Module 3
Linear Differential Equations

1) Find the general solution of $\frac{d^3y}{dx^3}+y=0$

2) Find the wronskian of $e^x\cos 2x$ & $e^x\sin 2x$

3) Find the particular integral of $y''-4y'-5y=4\cos 2x$

4) Find the P.I. of $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=\sinh 2x$

5) Solve the initial value problem
$y''+4y'+5y=0,\ y(0)=2,\ y'(0)=-5$

6) Find the general solution of the differential equation $y'''-y''+4y'=0$

7) If $y_1(x)=x$ is a solution to the differential equation $(1+x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$, find the general solution

8) Solve the ordinary differential equation
$y'''-3y''-4y'+6y=0$

9) Solve $(D^4+2D^2+1)y=x^4$

10) Use method of variation of parameters to solve $\frac{d^2y}{dx^2}+4y=\tan 2x$

11) Solve $(D^2-4D+4)y=\sin^2 x$

1) A.E. is $m^3+1=0$

$m = -1, \frac{1 \pm \sqrt{3}i}{2}$

$y = c_1 e^{-x} + e^{x/2}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$

2) $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} e^x \cos 2x & e^x \sin 2x \\ e^x(\cos 2x - 2\sin 2x) & e^x(\sin 2x + 2\cos 2x) \end{vmatrix}$

$= 2e^{2x}$

3) $y = A\cos 2x + B\sin 2x$

$y' = -2A\sin 2x + 2B\cos 2x$

$y'' = -4A\cos 2x - 4B\sin 2x$

$y'' - 4y' - 5y = 4\cos 2x$

$\Rightarrow -4A\cos 2x - 4B\sin 2x - 4(-2A\sin 2x + 2B\cos 2x)$

$-5(A\cos 2x + B\sin 2x) = 4\cos 2x$

$\cos 2x(-4A - 8B - 5A) + \sin 2x(-4B + 8A - 5B)$

$= 4\cos 2x$

$\Rightarrow -4A - 8B - 5A = 4 \Rightarrow -9A - 8B = 4$ ①

$\cancel{-4B+8A}$

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$-4B+8A-5B=0$

$8A-9B=0$ — (2)

$-9A-8B=4$

$B=\frac{-32}{145}$, $A=\frac{-36}{145}$

$\therefore P.I.=\frac{-4}{145}(8\sin 2x+9\cos 2x)$

4) $y''+4y'+4y=\sinh 2x$ — (1)

$=\frac{e^{2x}-e^{-2x}}{2}=\frac{e^{2x}}{2}-\frac{e^{-2x}}{2}$

Let $y=Ae^{2x}$

$y'=2Ae^{2x}$, $y''=4Ae^{2x}$

$4Ae^{2x}+8Ae^{2x}+4Ae^{2x}=\frac{e^{2x}}{2}$

$16Ae^{2x}=\frac{e^{2x}}{2} \Rightarrow A=\frac{1}{32}$

$y=Be^{-2x}$

$y'=-2Be^{-2x}$, $y''=4Be^{-2x}$

(1) $\Rightarrow$ $4Be^{-2x}-8Be^{-2x}+4Be^{-2x}=\frac{-e^{-2x}}{2}$

$0=\frac{e^{-2x}}{2}$

$\therefore$ Let $y=Bxe^{-2x}$

$$y' = B(-2xe^{-2x} + e^{-2x})$$

$$y'' = B(-2(-2xe^{-2x} + e^{-2x}) + -2e^{-2x})$$

$$= B(4xe^{-2x} - 2e^{-2x} - 2e^{-2x})$$

$$= B(4xe^{-2x} - 4e^{-2x})$$

① ⇒ $B(4xe^{-2x} - 4e^{-2x}) + 4B(-2xe^{-2x} + e^{-2x}) +$

$4Bxe^{-2x} = -\dfrac{e^{-2x}}{2}$

$$B(4xe^{-2x} - 4e^{-2x} - 8xe^{-2x} + 4e^{-2x} + 4xe^{-2x}) = -\frac{e^{-2x}}{2}$$

$$0 = \frac{e^{-2x}}{2}$$

So let $y = Bx^2e^{-2x}$

$$y' = B(-2x^2e^{-2x} + 2xe^{-2x})$$

$$y'' = B[-2(-2x^2e^{-2x} + 2xe^{-2x}) + 2(-2xe^{-2x} + e^{-2x})]$$

$$= B[4x^2e^{-2x} - 4xe^{-2x} - 4xe^{-2x} + 2e^{-2x}]$$

$$= B[4x^2e^{-2x} - 8xe^{-2x} + 2e^{-2x}]$$

① ⇒ $B[4x^2e^{-2x} - 8xe^{-2x} + 2e^{-2x} - 8x^2e^{-2x} + 8xe^{-2x}$

$+ 4x^2e^{-2x}] = -\dfrac{e^{-2x}}{2}$

$$B[2e^{-2x}] = -\frac{e^{-2x}}{2}$$

$$2B = -\frac{1}{2} \Rightarrow B = -\frac{1}{4}$$

$$\therefore P.I. = \frac{1}{32}e^{2x} - \frac{1}{4}x^2e^{-2x}$$

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5) A.E. is $m^2+4m+5=0$

$m = -2 \pm i$

$\therefore y = e^{-2x}(c_1 \cos x + c_2 \sin x)$

$y(0) = 2 \Rightarrow c_1 \cos 0 + c_2 \sin 0 = 2$

$c_1 = 2$

$y' = e^{-2x}(-c_1 \sin x + c_2 \cos x) + -2e^{-2x}(c_1 \cos x + c_2 \sin x)$

$y'(0) = 5 \Rightarrow$

$c_2 - 2c_1 = 5$

$\cancel{+c_2 = 5} \Rightarrow -6 \cancel{+c_2} = 5$

$\cancel{c_2 = 5+6 = 11}$

$c_2 - 4 = 5 \Rightarrow c_2 = -1$

$\therefore y = e^{-2x}(2\cos x - \sin x)$

6) A.E. is $m^3 - m^2 + 4m = 0$

$m = 0, \frac{1}{2}(1 \pm \sqrt{15}\, i)$

$\therefore y = c_1 + e^{\frac{1}{2}x}\left[c_2 \cos \frac{\sqrt{15}}{2}x + c_3 \sin \frac{\sqrt{15}}{2}x\right]$

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7) $y_1(x) = x$ is given then $y_2 = u y_1$

$$u = \int \frac{1}{y_1^2} e^{-\int P dx} dx$$

$$(1+x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{2x}{1+x^2}\frac{dy}{dx} + \frac{2y}{1+x^2} = 0$$

$$P = \frac{-2}{1+x^2}$$

$$u = \int \frac{1}{x^2} e^{-\int \frac{-2}{1+x^2} dx} dx$$

$$= \int \frac{1+x^2}{x^2} dx$$

$$= x - \frac{1}{x}$$

$$\therefore y_2 = x\left(x - \frac{1}{x}\right) = \underline{\underline{x^2 - 1}}$$

8) A.E. is $m^3 - 3m^2 - 4m + 6 = 0$

$m = 1, 1 \pm \sqrt{7}$.

$$\therefore y = c_1 e^x + c_2 e^{(1+\sqrt{7})x} + c_3 e^{(1-\sqrt{7})x}$$

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9) Let $y = Ax^4 + Bx^3 + Cx^2 + Dx + E$

$y' = 4Ax^3 + 3Bx^2 + 2Cx + D$

$y'' = 12Ax^2 + 6Bx + 2C$

$y''' = 24Ax + 6B$

$y^{iv} = 24A$

$y^{iv} + 2y'' + y = x^4$

$\Rightarrow 24A + 2(12Ax^2 + 6Bx + 2C) + 1 = x^4$

$24A + 24Ax^2 + 12Bx + 4C + 1 = x^4$

$\cancel{24A + 4C + 1 = 0 \Rightarrow 6A + C + 1 = 0}$

$\cancel{24A = 0}$ $\quad 24A + 4C + 1 = 0$

$24A + 24Ax^2 + 12Bx + 4C + Ax^4 + Bx^3 + Cx^2 + Dx + E = x^4$

Equating the coefficients

$A = 1$

$B = 0$

$24A + C = 0 \Rightarrow C = -24$

$12B + D = 0 \Rightarrow D = 0$

$24A + 4C + E = 0 \Rightarrow 24 - 96 + E = 0$

$\Rightarrow E = 72$

$\therefore P.I. = x^4 - 24x^2 + 72$

A.E. is $D^4+2D^2+1=0$

$(1+D^2)^2=0$

$\Rightarrow (1+m^2)^2=0$

$\Rightarrow m=\pm i, \pm i$

$\therefore$ C.F. $=(c_1+c_2x)\cos x+(c_3+c_4x)\sin x$

$\therefore$ General soln. is $y=C.F.+P.I$

10) A.E. is $m^2+4=0$

$m=\pm 2i$

C.F. $= c_1\cos 2x+c_2\sin 2x$

$y_1=\cos 2x, \quad y_2=\sin 2x$

$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=2$

$$P.I.=-y_1\int \frac{y_2 R(x)}{W}dx+y_2\int\frac{y_1 R(x)}{W}dx$$

$$=-\cos 2x\int\frac{\sin 2x\tan 2x}{2}dx+\sin 2x\int\frac{\cos 2x\tan 2x}{2}dx$$

$$=-\frac{\cos 2x}{2}\int\frac{\sin^2 2x}{\cos 2x}dx+\frac{\sin 2x}{2}\int\sin 2x\,dx$$

$$=-\frac{\cos 2x}{2}\int(\sec 2x-\cos 2x)dx+\frac{1}{2}\sin 2x\times -\frac{\cos 2x}{2}$$

$$=\frac{-1}{4}\cos 2x\log(\sec 2x+\tan 2x)$$

$\therefore y=C.F.+P.I.$

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11) $(D^2-4D+4)y = \sin^2 x$

$$= \frac{1-\cos 2x}{2} = \frac{1}{2}e^{0x} - \frac{1}{2}\cos 2x$$

A.E. is $m^2-4m+4=0$

$m = 2, 2$

$\therefore$ C.F. $= (c_1 + c_2 x)e^{2x}$

$y = Ae^{0x}$

$y' = 0, \quad y'' = 0$

$y'' - 4y' + 4y = \frac{1}{2}e^{0x}$

$\Rightarrow 4A = \frac{1}{2}$

$A = \frac{1}{8}$

$y = A\cos 2x + B\sin 2x$

$y' = -2A\sin 2x + 2B\cos 2x$

$y'' = -4A\cos 2x - 4B\sin 2x$

$y'' - 4y' + 4y = -\frac{1}{2}\cos 2x$

$-4A\cos 2x - 4B\sin 2x - 4(-2A\sin 2x + 2B\cos 2x)$

$+ 4(A\cos 2x + B\sin 2x) = -\frac{1}{2}\cos 2x$

$\cos 2x(-4A - 8B + 4A) + \sin 2x(-4B + 8A + 4B) =$

$-\frac{1}{2}\cos 2x$

$-4A + 4A - 8B = -\frac{1}{2} \Rightarrow B = \frac{1}{16}$

$8A = 0 \Rightarrow A = 0$

$\therefore y = \frac{1}{8} + \frac{\sin 2x}{16}$

$\therefore$ General soln. is $y = \underline{\underline{C.F + P.I.}}$

12) Solve the Cauchy - Euler differential eqn.

$$(x^2D^2 - 3xD + 10)y = 0$$

13) Solve the Cauchy - Euler diffl. eqn.

$$x^2y'' - xy' + 5y = 0.$$

14) By the method of undetermined coeffs., solve

$$y'' - 4y' + 13y = e^{2x}\cos x$$

15) Using the method of variation of parameters solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = \frac{e^{2x}}{\sin x}$

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12) Let $y = x^m$ be the soln. of the eqn.

$x^2 y'' - 3xy' + 10y = 0$ — ①

$y = x^m$, $y' = m x^{m-1}$, $y'' = m(m-1) x^{m-2}$

① $\Rightarrow x^2(m(m-1)x^{m-2}) - 3x\, m x^{m-1} + 10x^m =$

$\Rightarrow m^2 - 4m + 10 = 0$

$m = 2 \pm i\sqrt{6}$

$y = x^2[A\cos(\sqrt{6}\ln x) + B\sin(\sqrt{6}\ln x)]$

13) Let $y = x^m$

$y' = m x^{m-1}$, $y'' = m(m-1)x^{m-2}$

$x^2 m(m-1) x^{m-2} - x\, m x^{m-1} + 5x^m = 0$

$\Rightarrow m^2 - 2m + 5 = 0$

$m = 1 \pm 2i$

$\therefore$ The general soln. is

$y = x[A\cos(2\ln x) + B\sin(2\ln x)]$

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13) ~~Let y~~ $y'' - 4y' + 13y = e^{2x}\cos x$ — (1)

A.E. is $m^2 - 4m + 13 = 0$

C.F. $= e^{2x}(C_1 \cos 3x + C_2 \sin 3x)$

Let $y_p = e^{2x}(A\cos x + B\sin x)$

$y_p' = 2e^{2x}(A\cos x + B\sin x) + e^{2x}(-A\sin x + B\cos x)$

$y_p'' = 4e^{2x}(A\cos x + B\sin x) + 2e^{2x}(-A\sin x + B\cos x)$
$+ 2e^{2x}(-A\sin x + B\cos x) + e^{2x}(-A\cos x - B\sin x)$

Substituting in eqn. (1) & equating the coefficients of $\cos x$ & $\sin x$, we get

$A = \frac{1}{8}$, $B = \left(\frac{-3}{64}\right)$

$\therefore y_p = e^{2x}\left(\frac{1}{8}\cos x - \frac{3}{64}\sin x\right)$

General soln. is $y = C.F + P.I$

15) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = \frac{e^{2x}}{\sin x}$

$(D^2 - 4D + 5)y = \frac{e^{2x}}{\sin x}$

A.E. is $m^2 - 4m + 5 = 0$

$m = 2 \pm i$

$\therefore$ C.F. $= e^{2x}(C_1 \cos x + C_2 \sin x)$

$y_1 = e^{2x}\cos x \; , \; y_2 = e^{2x}\sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x}\cos x & e^{2x}\sin x \\ -e^{2x}\sin x + 2\cos x e^{2x} & e^{2x}\cos x + 2\sin x e^{2x} \end{vmatrix}$$

$$= e^{4x}$$

$$y_p = -y_1 \int \frac{y_2 R}{W}\, dx + y_2 \int \frac{y_1 R}{W}\, dx$$

$$= -e^{2x}\cos x \int \frac{e^{2x}\sin x \cdot \frac{e^{2x}}{\sin x}}{e^{4x}}\, dx +$$

$$e^{2x}\sin x \int \frac{e^{2x}\cos x \, \frac{e^{2x}}{\sin x}}{e^{4x}}\, dx$$

$$= -e^{2x}\cos x \int -1\, dx + e^{2x}\sin x \int \cot x\, dx$$

$$= -x e^{2x}\cos x + e^{2x}\sin x \log(\sin x)$$

$\therefore \; y = C.F. + P.I.$