Appendix

vertex model (general)

energy

$$
\begin{aligned}
E_{(cell-edge)} &= \sum_{i=0}^{n-1} \left(\alpha^{(a)} l_i^{(a)} + \alpha^{(b)} l_i^{(b)} + \alpha^{(l)} l_i^{(l)} \right) \\
E_{(cell-area)} &= \sum_{i=0}^{n-1} \beta (A_i^{(c)} - A_e^{(r)})^2 \\
E_{(vit-rigid)} &= \sum_{i=0}^{n-1} \epsilon \Theta(D_i) w_i D_i^2 \\
E_{(yolk-area)} &= \gamma (A_Y^{(c)} - A_Y^{(r)})^2 \text{ [yolk rest-area fixed]} \\
&= -p^{(Y)} A_Y^{(c)} \text{ [yolk pressure constant]}
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
E &= E_{(cell-edge)} + E_{(cell-area)} + E_{(vit-rigid)} + E_{(yolk-area)} \\
&= \sum_{i=0}^{n-1} \left(\alpha^{(a)} l_i^{(a)} + \alpha^{(b)} l_i^{(b)} + \alpha^{(l)} l_i^{(l)} \right) + \sum_{i=0}^{n-1} \beta (A_i^{(c)} - A_e^{(r)})^2 + \sum_{i=0}^{n-1} \epsilon \Theta(D_i) w_i D_i^2 \\
&+ \gamma (A_Y^{(c)} - A_Y^{(r)})^2 \text{ [yolk rest-area fixed]} \\
E &= E_{(cell-edge)} + E_{(cell-area)} + E_{(vit-rigid)} + E_{(yolk-area)} \\
&= \sum_{i=0}^{n-1} \left(\alpha^{(a)} l_i^{(a)} + \alpha^{(b)} l_i^{(b)} + \alpha^{(l)} l_i^{(l)} \right) + \sum_{i=0}^{n-1} \beta (A_i^{(c)} - A_e^{(r)})^2 + \sum_{i=0}^{n-1} \epsilon \Theta(D_i) w_i D_i^2 \\
&- p^{(Y)} A_Y^{(c)} \text{ [yolk pressure constant]}
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
l_i^{(a)} &= \sqrt{\left(x_{i+1}^{(a)} - x_i^{(a)} \right)^2 + \left(y_{i+1}^{(a)} - y_i^{(a)} \right)^2} \\
l_i^{(b)} &= \sqrt{\left(x_{i+1}^{(b)} - x_i^{(b)} \right)^2 + \left(y_{i+1}^{(b)} - y_i^{(b)} \right)^2} \\
l_i^{(l)} &= \sqrt{\left(x_i^{(a)} - x_i^{(b)} \right)^2 + \left(y_i^{(a)} - y_i^{(b)} \right)^2}
\end{aligned}
$$

$$
\begin{aligned}
A_i^{(c)} = \frac{1}{2}(&x_{i+1}^{(a)}y_i^{(a)} + x_{i+1}^{(b)}y_{i+1}^{(a)} + x_i^{(b)}y_{i+1}^{(b)} + x_i^{(a)}y_i^{(b)} \\
&- x_i^{(a)}y_{i+1}^{(a)} - x_{i+1}^{(a)}y_{i+1}^{(b)} - x_{i+1}^{(b)}y_i^{(b)} - x_i^{(b)}y_i^{(a)})
\end{aligned}
$$

$$
\begin{aligned}
l_{i-1}^{(a)} &= \sqrt{\left(x_i^{(a)} - x_{i-1}^{(a)}\right)^2 + \left(y_i^{(a)} - y_{i-1}^{(a)}\right)^2} \\
l_{i-1}^{(b)} &= \sqrt{\left(x_i^{(b)} - x_{i-1}^{(b)}\right)^2 + \left(y_i^{(b)} - y_{i-1}^{(b)}\right)^2} \\
l_{i-1}^{(l)} &= \sqrt{\left(x_{i-1}^{(a)} - x_{i-1}^{(b)}\right)^2 + \left(y_{i-1}^{(a)} - y_{i-1}^{(b)}\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
A_{i-1}^{(c)} = \frac{1}{2}(&x_i^{(a)}y_{i-1}^{(a)} + x_i^{(b)}y_i^{(a)} + x_{i-1}^{(b)}y_i^{(b)} + x_{i-1}^{(a)}y_{i-1}^{(b)} \\
&- x_{i-1}^{(a)}y_i^{(a)} - x_i^{(a)}y_i^{(b)} - x_i^{(b)}y_{i-1}^{(b)} - x_{i-1}^{(b)}y_{i-1}^{(a)})
\end{aligned}
$$

$$
\begin{aligned}
A_Y^{(c)} &= \frac{1}{2}(x_1^{(b)}y_0^{(b)} + x_2^{(b)}y_1^{(b)} + \cdots + x_0^{(b)}y_{n-1}^{(b)} \\
&- x_0^{(b)}y_1^{(b)} - x_1^{(b)}y_2^{(b)} - \cdots - x_{n-1}^{(b)}y_0^{(b)}) \\
&= \frac{1}{2}\left(\sum_{i=1}^{n-1} x_i^{(b)}y_{i-1}^{(b)} - x_0^{(b)}y_{n-1}^{(b)} - \sum_{i=0}^{n-2} x_i^{(b)}y_{i+1}^{(b)} + x_{n-1}^{(b)}y_0^{(b)}\right) \\
&= \frac{1}{2}\left(\sum_{i=0}^{n-1} x_i^{(b)}y_{(i-1+n)\%n}^{(b)} - \sum_{i=0}^{n-1} x_i^{(b)}y_{(i+1+n)\%n}^{(b)}\right), \text{[closed curve]} \\
&= \frac{1}{2}\sum_{i=0}^{n-1} x_i^{(b)}\left(y_{(i-1+n)\%n}^{(b)} - y_{(i+1+n)\%n}^{(b)}\right)
\end{aligned} \tag{45}
$$

alternatively,

$$
\begin{aligned}
A_Y^{(c)} &= \frac{1}{2}(x_1^{(b)}y_0^{(b)} + x_2^{(b)}y_1^{(b)} + \cdots + x_{n-1}^{(b)}y_{n-2}^{(b)} + x_0^{(b)}y_{n-1}^{(b)} \\
&\quad - x_0^{(b)}y_1^{(b)} - x_1^{(b)}y_2^{(b)} - \cdots - x_{n-2}^{(b)}y_{n-1}^{(b)} - x_{n-1}^{(b)}y_0^{(b)}) \\
&= \frac{1}{2}\left(\sum_{i=0}^{n-2} x_{i+1}^{(b)}y_i^{(b)} - x_0^{(b)}y_{n-1}^{(b)} - \sum_{i=1}^{n-1} x_{i-1}^{(b)}y_i^{(b)} + x_{n-1}^{(b)}y_0^{(b)}\right) \\
&= \frac{1}{2}\left(\sum_{i=0}^{n-1} y_i^{(b)}x_{(i+1+n)\%n}^{(b)} - \sum_{i=0}^{n-1} y_i^{(b)}x_{(i-1+n)\%n}^{(b)}\right), \text{[closed curve]} \\
&= \frac{1}{2}\sum_{i=0}^{n-1} y_i^{(b)}\left(x_{(i+1+n)\%n}^{(b)} - x_{(i-1+n)\%n}^{(b)}\right)
\end{aligned} \tag{46}
$$

energy derivatives

Energy gradient at a vertex (x_i, y_i) can be calculated by,

$$
F_i = -\nabla_i E \rightarrow \begin{cases} F_{x_i} = \partial_{x_i} E \\ F_{y_i} = \partial_{y_i} E \end{cases} \tag{47}
$$

$$
\begin{aligned}
F_{x_i^{(a)}}|_{cell-edge} &= \sum_{j=0}^{n-1}\left(\alpha^{(a)}\partial_{x_i^{(a)}} l_j^{(a)} + \alpha^{(b)}\partial_{x_i^{(a)}} l_j^{(b)} + \alpha^{(l)}\partial_{x_i^{(a)}} l_j^{(l)}\right) \\
&= \sum_{j=i}^{i-1} \alpha^{(a)}\partial_{x_i^{(a)}} l_j^{(a)} + \alpha^{(l)}\partial_{x_i^{(a)}} l_i^{(l)} \\
&= \alpha^{(a)}\partial_{x_i^{(a)}}\left(l_i^{(a)} + l_{i-1}^{(a)}\right) + \alpha^{(l)}\partial_{x_i^{(a)}} l_i^{(l)} \\
&= \alpha^{(a)}\left(\frac{x_i^{(a)} - x_{i+1}^{(a)}}{l_i^{(a)}} + \frac{x_i^{(a)} - x_{i-1}^{(a)}}{l_{i-1}^{(a)}}\right) + \frac{\alpha^{(l)}(x_i^{(a)} - x_i^{(b)})}{l_i^{(l)}}
\end{aligned}
$$

$$
\begin{aligned}
F_{y_i^{(a)}}|_{cell-edge} &= \sum_{j=0}^{n-1} \left(\alpha^{(a)} \partial_{y_i^{(a)}} l_j^{(a)} + \alpha^{(b)} \partial_{y_i^{(a)}} l_j^{(b)} + \alpha^{(l)} \partial_{y_i^{(a)}} l_j^{(l)} \right) \\
&= \sum_{j=i}^{i-1} \alpha^{(a)} \partial_{y_i^{(a)}} l_j^{(a)} + \alpha^{(l)} \partial_{y_i^{(a)}} l_i^{(l)} \\
&= \alpha^{(a)} \partial_{y_i^{(a)}} \left(l_i^{(a)} + l_{i-1}^{(a)} \right) + \alpha^{(l)} \partial_{y_i^{(a)}} l_i^{(l)} \\
&= \alpha^{(a)} \left(\frac{y_i^{(a)} - y_{i+1}^{(a)}}{l_i^{(a)}} + \frac{y_i^{(a)} - y_{i-1}^{(a)}}{l_{i-1}^{(a)}} \right) + \frac{\alpha^{(l)} (y_i^{(a)} - y_i^{(b)})}{l_i^{(l)}}
\end{aligned}
$$

$$
\begin{aligned}
F_{x_i^{(b)}}|_{cell-edge} &= \sum_{j=0}^{n-1} \left(\alpha^{(a)} \partial_{x_i^{(b)}} l_j^{(a)} + \alpha^{(b)} \partial_{x_i^{(b)}} l_j^{(b)} + \alpha^{(l)} \partial_{x_i^{(b)}} l_j^{(l)} \right) \\
&= \sum_{j=i}^{i-1} \alpha^{(b)} \partial_{x_i^{(b)}} l_j^{(b)} + \alpha^{(l)} \partial_{x_i^{(b)}} l_i^{(l)} \\
&= \alpha^{(b)} \partial_{x_i^{(b)}} \left(l_i^{(b)} + l_{i-1}^{(b)} \right) + \alpha^{(l)} \partial_{x_i^{(b)}} l_i^{(l)} \\
&= \alpha^{(b)} \left(\frac{x_i^{(b)} - x_{i+1}^{(b)}}{l_i^{(b)}} + \frac{x_i^{(b)} - x_{i-1}^{(b)}}{l_{i-1}^{(b)}} \right) + \frac{\alpha^{(l)} (x_i^{(b)} - x_i^{(a)})}{l_i^{(l)}}
\end{aligned}
$$

$$
\begin{aligned}
F_{y_i^{(b)}}|_{cell-edge} &= \sum_{j=0}^{n-1} \left(\alpha^{(a)} \partial_{y_i^{(b)}} l_j^{(a)} + \alpha^{(b)} \partial_{y_i^{(b)}} l_j^{(b)} + \alpha^{(l)} \partial_{y_i^{(b)}} l_j^{(l)} \right) \\
&= \sum_{j=i}^{i-1} \alpha^{(b)} \partial_{y_i^{(b)}} l_j^{(b)} + \alpha^{(l)} \partial_{y_i^{(b)}} l_i^{(l)} \\
&= \alpha^{(b)} \partial_{y_i^{(b)}} \left(l_i^{(b)} + l_{i-1}^{(b)} \right) + \alpha^{(l)} \partial_{y_i^{(b)}} l_i^{(l)} \\
&= \alpha^{(b)} \left(\frac{y_i^{(b)} - y_{i+1}^{(b)}}{l_i^{(b)}} + \frac{y_i^{(b)} - y_{i-1}^{(b)}}{l_{i-1}^{(b)}} \right) + \frac{\alpha^{(l)} (y_i^{(b)} - y_i^{(a)})}{l_i^{(l)}}
\end{aligned}
$$

$$
\begin{aligned}
F_{x_i^{(a)}}|_{cell-area} &= \sum_{j=1}^{n-1} \beta \partial_{x_i^{(a)}} (A_j^{(c)} - A_e^{(r)})^2 \\
&= 2\beta \sum_{j=i}^{i-1} A_j^{(d)} \partial_{x_i^{(a)}} (A_j^{(c)} - A_e^{(r)}), [A_j^{(d)} = A_j^{(c)} - A_e^{(r)}] \\
&= 2\beta \left(A_i^{(d)} \partial_{x_i^{(a)}} (A_i^{(c)} - A_e^{(r)}) + A_{i-1}^{(d)} \partial_{x_i^{(a)}} (A_{i-1}^{(c)} - A_e^{(r)}) \right) \\
&= 2\beta \left(A_i^{(d)} \partial_{x_i^{(a)}} A_i^{(c)} + A_{i-1}^{(d)} \partial_{x_i^{(a)}} A_{i-1}^{(c)} \right) \\
&= \beta \left(A_i^{(d)} (y_i^{(b)} - y_{i+1}^{(a)}) + A_{i-1}^{(d)} (y_{i-1}^{(a)} - y_i^{(b)}) \right)
\end{aligned}
$$

$$
\begin{aligned}
F_{y_i^{(a)}}|_{cell-area} &= \sum_{j=1}^{n-1} \beta \partial_{y_i^{(a)}} (A_j^{(c)} - A_e^{(r)})^2 \\
&= 2\beta \sum_{j=i}^{i-1} A_j^{(d)} \partial_{y_i^{(a)}} (A_j^{(c)} - A_e^{(r)}), [A_j^{(d)} = A_j^{(c)} - A_e^{(r)}] \\
&= 2\beta \left(A_i^{(d)} \partial_{y_i^{(a)}} (A_i^{(c)} - A_e^{(r)}) + A_{i-1}^{(d)} \partial_{y_i^{(a)}} (A_{i-1}^{(c)} - A_e^{(r)}) \right) \\
&= 2\beta \left(A_i^{(d)} \partial_{y_i^{(a)}} A_i^{(c)} + A_{i-1}^{(d)} \partial_{y_i^{(a)}} A_{i-1}^{(c)} \right) \\
&= \beta \left(A_i^{(d)} (x_{i+1}^{(a)} - x_i^{(b)}) + A_{i-1}^{(d)} (x_i^{(b)} - x_{i-1}^{(a)}) \right)
\end{aligned}
$$

$$
\begin{aligned}
F_{x_i^{(b)}}|_{cell-area} &= \sum_{j=1}^{n-1} \beta \partial_{x_i^{(b)}} (A_j^{(c)} - A_e^{(r)})^2 \\
&= 2\beta \sum_{j=i}^{i-1} A_j^{(d)} \partial_{x_i^{(b)}} (A_j^{(c)} - A_e^{(r)}), [A_j^{(d)} = A_j^{(c)} - A_e^{(r)}] \\
&= 2\beta \left(A_i^{(d)} \partial_{x_i^{(b)}} (A_i^{(c)} - A_e^{(r)}) + A_{i-1}^{(d)} \partial_{x_i^{(b)}} (A_{i-1}^{(c)} - A_e^{(r)}) \right) \\
&= 2\beta \left(A_i^{(d)} \partial_{x_i^{(b)}} A_i^{(c)} + A_{i-1}^{(d)} \partial_{x_i^{(b)}} A_{i-1}^{(c)} \right) \\
&= \beta \left(A_i^{(d)} (y_{i+1}^{(b)} - y_i^{(a)}) + A_{i-1}^{(d)} (y_i^{(a)} - y_{i-1}^{(b)}) \right)
\end{aligned}
$$

$$
\begin{aligned}
F_{y_i^{(b)}}|_{cell-area} &= \sum_{j=1}^{n-1} \beta \partial_{y_i^{(b)}} (A_j^{(c)} - A_e^{(r)})^2 \\
&= 2\beta \sum_{j=i}^{i-1} A_j^{(d)} \partial_{y_i^{(b)}} (A_j^{(c)} - A_e^{(r)}), [A_j^{(d)} = A_j^{(c)} - A_e^{(r)}] \\
&= 2\beta \left(A_i^{(d)} \partial_{y_i^{(b)}} (A_i^{(c)} - A_e^{(r)}) + A_{i-1}^{(d)} \partial_{y_i^{(b)}} (A_{i-1}^{(c)} - A_e^{(r)}) \right) \\
&= 2\beta \left(A_i^{(d)} \partial_{y_i^{(b)}} A_i^{(c)} + A_{i-1}^{(d)} \partial_{y_i^{(b)}} A_{i-1}^{(c)} \right) \\
&= \beta \left(A_i^{(d)} (x_i^{(a)} - x_{i+1}^{(b)}) + A_{i-1}^{(d)} (x_{i-1}^{(b)} - x_i^{(a)}) \right)
\end{aligned}
$$

$$
\begin{aligned}
F_{x_i^{(a)}}|_{vit-rigid} &= \sum_{j=0}^{n-1} \partial_{x_i^{(a)}} \epsilon \Theta(D_j) w_j D_j^2 \\
&= \sum_{j=i-1}^{i+1} \partial_{x_i^{(a)}} \epsilon \Theta(D_j) w_j D_j^2 \\
&= \epsilon \partial_{x_i^{(a)}} \Theta(D_{i-1}) w_{i-1} D_{i-1}^2 + \epsilon \Theta(D_i) w_i D_i^2 + \epsilon \Theta(D_{i+1}) w_{i+1} D_{i+1}^2 \\
&= \epsilon \Theta(D_{i-1}) D_{i-1}^2 \partial_{x_i^{(a)}} w_{i-1} + \epsilon \Theta(D_i) D_i^2 \partial_{x_i^{(a)}} w_i \\
&+ 2\epsilon w_i \Theta(D_i) D_i \partial_{x_i^{(a)}} D_i + \epsilon \Theta(D_{i+1}) D_{i+1}^2 \partial_{x_i^{(a)}} w_{i+1} \\
&= 2\epsilon w_i \Theta(D_i) D_i \partial_{x_i^{(a)}} D_i + \epsilon \Theta(D_i) D_i^2 \partial_{x_i^{(a)}} w_i \\
&+ \epsilon \Theta(D_{i-1}) D_{i-1}^2 \partial_{x_i^{(a)}} w_{i-1} + \epsilon \Theta(D_{i+1}) D_{i+1}^2 \partial_{x_i^{(a)}} w_{i+1} \\
&= -2\epsilon w_i \Theta(D_i) D_i \frac{x_i^{(a)} - x_i^{(v)}}{v_w - D_i} + \frac{1}{2} \epsilon \Theta(D_i) D_i^2 \left(\frac{x_i^{(a)} - x_{i+1}^{(a)}}{l_i^{(a)}} + \frac{x_i^{(a)} - x_{i-1}^{(a)}}{l_{i-1}^{(a)}} \right) \\
&+ \frac{1}{2} \epsilon \Theta(D_{i-1}) D_{i-1}^2 \frac{x_i^{(a)} - x_{i-1}^{(a)}}{l_{i-1}^{(a)}} + \frac{1}{2} \epsilon \Theta(D_{i+1}) D_{i+1}^2 \frac{x_i^{(a)} - x_{i+1}^{(a)}}{l_i^{(a)}} \\
\\
&= 2\epsilon \Theta(D_i) w_i D_i \frac{x_i^{(v)} - x_i^{(a)}}{v_w - D_i} + \frac{\epsilon (x_i^{(a)} - x_{i+1}^{(a)})}{2 l_i^{(a)}} \left(\Theta(D_i) D_i^2 + \Theta(D_{i+1}) D_{i+1}^2 \right) \\
&+ \frac{\epsilon (x_i^{(a)} - x_{i-1}^{(a)})}{2 l_{i-1}^{(a)}} \left(\Theta(D_i) D_i^2 + \Theta(D_{i-1}) D_{i-1}^2 \right)
\end{aligned}
$$

$$
\begin{aligned}
F_{y_i^{(a)}}|_{vit-rigid} = 2\epsilon\Theta(D_i)w_iD_i\frac{y_i^{(v)} - y_i^{(a)}}{v_w - D_i} + \frac{\epsilon(y_i^{(a)} - y_{i+1}^{(a)})}{2l_i^{(a)}}\left(\Theta(D_i)D_i^2 + \Theta(D_{i+1})D_{i+1}^2\right) \\
+ \frac{\epsilon(y_i^{(a)} - y_{i-1}^{(a)})}{2l_{i-1}^{(a)}}\left(\Theta(D_i)D_i^2 + \Theta(D_{i-1})D_{i-1}^2\right)
\end{aligned}
$$

$$F_{x_i^{(b)}}|_{vit-rigid} = 0$$

$$F_{y_i^{(b)}}|_{vit-rigid} = 0$$

$$F_{x_i^{(a)}}|_{yolk-area} = 0$$

$$F_{y_i^{(a)}}|_{yolk-area} = 0$$

$$
\begin{aligned}
F_{x_i^{(b)}}|_{yolk-area} &= \gamma\partial_{x_i^{(b)}}(A_Y^{(c)} - A_Y^{(r)})^2 \\
&= 2\gamma A_Y^{(d)}\partial_{x_i^{(b)}}(A_Y^{(c)} - A_Y^{(r)}), [A_Y^{(d)} = A_Y^{(c)} - A_Y^{(r)}] \\
&= 2\gamma A_Y^{(d)}\partial_{x_i^{(b)}}A_Y^{(c)} \\
&= \gamma A_Y^{(d)}\left(y_{(i-1+n)\%n}^{(b)} - y_{(i+1+n)\%n}^{(b)}\right) \text{ [yolk rest-area fixed]}
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
F_{x_i^{(b)}}|_{yolk-area} &= -p^{(Y)}\partial_{x_i^{(b)}}A_Y^{(c)} \\
&= -\frac{p^{(Y)}}{2}\left(y_{(i-1+n)\%n}^{(b)} - y_{(i+1+n)\%n}^{(b)}\right) \\
&= \frac{p^{(Y)}}{2}\left(y_{(i+1+n)\%n}^{(b)} - y_{(i-1+n)\%n}^{(b)}\right) \text{ [yolk pressure constant]}
\end{aligned}
$$

$$
\begin{aligned}
F_{y_i^{(b)}}|_{yolk-area} &= \gamma \partial_{y_i^{(b)}} (A_Y^{(c)} - A_Y^{(r)})^2 \\
&= 2\gamma A_Y^{(d)} \partial_{y_i^{(b)}} (A_Y^{(c)} - A_Y^{(r)}), [A_Y^{(d)} = A_Y^{(c)} - A_Y^{(r)}] \\
&= 2\gamma A_Y^{(d)} \partial_{y_i^{(b)}} A_Y^{(c)} \\
&= \gamma A_Y^{(d)} \left(x^{(b)}_{(i+1+n)\%n} - x^{(b)}_{(i-1+n)\%n} \right) \text{ [yolk rest-area fixed]}
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
F_{y_i^{(b)}}|_{yolk-area} &= -p^{(Y)} \partial_{y_i^{(b)}} A_Y^{(c)} \\
&= -\frac{p^{(Y)}}{2} \left(x^{(b)}_{(i+1+n)\%n} - x^{(b)}_{(i-1+n)\%n} \right) \\
&= \frac{p^{(Y)}}{2} \left(x^{(b)}_{(i-1+n)\%n} - x^{(b)}_{(i+1+n)\%n} \right) \text{ [yolk pressure constant]}
\end{aligned}
$$