure, so users might opt instead to pass their problem directly to a super-polynomial algorithm, such as an integer linear programming solver. Thus, in this section we will pay particular attention to the running times of the algorithms we consider. Nevertheless, we will also mention several slower algorithms, since they offer different perspectives, and provide a deeper understanding of the respective domains. In particular, some (potentially slower) algorithms are *certifying*, i.e., they return a short certificate in case the input profile *does not* belong to the target domain.

4.1 Algorithms for Single-Peaked Preferences

We will present two approaches to recognizing single-peaked profiles. Historically, the first algorithm to solve the recognition problem uses a simple reduction to the *consecutive ones*

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Figure 9: A matrix with the consecutive ones property.

problem which was put forward by Bartholdi III and Trick (1986). This approach leads to an $O(m^2n)$ runtime. Later, faster runtimes were achieved by direct algorithms, which as a by-product also offer insight into the nature of single-peaked preferences. In particular, an algorithm by Doignon and Falmagne (1994) runs in $O(mn + m^2)$ time, which was further improved to $O(mn)$ time by Escoffier et al. (2008). Another linear-time algorithm, which is also applicable to certain profiles of weak orders, was subsequently described by Lackner (2014).

The Consecutive Ones Approach

Let us consider a binary matrix, i.e., a matrix consisting of zeros and ones. We say that this matrix possesses the *consecutive ones property* if its columns can be permuted so that in each row all ones appear consecutively; see Figure 9 for an example. This property gives rise to the following algorithmic problem.

CONSECUTIVE ONES	
Instance:	An $n \times m$ matrix M with $M(i, j) \in \{0, 1\}$
Question:	Does M have the consecutive ones property, i.e., is there a permutation of the columns of M such that in every row, all 1-entries appear consecutively?

The consecutive ones property was first defined by Fulkerson and Gross (1965), who also described an $O(m^2n)$ time algorithm for solving CONSECUTIVE ONES. Later, Booth and Lueker (1976) introduced the PQ-tree data structure, which enabled them to prove the following result.

Theorem 4.1 (Booth and Lueker, 1976). CONSECUTIVE ONES can be solved in time $O(m + n + f)$, where $f = \sum_{i,j} M(i, j)$ is the number of 1s occurring in M .

In the applications that we consider, the matrices are dense, so the time bound boils down to $O(mn)$. In addition to allowing for a linear-time algorithm, PQ-trees also have the property that they generate all permutations that witness the consecutive ones property. Subsequent work has improved this result in various ways (Meidanis et al., 1998; Habib et al., 2000; McConnell, 2004), and a survey of the literature on the consecutive ones problem is provided by Dom (2009).

We will now explain how to reduce the problem of recognizing single-peaked preference profiles to the consecutive ones problem; our exposition follows Bartholdi III and Trick (1986). As shown in Proposition 3.3, a vote v_i is single-peaked with respect to an order \triangleleft if and only if for each $c \in A$, the set $\{a \in A : a \succ_i c\}$ is an interval of \triangleleft , or, equivalently, for all $\ell \in [m]$ the top ℓ alternatives in v_i form an interval of \triangleleft . This observation enables us map a profile P to a binary matrix M so that P is single-peaked if and only if M has the consecutive ones property. The reduction proceeds as follows.

Assume that $A = [m]$; we create a matrix M with m columns and nm rows, so that each column corresponds to an alternative and each group of m rows corresponds to a voter. Specifically, for each $i \in [n]$, $\ell \in [m]$ the row $(i-1)m + \ell$ encodes the top ℓ alternatives in the preferences of voter i , i.e., for each $j \in [m]$ we set

$$M((i-1)m + \ell, j) = \begin{cases} 1 & \text{if } j \text{ is among the top } \ell \text{ alternatives of voter } i \\ 0 & \text{otherwise.} \end{cases}$$

Now, observe that each permutation of columns that witnesses the consecutive ones property of M corresponds to an ordering of alternatives such that every prefix of every vote forms an interval of that ordering, and vice versa. By [Proposition 3.3](#), this establishes that our reduction is correct.

Example 25: *Reduction of the single-peakedness recognition problem to the consecutive ones problem*

v_1	v_2		a	b	c	d	
b	c		0	1	0	0	$\{b\}$
c	d		0	1	1	0	$\{b, c\}$
a	a		1	1	1	0	$\{a, b, c\}$
d	b		1	1	1	1	$\{a, b, c, d\}$
			0	0	1	0	$\{c\}$
			0	0	1	1	$\{c, d\}$
			1	0	1	1	$\{a, c, d\}$
			1	1	1	1	$\{a, b, c, d\}$

Thus, the problem of recognizing single-peaked profiles is polynomial-time solvable. More precisely, since we can translate a profile into a binary matrix in $O(m^2n)$ time and solve the resulting consecutive ones problem in $O(m^2n)$ time ([Theorem 4.1](#)), we obtain a total runtime of $O(m^2n)$. The PQ-tree data structure behind this approach yields a compact representation of all orderings that witness the single-peaked property.

A Direct Approach

We now turn to a direct, linear-time algorithm for recognizing single-peaked profiles. Our presentation builds on the algorithm by [Doignon and Falmagne \(1994\)](#), infused with ideas from the linear-time algorithm by [Escoffier et al. \(2008\)](#). We follow the exposition of [Brandt \(2024\)](#). The main idea, which is shared by both algorithms, is to build an axis \triangleleft by starting from the outside and iteratively placing alternatives proceeding inwards. At each stage, we consider the set of alternatives that are ranked last in some vote. If there are one or two such alternatives, those are the alternatives that we place. If there are three or more, the profile is not single-peaked: one of these three has to be placed in between the others on \triangleleft , creating a valley in the vote where it is ranked below the other two (cf. [Proposition 3.3 \(3\)](#)).

Example 26: *Placing outermost alternatives*

v_1	v_2	v_3
e	e	d
d	d	e
b	c	c
c	b	a
a	a	b

Consider the profile P on the left. There are two alternatives that appear in the bottom-most position: alternative a in v_1 and v_2 and alternative b in v_3 . These two alternatives have to be at outermost positions in any order of alternatives witnessing the single-peaked property. As axes can be reversed without affecting single-peakedness, we can restrict our attention to orders \triangleleft satisfying $a \triangleleft \{c, d, e\} \triangleleft b$.

Let B denote the set of alternatives that are ranked last by at least one voter (among the alternatives that have not yet been placed). If at any point in the execution of the algorithm we have $|B| \geq 3$, the algorithm returns ‘no’ since these alternatives would form a valley in at least one vote, no matter how we place them on \triangleleft ; thus, from now on, when describing the algorithm, we assume that $|B| \leq 2$.

The algorithm proceeds in two stages. The goal of the first stage is to ensure that the leftmost position and the rightmost position on \triangleleft are filled. Thus, if initially $|B| = 1$, we place the unique alternative in B in the leftmost unfilled position on \triangleleft and remove this alternative from the profile. We repeat this step until $|B| = 2$. If $|B| = 2$, so that $B = \{a, b\}$, we place a into the leftmost unfilled position on \triangleleft and b into the rightmost unfilled position on \triangleleft . We are now guaranteed that there is at least one alternative on the left-hand side of the partial axis and at least one alternative on its right-hand side.

At the second stage, we proceed as follows. At each iteration, we first check that there are at least two alternatives that have not been placed yet; otherwise, we can terminate. Let ℓ denote the alternative last placed on the left-hand side and let r denote the alternative last placed on the right-hand side, so that the partial axis is of the form $\ell' \triangleleft \dots \triangleleft \ell \triangleleft \dots \triangleleft r \triangleleft \dots \triangleleft r'$, where the positions between ℓ and r are unfilled. For each $x \in B$ let N_x be the set of voters that do not rank x first among the remaining alternatives. Suppose that for some $a \in B$ there is a voter $i \in N_a$ with $\ell \succ_i a \succ_i r$. Then a has to be placed into the rightmost available spot, i.e., just to the left of r . Indeed, if, instead, we place a just to the right of ℓ , the top remaining alternative of voter i (which is distinct from a since $i \in N_a$) would create a valley together with ℓ and a . Similarly, if there is a voter $j \in N_a$ with $r \succ_j a \succ_j \ell$, then a has to be placed just to the right of ℓ .

Example 26: *continued — placing inner alternatives*

v_1	v_2	v_3
e	e	d
d	d	e
b	c	c
c	b	a
a	a	b

We consider the restriction of our profile P to alternatives not yet placed, i.e., the set $\{c, d, e\}$. The set B of bottom-ranked alternatives is $\{c\}$. Our current partial axis is $a \triangleleft \dots \triangleleft b$, so $\ell = a$ and $r = b$. Since $b \succ_1 c \succ_1 a$, and voter 1 does not rank c first among $\{c, d, e\}$, alternative c has to be placed next to a . Otherwise, if we chose the partial axis $a \triangleleft \dots \triangleleft c \triangleleft b$, the alternatives e, c, b would form a valley for vote v_1 . Votes v_2 and v_3 rank c above a and b , so they impose no constraints on the placement of c . We thus continue with the partial axis $a \triangleleft c \triangleleft \dots \triangleleft b$.

If at some point $|B| = 2$ and both alternatives in B have to be placed next to ℓ or both have to be placed next to r , the profile is not single-peaked. It may also be the case that all alternatives in B can be placed in either position, in which case we make the choice arbitrarily.

Example 26: *continued — placing inner alternatives*

It remains to place the alternatives $\{d, e\}$. Since our partial axis is $a \triangleleft c \triangleleft \dots \triangleleft b$, we have $\ell = c$ and $r = b$. Observe that there does not exist a voter i with $c \succ_i d \succ_i b$; nor is there a voter j with $b \succ_j d \succ_j c$. The same holds for alternative e . Thus, both alternatives can be placed arbitrarily. The algorithm concludes that both $a \triangleleft c \triangleleft e \triangleleft d \triangleleft b$ and $a \triangleleft c \triangleleft d \triangleleft e \triangleleft b$ are single-peaked axes for P , and returns one of those; the reader can verify that this is correct.

A more precise description of the algorithm is given by the pseudocode in [Algorithm 1](#). In the pseudocode, $N_x(A')$ denotes the set of voters that do not rank x first among the alternatives in $A' \subseteq A$. Furthermore, we use the \oplus operator to concatenate two lists. We also concatenate lists and sets with \oplus if the sets are guaranteed to contain at most one element.

Algorithm 1: Recognizing single-peaked profiles in $O(mn)$ time

Input: A profile P over A
Output: An axis \triangleleft on A such that P is single-peaked with respect to \triangleleft , if one exists

```

1 left, right  $\leftarrow$  empty lists
2  $A' \leftarrow A$  //  $A'$  is the set of alternatives that still need to be placed
3 while  $A' \neq \emptyset$  and right is empty do // first stage
4    $B \leftarrow$  the set of bottom-ranked alternatives in  $P|_{A'}$ 
5   if  $|B| > 2$  then return “ $P$  is not single-peaked”
6   if  $|B| = \{x\}$  then left  $\leftarrow$  left  $\oplus \{x\}$ 
7   if  $|B| = \{x, y\}$  then
8     left  $\leftarrow$  left  $\oplus \{x\}$ 
9     right  $\leftarrow \{y\} \oplus$  right
10   $A' \leftarrow A' \setminus B$ 
11 while  $|A'| \geq 2$  do // second stage
12    $\ell \leftarrow$  rightmost element of left
13    $r \leftarrow$  leftmost element of right
14    $B \leftarrow$  the set of bottom-ranked alternatives in  $P|_{A'}$ 
15   if  $|B| > 2$  then return “ $P$  is not single-peaked”
16    $L \leftarrow \{x \in B : \text{there is } i \in N_x(A') \text{ with } r \succ_i x \succ_i \ell\}$ 
17    $R \leftarrow \{x \in B : \text{there is } i \in N_x(A') \text{ with } \ell \succ_i x \succ_i r\}$ 
18   if  $|L| > 1$  or  $|R| > 1$  then return “ $P$  is not single-peaked”
19   if  $L \cap R \neq \emptyset$  then return “ $P$  is not single-peaked”
20   if  $L \cup R \neq B$  then
21     Arbitrarily assign elements of  $B \setminus (L \cup R)$  to  $L$  and  $R$  so that  $|L| \leq 1$  and  $|R| \leq 1$ .
22   left  $\leftarrow$  left  $\oplus L$ 
23   right  $\leftarrow R \oplus$  right
24    $A' \leftarrow A' \setminus B$ 
25 return left  $\oplus A' \oplus$  right //  $A'$  contains at most one element

```

Theorem 4.2. *Algorithm 1 recognizes single-peaked profiles in time $O(mn)$.*

Proof. We proceed in three steps. First, we argue that if the algorithm outputs an axis \triangleleft then the input profile P is single-peaked with respect to \triangleleft . Second, we show that if the algorithm fails, then the profile is not single-peaked. Finally, we analyze the running time.

Claim 1. *If Algorithm 1 outputs an axis \triangleleft then P is single-peaked with respect to \triangleleft .*

Proof. Assume towards a contradiction that there is a vote v_i in P that is not single-peaked on \triangleleft . Then there exists a valley, i.e., three alternatives a, b, c such that $a \triangleleft b \triangleleft c$, $a \succ_i b$, and $c \succ_i b$. We can assume that a, b , and c appear consecutively on \triangleleft , i.e., the axis is of the form $\dots \triangleleft a \triangleleft b \triangleleft c \triangleleft \dots$. (If there exists a valley, a valley also exists on consecutive candidates.) Let us consider the order in which a, b , and c were placed on the axis. We distinguish with four cases which candidates were placed first on the axis: $\{a, b\}$ or $\{b, c\}$ were placed simultaneously first, $\{a, c\}$ were placed simultaneously first, only $\{b\}$ was placed first, and either $\{a\}$ or $\{c\}$ was placed first.

It cannot be the case that a and b were placed simultaneously, and before c (or that b and c were placed simultaneously, and before a), since whenever $|B| = 2$, we place the alternatives in B at the opposite ends of the unfilled part of the axis. Also, it cannot be the case that a and c were placed simultaneously, and before b : if b has not been placed yet, we cannot have $B = \{a, c\}$, since voter i ranks both a and c above b . Thus, there was an iteration where all three of a, b and c were available, and we placed exactly one of them. It could not have been b , as the unfilled positions should always form an interval of the axis.

The only remaining possibility is that one of a and c was placed strictly before the other two alternatives in $\{a, b, c\}$; assume without loss of generality that it was a . Further, we note that b and c were placed during the second stage. This can be seen as follows: since $a \succ_i b$, a was not placed as the only candidate in this round, i.e., in this round $|B| = 2$. Thus, a was placed either in the final iteration of the first stage (line 7) or in the second stage, and consequently b and c were placed during the second stage. Now, suppose that c was placed strictly before b ; we will show that this leads to a contradiction. Indeed, given that a, b and c form a contiguous segment of \triangleleft , it follows that in the iteration in which c was placed we had to have $A' = \{b, c\}$. As $c \succ_i b$, it follows that we had $B = \{b, c\}$ at that point, so b would have to be placed in the same iteration.

Thus, it has to be the case that b is placed before c or simultaneously with c . Consider the iteration in which b is placed. At the start of this iteration, the partial axis is of the form $\dots \triangleleft a \triangleleft \dots \triangleleft r \triangleleft \dots$, with positions between a and r unfilled. Note that the alternatives placed in the previous iteration form a subset of $\{a, r\}$; as $a \succ_i b$, it follows that the bottom-most alternative in i 's vote in the previous iteration was r , so we have $a \succ_i b \succ_i r$. As c has not been placed yet and voter i prefers c to b , we have $i \in N_b(A')$. Together with $a \succ_i b \succ_i r$ this implies (line 17) $b \in R$. As $|A'| \geq 2$, this means that b cannot be placed next to a , a contradiction. We conclude that every vote in P is single-peaked on \triangleleft . \square

To prove the converse claim, i.e., that Algorithm 1 only returns ‘no’ on profiles that are not single-peaked, we need a technical lemma.

Lemma 4.3. *Let P be a profile. If there exist alternatives $a, b, c \in A$ and voters $i, j, k \in N$ such that*

$$\{b, c\} \succ_i a, \quad \{a, c\} \succ_j b, \quad \text{and} \quad \{a, b\} \succ_k c. \quad (1)$$

then P is not single-peaked. If there exist alternatives $a, b, c, d \in A$ and voters $i, j \in N$ such that

$$\{a, d\} \succ_i b \succ_i c, \quad \text{and} \quad \{c, d\} \succ_j b \succ_j a, \quad (2)$$

then P is not single-peaked.

Proof. Assume for a contradiction that P is single-peaked with respect to \triangleleft , but condition (1) holds. One of the alternatives a, b, c appears between the other two on \triangleleft . This creates a valley in the preferences of the voter who ranks this alternative last, a contradiction.

Suppose next that condition (2) holds. Since P is single-peaked, so is its restriction to $A' = \{a, b, c, d\}$. Thus we can assume without loss of generality that $A = \{a, b, c, d\}$. From [Proposition 3.3](#) (4), get that $\{a, d\}$ and $\{c, d\}$ must be intervals of \triangleleft . Since the alternatives a and c occur bottom-ranked, they must appear at the ends of \triangleleft , so without loss of generality we have $a \triangleleft \{b, d\} \triangleleft c$. But if $a \triangleleft b \triangleleft d \triangleleft c$, then $\{a, d\}$ is not an interval, and if $a \triangleleft d \triangleleft b \triangleleft c$, then $\{c, d\}$ is not an interval. This is a contradiction. \square

Claim 2. *If [Algorithm 1](#) returns ‘no’, then P is not single-peaked.*

Proof. We show that whenever [Algorithm 1](#) returns ‘no’, then either situation (1) or (2) from [Lemma 4.3](#) occurs and hence P is not single-peaked.

If [Algorithm 1](#) fails in Line 5 or in Line 15, so that $|B| \geq 3$, then we can take three alternatives $a, b, c \in B$ and deduce that P is not single-peaked by (1).

Next, assume that [Algorithm 1](#) fails in Line 18, i.e., either $|L| = 2$ or $|R| = 2$. Without loss of generality, suppose $|L| = 2$ and write $L = \{a, b\}$. This means that there are voters $i \in N_a(A')$ with $r \succ_i a \succ_i \ell$ and $j \in N_b(A')$ with $r \succ_j b \succ_j \ell$.

Suppose first that we can choose such voters to be the same ($i = j$), so there is a single voter $i \in N_a(A') \cap N_b(A')$ with $r \succ_i \{a, b\} \succ_i \ell$. Let x be i ’s most-preferred alternative among the set A' of unplaced alternatives, which is different from a and b . Assume without loss of generality that $a \succ_i b$, and thus $\{x, r\} \succ_i a \succ_i b \succ_i \ell$. Since $a \in B$ and $x, b \in A'$, there is a voter k with $\{x, b\} \succ_k a$. If $a \succ_k r$, then we have

$$\{x, r\} \succ_i a \succ_i b \text{ and } \{x, b\} \succ_k a \succ_k r,$$

which is an instance of (2). Otherwise, $r \succ_k a$. In that case, take a voter t with $A' \succ_t r$, where such a voter exists because alternative r was placed in the previous iteration. Then we have

$$r \succ_i a \succ_i b \text{ and } b \succ_k r \succ_k a \text{ and } \{a, b\} \succ_t r,$$

which is an instance of (1).

Otherwise, the voters i and j cannot be chosen to be the same. Thus, for neither voter do we have $r \succ \{a, b\} \succ \ell$. On the other hand, we do have $\{a, b\} \succ \ell$ for both i and j . This can be seen as follows: Consider voter i , the argument for j is the same. We have $r \succ_i a \succ_i \ell$ but not $r \succ_i \{a, b\} \succ_i \ell$. So the remaining possibilities are $b \succ_i r \succ_i a \succ_i \ell$ and $r \succ_i a \succ_i \ell \succ_i b$. The latter is not possible because then b would have been placed before (or together with) either r or ℓ . Consequently, $b \succ_i r \succ_i a \succ_i \ell$ and $a \succ_j r \succ_j b \succ_j \ell$. Now, take again a voter t with $A' \succ_t r$, who exists because r was placed in a previous iteration. Then we have

$$b \succ_i r \succ_i a \text{ and } a \succ_j r \succ_j b \text{ and } \{a, b\} \succ_t r,$$

which is an instance of (1).

Finally, suppose the algorithm fails in Line 19 (and thus did not fail in Line 18), and let $a \in L \cap R$. Thus there are voters $i, j \in N_a(A')$ with $r \succ_i a \succ_i \ell$ and $\ell \succ_j a \succ_j r$. If $|B| = 1$, let $b \neq a$ be an arbitrary alternative in A' (noting that $|A'| \geq 2$ by the condition of the while-loop). Then by definition of B , we have $b \succ_i a$ and $b \succ_j a$. If $|B| = 2$, let us write $B = \{a, b\}$. In that case we also have $b \succ_i a$ and $b \succ_j a$, because otherwise we would have, say, $a \succ_i b$ and thus $r \succ_i \{a, b\} \succ_i \ell$ and then we would have already failed in Line 18, contradiction. Hence we have

$$\{b, r\} \succ_i a \succ_i \ell \text{ and } \{b, \ell\} \succ_j a \succ_j r,$$

which is an instance of (2). □

Claim 3. *Algorithm 1 runs in time $O(nm)$.*

Proof. Observe that both while loops induce at most m iterations. Thus, to prove the runtime bound, we have to show that all lines within these loops require at most $O(n)$ time. This is straightforward for most lines, but the computation of the sets B , L , and R requires some attention. The set B contains all bottom-ranked alternatives in $P|_{A'}$. Iterating through each vote until we found an alternative that is contained in A' would require $O(m)$ time per voter. This issue can be circumvented by storing—for each voter—a pointer to the bottom-ranked alternative in A' . Similarly, we can store a pointer to the top-ranked alternative in A' for each voter; this way, for each $x \in A'$ and $i \in [n]$ we can easily check whether $i \in N_x(A')$. We update these pointers at the end of each while loop, by iterating through all votes; importantly, each pointer needs to be moved by at most two positions.

Recall that $L = \{x \in B : \text{there is } i \in N_x(A') \text{ with } r \succ_i x \succ_i \ell\}$. At the time of computing L and R , we know that $|B| \leq 2$. Thus, under the assumption that $r \succ_i x \succ_i \ell$ can be checked in constant time, computing L takes $O(n)$ time. Satisfying this assumption is not entirely trivial: e.g., if we stored votes as lists, comparing two alternatives would require $O(m)$ time, as we would have to locate these alternatives in the list. Thus, we precompute (outside of the while loops) for each vote a function from alternatives to positions that maps each alternative c to its position in the vote; this can be done by scanning each vote once, so in $O(m)$ time per vote. Then we can decide if, e.g., $r \succ_i x$, by comparing the positions of r and x in vote v_i . Hence the set L —and similarly the set R —can be computed in $O(n)$ time. □

Together, the three claims establish **Theorem 4.2**. □

Finding All Axes

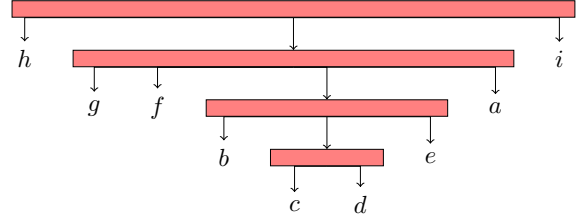
When **Algorithm 1** succeeds, it returns a single axis on which the input profile is single-peaked. We will now discuss how to modify the algorithm so that it can return all axes in a compact representation. It turns out that this is possible within the $O(mn)$ runtime bound.

A set $A' \subseteq A$ is a *common prefix* of a profile P if $A' \succ_i A \setminus A'$ for all $i \in N$, i.e., every voter in P ranks the alternatives in A' above all other alternatives. Suppose that P is single-peaked on \triangleleft , and A' is a common prefix of P . Since P is single-peaked on \triangleleft , every prefix of every vote corresponds to an interval of \triangleleft . Hence A' is an interval of \triangleleft . Now consider the axis \triangleleft' obtained from \triangleleft by reversing the order of alternatives in A' , and observe that every prefix of every vote is still an interval of \triangleleft' . Thus, the collection of axes is closed under reversing common prefixes. In fact, the set of axes obtained from \triangleleft in this way is complete, in the sense that it contains all valid axes. This can be seen by analyzing **Algorithm 1**. During the first stage, each iteration except for the last one corresponds to the case where the unranked alternatives form a common prefix. While during these iterations **Algorithm 1** places the alternative it considers on the left-hand side of the axis, this choice is only made for convenience, as it allows us to quickly decide when to move to the second stage; placing any of these alternatives on the right (which is equivalent to reversing the associated common prefix) would result in a valid axis as well. During the second stage, the algorithm can only make choices in case $L = R = \emptyset$. In this case, it then follows from the definition of L and R that the remaining set of alternatives A' is a common prefix of P .

Proposition 4.4 (**Doignon and Falmagne, 1994, Prop. 7**). *Suppose a profile P is single-peaked on \triangleleft . Let $\emptyset \neq A_1 \subset A_2 \subset \dots \subset A_t = A$ be the collection of common prefixes of P . Then P is single-peaked on an axis \triangleleft' if and only if \triangleleft' can be obtained from \triangleleft by the following process:*

for $j = 1, \dots, t$ sequentially, reverse or do not reverse the interval A_j in \triangleleft .

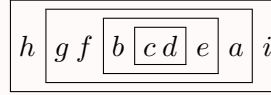
Another way to obtain a concise representation of the set of all axes a given profile is single-peaked on is by leveraging the connection between single-peakedness and the consecutive ones property of the associated 0-1 matrix (see [Example 25](#)). [Booth and Lueker \(1976\)](#) show that the collection of column reorderings making a 0-1 matrix exhibit the consecutive ones property can be represented by a data structure known as a *PQ-tree*. In this context, [Proposition 4.4](#) can be viewed as a statement about the structure of this PQ-tree: all its non-leaf nodes are Q-nodes, and each Q-node has at most one non-leaf child. The pictured PQ-tree captures the structure of [Example 27](#).



Example 27: All single-peaked axes of a profile ([Doignon and Falmagne, 1994, Example 9](#))

v_1	v_2	v_3
c	d	c
d	c	d
b	b	e
e	e	b
f	a	a
g	f	f
a	g	g
h	i	h
i	h	i

The profile on the left is single-peaked on $h \triangleleft g \triangleleft f \triangleleft b \triangleleft c \triangleleft d \triangleleft e \triangleleft a \triangleleft i$. This profile has four non-empty common prefixes, indicated using horizontal lines: $\{c, d\}$, $\{b, c, d, e\}$, $\{a, b, c, d, e, f, g\}$, $\{a, b, c, d, e, f, g, h, i\}$. One can arrange these in nested ‘boxes’, where the order in each box follows \triangleleft , as below:



Then, each axis on which the profile is single-peaked can be obtained by deciding, for each box, whether to reverse it. Thus, there are $2^4 = 16$ different axes.

As we have seen, the problem of recognizing single-peaked preferences is very well understood. However, this does not extend to multidimensional single-peaked preferences ([Definition 11](#)). Indeed, no algorithms or complexity classifications for two or more dimensions are known.

Open Problem 1. What is the complexity of recognizing d -dimensional (hereditary) single-peaked preferences for $d \geq 2$?

4.2 Algorithms for Single-Crossing Preferences

For single-crossing preferences, we can use a reduction to the consecutive ones problem to obtain an $O(nm^2)$ algorithm; a more direct combinatorial approach leads to an $O(nm \log m)$ algorithm. However, in contrast to single-peaked preferences, a linear-time recognition algorithm for this domain is not known.

Reduction to Consecutive Ones

We have already seen that for single-peaked preferences the recognition problem can be reduced to the consecutive ones problem. This is also the case for single-crossing preferences ([Bredereck et al., 2013](#)).

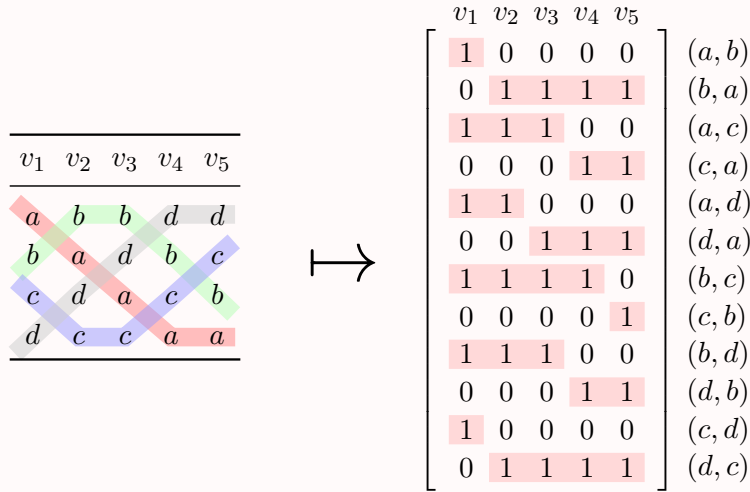
Given a profile P with m alternatives and n voters, we construct an $m^2 \times n$ matrix A as follows: we introduce a column for each voter, and one row for each ordered pair $(a, b) \in A \times A$.

We set

$$A((a, b), i) = \begin{cases} 1 & \text{if } a \succ_i b, \\ 0 & \text{if } b \succ_i a. \end{cases}$$

Then any ordering of the columns of A witnessing the consecutive ones property corresponds to a permutation of the profile P making it single-crossing in the given order (see [Example 28](#)). Since we can check the consecutive ones property in linear time ([Theorem 4.1](#)), this gives an $O(nm^2)$ time algorithm for recognizing single-crossing preferences.

Example 28: *Reducing the recognition of single-crossing profiles to the consecutive ones problem.*



Refining Partitions

[Bredereck et al. \(2013\)](#) also give a direct recognition algorithm, which stores the set of voters in an ordered partition. Initially, all the voters are in a single partition cell. Then, the algorithm iterates through all pairs (a, b) of alternatives, at each step separating the voters with $a \succ b$ from the voters with $b \succ a$. If the algorithm does not discover a contradiction, it then returns a linear ordering of the voters that is compatible with the ordered partition. This algorithm takes $O(m^2n)$ time and can be implemented in a certifying way: if the given profile is not single-crossing, the algorithm returns a small *forbidden subprofile* to witness this (see [Section 5](#)). Another advantage of this approach is that it extends naturally to preferences that are single-crossing on trees (see [Kung \(2015\)](#) and [Section 4.6](#)).

Recognizing Profiles That Are Single-Crossing in the Given Order

Before we go into describing faster and more involved recognition algorithms, it will be useful to carefully consider the problem of determining whether a profile $P = (v_1, \dots, v_n)$ is single-crossing *in the given order*, that is, whether we are already done. This task can be accomplished by a straightforward $O(m^2n)$ time algorithm: we transform each vote into an $m \times m$ pairwise comparison matrix in time $O(m^2)$ per vote (so that we can decide whether $a \succ_i b$ in constant time), and then, for each of the m^2 pairs $(a, b) \in A \times A$, go through the profile in the given order and check that there is at most one crossing. We will now show that, with a bit more sophistication, this check can actually be performed in $O(nm \log m)$ time.

Definition 15. The *Kendall-tau distance* between two rankings \succ_1 and \succ_2 over A is the number of pairwise comparisons that they disagree on: $K(\succ_1, \succ_2) = |\{(a, b) \in A \times A : a \succ_1 b \text{ and } b \succ_2 a\}|$. Let us write $\Delta(\succ_1, \succ_2) = \{(a, b) \in A \times A : a \succ_1 b \text{ and } b \succ_2 a\}$, so that $K(\succ_1, \succ_2) = |\Delta(\succ_1, \succ_2)|$.

In what follows, when considering the Kendall-tau distance between two votes v_i and v_j in a profile P , for brevity, we will write $K[i, j]$ instead of $K(v_i, v_j)$.

Our algorithms make use of the observation that if P is single-crossing in the given order, then, as we go from left to right, the Kendall-tau distance $K[1, i]$ to the leftmost voter increases. To see this, observe that if voter i disagrees with voter 1 on some pair of alternatives (a, b) then so does each voter j with $j > i$, i.e., $\Delta[v_1, v_i] \subseteq \Delta[v_1, v_j]$ and hence $K[1, i] \leq K[1, j]$; further, $K[1, i] = K[1, j]$ is only possible if $v_i = v_j$. We will also need the following two results:

Proposition 4.5. *The Kendall-tau distance between two linear orders u and v over m alternatives can be computed in $O(m \log m)$ time.*

Proof. Treat u as the ‘correct’ ordering of the alternatives and use merge sort to sort the list v into the ordering u . While doing this, we can keep track of the number of swaps required to do so. \square

The fastest known algorithm for computing the Kendall-tau distance takes $O(m\sqrt{\log m})$ time (Chan and Pătraşcu, 2010).

Proposition 4.6. *The Kendall-tau distance satisfies the triangle inequality*

$$K(u, w) \leq K(u, v) + K(v, w),$$

with equality if and only if $\Delta(u, w) \supseteq \Delta(u, v)$.

Proof. The Kendall-tau distance can be rewritten as

$$K(u, w) = \sum_{(a,b) \in A \times A} \mathbf{1}_{(a,b) \in \Delta(u,w)},$$

where $\mathbf{1}_\phi$ is the indicator of whether ϕ is true.

To see the triangle inequality, note that whenever u and w disagree on a pair (a, b) , this disagreement must be present either between u and v , or between v and w . Thus, $\mathbf{1}_{(a,b) \in \Delta(u,w)} \leq \mathbf{1}_{(a,b) \in \Delta(u,v)} + \mathbf{1}_{(a,b) \in \Delta(v,w)}$ for all $(a, b) \in A \times A$. Summing over all pairs, we obtain the triangle inequality.

Given the inequality shown above, equality occurs if and only if $\mathbf{1}_{(a,b) \in \Delta(u,w)} = \mathbf{1}_{(a,b) \in \Delta(u,v)} + \mathbf{1}_{(a,b) \in \Delta(v,w)}$ for all $(a, b) \in A \times A$.

(\Rightarrow): Assume these equalities hold, and let $(a, b) \in \Delta(u, v)$. Then we must have $\mathbf{1}_{(a,b) \in \Delta(u,w)} = 1$, so that $(a, b) \in \Delta(u, w)$. Hence $\Delta(u, w) \supseteq \Delta(u, v)$.

(\Leftarrow): Suppose $\Delta(u, w) \supseteq \Delta(u, v)$. Let $(a, b) \in A \times A$ be a pair of alternatives. We show that $\mathbf{1}_{(a,b) \in \Delta(u,w)} = \mathbf{1}_{(a,b) \in \Delta(u,v)} + \mathbf{1}_{(a,b) \in \Delta(v,w)}$. Suppose first that $(a, b) \in \Delta(u, v)$. By assumption, $(a, b) \in \Delta(u, w)$. Since both v and w disagree with u on (a, b) , they must agree with each other, so $(a, b) \notin \Delta(v, w)$. Hence the equality holds. Alternatively, suppose that $(a, b) \notin \Delta(u, v)$, so that u and v agree on (a, b) . Then u and w disagree on (a, b) if and only if v and w disagree, again confirming the equality. \square

The statement about equality in Proposition 4.6 suggests a way of reasoning about the set

of up to $O(m^2)$ disagreements without having to store a set of this size explicitly. This gives us [Algorithm 2](#), which achieves the promised time bound: for each voter, it calculates two Kendall-tau distances. The correctness of this algorithm follows immediately from the following proposition.

Proposition 4.7. *Let $P = (v_1, \dots, v_n)$ be a profile. The following statements are equivalent:*

1. P is single-crossing in the given order.
2. $\Delta[1, i] \subseteq \Delta[1, i + 1]$ for all $1 \leq i \leq n - 1$.
3. $K[1, i + 1] = K[1, i] + K[i, i + 1]$ for all $1 \leq i \leq n - 1$.

Algorithm 2: Recognizing profiles single-crossing in the given order in $O(nm \log m)$ time

Input: A profile $P = (v_1, \dots, v_n)$ over A
Output: Is P single-crossing in the given order?
for each vote v_i in P with $i \geq 2$ **do**
 calculate the distances $K[1, i]$, $K[i, i + 1]$, and $K[1, i + 1]$
 if $K[1, i] + K[i, i + 1] \neq K[1, i + 1]$ **then**
 return “ P is not single-crossing in the given order”
return “ P is single-crossing in the given order”

Guessing the Leftmost Voter

Based on the insights of the above procedure, we can give a very simple (but slow) recognition algorithm, similar to an algorithm proposed by [Elkind et al. \(2012\)](#). It proceeds by guessing which voter will appear in the leftmost position in the single-crossing order. For each of the n possible guesses, we then sort the remaining voters in increasing order of their Kendall-tau distance to the leftmost voter, and check whether the resulting profile is single-crossing in the given order.

In each iteration, we calculate Kendall-tau distances in $O(nm \log m)$ time, sort the voters in $O(n \log n)$ time, and spend $O(nm \log m)$ time invoking [Algorithm 2](#). Thus, overall this process can be implemented in $O(n(n \log n + nm \log m))$ time. If we assume that all the votes in P are distinct (and hence $n = O(m^2)$ by [Proposition 3.11](#)), then this time bound can be simplified to $O(n^2 m \log m)$; the same time bound can be achieved without assuming distinctness of the votes using the approach we describe in the context of the next algorithm.

Algorithm 3: Recognizing single-crossing profiles in $O(n^2 m \log m)$ time

Input: A profile P over A , in which all votes are pairwise distinct
Output: A single-crossing ordering of P , if one exists
for each voter v_i in P **do**
 sort the voters in P according to increasing Kendall-tau distance from v_i
 if this ordering is single-crossing **then**
 return this ordering of P
return “ P is not single-crossing”

Fast Recognition by Sorting

The above naïve algorithm can be modified in a way that does not require us to guess the leftmost voter. The resulting algorithm runs in time $O(nm \log m)$, and is essentially a faster

implementation of an algorithm due to [Doignon and Falmagne \(1994\)](#) That algorithm repositions voters at each iteration, leading to a worse time bound of $O(n^2 + nm \log m)$. Our algorithm calculates a value $score[i]$ for each voter i that is based on Kendall-tau distances, and then reorders the input profile P in increasing order of the score. If P is single-crossing then the resulting ordering will be single-crossing.

Algorithm 4: Recognizing single-crossing profiles in $O(nm \log m)$ time

Input: A profile P over A
Output: A single-crossing ordering of P , if one exists

- 1 Ensure that voters 1 and 2 have different preference orders (otherwise relabel)
 Calculate $K[1, 2]$
 $score \leftarrow$ empty array indexed by the voters // will have $score[i] = \pm K[1, i]$
 $score[1] \leftarrow 0; score[2] \leftarrow +K[1, 2]$
- for each voter $i \in N \setminus \{1, 2\}$ do**
 - Calculate the distances $K[1, i]$ and $K[2, i]$
 - 2 **if $K[1, 2] = K[1, i] + K[i, 2]$ then**
 - $score[i] \leftarrow +K[1, i]$ // i goes between 1 and 2
 - 3 **else if $K[1, i] = K[1, 2] + K[2, i]$ then**
 - $score[i] \leftarrow +K[1, i]$ // i goes to the right of 2
 - 4 **else if $K[2, i] = K[1, i] + K[1, 2]$ then**
 - $score[i] \leftarrow -K[1, i]$ // i goes to the left of 1
 - else**
 - return** “ P is not single-crossing”
- // Order the voters by their score. Fastest way depends on whether $n < m$ or not.
- if $n < m$ then**
 - 5 $P' \leftarrow$ the list P sorted in order of $score[i]$ using $O(n \log n)$ time
- else if $n \geq m$ then**
 - $B \leftarrow$ an array indexed by $-m^2, \dots, 0, \dots, m^2$, each entry containing an empty list
 - for each voter $i \in N$ do**
 - add v_i to the end of the list $B[score[i]]$
 - 6 $P' \leftarrow B[-m^2] \oplus \dots \oplus B[m^2]$
- 7 **if P' is single-crossing in the given order (use [Algorithm 2](#)) then**
 - 8 **return** P'
- else**
 - 9 **return** “ P is not single-crossing”

Theorem 4.8. [Algorithm 4](#) recognizes single-crossing profiles in $O(nm \log m)$ time.

Proof. First we establish the time bound. A voter whose vote is distinct from v_1 (line 1) can be found in $O(nm)$ time by starting at voter 1 and then scanning the profile until we find a v_i such that $v_i \neq v_1$ (if no such voter can be found, the profile is trivially single-crossing). Now, for each voter in P we calculate the Kendall-tau distances to v_1 and v_2 , taking $O(nm \log m)$ time. If $n < m$, creating the profile P' takes $O(n \log n)$ time, which is in $O(m \log m)$ by the assumption that $n < m$. If $n \geq m$, creating the profile P' takes $O(m^2)$ time which is in $O(nm)$ by the assumption that $n \geq m$. In either case this step was performed in time $O(nm \log m)$. Finally, we check if the profile is single-crossing in the given order in $O(nm \log m)$ time using [Algorithm 2](#).

For correctness, we show that the algorithm reports that P is single-crossing if and only

if P is indeed single-crossing. One direction is easy: if the algorithm returns ‘yes’ in line 8, then P' is single-crossing in the returned order by the check in line 9 and the correctness of Algorithm 2, and P' is a single-crossing reordering of the votes in P .

Conversely, assume that the input profile P is single-crossing. Fix some reordering \hat{P} of P that is single-crossing in the given order and such that voter 2 is placed to the right of voter 1. As in the algorithm, write $K[i, j]$ for the Kendall-tau distance between v_i and v_j . We claim that, after running the for-loop, it holds that for all $i \neq 1$ that are placed to the right of 1 in \hat{P} , we have $score[i] = K[1, i]$, and for all $i \neq 1$ placed to the left of 1 in \hat{P} , we have $score[i] = -K[1, i]$.

Clearly this holds for voter 2. For a voter i with the same vote as voter 1, the conditions of all three if-conditions are satisfied, but in each case the algorithm sets $score[i] = 0$, which is compatible with our claim no matter if i is placed to the left or to the right of voter 1. Next, for a voter i with the same vote as voter 2, the first two if-conditions are satisfied (but not the third since $K[1, 2] > 0$) and in either case the algorithm sets $score[i] = K[1, 2]$, which agrees with our claim since i must be placed to the right of 1 in \hat{P} . For all other voters i , at most one of the conditions in lines 2, 3, and 4 can be true, because in each case the value on the left-hand side needs to be the uniquely largest of the three. We will now do a case analysis on the position of voter i in \hat{P} .

- If i is between 1 and 2 in \hat{P} , then we have $\Delta(v_1, v_i) \subseteq \Delta(v_1, v_2)$ by the single-crossing property, and hence $K[1, 2] = K[1, i] + K[i, 2]$, which is the condition of line 2. Thus the algorithm sets $score[i] = K[1, 2]$ as required by the claim.
- If i is to the right of 2 in \hat{P} , then we have $\Delta(v_1, v_2) \subseteq \Delta(v_1, v_i)$ by the single-crossing property, and hence $K[1, i] = K[1, 2] + K[2, i]$, which is the condition of line 3. Thus the algorithm sets $score[i] = K[1, 2]$ as required by the claim.
- If i is to the left of 1 in \hat{P} , then we have $\Delta(v_i, v_1) \subseteq \Delta(v_i, v_2)$ by the single-crossing property, and hence $K[i, 2] = K[i, 1] + K[1, 2]$, which is the condition of line 4. Thus the algorithm sets $score[i] = -K[1, 2]$ as required by the claim.

In each case, by choosing the sign of $score[i]$, the algorithm correctly decides whether to place i to the right of voter 1 (setting $score[i] > 0$) or to the left (setting $score[i] < 0$). Now, because \hat{P} is in single-crossing order, we have that the Kendall-tau distance to voter 1 increases as we scan from 1 to the right or as we scan from 1 to the left. It follows that we have $score[i] = score[j]$ if and only if $v_i = v_j$ (because by single-crossingness two voters on the same side of voter 1 with the same Kendall-tau distance to voter 1 must be identical) and that we have $score[i] < score[j]$ if and only if $v_i \neq v_j$ and i is positioned to the left of j in \hat{P} . Thus, any list P' obtained by ordering votes in order of increasing $score[i]$ coincides with \hat{P} (up to reordering voters with the same vote). Hence, the profile P' constructed by the algorithm in line 5 or line 6 is single-crossing in the given order, and hence P' passes the check in line 7 and the algorithm correctly returns P' in line 8. \square

Open Problem 2. *Does there exist an $O(mn)$ time algorithm for recognizing single-crossing profiles?*

4.3 Algorithms and Complexity Results for Euclidean Preferences

We start by sketching an algorithm for recognizing 1-Euclidean profiles; subsequently, we will discuss complexity results for d -Euclidean preferences, $d > 1$.

1-Euclidean

For $d = 1$, we can capture the recognition problem by $n \cdot \binom{m}{2}$ constraints over $n + m$ variables: the embedding $x : N \cup A \rightarrow \mathbb{R}$ witnesses that the input profile is 1-Euclidean if for each voter i and every pair of alternatives (a, b) with $a \succ_i b$ we have $|x(i) - x(a)| < |x(i) - x(b)|$. While these constraints are not linear, they can be transformed into linear constraints if the order of the points $\{x(a) : a \in A\}$ on the real line is known: the constraint $|x(i) - x(a)| < |x(i) - x(b)|$ can be replaced with $x(i) < \frac{1}{2}(x(a) + x(b))$ if $x(a) < x(b)$ and $x(i) > \frac{1}{2}(x(a) + x(b))$ if $x(a) > x(b)$. The problem of deciding if the input profile is 1-Euclidean then boils down to finding a feasible solution of the resulting linear program (the reader may worry that our constraints involve strict inequalities, but this can be handled by replacing each constraint of the form $z > z'$ with $z \geq z' + 1$, as discussed, e.g., by [Elkind and Faliszewski \(2014\)](#)).

It remains to figure out how to order the alternatives on the line. One may be tempted to use an algorithm from [Section 4.1](#) for this purpose: indeed, a profile is 1-Euclidean only if it is single-peaked, so we can reject the input if it is not single-peaked, and otherwise use the ordering provided by a single-peaked axis. The problem with this approach is that a 1-Euclidean profile may be single-peaked with respect to several different axes (indeed, potentially exponentially many of them), and some of these axes may be unsuitable for our purposes. This issue is illustrated by the following example.

Example 29: Recognizing 1-Euclidean preferences: choosing an axis

v_1	v_2	
b	c	The profile P on the left is single-peaked with respect to the axis \triangleleft_1 given by $a \triangleleft_1 b \triangleleft_1 c \triangleleft_1 d$ as well as the axis \triangleleft_2 given by $d \triangleleft_2 b \triangleleft_2 c \triangleleft_2 a$. There is a mapping $x : N \cup A \rightarrow \mathbb{R}$ that is consistent with \triangleleft_1 and witnesses that P is 1-Euclidean: e.g., we can take $x(a) = -4$, $x(b) = -1$, $x(c) = 1$, $x(d) = 4$ and co-locate each voter with its top alternative (i.e., $x(v_1) = -1$, $x(v_2) = 1$). However, there is no mapping $x' : N \cup A \rightarrow \mathbb{R}$ that is consistent with \triangleleft_2 and witnesses that P is 1-Euclidean.
c	b	
a	d	
d	a	

To see this, consider a mapping $x' : N \cup A \rightarrow \mathbb{R}$ that is consistent with \triangleleft_2 , i.e., satisfies $x'(d) < x'(b) < x'(c) < x'(a)$. Let x_{bc} be the midpoint of the segment $[x'(b), x'(c)]$, and observe that $b \succ_1 c$, $c \succ_2 b$ implies that v_1 needs to be placed to the left of x_{bc} , while v_2 needs to be placed to the right of x_{bc} , so $x'(v_1) < x'(v_2)$. On the other hand, $a \succ_1 d$, $d \succ_2 a$, so, by considering the positions of v_1 and v_2 with respect to the midpoint of the segment $[x'(d), x'(a)]$, we obtain $x'(v_1) > x'(v_2)$, a contradiction.

The history of recognition algorithms for 1-Euclidean preferences is colorful. The first polynomial-time algorithm appears in the paper by [Doignon and Falmagne \(1994\)](#), who start by developing an elegant theory of single-peaked and single-crossing preferences, and then use it to reason about 1-Euclidean preferences. However, their work used the terminology of the field of psychometrics, such as ‘unidimensional unfolding’ ([Coombs, 1950](#)). Perhaps as a consequence of that, until recently, economists and computational social choice theorists were not aware of this work. [Knoblauch \(2010\)](#) proposed a recognition algorithm that reasoned about single-peaked axes before turning to a linear program; in 2014, [Elkind and Faliszewski](#) proposed a recognition algorithm that used a single-crossing order of the voters before, again, turning to a linear program. Neither of these two papers cites [Doignon and Falmagne \(1994\)](#).

Out of all these algorithms, [Doignon and Falmagne](#)’s way of *combining* information derived from a single-peaked order of alternatives and a single-crossing order of voters is particularly elegant; below, we provide an outline of their proof.

Proposition 4.9 (Doignon and Falmagne, 1994). *Suppose $P = (v_1, \dots, v_n)$ is a profile that is single-crossing in the given order, as well as single-peaked. Then there exists an axis \triangleleft on which P is single-peaked such that the profile $P' = (\triangleleft, v_1, \dots, v_n, \text{reverse}(\triangleleft))$ is still single-crossing. Moreover, such an axis can be found in polynomial time.*

We will say that an axis whose existence is established in [Proposition 4.9](#) is *compatible* with P . To prove existence, [Doignon and Falmagne](#) use an inductive argument. Given the knowledge that a compatible axis exists, finding one is straightforward. First ensure that the input profile P is ordered to be single-crossing, using an algorithm from [Section 4.2](#). Then use [Algorithm 1](#) to obtain a single-peaked axis for P , but whenever we can make an arbitrary decision (in line 20 of the algorithm), we make it in a way that allows $(\triangleleft, v_1, v_n, \text{reverse}(\triangleleft))$ to be single-crossing.

To see the relevance of [Proposition 4.9](#), suppose we are considering a profile $P = (v_1, \dots, v_n)$ that is 1-Euclidean with embedding $x : N \cup A \rightarrow \mathbb{R}$. As we saw in the proof of [Proposition 3.13](#), the left-to-right ordering of the alternatives induced by x is an axis \triangleleft that P is single-peaked on. Moreover, \triangleleft is in fact compatible with P : just place a new voter ℓ to the left of the leftmost alternative in x , and place another voter r to the right of the rightmost alternative. The profile described by this new embedding is exactly $(\triangleleft, v_1, \dots, v_n, \text{reverse}(\triangleleft))$.

Thus, any ordering of the alternatives that can be induced by a 1-Euclidean embedding is a compatible axis. Very conveniently, the converse is also true.

Theorem 4.10 (Doignon and Falmagne, 1994). *Suppose P is a 1-Euclidean profile (and hence it is single-peaked and single-crossing). Then for any axis \triangleleft that is compatible with P , there exists a 1-Euclidean embedding $x : N \cup A \rightarrow \mathbb{R}$ of P that satisfies*

$$x(a) < x(b) \iff a \triangleleft b \quad \forall a, b \in A.$$

[Theorem 4.10](#) tells us that, when ordering the alternatives on a line for the purpose of constructing a linear program, we can take *any* compatible axis \triangleleft : If our profile does not admit a Euclidean embedding agreeing with \triangleleft , then no other axis \triangleleft' will work either. We can now see how to proceed: check that the input profile is single-peaked and single-crossing, identify a compatible axis, and then use linear programming to find the positions of voters and alternatives.

Theorem 4.11. *There is a polynomial-time algorithm that recognizes whether a given profile is 1-Euclidean.*

Proof. First, using the algorithms from [Sections 4.1](#) and [4.2](#), check whether the given profile P is single-peaked and single-crossing. If not, P is not 1-Euclidean. If yes, use the algorithm from [Proposition 4.9](#) to produce an axis \triangleleft compatible with P . Then construct the following linear feasibility program with variables $\{x(i) : i \in N\} \cup \{x(a) : a \in A\}$ over \mathbb{R} to check for the existence of an embedding $x : N \cup A \rightarrow \mathbb{R}$ respecting \triangleleft .

$$\begin{aligned} x(i) + 1 &\leq \frac{x(a) + x(b)}{2} && \text{for all } a \triangleleft b \text{ and } i \in N \text{ with } a \succ_i b, \\ \frac{x(a) + x(b)}{2} + 1 &\leq x(i) && \text{for all } a \triangleleft b \text{ and } i \in N \text{ with } b \succ_i a, \\ x(a) + 1 &\leq x(b) && \text{for all } a \triangleleft b. \end{aligned}$$

□

Open Problem 3. *Can 1-Euclidean profiles be recognized by a purely ‘combinatorial’ algorithm, that is, an algorithm that does not depend on solving a linear program?*

It should be noted that [Theorem 5.5](#) from [Section 5](#) is, perhaps, bad news for this endeavor.

Higher Dimensions

Complexity While the one-dimensional case admits a polynomial-time recognition algorithm, in higher dimensions this is unlikely to be the case. Indeed, by [Theorem 4.13](#), we might need exponentially many bits to write down a Euclidean embedding when one exists, so it is not even clear if the decision version of our problem is in NP. The right complexity class for this recognition problem turns out to be the little-known class $\exists\mathbb{R}$ (see [Schaefer, 2013](#) and [Schaefer and Štefankovič, 2015](#)).

Theorem 4.12 ([Peters, 2017](#)). *For fixed $d \geq 2$, it is NP-hard to recognize whether a given profile is d -Euclidean. In fact, this problem is equivalent to the existential theory of the reals (ETR), and thus $\exists\mathbb{R}$ -complete. It is contained in PSPACE.*

Suppose P is a d -Euclidean profile. Then, because [Definition 7](#) only uses strict inequalities, there must be a Euclidean embedding that only uses *rational* coordinates; and by multiplying by the denominators we can assume that only integral coordinates are used. Thus, there is a Euclidean embedding $x : N \cup A \rightarrow \mathbb{Z}^d$. Now we can ask how big the integers in this representation need to be: do numbers that are polynomial in n , m and d suffice? Do singly exponential numbers, that is, numbers with polynomially many bits, suffice? The answer turns out to be ‘no’:

Theorem 4.13 ([Peters, 2017](#), based on [McDiarmid and Müller, 2013](#)). *For each fixed $d \geq 2$, there are d -Euclidean profiles with n voters and m alternatives such that every integral Euclidean embedding uses at least one coordinate that is $2^{2^{\Omega(n+m)}}$. On the other hand, every d -Euclidean profile can be realized by an integral Euclidean embedding whose coordinates are at most $2^{2^{O(n+m)}}$.*

Heuristic Algorithms While the recognition problem is hard even for $d = 2$, heuristic algorithms have been proposed for this problem that categorize profiles into three groups: *yes*-instances (together with an embedding), *no*-instances (together with a forbidden subprofile), and *unknown*. The first such algorithm was proposed by [Escoffier et al. \(2023\)](#). It starts by searching for an occurrence of some known subprofiles with 4 candidates that are not 2-Euclidean. If it does not find such a *no*-witness, it then attempts to find an embedding by repeatedly sampling potential locations for the alternatives using different probability distributions, until it reaches a time limit. [Dvořák et al. \(2025\)](#) developed an improved heuristic algorithm using additional forbidden subprofiles, applying some reduction rules, and using ILP and QCP solvers. The resulting heuristic was able to classify all but 60 instances from PrefLib, with 98.7% of PrefLib instances resolved in under 1 second.

Other Metrics When we defined d -Euclidean preferences in [Section 3.4](#), we used the Euclidean ℓ_2 -metric. When using the arguably very natural ℓ_1 - or ℓ_∞ -metrics for the definition, [Peters \(2017\)](#) noted that the recognition problem is contained in NP.

Open Problem 4. *Is the recognition problem for d -Euclidean preferences with the ℓ_1 - or ℓ_∞ -metric NP-hard?*

4.4 Algorithms for Preferences Single-Peaked on a Tree

The linear-time algorithm for recognizing single-peaked profiles ([Section 4.1](#)) proceeds by building the underlying axis from the outside in, that is, starting at the extremes. We will now see that a similar approach is helpful for recognizing preferences that are single-peaked on a tree.

The algorithm we present is due to [Trick \(1989\)](#); it can decide whether the input profile is single-peaked on a tree in time $O(m^2n)$. While it is not as fast as the linear-time algorithm for preferences single-peaked on a line, it is very intuitive and easy to describe. The exposition in this section follows [Peters et al. \(2022\)](#).

We have observed that if a profile is single-peaked on some axis \triangleleft , then every alternative that is ranked last by some voter must be placed at one of the endpoints of \triangleleft . The analog of this claim for a profile that is single-peaked on a tree can be formulated as follows.

Proposition 4.14. *Suppose P is single-peaked on T , and suppose a occurs as a bottom-most alternative of some voter i . Then a is a leaf of T .*

Proof. The set $A \setminus \{a\}$ is a prefix of v_i and hence must be connected in T . This can only be the case if a is a leaf of T . \square

Note that this observation potentially allows for many different alternatives to appear in the bottom-most position. This is in contrast to preferences single-peaked on a line, where at most two alternatives can be bottom-ranked. Indeed, a path only has two leaves, whereas an arbitrary tree can have up to $m - 1$ many leaves.

Consider a bottom-ranked alternative a ; we know that if our profile is single-peaked on some tree T , then a is a leaf of T . Now, being a leaf, a must have exactly one adjacent vertex. Which vertex could this be? The following simple observation provides an answer.

Proposition 4.15. *Suppose a vote v_i is single-peaked on the tree T , and $a \in A$ is a leaf of T , adjacent to $b \in A$. Then either*

(i) $b \succ_i a$, or

(ii) a is i 's top-ranked alternative and b is i 's second-ranked alternative.

Proof. Suppose first that a is not i 's top-ranked alternative, and let $c = \text{top}(v_i)$. Let π be the (unique) path from c to a ; note that π passes through b . Since v_i is single-peaked on T , it is single-peaked on π , and hence i 's preferences decrease along π . Since π visits b before a , it follows that $b \succ_i a$.

On the other hand, suppose that a is i 's top-ranked alternative, and c is i 's second-ranked alternative. Then $\{a, c\}$ is a prefix segment of v_i . Since v_i is single-peaked on T , $\{a, c\}$ is connected in T , and hence forms an edge. Thus, a is adjacent c , so $c = b$, as required. \square

Thus, in our search for a neighbor of the leaf a , we can restrict our attention to alternatives b such that for every voter i one of the conditions (i) or (ii) in the proposition above is satisfied. Let us write this down more formally. For each $i \in N$, we write $\text{second}(v_i)$ for the alternative that is ranked second in v_i . For each $c \in A$ and $i \in N$, define

$$B(v_i, c) = \begin{cases} \{c' : c' \succ_i c\} & \text{if } \text{top}(v_i) \neq c, \\ \{\text{second}(v_i)\} & \text{if } \text{top}(v_i) = c. \end{cases}$$

Applying [Proposition 4.15](#) to all voters gives us the following constraint for our choice of b .

Corollary 4.16. *Suppose a profile is single-peaked on T , and $a \in A$ is a leaf of T . Then a must be adjacent to an element of $B(a) := \bigcap_{i \in N} B(v_i, a)$.*

We have established that it is necessary for a leaf a to be adjacent to some alternative in $B(a)$. It turns out that if the profile is single-peaked on a tree, we can actually attach a to *any* of the alternatives in $B(a)$.

Proposition 4.17. *If a profile P is single-peaked on some tree, and a occurs bottom-ranked, then for each $b \in B(a)$ there exists a tree T in which a is a leaf adjacent to b , and P is single-peaked on T .*

These observations suggest that a recognition algorithm can proceed recursively: we identify a leaf a , compute $B(a)$, then obtain a tree T for a restriction of P to $A \setminus \{a\}$, pick an element $b \in B(a)$ in this tree, and connect a to b . This is in fact precisely what [Trick's](#) algorithm ([Algorithm 5](#)) does (we present it in non-recursive form to simplify the analysis of the running time). However, to argue that [Algorithm 5](#) is correct, we need to be a bit more careful. Namely, in [Section 3.5](#) we have seen that the class of profiles single-peaked on trees is not closed under alternative deletion, so deleting a may potentially result in a profile that is not single-peaked on a tree. However, since a is chosen to be a leaf, this is not an issue:

Proposition 4.18. *A profile P over A is single-peaked on a tree T with leaf $a \in A$ if and only if the profile $P|_{A \setminus \{a\}}$ is single-peaked on $T - \{a\}$.*

Putting all these ideas together, we obtain [Trick's](#) algorithm.

Algorithm 5: Recognizing profiles single-peaked on trees in $O(m^2n)$ time ([Trick, 1989](#))

Input: A profile P over A

Output: A tree on A such that P is single-peaked on T , if one exists

$T \leftarrow$ the empty tree on vertex set A

$A' \leftarrow A$

while $|A'| \geq 3$ **do**

$L \leftarrow$ the set of last-ranked alternatives in $P|_{A'}$

for each alternative $a \in L$ **do**

$B_a \leftarrow \bigcap_{i \in N} B(v_i|_{A'}, a)$

if $B_a \neq \emptyset$ **then**

$b \leftarrow$ an arbitrary alternative from B_a

 add edge $\{b, a\}$ to T

$A' \leftarrow A' \setminus \{a\}$

else

return P is not single-peaked on any tree

if $|A'| = 2$ with $A' = \{c, d\}$ **then**

 add edge $\{c, d\}$ to T

return T

Recall that a profile is *narcissistic* if every alternative is ranked first by some voter, that is, if for each $a \in A$ there is an $i \in N$ with $\text{top}(v_i) = a$. Interestingly, for narcissistic profiles the running time of [Trick's](#) recognition algorithm can be improved to $O(nm)$. Intuitively, this is because for narcissistic profiles, we can make heavy use of condition (ii) in [Proposition 4.15](#).

Theorem 4.19 ([Trick, 1989](#)). *For a narcissistic profile P , we can decide in time $O(mn)$ whether P is single-peaked on a tree T . Moreover, if P is single-peaked on a tree, this tree is unique.*

In [Section 6](#), we will see that many hard voting problems remain hard even for profiles single-peaked on a tree. However, one can find algorithms that perform well when given a profile that is single-peaked on a tree T that satisfies some additional properties, such as having few leaves. In order to use these algorithms, we need to be able to find such a ‘nice’ tree if it exists. It turns out that, by studying [Algorithm 5](#) more closely, we can identify ‘nice’ trees for various notions of niceness.

Theorem 4.20 (Peters et al., 2022). *Suppose a profile P is single-peaked on some tree, and let $\mathcal{T}(P)$ be the set of all such trees. Then we can find in polynomial time an element of $\mathcal{T}(P)$ with (i) a minimum number of leaves, (ii) a minimum number of internal vertices, (iii) a minimum diameter, (iv) a minimum pathwidth, (v) a minimum max-degree. We can also decide in polynomial time whether a profile is single-peaked on a star, caterpillar, lobster, or subdivision of a star.*

Interestingly, Theorem 4.20 does not extend to *all* questions of this type.

Theorem 4.21 (Peters et al., 2022). *Given a profile P over A and a tree T with $|V(T)| = |A|$, it is NP-complete to decide whether the vertices of T can be labeled with alternatives so that P becomes single-peaked on that (labeled) tree.*

Peters et al. (2022) also show that it is NP-complete to decide whether a profile is single-peaked on a tree whose non-leaf vertices all have degree 4.

Escoffier et al. (2024b) consider the recognition problem in the framework of single-peaked preferences on general graphs. One of their contributions is a linear program that can be used to recognize preferences single-peaked on a tree. More generally, they describe an integer linear program for finding a graph G with a minimum number of edges or a minimum degree such that the input profile is single-peaked on G , but prove that these problems are NP-hard for general graphs.

4.5 Algorithms for Preferences Single-Peaked on a Circle

In the beginning of Section 4.1, we saw that single-peaked preferences can be recognized using a straightforward reduction to the consecutive ones problem, which is known to be polynomial-time solvable. For the task of recognizing preference profiles that are single-peaked on a circle, the same reduction yields an instance of the *circular* ones problem, which is also polynomial-time solvable by reduction to the consecutive ones problem (Booth and Lueker, 1976; Dom, 2009). The resulting time complexity is $O(m^2n)$. This approach also works for weak orders. For the case of linear orders, Peters and Lackner (2020) develop a linear-time algorithm that runs in $O(mn)$ time, using a reduction to the problem of deciding whether certain profiles of weak orders are single-peaked.

4.6 Algorithms for Preferences Single-Crossing on a Tree

The observations that helped us develop efficient algorithms for recognizing single-crossing preference profiles are also useful for recognizing preference profiles that are single-crossing on a tree. The approach described below can be viewed as a specialization of the algorithm by Clearwater et al. (2015) (which works for all median graphs) to the special case of trees; for trees, this approach results in a particularly simple algorithm.

It will be convenient to assume that in the input profile every vote occurs at most once; the following observation shows that this assumption is without loss of generality.

Proposition 4.22. *Consider a profile P , and let \bar{P} be a profile that contains exactly one copy of each linear order that appears in P . Then P is single-crossing on a tree if and only if \bar{P} is single-crossing on a tree.*

Proof. If \bar{P} is single-crossing on a tree \bar{T} , we can transform \bar{T} into a tree T such that P is single-crossing on T , as follows. Consider a vote v that occurs k times in P (and once in \bar{P});

assume for convenience that $v_1 = \dots = v_k = v$. We create a path of length $k - 1$, label its vertices as $1, \dots, k - 1$, connect $k - 1$ to the node of \bar{T} that is associated with the unique voter in \bar{P} whose vote is equal to v , and label that node as k . By repeating this step for each vote that occurs in \bar{P} , we obtain a tree for P .

Conversely, suppose that P is single-crossing on a tree T . Then each set of nodes of T that corresponds to a group of voters with identical preferences must form a subtree of T . By contracting this set of nodes into a single node, we obtain a tree \bar{T} such that \bar{P} is single-crossing on \bar{T} . \square

Now, given a profile P in which each linear order occurs at most once, we construct a graph G_P with vertex set N so that there is an edge between i and j if and only if P does not contain a vote v_k such that $K[i, k] + K[k, j] = K[i, j]$. We will now argue that if this graph is a tree, it is exactly the ‘right’ tree for our profile.

First, we observe that G_P is connected.

Proposition 4.23. *The graph G_P is connected.*

Proof. Suppose for the sake of contradiction that G_P has $s > 1$ connected components, namely, G_1, \dots, G_s . Let q be the minimum Kendall-tau distance between two votes that lie in different connected components. Fix a pair of voters i and j that lie in different connected components (say, G_ℓ and G_r) and satisfy $K[i, j] = q$. Since there is no edge between i and j , the profile P contains a vote v_k with $K[i, k] + K[k, j] = K[i, j]$. Since all votes in P are distinct, all distances are positive, so $K[i, k] < q$, $K[k, j] < q$. By our choice of q , the first of these inequalities means that k must be in G_ℓ , while the second of these inequalities means that k must be in G_r , a contradiction. \square

Given this observation, we know that G_P is a tree if and only if it is acyclic. The following observation provides additional insights into the structure of G_P .

Proposition 4.24. *Let i and j be two non-adjacent vertices of G_P . Then there is a path in G_P between i and j such that for each vertex k on this path we have $K[i, k] + K[k, j] = K[i, j]$.*

Proof. Suppose for the sake of contradiction that this is not the case; among all pairs (i, j) that violate this property, pick one with the minimum Kendall-tau distance $K[i, j]$ and let $q = K[i, j]$. Since the edge $\{i, j\}$ is not in G_P , there exists a voter v_k in P such that $K[i, k] + K[k, j] = K[i, j]$. Since all votes in P are distinct, we have $K[i, k] < q$. Hence there exists an i - k path in G_P such that for each vertex ℓ on this path we have $K[i, \ell] + K[\ell, k] = K[i, k]$. By the triangle inequality, we have

$$K[i, \ell] + K[\ell, j] \leq K[i, \ell] + K[\ell, k] + K[k, j] = K[i, k] + K[k, j] = K[i, j].$$

On the other hand, by the triangle inequality we obtain $K[i, j] \leq K[i, \ell] + K[\ell, j]$ and hence $K[i, \ell] + K[\ell, j] = K[i, j]$. Similarly, since $K[k, j] < q$, there exists a k - j path in G_P such that for each vertex ℓ' on this path we have $K[k, \ell'] + K[\ell', j] = K[k, j]$; arguing as above, we conclude that this implies $K[i, \ell'] + K[\ell', j] = K[i, j]$. By combining these two paths, we obtain an i - j path with the desired property, a contradiction. \square

We will now show that cycles in G_P correspond to violations on the single-crossing property.

Theorem 4.25. *If P is single-crossing on a tree then G_P is a tree, and P is single-crossing on G_P . Conversely, if G_P is a tree, then P is single-crossing on G_P .*

Proof. Suppose that P is single-crossing on some tree T . Let i and j be two adjacent vertices of T . We claim that i and j are adjacent in G_P . Indeed, suppose for the sake of contradiction that $K[i, j] = K[i, k] + K[k, j]$ for some $k \in N$. Consider the path connecting i , j and k in T . Since i and j are adjacent, this path is of the form $i-j-\dots-k$ or $j-i-\dots-k$; without loss of generality, assume the former is true. But then the profile (v_i, v_j, v_k) must be single-crossing in the given order, and hence $K[i, j] + K[j, k] = K[i, k]$. Substituting $K[i, j] = K[i, k] + K[k, j]$, we obtain $K[j, k] = 0$, a contradiction with the assumption that all votes in P are distinct. Thus, T is a subgraph of G_P .

On the other hand, suppose that i and j are adjacent in G_P . We claim that if P is single-crossing on some tree T then i and j are adjacent in T . Indeed, suppose some node k appears on the (unique) path from i to j in T . Then the profile (v_i, v_k, v_j) is single-crossing in the given order and hence $K[i, k] + K[k, j] = K[i, j]$, a contradiction with i and j being adjacent in G_P . Thus, G_P must be a subgraph of every tree T such that P is single-crossing on T .

We conclude that if P is single-crossing on a tree T , then $T = G_P$. In particular, this means that if G_P contains a cycle then P is not single-crossing on a tree.

Conversely, suppose that G_P is a tree. We will argue that in this case P is single-crossing on G_P . Suppose for the sake of contradiction that this is not the case, and let $i-\dots-j$ be a minimum-length path in G_P such that the profile (v_i, \dots, v_j) obtained by restricting P to this path is not single-crossing in the given order. Let I denote the set of internal vertices of this path. By minimality of this path, there exists a pair of alternatives (a, b) such that each vote v_ℓ with $\ell \in I$ disagrees with both v_i and v_j on this pair. Hence, $K[i, \ell] + K[\ell, j] > K[i, j]$ for each $\ell \in I$. However, by [Proposition 4.24](#), there is an $i-j$ path in G_P such that $K[i, \ell] + K[\ell, j] = K[i, j]$ for every vertex ℓ of this path. Since G_P is a tree, and hence there is a unique $i-j$ path in G_P , we obtain a contradiction. \square

The proof of [Theorem 4.25](#) establishes that if P is single-crossing on a tree and contains at most one occurrence of each linear order then it is single-crossing on a unique tree, namely, G_P . Of course, this is no longer the case if P may contain multiple copies of the same linear order, as the respective voters can be arranged in many different ways (with the constraint that they form a subtree of the resulting tree).

Further, [Theorem 4.25](#) provides an efficient way to decide if the input profile is single-crossing on a tree and, if yes, to construct a suitable tree: all we need to do is to build the graph G_P and check whether it is acyclic. A straightforward implementation of this approach results in an algorithm that runs in time $O(n^3 + n^2m \log m)$.

We note that there are other approaches to detecting profiles that are single-crossing on trees. For instance, [Kung \(2015\)](#) describes a recursive algorithm, which works by splitting voters into cells based on their preferences. That is, at the first step it picks a pair of alternatives $\{a, b\}$ and creates two cells—one containing voters who prefer a to b and one containing voters who prefer b to a . It proceeds recursively in this manner until each cell contains exactly one voter; the sequence of ‘cuts’ determines the edges of the tree. Yet another approach is proposed by [Clearwater et al. \(2014\)](#), who start by identifying a leaf of the tree, and then proceed recursively. Both procedures can be implemented in polynomial time.

4.7 Algorithms for Group-Separable Preferences

Recall the characterization of group-separable profiles in terms of their canonical clone tree decompositions: a profile is group-separable if and only if its canonical clone tree decomposition does not contain P-nodes ([Karpov, 2019a](#)). Since a canonical clone tree decomposition of a given profile can be computed in linear time ([Elkind et al., 2012](#)), we obtain a linear-time algorithm for

recognizing group-separable preferences. We will now describe a simpler (but slower) algorithm, which does not require the full power of PQ-trees.

Theorem 4.26. *Given a profile P over a set of alternatives A , we can decide in polynomial time whether P is group-separable.*

Proof. We can assume that the first voter in P ranks the alternatives as $a_1 \succ \dots \succ a_m$. For each $i \in [m]$, let $A_i = \{a_1, \dots, a_i\}$. We first check if there is an alternative a_i , $1 \leq i \leq m-1$, such that each voter in P either ranks A_i above $A \setminus A_i$ or ranks $A \setminus A_i$ above A_i . If no, then P is clearly not group-separable. If yes, we recurse on A_i and $A \setminus A_i$; we declare a success if at each level of recursion we manage to split each non-singleton set into two clone sets.

Now, clearly, if at any step our recursive procedure fails to split a set of alternatives into two clone sets, then the input profile is not group-separable. To prove the converse, we need to argue that, if the algorithm succeeds, then any non-singleton set $A' \subseteq A$ —and not just the sets explicitly considered by our algorithm—can be split into two clone sets. Consider some such set A' . Let C be the minimal set among the sets considered by the algorithm such that $A' \subseteq C$ (we may have $C = A$). Suppose that our algorithm partitions C as B and $C \setminus B$. Note that by our choice of C we have $A' \cap B \neq \emptyset$ and $A' \cap (C \setminus B) \neq \emptyset$. But then each voter who ranks B above $C \setminus B$ places $A' \cap B$ above $A' \cap (C \setminus B)$, and each voter who ranks $C \setminus B$ above B places $A' \cap (C \setminus B)$ above $A' \cap B$, i.e., both $A' \cap B$ and $A' \cap (C \setminus B)$ form clone sets in $P|_{A'}$. This completes the proof. \square

Note that the procedure in the proof of [Theorem 4.26](#) implicitly constructs a binary tree T whose set of leaves is A . We can interpret this tree as a PQ-tree; however, T need not be a clone decomposition tree of P , as there may exist sets of alternatives that are clone sets with respect to P , but do not correspond to any of the nodes of T . Moreover, the tree T is not unique. For instance, suppose that P consists of a single vote, namely, $a \succ b \succ c \succ d$. In the first iteration, the algorithm may split $A = \{a, b, c, d\}$ as $\{a, b\}$ and $\{c, d\}$, and then split both of these sets into singletons, resulting in a balanced binary tree. Alternatively, in the first iteration it may split A as $\{a\}$, and $\{b, c, d\}$, then split $\{b, c, d\}$ as $\{b\}$ and $\{c, d\}$, and finally split $\{c, d\}$, as $\{c\}$ and $\{d\}$; the corresponding tree is a caterpillar. In either case, there are clones in P that are not represented by the resulting tree. In fact, the canonical clone decomposition tree for this profile is not binary: it is simply a tree that has a Q-node as a root and all four alternatives as its children.

4.8 Nearly Structured Preferences

Most preference profiles are not structured according to the notions that we have considered so far. This is true both in a probabilistic sense (see [Section 8](#) on counting and probability) and for preferences observed in practice ([Mattei et al., 2012](#)). For example, PrefLib, the widely used reference library of preference data, does not contain any profiles of linear orders that are single-peaked ([Mattei and Walsh, 2017](#)). Still, we might hope that preference data is *almost* structured, in the sense of being very close to a structured profile according to some metric. In this section, we will consider several notions of being almost structured, and discuss the computational complexity of finding a structured profile that is as close as possible to an input profile.

4.8.1 Voter and Alternative Deletion

Suppose we hold an election over some numerical quantity, such as a tax rate. As we discussed in [Section 3.2](#), we may expect everyone's preferences to be single-peaked with respect to the natural axis. Yet we find this is not the case; the actual preferences are not single-peaked. What happened? A first suspicion could be that one of the voters got confused while voting, or maybe that a few of the voters do not understand what a tax is, or that some voters hold non-standard theories of taxation. If this is the reason, then we would expect that most of the voters have submitted single-peaked preferences, and we only need to delete very few voters from the input profile to obtain a single-peaked profile. This leads us to the following computational problem, defined for any preference domain Γ .

Γ VOTER DELETION

Instance: A preference profile P , and an integer $k \geq 0$.

Question: Can we delete at most k voters from P to obtain a profile that satisfies Γ ?

Another reason why our profile may fail to be single-peaked could be the presence of a small number of alternatives that do not quite fit onto our one-dimensional axis, or, alternatives for which insufficient information is available for voters to place them on the axis. For example, voters may reject the tax rate 20% out of principle as it corresponds to the undesirable status quo; deleting this option could reveal a single-peaked profile. This example suggests the following problem.

Γ ALTERNATIVE DELETION

Instance: A preference profile P over A , and an integer $k \geq 0$.

Question: Can we delete at most k alternative from A so that $P|_{A'}$ satisfies Γ ?

The complexity of these problems has been studied for several domain restrictions. In particular, for single-peaked and single-crossing preferences we obtain very intuitive results: for the former domain (which is defined in terms of an order of alternatives) the alternative deletion problem is easy and the voter deletion problem is hard, whereas for the latter domain (which is defined in terms of an ordering of the voters), the alternative deletion problem is hard and the voter deletion problem is easy. In what follows, let Γ_{SP} denote the domain of single-peaked preference profiles, and let Γ_{SC} denote the domain of single-crossing preference profiles.

Theorem 4.27 ([Erdélyi et al., 2017](#), Theorem 6.16, [Przedmojski, 2016](#)). Γ_{SP} alternative deletion can be solved in time $O(m^3n)$.

The algorithm of [Erdélyi et al. \(2017\)](#) proceeds by dynamic programming, building up 'good' axes; it is based on ideas described in [Section 4.1](#). The original time bound of $O(m^6n)$ was subsequently improved to $O(m^3n)$ by [Przedmojski \(2016\)](#).

Theorem 4.28 ([Bredereck et al., 2016](#), Theorem 6). Γ_{SC} voter deletion can be solved in time $O(m^2n^3)$.

Proof. The algorithm constructs a single-crossing subprofile P' of P with the maximum number of voters. It starts by guessing the voter i that will be the leftmost voter of P' in its single-crossing order, and then constructs an acyclic digraph D on the voter set of P . The arcs of D are defined as follows: there is an arc from j to k if k disagrees on more pairs with i than j

does, i.e., if $\Delta(v_i, v_j) \subseteq \Delta(v_i, v_k)$. The voters on a path in D starting at i then correspond to single-crossing subprofiles of P that start with v_i . Since D is acyclic, the algorithm then just needs to output a longest path of D starting at i ; this can be accomplished by means of dynamic programming. \square

Theorem 4.29 (Erdélyi et al., 2017, Theorem 6.4; Bredereck et al., 2016, Corollary 1). Γ_{SP} voter deletion is NP-complete.

Proof. We sketch Bredereck et al.’s proof, which is a reduction from VERTEX COVER. Let $G = (V, E)$ be a graph, where $V = \{\nu_1, \dots, \nu_n\}$ and $E = \{e_1, \dots, e_m\}$, and let $k \geq 1$ be the target size of the vertex cover.

For each edge $e_j \in E$, we introduce three edge alternatives a_j, b_j, c_j . For each vertex $\nu_i \in V$ we construct a voter i and set

$$\{a_1, b_1, c_1\} \succ_i \dots \succ_i \{a_m, b_m, c_m\}.$$

Further, for each edge $e_j = \{\nu_r, \nu_s\}$ with $r < s$, we set

$$\begin{aligned} c_j &\succ_r a_j \succ_r b_j, \\ b_j &\succ_s c_j \succ_s a_j, \\ a_j &\succ_i b_j \succ_i c_j \quad \text{for all } i \notin \{r, s\}. \end{aligned}$$

This completes the description of the constructed profile P . Observe that, for $e_j = \{\nu_r, \nu_s\} \in E$, $P|_{\{a_j, b_j, c_j\}}$ is not single-peaked, but can be made single-peaked by (1) deleting voter r , (2) deleting voter s , or (3) deleting all voters except r and s . As a consequence, it can be shown that G has a vertex cover of size at most k if and only if we can delete at most k voters to make P single-peaked. \square

The hardness proof for Γ_{SC} alternative deletion is substantially more complicated and proceeds by a reduction from MAX2SAT.

Theorem 4.30 (Bredereck et al., 2016). Γ_{SC} alternative deletion is NP-complete.

Theorem 4.31 (Bredereck et al., 2016). Γ voter deletion and Γ alternative deletion are NP-complete for value-restricted, group-separable, and best-, medium-, and worst-restricted preferences.

In Table 2 we give an overview of these complexity results as well as list approximation results obtained by Elkind and Lackner (2014): ‘ α -approx.’ refers to a constant-factor approximation algorithm for the given problem, i.e., a polynomial-time algorithm that constructs a profile in the target domain by performing at most α times the optimal number of voter/alternative deletions. These approximation algorithms rely on characterizations of domain restrictions via forbidden subprofiles (cf. Section 5).

Alternative deletion has the drawback that even a small number of removed alternatives yields a significant loss of information about the voters’ preferences. In the context of single-peaked preferences, Erdélyi et al. (2017) propose *local alternative deletion*, where in each vote a small number of alternatives may be removed.

Domain Γ	Voter Deletion	Alternative Deletion
Single-peaked	NP-complete (2-approx.)	polynomial time $O(m^3n)$
Single-crossing	polynomial time $O(n^3m^2)$	NP-complete (6-approx.)
Value-restricted	NP-complete (3-approx.)	NP-complete (3-approx.)
Best/medium/worst-restricted	NP-complete (2-approx.)	NP-complete (3-approx.)
Group-separable	NP-complete (2-approx.)	NP-complete (4-approx.)
1-Euclidean	open	open
Single-peaked on a tree	open	open
Single-peaked and single-crossing	open	open
d -dimensional single-peaked	open	open

Table 2: Complexity results for various voter and alternative deletion problems.

Γ_{SP} LOCAL ALTERNATIVE DELETION	
Instance:	A preference profile P over A , and an integer $k \geq 0$.
Question:	Does there exist an axis \triangleleft such that for each vote $v \in P$ there exists a subset $A' \subseteq A$ of size at least $ A - k$ with the property that $v _{A'}$ is single-peaked on $\triangleleft _{A'}$?

Note that this is a more flexible notion than alternative deletion, as voters do not have to ‘agree’ which alternatives are the outliers. Sui et al. (2013) successfully employ local alternative deletion to argue that two sets of Irish voting data are close to being two-dimensional single-peaked. Erdélyi et al. (2017, Theorem 6.7) show that Γ_{SP} LOCAL ALTERNATIVE DELETION is NP-complete. This result is somewhat surprising, given the easiness result for Γ_{SP} ALTERNATIVE DELETION.

Generalizing Γ_{SP} LOCAL ALTERNATIVE DELETION from Γ_{SP} to other domains is not entirely straightforward. For example, for the domain of preferences single-peaked on a tree, we cannot simply replace the axis \triangleleft in the definition with a tree T , as a restriction of T to a subset of alternatives A' need not be a tree. Indeed, to the best of our knowledge, this problem has not been formally defined or studied for domains other than Γ_{SP} .

4.8.2 Clone Sets and Width Measures

Suppose a city is planning to open a new library somewhere along Main Street, and is asking its residents for their opinions on where and how to build the library. There are several potential sites available for the new building, and for each site, several designs have been proposed. Voters are asked to rank these plans. Should we expect the voters to provide single-peaked rankings?

The answer is not clear. On the one hand, most people will prefer the library to be built as close to them as possible, and thus it seems likely that preferences over the *locations* on Main Street will be single-peaked. On the other hand, the choice of design is merely a question of taste, and so single-peakedness seems unlikely.

If the voters view the location of the library as the more important feature, then we can think of each vote as consisting of ‘blocks’: for each location, the different proposals for this location are ranked consecutively (in some order) within the preference ranking. Formally, we will say that the proposals form an *interval*.

Definition 16. A set $I \subseteq A$ of alternatives forms an *interval* (also known as a *clone set*) of the profile P if for every vote $v_i \in P$, every pair of alternatives $a, b \in I$ and every alternative $c \in A \setminus I$ it is not the case that $a \succ_i c \succ_i b$.

Notice that the entire set A is always an interval, and so are all the singletons. If I is an interval of P , then we can *contract* this interval by replacing all the alternatives in I by a single super-alternative. Using this operation, we can define a structural restriction that captures our library example.

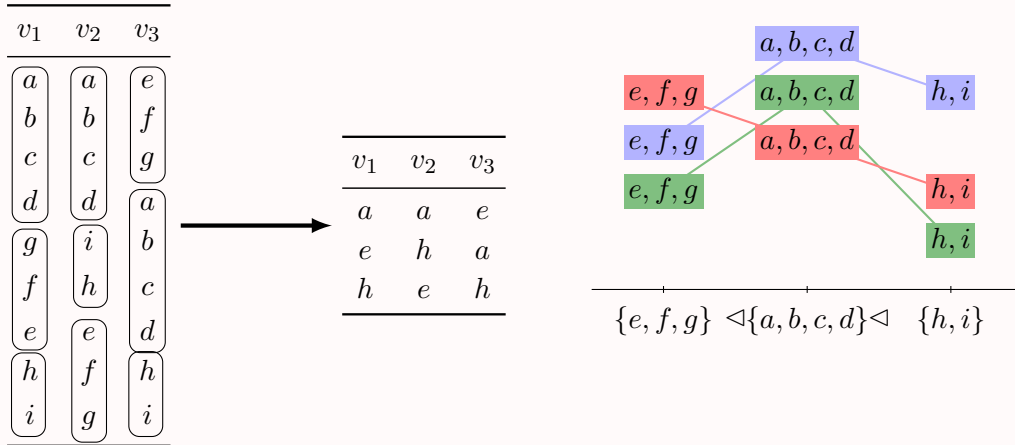
Definition 17 (Cornaz et al., 2012). A profile P over A has *single-peaked width* k if the set A can be partitioned into intervals I_1, \dots, I_q so that $|I_j| \leq k$ for each $j \in [q]$ and the profile P' obtained by contracting each of the intervals I_j , $j \in [q]$, is single-peaked.

While **Definition 17** is formulated for the single-peaked domain, it extends naturally to other domain restrictions.

Definition 17 (continued). Let Γ be a domain restriction. A profile P over A has Γ -width k if the set A can be partitioned into intervals I_1, \dots, I_q so that $|I_j| \leq k$ for each $j \in [q]$ and the profile P' obtained by contracting each of the intervals I_j , $j \in [q]$, is in Γ .

As we will see in **Section 6**, some winner determination algorithms that work for single-peaked (respectively, single-crossing) input profiles continue to work for profiles of bounded single-peaked (respectively, single-crossing) width, i.e., Γ -width is a very useful concept from an algorithmic point of view. Moreover, in contrast to most problems concerning voter and alternative deletion, it is possible to determine the single-peaked width or the single-crossing width of a profile in polynomial time.

Example 30: Single-peaked width



Theorem 4.32 (Cornaz et al., 2013). The single-peaked width and the single-crossing width of a profile with n voters and m alternatives can be computed in $O(nm^3)$ time.

The algorithms of Cornaz et al. (2013) proceed by enumerating all intervals of the input profile, and then calculating a PQ-tree that encodes all rankings for which every interval of

the profile forms an interval of the ranking (following Elkind et al., 2012). The algorithm then manipulates this PQ-tree, deciding which of the nodes need to be collapsed so as to make the profile single-peaked or single-crossing, respectively.³ For domain restrictions Γ other than Γ_{SP} and Γ_{SC} , the concept of Γ -width and the associated computational challenges have not yet been explored.

Another closeness measure that is based on intervals was proposed by Elkind et al. (2012). They wish to obtain a structured profile by contracting as few alternatives as possible, which they call ‘optimal decloning’.

Theorem 4.33 (Elkind et al., 2012, Theorems 5.8 and 6.4). *Given a profile P and an integer $k \geq 1$, it is NP-complete to decide whether we can obtain a single-crossing profile P' with at least k alternatives from P by contracting intervals. In contrast, it is decidable in polynomial time whether P can be transformed into a single-peaked profile P' with at least k alternatives by contracting intervals.*

4.8.3 Swap Distance

A well-known measure of distance between two linear orders is the Kendall-tau distance K (cf. Definition 15), which counts how many swaps of adjacent alternatives are necessary to transform one linear order into another. The Kendall-tau distance can also be used to evaluate closeness to preference domains; an attractive feature of this measure is that it offers a fine-grained perspective on how much a given profile has to be modified—in contrast to ‘coarser’ measures such as deleting voters or alternatives.

Faliszewski et al. (2014) and Erdélyi et al. (2017) present two natural notions of closeness that are based on the Kendall-tau distance: we can count the overall number of required swaps (global swap distance) or the number of required swaps per vote (local swap distance).

Theorem 4.34 (Erdélyi et al., 2017, Theorems 6.8 and 6.9). *Given a profile P and an integer $k \geq 1$, it is NP-complete to decide whether there exists a single-peaked profile P' whose global swap distance from P is at most k . The same holds for the local swap distance.*

These problems are also NP-complete for the single-crossing domain (Lakhani et al., 2019, Theorems 1 and 2). Jaeckle et al. (2018) show that the local swap problem for single-crossing preferences can be solved in XP time with respect to the parameter k .

Open Problem 5. *What is the computational complexity of determining the global/local swap distance to other preference domains? Are there FPT algorithms for these problems?*

For single-peakedness specifically, another fine-grained distance measure that was proposed by Escoffier et al. (2021) is the *forbidden triples* distance that counts the number of valleys (see Figure 3) induced by a given axis. Escoffier et al. (2021) show that this distance is well-behaved in experiments, but like other fine-grained distances, it is NP-hard to compute.

4.8.4 Voter and Alternative Partition

Another set of closeness measures is based on partitioning voters or alternatives (Erdélyi et al., 2017). Consider a situation where each alternative can be characterized by a pair of real-valued parameters. Some voters consider the first parameter to be more important, the others the second parameter. Then it may be possible to split the set of voters into two sets so that each

³The published paper omits some proof details.

set is single-peaked with respect to its own axis. The underlying computational problem is the following:

Γ VOTER PARTITION

Instance: A preference profile P , and an integer $k \geq 1$.

Question: Can we partition the set of voters N into k sets N_1, \dots, N_k so that for all $j \in [k]$ the profile $(v_i : i \in N_j)$ belongs to Γ ?

For the single-peaked domain, this problem is NP-hard for $k \geq 3$, but can be solved in polynomial time for $k = 2$ via a reduction to 2SAT, which uses the forbidden subprofile characterization of single-peaked profiles.

Theorem 4.35 (Erdélyi et al., 2017, Theorem 6.5). Γ_{SP} VOTER PARTITION is NP-complete, for each fixed $k \geq 3$.

Theorem 4.36 (Yang, 2020, Theorem 8). Γ_{SP} VOTER PARTITION can be solved in polynomial time for $k = 2$.

The 2SAT technique also works for some other domains.

Theorem 4.37 (Kraiczky and Elkind, 2022). For the domains Γ of value-restricted or of group-separable preferences, Γ VOTER PARTITION can be solved in polynomial time for $k = 2$.

A similar question can be posed for partitioning alternatives.

Γ ALTERNATIVE PARTITION

Instance: A preference profile $P = (v_1, \dots, v_n)$, and an integer $k \geq 1$.

Question: Can we partition the set of alternatives A into k sets A_1, \dots, A_k so that for all $j \in [k]$ the profile $P|_{A_j}$ belongs to Γ ?

The computational complexity of this problem for single-peaked preferences is unknown; in contrast, for single-crossing preferences this problem is known to be hard even for $k = 3$.

Theorem 4.38 (Jaeckle et al., 2018). Γ_{SC} ALTERNATIVE PARTITION is NP-complete, for each $k \geq 3$.

Open Problem 6. What is the computational complexity of Γ_{SP} ALTERNATIVE PARTITION? Furthermore, what is the complexity of Γ VOTER PARTITION and Γ VOTER PARTITION for other preference domains Γ ?

To conclude, we briefly discuss two notions of closeness that can be viewed as local versions of alternative/voter partition, but are defined for specific restricted domains and do not generalize easily to other domains.

Yang and Guo (2018) consider k -peaked profiles: a profile P over a set of alternatives A is said to be k -peaked if there is an axis \triangleleft such that each voter i can partition A into $k_i \leq k$ pairwise disjoint subsets A_1, \dots, A_{k_i} so that for each $j = 1, \dots, k_i$ the set A_j forms a contiguous subset of \triangleleft and the restriction of v_i to A_j is single peaked on $\triangleleft|_{A_j}$. This notion can be seen as a local analog of alternative partition, in that each voter is allowed to choose their own partition of \triangleleft . However, Yang and Guo (2018) focus on the complexity of election control in this domain rather than the recognition problem, and leave the recognition problem open.

In a similar spirit, Misra et al. (2017) discuss k -crossing profiles: a profile P over a set of alternatives A is said to be k -crossing if the voters can be reordered so that in the reordered profile P' each pair of alternatives ‘crosses’ at most k times, i.e., for each pair of alternatives (a, b) the voters can be split into $k + 1$ groups so that the voters in each group form a contiguous block in P' and agree on (a, b) . This notion can be viewed as a local variant of the voter partition problem, in the sense that for each pair of alternatives we may choose a different partition of the reordered profile P' into $k + 1$ groups. Again, Misra et al. (2017) study the complexity of multi-winner voting in this domain rather than the recognition problem for the domain itself.

4.8.5 Fixed Order of Voters or Alternatives

The domain Γ_{SP} consists of all profiles that are single-peaked on *some* axis. Alternatively, we can fix an axis \triangleleft and consider the domain Γ_{\triangleleft} of all profiles that are single-peaked on \triangleleft . We can then ask if a given profile P is close to being in Γ_{\triangleleft} , i.e., whether we can make a small number of changes to P to obtain a profile in Γ_{\triangleleft} . In a similar fashion, we can ask if we can make a small number of changes to P in order to obtain a profile that is single-crossing in the given order.

Erdélyi et al. (2017) study the complexity of making a given profile P single-peaked on a fixed axis \triangleleft , for many of the distance notions discussed earlier in this chapter, and obtain polynomial-time algorithms for each distance measure they consider (Section 6.3 of their paper). For voter deletion, the algorithm is trivial: one can simply delete the voters whose preferences are not single-peaked on \triangleleft . For the alternative deletion problem, Erdélyi et al. (2017) argue that this problem can be reduced to Γ_{SP} ALTERNATIVE DELETION by adding two votes that rank the alternatives according to \triangleleft and its reverse to the input profile: this ensures that the algorithm for Γ_{SP} ALTERNATIVE DELETION does not benefit from considering axes other than \triangleleft . However, this problem also admits a direct $O(nm^3)$ algorithm,⁴ which we describe below.

Theorem 4.39. *Given an n -voter profile P over an alternative set A , $|A| = m$, and an axis \triangleleft , we can compute a maximum-size set A' such that $P|_{A'}$ is single-peaked on $\triangleleft|_{A'}$ in time $O(nm^3)$.*

Proof. Assume without loss of generality that the axis \triangleleft is given by $a_1 \triangleleft \dots \triangleleft a_m$. We proceed by dynamic programming. For each pair of indices j, ℓ with $1 \leq j < \ell \leq m$, let $s(j, \ell)$ be the maximum size of a set of alternatives C such that $C \subseteq \{a_1, \dots, a_\ell\}$, $a_j, a_\ell \in C$ and $P|_C$ is single-peaked on $\triangleleft|_C$. Clearly, we have $s(1, 2) = 2$, and the size of the target set A' is given by $\max_{1 \leq j < \ell \leq m} s(j, \ell)$; the set A' itself can be computed by standard dynamic programming techniques.

We will now explain how to compute $s(j, \ell)$ when $\ell > 2$. We say that an index k , $1 \leq k < j$ is (j, ℓ) -good if for each voter $i \in N$ the restriction of v_i to $\{a_k, a_j, a_\ell\}$ does not form a valley with respect to \triangleleft , i.e., if $a_k \succ_i a_j$ or $a_\ell \succ_i a_i$. We then have

$$s(j, \ell) = 1 + \max_{k: k \text{ is } (j, \ell)\text{-good}} s(k, j).$$

Indeed, our definition of a good index ensures that no triple of alternatives that appear consecutively in the set that is implicitly constructed by the algorithm forms a valley in any of the voter’s preferences, i.e., the restriction of P onto this set has the ‘no local valleys’ property (see Proposition 3.3 (2)). \square

For local alternative deletion, the votes are processed one by one; for each vote, the problem

⁴We are grateful to Andrei Constantinescu for this observation.

reduces to guessing a new peak, splitting the axis \triangleleft at this peak, and finding a maximum-length increasing subsequence in the left part and a maximum-length decreasing subsequence in the right part.

To evaluate the swap-based distance measures, a key step is to compute the minimum number of swaps required to make a given vote single-peaked on \triangleleft . [Faliszewski et al. \(2014\)](#) and [Erdélyi et al. \(2017\)](#) describe dynamic programming algorithms for this problem, whose running time is, respectively, $O(m^4)$ and $O(m^3)$; below, we present a slightly modified version of the procedure proposed by [Erdélyi et al. \(2017\)](#).

Theorem 4.40. *Given a vote v over a set of alternatives $A = \{a_1, \dots, a_m\}$ and an axis \triangleleft , we can compute $\min_{u \in \Gamma_{\triangleleft}} K(u, v)$ in time $O(m^3)$.*

Proof. Assume without loss of generality that the axis \triangleleft is given by $a_1 \triangleleft \dots \triangleleft a_m$.

For each i, j with $1 \leq i < j \leq m$, let $A_{i,j} = \{a_1, \dots, a_i, a_j, \dots, a_m\}$, let $S_L(i, j)$ (respectively, $S_R(i, j)$) be the set of all preference orders over $A_{i,j}$ that are single-peaked on the restriction of \triangleleft to $A_{i,j}$ and rank a_i first (respectively, rank a_j first). For $X \in \{L, R\}$, let $s_X(i, j) = \min_{u \in S_X(i, j)} K(v|_{A_{i,j}}, u)$; it will be convenient to set $s_X(i, j) = +\infty$ if $i = 0$ or $j = m + 1$.

Note that for each $i \in [m - 1]$ the set $S_L(i, i + 1)$ consists of all preference orders over A that are single-peaked on \triangleleft and rank a_i first, whereas the set $S_R(m - 1, m)$ consists of the (unique) preference order over A that is single-peaked on \triangleleft and ranks a_m first, so

$$\min_{u \in \Gamma_{\triangleleft}} K(u, v) = \min\{s_R(m - 1, m), \min_{i \in [m - 1]} s_L(i, i + 1)\}.$$

We will now explain how to compute $s_X(i, j)$ for all $X \in \{L, R\}$ and $1 \leq i < j \leq m$. For $i = 1, j = m$, we have $s_L(i, j) = 0$, $s_R(i, j) = 1$ if v ranks a_1 above a_m and $s_L(i, j) = 1$, $s_R(i, j) = 0$ otherwise. Now, fix i, j with $1 \leq i < j \leq m$, and suppose we have computed $s_X(k, \ell)$ for all $X \in \{L, R\}$ and all k, ℓ with $\ell - k > j - i$. Let $v' = v|_{A_{i,j}}$, and let \triangleleft' be the restriction of \triangleleft to $A_{i,j}$. To compute $s_L(i, j)$, we first compute the cost of moving a_i to the top position in v' ; if a_i is ranked in position r in v' , this requires $r - 1$ swaps. The remaining alternatives in v' are now ordered according to $v|_{A_{i-1,j}}$; we need to reorder them to obtain a vote over $A(i, j)$ that is single-peaked on \triangleleft' . Note that the length-2 prefix of this vote must form a contiguous segment of \triangleleft' , so its second position must be occupied by a_{i-1} or a_j . Thus, we obtain

$$s_L(i, j) = r - 1 + \min\{s_L(i - 1, j), s_R(i - 1, j)\}.$$

By a similar argument,

$$s_R(i, j) = r' - 1 + \min\{s_L(i, j + 1), s_R(i, j + 1)\},$$

where r' is the position of a_j in v' . Altogether, we need to compute $O(m^2)$ quantities, and each of them can be computed in time $O(m)$, which implies our bound on the running time. \square

[Lakhani et al. \(2019\)](#) consider the problem of modifying a given profile to make it single-crossing in the given order. This problem turns out to be quite challenging: they obtain NP-hardness results for alternative deletion, alternative partition, and both local and global swaps.

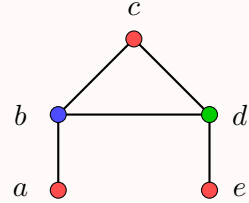
[Lakhani et al.](#)'s hardness results for alternative deletion and alternative partition are based on the notion of *crossing graph* ([Cohen et al., 2019](#)). The *crossing graph* of a profile P on a set of alternatives A is an undirected graph $G(P)$ that has A as its set of vertices; there is an edge connecting a and b if the pair (a, b) violates the single-crossing condition. [Cohen et al. \(2019\)](#) describe an efficient algorithm that, given an undirected m -vertex graph G with no loops

and parallel edges, builds a profile P with at most $2m + 1$ voters such that $G(P) = G$. This construction immediately implies hardness of alternative deletion and partition: deleting a set of k alternatives to make the profile P single-crossing corresponds to finding a vertex cover of size at most k in $G(P)$, and partitioning A into at most k sets so that the restriction of A to each set is single-crossing corresponds to coloring $G(P)$ with at most k colors.

Example 31: The crossing graph.

v_1	v_2	v_3	v_4
a	a	b	a
b	d	a	b
c	c	c	c
d	b	e	d
e	e	d	e

The reader can verify that the graph G on the right is the crossing graph of the profile on the left: e.g., G contains the edge $\{a, b\}$ and we have $a \succ_1 b$, $b \succ_3 a$ and $a \succ_4 b$, i.e., a and b cross more than once.



The set $\{b, d\}$ forms a vertex cover of G ; thus, if we delete b and d from G , the induced graph on $\{a, c, e\}$ has no edges. Accordingly, the restriction of P to $\{a, c, e\}$ is single-crossing. Further, as G contains a cycle of length 3, it is not 2-colorable, but it admits a 3-coloring, as shown in the figure. This corresponds to partitioning A into three sets ($\{a, c, e\}$, $\{b\}$ and $\{d\}$) so that the restriction of P to each of these sets is single-crossing.

On the positive side, the problem of finding the minimum number of voters to remove to make the profile single-crossing in the given order turns out to be easy: one can use (a simplification of) [Bredereck et al. \(2016\)](#)'s algorithm for Γ_{SC} VOTER DELETION, which is based on finding the longest path in a directed acyclic graph. [Lakhani et al. \(2019\)](#) also propose a polynomial-time algorithm for the variant of the voter partition problem where each part has to form a contiguous subprofile of the input profile P , as well as for the problem of finding the minimum number of voter swaps required to make P single-crossing in the given order.

4.8.6 Axiomatic Approach to Nearly Structured Preferences

As we have seen, there are many possible measures for a profile to be close to being structured. Which of these measures is the most appropriate? [Escoffier et al. \(2021\)](#) (extended in the thesis of [Tydrichová, 2023](#), Sec. 4.4) consider this question from an axiomatic perspective. They focus on single-peakedness, and conceptualize the problem through *axis selection rules*, which given a profile output an axis \triangleleft that “best” explains the profile. For example, the voter deletion rule outputs an axis on which the maximum number of voters is single-peaked. [Tydrichová \(2023\)](#) compares different such rules through their axiomatic behavior. For example, she shows that the voter deletion rule satisfies a reinforcement axiom as well as a stability axiom. [Delemazure et al. \(2024\)](#) perform a similar analysis for approval preferences based on the CI property (see [Section 8.2](#)). The same kind of approach could fruitfully be employed for many other notions of structure, such as (multidimensional) Euclidean preferences and preferences single-peaked on trees, or even to single-crossing preferences (for rules that select an ordering of the voters).

5 Characterizations by Forbidden Subprofiles

A classic result in graph theory is Kuratowski's theorem, which characterizes planar graphs as graphs that do not contain subdivisions of K_5 and $K_{3,3}$ as their subgraphs (Kuratowski, 1930). In a similar fashion, some restricted domains can be defined by a set of forbidden subprofiles: a profile belongs to a restricted domain if and only if it does not contain these subprofiles. In this section, we introduce several characterization results that are based on forbidden subprofiles, and discuss some applications of this theory.

5.1 Single-Peaked Preferences

We start with the following theorem, which characterizes the single-peaked domain in terms of forbidden substructures.

Theorem 5.1 (Ballester and Haeringer, 2011). *A profile P over A is single-peaked if and only if there do not exist alternatives $a, b, c, d \in A$ and voters $i, j \in N$ such that*

$$\{a, d\} \succ_i b \succ_i c, \quad \text{and} \quad \{c, d\} \succ_j b \succ_j a, \quad (3)$$

and there do not exist alternatives $a, b, c \in A$ and voters $i, j, k \in N$ such that

$$\{b, c\} \succ_i a, \quad \{a, c\} \succ_j b, \quad \text{and} \quad \{a, b\} \succ_k c. \quad (4)$$

Proof. We will first argue that if a profile P violates condition (1) or condition (2) then it is not single-peaked. Suppose for the sake of contradiction that P is single-peaked with respect to some axis \triangleleft , but violates condition (1). We can assume without loss of generality that $A = \{a, b, c, d\}$: otherwise we can consider the restriction of P to $\{a, b, c, d\}$, which is single-peaked as long as P is. By Proposition 3.3 (4), we have $a \triangleleft \{b, d\} \triangleleft c$ or $c \triangleleft \{b, d\} \triangleleft a$; we can assume without loss of generality that $a \triangleleft \{b, d\} \triangleleft c$. But if $a \triangleleft b \triangleleft d \triangleleft c$ then \succ_i is incompatible with \triangleleft (as a, d form a prefix of \succ_i , but do not form an interval with respect to \triangleleft), and if $a \triangleleft d \triangleleft b \triangleleft c$ then \succ_j is incompatible with \triangleleft (as c, d form a prefix of \succ_j , but do not form an interval with respect to \triangleleft). We obtain a contradiction. Now, suppose that P is single-peaked with respect to \triangleleft , but violates condition (2). Again, we can assume that $A = \{a, b, c\}$. One of the alternatives a, b, c appears between the other two on \triangleleft . This creates a valley in the preferences of the voter who ranks this alternative last, a contradiction again.

For the converse direction, we consider the execution of Algorithm 1, which recognizes single-peaked profiles. We will show that if this algorithm decides that the input profile P is not single-peaked, then P fails condition (1) or condition (2).

It is straightforward to see that if Algorithm 1 terminates in line 5 or 15 then condition (2) is not satisfied.

If Algorithm 1 terminates in line 19, let x be the alternative that is contained in both L and R . Thus, there exists a voter i with $\ell \succ_i x \succ_i r$ and a voter j with $r \succ_j x \succ_j \ell$. Furthermore, since $|A'| > 1$ and $i, j \in N_x(A')$, there exists an alternative y with $y \succ_i x$ and $y \succ_j x$. Thus, the alternatives ℓ, r, x, y and voters i, j witness that condition (1) is not satisfied.

Now suppose that Algorithm 1 terminates in line 18. Without loss of generality, we can assume that $|L| = \{x, y\}$. Then there exist voters i, j with $r \succ_i x \succ_i \ell$ and $r \succ_j y \succ_j \ell$. Since $i \in N_x(A')$, we have $y \succ_i x$, and since $j \in N_y(A')$, we have $x \succ_j y$. Since r has already been placed during an earlier step, there has to exist a voter k with $\{x, y\} \succ_k r$. Thus, candidates r, x, y and voters i, j, k witness that condition (2) is violated. We conclude that if a profile is not single-peaked, then it violates condition (1) or condition (2). \square

While [Theorem 5.1](#) does not explicitly list forbidden subprofiles, it defines them implicitly: these are all profiles with two voters and four alternatives witnessing a violation of the first condition and all profiles with three voters and three alternatives witnessing a violation of the second condition. This list is essentially finite. Formally, we say that a profile P over A is *isomorphic* to a profile P' over A' if P' can be obtained from P by renaming alternatives and permuting the votes. In particular, if P is isomorphic to P' then $|A| = |A'|$ and P and P' contain the same number of voters. Abusing the terminology somewhat, we say that a profile P *contains* P' *as a subprofile* if there exists a profile P'' that is isomorphic to P' and can be obtained from P by alternative and voter deletion. With these definitions in hand, we can restate [Theorem 5.1](#) purely in terms of forbidden subprofiles.

Theorem 5.2. *A profile P over A is single-peaked if and only if P does not contain the following profiles as subprofiles:*

v_1	v_2	v_3	v_1	v_2	v_3	v_1	v_2	v_1	v_2	v_1	v_2
a	b	c	a	c	a	d	d	d	c	a	c
b	c	a	b	b	c	a	c	a	d	d	d
c	a	b	c	a	b	b	b	b	b	b	b
						c	a	c	a	c	a

[Theorems 5.1](#) and [5.2](#) admit natural analogs for the single-caved domain (just reverse all the forbidden subprofiles). By analyzing their recognition algorithm, [Peters and Lackner \(2020\)](#) give a characterization of preferences single-peaked on a circle by forbidden subprofiles. These are structurally similar to the ones in [Theorems 5.1](#) and [5.2](#), but are slightly larger.

5.2 Single-Crossing Preferences

The single-crossing domain, too, has been characterized in terms of forbidden subprofiles ([Bredereck et al., 2013](#)). Note that in the theorem statement, the alternatives and voters mentioned are not necessarily distinct.

Theorem 5.3 ([Bredereck et al., 2013](#)). *A profile P over A is single-crossing if and only if there do not exist alternatives $a, b, c, d, e, f \in A$ and voters $i, j, k \in N$ such that*

$$b \succ_i a, c \succ_i d, e \succ_i f, \quad a \succ_j b, d \succ_j c, e \succ_j f, \quad \text{and} \quad a \succ_k b, c \succ_k d, f \succ_k e,$$

and there do not exist alternatives $a, b, c, d \in A$ and voters $i, j, k, \ell \in N$ such that

$$a \succ_i b, c \succ_i d, \quad b \succ_j a, c \succ_j d, \quad a \succ_k b, d \succ_k c, \quad \text{and} \quad b \succ_\ell a, d \succ_\ell c.$$

This theorem, too, can be phrased purely in terms of forbidden subprofiles (similarly to [Theorem 5.2](#)). However, in the case of the single-crossing domain there are 30 non-isomorphic forbidden subprofiles, and therefore we do not list them all.

5.3 Value-Restricted and Group-Separable Preferences

As we have seen in [Proposition 3.22](#), the value-restricted domain also allows a characterization via forbidden subprofiles: As it is characterized by excluding the Condorcet cycle on three candidates, it follows that the value-restricted domain is the largest domain definable by forbidden subprofiles that guarantees a Condorcet winner (for an odd number of candidates, cf. [Proposition 3.23](#)).

The group-separable domain has also been characterized in terms of forbidden subprofiles.

Theorem 5.4 (Ballester and Haeringer, 2011). *A profile P over A is group-separable if and only if there do not exist alternatives $a, b, c, d \in A$ and voters $i, j \in N$ such that*

$$a \succ_i b \succ_i c \succ_i d \quad \text{and} \quad b \succ_j d \succ_j a \succ_j c,$$

and there do not exist alternatives $a, b, c \in A$ and voters $i, j, k \in N$ such that

$$(a \succ_i b \succ_i c, a \succ_j c \succ_j b, b \succ_k a \succ_k c) \quad \text{or} \quad (a \succ_i b \succ_i c, b \succ_j c \succ_j a, c \succ_k a \succ_k b).$$

The second condition ensures that the domain is medium-restricted, so that for each triple of alternatives, at least one of those alternatives is never ranked in between the other two.

5.4 Euclidean Preferences

So far we have seen four examples of domains that can be characterized via a finite set of forbidden subprofiles, namely, the single-peaked domain, single-peaked on circles, the single-crossing domain, and the value-restricted domain. In contrast, a recent result of Chen et al. (2017) shows that the 1-Euclidean domain cannot be characterized in this manner.

Theorem 5.5 (Chen et al., 2017). *There is no finite set of forbidden subprofiles that characterizes the 1-Euclidean domain.*

Proof idea. Chen et al. (2017) construct an infinite family of profiles that are minimally non-1-Euclidean. For each n , they build an n -voter profile P_n over the set of alternatives $A = \{a_1, b_1, \dots, a_n, b_n\}$. The preference order of voter i ensures that for every potential embedding $x : N \cup A \rightarrow \mathbb{R}$ witnessing that P is 1-Euclidean we have $\|x(a_i) - x(b_i)\| < \|x(a_{i+1}) - x(b_{i+1})\|$, where subscripts are taken modulo n . Taken together, these inequalities imply $\|x(a_1) - x(b_1)\| < \|x(a_2) - x(b_2)\| < \dots < \|x(a_n) - x(b_n)\| < \|x(a_1) - x(b_1)\|$, which is impossible. Hence the constructed profile is not 1-Euclidean. However, whenever we delete a voter, one of the constraints of the cycle vanishes, allowing us to embed the profile into \mathbb{R} . These profiles are thus minimally non-1-Euclidean.

Now, suppose that the 1-Euclidean domain can be characterized by a finite set of forbidden subprofiles \mathcal{S} . There exist an $n \geq 0$ such that each profile in \mathcal{S} has fewer than n voters. However, $P_n \notin \mathcal{S}$, since it contains n voters, and every subprofile of P_n is not in \mathcal{S} , because it is 1-Euclidean, a contradiction. \square

However, the 1-Euclidean domain can be characterized by an *infinite* set of forbidden subprofiles. In fact, this is the case for every hereditary domain (i.e., a domain closed under deletion of voters and alternatives; see Definition 3).

Proposition 5.6 (Lackner and Lackner 2017, Proposition 5). *A domain is hereditary if and only if it can be characterized by a (possibly infinite) set of forbidden subprofiles.*

Proof. Suppose that \mathcal{D} is not a hereditary domain. Then there is a profile $P \in \mathcal{D}$ such that some subprofile P' of P does not belong to \mathcal{D} . However, every subprofile of P' is also a subprofile of P , which means that \mathcal{D} cannot be characterized via forbidden subprofiles.

Conversely, consider a hereditary domain \mathcal{D} , and let \mathcal{S} be the (infinite) set of all profiles that do not belong to \mathcal{D} . Note that if a profile belongs to \mathcal{D} , then none of its subprofiles is in \mathcal{S} , exactly because \mathcal{D} is a hereditary domain. Thus, \mathcal{S} offers the desired characterization. \square

Open Problem 7. *Give an explicit characterization of the 1-Euclidean domain by infinitely many forbidden subprofiles.*

Proposition 5.6 offers a straightforward way to identify domains that are characterized by a (possibly infinite) set of forbidden subprofiles. For instance, it immediately implies that the domain of preferences single-peaked on a tree does not admit such a characterization, whereas the domain of d -Euclidean preferences does, for every $d \geq 1$. In contrast, we are not aware of a general technique that can distinguish between domains that can be characterized by finitely many forbidden subprofiles (such as the single-peaked domain) and those that can only be characterized by an infinite set (such as the 1-Euclidean domain). A complexity-theoretic approach offers a partial solution: a domain that can be characterized by a finite number of forbidden subprofiles is polynomial-time recognizable (e.g., by a simple brute-force algorithm), so if recognizing a restricted domain is known to be computationally hard, we can conclude that this domain does not admit a finite characterization (subject to a complexity assumption).

For example, **Theorem 4.12** shows that for $d \geq 2$ recognizing whether a profile is d -Euclidean is NP-hard. Thus, unless $P=NP$, we know that for each $d \geq 2$ the d -Euclidean domain is not characterizable by a finite set of forbidden subprofiles. We note that [Peters \(2017\)](#) proves the same result without referring to any complexity assumption.

Let us end this section by briefly mentioning some applications of characterizations via forbidden subprofiles. Such characterizations have been shown to be useful for detecting profiles that are close to being in the respective restricted domain ([Elkind and Lackner, 2014](#); [Bredereck et al., 2016](#)); more details can be found in [Section 4.8](#). Forbidden subprofiles can also be used to prove general results about arbitrary domain restrictions: [Lackner and Lackner \(2017\)](#) obtain a combinatorial result counting the number of profiles of a given size that belong to a restricted domain characterized by a small forbidden subprofile.

6 Winner Determination

Ordinal preferences are often used as inputs to group decision problems. In *voting*, voters submit preference rankings, and the aim is to identify an alternative, a set of alternatives, or a ranking of the alternatives that best represent the voters' joint preferences. Over time, many variations of this setting have been studied, and many voting rules have been proposed. For each such voting rule, the key computational problem is *winner determination*, i.e., computing the output of the rule given the input preferences.

For many popular voting rules, this problem is straightforward to solve, and indeed many voting rules are *defined* by a specification of a winner determination algorithm. For example, the *Plurality* rule computes the score of each alternative as the number of voters that rank this alternative first, and outputs the alternative(s) with the highest score; clearly, the running time of this procedure is linear in $n + m$. However, as observed by Bartholdi III et al. (1989), some voting rules only specify their winning alternatives implicitly, and naïve winner determination algorithms require exponential time. Indeed, Bartholdi III et al. prove that it is NP-hard to decide whether a given alternative is winning under the *Dodgson voting rule*, and similar hardness results have subsequently been obtained for many (otherwise) very attractive voting rules.

As Bartholdi III et al. argue, a good voting rule should admit an efficient winner determination algorithm, for otherwise “a candidate’s mandate might have expired before it was ever recognized”! Indeed, in practice, voters demand the votes to be counted within days or hours. If we nevertheless wish to use voting rules for which computing the output is hard, we need to find ways to mitigate this computational complexity. Several popular strategies exist for this purpose. Bartholdi III et al. (1989) suggest an integer linear programming formulation that captures the winner determination problem for the Dodgson voting rule, and we may hope that practical instances will be solved quickly by modern solvers. We could employ parameterized analysis to develop algorithms whose running time is exponential, say, in the number of alternatives, but not the number of voters; this technique can be useful in elections with a small number of alternatives. More controversially, we may use approximation algorithms: if a voting rule is defined in terms of a scoring procedure so that it outputs the alternative(s) with the maximum/minimum score, we may be able to design an efficient algorithm that finds an alternative whose score is close to optimal. This approach may be acceptable in low-stake elections; moreover, we can view the resulting approximation algorithms as new voting rules, and some of these rules are quite attractive (Caragiannis et al., 2014; Skowron et al., 2015a).

In this section, we will explore yet another strategy: we demonstrate that for many voting rules the winner determination problem becomes easy when voters' preferences belong to one of the restricted preference domains discussed in this survey. This approach fits within the framework of identifying “islands of tractability”, i.e., classes of inputs on which a given problem can be solved in polynomial time: just like many hard graph-theoretic problems become computationally easy if the input graph is a tree, many preference aggregation problems become easy if the input profile is single-peaked or single-crossing. Also, while real-life preferences are rarely single-peaked or single-crossing, they are often not that far from belonging to some restricted domain, in terms of distance measures discussed in Section 4.8 (see, e.g., Sui et al., 2013), and some of the positive results for structured profiles extend to almost structured profiles; we will mention several results of this type later in this section.

In the remainder of this section, we will focus on single-winner voting rules, which aim to output a single winning alternative, and multi-winner rules, which elect a fixed-size set of winners. We will also briefly mention some results for social welfare functions, i.e., mappings that output a ranking of the alternatives.

6.1 Single-Winner Rules

This section studies single-winner voting rules, so-called *social choice functions*. Formally, a social choice function is a mapping f that for every preference profile P selects a non-empty set $f(P) \subseteq A$ of *winning alternatives*. The interpretation is that the alternatives in $f(P)$ are tied for winning, and some tie-breaking mechanism will later decide the winner. For instance, under Plurality each alternative gets 1 point from each voter that ranks it first, and under the Borda rule, each alternative gets $m - i$ points from each voter who ranks it in position i ; in both cases, the alternative(s) with the largest number of points are considered to be the election winners.

A large variety of social choice functions have been discussed in the literature (see, e.g., [Zwicker, 2016](#)). For most of them, an element of the set $f(P)$ can be computed in polynomial time. Indeed, for many rules, a stronger statement is true: given an alternative, we can decide in polynomial time whether it belongs to the set $f(P)$. This holds, for instance, for all scoring rules (a large class of rules that includes both the Borda rule and Plurality), for Plurality with Runoff, for Copeland's rule, for Minimax and for Schulze's method.

However, as observed by [Bartholdi III et al. \(1989\)](#), there are appealing social choice functions for which one cannot find an element of $f(P)$ in polynomial time unless $P=NP$. The key examples are the Dodgson rule ([Hemaspaandra et al., 1997](#)), the Young rule ([Rothe et al., 2003](#)), and the Kemeny winner rule ([Hemaspaandra et al., 2005](#)). For each of these rules, deciding whether a given alternative is a winner is Θ_2^P -complete. Further examples are provided by some *tournament solutions*, i.e., social choice functions that only depend on the majority relation induced by a profile: specific tournament solutions that have a hard winner determination problem are the Banks set, the minimal extending set, and the tournament equilibrium set ([Brandt et al., 2016](#)). We will now argue that for many of these rules, winner determination becomes much easier if voters' preferences belong to one of the restricted domains discussed in this survey.

6.1.1 Condorcet Extensions

Recall the definitions of weak and strong Condorcet winners from [Section 2](#): these are the alternatives preferred to every other alternative by a weak (respectively, strong) majority of voters. In many settings, it is desirable to elect Condorcet winners whenever they exist. Formally, we say that a voting rule f is a *Condorcet extension* if given a profile P with a strong Condorcet winner c , f outputs c as the unique winner at P . A *weak-Condorcet extension* is a voting rule that outputs the set of all weak Condorcet winners whenever this set is non-empty. By construction, every weak-Condorcet extension is also a Condorcet extension.

Many of the social choice functions commonly discussed in the literature are Condorcet extensions. Below we define three such rules; see also [Fishburn \(1977\)](#) for an overview.

The Dodgson rule Charles Dodgson, better known by his pen name Lewis Carroll, proposed a Condorcet extension in 1876, though apparently unaware of Condorcet's work. According to Dodgson's method, the score of an alternative is defined as the minimum number of swaps of adjacent alternatives in the preference profile needed to make this alternative a strong Condorcet winner, and the winners are the alternatives with the smallest score. While this rule is not particularly attractive from an axiomatic perspective ([Brandt, 2009](#)), it offers an interesting approach to winner determination, and can be seen as an instantiation of the *distance rationalizability* framework ([Elkind and Slinko, 2016](#)). Moreover, it is an important example of a voting rule with a hard winner determination problem: [Bartholdi III et al. \(1989\)](#) proved that checking whether a given alternative is a Dodgson winner is NP-hard,

and subsequently Hemaspaandra et al. (1997) proved that this problem is Θ_2^P -complete, which was the first Θ_2^P -completeness result in the computational social choice literature.

The Young rule Young (1977) proposed a voting method that is somewhat similar in spirit to the Dodgson rule. According to the Young rule, an alternative's score is the number of voters that need to be removed to make that alternative a weak Condorcet winner; the alternatives with the smallest score are election winners. Again, deciding whether a given alternative is a Young winner is Θ_2^P -complete (Rothe et al., 2003; Brandt et al., 2015). There is also a variant of this rule, referred to as strongYoung (Brandt et al., 2015), where the score of an alternative a is the number of voters that need to be removed to make a a *strong* Condorcet winner (this number may be $+\infty$ for some $a \in A$, but there is always an alternative whose strongYoung score is finite).

The Kemeny rule The Kemeny rule (Kemeny, 1959) is a *social welfare function*, i.e., a mapping that returns a set of linear orders over A . This mapping can then be transformed into a social choice function: we say that an alternative is a *Kemeny winner* if it is ranked first in some linear order produced by the Kemeny social welfare function.

Given a preference profile P , we define the *Kemeny score* of a linear order \succ as

$$\sum_{i \in N} |\succ_i \cap \succ| = \sum_{i \in N} |\{(a, b) \in A \times A : a \succ_i b \text{ and } a \succ b\}|,$$

and output the ranking(s) with the maximum score. Intuitively, these are the rankings that maximize agreement with the input profile. Equivalently, we can define the rule as minimizing the number of *disagreements* with the input profile. Computing a Kemeny ranking is equivalent to solving the feedback arc set problem in the weighted majority graph induced by the preference profile (Bartholdi III et al., 1989), and therefore it is NP-hard to find a Kemeny ranking. Deciding whether a given alternative is a Kemeny winner is Θ_2^P -complete (Hemaspaandra et al., 2005).

For weak-Condorcet extensions, winner determination is easy as long as the input profile belongs to a restricted domain that guarantees the existence of a weak Condorcet winner: one can simply output all weak Condorcet winners. If the majority relation is antisymmetric (this happens, for instance, if the number of voters is odd), every weak Condorcet winner is also a strong Condorcet winner, so in this case we also get positive results for Condorcet extensions. These observations are due to Brandt et al. (2015, Theorem 3.2).

Proposition 6.1. *Consider a social choice function f , a profile P over A that is single-peaked on a tree, single-crossing on a tree, or group-separable, and an alternative $a \in A$. Then if f is a weak-Condorcet extension, or if f is a Condorcet extension and the majority relation associated with P is antisymmetric, we can decide in polynomial time whether $a \in f(P)$.*

The Kemeny rule and the Young rule are weak-Condorcet extensions, and therefore by Proposition 6.1 the winner determination problem is easy for these rules, as long as voters' preferences are single-peaked on a tree, single-crossing on a tree, or group-separable. Also, tournament solutions are defined on tournament graphs, which correspond to profiles with antisymmetric majority relation, and hence Proposition 6.1 applies to many tournament solutions as well. However, some well-studied Condorcet extensions fail to be weak-Condorcet extensions; for example, this is the case for the Dodgson rule. For these rules, Proposition 6.1 only offers an easiness result for an odd number of voters, and the case of the even number of voters has to be handled separately. In the following sections, we discuss the Dodgson rule, the Young rule, and the Kemeny rule in more detail.

6.1.2 The Dodgson Rule

While the Dodgson rule is clearly a Condorcet extension, it is not a weak-Condorcet extension (Brandt et al., 2015). Thus, Proposition 6.1 does not tell us whether we can efficiently compute Dodgson winners for single-peaked profiles with an even number of voters. Brandt et al. (2015) answer this question in the positive, presenting a greedy algorithm that returns all Dodgson winners when given a single-peaked profile. The correctness proof for this algorithm is somewhat involved. The key insights behind the algorithm are that every Dodgson winner must also be a weak Condorcet winner, and that there is always an optimal sequence of swaps that only modifies the preference relations of at most two voters. Fitzsimmons and Hemaspaandra (2020, Theorem 16) show that we can efficiently compute Dodgson winners for single-crossing preferences, using a similar algorithm.

Beyond this result, it is natural to ask whether it is possible to compute the Dodgson *score* of a given alternative in polynomial time, for profiles that belong to restricted domains (this problem is NP-hard in the general case). Being able to calculate scores is useful if one would like to use the Dodgson rule in order to *rank* the alternatives. Fitzsimmons and Hemaspaandra (2020, Theorem 13) prove that calculating Dodgson scores is easy for single-peaked preferences, which provides an alternative algorithm for computing Dodgson winners for single-peaked profiles.

Open Problem 8. *Can one efficiently compute Dodgson scores for other restricted domains, such as single-crossing preferences? Can one efficiently find Dodgson winners for profiles that are single-crossing on a tree or single-peaked on a tree, when the number of voters is even?*

6.1.3 The Young Rule

Since the Young rule is a weak Condorcet extension, its winner determination problem becomes easy for all the domain restrictions mentioned in Proposition 6.1. While strongYoung is not a weak-Condorcet extension, the winners under this rule can also be computed in polynomial-time for single-peaked or single-crossing profiles. This follows from the observation that for these domains the strongYoung score of the winning alternative does not exceed 1: it suffices to remove at most one voter to engineer a situation where there is a unique median voter.

A related problem of interest is to compute the Young (strongYoung) score of a given alternative. For preferences that are single-peaked (or even single-peaked on a circle), Peters and Lackner (2020) present a simple counting-based algorithm that solves this problem in polynomial time. For single-crossing preferences, this problem is also polynomial-time solvable, because the effect of deleting voters is simply to shift around the position of the median voter (Magiera and Faliszewski, 2017). In contrast, for group-separable preferences both variants of this problem are NP-hard, though they admit an FPT algorithm with respect to the height of the clone decomposition tree (Faliszewski et al., 2022); see the discussion in Section 6.2.1.

6.1.4 The Kemeny Rule

As Proposition 6.1 captures the complexity of computing Kemeny *winners* for many restricted domains, in what follows we focus on computing Kemeny *rankings*.

When the majority relation is transitive, the Kemeny ranking is unique and coincides with the majority relation: every pair of alternatives (a, b) contributes either $n_{ab} = |\{i \in N : a \succ_i b\}|$ or $n_{ba} = |\{i \in N : b \succ_i a\}|$ to the score of a ranking \succ , and the majority relation is the only ranking for which every pair (a, b) contributes $\max\{n_{ab}, n_{ba}\}$. Thus, if the number of voters is odd and voters' preferences are single-peaked or single-crossing, the Kemeny ranking can be computed easily. For an even number of voters and single-peaked or single-crossing preferences,

the strict part of the majority relation is transitive, and, by the same argument as above, the set of Kemeny rankings is exactly the set of all its linearizations.

For preferences single-peaked on trees, computing Kemeny rankings remains NP-hard. Indeed, we can reduce the problem of computing a Kemeny ranking for general preferences to that of computing a Kemeny ranking for preferences single-peaked on a star, by adding a new alternative that is ranked first by all voters: this transformation makes the profile single-peaked on a star, and there is a one-to-one correspondence between Kemeny rankings for the original profile and Kemeny rankings for the modified profile (Peters et al., 2022, p. 245). Hardness results also hold for preferences that are single-peaked on a circle (Peters and Lackner, 2020) because this preference domain does not impose any restriction on the majority relation (see Section 3.6).

Open Problem 9. *Can Kemeny rankings be computed efficiently for preferences that are single-peaked on “nice” trees, e.g., trees with few leaves?*

For multidimensional d -Euclidean preferences, $d \geq 2$, Escoffier et al. (2022) prove that deciding whether there exists a ranking with at most a given Kemeny score is NP-complete, and hence finding an optimum ranking is NP-hard. They assume that a d -Euclidean embedding is provided in the input. Their result also holds for d -Euclidean preferences defined with respect to the ℓ_1 and ℓ_∞ metrics, as well as for Slater rankings. This result is obtained by showing that McGarvey’s theorem continues to hold for these preference domains. Hamm et al. (2021) consider a variant of this problem, where given as input a preference profile and a d -Euclidean embedding (with respect to the ℓ_2 metric), we are asked to find the ranking with the highest Kemeny score among all rankings that are compatible with the given embedding. They show that this problem can be solved in polynomial time for each fixed d .

Open Problem 10. *What is the complexity of winner determination for other aggregation rules (such as Dodgson, Young, or multi-winner rules) under the d -Euclidean domain?*

The Kemeny rule behaves like a median. Kemeny (1959) also defined a second rule that behaves like a *mean*. Lederer et al. (2024) study this rule under the name *Squared Kemeny*, since it is obtained by squaring the distances appearing in the definition of the normal Kemeny rule. Lederer et al. (2024) show that Squared Kemeny is also NP-hard. Since Squared Kemeny is not a function of the weighted majority margins, it is not clear that it would become easy to compute on restricted classes of preferences that induce a transitive majority relation, such as single-peakedness. Thus, it is an open problem to identify domains where Squared Kemeny rankings becomes easy to compute. The same problem is also open for the *equalitarian Kemeny rule*, which find the ranking that minimizes the maximum distance to any voter, which is again NP-hard (Biedl et al., 2009).

6.1.5 Sequential Rules

An important class of single-winner rules that we have not discussed so far is that of sequential, or multi-step, rules. These are rules defined by sequential procedures, where one may have to break ties at each step: examples of such rules include Ranked Pairs, Instant Runoff Voting (also known as Single Transferable Vote), the Baldwin rule, and the Nanson rule.

If these “intermediate” ties are always broken according to a fixed order of alternatives, there is a unique winner for each profile, and it can be computed in polynomial time; however, the resulting rule is not neutral. On the other hand, if we define $f(P)$ to be the set of all alternatives that win for some way of breaking intermediate ties (“parallel-universe tie-breaking”), then it is still easy to compute *some* element of $f(P)$, but for many of these rules checking whether a

given alternative is in $f(P)$ becomes NP-hard (Conitzer et al., 2009; Brill and Fischer, 2012; Mattei et al., 2014). Thus, in terms of their computational complexity, these rules are positioned between “easy” rules, such as Plurality, and “hard” rules, such as the Dodgson rule. It remains open whether computing all winners under Instant Runoff Voting or Ranked Pairs becomes easy when voters’ preferences belong to one of the restricted domains considered in this survey; no results of this type are known so far.

Open Problem 11. *What is the complexity of checking whether an alternative is a winner under Instant Runoff Voting or Ranked Pairs under parallel-universe tie-breaking for profiles that are single-peaked, or that have some other structure?*

Wang et al. (2019) study Instant Runoff Voting experimentally and find that on random single-peaked profiles, it is likely to select the Condorcet winner as the unique winner. Tomlinson et al. (2023) show that Instant Runoff Voting is better behaved under ballot truncation on single-peaked, single-crossing, and 1-Euclidean profiles. Tomlinson et al. (2024) show that on 1-Euclidean profiles, Instant Runoff Voting never elects candidates in extreme positions, if there exists at least one candidate in a location bounded away from the extremes; Tomlinson et al. (2025) show that a similar result need not hold in higher dimensions.

Two sequential elimination methods that are similar to Instant Runoff Voting are the Baldwin rule (which at each step eliminates an alternative with the lowest Borda score) and the Coombs method (which at each step eliminates an alternative that is bottom-ranked by the highest number of voters). Both have NP-hard winner determination problems under parallel-universe tie-breaking and unrestricted preferences (Mattei et al., 2014). The Baldwin rule is a weak-Condorcet extension (because a weak Condorcet winner is never a Borda loser unless all remaining alternatives are weak Condorcet winners, see Zwicker, 2016, Footnote 38), and hence it is easy to compute for preference domains such as single-peaked preferences. The Coombs method is not a Condorcet extension, but on single-peaked profiles with an odd number of voters, it always selects the Condorcet winner (Grofman and Feld, 2004, Prop. 2) and thus is easy to compute in this case.

6.2 Multi-Winner Rules

In many group decision problems the goal is to select multiple alternatives rather than a single winner. Voting rules used for this purpose are called *multi-winner voting rules*, or *committee selection rules*. The input to such rules is a preference profile P over A and an integer k ; the output is a non-empty set of *committees*, which are size- k subsets of A . Multi-winner voting rules can be used to select governing bodies, decide how to allocate limited advertising space, or to shortlist applicants to be interviewed for a job (Elkind et al., 2017a). The desiderata for committee selection rules depend on the target application: choosing movies to be included in the in-flight entertainment system is very different from deciding which applicants to accept into a selective PhD program. Accordingly, the literature has proposed a wide variety of committee selection rules. For a general overview on this research topic, we refer the reader to the surveys by Faliszewski et al. (2017) and Lackner and Skowron (2023).

While for some multi-winner voting rules one can compute a winning committee in polynomial time, for others this problem is computationally hard. In the rest of this section, we focus on two committee selection rules whose aim is to select a *representative committee* (the Chamberlin–Courant rule and the Monroe rule), and discuss the complexity of computing winning committees when voters’ preferences belong to a restricted domain.

6.2.1 The Chamberlin–Courant Rule

Chamberlin and Courant (1983) proposed a rule that aims to identify a committee that is maximally representative. Under their approach, each size- k committee is assigned a score, and committees with the highest score are declared to be the election winners. The score of a given committee W is computed by asking each voter to evaluate W , and aggregating the resulting scores; under the utilitarian variant of this rule (this is the variant that was proposed by Chamberlin and Courant, 1983) the individual voters' scores are added up, and under the egalitarian variant (Betzler et al., 2013) the score of a committee W is computed as the minimum of voters' scores. It remains to explain how voters evaluate committees. In the classic variant of the rule, the evaluations are based on Borda scores: if a voter ranks her most preferred member of W in position i in her vote, she assigns a score of $m - i$ to W . More generally, each vector (w_1, \dots, w_m) with $w_1 \geq \dots \geq w_m$ can be used to define a committee selection rule, which we will call **w-Chamberlin–Courant**: a voter's score for a committee W is w_i if her most preferred member of W appears in the i -th position in her preference ranking. Formally, the utilitarian and the egalitarian score of a committee W under the **w-Chamberlin–Courant** rule are defined as, respectively,

$$\text{CC}_{\text{util}}(W) = \sum_{i \in N} \max\{w_{\text{rank}_i(a)} : a \in W\}, \quad \text{CC}_{\text{egal}}(W) = \min_{i \in N} \max\{w_{\text{rank}_i(a)} : a \in W\}.$$

Intuitively, under both variants of the Chamberlin–Courant rule, each voter is represented by a single member of the committee. She can choose her representative within the committee, and thus will choose the committee member she prefers most. Her satisfaction is measured by how much she likes her representative.

The winner determination problem associated with the Chamberlin–Courant rule is usually translated into a decision problem by considering it as an optimization problem, so that we ask: “given a profile P , a target committee size k , and integer B , does there exist a committee W whose Chamberlin–Courant score is at least B ?”. This problem is NP-complete when using Borda scores (Lu and Boutilier, 2011; see also Peters et al., 2022). Thus, we will analyze restrictions of this problem to structured preference profiles.

Single-Peaked Preferences Betzler et al. (2013) show that a winning committee under the utilitarian Chamberlin–Courant rule can be identified in polynomial time if the input preferences are single-peaked. Their algorithm is a dynamic program which builds up an optimal committee by moving along the axis from left to right.

Theorem 6.2 (Betzler et al., 2013, Theorem 8). *For any non-increasing scoring vector \mathbf{w} , any target committee size k , and a single-peaked preference profile, we can find a winning committee according to the utilitarian variant of the Chamberlin–Courant rule in $O(m^2n)$ time.*

Proof. Let P be a profile that is single-peaked with respect to the axis $a_1 \triangleleft a_2 \triangleleft \dots \triangleleft a_m$. We can find this axis in linear time using the algorithm in Section 4.1. For each alternative $a_t \in A$, and every integer j with $j \leq \min\{k, t\}$, define

$$z(a_t, j) := \max\{\text{CC}_{\text{util}}(W) : W \subseteq \{a_1, \dots, a_t\}, |W| = j, \text{ and } a_t \in W\}$$

to be the highest Chamberlin–Courant score attainable by a size- j committee that only uses alternatives that appear in the first t positions on the axis and includes a_t . Then, the Chamberlin–Courant score of an optimum committee is given by $\max_{a_t \in A} z(a_t, k)$. The values

$z(a_t, j)$ can be computed using the following recurrence, valid for $t \geq 2$, $2 \leq j \leq k$, $j \leq t$:

$$z(a_t, j) = \max_{j-1 \leq p \leq t-1} \left\{ z(a_p, j-1) + \sum_{i \in N} \max\{0, w_{\text{rank}_i(a_t)} - w_{\text{rank}_i(a_p)}\} \right\}.$$

To see that this recurrence is correct, consider a size- j committee W whose rightmost member is a_t . Let $W' = W \setminus \{a_t\}$, and let a_p be the rightmost member of W' . Now, each voter i who was represented by an alternative a_q with $q < p$ in W' is represented by the same alternative in W : if she preferred a_t to a_q , then $a_q \triangleleft a_p \triangleleft a_t$ would form a valley in her preferences. In particular, we have $w_{\text{rank}_i(a_t)} - w_{\text{rank}_i(a_p)} \leq 0$. On the other hand, a voter i that was represented by a_p in W' may be represented by either a_p or a_t in W ; her gain from switching from a_p to a_t is $w_{\text{rank}_i(a_t)} - w_{\text{rank}_i(a_p)}$ (if this value is negative, she prefers not to switch). Hence the increase in the Chamberlin–Courant score of W over W' is just $w_{\text{rank}_i(a_t)} - w_{\text{rank}_i(a_p)}$ summed over all voters i who prefer a_t to a_p . For implementation details, see [Betzler et al. \(2013, Thm. 8\)](#). \square

[Sornat et al. \(2022, Theorem 6\)](#) provide a faster algorithm for this problem, based on the Minimum Weight t -Link Path problem, running in $O(nm \log n + m^{1+o(1)})$, which is almost linear. Yet another algorithm (with worse running time) involves formulating the winner determination problem under this rule as a totally unimodular integer linear program ([Peters, 2018](#); [Peters and Lackner, 2020](#)).

There also exist tractability results for preferences that are nearly single-peaked. [Cornaz et al. \(2012\)](#) show that the dynamic programming approach of [Theorem 6.2](#) can be extended to profiles of bounded single-peaked width. [Misra et al. \(2017\)](#) give algorithms for computing a Chamberlin–Courant winning committee for profiles that can be made single-peaked by deleting a few voters or alternatives: their algorithms run in FPT time with respect to the number of alternatives to be deleted, and in XP time with respect to the number of voters to be deleted. Their algorithm for alternative deletion runs in time $O^*(2^d)$ where d is the number of alternatives to be deleted; [Sornat et al. \(2022\)](#) show that this dependence on d is best-possible under the Strong Exponential Time Hypothesis. For voter deletion, [Chen et al. \(2023\)](#) improve the XP result by giving an FPT algorithm. While the deletion-based measures for near single-peakedness lead to tractability results, [Misra et al. \(2017\)](#) show NP-hardness for a generalization of single-peaked preferences where up to three peaks are allowed.

[Sonar et al. \(2020\)](#) study the problem of deciding whether a given alternative appears in some optimal Chamberlin–Courant committee. They show that this problem is Θ_2^P -complete in general, but becomes polynomial-time solvable for single-peaked preferences. They leave it as an open question to determine whether the same is true for other preference domains.

For the egalitarian version of the Chamberlin–Courant rule, [Betzler et al. \(2013\)](#) show that it can be evaluated in polynomial time using a simple greedy algorithm. To see why this is the case, suppose we wish to decide whether there is a committee whose egalitarian Chamberlin–Courant score is at least B . This is equivalent to looking for a committee where each voter’s representative is ranked in position b or higher, for an appropriate value of b . As voters’ preferences are single-peaked, for each voter the set of alternatives ranked in position b or higher forms an interval of the axis. So our problem is equivalent to asking whether there are k alternatives that together cover (or “stab”) all the intervals. As [Betzler et al. \(2013\)](#) note, this interval stabbing problem is equivalent to a clique cover problem, which can be solved in linear time ([Golumbic, 2004](#)). Explicitly, the algorithm proceeds from left to right along the axis; if it reaches the rightmost point of the interval of some voter who is not yet represented by any of the already-selected alternatives, it adds the alternative associated with this point to the

committee. The algorithm reports success if it reaches the end of the axis without selecting more than k alternatives. [Elkind and Ismaili \(2015\)](#) show tractability for single-peaked preferences for a family of Chamberlin–Courant variants that interpolate between the utilitarian and the egalitarian objective.

Single-Crossing Preferences [Skowron et al. \(2015b\)](#) study the complexity of the Chamberlin–Courant rule for single-crossing preferences, and obtain a positive result for this setting: one can find an optimal committee in polynomial time if the input profile is single-crossing. Their approach is based on an interesting structural observation concerning the allocation of representatives to voters. For each member a of a committee W obtained under the Chamberlin–Courant rule, we can think of the voters represented by a as a ’s “district”. For single-crossing preferences, the voter space has a one-dimensional structure, so a natural shape for each such district would be an interval of the voter ordering. [Skowron et al. \(2015b\)](#) find that there always exists an optimal Chamberlin–Courant committee whose districts have this shape. All of their results hold both for the utilitarian and for the egalitarian version of this rule, so in what follows we do not distinguish between the two versions.

Proposition 6.3 ([Skowron et al., 2015b](#), Lemma 5). *Let $P = (v_1, \dots, v_n)$ be a profile that is single-crossing with respect to the given order, and fix a non-increasing scoring vector \mathbf{w} and a target committee size k . Then for any committee W that is optimal for the Chamberlin–Courant rule and for each $a \in W$, the set $\{i \in N : a \succ_i b \text{ for all } b \in W\}$ of voters who are represented by a forms an interval of $(1, \dots, n)$.*

Proof. Suppose not, and alternative $a \in W$ represents voters i and ℓ , but does not represent voter j , where $i < j < \ell$. Suppose that j is represented by $b \in W$. Then we must have $a \succ_i b$, $b \succ_j a$, and $a \succ_\ell b$, a contradiction with P being single-crossing with respect to the given order. \square

Proposition 6.3 serves as a basis for a dynamic programming algorithm for the Chamberlin–Courant rule that runs in time $O(n^2mk)$ ([Skowron et al., 2015b](#), Theorem 6). [Constantinescu and Elkind \(2021\)](#) propose a different dynamic programming formulation that results in an $O(nmk)$ algorithm; for the utilitarian version of the Chamberlin–Courant rule, they further improve the running time to $nm2^{O(\sqrt{\log k \log \log n})}$ by interpreting the winner determination problem as a k -link path problem with concave Monge weights.

[Skowron et al. \(2015b\)](#) also show that their approach extends to profiles with bounded single-crossing width ([Cornaz et al., 2013](#)), by proving that the problem of finding an optimal Chamberlin–Courant committee is in FPT with respect to the single-crossing width. In particular, for profiles with bounded single-crossing width they establish an analog of **Proposition 6.3** saying that the voters represented by the alternatives within a given clone set again form an interval; with this result in hand, one can again apply dynamic programming to find an optimal committee.

Just as for single-peaked preferences, [Misra et al. \(2017\)](#) give algorithms for computing a winning committee under the Chamberlin–Courant rule for profiles that can be made single-crossing by deleting a few voters or alternatives: their algorithms run in FPT time with respect to the number of alternatives to be deleted, and in XP time with respect to the number of voters to be deleted. [Chen et al. \(2023\)](#) give an FPT algorithm for voter deletion. [Misra et al. \(2017\)](#) also show an NP-hardness result for a generalization of the single-crossing domain where each pair of alternatives is allowed to cross up to three times.

Preferences Single-Peaked on Trees For preferences single-peaked on trees, the egalitarian version of the Chamberlin–Courant rule remains polynomial-time computable. The argument is similar to the interval-stabbing argument presented above, except that we now need to stab subtrees rather than subintervals (Peters et al., 2022).

However, in contrast to the situation for single-peaked or single-crossing profiles, the utilitarian version of the Chamberlin–Courant rule remains NP-complete for preferences that are single-peaked on a tree, even for Borda scores (Peters et al., 2022). One way around this result is to consider profiles that are single-peaked on trees that have additional restrictions imposed on them, such as trees that have few non-leaf vertices.

For example, consider the Chamberlin–Courant rule with Borda scores for a profile P that is single-peaked on a star with center vertex $c \in A$. As we saw in Example 16, all voters in P rank c in one of the first two positions in their vote. With this restriction, it is easy to find an optimal Chamberlin–Courant committee: including alternative c in the committee guarantees very good representation to every voter, and the remaining spots in the committee can be filled greedily, by counting the number of alternatives’ appearances in the top position (for details, see Peters et al., 2022). Interestingly, there are some non-Borda scoring vectors for which the Chamberlin–Courant rule remains hard even for preferences single-peaked on a star (Peters et al., 2022).

Generalizing the above argument, Peters et al. (2022) show that the Chamberlin–Courant rule with Borda scores admits a polynomial-time algorithm for preferences single-peaked on trees that have a bounded number of internal (non-leaf) vertices. They also give a polynomial-time algorithm for trees that have a bounded number of leaves, using a generalization of the dynamic program of Betzler et al. (2013) for the case of the line. It would be interesting to identify further classes of trees that admit efficient algorithms for the Chamberlin–Courant rule.

Preferences Single-Crossing on Trees In contrast to the case of preferences single-peaked on trees, for preferences single-crossing on trees the Chamberlin–Courant rule admits a polynomial-time winner determination algorithm irrespective of the structure of the underlying tree. This was first claimed by Clearwater et al. (2015), who put forward a dynamic programming algorithm for this problem. However, Constantinescu and Elkind (2021) observed that the dynamic program of Clearwater et al. (2015) may have exponentially many variables, and proposed a different dynamic programming formulation, which results in an $O(nmk)$ algorithm, both for the utilitarian and for the egalitarian version of the Chamberlin–Courant rule. Their approach is based on an analog of Proposition 6.3 for trees, showing that the ‘district’ of each representative can be assumed to form a subtree of the underlying tree.

Group-Separable Preferences Recall the group-separable profile $P_{\text{gs-max}}^m$ over a set of m alternatives constructed in the proof of Proposition 3.26, and the bijection between the votes in this profile and the subsets of $A \setminus \{a_1\}$. Faliszewski et al. (2022) use this bijection to show that the group-separable domain is sufficiently rich for the Chamberlin–Courant rule to remain computationally difficult on profiles from this domain. Below, we use the same approach to derive a simple hardness proof for the Chamberlin–Courant rule with the 2-approval scoring vector, i.e., $\mathbf{w} = (1, 1, 0, \dots, 0)$; essentially, we show that (a variant of) the hardness proof of Procaccia et al. (2008) for this rule goes through for the group-separable domain.

Theorem 6.4. *Given a group-separable profile P with n voters and a parameter k , it is NP-hard to decide if there exists a committee of size k whose utilitarian \mathbf{w} -Chamberlin–Courant score for $\mathbf{w} = (1, 1, 0, \dots, 0)$ is at least n .*

Proof. Recall that an instance of VERTEX COVER is a pair $\langle G, k \rangle$, where G is an undirected graph and k is a positive integer; it is a yes-instance if G admits a vertex cover of size k . Given an instance $\langle G, k \rangle$ of VERTEX COVER with vertex set V , $|V| = m$, and edge set E , $|E| = n$, we construct a group-separable profile P over the set of alternatives V that contains one voter for each edge $\{v_i, v_j\}$; this voter ranks v_i and v_j in top two positions. Such a profile can be constructed by picking the required voters from $P_{\text{gs-max}}^m$; as the group-separable domain is hereditary, we are guaranteed to obtain a group-separable profile. Now, a voter that corresponds to an edge $e = \{v_i, v_j\}$ assigns a score of 1 to a committee $W \subseteq V$ if and only if the set of vertices W covers e . Thus, the Chamberlin–Courant score of a committee of size k is n if and only if this committee corresponds to a vertex cover of G . \square

Note that for the weight vector $(1, 1, 0, \dots, 0)$ the utilitarian Chamberlin–Courant score is equal to n if and only if the egalitarian Chamberlin–Courant score is strictly positive. We obtain the following corollary.

Corollary 6.5. *Given a group-separable profile P with n voters and a parameter k , it is NP-hard to decide if there exists a committee of size k whose egalitarian \mathbf{w} -Chamberlin–Courant score for $\mathbf{w} = (1, 1, 0, \dots, 0)$ is at least 1.*

The clone decomposition tree of $P_{\text{gs-max}}^m$ is a caterpillar, so its height is m . Faliszewski et al. (2022) show that the problem of computing an optimal utilitarian Chamberlin–Courant committee for group-separable preferences is in FPT with respect to the height of the clone decomposition tree (recall that the clone decomposition tree is essentially unique and polynomial-time computable). Specifically, they describe a dynamic programming-based algorithm that runs in time that is linear in 2^h , where h is the height of the clone decomposition tree, and polynomial in the number of alternatives m , the number of voters n , and the committee size k .

Euclidean preferences For 1-Euclidean preferences, we can compute both the utilitarian and the egalitarian Chamberlin–Courant rule in polynomial time using the above results on single-peaked preferences. For d -Euclidean preferences, $d \geq 2$, Sonar et al. (2022) show that computing the outcome of the egalitarian Chamberlin–Courant rule becomes NP-hard and hard to approximate up to a constant factor. They also give some polynomial-time approximation algorithms. The complexity of the utilitarian variant for 2-Euclidean preferences remains open, though Godziszewski et al. (2021) show that an approval-based variant of that rule is NP-hard.

Variants of the Chamberlin–Courant Rule Under the Chamberlin–Courant rule, each voter only obtains utility from their most-preferred committee member. Alternatively, we could allow voters to also obtain utility from their second-most-preferred committee member, etc. This approach leads to the class of *OWA-based rules* (Skowron et al., 2016) (where ‘OWA’ stands for ‘ordered weighted average’), which generalize the Chamberlin–Courant rule. To calculate a voter i ’s utility, an OWA rule with a list of weights $(\lambda_1, \lambda_2, \dots)$ first orders (sorts) the numbers $(w_{\text{rank}_i(a)})_{a \in W}$ and then takes a weighted average with weights $\lambda_1, \lambda_2, \dots$; the goal is to find a committee that maximizes the sum of voters’ utilities. Peters and Lackner (2020) show that these rules are polynomial-time computable for single-peaked preferences.

6.2.2 The Monroe Rule

While the Chamberlin–Courant rule excels at providing good representation to as many voters as possible, it may fail to represent the voters *proportionally*. Indeed, in many situations some committee members will have to represent a large fraction of the voters, while other committee

members will be responsible for a few voters only. Consider, for example, a profile in which five voters rank $a \in A$ first, whereas each of the alternatives $b, c, d, e, f \in A$ is ranked first by a single voter, so that there are ten voters in total. If we aim for a committee of size $k = 6$, then the optimal Chamberlin–Courant committee will be $W = \{a, b, c, d, e, f\}$. Notice that a represents five times as many voters as the other alternatives. This may be undesirable: on the one hand, alternative a may complain that they have excessive responsibilities, and, on the other hand, the five voters who rank a first may justifiably demand that half of the six committee seats be filled with alternatives that they rank highly—it is possible, for instance, that these five voters place b, \dots, f at the bottom of their rankings.

Chamberlin and Courant (1983) address the latter problem by suggesting that the committee W use *weighted voting* to make internal decisions, and then a would receive as much voting weight as the other five committee members together. (They also consider the effect of using power indices to determine the voting weights.) A different proposal, which directly ensures proportional representation, is due to Monroe (1995).

The Monroe rule searches for committees maximizing essentially the same objective function as in the Chamberlin–Courant scheme. However, Monroe requires that (up to rounding issues) every committee member represents the same number of voters.

To define the Monroe rule formally, we will need to make explicit the assignment of voters to representatives. For a committee W , such an assignment is just a function $\Phi : N \rightarrow W$, so that $\Phi(i)$ is the representative of voter i ; note that $\Phi(i)$ need not be i ’s most preferred alternative in W . An assignment is *balanced* if $\lfloor n/k \rfloor \leq |\Phi^{-1}(a)| \leq \lceil n/k \rceil$ for each committee member $a \in W$, so that every committee member represents essentially the same number of voters in Φ . Given a scoring vector \mathbf{w} , the Monroe score of an assignment Φ is $\sum_{i \in N} w_{\text{rank}(\Phi(i))}$. The Monroe score of a committee W is the Monroe score of the best balanced assignment of voters to the members of W . The Monroe rule then returns all committees with the highest score.

As before, we can replace the sum in the definition of Monroe scores by a minimum over all voters $i \in N$ to obtain an egalitarian variant of the Monroe rule.

Unsurprisingly, given that evaluating the Chamberlin–Courant rule is hard, it is also hard to find an optimal committee under the Monroe rule for general preferences, for both its utilitarian and its egalitarian variant (Betzler et al., 2013). As we will see, the addition of the balancedness constraint is an additional challenge from an algorithmic perspective, even if the input preferences are structured.

Single-Peaked Preferences Betzler et al. (2013) show that the *egalitarian* version of the Monroe rule admits a polynomial-time algorithm for single-peaked preferences and any choice of the scoring vector \mathbf{w} . Their algorithm is rather involved and exploits a connection to the 1-dimensional rectangle stabbing problem. The problem admits an FPT (resp. XP) algorithm for nearly single-peaked profiles with respect to voter (resp. alternative) deletion (Chen et al., 2023).

For the utilitarian version of the Monroe rule, Betzler et al. (2013) give an example of a scoring vector \mathbf{w} for which evaluating the Monroe rule remains hard even for single-peaked preferences. However, it remains a challenging open problem to determine whether hardness also holds for Borda scores:

Open Problem 12. *What is the computational complexity of computing a winning committee under the Monroe rule with Borda scores if voters’ preferences are single-peaked?*

Single-Crossing Preferences Skowron et al. (2015b) show that the Monroe rule with Borda scores remains NP-complete to evaluate for single-crossing profiles, by giving a very involved

reduction from the *unrestricted* version of the winner determination problem under the Monroe rule. For the egalitarian version, the complexity is open. However, Skowron et al. (2015b) show that for *narcissistic* single-crossing profiles, an efficient algorithm is available, and Elkind et al. (2014) extend this result to profiles that are both single-crossing and single-peaked. These two algorithms are faster than the algorithm by Betzler et al. (2013) for single-peaked profiles (see the proof of Theorem 6.2).

Other Restricted Domains For other domain restrictions, such as preferences single-peaked on trees, no results for the Monroe rule are known.

Open Problem 13. *What is the complexity of computing the utilitarian or egalitarian version of the Monroe rule for preferences single-peaked on trees?*

A hardness result for the utilitarian version of the Monroe rule over this domain should be easier to obtain than for the elusive case of single-peakedness on a line.

7 Manipulation and Control

Algorithmic aspects of strategic behavior in elections are a major topic within computational social choice. As with many voting problems, the computational complexity of determining how to influence an election often decreases if structure in preferences is assumed. However, in contrast to the results considered in [Section 6](#), here it is not necessarily the case that a decrease in complexity is desirable. Indeed, NP-hardness results for manipulation are often interpreted as ‘barriers’ to strategic behavior, so polynomial-time algorithms for structured preferences are viewed as negative results in this context, showing that for such preferences these barriers may disappear.

[Walsh \(2007\)](#) was the first to consider the complexity of strategic behavior for restricted preference domains, showing that a popular voting rule known as Single Transferable Vote remains NP-hard to manipulate when voters’ preferences are single-peaked; this result holds if there are at least three alternatives and the voters are weighted, with weights given in binary. The first systematic study of the complexity of manipulation and control in single-peaked elections was undertaken by [Faliszewski et al. \(2011\)](#), and by now there is a large body of research that considers various forms of strategic behavior, focusing primarily (though not exclusively) on single-peaked and nearly single-peaked preferences.

In the following we will focus on two computational problems that have been fundamental in the study of strategic voting: coalitional manipulation and control. There is a large research literature on these and related problems (such as cloning and bribery); we refer the reader to the surveys of [Faliszewski and Procaccia \(2010\)](#), [Faliszewski et al. \(2010\)](#), [Conitzer and Walsh \(2016\)](#), [Faliszewski and Rothe \(2016\)](#), and [Hemaspaandra et al. \(2016\)](#). Furthermore, we restrict ourselves to two simple voting rules: Borda and Veto. Both Borda and Veto are scoring rules, i.e., alternatives receive points depending on their positions in voters’ rankings and the alternative(s) with the largest number of points win the election. For Borda, each alternative gets $m - i$ points from each voter who ranks it in position i . For Veto, alternatives get 1 point from each voter who ranks them in position $1, \dots, m - 1$, and 0 points from voters that rank them last. Towards the end of this section, we provide a brief summary of results for other voting rules and other forms of strategic behavior.

Coalitional Manipulation As a consequence of the seminal result of [Gibbard \(1973\)](#) and [Satterthwaite \(1975\)](#), any reasonable voting rule is manipulable, in particular Borda and Veto. Hence, there are preference profiles in which some voter can change the outcome of the election in their favor if they misreport their true preferences. If a group of voters form a coalition with the intention of manipulation, they may be able to influence the outcome of the election even if no single voter is pivotal. This form of *coalitional manipulation* is captured by the following computational problem.

UNWEIGHTED \mathcal{R} COALITIONAL MANIPULATION

Instance: A preference profile P , a distinguished alternative c , and an integer $k \geq 0$.

Question: Can we add k new voters (manipulators) to P so as to make c the winner according to voting rule \mathcal{R} ?

The WEIGHTED COALITIONAL MANIPULATION is defined analogously, but with both voters in the original profile P and the manipulators having non-negative integer weights (given in binary).

Note that these two problems can also be viewed from the perspective of winner determination given incomplete information: given a profile in which k voters have not yet declared their preferences, is it still possible that alternative c wins?

Let us first consider the case of unrestricted preferences, i.e., arbitrary preference profiles. UNWEIGHTED \mathcal{R} COALITIONAL MANIPULATION is solvable in polynomial time for the Veto rule (Zuckerman et al., 2009) and is NP-complete for Borda (Betzler et al., 2011; Davies et al., 2014). The WEIGHTED COALITIONAL MANIPULATION problem is NP-complete for both Veto and Borda; indeed, it is NP-complete for essentially all scoring rules except for Plurality (Conitzer et al., 2007; Hemaspaandra and Hemaspaandra, 2007; Procaccia and Rosenschein, 2007).

We have already seen many examples of hard computational social choice problems that become polynomial-time solvable for single-peaked preferences. For coalitional manipulation we have to be a bit careful in defining what the corresponding computational question actually is. The approach that is common in the literature (Walsh, 2007; Faliszewski et al., 2011) is to assume that a single-peaked axis \triangleleft is part of the input, so that P is single-peaked with respect to \triangleleft and the manipulators must submit votes that are also single-peaked with respect to \triangleleft . In the following, if we consider manipulation problems given single-peaked profiles, we assume this model and the associated extra input.

It is not hard to see that UNWEIGHTED VETO COALITIONAL MANIPULATION for single-peaked profiles is in P: observe that only the two outermost alternatives on \triangleleft can be vetoed. For Borda elections, Yang and Guo (2016) show that UNWEIGHTED BORDA COALITIONAL MANIPULATION for single-peaked profiles can be solved in polynomial time for one or two manipulators; the general question remains as an interesting open problem.

Open Problem 14. *What is the computational complexity of UNWEIGHTED BORDA COALITIONAL MANIPULATION for single-peaked profiles with three or more manipulators?*

For the weighted problem, the computational landscape is far better explored: WEIGHTED \mathcal{R} COALITIONAL MANIPULATION is in P for Veto and NP-complete for Borda. This follows from a very general theorem by Brandt et al. (2015), which provides a complexity dichotomy for all scoring rules.

One may then wonder if this easiness result for Veto extends to preferences that are nearly single-peaked. The impact of nearly single-peaked preferences on coalitional manipulation and control was first studied by Faliszewski et al. (2014). They obtain a dichotomy results for profiles that can be made single-peaked by deleting voters (cf. Section 4.8).

Theorem 7.1 (Faliszewski et al., 2014). *Let $m \geq 3$, and consider m -alternative profiles that can be made single-peaked by deleting at most ℓ voters. For such profiles UNWEIGHTED VETO COALITIONAL MANIPULATION can be solved in polynomial time if $\ell \leq m - 3$ and is NP-complete for $\ell > m - 3$.*

We see that manipulating the Veto rule is easy as long as the profile is close to being single-peaked, and becomes computationally hard if we need to delete many voters to obtain a single-peaked profile. Interestingly, the statement of Theorem 7.1 extends to alternative deletion, i.e., we have polynomial-time solvability for profiles that can be made single-peaked by deleting ℓ -alternatives if $\ell \leq m - 3$ and NP-completeness for $\ell > m - 3$ (Erdélyi et al., 2015). Erdélyi et al. (2015) further explore these questions for many of the closeness measures introduced in Section 4.8 and confirm the intuition that easiness results only hold for almost single-peaked profiles. Further work on almost single-peaked profiles has been done by Menon and Larson (2016), who study the complexity of coalitional manipulation when voters are allowed to submit partial orders of a certain form, namely *top-truncated ballots*, and Yang (2015), who focuses on

single-peaked width (cf. [Section 4.8](#)).

Control Strategic behavior in elections may also originate from an authority with the power to directly influence the election, which is referred to as *control*. In particular, this authority might be able to exclude some voters from participating in the election so as to make a certain alternative win. This motivates the following computational problem:

\mathcal{R} CONSTRUCTIVE CONTROL BY DELETING VOTERS (\mathcal{R} -CCDV)

Instance: A preference profile P , a distinguished alternative c , and an integer $k \geq 0$.

Question: Can we delete k voters from P so as to make c the winner according to voting rule \mathcal{R} ?

Constructive control by deleting alternatives is defined in a similar manner. One can also consider control by adding voters or alternatives. In this case, an instance of the problem consists of a profile P over A and a pool of additional voters/alternatives that can be added; for adding voters, we assume that each voter in the pool has a ranking of the alternatives in A , and for adding alternatives, it is assumed that each voter in the original profile has a ranking over the set $A \cup A'$, where A' is the set of potential new alternatives, and if we choose to add a subset $B \subseteq A'$ of the alternatives, each voter forms their vote by restricting of their ranking of $A \cup A'$ to $A \cup B$.

CCDV is NP-complete both for Borda ([Russell, 2007](#)) and for Veto elections ([Lin, 2012](#)). If we now turn to single-peaked profiles, we encounter an interesting situation: [Yang \(2017\)](#) shows that BORDA-CCDV remains hard even for single-peaked preferences, but can be solved in polynomial time for single-caved preferences (cf. [Definition 5](#)). In contrast, VETO-CCDV is solvable in polynomial time for single-peaked preferences; single-caved preferences have not been studied in this context so far. For Copeland $^\alpha$ -rules ($\alpha \in [0, 1)$) the complexity of the CCDV (and the CCAV problem about adding voters) is open.

Further work has established a detailed complexity map of control problems for almost single-peaked preferences ([Faliszewski et al., 2014](#); [Yang and Guo, 2017, 2014, 2018](#); [Yang, 2020](#)).

[Magiera and Faliszewski \(2017\)](#) study the complexity of control for single-crossing preferences. They consider adding/deleting voters/alternatives, and investigate both constructive and destructive control, for the Plurality rule and the Condorcet rule (i.e., the rule that outputs the Condorcet winner if it exists and no winners otherwise). They identify a number of settings where the respective control problem is NP-hard for general preferences, but becomes polynomial-time solvable for single-crossing preferences. [Bulteau et al. \(2015\)](#) extend some of these results to the more general problem of combinatorial voter control, where the attacker's cost for adding/deleting a group of voters is not necessarily equal to the sum of her costs for adding/deleting individual voters in that group.

For group-separable preferences, [Faliszewski et al. \(2022\)](#) investigate the complexity of constructive control by adding/deleting voters/alternatives; again, they focus on the Plurality rule and the Condorcet rule. Just as for the winner determination problems discussed in [Section 6](#), for each of the settings they consider, they obtain an NP-hardness result for general group-separable preferences and an FPT result with respect to the height of the clone decomposition tree.

Bribery and Other Forms of Strategic Behavior The *bribery* problem was introduced by [Faliszewski et al. \(2009\)](#); in this problem, an external party (briber) can pay each voter to change their vote according to the briber's instructions. [Brandt et al. \(2015\)](#) find that this problem,

while NP-hard for several voting rules, becomes polynomial-time solvable for single-peaked elections. Subsequently, [Elkind et al. \(2009\)](#) proposed two more fine-grained models, where the price of changing each vote depends on the nature of the changes requested; these are called swap bribery and shift bribery. The computational complexity of these two models have been analysed by [Elkind et al. \(2020b\)](#) for single-peaked and single-crossing preferences, and they obtained polynomial-time algorithms for a number of rules, including Plurality, Borda and Condorcet-consistent rules.

8 Further Topics

In the following we list topics that concern domain restrictions and their computational aspects, but have not been discussed elsewhere in this survey.

8.1 Weak Orders

Single-Peaked Preferences Extending the definition of single-peaked preferences to weak orders is not as straightforward as one might believe. Indeed, there are several natural ways of defining single-peaked weak orders. All these definitions collapse to the usual definition of single-peaked preferences for linear orders, but differ in their treatment of ties.

- (a) In his original definition, Black (1948, 1958) allows ties ‘across’ the peak, but no ties at the peak⁵ and no ties between two alternatives on the same side of the peak. For example, a voter may be indifferent between the alternatives immediately to the left and immediately to the right of the voter’s peak. Written down formally, this is exactly the same as Definition 4. Note that a voter can never report a tie between more than two alternatives.
- (b) In *single-plateaued preferences* (Black, 1958; Moulin, 1984), the vote may have ties among several most-preferred alternatives, but no other ties are allowed. The set of most-preferred alternatives must form an interval of the axis \triangleleft . Some other authors allow additional ties between two alternatives on opposite sides of the top plateau.
- (c) Finally, *possibly single-peaked preferences* (Lackner, 2014) are profiles of weak orders where it is possible to break ties so as to obtain a profile of linear orders that is single-peaked. Equivalently, a profile of weak orders is possibly single-peaked if there is an axis \triangleleft such that for each vote \succsim and each $c \in A$, the set $\{a \in A : a \succsim c\}$ is an interval of \triangleleft ; this is the same condition that appears in Proposition 3.3.

The last notion is the most general among these three definitions: every profile that is single-peaked according to (a) or (b) is also single-peaked according to (c). In addition, definition (c) naturally generalizes to a notion of single-peakedness for profiles of partial orders, which are single-peaked if they can be extended to a single-peaked profile of linear orders (Lackner, 2014).

See Figure 10 for examples.

To recognize profiles that are single-peaked in one of these senses, Fitzsimmons and Lackner (2020) present reductions to the consecutive ones problem, meaning that the recognition problems can be solved in polynomial time. For profiles of partial orders, Fitzsimmons and Lackner (2020) prove that the recognition problem of possible single-peakedness is NP-complete, though they show that for a *given* axis \triangleleft , one can check in polynomial time whether a profile of partial orders can be extended to linear orders that are single-peaked on \triangleleft .

Some of the desirable properties of single-peaked preferences extend to these generalizations. For example, Condorcet winners exist for profiles satisfying Black’s condition (a), by the same proof as for linear orders (see Proposition 3.4). The case for single-plateaued preferences is more complicated; Barberà (2007) provides a detailed discussion. For possibly single-peaked preferences, Condorcet winners definitely do not exist. Fishburn (1973, Table 9.1) provides an example with 5 voters, where $b \succ_1 a \succ_1 c$, $c \succ_2 b \succ_2 a$, $c \succ_3 b \succ_3 a$, $a \succ_4 b \sim_4 c$, and

⁵Arrow (1951) states Black’s definition slightly differently. In Arrow’s definition, we are allowed to tie two top alternatives; cf. the discussion by Dummett and Farquharson (1961).

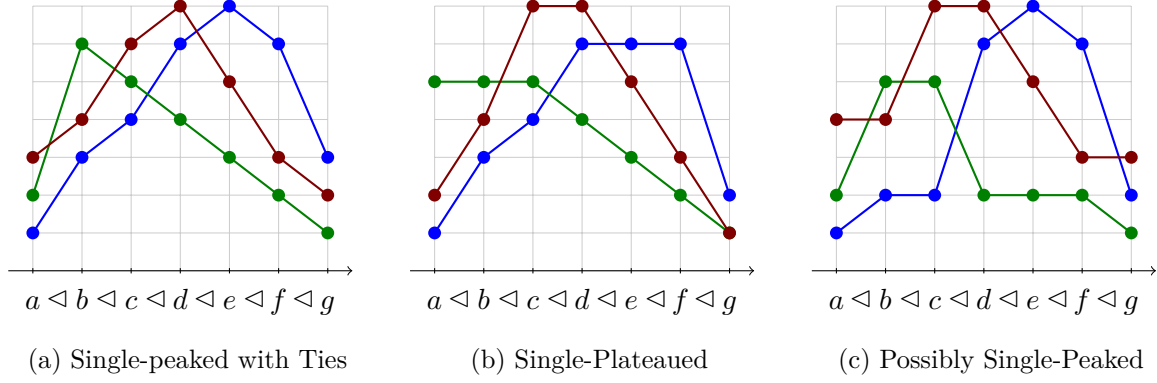


Figure 10: Different definitions for single-peakedness on weak orders

$a \succ_5 b \sim_5 c$. This profile is possibly single-peaked on the axis $a \triangleleft b \triangleleft c$. Its majority relation satisfies $a \succ_{\text{maj}} c \succ_{\text{maj}} b \succ_{\text{maj}} a$, which is not transitive.

Regarding winner determination, the algorithms based on total unimodularity for multi-winner voting rules discussed by [Peters and Lackner \(2020\)](#) continue to work for possibly single-peaked profiles of weak orders. [Fitzsimmons and Hemaspaandra \(2015, 2016\)](#) and [Menon and Larson \(2016\)](#) investigated the complexity of manipulation and bribery when the input preference profile consists of weak orders and is single-peaked.

Single-Crossing Preferences Like for single-peaked preferences, there are several ways to generalize single-crossing preferences to weak orders. [Elkind et al. \(2015\)](#) discuss three natural definitions. They say a profile $P = (v_1, \dots, v_n)$ of weak orders is

- (a) *weakly single-crossing* in the given order if for every pair $a, b \in A$, the sets $I_1 = \{i \in [n] : a \succ_i b\}$, $I_2 = \{i \in [n] : a \sim_i b\}$, and $I_3 = \{i \in [n] : b \succ_i a\}$ form intervals of $[n]$, and I_2 is between I_1 and I_3 .
- (b) *seemingly single-crossing* in the given order if for every pair $a, b \in A$, if $i < k$ are such that $a \succ_i b$ and $a \succ_k b$, then for every $i \leq j \leq k$ we have either $a \succ_j b$ or $a \sim_j b$. In other words, as we scan from left to right, ignoring voters indifferent between a and b , we see that the voters with $a \succ b$ are on one end, and the voters with $b \succ a$ are on the other.
- (c) *possibly single-crossing* if there is some way to break ties in P so as to obtain a profile of linear orders that is single-crossing.

Then P is weakly (resp. seemingly or possibly) single-crossing if there exists a permutation of the votes in P such that the resulting profile is weakly (resp. seemingly or possibly) single-crossing in the given order.

[Elkind et al. \(2015\)](#) consider the computational complexity of recognizing profiles satisfying these conditions. They find that weakly single-crossing profiles can be recognized in polynomial time using similar techniques as in the case for linear orders. On the other hand, by reduction from the betweenness problem, they show that recognizing seemingly and possibly single-crossing profiles is NP-complete. Intriguingly, they find that these problems are not even necessarily easy if we fix the ordering of the voters.

Single-Peaked on a Tree or Circle It is possible to define single-peakedness for trees and circles (and other graphs) using the same recipe that we saw for possibly single-peaked preferences:

require that any upper contour set $\{a \in A : a \succ c\}$ is connected in the underlying graph. To recognize profiles that are single-peaked in such a sense, for circles we can use algorithms for the circular ones property (Booth and Lueker, 1976; Peters and Lackner, 2020), and for trees we can use one of a number of different algorithms for “tree convexity” or “hypergraph acyclicity” (Trick, 1988; Conitzer et al., 2004; Tarjan and Yannakakis, 1984; Sheng Bao and Zhang, 2012). For the case of trees, an interesting open problem is whether we can decide whether a profile of weak orders is single-peaked on a tree with additional structural properties, in the spirit of Peters and Elkind (2016).

Open Problem 15. *For a profile of weak orders that is single-peaked on a tree, can we efficiently find such a tree with the minimum or maximum number of leaves? With minimum max-degree?*

For circles, the algorithms based on total unimodularity for multi-winner voting rules discussed by Peters and Lackner (2020) continue to work for weak orders.

8.2 Dichotomous Preferences

An important special case of weak preferences are dichotomous preferences, which distinguish between approved and disapproved alternatives. Consequently, a vote based on dichotomous preferences corresponds to a subset of (approved) alternatives. The concepts of single-peaked and single-crossing preferences have been adapted to dichotomous preferences (Dietrich and List, 2010; List, 2003; Faliszewski et al., 2011; Elkind and Lackner, 2015), and Yang (2019) has considered dichotomous preferences that are single-peaked or single-crossing on trees. The most prominent definitions are the candidate interval and voter interval domains:

Definition 18. Consider a profile $P = (v_1, \dots, v_n)$, where $v_i \subseteq A$ for $i \in [n]$. We say that P satisfies *Candidate Interval (CI)* if alternatives can be ordered so that each of the sets v_i forms an interval of that ordering. We say that P satisfies *Voter Interval (VI)* if the voters in P can be reordered so that for every alternative c the voters that approve c form an interval of that ordering.

Both CI and VI preferences can be recognized in polynomial time via a reduction to the CONSECUTIVE ONES problem (Faliszewski et al., 2011; Elkind and Lackner, 2015), see also Section 4.1. Terzopoulou et al. (2021) provide forbidden subprofile characterizations for these and other domains, and consider the computational complexity of deciding whether a partially specified profile can be completed so as to fall into one of these domains. Constantinescu and Wattenhofer (2023) give an algorithm for recognizing profiles that are possibly single-crossing.

These domains also prove to be useful for algorithmic purposes. Proportional Approval Voting (PAV), proposed by Thiele (1895), is a multi-winner voting rule defined as follows: Given a desired committee size k , PAV outputs all committees W of size k that maximize $\sum_{i \in N} h(|W \cap v_i|)$, where h is the harmonic series $h(j) = 1 + \frac{1}{2} + \dots + \frac{1}{j}$. PAV is known to satisfy desirable proportionality axioms (Aziz et al., 2017; Lackner and Skowron, 2021), but is NP-hard to compute (Aziz et al., 2015; Skowron et al., 2016). However, for CI preferences a winning committee under PAV can be computed in polynomial time. This was shown by Peters (2018) using an approach based on totally unimodular matrices (see also Peters and Lackner 2020 for a generalization to circular preferences, and Sornat et al. 2022 for a generalization to nearly CI preferences). Interestingly, this approach does not extend to VI preferences.

Open Problem 16. *What is the computational complexity of computing a PAV winner given VI preferences?*

Elkind and Lackner (2015) do, however, show that VI preferences are useful from a parameterized complexity perspective: they identify two natural parameters for which computing PAV winners is para-NP-hard when not restricting the preference domain, but obtain XP and FPT algorithms with respect to these parameters for VI preferences. Liu and Guo (2016) obtain easiness results for another prominent approval-based committee selection rule both for CI and for VI preferences. For a discussion of other structured domains of dichotomous preferences, we refer the reader to the overview by Elkind and Lackner (2015) and, for analogs to multidimensional single-peakedness, to the work of Peters (2017). Lackner and Skowron (2023) review further algorithmic results for these domains.

Pierczyński and Skowron (2022) show that for CI and for VI preferences, there always exists a committee that is in the *core*, which is a stability notion guaranteeing proportional representation. Whether such committees exist without any domain restrictions is a famous open problem (Lackner and Skowron, 2023).

8.3 Elicitation

Preference elicitation is concerned with the acquisition of preference data from (typically human) agents, a topic clearly relevant to computational social choice. An overview of this research agenda can be found in a handbook chapter by Boutilier and Rosenschein (2016). The general goal of preference elicitation is to design efficient elicitation protocols and an often-used measure for the efficiency of a protocol is its communication complexity.

It is natural to expect that structured preferences are easier to elicit. Indeed, it takes $\Theta(\log m!) = \Theta(m \log m)$ comparison queries to elicit an arbitrary ranking of m alternatives. In contrast, if we know that the ranking being elicited is single-peaked with respect to a given axis \triangleleft , we can elicit it using $m - 1$ queries in a bottom-up fashion: we ask the voter to compare the two endpoints of the axis, place the less preferred of the two alternatives in the last position in the voter's ranking, and continue recursively.

Positive communication complexity results are not limited to single-peaked preferences: if preferences are single-peaked (on trees), single-crossing (on trees), or d -dimensional Euclidean, communication complexity of preference elicitation is considerably lower than in the general case (Conitzer, 2009; Jamieson and Nowak, 2011; Dey and Misra, 2016b,a; Dey, 2016); this line of research has been surveyed by Elkind et al. (2017b).

8.4 Counting and Probability

Domain restrictions impose constraints on the mathematical structure of preference profiles. From a combinatorial viewpoint, it is thus natural to ask how many preference profiles (of a given size) belong to a given restricted domain. Assuming that all preference profiles are equally likely, such a result yields the probability that a profile belongs to this restricted domain. Lackner and Lackner (2017) study this question for the single-peaked domain and obtain an asymptotically tight result counting the number of single-peaked profiles (see also Durand, 2003, pp. 581–585). Chen and Finnendahl (2018) obtain an exact enumeration result for single-peaked profiles where the sets of voters and candidates coincide and voters rank themselves first, i.e., voters are narcissistic (cf. Section 3.8). Related results are obtained by Brown et al. (2014) for single-peaked preferences over a multi-attribute domain. Lackner and Lackner (2017) further study the likelihood that a profile is single-peaked if drawn according to the Pólya urn or the Mallows model, and Karpov (2020) considers the likelihood for several natural distributions related to the impartial culture. Karpov (2019b) analyses the number of group-separable profiles.

8.5 Matching and Assignment

Stable Roommates In the *stable roommates* problem, a set of $2n$ agents need to be matched into pairs. Each agent has a preference over potential matching partners. A matching is *stable* if there is no pair of agents who both strictly prefer this pair over their pair in the matching. In contrast to the stable marriage setting (where agents come in two types), stable matchings need not exist for the stable roommates problem (Gale and Shapley, 1962). If there are no ties in the preferences, one can decide in $O(n^2)$ time whether one exists and if so find one (Gusfield and Irving, 1989). In the presence of ties, the problem becomes NP-complete (Ronn, 1990).

Bartholdi III and Trick (1986) studied the stable roommates problem with single-peaked preferences. They assume that there is a common ordering \triangleleft of the set of agents, that each agent's preferences over matching partners is single-peaked with respect to \triangleleft , and that each agent i 's peak is located at i 's position in the axis ("narcissistic" single-peaked preferences). They show that in this case, there always exists a stable matching and that it is unique. They also give a simple algorithm for finding this matching. Bredereck et al. (2020) show that this algorithm runs in time $O(n^2)$. (Bartholdi III and Trick (1986) had claimed an $O(n)$ runtime.)

For narcissistic single-crossing preferences, Bredereck et al. (2020) show that the same result holds: there exists a unique stable matching, and it can be found in $O(n^2)$ time. They also extend the results for single-peaked and single-crossing preferences to the case with ties allowed, for which the same results hold except that uniqueness is not guaranteed. For incomplete preferences with ties, they show NP-completeness results.

Assignment In the assignment problem, each of n agents needs to receive exactly one of n objects. Each agent has preferences over these objects. In *random assignment*, we look for methods that, given the preferences, select a probability distribution of assignments. A famous impossibility theorem of Bogomolnaia and Moulin (2001) states that there is no randomized method that satisfies efficiency, strategyproofness, and equal treatment of equals. The former two axioms are defined via stochastic dominance, and the third is a weakening of anonymity. Kasajima (2013) proved that this impossibility holds even for single-peaked preferences over objects. Chang and Chun (2017) show that it holds even for single-peaked preference profiles in which all agents have the same peak.

A variant of the (deterministic) assignment problem, called the *house assignment problem* (Shapley and Scarf, 1974), starts with an initial assignment of houses to agents (endowments). These can then be traded among the agents. For unrestricted preferences, the Top Trading Cycle algorithm (Shapley and Scarf, 1974) is the only one that satisfies Pareto optimality, individual rationality, and strategyproofness. For single-peaked preferences, however, other rules satisfying these properties are known (Bade, 2019; Beynier et al., 2020), and swap dynamics have been studied for this restricted domain (Beynier et al., 2021; Brandt and Wilczynski, 2024).

8.6 Other Domain Restrictions

There are a number of domain restrictions that have received little or no attention in the computational social choice literature. Among those are Level-1 Consensus preferences (Nitzan et al., 2018) and intermediate preferences on a median graph (Grandmont, 1978; Demange, 2012); see also the book by Gaertner (2001) and the surveys by Puppe and Slinko (2024) and Karpov (2022) for additional domains. A particularly interesting domain restriction is top monotonicity (Barberà and Moreno, 2011), which generalizes both single-peaked and single-crossing preferences (see also Pierczyński and Skowron, 2022). Magiera and Faliszewski (2019) show, using a reduction to 2-SAT, that the recognition problem for top monotonicity is solvable

in polynomial time; previously [Aziz \(2014\)](#) had shown that the problem is NP-hard for partial orders. It remains an open question whether these domains are algorithmically useful, for example for winner determination problems.

9 Research Directions

Throughout the survey, we have mentioned open problems, which pose gaps in our understanding of preference restrictions and their computational benefits. Now, we would like to highlight a more general research directions that we consider promising.

Multidimensional Restrictions Most of this survey has focused on domain restrictions that are in some sense one-dimensional: single-peaked, single-crossing and 1-Euclidean preferences are all defined by a linear order or an embedding into the real line. Multidimensional analogs of these notions have received much less attention in the computational social choice literature. In particular, little is known about computational benefits of such higher-dimensional restrictions. For example, it is not known whether the Kemeny rule is computable in polynomial time on two-dimensional single-peaked profiles (cf. [Section 3.9](#)) or on 2-Euclidean profiles (cf. [Section 3.4](#)). Even if NP-hard voting problems remain hard for these domains, it might be that better approximation algorithms can be found than for general preferences. Multidimensional domain restrictions offer many challenging research questions, but faster algorithms for these classes are very desirable: these algorithms would be applicable to a much larger class of preferences than algorithms for one-dimensional restrictions.

New Preference Restrictions Another direction is to consider completely new restrictions. Domains suggested in the social choice literature usually guarantee the existence of a Condorcet winner, but this is not a necessarily relevant property for algorithmic purposes. Inspiration could be found by adapting structural concepts from graph theory, such as restrictions resembling treewidth. For a systematic study of domain restrictions, the framework of forbidden subprofiles could prove to be valuable, as well as related mathematical theories such as 0-1-matrix containment ([Füredi and Hajnal, 1992](#); [Klinz et al., 1995](#)) or permutation patterns ([Kitaev, 2011](#); [Vatter, 2015](#)) (the connection between forbidden subprofiles and permutation patterns has been established in [Lackner and Lackner \(2017\)](#)). Preference profiles, seen as tuples of linear orders, are mathematically rich structures and there is hope for a similarly diverse and powerful classification of structure as exists for graph classes—along with algorithmic applications of these structural restrictions.

Experiments and Real-World Data A pressing issue is the compatibility of theoretical structural restrictions (such as those discussed in this survey) and structural properties of real-world data sets. Little work has been done on tackling this practical challenge. It can be expected that real-world data such as the data sets found at [preflib.org](#) ([Mattei and Walsh, 2013](#)) rarely belong to a strict mathematical restriction such single-peaked or single-crossing. However, the introduction of distances to these restrictions (as discussed in [Section 4.8](#)) may yield practically useful notions of structure. First analyses in this respect have been performed by [Sui et al. \(2013\)](#) and [Przedmojski \(2016\)](#). As can be expected, the conclusions drawn from experimental work depend on the chosen data set: [Sui et al. \(2013\)](#) study Irish election data and conclude that their data set are almost two-dimensional single-peaked preferences. [Przedmojski \(2016\)](#) finds some data sets that are close to single-peaked preferences, although these typically have few alternatives.

[Boehmer and Schaar \(2023\)](#) report results from a larger-scale analysis of real-world preference data. In their collection of 7582 profiles, only very few belong to a restricted domain (1.3% are single-peaked, 2.3% are single-crossing, and 1.6% are group-separable). They report that only some profiles are close to these domains under the voter or alternative deletion nearness

measures. In addition, profiles that are close to one domain are typically also close to another. They report that these profiles are typically quite degenerate (votes are very similar to each other; single-peaked profiles seems to be single-peaked with respect to many different axis and voters have only few different top-choices; in single-crossing profiles only few candidates ever change their pairwise ordering). 6.3% of their profiles are value-restricted.

Szufa et al. (2020) and Boehmer et al. (2021) propose a new approach to compare *statistical cultures* (probability distributions over profiles), including statistical cultures that generate preferences within restricted domains (e.g., single-peaked profiles). They generate “maps” that visualize the similarity of statistical cultures and domain restrictions, and thereby offer new insights into the structure of preferences. In particular, this approach offers another way to evaluate the similarity between real-world data sets and restricted preference domains. In conclusion, it is of high importance to continue to analyze structural properties of available data sets and statistical cultures, and thus enrich the theoretical research done on this topic.

Beyond Voting Finally, the work on structured preferences has mostly focused on voting-related topics: winner determination, manipulation, control, etc. Given the advances that have been made in these fields, it could prove to be worthwhile to investigate the impact of structured preferences in other fields of social choice; matching, fair division and judgment aggregation are natural candidates.

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