485 - a new upper bound for Morpion Solitaire

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Morpion Solitaire

- A single-player paper-and-pencil game

  The goal is to find longest possible sequence of valid moves.

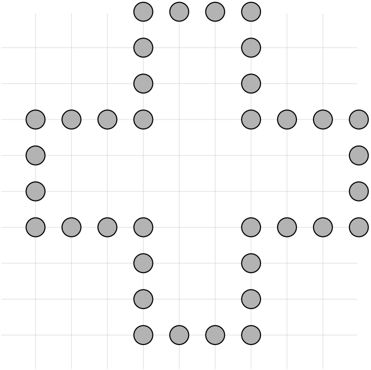
- Popularized in France in 70's by *Science & Vie* magazine

  The magazine published long sequences found by its readers.

- Difficult for computers.

  The human record was obtained by C.-H. Bruneau in 1976.
  It was beat by NRPA (Monte Carlo) algorithm by Ch. Rosin in 2010.
  Rosin got best paper award at IJCAI 2011 for this work.

Rules of Morpion Solitaire



Initial position: 36 dots arranged in a cross on a square grid.

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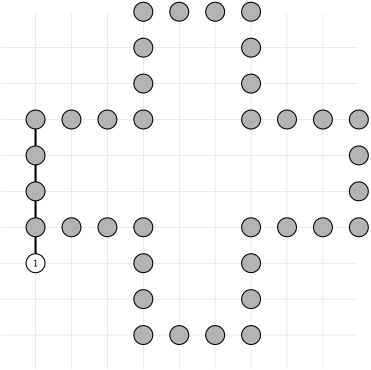
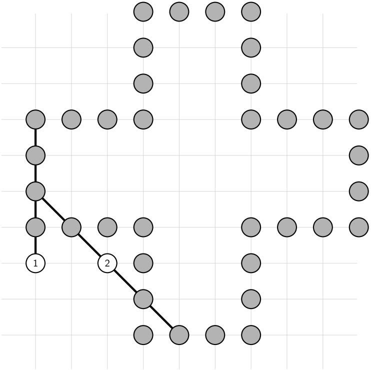


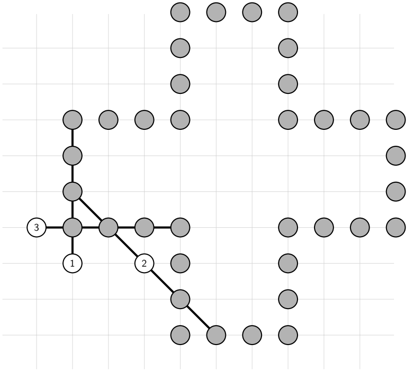
A move: place a dot and draw a line (diagonal, horizontal or vertical) passing through the placed dot and four others.

Rules of Morpion Solitaire



Two lines may cross, but they may not overlap.

Rules of Morpion Solitaire



Vertical, diagonal and horizontal moves allowed.

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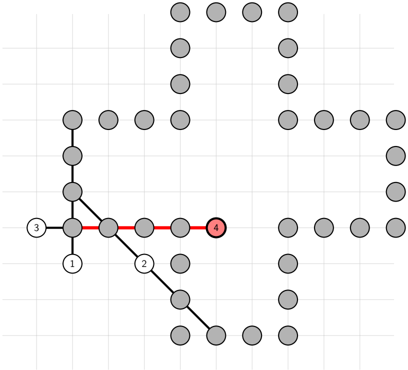
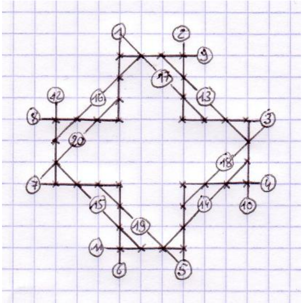


Moves may not overlap.

The goal

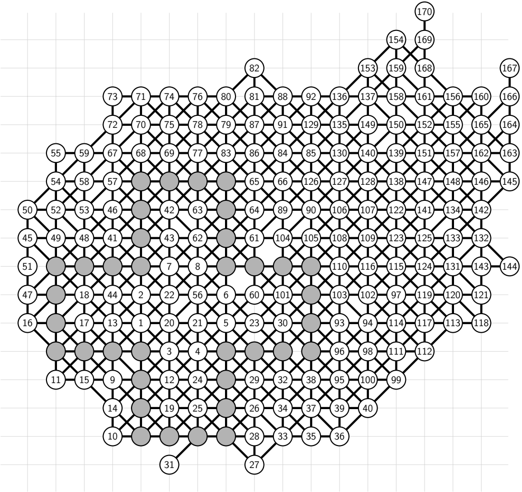
The goal is to find the longest sequence of moves.

Example of shortest possible sequence (20 moves):

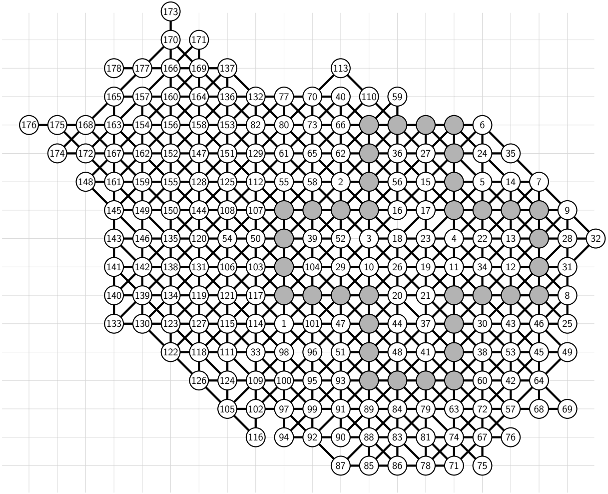


Ch. Boyer, morpionsolitaire.com

Record sequence of Bruneau (170 moves, 1974)



Record sequence of Rosin (178 moves, 2011)



Finiteness of the game

Is there and upper bound on the length of the sequence?

In 70's *Science & Vie* published different bounds (ranging from 540 to 20736) submitted by its readers, but without detailed and/or valid proofs.

The rigorous proof was published by Demaine et al in 2006. They obtained an upper bound of 705 moves.

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Our results

The rigorous proof was published by Demaine et al in 2006. They obtained an upper bound of 705 moves.

We prove an upper bound of 485.

The proof has two main ingredients:

- linearization of the problem;
- use of isoperimetric inequality to limit its size.

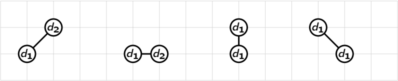
Linearization

The board

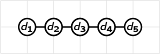
We fix part of the $\mathbb{Z}^2$ grid as the board on which Morpion Solitiare is played.

Vocabulary

- Dot, $d = (x, y)$. Let $\mathcal{D}$ denote set of all (possible) dots on the board.
- Segment, $s = \{d_1, d_2\}$ with d_1, d_2 adjacent dots. Let $\mathcal{S}$ denote set of all (possible) segments on the board.

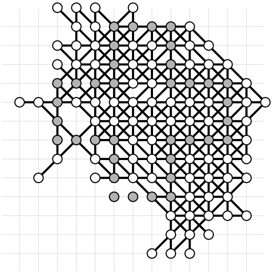


- Move, $m = \{\{d_1, d_2\}, \{d_2, d_3\}, \{d_3, d_4\}, \{d_4, d_5\}\}$. Let $\mathcal{M}$ denote set of all (possible) moves on the board.



Morpion position graph

For each dot d on board we define a binary variable dot_d.
For each move m on board we define a binary variable move_m.



Every Morpion position graph can be encoded using these variables:

- set dot_d to 1 iff the dot is a vertex of the graph
- set move_m to 1 iff move m was in the playout.

Properties of Morpion positions

We start with 36 dots and every move adds 1 dot.

$$36 + \sum_{m \in \mathcal{M}} \text{move}_m = \sum_{d \in \mathcal{D}} \text{dot}_d \ .$$

Moves do not overlap. For each segment $s \in \mathcal{S}$:

$$\sum_{m \in \mathcal{M}, s \in m} \text{move}_m \leq 1.$$

We can draw a line only through dots that are placed on board. For $m = \{\{d_1, d_2\}, \{d_2, d_3\}, \ldots, \{d_4, d_5\}\}$ and each $1 \leq i \leq 5$:

$$\text{move}_m \leq \text{dot}_{d_i}$$

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Linear Programming Problem

Variable bounds.

$$\forall_{m \in \mathcal{M}} \, \mathrm{move}_m \in [0,1], \forall_{d \in \mathcal{D}} \, \mathrm{dot}_d \in [0,1]$$

Constraints.

$$36 + \sum_{m \in \mathcal{M}} \mathrm{move}_m = \sum_{d \in \mathcal{D}} \mathrm{dot}_d$$

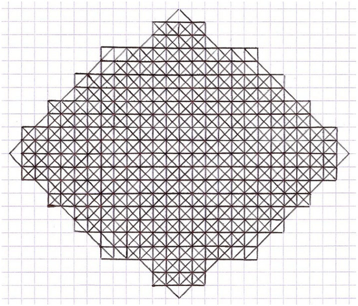
$$\forall_{s \in \mathcal{S}} \sum_{m \in \mathcal{M}, s \in m} \mathrm{move}_m \leq 1$$

$$\forall_{m \in \mathcal{M}, s \in m, d \in s} \, \mathrm{move}_m \leq \mathrm{dot}_d$$

Objective. Maximize $\sum_{m \in \mathcal{M}} \mathrm{move}_m$.

A note on solutions

The class of graphs that can be obtained as solutions of the above MIP contain all Morpion positions on the given board.
The converse is not true.



Grid of 317 moves found by Ch. Boyer

Problem of infinite board

The objective value of the LPP strongly depends on the size of the board.

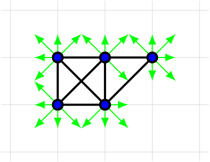
10	20	30	40	50	60	70	80	90	100
64.00	278.50	619.53	876.55	1130.01	1387.54	1641.74	1898.13	2152.86	2408.54

The top row contains the length n of the edge of a given square and the bottom row contains solutions to the relaxed problem on the $n \times n$ board.

We need to prove that every Morpion position is containted in a board that is not too big.

Isoperimetric Inequality

Area and boundary of a lattice graph



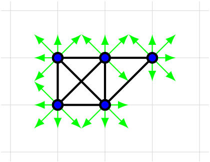
The area of the graph is the number of its vertices.

The boundary size of the graph is the number of **directed** segments with start point in a graph vertex and the endpoint in its complement.

Example: area 5, boundary size 24.

What we call boundary here is known in the literature as **potential** (and was used by Demaine et al in their proof of the upper bound).

Area and boundary of a lattice graph



In Morpion graphs, we have the equality (boundary size is constant)

$$\text{boundary } G = 36 \cdot 8.$$

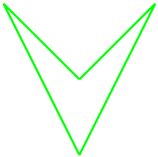
In each move:

- draw a dot (increases area by 1 and boundary by 8)
- draw a 4 (8 directed) segment line (decreases boundary by 8)

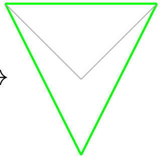
Isoperimetric problem

We fix the length of boundary. What is the maximal possible area?

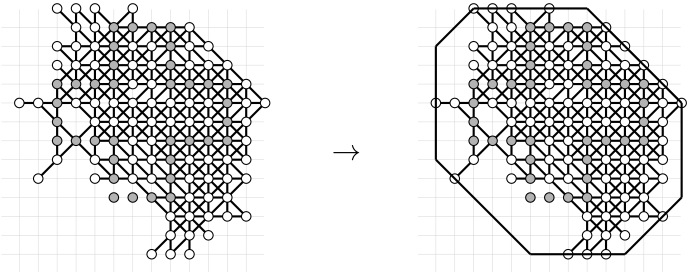
First step in the usual case: pass to the convex hull.



$\rightarrow$



Octagonal hull



We proved that passing to the octagonal hull of a Morpion position graph can increase its boundary size by up to 4.

Bound on the size of the board

Hence every Morpion position graph is contained in an octagonal board with boundary size up to 288 / 290 / 292.

You can enumerate all such boards - there are 126912 of them.

The largest one has area of 741 - with 36 starting dots this gives upper bound of 705 of Demaine et al.

The boards are small enough to get good linear relaxation bounds.

Linear Relaxation Bound

There are 126912 octagonal boards that contain all possible Morpion Solitaire positions.

The largest linear relaxation bound of 586.82353 was obtained for octagon with side lengths $10, 8, 10, 12, 10, 8, 10, 12$.

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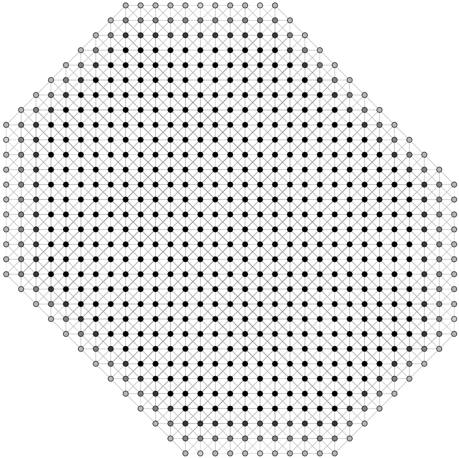
Linear relaxation bound



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The largest linear relaxation bound of 586.82353 was obtained for octagon with side lengths $10, 8, 10, 12, 10, 8, 10, 12$.

The total computation time was less than 24 hours on a single core of a modern processor.

The linear programming problems contain around 3000 variables and 6500 constraints each.

MIP Bound

Gurobi solver output for MIP version of the problem for the "hardest" case.

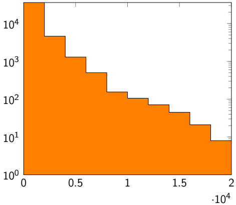
```
Root relaxation: objective 5.868235e+02, 6386 iterations, 7.81 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0  586.82353    0  228          -  586.82353      -      -    9s
     0     0  585.98492    0  227          -  585.98492      -      -   12s
     0     0  584.78440    0  235          -  584.78440      -      -   17s
     0     0  583.89727    0  236          -  583.89727      -      -   20s
     0     0  583.78764    0  236          -  583.78764      -      -   22s
     0     0  583.77255    0  236          -  583.77255      -      -   23s
     0     0  583.77255    0  236          -  583.77255      -      -   24s
     0     2  583.77255    0  236          -  583.77255      -      -   35s
     2     4  578.83917    1  271          -  582.45157      -  5671   67s
     3     5  578.73120    2  532          -  582.45106      -  6229   90s
     4     6  571.89240    2  528          -  578.83917      -  6442  114s
```

We can push the bound down by partially solving MIP versions of the linear problems with highest objectives.

MIP Bound



The vertical axis shows the number of cases and the horizontal axis shows the computation time in seconds.

Out of 126912 problems 42889 have relaxed bound bigger or equal to 485.

For 8 problems the computation time exceeded 5 hours.

The total computation time was approximately 310 days on a single core of a modern processor.

Source code

A lower upper bound may be found but the whole computation has to be repeated.

The source code is available on github.

http://github.com/anagorko/morpion-lpp/

Allows generation of four different relaxations (includes MIP problems whose all solutions are valid Morpion Solitaire games - much more difficult to solve), different variants of Morpion Solitaire, symmetry constraints, etc.

Thank you for your attention!