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The Concept of a Linguistic Variable and its Application to Approximate Reasoning–I

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ABSTRACT

By a *linguistic variable* we mean a variable whose values are words or sentences in a natural or artificial language. For example, *Age* is a linguistic variable if its values are linguistic rather than numerical, i.e., *young, not young, very young, quite young, old, not very old and not very young*, etc., rather than 20, 21, 22, 23, ... In more specific terms, a linguistic variable is characterized by a quintuple $(\mathscr{X}, T(\mathscr{X}), U, G, M)$ in which $\mathscr{X}$ is the name of the variable; $T(\mathscr{X})$ is the *term-set* of $\mathscr{X}$, that is, the collection of its linguistic values; U is a universe of discourse; G is a *syntactic rule* which generates the terms in $T(\mathscr{X})$; and M is a *semantic rule* which associates with each linguistic value X its *meaning*, $M(X)$, where $M(X)$ denotes a fuzzy subset of U. The meaning of a linguistic value X is characterized by a *compatibility function*, $c : U \rightarrow [0, 1]$, which associates with each u in U its compatibility with X. Thus, the compatibility of age 27 with *young* might be 0.7, while that of 35 might be 0.2. The function of the semantic rule is to relate the compatibilities of the so-called *primary* terms in a composite linguistic value–e.g., *young* and *old* in *not very young and not very old*–to the compatibility of the composite value. To this end, the hedges such as *very, quite, extremely*, etc., as well as the connectives *and* and *or* are treated as nonlinear operators which modify the meaning of their operands in a specified fashion. The concept of a linguistic variable provides a means of approximate characterization of phenomena which are too complex or too ill-defined to be amenable to description in conventional quantitative terms. In particular, treating *Truth* as a linguistic variable with values such as *true, very true, completely true, not very true, untrue*, etc., leads to what is called *fuzzy logic*. By providing a basis for *approximate reasoning*, that is, a mode of reasoning which is not exact nor very inexact, such logic may offer a more realistic framework for human reasoning than the traditional two-valued logic. It is shown that probabilities, too, can be treated as linguistic variables with values such as *likely, very likely, unlikely*, etc. Computation with linguistic probabilities requires the solution of

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nonlinear programs and leads to results which are imprecise to the same degree as the underlying probabilities. The main applications of the linguistic approach lie in the realm of humanistic systems–especially in the fields of artificial intelligence, linguistics, human decision processes, pattern recognition, psychology, law, medical diagnosis, information retrieval, economics and related areas.

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1. INTRODUCTION

One of the fundamental tenets of modern science is that a phenomenon cannot be claimed to be well understood until it can be characterized in quantitative terms.[1] Viewed in this perspective, much of what constitutes the core of scientific knowledge may be regarded as a reservoir of concepts and techniques which can be drawn upon to construct mathematical models of various types of systems and thereby yield quantitative information concerning their behavior.

Given our veneration for what is precise, rigorous and quantitative, and our disdain for what is fuzzy, unrigorous and qualitative, it is not surprising that the advent of digital computers has resulted in a rapid expansion in the use of quantitative methods throughout most fields of human knowledge. Unquestionably, computers have proved to be highly effective in dealing with *mechanistic* systems, that is, with inanimate systems whose behavior is governed by the laws of mechanics, physics, chemistry and electromagnetism. Unfortunately, the same cannot be said about *humanistic* systems,[2] which–so far at least–have proved to be rather impervious to mathematical analysis and computer simulation. Indeed, it is widely agreed that the use of computers has not shed much light on the basic issues arising in philosophy, psychology, literature, law, politics, sociology and other human-oriented fields. Nor have computers added significantly to our understanding of human thought processes–excepting, perhaps, some examples to the contrary that can be drawn from artificial intelligence and related fields [2, 3, 4, 5, 51].

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[1] As expressed by Lord Kelvin in 1883 [1], "In physical science a first essential step in the direction of learning any subject is to find principles of numerical reckoning and practicable methods for measuring some quality connected with it. I often say that when you can measure what you are speaking about and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind: it may be the beginning of knowledge but you have scarcely, in your thoughts, advanced to the state of *science*, whatever the matter may be."

[2] By a *humanistic* system we mean a system whose behavior is strongly influenced by human judgement, perception or emotions. Examples of humanistic systems are: economic systems, political systems, legal systems, educational systems, etc. A single individual and his thought processes may also be viewed as a humanistic system.

It may be argued, as we have done in [6] and [7], that the ineffectiveness of computers in dealing with humanistic systems is a manifestation of what might be called the *principle of incompatibility*—a principle which asserts that high precision is incompatible with high complexity.[3] Thus, it may well be the case that the conventional techniques of system analysis and computer simulation—based as they are on precise manipulation of numerical data—are intrinsically incapable of coming to grips with the great complexity of human thought processes and decision-making. The acceptance of this premise suggests that, in order to be able to make significant assertions about the behavior of humanistic systems, it may be necessary to abandon the high standards of rigor and precision that we have become conditioned to expect of our mathematical analyses of well-structured mechanistic systems, and become more tolerant of approaches which are approximate in nature. Indeed, it is entirely possible that only through the use of such approaches could computer simulation become truly effective as a tool for the analysis of systems which are too complex or too ill-defined for the application of conventional quantitative techniques.

In retreating from precision in the face of overpowering complexity, it is natural to explore the use of what might be called *linguistic* variables, that is, variables whose values are not numbers but words or sentences in a natural or artificial language. The motivation for the use of words or sentences rather than numbers is that linguistic characterizations are, in general, less specific than numerical ones. For example, in speaking of age, when we say "John is young," we are less precise than when we say, "John is 25." In this sense, the label *young* may be regarded as a *linguistic value* of the variable *Age*, with the understanding that it plays the same role as the numerical value 25 but is less precise and hence less informative. The same is true of the linguistic values *very young, not young, extremely young, not very young,* etc. as contrasted with the numerical values 20, 21, 22, 23, ...

If the values of a numerical variable are visualized as points in a plane, then the values of a linguistic variable may be likened[3] to ball parks with fuzzy boundaries. In fact, it is by virtue of the employment of ball parks rather than points that linguistic variables acquire the ability to serve as a means of approximate characterization of phenomena which are too complex or too ill-defined to be susceptible of description in precise terms. What is also important, however, is that by the use of a so-called *extension principle,* much of the existing mathematical apparatus of systems analysis can be adapted to the manipulation of linguistic variables. In this way, we may be able to develop an approximate calculus of linguistic variables which could be of use in a wide variety of practical applications.

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[3] Stated somewhat more concretely, the complexity of a system and the precision with which it can be analyzed bear a roughly inverse relation to one another.

The totality of values of a linguistic variable constitute its *term-set*, which in principle could have an infinite number of elements. For example, the term-set of the linguistic variable *Age* might read

$$T(Age) = young + not\ young + very\ young + not\ very\ young + very\ very\ young + \cdots + old + not\ old + very\ old + not\ very\ old + \cdots + not\ very\ young\ and\ not\ very\ old + \cdots + middle\text{-}aged + not\ middle\text{-}aged + \cdots + not\ old\ and\ not\ middle\text{-}aged + \cdots + extremely\ old + \cdots, \tag{1.1}$$

in which + is used to denote the union rather than the arithmetic sum. Similarly, the term-set of the linguistic variable *Appearance* might be

$$T(Appearance) = beautiful + pretty + cute + handsome + attractive + not\ beautiful + very\ pretty + very\ very\ handsome + more\ or\ less\ pretty + quite\ pretty + quite\ handsome + fairly\ handsome + not\ very\ attractive\ and\ not\ very\ unattractive + \cdots$$

In the case of the linguistic variable *Age*, the numerical variable *age* whose values are the numbers 0, 1, 2, 3, . . . , 100 constitutes what may be called the *base variable* for *Age*. In terms of this variable, a linguistic value such as *young* may be interpreted as a label for a *fuzzy restriction* on the values of the base variable. This fuzzy restriction is what we take to be the *meaning* of *young*.

A fuzzy restriction on the values of the base variable is characterized by a *compatibility function* which associates with each value of the base variable a number in the interval [0, 1] which represents its *compatibility* with the fuzzy restriction. For example, the compatabilities of the numerical ages 22, 28 and 35 with the fuzzy restriction labeled *young* might be 1, 0.7 and 0.2, respectively. The meaning of *young*, then, would be represented by a graph of the form shown in Fig. 1, which is a plot of the compatibility function of *young* with respect to the base variable *age*.

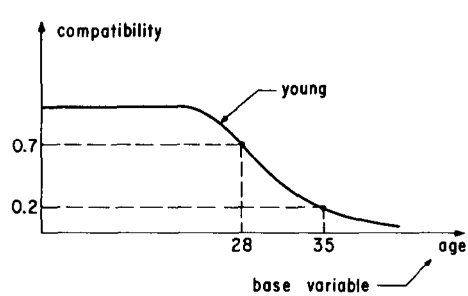


Fig. 1. Compatibility function for *young*.

The conventional interpretation of the statement "John is young," is that John is a member of the class of young men. However, considering that the class of young men is a fuzzy set, that is, there is no sharp transition from being young to not being young, the assertion that John is a member of the class of young men is inconsistent with the precise mathematical definition of "is a member of." The concept of a linguistic variable allows us to get around this difficulty in the following manner.

The name "John" is viewed as a name of a composite linguistic variable whose components are linguistic variables named *Age, Height, Weight, Appearance,* etc. Then the statement "John is young" is interpreted as an *assignment equation* (Fig. 2).

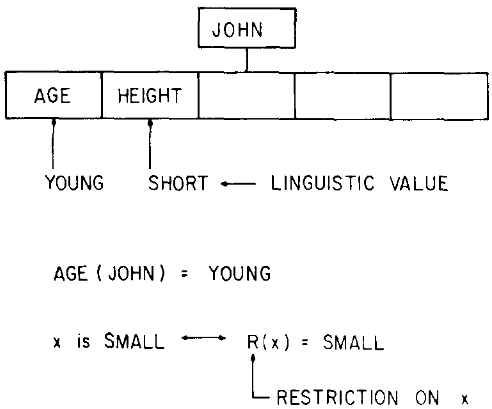


Fig. 2. Assignment of linguistic values to attributes of John and x.

$$Age = young$$

which assigns the value *young* to the linguistic variable *Age*. In turn, the value *young* is interpreted as a label for a fuzzy restriction on the base variable *age*, with the meaning of this fuzzy restriction defined by its compatibility function. As an aid in the understanding of the concept of a linguistic variable, Fig. 3 shows the hierarchical structure of the relation between the linguistic variable *Age*, the fuzzy restrictions which represent the meaning of its values, and the values of the base variable *age*.

There are several basic aspects of the concept of a linguistic variable that are in need of elaboration.

First, it is important to understand that the notion of compatibility is distinct from that of probability. Thus, the statement that the compatibility of, say, 28 with *young* is 0.7, has no relation to the probability of the age-value 28. The correct interpretation of the compatibility-value 0.7 is that it

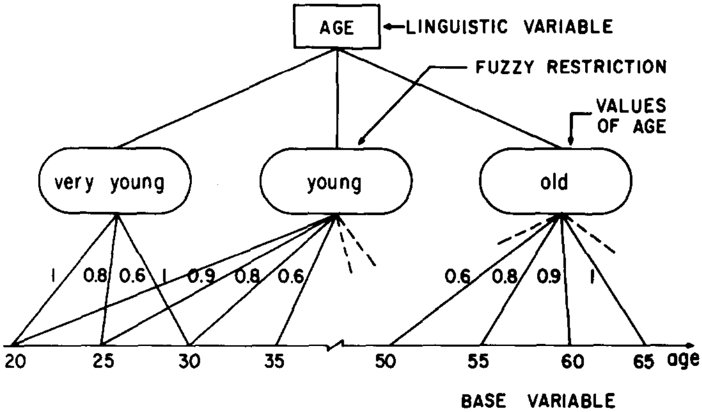


Fig. 3. Hierarchical structure of a linguistic variable.

is merely a subjective indication of the extent to which the age-value 28 fits one's conception of the label *young*. As we shall see in later sections, the rules of manipulation applying to compatibilities are different from those applying to probabilities, although there are certain parallels between the two.

Second, we shall usually assume that a linguistic variable is *structured* in the sense that it is associated with two rules. Rule (i), a *syntactic rule*, specifies the manner in which the linguistic values which are in the term-set of the variable may be generated. In regard to this rule, our usual assumption will be that the terms in the term-set of the variable are generated by a context-free grammar.

The second rule, (ii), is a *semantic rule* which specifies a procedure for computing the meaning of any given linguistic value. In this connection, we observe that a typical value of a linguistic variable, e.g., *not very young and not very old*, involves what might be called the *primary* terms, e.g., *young* and *old*, whose meaning is both subjective and context-dependent. We assume that the meaning of such terms is specified *a priori*.

In addition to the primary terms, a linguistic value may involve connectives such as *and, or, either, neither*, etc.; the negation *not*; and the hedges such as *very, more or less, completely, quite, fairly, extremely, somewhat*, etc. As we shall see in later sections, the connectives, the hedges and the negation may be treated as operators which modify the meaning of their operands in a specified, context-independent fashion. Thus, if the meaning of *young* is defined by the compatibility function whose form is shown in Fig. 1, then the meaning of *very young* could be obtained by squaring the compatibility function of *young*, while that of *not young* would be given by subtracting the compatibility function of *young* from unity (Fig. 4). These two rules are special instances of a more general semantic rule which is described in Part II, Sec. 2.

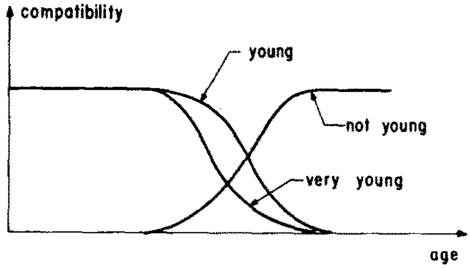


Fig. 4. Compatibilities of *young, not young,* and *very young.*

Third, when we speak of a linguistic variable such as *Age*, the underlying base variable, *age*, is numerical in nature. Thus, in this case we can define the meaning of a linguistic value such as *young* by a compatibility function which associates with each age in the interval [0, 100] a number in the interval [0, 1] which represents the compatibility of that age with the label *young*.

On the other hand, in the case of the linguistic variable *Appearance*, we do not have a well-defined base variable; that is, we do not know how to express the degree of beauty, say, as a function of some physical measurements. We could still associate with each member of a group of ladies, for example, a grade of membership in the class of beautiful women, say 0.9 with Fay, 0.7 with Adele, 0.8 with Kathy and 0.9 with Vera, but these values of the compatibility function would be based on impressions which we may not be able to articulate or formalize in explicit terms. In other words, we are defining the compatibility function not on a set of mathematically well-defined objects, but on a set of labeled impressions. Such definitions are meaningful to a human but not—at least directly—to a computer.[4]

As we shall see in later sections, in the first case, where the base variable is numerical in nature, linguistic variables can be treated in a reasonably precise fashion by the use of the extension principle for fuzzy sets. In the second case, their treatment becomes much more qualitative. In both cases, however, some computation is involved—to a lesser or greater degree. Thus, it should be understood that the linguistic approach is not entirely qualitative in nature. Rather, the computations are performed behind the scene, and, at the end, *linguistic approximation* is employed to convert numbers into words (Fig. 5).

A particularly important area of application for the concept of a linguistic variable is that of *approximate reasoning*, by which we mean a type of reasoning which is neither very precise nor very imprecise. As an illustration, the following inference would be an instance of approximate reasoning:

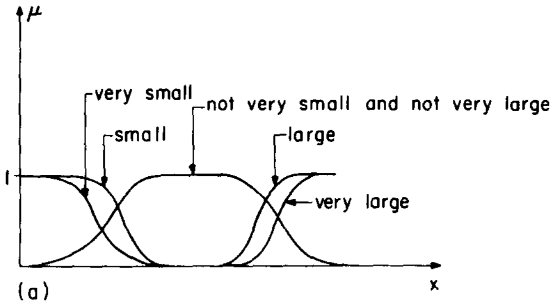
[4]The basic problem which is involved here is that of abstraction from a set of samples of elements of a fuzzy set. A discussion of this problem may be found in [8].

x is small,

x and y are approximately equal;

therefore,

y is more or less small.



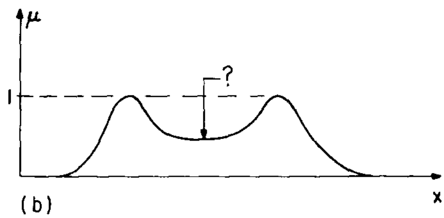


Fig. 5. (a) Compatibilities of *small, very small, large, very large* and *not very small and not very large.* (b) The problem of linguistic approximation is that of finding an approximate linguistic characterization of a given compatibility function.

The concept of a linguistic variable enters into approximate reasoning as a result of treating *Truth* as a linguistic variable whose truth-values form a term-set such as shown below:

T(*Truth*) = *true* + *not true* + *very true* + *completely true* + *more or less true* + *fairly true* + *essentially true* + $\cdots$ + *false* + *very false* + *neither true nor false* + $\cdots$.

The corresponding base variable, then, is assumed to be a number in the interval [0, 1], and the meaning of a primary term such as *true* is identified with a fuzzy restriction on the values of the base variable. As usual, such a restriction is characterized by a compatibility function which associates a number in the interval [0, 1] with each numerical truth-value. For example, the compatibility of the numerical truth-value 0.7 with the linguistic truth-value *very true* might be 0.6. Thus, in the case of truth-values, the compatibility function is a mapping from the unit interval to itself. (This will be shown in Part II, Fig. 13.)

Treating truth as a linguistic variable leads to a fuzzy logic which may well be a better approximation to the logic involved in human decision processes than the classical two-valued logic.[5] Thus, in fuzzy logic it is meaningful to assert what would be inadmissibly vague in classical logic, e.g.,

The truth-value of "Berkeley is close to San Francisco," is *quite true.*
The truth-value of "Palo Alto is close to San Francisco," is *fairly true.*

Therefore,

the truth-value of "Palo Alto is more or less close to Berkeley," is *more or less true.*

Another important area of application for the concept of a linguistic variable lies in the realm of probability theory. If probability is treated as a linguistic variable, its term-set would typically be:

$$T(\textit{Probability}) = \textit{likely} + \textit{very likely} + \textit{unlikely} + \textit{extremely likely} + \textit{fairly likely} + \cdots + \textit{probable} + \textit{improbable} + \textit{more or less probable} + \cdots .$$

By legitimizing the use of linguistic probability-values, we make it possible to respond to a question such as "What is the probability that it will be a warm day a week from today?" with an answer such as *fairly high,* instead of, say, 0.8. The linguistic answer would, in general, be much more realistic, considering, first, that *warm day* is a fuzzy event, and, second, that our understanding of weather dynamics is not sufficient to allow us to make unequivocal assertions about the underlying probabilities.

In the following sections, the concept of a linguistic variable and its applications will be discussed in greater detail. To place the concept of a linguistic variable in a proper perspective, we shall begin our discussion with a formalization of the notion of a conventional (nonfuzzy) variable. For our purposes, it will be helpful to visualize such a variable as a tagged valise with rigid (hard) sides (Fig. 6). Putting an object into the valise corresponds to assigning a value to the variable, and the restriction on what can be put in corresponds to a subset of the universe of discourse which comprises those points which can be assigned as values to the variable. In terms of this analogy, a *fuzzy variable,*

[5] Expositions of alternative approaches to vagueness may be found in [9 – 18].

which is defined in Part II, Sec. 1, may be likened to a tagged valise with soft rather than rigid sides (Part II, Fig. 1). In this case, the restriction on what can be put in is fuzzy in nature, and is defined by a compatibility function which associates with each object a number in the interval [0, 1] representing the degree of ease with which that object can be fitted in the valise. For example, given a valise named X, the compatibility of a coat with X would be 1, while that of a record-player might be 0.7.

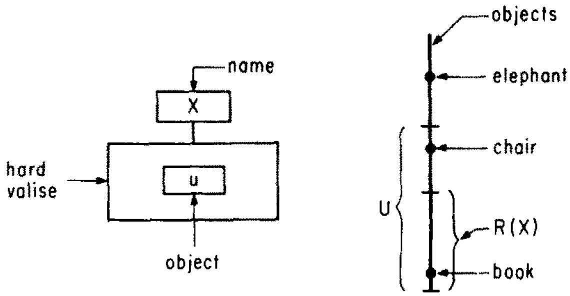


Fig. 6. Illustration of the valise analogy for a unary nonfuzzy variable.

As will be seen in Part II, Sec. 1, an important concept in the case of fuzzy variables is that of *noninteraction*, which is analogous to the concept of independence in the case of random variables. This concept arises when we deal with two or more fuzzy variables, each of which may be likened to a compartment in a soft valise. Such fuzzy variables are *interactive* if the assignment of a value to one affects the fuzzy restrictions placed on the others. This effect may be likened to the interference between objects which are put into different compartments of a soft valise (Part II, Fig. 3).

A linguistic variable is defined in Part II, Sec. 2 as a variable whose values are fuzzy variables. In terms of our valise analogy, a linguistic variable corresponds to a hard valise into which we can put soft valises, with each soft valise carrying a name tag which describes a fuzzy restriction on what can be put into that valise (Part II, Fig. 6).

The application of the concept of a linguistic variable to the notion of *Truth* is discussed in Part II, Sec. 3. Here we describe a technique for computing the conjunction, disjunction and negation for linguistic truth-values and lay the groundwork for fuzzy logic.

In Part III, Sec. 1, the concept of a linguistic variable is applied to probabilities, and it is shown that linguistic probabilities can be used for computational purposes. However, because of the constraint that the numerical probabilities must add up to unity, the computations in question involve the solution of nonlinear programs and hence are not as simple to perform as computations involving numerical probabilities.

The last section is devoted to a discussion of the so-called *compositional rule of inference* and its application to approximate reasoning. This rule of inference is interpreted as the process of solving a simultaneous system of so-called *relational assignment equations* in which linguistic values are assigned to fuzzy restrictions. Thus, if a statement such as "x is small" is interpreted as an assignment of the linguistic value *small* to the fuzzy restriction on x, and the statement "x and y are approximately equal" is interpreted as the assignment of a fuzzy relation labeled *approximately equal* to the fuzzy restriction on the ordered pair (x, y), then the conclusion "y is more or less small" may be viewed as a linguistic approximation to the solution of the simultaneous equations

$$R(x) = small,$$

$$R(x,y) = approximately\ equal,$$

in which $R(x)$ and $R(x,y)$ denote the restrictions on x and (x, y), respectively (Part III, Fig. 5).

The compositional rule of inference leads to a *generalized modus ponens*, which may be viewed as an extension of the familiar rule of inference: If A is true and A implies B, then B is true. The section closes with an example of a fuzzy theorem in elementary geometry and a brief discussion of the use of fuzzy flowcharts for the representation of definitional fuzzy algorithms.

The material in Secs. 2 and 3 and in Part II, Sec. 1 is intended to provide a mathematical basis for the concept of a linguistic variable, which is introduced in Part II, Sec. 2. For those readers who may not be interested in the mathematical aspects of the theory, it may be expedient to proceed directly to Part II, Sec. 2 and refer where necessary to the definitions and results described in the preceding sections.

2. THE CONCEPT OF A VARIABLE

In the preceding section, our discussion of the concept of a linguistic variable was informal in nature. To set the stage for a more formal definition, we shall focus our attention in this section on the concept of a conventional (nonfuzzy) variable. Then in Sec. 3 we shall extend the concept of a variable to fuzzy variables and subsequently will define a linguistic variable as a variable whose values are fuzzy variables.

Although the concept of a (nonfuzzy) variable is very elementary in nature, it is by no means a trivial one. For our purposes, the following formalization of the concept of a variable provides a convenient basis for later extensions.

DEFINITION 2.1. A variable is characterized by a triple $(X, U, R(X;u))$, in which X is the name of the variable; U is a universe of discourse (finite or infinite set); u is a generic[6] name for the elements of U; and $R(X;u)$ is a subset of U which represents a *restriction*[7] on the values of u imposed by X. For convenience, we shall usually abbreviate $R(X;u)$ to $R(X)$ or $R(u)$ or $R(x)$, where x denotes a generic name for the values of X, and will refer to $R(X)$ simply as the restriction *on* u or the restriction *imposed by* X.

In addition, a variable is associated with an *assignment equation*

$$x = u : R(X) \tag{2.1}$$

or equivalently

$$x = u, \qquad u \in R(X) \tag{2.2}$$

which represents the assignment of a value u to x subject to the restriction $R(X)$. Thus, the assignment equation is *satisfied* iff (if and only if) $u \in R(X)$.

Example 2.1. As a simple illustration consider a variable named *age*. In this case, U might be taken to be the set of integers $0, 1, 2, 3, \ldots$, and $R(X)$ might be the subset $0, 1, 2, \ldots, 100$.

More generally, let $X_1, \ldots, X_n$ be n variables with respective universes of discourse $U_1, \ldots, U_n$. The ordered n-tuple $X = (X_1, \ldots, X_n)$ will be referred to as an *n-ary composite* (or *joint*) *variable*. The universe of discourse for X is the Cartesian product

$$U = U_1 \times U_2 \times \cdots \times U_n, \tag{2.3}$$

and the restriction $R(X_1, \ldots, X_n)$ is an n-ary relation in $U_1 \times \cdots \times U_n$. This relation may be defined by its characteristic (membership) function $\mu_R : U_1 \times \cdots \times U_n \to \{0, 1\}$, where

$$\begin{aligned} \mu_R(u_1, \ldots, u_n) &= 1 \qquad \text{if } (u_1, \ldots, u_n) \in R(X_1, \ldots, X_n) \\ &= 0 \qquad \text{otherwise,} \end{aligned} \tag{2.4}$$

[6]A generic name is a single name for all elements of a set. For simplicity, we shall frequently use the same symbol for both a set and the generic name for its elements, relying on the context for disambiguation.

[7]In conventional terminology, $R(X)$ is the range of X. Our use of the term *restriction* is motivated by the role played by $R(X)$ in the case of fuzzy variables.

and u_i is a generic name for the elements of U_i, $i = 1, \ldots, n$. Correspondingly, the n-ary assignment equation assumes the form

$$(x_1, \ldots x_n) = (u_1, \ldots, u_n) : R(X_1, \ldots, X_n), \tag{2.5}$$

which is understood to mean that

$$x_i = u_i, \qquad i = 1, \ldots, n \tag{2.6}$$

subject to the restriction $(u_1, \ldots, u_n) \in R(X_1, \ldots, X_n)$, with x_i, $i = 1, \ldots, n$, denoting a generic name for values of X_i.

Example 2.2 Suppose that $X_1 \triangleq$ age of father,[8] $X_2 \triangleq$ age of son, and $U_1 \triangleq U_2 = \{1, 2, \ldots, 100\}$. Furthermore, suppose that $x_1 \geqslant x_2 + 20$ (x_1 and x_2 are generic names for values of X_1 and X_2). Then $R(X_1, X_2)$ may be defined by

$$\begin{aligned} \mu_R(u_1, u_2) &= 1 \qquad \text{for } 21 \leqslant u_1 \leqslant 100, \quad u_1 \geqslant u_2 + 20 \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{2.7}$$

MARGINAL AND CONDITIONED RESTRICTIONS

As in the case of probability distributions, the restriction $R(X_1, \ldots, X_n)$ imposed by $(X_1, \ldots, X_n)$ induces *marginal* restrictions $R(X_{i_1}, \ldots, X_{i_k})$ imposed by composite variables of the form $(X_{i_1}, \ldots, X_{i_k})$, where the index sequence $q = (i_1, \ldots, i_k)$ is a subsequence of the index sequence $(1, 2, \ldots, n)$.[9] In effect, $R(X_{i_1}, \ldots, X_{i_k})$ is the smallest (i.e., most restrictive) restriction imposed by $(X_{i_1}, \ldots, X_{i_k})$ which satisfies the implication

$$(u_1, \ldots, u_n) \in R(X_1, \ldots, X_n) \Rightarrow (u_{i_1}, \ldots, u_{i_k}) \in R(X_{i_1}, \ldots, X_{i_k}). \tag{2.8}$$

Thus, a given k-tuple $u_{(q)} \triangleq (u_{i_1}, \ldots, u_{i_k})$ is an element of $R(X_{i_1}, \ldots, X_{i_k})$ iff there exists an n-tuple $u \triangleq (u_1, \ldots, u_n) \in R(X_1, \ldots, X_n)$ whose i_1th, $\ldots$, i_kth components are equal to $u_{i_1}, \ldots, u_{i_k}$, respectively. Expressed in terms of the characteristic functions of $R(X_1, \ldots, X_n)$ and $R(X_{i_1}, \ldots, X_{i_k})$, this statement translates into the equation

[8] The symbol $\triangleq$ stands for "denotes" or is "equal by definition."

[9] In the case of a binary relation $R(X_1, X_2)$, $R(X_1)$ and $R(X_2)$ are usually referred to as the *domain* and *range* of $R(X_1, X_2)$.

$$\mu_{R(X_{i_1}, \ldots, X_{i_k})}(u_{i_1}, \ldots, u_{i_k}) = \bigvee_{u_{(q')}} \mu_{R(X_1, \ldots, X_n)}(u_1, \ldots, u_n), \tag{2.9}$$

or more compactly,

$$\mu_{R(X_{(q)})}(u_{(q)}) = \bigvee_{u_{(q')}} \mu_{R(X)}(u), \tag{2.10}$$

where q' is the complement of the index sequence $q = (i_1, \ldots, i_k)$ relative to $(1, \ldots, n)$, $u_{(q')}$ is the complement of the k-tuple $u_{(q)} \triangleq (u_{i_1}, \ldots, u_{i_k})$ relative to the n-tuple $u \triangleq (u_1, \ldots, u_n)$, $X_{(q)} \triangleq (X_{i_1}, \ldots, X_{i_k})$, and $\bigvee_{u_{(q')}}$ denotes the supremum of its operand over the u's which are in $u_{(q')}$. (Throughout this paper, the symbols $\vee$ and $\wedge$ stand for Max and Min, respectively; thus, for any real a, b

$$\begin{aligned} a \vee b = \mathrm{Max}(a, b) &= a \quad \text{if} \quad a \geqslant b \\ &= b \quad \text{if} \quad a < b \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} a \wedge b = \mathrm{Min}(a, b) &= a \quad \text{if} \quad a \leqslant b \\ &= b \quad \text{if} \quad a > b. \end{aligned}$$

Consistent with this notation, the symbol $\bigvee_z$ should be read as "supremum over the values of z.") Since μ_R can take only two values–0 or 1–(2.10) means that $\mu_{R(X_{(q)})}(u_{(q)})$ is 1 iff there exists a $u_{(q')}$ such that $\mu_{R(X)}(u) = 1$.

COMMENT 2.1. There is a simple analogy which is very helpful in clarifying the notion of a variable and related concepts. Specifically, a non-fuzzy variable in the sense formalized in Definition 2.1 may be likened to a tagged valise having rigid (hard) sides, with X representing the name on the tag, U representing a list of objects which could be put in a valise, and $R(X)$ representing a sublist of U which comprises those objects which can be put into valise X. [For example, an object like a boat would not be in U, while an object like a typewriter might be in U but not in $R(X)$, and an object like a cigarette box or a pair of shoes would be in $R(X)$.] In this interpretation, the assignment equation

$$x = u : R(X)$$

signifies that an object u which satisfies the restriction $R(X)$ (i.e., is on the list of objects which can be put into X) is put into X (Fig. 6).

An n-ary composite variable $X \triangleq (X_1, \ldots, X_n)$ corresponds to a valise, carrying the name-tag X, which has n compartments named $X_1, \ldots, X_n$ with adjustable partitions between them. The restrictions $R(X_1, \ldots, X_n)$ corresponds to a list of n-tuples of objects $(u_1, \ldots, u_n)$ such that u_1 can be put in compartment X_1, u_2 in compartment $X_2, \ldots$, and u_n in compartment X_n *simultaneously.* (see Fig. 7.) In this connection, it should be noted that n-tuples on this list could be associated with different arrangements of partitions. If $n = 2$, for example, then for a particular placement of the partition we could put a coat in compartment X_1 and a suit in compartment X_2, while for some other placement we could put the coat in compartment X_2 and a box of shoes in compartment X_1. In this event, both (coat, suit) and (shoes, coat) would be included in the list of pairs of objects which can be put in X simultaneously.

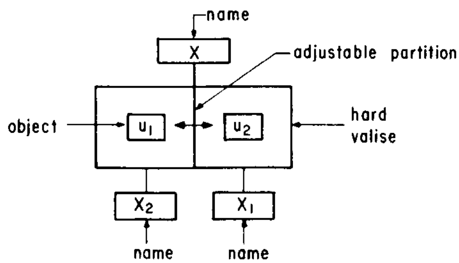


Fig. 7. Valise analogy for a binary nonfuzzy variable.

In terms of the valise analogy, the n-ary assignment equation

$$(x_1, \ldots, x_n) = (u_1, \ldots, u_n) : R(X_1, \ldots, X_n)$$

represents the action of putting u_1 in $X_1, \ldots$, and u_n in X_n simultaneously, under the restriction that the n-tuple of objects $(u_1, \ldots, u_n)$ must be on the $R(X_1, \ldots, X_n)$ list. Furthermore, a marginal restriction such as $R(X_{i_1}, \ldots, X_{i_k})$ may be interpreted as a list of k-tuples of objects which can be put in compartments $X_{i_1}, \ldots, X_{i_k}$ simultaneously, in conjunction with every allowable placement of objects in the remaining compartments.

COMMENT 2.2. It should be noted that (2.9) is analogous to the expression for a marginal distribution of a probability distribution, with $\vee$ corresponding to summation (or integration). However, this analogy should not be construed to imply that $R(X_{i_1}, \ldots, X_{i_k})$ is in fact a marginal probability distribution.

It is convenient to view the right-hand side of (2.9) as the characteristic function of the projection[10] of $R(X_1, \ldots, X_n)$ on $U_{i_1} \times \cdots \times U_{i_k}$. Thus, in symbols,

$$R(X_{i_1}, \ldots, X_{i_k}) = \text{Proj}\, R(X_1, \ldots, X_n) \text{ on } U_{i_1} \times \cdots \times U_{i_k}, \tag{2.12}$$

or more simply,

$$R(X_{i_1}, \ldots, X_{i_k}) = P_q R(X_1, \ldots, X_n),$$

where P_q denotes the operation of projection on $U_{i_1} \times \cdots \times U_{i_k}$ with $q = (i_1, \ldots, i_k)$.

Example 2.3. In the case of Example 2.2, we have

$$R(X_1) = P_1\, R(X_1, X_2) = \{21, \ldots, 100\},$$

$$R(X_2) = P_2\, R(X_1, X_2) = \{1, \ldots, 80\}.$$

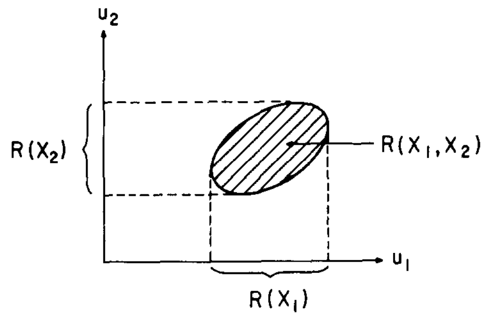


Fig. 8. Marginal restrictions induced by $R(X_1, X_2)$.

Example 2.4. Fig. 8 shows the restrictions on u_1 and u_2 induced by $R(X_1, X_2)$.

An alternative way of describing projections is the following. Viewing $R(X_1, \ldots, X_n)$ as a relation in $U_1 \times \cdots \times U_n$, let $q' = (j_1, \ldots, j_m)$ denote the index sequence complementary to $q = (i_1, \ldots, i_k)$, and let $R(X_{i_1}, \ldots, X_{i_k} \mid u_{j_1}, \ldots, u_{j_m})$—or, more compactly, $R(X_{(q)} \mid u_{(q')})$—denote a restriction in $U_{i_1} \times \cdots \times U_{i_k}$ which is *conditioned on* $u_{j_1}, \ldots, u_{j_m}$. The characteristic function of this conditioned restriction is defined by

[10]The term *projection* as used in the literature is somewhat ambiguous in that it could denote either the operation of projecting or the result of such an operation. To avoid this ambiguity in the case of fuzzy relations, we will occasionally employ the term *shadow* [19] to denote the relation resulting from applying an operation of projection to another relation.

$$\mu_{R(X_{i_1},\ldots,X_{i_k}|u_{j_1},\ldots,u_{j_m})}(u_{i_1},\ldots,u_{i_k})=\mu_{R(X_1,\ldots,X_n)}(u_1,\ldots,u_n), \tag{2.13}$$

or more simply [see (2.10)],

$$\mu_{R(X_{(q)}|u_{(q')})}(u_{(q)})=\mu_{R(X)}(u)$$

with the understanding that the arguments $u_{j_1},\ldots,u_{j_m}$ on the right-hand side of (2.13) are treated as parameters. In consequence of this understanding, although the characteristic function of the conditioned restriction is numerically equal to that of $R(X_1,\ldots,X_n)$, it defines a relation in $U_{i_1}\times\cdots\times U_{i_k}$ rather than in $U_1\times\cdots\times U_n$.

In view of (2.9), (2.12) and (2.13), the projection of $R(X_1,\ldots,X_n)$ on $U_{i_1}\times\cdots\times U_{i_k}$ may be expressed as

$$P_qR(X_1,\ldots,X_n)=\cup_{u(q')}R(X_{i_1},\ldots,X_{i_k}|u_{j_1},\ldots,u_{j_m}), \tag{2.14}$$

where $\cup_{u(q')}$ denotes the union of the family of restrictions $R(X_{i_1},\ldots,X_{i_k}|u_{j_1},\ldots,u_{j_m})$ parametrized by $u_{(q')}\triangleq(u_{j_1},\ldots,u_{j_m})$. Consequently, (2.14) implies that the marginal restriction $R(X_{i_1},\ldots,X_{i_k})$ in $U_{i_1}\times\cdots\times U_{i_k}$ may be expressed as the union of conditioned restrictions $R(X_{i_1},\ldots,X_{i_k}|u_{j_1},\ldots,u_{j_m})$, i.e.,

$$R(X_{i_1},\ldots,X_{i_k})=\cup_{u(q')}R(X_{i_1},\ldots,X_{i_k}|u_{j_1},\ldots,u_{j_m}), \tag{2.15}$$

or more compactly,

$$R(X_{(q)})=\cup_{u(q')}R(X_{(q)}|u_{(q')}).$$

Example 2.5. As a simple illustration of (2.15), assume that $U_1=U_2\triangleq\{3,5,7,9\}$ and that $R(X_1,X_2)$ is characterized by the following relation matrix. [In this matrix, the (i,j)th entry is 1 iff the ordered pair (ith element of U_1, jth element of U_2) belongs to $R(X_1,X_2)$. In effect, the relation matrix of a relation R constitutes a tabulation of the characteristic function of R.]

R	3	5	7	9
3	0	0	1	0
5	1	0	1	0
7	1	0	1	1
9	1	0	0	1

In this case,

$$R(X_1, X_2 \,|\, u_1 = 3) = \{7\},$$
$$R(X_1, X_2 \,|\, u_1 = 5) = \{3,7\},$$
$$R(X_1, X_2 \,|\, u_1 = 7) = \{3,7,9\},$$
$$R(X_1, X_2 \,|\, u_1 = 9) = \{3,9\},$$

and hence

$$\begin{aligned} R(X_2) &= \{7\} \cup \{3,7\} \cup \{3,7,9\} \cup \{3,9\} \\ &= \{3,7,9\}. \end{aligned}$$

INTERACTION AND NONINTERACTION

A basic concept that we shall need in later sections is that of the *interaction* between two or more variables—a concept which is analogous to the *dependence* of random variables. More specifically, let the variable $X = (X_1, \ldots, X_n)$ be associated with the restriction $R(X_1, \ldots, X_n)$, which induces the restrictions $R(X_1), \ldots, R(X_n)$ on $u_1, \ldots, u_n$, respectively. Then we have

DEFINITION 2.2. $X_1, \ldots, X_n$ are *noninteractive variables under* $R(X_1, \ldots, X_n)$ iff $R(X_1, \ldots, X_n)$ is *separable*, i.e.,

$$R(X_1, \ldots, X_n) = R(X_1) \times \cdots \times R(X_n), \tag{2.16}$$

where, for $i = 1, \ldots, n$,

$$\begin{aligned} R(X_i) &= \text{Proj } R(X_1, \ldots, X_n) \text{ on } U_i \\ &= \cup_{u_{(q')}} R(X_i \,|\, u_{(q')}), \end{aligned} \tag{2.17}$$

with $u_{(q)} \triangleq u_i$ and $u_{(q')} \triangleq$ complement of u_i in $(u_1, \ldots, u_n)$.

Example 2.6. Fig. 9(a) shows two noninteractive variables X_1 and X_2 whose restrictions $R(X_1)$ and $R(X_2)$ are intervals; in this case, $R(X_1, X_2)$ is the Cartesian product of the intervals in question. In Fig. 9(b), $R(X_1, X_2)$ is a proper subset of $R(X_1) \times R(X_2)$, and hence X_1 and X_2 are interactive. Note that in Example 2.3, X_1 and X_2 are interactive.

As will be shown in a more general context in Part II, Sec. 1, if $X_1, \ldots, X_n$ are noninteractive, then an n-ary assignment equation

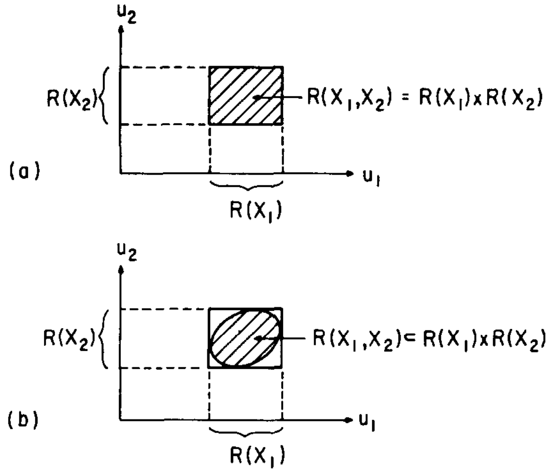


Fig. 9. (a) X_1 and X_2 are noninteractive. (b) X_1 and X_2 are interactive.

$$(x_1, \ldots, x_n) = (u_1, \ldots, u_n) : R(X_1, \ldots, X_n) \tag{2.18}$$

can be decomposed into a sequence of n unary assignment equations

$$\begin{aligned} x_1 &= u_1 : R(X_1), \\ x_2 &= u_2 : R(X_2), \\ &\cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \\ x_n &= u_n : R(X_n), \end{aligned} \tag{2.19}$$

where $R(X_i)$, $i = 1, \ldots, n$, is the projection of $R(X_1, \ldots, X_n)$ on U_i, and by Definition 2.2,

$$R(X_1, \ldots, X_n) = R(X_1) \times \cdots \times R(X_n). \tag{2.20}$$

In the case where $X_1, \ldots, X_n$ are interactive, the sequence of n unary assignment equations assumes the following form [see also Part II, Eq. (1.34)].

$$
\begin{aligned}
x_1 &= u_1 : R(X_1), \\
x_2 &= u_2 : R(X_2 \mid u_1), \\
&\cdot \quad \cdot \quad \cdot \\
&\cdot \quad \cdot \quad \cdot \\
&\cdot \quad \cdot \quad \cdot \\
x_n &= u_n : R(X_n \mid u_1, \ldots, u_{n-1}),
\end{aligned}
\tag{2.21}
$$

where $R(X_i \mid u_1, \ldots, u_{i-1})$ denotes the induced restriction on u_i conditioned on $u_1, \ldots, u_{i-1}$. The characteristic function of this conditioned restriction is expressed by [see (2.13)]

$$\mu_{R(X_i \mid u_1, \ldots, u_{i-1})}(u_i) = \mu_{R(X_1, \ldots, X_i)}(u_1, \ldots, u_i), \cdot \tag{2.22}$$

with the understanding that the arguments $u_1, \ldots, u_{i-1}$ on the right-hand side of (2.22) play the role of parameters.

COMMENT 2.3. In words, (2.21) means that, in the case of interactive variables, once we have assigned a value u_1 to x_1, the restriction on u_2 becomes dependent on u_1. Then, the restriction on u_3 becomes dependent on the values assigned to x_1 and x_2, and, finally, the restriction on u_n becomes dependent on $u_1, \ldots, u_{n-1}$. Furthermore, (2.22) implies that the restriction on u_i given $u_1, \ldots, u_{i-1}$ is essentially the same as the marginal restriction on $(u_1, \ldots, u_i)$, with $u_1, \ldots, u_{i-1}$ treated as parameters. This illustrated in Fig. 10.

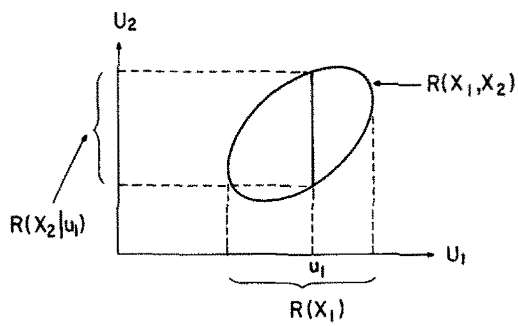


Fig. 10. $R(X_2 \mid u_1)$ is the restriction on u_2 conditioned on u_1.

In terms of the valise analogy (see Comment 2.1), $X_1, \ldots, X_n$ are noninteractive if the partitions between the compartments named $X_1, \ldots, X_n$ are not adjustable. In this case, what is placed in a compartment X_i has no influence on the objects that can be placed in the other compartments.

In the case where the partitions are adjustable, this is no longer true, and $X_1, \ldots, X_n$ become interactive in the sense that the placement of an object, say u_i, in X_i affects what can be placed in the complementary compartments. From this point of view, the sequence of unary assignment equations (2.21) describes the way in which the restriction on compartment X_i is influenced by the placement of objects $u_1, \ldots, u_{i-1}$ in $X_1, \ldots, X_{i-1}$.

Our main purpose in defining the notions of noninteraction, marginal restriction, conditioned restriction, etc. for nonfuzzy variables is (a) to indicate that concepts analogous to statistical independence, marginal distribution, conditional distribution, etc., apply also to nonrandom, nonfuzzy variables; and (b) to set the stage for similar concepts in the case of fuzzy variables. As a preliminary, we shall turn our attention to some of the relevant properties of fuzzy sets and formulate an extension principle which will play an important role in later sections.

3. FUZZY SETS AND THE EXTENSION PRINCIPLE

As will be seen in Part II, Sec. 1, a fuzzy variable X differs from a nonfuzzy variable in that it is associated with a restriction $R(X)$ which is a fuzzy subset of the universe of discourse.[11] Consequently, as a preliminary to our consideration of the concept of a fuzzy variable, we shall review some of the pertinent properties of fuzzy sets and state an extension principle which allows the domain of a transformation or a relation in U to be extended from points in U to fuzzy subsets of U.

FUZZY SETS–NOTATION AND TERMINOLOGY

A fuzzy subset A of a universe of discourse U is characterized by a *membership function* $\mu_A : U \to [0, 1]$ which associates with each element u of U a number $\mu_A(u)$ in the interval $[0, 1]$, with $\mu_A(u)$ representing the *grade of membership* of u in A.[12] The *support* of A is the set of points in U at which $\mu_A(u)$ is positive. The *height* of A is the supremum of $\mu_A(u)$ over U. A *crossover point* of A is a point in U whose grade of membership in A is 0.5.

Example 3.1. Let the universe of discourse be the interval $[0, 1]$, with u interpreted as *age*. A fuzzy subset of U labeled *old* may be defined by a membership function such as

[11] More detailed discussions of fuzzy sets and their properties may be found in the listed references. (A detailed exposition of the fundamentals together with many illustrative examples may be found in the recent text by A. Kaufmann [20]).

[12] More generally, the range of μ_A may be a partially ordered set (see [21], [22]) or a collection of fuzzy sets. The latter case will be discussed in greater detail in Sec. 6.

$$\mu_A(u) = 0 \qquad \text{for } 0 \leqslant u \leqslant 50, \tag{3.1}$$

$$\mu_A(u) = \left[1 + \left(\frac{u-50}{5}\right)^{-2}\right]^{-1} \quad \text{for } 50 \leqslant u \leqslant 100.$$

In this case, the support of *old* is the interval [50, 100]; the height of *old* is effectively unity; and the crossover point of *old* is 55.

To simplify the representation of fuzzy sets we shall employ the following notation.

A nonfuzzy finite set such as

$$U = \{u_1, \ldots, u_n\} \tag{3.2}$$

will be expressed as

$$U = u_1 + u_2 + \cdots + u_n \tag{3.3}$$

or

$$U = \sum_{i=1}^{n} u_i, \tag{3.4}$$

with the understanding that + denotes the union rather than the arithmetic sum. Thus, (3.3) may be viewed as a representation of U as the union of its constituent singletons.

As an extension of (3.3), a fuzzy subset A of U will be expressed as

$$A = \mu_1 u_1 + \cdots + \mu_n u_n \tag{3.5}$$

or

$$A = \sum_{i=1}^{n} \mu_i u_i, \tag{3.6}$$

where μ_i, $i = 1, \ldots, n$, is the grade of membership of u_i in A. In cases where the u_i are numbers, there might be some ambiguity regarding the identity of the μ_i and u_i components of the string $\mu_i u_i$. In such cases, we shall employ a separator symbol such as / for disambiguation, writing

$$A = \mu_1/u_1 + \cdots + \mu_n/u_n \tag{3.7}$$

or

$$A = \sum_{i=1}^{n} \mu_i / u_i . \tag{3.8}$$

Example 3.2. Let $U = \{a, b, c, d\}$ or, equivalently,

$$U = a + b + c + d . \tag{3.9}$$

In this case, a fuzzy subset A of U may be represented unambiguously as

$$A = 0.3a + b + 0.9c + 0.5d. \tag{3.10}$$

On the other hand, if

$$U = 1 + 2 + \cdots + 100, \tag{3.11}$$

then we shall write

$$A = 0.3/25 + 0.9/3 \tag{3.12}$$

in order to avoid ambiguity.

Example 3.3. In the universe of discourse comprising the integers $1, 2, \ldots,$ 10, i.e.,

$$U = 1 + 2 + \cdots + 10, \tag{3.13}$$

the fuzzy subset labeled *several* may be defined as

$$\textit{several} = 0.5/3 + 0.8/4 + 1/5 + 1/6 + 0.8/7 + 0.5/8. \tag{3.14}$$

Example 3.4. In the case of the countable universe of discourse

$$U = 0 + 1 + 2 + \cdots , \tag{3.15}$$

the fuzzy set labeled *small* may be expressed as

$$\textit{small} = \sum_{0}^{\infty} \left[1 + \left(\frac{u}{10}\right)^2\right]^{-1} / u. \tag{3.16}$$

Like (3.3), (3.5) may be interpreted as a representation of a fuzzy set as the union of its constituent fuzzy singletons $\mu_i u_i$ (or μ_i / u_i). From the definition of the union [see (3.34)], it follows that if in the representation of A we have $u_i = u_j$, then we can make the substitution expressed by

$$\mu_i u_i + \mu_j u_i = (\mu_i \vee \mu_j) u_i . \tag{3.17}$$

For example,

$$A \;=\; 0.3a + 0.8a + 0.5b \tag{3.18}$$

may be rewritten as

$$\begin{aligned} A &= (0.3 \vee 0.8)a \;+\; 0.5b \\ &= \;0.8a + 0.5b. \end{aligned} \tag{3.19}$$

When the support of a fuzzy set is a continuum rather than a countable or a finite set, we shall write

$$A = \int_U \mu_A(u)/u, \tag{3.20}$$

with the understanding that $\mu_A(u)$ is the grade of membership of u in A, and the integral denotes the union of the fuzzy singletons $\mu_A(u)/u,\ u \in U$.

Example 3.5. In the universe of discourse consisting of the interval [0, 100], with $u = age$, the fuzzy subset labeled *old* [whose membership function is given by (3.1)] may be expressed as

$$old = \int_{50}^{100} \left[1 + \left(\frac{u-50}{5}\right)^{-2} \right]^{-1} /u. \tag{3.21}$$

Note that the crossover point for this set, that is, the point u at which

$$\mu_{old}(u) \;=\; 0.5, \tag{3.22}$$

is $u = 55$.

A fuzzy set A is said to be *normal* if its height is unity, that is, if

$$\operatorname*{Sup}_u \; \mu_A(u) = 1. \tag{3.23}$$

Otherwise A is *subnormal.* In this sense, the set *old* defined by (3.21) is *normal,* as is the set *several* defined by (3.14). On the other hand, the subset of $U = 1 + 2 + \cdots + 10$ labeled *not small and not large* and defined by

$$\textit{not small and not large} = 0.2/2 + 0.3/3 + 0.4/4 + 0.5/5 + 0.4/6 + 0.3/7 + 0.2/8 \tag{3.24}$$

is subnormal. It should be noted that a subnormal fuzzy set may be *normalized* by dividing μ_A by $\operatorname{Sup}_u \mu_A(u)$.

A fuzzy subset of U may be a subset of another fuzzy or nonfuzzy subset of U. More specifically, A is a *subset of* B or is *contained in* B iff $\mu_A(u) \leqslant \mu_B(u)$ for all u in U. In symbols,

$$A \subset B \Leftrightarrow \mu_A(u) \leqslant \mu_B(u), \qquad u \in U. \tag{3.25}$$

Example 3.6. If $U = a + b + c + d$ and

$$\begin{aligned} A &= 0.5\,a + 0.8b + 0.3d, \\ B &= 0.7a + b + 0.3c + d, \end{aligned} \tag{3.26}$$

then $A \subset B$.

LEVEL-SETS OF A FUZZY SET

If A is a fuzzy subset of U, then an α-*level set* of A is a nonfuzzy set denoted by A_α which comprises all elements of U whose grade of membership in A is greater than or equal to α. In symbols,

$$A_\alpha = \{u \mid \mu_A(u) \geqslant \alpha\}. \tag{3.27}$$

A fuzzy set A may be decomposed into its level-sets through the *resolution identity*[13]

$$A = \int_0^1 \alpha A_\alpha \tag{3.28}$$

[13]The resolution identity and some of its applications are discussed in greater detail in [6] and [24].

or

$$A = \sum_{\alpha} \alpha A_{\alpha}, \tag{3.29}$$

where αA_{α} is the product of a scalar α with the set A_{α} [in the sense of (3.39)], and $\int_0^1$ (or $\sum_{\alpha}$) is the union of the A_{α}, with α ranging from 0 to 1.

The resolution identity may be viewed as the result of combining together those terms in (3.5) which fall into the same level-set. More specifically, suppose that A is represented in the form

$$A = 0.1/2 + 0.3/1 + 0.5/7 + 0.9/6 + 1/9. \tag{3.30}$$

Then by using (3.17), A can be rewritten as

$$\begin{aligned} A = 0.1/2 + 0.1/1 + 0.1/7 + 0.1/6 + 0.1/9 \\ + 0.3/1 + 0.3/7 + 0.3/6 + 0.3/9 \\ + 0.5/7 + 0.5/6 + 0.5/9 \\ + 0.9/6 + 0.9/9 \\ + 1/9 \end{aligned}$$

or

$$\begin{aligned} A = {} & 0.1\,(1/2 + 1/1 + 1/7 + 1/6 + 1/9) \\ & + 0.3\,(1/1 + 1/7 + 1/6 + 1/9) \\ & + 0.5\,(1/7 + 1/6 + 1/9) \\ & + 0.9\,(1/6 + 1/9) \\ & + 1(1/9), \end{aligned} \tag{3.31}$$

which is in the form (3.29), with the level-sets given by [see (3.27)]

$$A_{0.1} = 2 + 1 + 7 + 6 + 9,$$

$$A_{0.3} = 1 + 7 + 6 + 9,$$

$$A_{0.5} = 7 + 6 + 9,$$

$$A_{0.9} = 6 + 9,$$

$$A_1 \;\; = 9. \tag{3.32}$$

As will be seen in later sections, the resolution identity—in combination with the extension principle—provides a convenient way of generalizing various concepts associated with nonfuzzy sets to fuzzy sets. This, in fact, is the underlying basis for many of the definitions stated in what follows.

OPERATIONS ON FUZZY SETS

Among the basic operations which can be performed on fuzzy sets are the following.

1. The *complement* of A is denoted by $\neg A$ (or sometimes by A') and is defined by

$$\neg A = \int_U [1 - \mu_A(u)]/u. \tag{3.33}$$

The operation of complementation corresponds to negation. Thus, if A is a label for a fuzzy set, then *not* A would be interpreted as $\neg A$. (See Example 3.7 below.)

2. The *union* of fuzzy sets A and B is denoted by $A + B$ (or, more conventionally, by $A \cup B$) and is defined by

$$A + B = \int_U [\mu_A(u) \vee \mu_B(u)]/u. \tag{3.34}$$

The union corresponds to the connective *or*. Thus, if A and B are labels of fuzzy sets, then A *or* B would be interpreted as $A + B$.

3. The *intersection* of A and B is denoted by $A \cap B$ and is defined by

$$A \cap B = \int_U [\mu_A(u) \wedge \mu_B(u)]/u. \tag{3.35}$$

The intersection corresponds to the connective *and*; thus

$$A \textit{ and } B = A \cap B. \tag{3.36}$$

COMMENT 3.1. It should be understood that $\vee (\triangleq \text{Max})$ and $\wedge (\triangleq \text{Min})$ are not the only operations in terms of which the union and intersection can be defined. (See [25] and [26] for discussions of this point.) In this connection, it is important to note that when *and* is identified with Min, as in (3.36), it represents a "hard" *and* in the sense that it allows no trade-offs between its operands. By contrast, an *and* identified with the arithmetic product, as in (3.37) below, would act as a "soft" *and*. Which of these two and possibly other definitions is more appropriate depends on the context in which *and* is used.

4. The *product* of A and B is denoted by AB and is defined by

$$AB = \int_U \mu_A(u)\mu_B(u)/u. \tag{3.37}$$

Thus, A^α, where α is any positive number, should be interpreted as

$$A^\alpha = \int_U [\mu_A(u)]^\alpha/u. \tag{3.38}$$

Similarly, if α is any nonnegative real number such that $\alpha \operatorname{Sup}_u \mu_A(u) \leqslant 1$, then

$$\alpha A = \int_U \alpha\mu_A(u)/u. \tag{3.39}$$

As a special case of (3.38), the operation of *concentration* is defined as

$$\mathrm{CON}(A) = A^2, \tag{3.40}$$

while that of *dilation* is expressed by

$$\mathrm{DIL}(A) = A^{0.5} \tag{3.41}$$

As will be seen in Part II, Sec. 3, the operations of concentration and dilation are useful in the representation of linguistic hedges.

Example 3.7. If

$$\begin{aligned} U &= 1 + 2 + \cdots + 10, \\ A &= 0.8/3 + 1/5 + 0.6/6, \\ B &= 0.7/3 + 1/4 + 0.5/6, \end{aligned} \tag{3.42}$$

then

$$
\begin{aligned}
\neg A &= 1/1 + 1/2 + 0.2/3 + 1/4 + 0.4/6 + 1/7 + 1/8 + 1/9 + 1/10,\\
A + B &= 0.8/3 + 1/4 + 1/5 + 0.6/6,\\
A \cap B &= 0.7/3 + 0.5/6,\\
AB &= 0.56/3 + 0.3/6,\\
A^2 &= 0.64/3 + 1/5 + 0.36/6,\\
0.4A &= 0.32/3 + 0.4/5 + 0.24/6,\\
\mathrm{CON}(B) &= 0.49/3 + 1/4 + 0.25/6,\\
\mathrm{DIL}(B) &= 0.84/3 + 1/4 + 0.7/6.
\end{aligned}
\tag{3.43}
$$

5. If $A_1, \ldots, A_n$ are fuzzy subsets of U, and $w_1, \ldots, w_n$ are nonnegative weights adding up to unity, then a *convex combination* of $A_1, \ldots, A_n$ is a fuzzy set A whose membership function is expressed by

$$\mu_A = w_1 \mu_{A_1} + \cdots + w_n \mu_{A_n}, \tag{3.44}$$

where + denotes the arithmetic sum. The concept of a convex combination is useful in the representation of linguistic hedges such as *essentially, typically,* etc., which modify the weights associated with the components of a fuzzy set [27].

6. If $A_1, \ldots, A_n$ are fuzzy subsets of $U_1, \ldots, U_n$, respectively, the *Cartesian product* of $A_1, \ldots, A_n$ is denoted by $A_1 \times \cdots \times A_n$ and is defined as a fuzzy subset of $U_1 \times \cdots \times U_n$ whose membership function is expressed by

$$\mu_{A_1 \times \cdots \times A_n}(u_1, \ldots, u_n) = \mu_{A_1}(u_1) \wedge \cdots \wedge \mu_{A_n}(u_n). \tag{3.45}$$

Thus, we can write [see (3.52)]

$$A_1 \times \cdots \times A_n = \int_{U_1 \times \cdots \times U_n} [\mu_{A_1}(u_1) \wedge \cdots \wedge \mu_{A_n}(u_n)] / (u_1, \ldots, u_n). \tag{3.46}$$

Example 3.8. If $U_1 = U_2 = 3 + 5 + 7$, $A_1 = 0.5/3 + 1/5 + 0.6/7$ and $A_2 = 1/3 + 0.6/5$, then

$$
\begin{aligned}
A_1 \times A_2 = {}& 0.5/(3,3) + 1/(5,3) + 0.6/(7,3)\\
&+ 0.5/(3,5) + 0.6/(5,5) + 0.6/(7,5).
\end{aligned}
\tag{3.47}
$$

7. The operation of *fuzzification* has, in general, the effect of transforming a nonfuzzy set into a fuzzy set or increasing the fuzziness of a fuzzy set. Thus, a *fuzzifier* F applied to a fuzzy subset A of U yields a fuzzy subset $F(A;K)$ which is expressed by

$$F(A;K) = \int_U \mu_A(u)K(u), \tag{3.48}$$

where the fuzzy set $K(u)$ is the *kernel* of F, that is, the result of applying F to a singleton $1/u$:

$$K(u) = F(1/u;K); \tag{3.49}$$

$\mu_A(u)K(u)$ represents the product [in the sense of (3.39) of a scalar $\mu_A(u)$ and the fuzzy set $K(u)$; and $\int$ is the union of the family of fuzzy sets $\mu_A(u)K(u)$, $u \in U$. In effect, (3.48) is analogous to the integral representation of a linear operator, with $K(u)$ being the counterpart of the impulse response.

Example 3.9. Assume that U, A and $K(u)$ are defined by

$$\begin{aligned} U &= 1 + 2 + 3 + 4, \\ A &= 0.8/1 + 0.6/2, \\ K(1) &= 1/1 + 0.4/2, \\ K(2) &= 1/2 + 0.4/1 + 0.4/3. \end{aligned} \tag{3.50}$$

Then

$$\begin{aligned} F(A;K) &= 0.8(1/1 + 0.4/2) + 0.6(1/2 + 0.4/1 + 0.4/3) \\ &= 0.8/1 + 0.6/2 + 0.24/3. \end{aligned} \tag{3.51}$$

The operation of fuzzification plays an important role in the definition of linguistic hedges such as *more or less, slightly, somewhat, much,* etc. For example, if $A \triangleq$ *positive* is the label for the nonfuzzy class of positive numbers, then *slightly positive* is a label for a fuzzy subset of the real line whose membership function is of the form shown in Fig. 11. In this case, *slightly* is a fuzzifier which transforms *positive* into *slightly positive.* However, it is not always possible to express the effect of a fuzzifier in the form (3.48), and *slightly* is a case in point. A more detailed discussion of this and related issues may be found in [27].

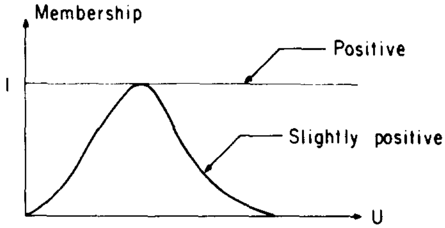


Fig. 11. Membership functions of *positive* and *slightly positive.*

FUZZY RELATIONS

If U is the Cartesian product of n universes of discourse $U_1, \ldots, U_n$, then an n-ary *fuzzy relation, R, in U* is a fuzzy subset of U. As in (3.20), R may be expressed as the union of its constituent fuzzy singletons $\mu_R(u_1, \ldots, u_n)/(u_1, \ldots, u_n)$, i.e.,

$$R = \int_{U_1 \times \cdots \times U_n} \mu_R(u_1, \ldots, u_n) / (u_1, \ldots, u_n), \tag{3.52}$$

where μ_R is the membership function of R.

Common examples of (binary) fuzzy relations are: *much greater than, resembles, is relevant to, is close to,* etc. For example, if $U_1 = U_2 = (-\infty, \infty)$, the relation *is close to* may be defined by

$$\textit{is close to} \triangleq \int_{U_1 \times U_2} e^{-a|u_1 - u_2|} / (u_1, u_2), \tag{3.53}$$

where a is a scale factor. Similarly, if $U_1 = U_2 = 1 + 2 + 3 + 4$, then the relation *much greater than* may be defined by the relation matrix

R	1	2	3	4
1	0	0.3	0.8	1
2	0	0	0	0.8
3	0	0	0	0.3
4	0	0	0	0

, (3.54)

in which the (i, j)th element is the value of $\mu_R(u_1, u_2)$ for the ith value of u_1 and jth value of u_2.

If R is a relation from U to V (or, equivalently, a relation in $U \times V$) and S is a relation from V to W, then the composition of R and S is a fuzzy relation from U to W denoted by $R \circ S$ and defined by[14]

$$R \circ S = \int_{U \times W} \vee_v \, [\mu_R(u, v) \wedge \mu_S(v, w)] \, /(u, w). \tag{3.55}$$

If U, V and W are finite sets, then the relation matrix for $R \circ S$ is the max-min product[15] of the relation matrices for R and S. For example, the max-min product of the relation matrices on the left-hand side of (3.56) is given by the right-hand side of (3.56):

$$\overset{R}{\begin{bmatrix} 0.3 & 0.8 \\ 0.6 & 0.9 \end{bmatrix}} \circ \overset{S}{\begin{bmatrix} 0.5 & 0.9 \\ 0.4 & 1 \end{bmatrix}} = \overset{R \circ S}{\begin{bmatrix} 0.4 & 0.8 \\ 0.5 & 0.9 \end{bmatrix}} \tag{3.56}$$

PROJECTIONS AND CYLINDRICAL FUZZY SETS

If R is an n-ary fuzzy relation in $U_1 \times \cdots \times U_n$, then its *projection (shadow)* on $U_{i_1} \times \cdots \times U_{i_k}$ is a k-ary fuzzy relation R_q in U which is defined by [compare with (2.12)]

$$\begin{aligned} R_q &\triangleq \text{Proj } R \text{ on } U_{i_1} \times \cdots \times U_{i_k} \\ &\triangleq P_q R \\ &\triangleq \int_{U_{i_1} \times \cdots \times U_{i_k}} [\vee_{u_{(q')}} \mu_R(u_1, \ldots, u_n)] \, /(u_{i_1}, \ldots, u_{i_k}), \end{aligned} \tag{3.57}$$

[14] Equation (3.55) defines the max-min composition of R and S. Max-product composition is defined similarly, except that $\wedge$ is replaced by the arithmetic product. A more detailed discussion of these compositions may be found in [24].

[15] In the max-min matrix product, the operations of addition and multiplication are replaced by $\vee$ and $\wedge$, respectively.

where q is the index sequence $(i_1, \ldots, i_k)$; $u_{(q)} \triangleq (u_{i_1}, \ldots, u_{i_k})$; q' is the complement of q; and $\bigvee_{u_{(q')}}$ is the supremum of $\mu_R(u_1, \ldots, u_n)$ over the u's which are in $u_{(q')}$. It should be noted that when R is a nonfuzzy relation, (3.57) reduces to (2.9).

Example 3.10. For the fuzzy relation defined by the relation matrix (3.54), we have

$$R_1 = 1/1 + 0.8/2 + 0.3/3$$

and

$$R_2 = 0.3/2 + 0.8/3 + 1/4.$$

It is clear that distinct fuzzy relations in $U_1 \times \cdots \times_n$ can have identical projections on $U_{i_1} \times \cdots \times U_{i_k}$. However, given a fuzzy relation R_q in $U_{i_1} \times \cdots \times U_{i_k}$, there exists a unique *largest*[16] relation $\bar{R}_q$ in $U_1 \times \cdots \times U_n$ whose projection on $U_{i_1} \times \cdots \times U_{i_k}$ is R_q. In consequence of (3.57), the membership function of $\bar{R}_q$ is given by

$$\mu_{\bar{R}_q}(u_1, \ldots, u_n) = \mu_{R_q}(u_{i_1}, \ldots, u_{i_k}), \tag{3.58}$$

with the understanding that (3.58) holds for all $u_1, \ldots, u_n$ such that the $i_1, \ldots, i_k$ arguments in $\mu_{\bar{R}_q}$ are equal, respectively, to the first, second, ..., kth arguments in μ_{R_q}. This implies that the value of $\mu_{\bar{R}_q}$ at the point $(u_1, \ldots, u_n)$ is the same as that at the point $(u'_1, \ldots, u'_n)$ provided that $u_{i_1} = u'_{i_1}, \ldots, u_{i_k} = u'_{i_k}$. For this reason, $\bar{R}_q$ will be referred to as the *cylindrical extension* of R_q, with R_q constituting the *base* of $\bar{R}_q$. (See Fig. 12.)

[16]That is, a relation which contains all other relations whose projection on $U_{i_1} \times \cdots \times U_{i_k}$ is R_q.

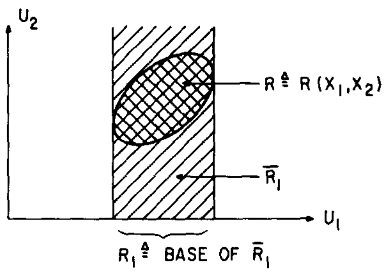


Fig. 12. R_1 is the base of the cylindrical set $\bar{R}_1$.

Suppose that R is an n-ary relation in $U_1 \times \cdots \times U_n$, R_q is its projection on $U_{i_1} \times \cdots \times U_{i_k}$, and $\bar{R}_q$ is the cylindrical extension of R_q. Since $\bar{R}_q$ is the largest relation in $U_1 \times \cdots \times U_n$ whose projection on $U_{i_1} \times \cdots \times U_{i_k}$ is R_q, it follows that R_q satisfies the *containment relation*

$$R \subset \bar{R}_q \tag{3.59}$$

for all q, and hence

$$R \subset \bar{R}_{q_1} \cap \bar{R}_{q_2} \cap \cdots \cap \bar{R}_{q_r} \tag{3.60}$$

for arbitrary $q_1, \ldots, q_r$ [index subsequences of $(1, 2, \ldots, n)$].

In particular, if we set $q_1 = 1, \ldots, q_r = n$, then (3.60) reduces to

$$R \subset \bar{R}_1 \cap \bar{R}_2 \cap \cdots \cap \bar{R}_n, \tag{3.61}$$

where $R_1, \ldots, R_n$ are the projections of R on $U_1, \ldots, U_n$, respectively, and $\bar{R}_1, \ldots, \bar{R}_n$ are their cylindrical extensions. But, from the definition of the Cartesian product [see (3.45)] it follows that

$$\bar{R}_1 \cap \cdots \cap \bar{R}_n = R_1 \times \cdots \times R_n, \tag{3.62}$$

which leads us to the

PROPOSITION 3.1. *If R is an n-ary fuzzy relation in $U_1 \times \cdots \times U_n$ and $R_1, \ldots, R_n$ are its projections on $U_1, \ldots, U_n$, then (see Fig. 13 for illustration)*

$$R \subset R_1 \times \cdots \times R_n. \tag{3.63}$$

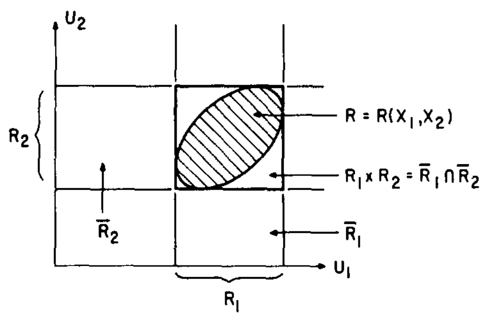


Fig. 13. Relation between the Cartesian product and intersection of cylindrical sets.

The concept of a cylindrical extension can also be used to provide an intuitively appealing interpretation of the composition of fuzzy relations. Thus, suppose that R and S are binary fuzzy relations in $U_1 \times U_2$ and $U_2 \times U_3$, respectively. Let $\bar{R}$ and $\bar{S}$ be the cylindrical extensions of R and S in $U_1 \times U_2 \times U_3$. Then, from the definition of $R \circ S$ [see (3.55)] it follows that

$$R \circ S = \text{Proj } \bar{R} \cap \bar{S} \text{ on } U_1 \times U_3. \tag{3.64}$$

If R and S are such that

$$\text{Proj } R \text{ on } U_2 = \text{Proj } S \text{ on } U_2, \tag{3.65}$$

then $\bar{R} \cap \bar{S}$ becomes the *join*[17] of R and S. A basic property of the join of R and S may be stated as

[17]The concept of the join of nonfuzzy relations was introduced by E. F. Codd in [28].

PROPOSITION 3.2. *If R and S are fuzzy relations in* $U_1 \times U_2$ *and* $U_2 \times U_3$*, respectively, and* $\bar{R} \cap \bar{S}$ *is the join of R and S, then*

$$R = \text{Proj } \bar{R} \cap \bar{S} \text{ on } U_1 \times U_2 \tag{3.66}$$

and

$$S = \text{Proj } \bar{R} \cap \bar{S} \text{ on } U_2 \times U_3. \tag{3.67}$$

Thus, R and S can be retrieved from the join of R and S.

Proof. Let μ_R and μ_S denote the membership functions of R and S, respectively. Then the right-hand sides of (3.66) and (3.67) translate into

$$\vee_{u_3} [\mu_R(u_1,u_2) \wedge \mu_S(u_2,u_3)] \tag{3.68}$$

and

$$\vee_{u_1} [\mu_R(u_1,u_2) \wedge \mu_S(u_2,u_3)]. \tag{3.69}$$

In virtue of the distributivity and commutativity of $\vee$ and $\wedge$, (3.68) and (3.69) may be rewritten as

$$\mu_R(u_1,u_2) \wedge [\vee_{u_3} \mu_S(u_2,u_3)] \tag{3.70}$$

and

$$\mu_S(u_2,u_3) \wedge [\vee_{u_1} \mu_R(u_1,u_2)]. \tag{3.71}$$

Furthermore, the definition of the join implies (3.65) and hence that

$$\vee_{u_1} \mu_R(u_1,u_2) = \vee_{u_3} \mu_S(u_2,u_3). \tag{3.72}$$

From this equality and the definition of $\vee$ it follows that

$$\mu_R(u_1,u_2) \leqslant \vee_{u_1} \mu_R(u_1,u_2) = \vee_{u_3} \mu_S(u_2,u_3) \tag{3.73}$$

and

$$\mu_S(u_2,u_3) \leqslant \vee_{u_3}\mu_S(u_2, u_3) = \vee_{u_1}\mu_R(u_1,u_2). \tag{3.74}$$

Consequently

$$\mu_R(u_1,u_2) \wedge [\vee_{u_3}\mu_S(u_2,u_3)] = \mu_R(u_1,u_2) \tag{3.75}$$

and

$$\mu_S(u_2,u_3) \wedge [\vee_{u_1}\mu_R(u_1,u_3)] = \mu_S(u_2,u_3), \tag{3.76}$$

which translate into (3.66) and (3.67). Q.E.D.

A basic property of projections which we shall have an occasion to use in Part II, Sec. 1 is the following.

PROPOSITION 3.3. *If R is a normal relation* [*see* (*3.23*)], *then so is every projection of R.*

Proof. Let R be an n-ary relation in $U_1 \times \cdots \times U_n$, and let R_q be its projection (shadow) on $U_{i_1} \times \cdots \times U_{i_k}$, with $q = (i_1, \ldots, i_k)$. Since R is normal, we have by (3.23),

$$\vee_{(u_1,\ldots,u_n)}\mu_R(u_1,\ldots,u_n) = 1, \tag{3.77}$$

or more compactly

$$\vee_u \mu_R(u) = 1.$$

On the other hand, by the definition of R_q [see (3.57)],

$$\mu_{R_q}(u_{i_1},\ldots,u_{i_k}) = \vee_{(u_{j_1},\ldots,u_{j_m})}\mu_R(u_1,\ldots,u_n),$$

or

$$\mu_{R_q}(u_{(q)}) = \vee_{u_{(q')}}\mu_R(u),$$

and hence the height of R_q is given by

$$\vee_{u_{(q)}} \mu_{R_q}(u_{(q)}) = \vee_{u_{(q)}} \vee_{u_{(q')}} \mu_R(u) \tag{3.78}$$

$$= \vee_u \mu_R(u)$$

$$= 1. \qquad \text{Q.E.D.}$$

THE EXTENSION PRINCIPLE

The *extension principle* for fuzzy sets is in essence a basic identity which allows the domain of the definition of a mapping or a relation to be extended from points in U to fuzzy subsets of U. More specifically, suppose that f is a mapping from U to V, and A is a fuzzy subset of U expressed as

$$A = \mu_1 u_1 + \cdots + \mu_n u_n. \tag{3.79}$$

Then the extension principle asserts that[18]

$$f(A) = f(\mu_1 u_1 + \cdots + \mu_n u_n) \equiv \mu_1 f(u_1) + \cdots + \mu_n f(u_n). \tag{3.80}$$

Thus, the image of A under f can be deduced from the knowledge of the images of $u_1, \ldots, u_n$ under f.

Example 3.11. Let

$$U = 1 + 2 + \cdots + 10,$$

and let f be the operation of squaring. Let *small* be a fuzzy subset of U defined by

$$small = 1/1 + 1/2 + 0.8/3 + 0.6/4 + 0.4/5. \tag{3.81}$$

Then, in consequence of (3.80), we have [19]

$$small^2 = 1/1 + 1/4 + 0.8/9 + 0.6/16 + 0.4/25. \tag{3.82}$$

If the support of A is a continuum, that is,

---

[18]The extension principle is implicit in a result given in [29]. In probability theory, the extension principle is analogous to the expression for the probability distribution induced by a mapping [30]. In the special case of intervals, the results of applying the extension principle reduced to those of interval analysis [31].

[19]Note that this definition of $small^2$ differs from that of (3.38).

$$A = \int_U \mu_A(u)/u, \tag{3.83}$$

then the statement of the extension principle assumes the following form:

$$f(A) = f\left(\int_U \mu_A(u)/u\right) \equiv \int_V \mu_A(u)/f(u), \tag{3.84}$$

with the understanding that $f(u)$ is a point in V and $\mu_A(u)$ is its grade of membership in $f(A)$, which is a fuzzy subset of V.

In some applications it is convenient to use a modified form of the extension principle which follows from (3.84) by decomposing A into its constituent level-sets rather than its fuzzy singletons [see the resolution identity (3.28)]. Thus, on writing

$$A = \int_0^1 \alpha A_\alpha, \tag{3.85}$$

where A_α is an α-level set of A, the statement of the extension principle assumes the form

$$f(A) = f\left(\int_0^1 \alpha A_\alpha\right) \equiv \int_0^1 \alpha f(A_\alpha) \tag{3.86}$$

when the support of A is a continuum, and

$$f(A) = f\left(\sum_\alpha \alpha A_\alpha\right) = \sum_\alpha \alpha f(A_\alpha) \tag{3.87}$$

when either the support of A is a countable set or the distinct level-sets of A form a countable collection.

COMMENT 3.2. Written in the form (3.84), the extension principle extends the domain of definition of f from points in U to fuzzy subsets of U. By contrast, (3.86) extends the domain of definition of f from nonfuzzy subsets of U to fuzzy subsets of U. It should be clear, however, that (3.84) and (3.86) are equivalent, since (3.86) results from (3.84) by a regrouping of terms in the representation of A.

COMMENT 3.3. The extension principle is analogous to the superposition principle for linear systems. Under the latter principle, if L is a linear system and $u_1, \ldots, u_n$ are inputs to L, then the response of L to any linear combination

$$u = w_1 u_1 + \cdots + w_n u_n, \tag{3.88}$$

where the w_i are constant coefficients, is given by

$$L(u) = L(w_1 u_1 + \cdots + w_n u_n) = w_1 L(u_1) + \cdots + w_n L(u_n). \tag{3.89}$$

The important point of difference between (3.89) and (3.80) is that in (3.80) + is the union rather than the arithmetic sum, and f is not restricted to linear mappings.

COMMENT 3.4. It should be noted that when $A = u_1 + \cdots + u_n$, the result of applying the extension principle is analogous to that of forming the n-fold Cartesian product of the algebraic system (U,f) with itself. (An extension of the multiplication table is shown in Table 3.1.)

X	1	2	3	4	1v2	2v4
1	1	2	3	4	1v2	2v4
2	2	4	6	8	1v4	4v8
3	3	6	9	12	3v6	6v12
4	1	8	12	16	4v8	8v16
1v2	1v2	2v4	3v6	4v8	1v2v4	2v4v8

3 v 5 v 6
x
2 v 4 v 6

---

6 v 10 v 12
12 v 20 v 24
18 v 30 v 36

---

6 v 10 v 12 v 18 v 20 v 24 v 30 v 36

Table 1. Extension of the multiplication table to subsets of integers. $1 \vee 2$ means 1 or 2.

In many applications of the extension principle, one encounters the following problem. We have an n-ary function, f, which is a mapping from a Cartesian product $U_1 \times \cdots \times U_n$ to a space V, and a fuzzy set (relation) A in $U_1 \times \cdots \times U_n$ which is characterized by a membership function $\mu_A(u_1, \ldots, u_n)$, with u_i, $i = 1, \ldots, n$, denoting a generic point in U_i. A direct application of the extension principle (3.84) to this case yields

$$f(A) = f\left(\int_{U_1 \times \cdots \times U_n} \mu_A(u_1, \ldots, u_n)/(u_1, \ldots, u_n)\right) \quad (3.90)$$

$$= \int_V \mu_A(u_1, \ldots, u_n)/f(u_1, \ldots, u_n).$$

However, in many instances what we know is not A but its projections $A_1, \ldots, A_n$ on $U_1, \ldots, U_n$, respectively [see (3.57)]. The question that arises, then, is: What expression for μ_A should be used in (3.90)?

In such cases, unless otherwise specified we shall assume that the membership function of A is expressed by

$$\mu_A(u_1, \ldots, u_n) = \mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2) \wedge \cdots \wedge \mu_{A_n}(u_n), \quad (3.91)$$

where μ_{A_i}, $i = 1, \ldots, n$, is the membership function of A_i. In view of (3.45), this is equivalent to assuming that A is the Cartesian product of its projections, i.e.,

$$A = A_1 \times \cdots \times A_n,$$

which in turn implies that A is the largest set whose projections on $U_1, \ldots, U_n$ are $A_1, \ldots, A_n$, respectively. [See (3.63).]

Example 3.12. Suppose that, as in Example 3.11,

$$U_1 = U_2 = 1 + 2 + 3 + \cdots + 10$$

and

$$A_1 = \underset{\sim}{2} \triangleq \textit{approximately } 2 = 1/2 + 0.6/1 + 0.8/3, \quad (3.92)$$

$$A_2 = \underset{\sim}{6} \triangleq \textit{approximately } 6 = 1/6 + 0.8/5 + 0.7/7 \quad (3.93)$$

and

$$f(u_1, u_2) = u_1 \times u_2 = \text{arithmetic product of } u_1 \text{ and } u_2.$$

Using (3.91) and applying the extension principle as expressed by (3.90) to this case, we have

$$\underset{\sim}{2} \times \underset{\sim}{6} = (1/2 + 0.6/1 + 0.8/3) \times (1/6 + 0.8/5 + 0.7/7)$$
$$= 1/12 + 0.8/10 + 0.7/14 + 0.6/6 + 0.6/5 + 0.6/7$$
$$+ 0.8/18 + 0.8/15 + 0.7/21$$
$$= 0.6/5 + 0.6/6 + 0.6/7 + 0.8/10 + 1/12 + 0.7/14$$
$$+ 0.8/15 + 0.8/18 + 0.7/21. \tag{3.94}$$

Thus, the arithmetic product of the fuzzy numbers *approximately* 2 and *approximately* 6 is a fuzzy number given by (3.94).

More generally, let $*$ be a binary operation defined on $U \times V$ with values in W. Thus, if $u \in U$ and $v \in V$, then

$$w = u * v, \qquad w \in W$$

Now suppose that A and B are fuzzy subsets of U and V, respectively, with

$$A = \mu_1 u_1 + \cdots + \mu_n u_n,$$
$$B = \nu_1 v_1 + \cdots + \nu_m v_m. \tag{3.95}$$

By using the extension principle under the assumption (3.91), the operation $*$ may be extended to fuzzy subsets of U and V by the defining relation

$$A * B = \left(\sum_i \mu_i u_i\right) * \left(\sum_j \nu_j v_j\right)$$
$$= \sum_{i,j} (\mu_i \wedge \nu_j)(u_i * v_j). \tag{3.96}$$

It is easy to verify that for the case where $A = \underset{\sim}{2}$, $B = \underset{\sim}{6}$ and $* = \times$, as in Example 3.12, the application of (3.96) yields the expression for $\underset{\sim}{2} \times \underset{\sim}{6}$.

COMMENT 3.5. It is important to note that the validity of (3.96) depends in an essential way on the assumption (3.91), that is,

$$\mu_{(A,B)}(u, v) = \mu_A(u) \wedge \mu_B(v).$$

The implication of this assumption is that u and v are noninteractive in the sense of Definition 2.2. Thus, if there is a constraint on (u, v) which is expressed as a relation R with a membership function μ_R, then the expression for $A * B$ becomes

$$A * B = \left[\left(\sum_i \mu_i u_i\right) * \left(\sum_j \nu_j v_j\right)\right] \cap R$$
$$= \sum_{i,j} [\mu_i \wedge \nu_j \wedge \mu_R(u_i, v_j)]\,(u_i * v_j). \tag{3.97}$$

Note that if R is a nonfuzzy relation, then the right-hand side of (3.97) will contain only those terms which satisfy the constraint R.

A simple illustration of a situation in which u and v are interactive is provided by the expression

$$w = z \times (x+y), \tag{3.98}$$

in which $+ \triangleq$ arithmetic sum and $\times \triangleq$ arithmetic product. If x, y and z are noninteractive, then we can apply the extension principle in the form (3.96) to the computation of $A \times (B + C)$, where A, B and C are fuzzy subsets of the real line. On the other hand, if (3.98) is rewritten as

$$w = z \times x + z \times y,$$

then the terms $z \times x$ and $z \times y$ are interactive by virtue of the common factor z, and hence

$$A \times (B + C) \neq A \times B + A \times C. \tag{3.99}$$

A significant conclusion that can be drawn from this observation is that the product of fuzzy numbers is not distributive if it is computed by the use of (3.96). To obtain equality in (3.99), we may apply the unrestricted form of the extension principle (3.96) to the left-hand side of (3.99), and must apply the restricted form (3.97) to its right-hand side.

REMARK 3.1. The extension principle can be applied not only to functions, but also to relations or, equivalently, to predicates. We shall not discuss this subject here, since the application of the extension principle to relations does not play a significant role in the present paper.

FUZZY SETS WITH FUZZY MEMBERSHIP FUNCTIONS

Our consideration of fuzzy sets with fuzzy membership functions is motivated by the close association which exists between the concept of a linguistic truth with truth-values such as *true, quite true, very true, more or less true,* etc., on the one hand, and fuzzy sets in which the grades of membership are specified in linguistic terms such as *low, medium, high, very low, not low and not high,* etc., on the other.

Thus, suppose that A is a fuzzy subset of a universe of discourse U, and the values of the membership function, μ_A, of A are allowed to be fuzzy subsets of the interval [0, 1]. To differentiate such fuzzy sets from those considered previously, we shall refer to them as fuzzy sets of *type* 2, with the fuzzy sets whose membership functions are mappings from U to [0, 1] classified as *type* 1. More generally:

DEFINITION 3.1. A fuzzy set is of type n, $n = 2, 3, \ldots$, if its membership function ranges over fuzzy sets of type n-1. The membership function of a fuzzy set of type 1 ranges over the interval [0, 1].

To define such operations as complementation, union, intersection, etc. for fuzzy sets of type 2, it is natural to make use of the extension principle. It is convenient, however, to accomplish this in two stages: first, by extending the type 1 definitions to fuzzy sets with interval-valued membership functions; and second, generalizing from intervals to fuzzy sets[20] by the use of the level-set form of the extension principle [see (3.86)]. In what follows, we shall illustrate this technique by extending to fuzzy sets of type 2 the concept of intersection—which is defined for fuzzy sets of type 1 by (3.35).

Our point of departure is the expression for the membership function of the intersection of A and B, where A and B are fuzzy subsets of type 1 of U:

$$\mu_{A \cap B}(u) = \mu_A(u) \wedge \mu_B(u), \qquad u \in U.$$

Now if $\mu_A(u)$ and $\mu_B(u)$ are intervals in [0, 1] rather than points in [0, 1]—that is, for a fixed u,

$$\mu_A(u) = [a_1, a_2],$$

$$\mu_B(u) = [b_1, b_2],$$

where a_1, a_2, b_1 and b_2 depend on u—then the application of the extension principle (3.86) to the function $\wedge$(Min) yields

$$[a_1, a_2] \wedge [b_1, b_2] = [a_1 \wedge b_1, a_2 \wedge b_2]. \tag{3.100}$$

Thus, if A and B have interval-valued membership functions as shown in Fig. 14, then their intersection is an interval-valued curve whose value for each u is given by (3.100).

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[20] We are tacitly assuming that the fuzzy sets in question are convex, that is, have intervals as level-sets (see [29]). Only minor modifications are needed when the sets are not convex.

Next, let us consider the case where, for each u, $\mu_A(u)$ and $\mu_B(u)$ are fuzzy subsets of the interval [0, 1]. For simplicity, we shall assume that these subsets are *convex,* that is, have intervals as level-sets. In other words, we shall assume that, for each α in [0, 1], the α-level sets of μ_A and μ_B are interval-valued membership functions. (See Fig. 15).

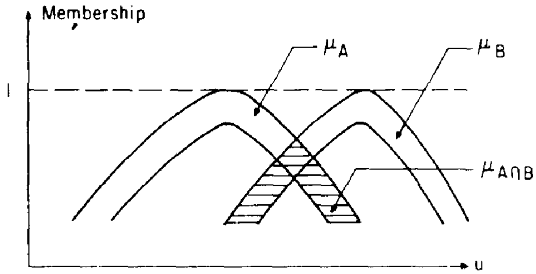

Fig. 14. Intersection of fuzzy sets with interval-valued membership functions.

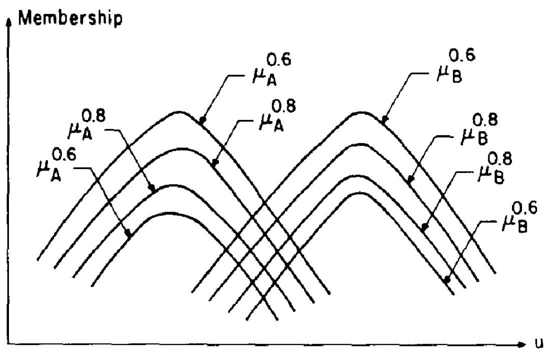

Fig. 15. Level-sets of fuzzy membership functions μ_A and μ_B.

By applying the level-set form of the extension principle (3.86) to the α-level sets of μ_A and μ_B we are led to the following definition of the intersection of fuzzy sets of type 2.

DEFINITION 3.2. Let A and B be fuzzy subsets of type 2 of U such that, for each $u \in U$, $\mu_A(u)$ and $\mu_B(u)$ are convex fuzzy subsets of type 1 of [0, 1], which implies that, for each α in [0, 1], the α-level sets of the fuzzy membership functions μ_A and μ_B are interval-valued membership functions μ_A^α and μ_B^α.

Let the α-level set of the fuzzy membership function of the intersection of A and B be denoted by $\mu_{A \cap B}^\alpha$, with the α-level sets μ_A^α and μ_B^β defined for each u by

$$\mu_A^\alpha \triangleq \{v \mid \nu_A(v) \geqslant \alpha\}, \tag{3.101}$$

$$\mu_B^\alpha \triangleq \{v \mid \nu_B(v) \geqslant \alpha\}, \tag{3.102}$$

where $\nu_A(v)$ denotes the grade of membership of a point v, $v \in [0, 1]$, in the fuzzy set $\mu_A(u)$, and likewise for μ_B. Then, for each u,

$$\mu_{A \cap B}^\alpha = \mu_A^\alpha \wedge \mu_B^\alpha \tag{3.103}$$

In other words, the α-level set of the fuzzy membership function of the intersection of A and B is the minimum [in the sense of (3.100)] of the α-level sets of the fuzzy membership functions of A and B. Thus, using the resolution identity (3.28), we can express $\mu_{A \cap B}$ as

$$\mu_{A \cap B} = \int_0^1 \alpha(\mu_A^\alpha \wedge \mu_B^\alpha). \tag{3.104}$$

For the case where μ_A and μ_B have finite supports, that is, μ_A and μ_B are of the form

$$\mu_A = \alpha_1 v_1 + \cdots + \alpha_n v_n, \qquad v_i \in [0, 1], \quad i = 1, \ldots, n \tag{3.105}$$

and

$$\mu_B = \beta_1 w_1 + \cdots + \beta_m w_m, \qquad w_j \in [0, 1], \quad j = 1, \ldots, m, \tag{3.106}$$

where α_i and β_j are the grades of membership of v_i and w_j in μ_A and μ_B, respectively, the expression for $\mu_{A \cap B}$ can readily be derived by employing the extension principle in the form (3.96). Thus, by applying (3.96) to the operation $\wedge$ ($\triangleq$ Min), we obtain at once

$$\begin{aligned} \mu_{A \cap B} &= \mu_A \wedge \mu_B \\ &= (\alpha_1 v_1 + \cdots + \alpha_n v_n) \wedge (\beta_1 w_1 + \cdots + \beta_m w_m) \qquad (3.107) \\ &= \sum_{i,j} (\alpha_i \wedge \beta_j)(v_i \wedge w_j) \end{aligned}$$

as the desired expression for $\mu_{A \cap B}$.[21]

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[21] Actually, Definition 3.2 can be deduced from (3.90).

Example 3.13. As a simple illustration of (3.104), suppose that at a point u the grades of membership of u in A and B are labeled as *high* and *medium*, respectively, with *high* and *medium* defined as fuzzy subsets of $V = 0 + 0.1 + 0.2 + \cdots + 1$ by the expressions

$$high = 0.8/0.8 + 0.8/0.9 + 1/1, \tag{3.108}$$

$$medium = 0.6/0.4 + 1/0.5 + 0.6/0.6. \tag{3.109}$$

The level sets of *high* and *medium* are expressed by

$$\begin{aligned} high_{0.6} &= 0.8 + 0.9 + 1, \\ high_{0.8} &= 0.8 + 0.9 + 1, \\ high_{1} &= 1, \\ medium_{0.6} &= 0.4 + 0.5 + 0.6, \\ medium_{1} &= 0.5, \end{aligned}$$

and consequently the α-level sets of the intersection are given by

$$\begin{aligned} \mu^{0.6}_{A \cap B}(u) &= high_{0.6} \wedge medium_{0.6} \\ &= (0.8 + 0.9 + 1) \wedge (0.4 + 0.5 + 0.6) \\ &= 0.4 + 0.5 + 0.6, \end{aligned} \tag{3.110}$$

$$\begin{aligned} \mu^{0.8}_{A \cap B}(u) &= high_{0.8} \wedge medium_{0.8} \\ &= (0.8 + 0.9 + 1) \wedge 0.5 \\ &= 0.5 \end{aligned} \tag{3.111}$$

and

$$\begin{aligned} \mu^{1}_{A \cap B}(u) &= high_{1} \wedge medium_{1} \\ &= 1 \wedge 0.5 \\ &= 0.5 \end{aligned} \tag{3.112}$$

Combining (3.110), (3.111) and (3.112), the fuzzy set representing the grade of membership of u in the intersection of A and B is found to be

$$\mu_{A \cap B}(u) = 0.6/(0.4 + 0.5 + 0.6) + 1/0.5$$

$$= medium, \qquad (3.113)$$

which is equivalent to the statement

$$high \wedge medium = medium. \qquad (3.114)$$

The same result can be obtained more expeditiously by the use of (3.107). Thus, we have

$$high \wedge medium = (0.8/0.8 + 0.8/0.9 + 1/1) \wedge (0.6/0.4 + 1/0.5 + 0.6/0.6)$$

$$= 0.6/0.4 + 1/0.5 + 0.6/0.6$$

$$= medium. \qquad (3.115)$$

In a similar fashion, we can extend to fuzzy sets of type 2 the operations of complementation, union, concentration, etc. This will be done in Part II, Sec. 3, in conjunction with our discussion of a fuzzy logic in which the truth-values are linguistic in nature.

REMARK 3.2. The results derived in Example 3.13 may be viewed as an instance of a general conclusion that can be drawn from (3.100) concerning an extension of the inequality $\leqslant$ from real numbers to fuzzy subsets of the real line. Specifically, in the case of real numbers a, b, we have the equivalence

$$a \leqslant b \quad \Leftrightarrow \quad a \wedge b = a. \qquad (3.116)$$

Using this as a basis for the extension of $\leqslant$ to intervals, we have in virtue of (3.100),

$$[a_1, a_2] \leqslant [b_1, b_2] \quad \Leftrightarrow \quad a_1 \leqslant b_1 \quad \text{and} \quad a_2 \leqslant b_2. \qquad (3.117)$$

This, in turn, leads us to the following definition.

DEFINITION 3.3. Let A and B be convex fuzzy subsets of the real line, and let A_α and B_α denote the α-level sets of A and B, respectively. Then an extension of the inequality $\leqslant$ to convex fuzzy subsets of the real line is expressed by[22]

$$A \leqslant B \Leftrightarrow A \wedge B = A \tag{3.118}$$

$$\Leftrightarrow A_\alpha \wedge B_\alpha = A_\alpha \qquad \text{for all } \alpha \text{ in } [0, 1], \tag{3.119}$$

where $A_\alpha \wedge B_\alpha$ is defined by (3.100).

In the case of Example 3.13, it is easy to verify by inspection that

$$\mathit{medium}_\alpha \leqslant \mathit{high}_\alpha \qquad \text{for all } \alpha \tag{3.120}$$

in the sense of (3.119), and hence we can conclude at once that

$$\mathit{medium} \wedge \mathit{high} = \mathit{medium}, \tag{3.121}$$

which is in agreement with (3.114).

[22]It can be readily be verified that $\leqslant$ as defined by (3.117) constitutes a partial ordering.

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