Literature Review Notes

January 25, 2026

Contents

1 Fair Division under Ordinal Preferences: Computing Envy-Free Allocations of Indivisible Goods

Bibliographic Information

- **Authors:** Sylvain Bouveret, Ulle Endriss, Jérôme Lang
- **Year:** 2010
- **Domain:** Fair Division, Computational Social Choice
- **DOI:** 10.3233/978-1-60750-606-5-387

Main Ideas

The paper studies the fair allocation of indivisible goods among agents when only partial ordinal preferences over individual goods are available with certainty, rather than complete preferences over bundles. Agents report strict rankings over individual goods, from which incomplete preferences over bundles are induced. The central question is whether envy-free and/or Pareto efficient allocations can be guaranteed despite this incomplete information.

Research Problem

Given:

- A finite set of indivisible goods
- A finite set of agents with strict ordinal preferences over individual goods
- Induced incomplete preferences over bundles of goods via SCI-nets

Does there exist an allocation that is possibly (necessarily) envy-free?

- **Envy-freeness:** Each agent prefers the bundle she received at least as much as any of the bundles received by others.
- **Pareto efficiency:** No other allocation makes some agents better and none worse off.
- **Possible envy-freeness (PEF):** An allocation is possibly envy-free if there exists at least one completion of the agents' preferences over bundles that makes the allocation envy-free. More precisely, the allocation is possibly envy-free if there exists a set of complete preferences that are consistent with the incomplete preferences

and can complete it to makes the allocation envy-free. For instance, if Alice prefers bundle A and Bob prefers bundle B and C, then an allocation giving A to Alice and B, C to Bob is possibly envy-free as long as Alice did not express her preference for C.

- **Necessary envy-freeness (NEF):** An allocation is necessarily envy-free if it is envy-free under all possible completions of the agents' preferences, no matter how the incomplete preferences are completed. For example, if Alice prefers bundle A to B and Bob prefers B, then the allocation "Alice gets A, Bob gets B" is necessarily envy-free since Alice will never envy Bob regardless of her preferences over other bundles.

Related Questions and Issues:

- Combinatorial explosion: The space of possible bundles grows exponentially with the number of goods, making full preference elicitation infeasible. Indeed, the full elicitation costs cognitively and can lead to uncertainty on the value of the information an agent transmits.
- Computational complexity: What are the computational complexities of checking PEF and NEF? Proof of intractability and tractable cases.
- Algorithmic characterization: Proposals for simple algorithms and hypothesis under which PEF and NEF can be guaranteed.

Hypothesis

- Finite set of indivisible goods and agents
- Strict ordinal preferences over individual goods
- Strict partial ordinal preferences over bundles induced via SCI-nets (irreflexive and transitive)
- Strict linear ordinal preferences over bundles in reality (unknown to the mechanism)

Key Words

- **Indivisible goods**
- **Fair division**
- **Pairwise dominant**
- **Incomplete Ordinal preferences**
- **Envy-freeness**
- **Pareto efficiency**

- **Possibly envy-free (PEF)**
- **Necessarily envy-free (NEF)**
- **SCI-nets**
- **Monotonicity**

Notations

- $G = \{x_1, ..., x_m\}$: Set of indivisible goods $m \geq 1$.
- N: Number of agents.
- $A = \{1, ..., n\}$: finite set of n agents $n \geq 2$.
- $\rhd_N$: linear order on G representing the common ranking of goods by all agents.
- $\succ_i^*$: preference relation of agent i over bundles she might receive.
- $\pi : A \rightarrow 2^G$: allocation function assigning to each agent a bundle of indivisible goods.
  - $\pi(i) \cap \pi(j) = \emptyset$ for all $i, j \in A$ with $i \neq j$: no good is assigned to more than one agent.
  - $\bigcup_{i \in A} \pi(i) = G$: complete allocation of all goods.
- Binary relation $\succ$ or $\rhd$: relation between goods or bundles.
- strict partial order $\succ$: binary relation that is irreflexive and transitive.
- linear order $\succ$: strict partial order that is complete, i.e $X \succ Y or Y \succ X$ whenever $X \neq Y$ for all $X, Y \in G$.
- Monotonicity: if $Y \subset X$, it implies $X \succ Y$
- Reflexive closure $\succeq$ or $\unrhd$:
  - $X \succeq Y$ if and only if $X \succ Y$ or $X = Y$.
  - For two binary relations R and R', R' refines R if $R \subseteq R'$.
- Compliant: A strict partial order $\succ$ is compliant with N, if:
  - $\succ$ is monotonic.
  - $S \cup \{x\} \succ S \cup \{y\}$ for any x, y such that $x \rhd_N y$ and $S \subseteq G \setminus \{x, y\}$.
- $\succ_N$: is the smallest strict partial order that complies with N.

Methodology

- Expression of preferences is restricted to prevent combinatorial explosion.
- Preferences over bundles are not elicited directly.
- Agents provide strict rankings over individual goods (strict linear order).
  - Instead of the actual preferences of an agent i on all bundles, we have a strict partial order $\succ_i$, which represents a partial knowledge of $\succ_i^*$ obtained by inference.
- These rankings induce a partial order over bundles. Inference rule (Brams and King, Brams et al): A set A is preferred to a set B if each preference in B not in A is pairwise dominated by a preference of A not in B.
  - For instance, for $A = \{\text{apple}, \text{banana}, \text{orange}\}$ and $B = \{\text{apple}, \text{strawberry}, \text{kiwi}\}$ we have $A \setminus B = \{\text{banana}, \text{orange}\}$ and $B \setminus A = \{\text{strawberry}, \text{kiwi}\}$. If the agent i prefer banana $\succ_i$ strawberry and orange $\succ_i$ kiwi, then each items in A, not in B pairwise dominates each items in B, not in A.
- SCI-nets provide a compact and structured representation of all compatible complete preferences.
  - An SCI-net is a special type of CI-net (conditional importance network), a compact representation with no preconditions, where comparisons are only made on individual goods (singletons).
- Envy-freeness is evaluated across all possible completions.

Key Results

1. Determining whether a possibly envy-free allocation exists is computationally easy.
2. Determining whether a necessarily envy-free allocation exists is NP-hard.
3. For two agents, the NEF problem is solvable in polynomial time.
4. Necessary envy-freeness requires strong conditions, such as distinct top-ranked goods.

Limitations

- Preferences are assumed to be strict and monotonic.
- Indifference between goods is not allowed. In this article, the authors consider strict order preferences over individual goods, i.e no equality in the order is possible.
- Only ordinal preferences are considered.
- No comparison with cardinal utility-based fairness notions.

Open Questions

- Determining the existence of an envy-freeness allocation which is also necessarily Pareto efficient.
- Extending results to nonstrict SCI-nets to allow indifferences.
- Studying possible/necessary fairness under cardinal preferences.
- Bridging ordinal and cardinal approaches.

Key Ideas for My Research

- Partial preferences are a realistic modelling assumption.
- A necessarily envy-free allocation is stable to preference incompleteness, as it remains envy-free under all possible completions of the agents' unknown preferences. This fairness property is strong for partial preferences.
- SCI-nets provide a formal framework for reasoning about incomplete preferences.

Summary

This paper provides a foundational framework (definitions, propositions, proofs) for studying fair division under incomplete ordinal preferences. It highlights the trade-off between expressive power, computational tractability, and robustness of fairness guarantees.

Dimension	**Options**
Preference model	ordinal / cardinal
Level of information	complet / partiel
Fairness criteria	envy-free / Pareto
Robustness	possible / necessary

2 Handbook of Computational Social Choice (2016) - Chapter 10: Incomplete Information and Communication in Voting

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Main Ideas

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Given:

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Question: ?

Definitions:

Related Questions and Issues:

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Hypothesis

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Key Words

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Notations

Under construction

- $A = \{a_1, ..., a_m\}$: Set of alternatives representing the space of possible choices.
- $N = \{1, ..., n\}$: Set of n agents/voters.
- $\succ_i$: (total) preference order of each voter i over alternatives in A.
  - $a_i \succ_i a_j$: voter i prefers alternative a_i to a_j.
  - Compared to a permutation σ_i of A.
  - $\sigma_i(j)$ denotes the position of alternative a_j in the ranking of voter i. Give position j.
  - $\sigma_i^{-1}(j)$ denotes the alternative ranked at position j by voter i. Give alternative a_j.
- $R(A)$: set of all preference orders over A.
- $R = \langle \succ_1, ..., \succ_n \rangle$: preference profile, i.e. a tuple of preference orders, one for each voter.
- voting rule or social choice function $f(R) \in A$: selects a winning alternative based on the preference profile R. $f(R) \in \arg\max_{a \in A} s(a, R)$.
- Scheme $s(a, R)$: score each alternative $a \in A$ given a preference profile R. Measures the quality (social welfare, etc) of an alternative given the profile.
- co-winner: any alternative a with maximum score, i.e. $s(a, R) = \max_{b \in A} s(b, R)$.
- π_i: partial preference of voter i / partial order of voter i over A / transitive closure of a consistent collection of pairwise comparisons $a_i \succ_i a_k$
- $\Pi = \langle \pi_1, ..., \pi_n \rangle$: partial profile, collection of partial votes.
- completion: any vote $\succ_i$ that extends π_i.
- $C(\pi_i)$: set of all completions of partial order π_i.
- $C(\Pi) = C(\pi_1) \times ... \times C(\pi_n)$: set of all completions of partial profile Π.

Methodology

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Key Results

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Limitations

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Open Questions

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Key Ideas for My Research

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3 Article 3 Title

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Notations

Under construction

- G: Set of indivisible goods

Methodology

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Key Results

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Limitations

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Open Questions

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Key Ideas for My Research

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4 Vocabulary

- **Term 1:** Definition
- **Term 2:** Definition
- **Term 3:** Definition